ON THE SURJECTIVITY OF THE SYMPLECTIC REPRESENTATION OF THE MAPPING CLASS GROUP

HYUNGRYUL BAIK, INHYEOK CHOI, AND DONGRYUL M. KIM

ABSTRACT. In this note, we study the symplectic representation of the mapping class group. In particular, we discuss the surjecivity of the representation restricted to certain mapping classes. It is well-known that the representation itself is surjective. In fact the representation is still surjective after restricting on pseudo-Anosov mapping classes. However, we show that the surjectivity does not hold when the representation is restricted on orientable pseudo-Anosovs, even after reducing its codomain to integer symplectic matrices with a bi-Perron leading eigenvalue. In order to prove the non-surjectivity, we explicitly construct an infinite family of symplectic matrices with a bi-Perron leading eigenvalue which cannot be obtained as the symplectic representation of an orientable pseudo-Anosov mapping class.

1. INTRODUCTION

Let S_g be the closed connected orientable surface of genus g. It is well known that the algebraic intersection number on $H_1(S_g; \mathbb{Z})$ extends to a symplectic form on $H_1(S_g; \mathbb{R})$ which is preserved under the action of the mapping class group $\mathrm{Mod}(S_g)$. Since $\mathrm{Mod}(S_g)$ preserves the lattice $H_1(S_g; \mathbb{Z})$ in $H_1(S_g; \mathbb{R})$, this gives us a representation $\Psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$, which is often called the symplectic representation of the mapping class group. The representation Ψ is surjective and the kernel is called the Torelli subgroup $\mathcal{I}_g$. For the proof of this fact and general background on the symplectic representation, consult Chapter 6 of [FM12].

In fact, the representation Ψ is still surjective even after restricting to the set of pseudo-Anosov elements (we will provide one argument in Section 3). On the other hand, it is not a priori clear wether Ψ remains surjective even when it is further restricted to the set of orientable pseudo-Anosov elements. Here we say a pseudo-Anosov mapping class orientable if its invariant measured foliation is orientable. Namely, one can ask the following question. [1]

Question 1.1. *When the symplectic representation Ψ of $\mathrm{Mod}(S)$ is restricted to the set of orientable pseudo-Anosovs, is Ψ surjective onto the set of symplectic matrices with a bi-Perron leading eigenvalue?*

---

[1] Question 1.1 was asked to the first author by Ursula Hamenstädt [Ham].

The reason for focusing on matrices with a bi-Perron leading eigenvalue is the following: for an orientable pseudo-Ansoov mapping class φ in $\mathrm{Mod}(S_g)$, the stretch factor (a.k.a. dilatation) λ_φ of φ coincides with the leading eigenvalue of the symplectic matrix $\Psi(\varphi)$ (Lemma 3.2). Since it is well-known result [Fri85] of Fried that stretch factor of pseudo-Anosov is a bi-Perron algebraic integer defined below (cf. [Do16]), the restriction

$$\left\{\varphi \in \mathrm{Mod}(S_g) : \begin{array}{c} \varphi \text{ is an orientable} \\ \text{pseudo-Anosov} \end{array}\right\} \xrightarrow{\Psi} \left\{A \in \mathrm{Sp}(2g, \mathbb{Z}) : \begin{array}{c} A \text{ has a bi-Perron} \\ \text{leading eigenvalue} \end{array}\right\}$$

is well-defined and thus we consider this restriction.

Definition 1.2 (bi-Perron algebraic integer)**.** *An algebraic integer $\lambda > 1$ is called* bi-Perron *if all the Galois conjugates of λ are contained in an annulus $\{z \in \mathbb{C} : 1/\lambda \le |z| \le \lambda\}$.*

Restriction of codomain in Question 1.1 is necessary. Indeed, it is remarked in [MS07] that Leininger has asserted the existence of cosets of $\mathcal{I}_g$ without orientable pseudo-Anosov representative. Let us consider the collection P of integer polynomials

$$q(x) = x^{2g} + a_{2g-1}x^{2g-1} + \cdots a_1 x + 1$$

which are palindromic (i.e. $a_k = a_{2g-k}$). We observe that P contains a polynomial with no real root. Indeed, for fixed $a_1, a_3, \cdots, a_{2g-3}$ and a_{2g-1}, $q(x) = 1 + x^{2g} + \sum_{k \ne 2, 2g-2} a_k x^k > 0$ outside of a compact subset of $\mathbb{R} - \{0\}$. Accordingly we can take $a_2 = a_{2g-2}$ large enough so that $q > 0$ on $\mathbb{R}$. For instance, $q(x) = x^4 + 10x^3 + 30x^2 + 10x + 1$ will do.

