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Minimal Asymptotic Translation Lengths on Curve Complexes and Homology of Mapping Tori

Hyungryul Baik, Dongryul M. Kim, & Chenxi Wu

Abstract. Let S_g be a closed orientable surface of genus $g > 1$. Consider the minimal asymptotic translation length $L_{\mathcal{T}}(k, g)$ on the Teichmüller space of S_g, among pseudo-Anosov mapping classes of S_g acting trivially on k-dimensional subspaces of $H_1(S_g)$, $0 \leq k \leq 2g$. The asymptote of $L_{\mathcal{T}}(k, g)$ for extreme cases $k = 0, 2g$ have been shown by several authors. Jordan Ellenberg asked whether there is a lower bound for $L_{\mathcal{T}}(k, g)$ interpolating the known results on $L_{\mathcal{T}}(0, g)$ and $L_{\mathcal{T}}(2g, g)$, which was affirmatively answered by Agol, Leininger, and Margalit.

In this paper, we study an analogue of Ellenberg's question, replacing Teichmüller spaces with curve complexes. We provide lower and upper bound on the minimal asymptotic translation length $L_{\mathcal{C}}(k, g)$ on the curve complex, whose lower bound interpolates the known results on $L_{\mathcal{C}}(0, g)$ and $L_{\mathcal{C}}(2g, g)$.

Finally, for each g, we construct a non-Torelli pseudo-Anosov $f_g \in \mathrm{Mod}(S_g)$ which does not normally generate $\mathrm{Mod}(S_g)$, so that the asymptotic translation length of f_g on the curve complex decays faster than a constant multiple of $1/g$ as $g \to \infty$. From this, we provide a restriction on how small the asymptotic translation lengths on curve complexes should be if the similar phenomenon as in the work of Lanier and Margalit on Teichmüller spaces holds for curve complexes.

1. Introduction

Let S_g be a closed connected orientable surface of genus $g > 1$, $\mathrm{Mod}(S_g)$ be its mapping class group, and $\mathcal{C}(S_g)$ be its curve complex. Then $\mathrm{Mod}(S_g)$ isometrically acts on $\mathcal{C}(S_g)$, hence the *asymptotic translation length* $\ell_{\mathcal{C}}(f)$ of $f \in \mathrm{Mod}(S_g)$ on $\mathcal{C}(S_g)$ is defined as follows:

$$\ell_{\mathcal{C}}(f) := \liminf_{n \to \infty} \frac{d_{\mathcal{C}}(x, f^n(x))}{n}$$

for any $x \in \mathcal{C}(S_g)$ where $d_{\mathcal{C}}$ is the standard metric on $\mathcal{C}(S_g)$. The asymptotic translation length is also called *stable translation length.*

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Note that $\mathrm{Mod}(S_g)$ also acts on $H_1(S_g)$, the first homology group of S_g with real coefficients. For $f \in \mathrm{Mod}(S_g)$, we denote the dimension of a maximal subspace of $H_1(S_g)$ on which f is trivial by $m(f)$. In particular, $m(f) = 2g$ if and only if f is in the *Torelli group* $\mathcal{I}_g < \mathrm{Mod}(S_g)$, the subgroup consisting of elements that act trivially on $H_1(S_g)$. As an application of Mayer–Vietoris sequence, one can observe that $m(f) + 1$ is the same as the first Betti number of the mapping torus of f, which is hyperbolic if and only if f is pseudo-Anosov by Thurston [Thu98].

In this paper, we mainly study the *minimal asymptotic translation lengths* among pseudo-Anosov mapping classes acting trivially on some subspaces of homology groups. Namely, for $0 \le k \le 2g$, we define

$$L_{\mathcal{C}}(k, g) := \inf\{\ell_{\mathcal{C}}(f) : f \in \mathrm{Mod}(S_g),\ f \text{ is pseudo-Anosov},\ m(f) \ge k\}.$$

Then we investigate asymptotes of $L_{\mathcal{C}}(k, g)$ with varying k and g.

By replacing the curve complex $\mathcal{C}(S_g)$ with Teichmüller space $\mathcal{T}(S_g)$, one can also define $\ell_{\mathcal{T}}(\cdot)$ and $L_{\mathcal{T}}(k, g)$ analogously. Note that $\ell_{\mathcal{T}}(f)$ for a pseudo-Anosov element f is the same as the logarithm of the stretch factor [L+78], hence coincides with the topological entropy of f [FLP12, Exposé Ten].

In each setting, there are two extreme cases: the first extreme is the case $k = 0$ that the minimal asymptotic translation length is considered in the *entire mapping class group* $\mathrm{Mod}(S_g)$. The other extreme is $k = 2g$, which means that the minimal asymptotic translation length is considered in the *Torelli subgroup* $\mathcal{I}_g < \mathrm{Mod}(S_g)$. These four cases have been resolved by various authors as in Table 1.