According to [Kir69], the map $\chi : \mathrm{Sp}(2g, \mathbb{Z}) \to P$ sending a matrix to its characteristic polynomial is surjective. Together with the surjectivity of $\Psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$, we conclude that q is a characteristic polynomial of $\Psi(f)$ for some $f \in \mathrm{Mod}(S_g)$. On the other hand, since q has no real root, none of representatives of a coset $f\mathcal{I}_g$ is an orientable pseudo-Anosov. Therefore, it concludes that there is no orientable pseudo-Anosov whose image is $\Psi(f)$.

However, this argument is based on the observation that there is a mapping class $f \in \mathrm{Mod}(S_g)$ such that $\Psi(f)$ has no real eigenvalue, which does not hold for typical characteristic polynomial of $\Psi(f)$ especially when $g = 2$.(Appendix A) As such, it is still unclear whether the symplectic representation is surjective on the set of orientable pseudo-Anosovs after restricting the codomain as above, and which symplectic matrices with a bi-Perron leading eigenvalue cannot arise as a representation of orientable pseudo-Anosovs.

Regarding bi-Perron algebraic integers and orientable pseudo-Anosovs, [BRW19] deals with a question whether typical bi-Perron algebraic integer comes from the stretch factor of orientable pseudo-Anosov. They proved

that

$$\lim_{R \to \infty} \frac{\left| \left\{ \lambda_\varphi : \begin{array}{c} \lambda_\varphi \leq R \text{ is the stretch factor} \\ \text{of orientable pseudo-Anosov } \varphi \in \mathrm{Mod}(S_g) \end{array} \right\} \right|}{\left| \left\{ \lambda : \begin{array}{c} \lambda \leq R \text{ is a bi-Perron algebraic integer} \\ \text{with characteristic polynomial of degree} \leq 2g \end{array} \right\} \right|} = 0$$

for fixed $g \geq 0$, suggesting that typical bi-Perron algebraic integer may not be realized as the stretch factor of orientable pseudo-Anosovs for fixed g. Based on this work, it is somewhat expected that Question 1.1 is not affirmative.

The main result of this note is that Question 1.1 indeed has negative answer in every genus ≥ 2. Furthermore, because our proof is constructive, we give a concrete way in Section 3 to construct an infinite family of symplectic matrices which have a bi-Perron leading eigenvalue but cannot be an image of orientable pseudo-Anosovs.

Theorem A. *For each genus $g \geq 2$, when the symplectic representation Ψ is restricted to the set of orientable pseudo-Anosovs, Ψ is not surjective onto the set of elements in $\mathrm{Sp}(2g, \mathbb{Z})$ whose leading eigenvalue is bi-Perron.*

There are some studies on algebraic properties of $H_1(S_g; \mathbb{R}) \to H_1(S_g; \mathbb{R})$ induced from an orientable pseudo-Anosov. For instance, McMullen and Thurston proved that its leading eigenvalue is simple.(Proposition 3.4) Our strategy is to show that there exists a symplectic matrix in $\mathrm{Sp}(2g, \mathbb{Z})$ whose leading eigenvalue is bi-Perron and is not simple.

In fact we show the following

Theorem B. *For each genus $g \geq 2$, there exists $A \in \mathrm{Sp}(2g, \mathbb{Z})$ with bi-Perron leading eigenvalue but none of its eigenvalues is simple. Indeed, there are infinitely many such matrices.*

As we described above, Theorem A immediately follows from Theorem B.

Remark 1.3. *Fried conjectured that the converse of Proposition 3.3 holds true. That is, every bi-Perron algebraic integer is obtained as the stretch factor of some pseudo-Anosov mapping classes. While Theorem A does not directly disprove the Fried's conjecture, it suggests that investigating the symplectic matrices with a fixed bi-Perron leading eigenvalue can lead to partial (negative) answer to Fried's conjecture, restricted on orientable pseudo-Anosovs. For another negative point of view toward Fried's conjecture, one can refer [BRW19].*

2. Symplectic representation of the mapping class group

In this section, we briefly review the symplectic representation of the mapping class group. Detailed discussion about it can be found in Chapter 6 of [FM12].

In order to study an algebraic aspect of $\mathrm{Mod}(S_g)$, we can observe its action on an appropriate space. Although it is still open whether $\mathrm{Mod}(S_g)$ is linear or not, regarding mapping classes as automorphisms of $H_1(S_g; \mathbb{R})$ allows us to have a sort of linear approximation

$$\Psi : \mathrm{Mod}(S_g) \to \mathrm{Aut}(H_1(S_g; \mathbb{R})).$$

As an algebraic intersection number $\hat{i} : H_1(S_g; \mathbb{Z}) \times H_1(S_g; \mathbb{Z}) \to \mathbb{Z}$ extends uniquely to a nondegenerate alternating bilinear form

$$\hat{i} : H_1(S_g; \mathbb{R}) \times H_1(S_g; \mathbb{R}) \to \mathbb{R},$$

we finally obtain a symplectic vector space $(H_1(S_g; \mathbb{R}), \hat{i})$.

Since $\mathrm{Mod}(S_g)$ acts on $H_1(S_g; \mathbb{R})$ preserving an algebraic intersection number, image of $\mathrm{Mod}(S_g)$ under Ψ lies in real symplectic group $\mathrm{Sp}(2g, \mathbb{R})$. In terms of matrices, the real symplectic group is described as follows:

$$\mathrm{Sp}(2g, \mathbb{R}) = \{A \in \mathrm{GL}(2g, \mathbb{R}) : A^t J A = J\}$$

where $J_{ij} = \delta_{i(j-1)} - \delta_{(i-1)j}$.

Moreover, $\mathrm{Mod}(S_g)$ preserves the integral lattice $H_1(S_g; \mathbb{Z})$ of $H_1(S_g; \mathbb{R})$, and thus the image of $\mathrm{Mod}(S_g)$ under Ψ should lies in $\mathrm{GL}(2g, \mathbb{Z})$. Denoting $\mathrm{Sp}(2g, \mathbb{Z}) = \mathrm{Sp}(2g, \mathbb{R}) \cap \mathrm{GL}(2g, \mathbb{Z})$, we finally obtain the following symplectic representation of the mapping class group.

$$\Psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$$

Having Ψ, one can further ask what can we say about its kernel and image. We say $\ker \Psi \leq \mathrm{Mod}(S_g)$ a Torelli (sub)group and denote $\mathcal{I}_g$. Then it is natural to ask whether $\mathcal{I}_g$ is trivial or not. Regarding this question, Thurston proved that $\mathcal{I}_g$ is not only nontrivial but also containing pseudo-Anosov elements.

Theorem 2.1 (Thurston, [T$^+$88]). *There exists a pseudo-Anosov element in the Torelli group $\mathcal{I}_g$.*

Proof. It is a direct application of Thurston construction of pseudo-Anosovs.
$\square$

While injectivity of Ψ does not hold, it is affirmative that Ψ is surjective.

Theorem 2.2. $\Psi : \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ *is surjective.*

Further question about surjectivity is whether it is still surjective after we restrict Ψ on the set of pseudo-Anosovs. If yes, one can further ask whether the surjectivity remains true after restricting Ψ on a particular subset of pseudo-Anosovs. These are major concerns in the next section.

3. Surjectivity on pseudo-Anosovs and Non-surjetivity on orientable pseudo-Anosovs

Before we discuss our proof of Theorem A, we first discuss the following proposition [2] which was mentioned in the introduction.