Ellenberg [Ell10] asked if $L_{\mathcal{T}}(k, g)$ interpolates $L_{\mathcal{T}}(0, g)$ and $L_{\mathcal{T}}(2g, g)$ in the sense that there exists $C > 0$ such that

$$L_{\mathcal{T}}(k, g) \ge C(k + 1)/g \tag{1.1}$$

for all $g > 1$ and $0 \le k \le 2g$. This was answered affirmatively by Agol, Leininger, and Margalit in [ALM16]. Indeed, they actually showed $L_{\mathcal{T}}(k, g) \asymp (k + 1)/g$.

We ask an analogous question whether $L_{\mathcal{C}}(k, g)$ interpolates $L_{\mathcal{C}}(0, g)$ and $L_{\mathcal{C}}(2g, g)$ in a similar sense as Ellenberg's question (1.1). We show that this is indeed the case, and more concretely we obtain the following.

Table 1 Four extreme cases of $L_{\mathcal{T}}(k, g)$ and $L_{\mathcal{C}}(k, g)$.[1]

	Teichmüller spaces	Curve complexes
$\mathrm{Mod}(S_g)$	(Penner [Pen91]) $L_{\mathcal{T}}(0, g) \asymp 1/g$	(Gadre–Tsai [GT11]) $L_{\mathcal{C}}(0, g) \asymp 1/g^2$
$\mathcal{I}_g$	(Farb–Leininger–Margalit [FLM08]) $L_{\mathcal{T}}(2g, g) \asymp 1$	(Baik–Shin [BS20]) $L_{\mathcal{C}}(2g, g) \asymp 1/g$

[1] Throughout the paper, we write $A(x) \gtrsim B(x)$ if there exists a uniform constant $C > 0$ such that $A(x) \le CB(x)$ for all x in the domain. We also write $A(x) \asymp B(x)$ if $A(x) \gtrsim B(x)$ and $B(x) \gtrsim A(x)$.

THEOREM 1.1. *There exist $C, C' > 0$ such that*

$$\frac{C}{g(2g-k+1)} \le L_{\mathcal{C}}(k,g) \le C' \frac{k+1}{g \log g}$$

for all $g > 1$ and $0 \le k \le 2g$.

From the statement, if k grows at least $2g - C'$ for some constant $C' > 0$, then $L_{\mathcal{C}}(k,g) \gtrsim 1/g$ while $L_{\mathcal{C}}(0,g) \asymp 1/g^2$. Observing this, we ask about minimal k with $L_{\mathcal{C}}(k,g) \asymp 1/g$. For this discussion, see Section 4.

Although the lower bound in Theorem 1.1 interpolates $L_{\mathcal{C}}(0,g) \asymp 1/g^2$ and $L_{\mathcal{C}}(2g,g) \asymp 1/g$, the upper bound in Theorem 1.1 does not interpolate these two values well. Indeed, we construct some values of k and g showing that $\frac{k+1}{g \log g}$ is larger than the actual asymptote. We also show that k/g^2 works as an upper bound for some choices of (k,g), which interpolates $L_{\mathcal{C}}(0,g) \asymp 1/g^2$ and $L_{\mathcal{C}}(2g,g) \asymp 1/g$.

THEOREM 1.2. *There is a uniform constant $C > 0$ satisfying the following: for any integers $g, k \ge 0$, there exists a pseudo-Anosov $f : S_{g'} \to S_{g'}$ such that $g' > g$, $m(f) = k' > k$, and*

$$\ell_{\mathcal{C}}(f) \le C \frac{k'}{g'^2}.$$

Applying Theorem 1.2 inductively, it follows that there is a diverging sequence $(k_j, g_j) \to \infty$ so that $L_{\mathcal{C}}(k_j, g_j) \lesssim k_j/g_j^2$. See Corollary 3.1. Based on Table 1, we conjecture that the upper bound in Theorem 1.2 is actually the asymptote for $L_{\mathcal{C}}(k,g)$.

CONJECTURE 1.3. *We have*

$$L_{\mathcal{C}}(k,g) \asymp \frac{k}{g^2}$$

for $g > 1$ and $0 \le k \le 2g$.