Proposition 3.1. *For each genus $g \geq 2$, the symplectic representation Ψ is still surjective when restricted to the set of pseudo-Anosov elements.*

Proof. Let $g \geq 2$ be fixed and pick an arbitrary element A of $\mathrm{Sp}(2g, \mathbb{Z})$. We want to show that there exists a pseudo-Anosov element of $\mathrm{Mod}(S_g)$ which gets mapped to A under Ψ.

Since Ψ is surjective, there exists $h \in \mathrm{Mod}(S_g)$ such that $\Psi(h) = A$. We will use the fact that even if we compose h with any element in the Torelli subgroup $\mathcal{I}_g$, it still gets mapped to A under Ψ.

Let f be any pseudo-Anosov element in $\mathcal{I}_g$, and let p, q be fixed points of f on the Thurston boundary of the Teichmüller space. We may assume that $h(p) \neq q$ (if not, just postcompose h with the Dehn twist along a separating curve so that $h(p)$ is no longer q but its image under Ψ is still A).

With out loss of generality, we may assume that p is the attracting fixed point of f. Take disjoint convex connected neighborhoods U of p and V of $h^{-1}(q)$. Then for large enough n,

$$(f^n \circ h)(U) \subseteq U \quad \text{and} \quad (f^n \circ h)(V) \supseteq V.$$

For detailed discussion on such dynamics, one can refer [Kid08]. Hence, by the Brouwer fixed point theorem, one gets fixed points $u \in U$ of $f^n \circ h$ and $v \in V$ of $(f^n \circ h)^{-1}$ and thus of $f^n \circ h$.

Since U and V are disjoint, $u \neq v$ so $f^n \circ h$ has (at least) two fixed points on the Thurston boundary, concluding that it is a pseudo-Anosov. As $f \in \mathcal{I}_g$, $\Psi(f^n \circ h) = \Psi(h) = A$ as desired. $\square$

From the previous proposition, our next question is whether the symplectic representation Ψ remains surjective after we restrict Ψ on the set of particular pseudo-Anosov mapping classes, orientable pseudo-Anosovs. Those pseudo-Anosovs have a nice algebraic property as automorphisms of the first homology group $H_1(S_g; \mathbb{R})$.

Lemma 3.2. *Let $g \geq 2$. For an orientable pseudo-Anosov $\varphi \in \mathrm{Mod}(S_g)$ with stretch factor $\lambda_\varphi > 1$, λ_φ is an eigenvalue of its symplectic representation $\Psi(\varphi)$.*

Proof. We can regard φ as its representative diffeomorphism. Then as φ is orientable, it has a invariant measured foliation induced from a closed 1-form ω, satisfying the following equation.

$$\varphi^* \omega = \lambda_\varphi^{-1} \omega$$

---

[2]Proposition 3.1 is well known to experts and there are many ways to prove it. For the sake of completeness, we provide one possible proof here.

One can show that its cohomology class $[\omega] \in H^1(S_g; \mathbb{R})$ is nontrivial. Henceforth, we have that $\varphi^* : H^1(S_g; \mathbb{R}) \to H^1(S_g; \mathbb{R})$ has an eigenvector $[\omega]$ with eigenvalue λ_φ^{-1}.

Now a naturality of Poincaré dual yields the following commutative diagram

$$
\begin{array}{ccc}
H^1(S_g; \mathbb{R}) & \xleftarrow{\varphi^*} & H^1(S_g; \mathbb{R}) \\
\downarrow{\scriptstyle PD} & & \downarrow{\scriptstyle PD} \\
H_1(S_g; \mathbb{R}) & \xrightarrow{\varphi_*} & H_1(S_g; \mathbb{R})
\end{array}
$$

where vertical map PD denotes the Poincaré dual. Therefore, we conclude that $PD[\omega]$ is an eigenvector of $\varphi_* : H_1(S_g; \mathbb{R}) \to H_1(S_g; \mathbb{R})$ with an eigenvalue λ_φ. Recalling that φ_* preserves the lattice $H_1(S_g; \mathbb{Z})$, it follows that λ_φ is an eigenvalue of $\Psi(\varphi)$. $\qquad\qquad\square$

Regarding the stretch factor of a pseudo-Anosov mapping class, its algebraic properties have also been studied. One well-known result is the following work of Fried, asserting that every such stretch factor should be a bi-Perron algebraic integer.

Proposition 3.3 (Fried, [Fri85])**.** *For $g \geq 2$ and a pseudo-Anosov mapping class $\varphi \in \mathrm{Mod}(S_g)$ with stretch factor $\lambda_\varphi > 1$, λ_φ is a bi-Perron algebraic integer.*

Combining above lemma and proposition, it follows that $\Psi(\varphi)$ should have a bi-Perron eigenvalue for orientable pseudo-Anosov φ. As such, it is clear that elements of $\mathrm{Sp}(2g, \mathbb{Z})$ without bi-Perron eigenvalue cannot be an image of orientable pseudo-Anosov under Ψ. This observation indicates the following.

(1) Our question is reduced to one pertaining to whether every element of $\mathrm{Sp}(2g, \mathbb{Z})$ with bi-Perron eigenvalue is $\Psi(\varphi)$ for some orientable pseudo-Anosov φ.
(2) To answer this question, we can investigate eigenvalues of elements in $\mathrm{Sp}(2g, \mathbb{Z})$.

Regarding the stretch factor $\lambda_\varphi > 1$ of orientable pseudo-Anosov, the following proposition is well-known. (cf. [LT11], [Kob11]) It was proved by Thurston and McMullen, and let us sketch here the idea of the proof, following [McM03].

Proposition 3.4 (McMullen)**.** *For $g \geq 2$ and an orientable pseudo-Anosov $\varphi \in \mathrm{Mod}(S_g)$, its stretch factor λ_φ is a simple eigenvalue of $\Psi(\varphi)$.*

Sketch of the Proof. It is a basic fact in linear algebra that A^t and A^{-1} are similar for $A \in \mathrm{Sp}(2g, \mathbb{R})$. Hence, A and A^{-1} have the same characteristic polynomial. Together with the commutative diagram induced from Poincaré

duality, as in the proof of Lemma 3.2, it suffices to show that λ_φ is a simple eigenvalue of $\varphi^* : H^1(S_g; \mathbb{R}) \to H^1(S_g; \mathbb{R})$.

In the same line of thought as Lemma 3.2, we have a cohomology class $[\alpha] \in H^1(S_g; \mathbb{R})$ such that

$$\varphi^*[\alpha] = \lambda_\varphi[\alpha].$$

Let $[\gamma] \in H^1(S_g; \mathbb{Z})$ be a cohomology class in the lattice whose Poincaré dual has a simple closed curve as its representative. Then from the choice of $[\gamma]$,

$$\frac{1}{\lambda_\varphi^n}(\varphi^*)^n[\gamma] \to a[\alpha] \qquad \text{as } n \to \infty$$

for some $a \in \mathbb{R}$. Since such $[\gamma]$'s span $H^1(S_g; \mathbb{Z})$, it follows that $[\alpha]$ is the unique eigenvector (up to scalar multiplication) with eigenvalue $\geq \lambda_\varphi$.

So far, we have proved that λ_φ is an eigenvalue of geometric multiplicity 1. Even though it is sufficient for our purpose, simpleness of λ_φ follows from observing the Jordan form for φ^*. Indeed, if λ_φ is not simple, the Jordan form yields a contradiction to $\lambda_\varphi^{-n}(\varphi^*)^n[\gamma] \to a[\alpha]$. $\qquad \square$

From the proof of Proposition 3.4, one may observe that the stretch factor $\lambda_\varphi > 1$ of an orientable pseudo-Anosov $\varphi \in \mathrm{Mod}(S_g)$ is indeed realized as the leading eigenvalue of $\Psi(\varphi)$, as mentioned in the introduction. Now we can prove the following, in order to show the desired non-surjectivity.