We focus on specific dimensions of maximal invariant subspaces. In [BS20], Torelli pseudo-Anosovs are constructed in a concrete way based on Penner's or Thurston's construction. In a similar line of thought, we utilize finite cyclic covers of S_2 so that we get pseudo-Anosovs $f \in \mathrm{Mod}(S_g)$ with $m(f) = 2g - 1$ and satisfying the upper bound in Theorem 1.2. As a consequence, this yields the asymptote of $L_{\mathcal{C}}(2g-1,g)$; only two extreme cases $\mathrm{Mod}(S_g)$ and $\mathcal{I}_g$ were previously known. It is also interesting to figure out the asymptote $L_{\mathcal{C}}(k,g)$ for other values (k,g):

QUESTION 1.4. *Can we give a sequence (k_j, g_j), other than $(0,g)$ and $(2g,g)$, with explicit asymptote for $L_{\mathcal{C}}(k_j, g_j)$ as $j \to \infty$?*

We give one such example in the following.

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THEOREM 1.5. *There exist a uniform constant $C > 0$ and pseudo-Anosovs $f_g \in \mathrm{Mod}(S_g)$ such that*

$$m(f_g) = 2g - 1 \quad \textit{and} \quad \ell_{\mathcal{C}}(f_g) \le \frac{C}{g}$$

for all $g > 1$. Moreover, the following asymptote holds:

$$L_{\mathcal{C}}(2g - 1, g) \asymp \frac{1}{g}.$$

The construction involved in Theorem 1.5 can be modified to deal with the Torelli case. Such a modification gives an asymptote for $L_{\mathcal{C}}(2g, g)$, which was already shown by [BS20] in a different way. See Remark 4.1. Further, only the last assertion can also be deduced from Theorem 1.1 and [BS20]. See Section 4 for details.

In [LM22], Lanier and Margalit showed that a pseudo-Anosov with small asymptotic translation length on the Teichmüller space has the entire mapping class group as its normal closure. The first and the third authors, Kin, and Shin, made an analogous question for asymptotic translation lengths on curve complexes in [B+23] (see [B+23, Question 1.2]). We later show that pseudo-Anosovs f_g constructed in Theorem 1.5 never normally generate the mapping class groups. Since $\ell_{\mathcal{C}}(f_g)$ is concretely estimated in Section 4, it provides how small the asymptotic translation length should be to observe the similar phenomenon as in [LM22]. In other words, we prove the following.

THEOREM 1.6. *Suppose that there exists a universal constant C so that if a non-Torelli pseudo-Anosov $f \in \mathrm{Mod}(S_g)$ has $\ell_{\mathcal{C}}(f) < C/g$, then f normally generates $\mathrm{Mod}(S_g)$ for large g. Then*

$$C \le 1{,}152.$$

Organization

In Section 2, we prove Theorem 1.1. Theorem 1.2 is proved in Section 3. In Section 4, explicit construction of pseudo-Anosovs realizing the asymptote of $L_{\mathcal{C}}(2g - 1, g)$ is provided, implying Theorem 1.5. The discussion on small asymptotic translation lengths on curve complexes and normal generation of mapping class groups is provided in Section 5.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

Lower Bound

The main idea of showing the lower bound is similar to the one used in the proof in [BS20] of $L_{\mathcal{C}}(2g, g) \ge C/g$ for some constant $C > 0$ and for all $g > 1$. First note that for any homeomorphism $f : S_g \to S_g$, the Lefschetz number $L(f)$ is $2 - \mathrm{Tr}(f_*)$, where $\mathrm{Tr}(f_*)$ is the trace of the induced map $f_* : H_1(S_g) \to H_1(S_g)$.

Let us fix a pseudo-Anosov $f : S_g \to S_g$ whose restriction onto a k-dimensional subspace of $H_1(S_g)$ is the identity.

Fixing a suitable basis for $H_1(S_g)$, the matrix for f_* can be written as

$$\begin{pmatrix} I_k & * \\ 0 & M \end{pmatrix}.$$

Suppose first that $k > 0$. When k is odd, let $m = 2g - k$, and when k is even, let $m = 2g - (k-1)$. By taking the upper left block to be I_{k-1} in case k is even, one may assume M is an $m \times m$ square matrix with determinant 1 and m is odd (determinant 1 comes from the fact that f_* is actually a symplectic matrix).

Recall that there is a relation between trace and determinant as follows.

LEMMA 2.1 ([KK92, Appendix B]). *For any $m \times m$ matrix A,*

$$(-1)^m \det A = \sum_{\substack{c_1,\dots,c_m \geq 0, \\ c_1+2c_2+\cdots+mc_m=m}} \prod_{i=1}^{m} \frac{1}{c_i!}\left(-\frac{\operatorname{Tr}(A^i)}{i}\right)^{c_i}.$$

Observe that at least one of the matrices $M, M^2, \dots, M^m$ must have positive trace. Otherwise the right-hand side of the equality in Lemma 2.1 is always nonnegative when we plug in M in the place of A in the lemma. On the other hand, since $\det(M) = 1$ and m is always odd by our choice, the left-hand side is -1, a contradiction.