Theorem B. *For each genus $g \geq 2$, there exists $A \in \mathrm{Sp}(2g, \mathbb{Z})$ with bi-Perron leading eigenvalue but none of its eigenvalues is simple. Indeed, there are infinitely many such matrices.*

Proof. Recall that

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{A \in \mathrm{GL}(2g, \mathbb{Z}) : A^t J A = J\}.$$

Since J is similar to $\left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right]$ by orthogonal matrix, where I_g is a $g \times g$ identity matrix, it suffices to deal with matrix A satisfying

$$A^t \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right] A = \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right].$$

First of all, let Y be a $g \times g$ integer symmetric matrix. Then as

$$\left[\begin{array}{c|c} I_g + Y^2 & Y \\ \hline Y & I_g \end{array}\right]^t \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right] \left[\begin{array}{c|c} I_g + Y^2 & Y \\ \hline Y & I_g \end{array}\right] = \left[\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right],$$

we have $A := \left[\begin{array}{c|c} I_g + Y^2 & Y \\ \hline Y & I_g \end{array}\right] \in \mathrm{Sp}(2g, \mathbb{Z})$. In order to figure out eigenvalues of A, let us compute the characteristic polynomial $p_A(x)$ of A:

$$p_A(x) = \det \left[\begin{array}{c|c} (x-1)I_g - Y^2 & -Y \\ \hline -Y & (x-1)I_g \end{array}\right]$$

$$= \det \left[(x-1)^2 I_g - (x-1)Y^2 - Y^2\right]$$

$$= x^g \det \left[\frac{(x-1)^2}{x} I_g - Y^2\right] = x^g p_{Y^2}\left(\frac{(x-1)^2}{x}\right)$$

As a result, characteristic polynomial of A is completely determined by one of Y^2, and thus of Y. Now set $a, b \in \mathbb{Z}$ and

$$Y := \left[\begin{array}{cc|ccc} a & b & 0 & \cdots & 0 \\ b & -a & & & \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right]$$

whose characteristic polynomial is $p_Y(x) = x^{g-2}(x^2 - a^2 - b^2)$. Letting $\lambda^2 = a^2 + b^2$ and $\lambda > 0$, we have $p_{Y^2}(x) = x^{g-2}(x - \lambda^2)^2$. Therefore,

$$p_A(x) = x^g p_{Y^2}\left(\frac{(x-1)^2}{x}\right) = (x-1)^{2g-4}(x^2 - (\lambda^2 + 2)x + 1)^2.$$

Now, as $\lambda > 0$, $x^2 - (\lambda^2 + 2)x + 1 = 0$ has two roots $\mu > 1 > \frac{1}{\mu}$. It implies that $A \in \mathrm{Sp}(2g, \mathbb{Z})$ has a leading eigenvalue μ which is bi-Perron. However, it follows from $p_A(x)$ above that every eigenvalue of A is not simple as desired. The last assertion is straightforward since we are free to choose $a, b \in \mathbb{Z}$. □

In fact, we can obtain more general form of such matrices by modifying an integer symmetric matrix Y, setting

$$Y := \left[\begin{array}{cc|ccc} a & b & 0 & \cdots & 0 \\ b & -a & & & \\ \hline 0 & & & & \\ \vdots & & & Z & \\ 0 & & & & \end{array}\right]$$

for an integer symmetric matrix Z whose all eigenvalues are contained in $[-\lambda, \lambda]$ where $\lambda^2 = a^2 + b^2$. Then by the same argument above, the leading eigenvalue of $A = \left[\begin{array}{c|c} I_g + Y^2 & Y \\ \hline Y & I_g \end{array}\right] \in \mathrm{Sp}(2g, \mathbb{Z})$ is a root $\mu > 1$ of a quadratic equation $x^2 - (\lambda^2 + 2)x + 1 = 0$, and is not a simple eigenvalue.

Since A is symmetric, all Galois conjugates of $\mu > 1$ are realized as eigenvalues of A. Henceforth, $|\alpha| \leq \mu$ for any Galois conjugate α of μ.

Furthermore, as $A \in \mathrm{Sp}(2g, \mathbb{Z})$, α is an eigenvalue of A if and only if $1/\alpha$ is so. Accordingly we have $1/|\alpha| \leq \mu$, which concludes that

$$\frac{1}{\mu} \leq |\alpha| \leq \mu$$

for any Galois conjugate α of μ. Therefore, the leading eigenvalue μ of $A \in \mathrm{Sp}(2g, \mathbb{Z})$ is bi-Perron as desired.

Remark 3.5. *In order to construct the desired counterexample $A \in \mathrm{Sp}(2g, \mathbb{Z})$, one can also start with a block diagonal matrix*

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_N \end{bmatrix}$$

where $A_i \in \mathrm{Sp}(2n_i, \mathbb{Z})$ and $\sum n_i = g$. Recalling that $J_{ij} = \delta_{i(j-1)} - \delta_{(i-1)j}$, we have $A^t J A = J$ so that $A \in \mathrm{Sp}(2g, \mathbb{Z})$. Then, setting $A_1 = A_2$ and an appropriate choice of $A_3, \cdots, A_N$ can make A to have a non-simple bi-Perron leading eigenvalue. For instance, set $n_1 = n_2 = 1$ and $A_3 = I_{2g-2}$.

However, we focused on symmetric matrices to make argument concrete and to provide a large class of explicit examples. In particular, to have a bi-Perron leading eigenvalue, we can just consider a diagonal matrix with the desired diagonal and then conjugate by orthogonal matrix resulting in a symmetric matrix, instead of comparing moduli of complex roots of a polynomial.

Combining it with Lemma 3.2 and Proposition 3.4, we conclude the following result as a corollary.

Theorem A. *For each genus $g \geq 2$, when the symplectic representation Ψ is restricted to the set of orientable pseudo-Anosovs, Ψ is not surjective onto the set of elements in $\mathrm{Sp}(2g, \mathbb{Z})$ whose leading eigenvalue is bi-Perron.*

Appendix A. Typical characteristic polynomial for $\mathrm{Sp}(4, \mathbb{Z})$ has a real root

In the introduction, we pointed out the rarity of characteristic polynomial for $\mathrm{Sp}(4, \mathbb{Z})$ without real root, while it works as an evidence for non-surjectivity of Ψ from the set of orientable pseudo-Anosovs onto $\mathrm{Sp}(4, \mathbb{Z})$. For the sake of completeness, in this appendix, we show that typical characteristic polynomials for $\mathrm{Sp}(4, \mathbb{Z})$ have a real root.

When the surface S_2 is of genus 2, we can measure the portion of characteristic polynomials for $\mathrm{Sp}(4, \mathbb{Z})$ without real root, based on the fact that roots of degree 4 real polynomial are explicitly determined by coefficients. For $(n, m) \in \mathbb{Z}^2$, let

$$q_{(n,m)}(x) = x^4 + nx^3 + mx^2 + nx + 1,$$

and
$$Q = \{(n, m) : q_{(n,m)} \text{ has no real root}\}.$$
In order to measure the typicality, we define the (upper) asymptotic density $\rho(L)$ of $L \subseteq \mathbb{Z}^2$ by
$$\rho(L) = \limsup_{K \to \infty} \frac{|\{(n, m) \in L : \|(n, m)\| \leq K\}|}{|\{(n, m) \in \mathbb{Z}^2 : \|(n, m)\| \leq K\}|}$$
where $\|(n, m)\| = \max\{|n|, |m|\}$ is a norm on $\mathbb{Z}^2$. Since the set of $q_{(n,m)}$'s coincides with the set of characteristic polynomials for $\text{Sp}(4, \mathbb{Z})$, the desired typicality follows from showing that
$$\rho(Q) = 0.$$
In other words, we say typical elements lie in Q^c if $\rho(Q) = 0$.

To do this, note that the following are all roots of $q_{(n,m)}$.

$$\frac{1}{4}\left(-\sqrt{n^2 - 4m + 8} - \sqrt{2}\sqrt{n\sqrt{n^2 - 4m + 8} + n^2 - 2(m + 2)} - n\right)$$

$$\frac{1}{4}\left(-\sqrt{n^2 - 4m + 8} + \sqrt{2}\sqrt{n\sqrt{n^2 - 4m + 8} + n^2 - 2(m + 2)} - n\right)$$

$$\frac{1}{4}\left(\sqrt{n^2 - 4m + 8} - \sqrt{2}\sqrt{-n\sqrt{n^2 - 4m + 8} + n^2 - 2(m + 2)} - n\right)$$

$$\frac{1}{4}\left(\sqrt{n^2 - 4m + 8} + \sqrt{2}\sqrt{-n\sqrt{n^2 - 4m + 8} + n^2 - 2(m + 2)} - n\right)$$