This implies that for some j satisfying $1 \leq j \leq m \leq 2g - k + 1$, $\operatorname{Tr}(M^j)$ is positive, that is, at least 1 since it is an integral matrix. $\operatorname{Tr}(f_*^j)$ is the sum of $\operatorname{Tr}(M^j)$ and the trace of the upper left block, which is $2g - m \geq 1$. Therefore, $\operatorname{Tr}(f_*^j)$ is at least 2 in general. But in fact $2g - m \geq 3$ as long as $k \geq 3$.

Assume $k \geq 3$. Now we have that $L(f^j) = 2 - \operatorname{Tr}(f_*^j) < 0$, and we can apply a result of Tsai [Tsa09]. Then $\ell_{\mathcal{C}}(f^j) \geq C/g$ for some constant $C > 0$ and consequently,

$$\ell_{\mathcal{C}}(f) \geq \frac{C}{gj} \geq \frac{C}{g(2g-k+1)}.$$

Recall that $L_{\mathcal{C}}(0, g) \asymp 1/g^2$. Hence for each $k \geq 0$, there exists C_k such that $L_{\mathcal{C}}(k, g) \geq \frac{C_k}{g(2g-k+1)}$ for all $g > 1$. Since the above argument works for any $k \geq 3$, replacing C by $\min\{C, C_0, C_1, C_2\}$, we obtain the lower bound in Theorem 1.1.

Upper Bound

We now prove the upper bound provided in Theorem 1.1.

Recall that the Teichmüller space $\mathcal{T}(S_g)$ is the space of marked hyperbolic structures on S_g, and vertices of the curve complex $\mathcal{C}(S_g)$ are isotopy classes of essential simple closed curves on S_g. Hence, we can associate each point $x \in \mathcal{T}(S_g)$ with systoles on S_g, the shortest simple closed geodesics, in the hyperbolic structure x. Because systoles are within a uniformly bounded distance in the curve complex, it gives a coarsely well-defined map $\pi_g : \mathcal{T}(S_g) \to \mathcal{C}(S_g)$.

Masur and Misnky studied $\pi_g : \mathcal{T}(S_g) \to \mathcal{C}(S_g)$ in [MM99] and showed that π_g is coarsely Lipschitz.

Proposition 2.2 ((K_g, D_g)-coarsely Lipschitz, [MM99]). *There exist constants $K_g, D_g > 0$ such that for any $x, y \in \mathcal{T}(S_g)$ we have*

$$d_{\mathcal{C}}(\pi_g(x), \pi_g(y)) \leq K_g d_{\mathcal{T}}(x, y) + D_g,$$

where $d_{\mathcal{T}}$ is the Teichmüller metric.

Furthermore, π_g is coarsely $\mathrm{Mod}(S_g)$-equivariant in the sense that there exists a constant A_g such that $d_{\mathcal{C}}((\pi_g \circ f)(x), (f \circ \pi_g)(x)) \leq A_g$ for any $x \in \mathcal{T}(S_g)$ and $f \in \mathrm{Mod}(S_g)$. Then, for $f \in \mathrm{Mod}(S_g)$, $n > 0$, and $x \in \mathcal{T}(S_g)$, we have

$$\begin{aligned} d_{\mathcal{C}}(\pi_g(x), f^n(\pi_g(x))) &\leq d_{\mathcal{C}}(\pi_g(x), \pi_g(f^n(x))) + A_g \\ &\leq K_g d_{\mathcal{T}}(x, f^n(x)) + D_g + A_g. \end{aligned}$$

Hence, we now have the comparison between asymptotic translation lengths of $f \in \mathrm{Mod}(S_g)$ measured on $\mathcal{C}(S_g)$ and $\mathcal{T}(S_g)$:

$$\ell_{\mathcal{C}}(f) \leq K_g \ell_{\mathcal{T}}(f).$$

In particular, we have

$$L_{\mathcal{C}}(k, g) \leq K_g L_{\mathcal{T}}(k, g). \tag{2.1}$$

Due to the work [ALM16] of Agol, Leininger, and Margalit, we already know the asymptote of $L_{\mathcal{T}}(k, g)$. Hence, it remains to figure out the asymptote of K_g.

In [G+13], Gadre, Hironaka, Kent, and Leininger considered the minimal possible Lipschitz constant K_g, which is defined as

$$\kappa_g := \inf\{K_g \geq 0 : \pi_g \text{ is } (K_g, D_g)\text{-coarsely Lipschitz for some } D_g > 0\}.$$

Then they showed that

$$\kappa_g \asymp \frac{1}{\log g}.$$

Combining this with [ALM16] and inequality (2.1), we deduce the upper bound in Theorem 1.1.