From this observation, we can prove the lemma:

Lemma A.1. *There exists a finite set $\tilde{Q} \subseteq \mathbb{Z}^2$ such that for $(n, m) \in Q \setminus \tilde{Q}$,*
$$n^2 - 4m + 8 \leq 0.$$

Proof. Let $(n, m) \in Q$, and we may assume that $n \geq 0$. Suppose first that $n^2 - 4m + 8 > 0$ and $n^2 - 2(m + 2) \geq 0$. Then we have
$$n\sqrt{n^2 - 4m + 8} + n^2 - 2(m + 2) \geq 0$$
and thus $q_{(n,m)}$ has a real root according to the explicit formula above. It contradicts to $(n, m) \in Q$.

Therefore, $n^2 - 4m + 8 > 0$ implies that $n^2 - 2(m + 2) < 0$. However, combining them deduces that
$$n^2 < 2m + 4 < \frac{1}{2}n^2 + 8$$
which holds only for finitely many (n, m), completing the proof. $\qquad \square$

From the lemma, we now have
$$\rho(Q) \leq \rho((n, m) : n^2 - 4m + 8 \leq 0).$$

We can estimate the right-hand-side as follows:

$$
\begin{aligned}
\rho((n,m) : n^2 - 4m + 8 \leq 0) &\leq \limsup_{K \to \infty} \frac{1}{(2K+1)^2} \cdot 2 \int_2^{K+1} \sqrt{4m-8} \; dm \\
&\leq \limsup_{K \to \infty} \frac{(4K-4)^{3/2}}{3(2K+1)^2} \\
&= 0 \\
&\therefore \rho(Q) = 0
\end{aligned}
$$

References

[BRW19] Hyungryul Baik, Ahmad Rafiqi, and Chenxi Wu. Is a typical bi-perron algebraic unit a pseudo-anosov dilatation? *Ergodic Theory and Dynamical Systems*, 39(7):1745–1750, 2019.

[Do16] Hieu Trung Do. New families of pseudo-anosov homeomorphisms with vanishing sah-arnoux-fathi invariant. 2016.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Fri85] David Fried. Growth rate of surface homeomorphisms and flow equivalence. *Ergodic Theory and Dynamical Systems*, 5(4):539–563, 1985.

[Ham] Urusla Hamenstädt. personal communication.

[Kid08] Yoshikata Kida. Classification of certain generalized bernoulli actions of mapping class groups. *preprint*, 2008.

[Kir69] David Kirby. Integer matrices of finite order. *Rend. Mat.*, 2(6):403–408, 1969.

[Kob11] Thomas Koberda. Teichmüller polynomials, Alexander polynomials and finite covers of surfaces. *arXiv preprint arXiv:1110.3746*, 2011.

[LT11] Erwan Lanneau and Jean-Luc Thiffeault. On the minimum dilatation of pseudo-anosov homeromorphisms on surfaces of small genus. In *Annales de l'Institut Fourier*, volume 61, pages 105–144, 2011.

[McM03] Curtis McMullen. Billiards and Teichmüller curves on hilbert modular surfaces. *Journal of the American Mathematical Society*, 16(4):857–885, 2003.

[MS07] Dan Margalit and Steven Spallone. A homological recipe for pseudo-anosovs. *Mathematical Research Letters*, 14(5):853–863, 2007.

[T$^+$88] William P Thurston et al. On the geometry and dynamics of diffeomorphisms of surfaces. *Bulletin (new series) of the american mathematical society*, 19(2):417–431, 1988.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu, Daejeon, 34141, South Korea
  Email address: hrbaik@kaist.ac.kr

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu, Daejeon, 34141, South Korea
  Email address: inhyeokchoi@kaist.ac.kr

Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu, Daejeon, 34141, South Korea
  Email address: dongryul.kim@kaist.ac.kr