3. Upper Bound Interpolates $L_{\mathcal{C}}(0, g)$ and $L_{\mathcal{C}}(2g, g)$

The upper bound provided in Theorem 1.1 does not interpolate $L_{\mathcal{C}}(0, g)$ and $L_{\mathcal{C}}(2g, g)$, and it is not sharp enough as one can see in Section 4. As stated in Theorem 1.2, the upper bound conjectured in Conjecture 1.3 can be observed along a certain sequence $(k_j, g_j) \to \infty$. This section is devoted to proving Theorem 1.2.

Proof of Theorem 1.2. Let f_0 be a pseudo-Anosov map in the Torelli group of genus $g_0 > 1$. Let M be its mapping torus, $\alpha \in H^1(M)$ be the first cohomology class of M corresponding to f_0, β be an element in $H^1(M)$, which is restricted to a cohomology class dual to a simple closed curve γ on S_{g_0}. For large enough $n > g + k$, let f_n be the pseudo-Anosov monodromy corresponding to $2^n\alpha + \beta$. Then f_n has the fiber of genus $O(2^n)$, and $\ell_{\mathcal{C}}(f_n)$ is $O(2^{-2n})$ (cf. [BSW21]).

A way to construct the surface S_n and map f_n corresponding to $2^n\alpha+\beta$ is as follows: let $\widehat{S}$ be the $\mathbb{Z}$-fold cover corresponding to β restricted to S_{g_0}, $\widehat{f}$ be a lift of f_0, and h be the deck transformation; then, with a suitable choice of $\widehat{f}$, we have $S_n = \widehat{S}/(h^{2^n}\widehat{f})$ and f_n is lifted to h. Now consider a simple closed curve on a fundamental domain of $\widehat{S}$ that is not homologous to the boundary, such that the homology class c represented by this curve γ is preserved by $\widehat{f}$. The existence of such a homology class is due to the construction in Baik and Shin [BS20]. Then $\sum_{i=0}^{2^n-1} f_n^i c$ is invariant under f_n, and for $k<n$, let $c_k = \sum_{i=0}^{2^{n-k}-1} f_n^{i2^k} c$. Now $\mathrm{Span}\{c_k, f_n c_k, \dots f_n^{2^k-1} c_k\}$ is a 2^k dimensional invariant subspace of $f_n^{2^k}$. This proves Theorem 1.2. □

Since the constant C in Theorem 1.2 does not depend on the choice of given g and k, we can apply the theorem inductively: at each jth step with g_j and k_j, Theorem 1.2 applied to g_j and k_j gives $g' > g_j$, $k' > k_j$, and a pseudo-Anosov $f_{j+1}: S_{g'_j} \to S_{g'_j}$ with $\ell_{\mathcal{C}}(f_{j+1}) \le Ck'_j/g'^2_j$. Then we set $g_{j+1} := g'_j$ and $k_{j+1} := k'_j$. As a consequence, we obtain the following corollary that interpolates $L_{\mathcal{C}}(0,g)$ [GT11] and $L_{\mathcal{C}}(2g,g)$ [BS20] in a partial way.

COROLLARY 3.1. *There are a constant C and a diverging sequence $(k_j, g_j) \to \infty$ as $j \to \infty$ such that*

$$L_{\mathcal{C}}(k_j, g_j) \le C\frac{k_j}{g_j^2}.$$

Corollary 3.1 can be regarded as an evidence for Conjecture 1.3 because it has a similar form to the desired asymptote. On the other hand, due to the inexplicit choice made in the proof of Theorem 1.2, it is hard to explicitly understand from which diverging sequence (k_j, g_j) we can deduce the desired asymptote. Hence it may require different approaches to make a concrete progress towards Conjecture 1.3.

However, pseudo-Anosov mapping classes we construct in the later section (Section 4) satisfy the asymptotes in Theorem 1.2 and Corollary 3.1.

4. Pseudo-Anosovs with Specified Invariant Homology Dimension

To the best of the authors' knowledge, asymptotes of $L_{\mathcal{C}}(k,g)$ are known only when $k=0$ (whole mapping class groups) and $k=2g$ (Torelli groups). In this section, we construct pseudo-Anosovs $f_g \in \mathrm{Mod}(S_g)$ with $m(f_g)=2g-1$ and realizing the asymptote of $L_{\mathcal{C}}(2g-1,g)$.

From the definition of $L_{\mathcal{C}}(k,g)$, $L_{\mathcal{C}}(k,g) \le L_{\mathcal{C}}(k',g)$ if $k \le k'$. Since $L_{\mathcal{C}}(2g,g) \asymp 1/g$ from [BS20], the lower bound in Theorem 1.1 implies that $L_{\mathcal{C}}(k,g) \asymp 1/g$ if k behaves like $2g$; for instance, $k \ge 2g-C$ for some constant $C>0$. However, $L_{\mathcal{C}}(0,g) \asymp 1/g^2$ by [GT11]. In this regard, we ask whether there is a sort of threshold for k that $L_{\mathcal{C}}(k,g)$ becomes strictly smaller than $1/g$, such as $1/g^2$.

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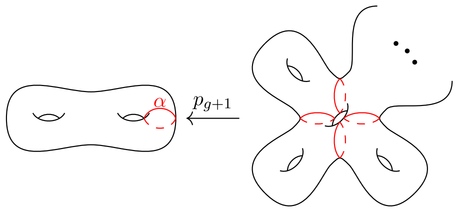

Figure 1 g-fold finite cyclic covering.

As a potential approach for this question, we think of constructing pseudo-Anosovs of specified maximal invariant homology dimensions on surfaces of large genera with small asymptotic translation lengths. In order to get pseudo-Anosov maps on surfaces of large genera, some previous results ([ALM16; BSW21; B+23]) employ a fixed hyperbolic mapping torus and consider its monodromy obtained from a fibered cone. Since the first Betti number of a mapping torus of f is the same as $m(f)+1$, such monodromies in a fibered cone share the same m-value.

In contrast, we come up with finite cyclic covers of a genus 2 surface to get the desired pseudo-Anosov maps on large genera surfaces as lifts of a fixed map. From the concrete estimation on how covering maps distort the distances on curve complexes [APT22], asymptotic translation length of the lift via degree g covering would be at least $1/g$, up to a constant multiple. We believe that constructing such lifts with specified maximal invariant homology dimensions would help to figure out the minimal $k = k(g)$ with $L_{\mathcal{C}}(k, g) \asymp 1/g$.

We start with a nonseparating simple closed curve α on the genus 2 surface S_2, and take g copies of $S_2 \setminus \alpha$ for $g > 1$. Gluing two different copies of $S_2 \setminus \alpha$ along one boundary component in a cyclic way, we obtain the finite cyclic cover p_{g+1} of degree g as in Figure 1. Let us denote the resulting cover by S_{g+1} since it is of genus $g+1$.

This cover p_{g+1} corresponds to the kernel of the composed map

$$\pi_1(S_2) \xrightarrow{\hat{i}(\cdot,\alpha)} \mathbb{Z} \xrightarrow{\text{mod } g} \mathbb{Z}/g\mathbb{Z},$$

where $\hat{i}(\cdot,\cdot)$ stands for the algebraic intersection number. To see this, one can observe that an element of $\pi_1(S_2)$ can be lifted to $\pi_1(S_{g+1})$ via p_{g+1} if and only if its lift departs one copy of $S_2 \setminus \alpha$ and then returns to the same copy. If the lift departs and returns through the same boundary component of $S_2 \setminus \alpha$, then the element of $\pi_1(S_2)$ has the algebraic intersection number 0 with α. Otherwise, if the lift departs and returns through different boundary components, then the algebraic intersection number is an integer multiple of g.

In [BS20], the first author and Shin directly constructed pseudo-Anosovs on S_g that are Torelli and of small asymptotic translation lengths on curve complexes. In the following, we construct pseudo-Anosovs with specific maximal invariant homology dimensions and satisfying the upper bound provided in Theorem 1.2

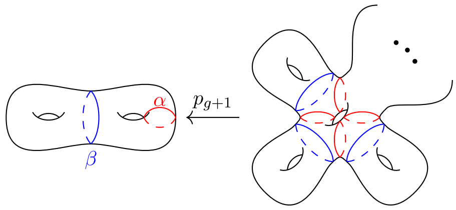

Figure 2 A nonseparating curve α and a separating curve β on S_2 with $\alpha \cap \beta = \emptyset$.

and Corollary 3.1. As a result, we obtain Theorem 1.5. Our strategy is to fix a suitable pseudo-Anosov on S_2 and then to lift it via p_{g+1}. Due to the symmetry of the covering, we can find a number of invariant homology classes proportional to the degree of the cover.

Proof of Theorem 1.5. The last assertion is a direct consequence of the first assertion and Theorem 1.1. By Theorem 1.1, there exists $C' > 0$ so that $L_{\mathcal{C}}(2g-1, g) \geq \frac{C'}{g}$ for all $g > 1$. Hence, it remains to show the existence of the desired pseudo-Anosovs.

Fix $g > 1$, and in the rest of the proof, we simply denote $p := p_{g+1}$. Let α be a nonseparating curve on S_2, and let β be a separating curve on S_2 disjoint from α. Then each lift of β through p is also separating. For instance, see Figure 2.

Now let ϕ be a Torelli pseudo-Anosov mapping class in $\mathrm{Mod}(S_2)$ such that $d_{\mathcal{C}}(\beta, \phi\beta) \geq 3$. Note that this means that β and $\phi\beta$ are separating simple closed curves on S_2 that fill the surface. Such a ϕ may be obtained as follows: let $\psi \in \mathrm{Mod}(S_2)$ be a pseudo-Anosov mapping class. Since a pseudo-Anosov mapping class acts on $\mathcal{C}(S_2)$ as a loxodromic isometry by [MM99], $\lim_{n\to\infty} d_{\mathcal{C}}(\beta, \psi^n\beta)/n = \ell_{\mathcal{C}}(\psi) > 0$. Hence replacing ψ with ψ^n for some large n if necessary, we may assume that $d_{\mathcal{C}}(\beta, \psi\beta) \geq 3$. We then consider a mapping class $T_\beta T_{\psi\beta}^{-1}$ that is pseudo-Anosov by Thurston [Thu88] or Penner [Pen88]. Again, since $T_\beta T_{\psi\beta}^{-1}$ acts on $\mathcal{C}(S_2)$ as a loxodromic isometry, we can set ϕ to be some power of $T_\beta T_{\psi\beta}^{-1}$ so that ϕ is a pseudo-Anosov mapping class with the property that $d_{\mathcal{C}}(\beta, \phi\beta) \geq 3$.

Now let $f = T_\beta T_{\phi\beta}^{-1} T_{\phi\alpha}^{-1}$. Since β, $\phi\beta$, and $\phi\alpha$ fill the surface and $\phi\beta \cap \phi\alpha = \emptyset$, f is pseudo-Anosov again by Thurston [Thu88] and Penner [Pen88]. Since β and $\phi\beta$ are separating, T_β and $T_{\phi\beta}^{-1}$ are Torelli, in particular, they preserve the homology class $[\alpha]$ of α. Furthermore, since ϕ is Torelli, $[\phi\alpha] = [\alpha]$, which implies that $T_{\phi\alpha}^{-1}$ also preserves $[\alpha]$. Hence, f preserves $[\alpha]$, and thus $\hat{i}(f(\cdot), \alpha) = \hat{i}(\cdot, \alpha)$. In particular, f preserves the kernel of $\pi_1(S_2) \xrightarrow{\hat{i}(\cdot,\alpha)} \mathbb{Z} \xrightarrow{\bmod g} \mathbb{Z}/g\mathbb{Z}$. Consequently, f can be lifted through p.

Let $\tilde{f} = T_{p^{-1}(\beta)} T_{p^{-1}(\phi\beta)}^{-1} T_{p^{-1}(\phi\alpha)}^{-1}$ be a lift of f via p. We now estimate $\ell_{\mathcal{C}}(\tilde{f})$. Our strategy to obtain the desired upper bound for $\ell_{\mathcal{C}}(\tilde{f})$ is to find a simple closed curve $\tilde{\alpha}$ such that $\tilde{\alpha}$ and its image under a sufficiently high power of $\tilde{f}$ do not fill

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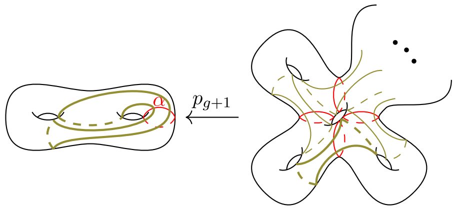

Figure 3 A curve intersecting a lift of the new curve (nonindexed one) in some X_j spreads into $X_{j-1} \cup X_{j+1}$ by twisting along the lift. It describes how the image of $\tilde{\alpha}$ under multitwists is trapped in the certain number of lifts of a subsurface, as in (4.1).

the surface, which means that they are within distance 2 in the curve complex. We do this by counting the number of intersections of images of $\tilde{\alpha}$ and lifts of subsurfaces.

Recall the construction of p: take g copies $X_1, \dots, X_g$ of $S_2 \setminus \alpha$ and glue X_i and X_{i+1} along one of their boundary components. Throughout, we write each index i modulo g. Let $\tilde{\alpha} = \partial X_0 \cap \partial X_1$. That is, let $\tilde{\alpha}$ be a boundary component of X_0 and X_1 where they are glued. Due to the construction, $\tilde{\alpha}$ is a lift of α.

Noting that $\hat{i}(\phi\alpha, \alpha) = 0$ since ϕ is Torelli, we get

$$T^{-1}_{p^{-1}(\phi\alpha)}\tilde{\alpha} \subseteq \bigcup_{j=-i(\phi\alpha,\alpha)/2}^{i(\phi\alpha,\alpha)/2} X_j, \tag{4.1}$$

where $i(\cdot,\cdot)$ is the geometric intersection number (cf. Figure 3). Similarly, $\hat{i}(\phi\beta, \alpha) = 0$ and

$$T^{-1}_{p^{-1}(\phi\beta)} T^{-1}_{p^{-1}(\phi\alpha)}\tilde{\alpha} \subseteq \bigcup_{j=-\frac{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}{2}}^{\frac{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}{2}} X_j.$$

Since $T_{p^{-1}(\beta)}$ fixes each X_j, we have

$$\tilde{f}\tilde{\alpha} \subseteq \bigcup_{j=-\frac{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}{2}}^{\frac{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}{2}} X_j.$$

Conducting this procedure inductively, we finally get

$$\tilde{f}^n\tilde{\alpha} \subseteq \bigcup_{j=-n\cdot\frac{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}{2}}^{n\cdot\frac{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}{2}} X_j.$$

Hence, for large enough g, there exists $\tilde{j}$ such that $\tilde{f}^{\lfloor\frac{g-2}{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}\rfloor}\tilde{\alpha} \cap X_{\tilde{j}} = \emptyset$. Since there exists an essential simple closed curve in $X_{\tilde{j}}$, which is a 2-holed torus, we have $d_{\mathcal{C}}(\tilde{\alpha}, \tilde{f}^{\lfloor\frac{g-2}{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}\rfloor}\tilde{\alpha}) \le 2$ so $\ell_{\mathcal{C}}(\tilde{f}^{\lfloor\frac{g-2}{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)}\rfloor}) \le 2$. This implies the

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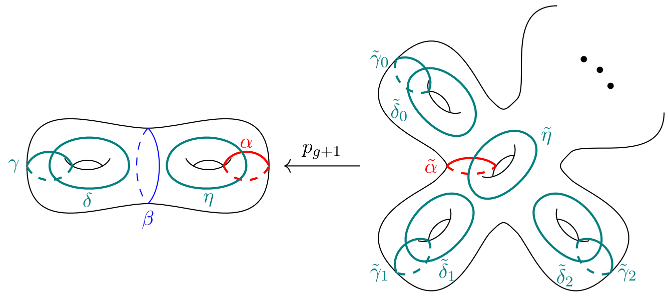

Figure 4 Basis for $H_1(S_{g+1})$.

following estimate. Note that ϕ, $i(\phi\beta,\alpha)$, and $i(\phi\alpha,\alpha)$ are universal quantities independent on p and g.

$$\ell_{\mathcal{C}}(\tilde{f}) \le \frac{2}{\lfloor \frac{g-2}{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)} \rfloor} \tag{4.2}$$

We now show that $m(\tilde{f}) = 2g+1$. Recall that $\beta \subseteq S_2$ is a separating curve and $\alpha \subseteq S_2$ is a nonseparating curve disjoint from β. Temporarily, let us denote by Y the component of $S_2 \setminus \beta$ that does not contain α. Let γ and δ be nonseparating simple closed curves whose homology classes form a basis for $H_1(Y) \cong \mathbb{R}^2$. Let us also denote η to be a nonseparating curve on $S_2 \setminus \beta$ with $i(\eta,\alpha) = 1$. Then $p^{-1}(\gamma)$, $p^{-1}(\delta)$, $p^{-1}(\eta)$, and one component of $p^{-1}(\alpha)$ form a basis for $H_1(S_{g+1}) \cong \mathbb{R}^{2g+2}$. See Figure 4 for instance.

Let $\tilde{\gamma}_j = p^{-1}(\gamma) \cap X_j$, $\tilde{\delta}_j = p^{-1}(\delta) \cap X_j$, and $\tilde{\eta} = p^{-1}(\eta)$. Further, let $\tilde{\alpha} = \partial X_0 \cap \partial X_1$, which is a component of $p^{-1}(\alpha)$. Since ϕ is Torelli, it has a lift $\tilde{\phi}$ through p. Hence, homology classes $\{[\tilde{\phi}\tilde{\gamma}_j], [\tilde{\phi}\tilde{\delta}_j], [\tilde{\phi}\tilde{\eta}], [\tilde{\phi}\tilde{\alpha}]\}$ also form a basis for $H_1(S_{g+1})$.

Recall that $\tilde{f} = T_{p^{-1}(\beta)} T^{-1}_{p^{-1}(\phi\beta)} T^{-1}_{p^{-1}(\phi\alpha)}$. Since $\gamma \cap (\alpha \cup \beta) = \emptyset$, we have $\tilde{\phi}\tilde{\gamma}_j \cap (p^{-1}(\phi\alpha) \cap p^{-1}(\phi\beta)) = \emptyset$. Here, note that $\tilde{\phi}\tilde{\gamma}_j$ is a lift of $\phi\gamma$ which is a component of $p^{-1}(\phi\gamma)$. Hence it follows that $[\tilde{f}\tilde{\phi}\tilde{\gamma}_j] = [T_{p^{-1}(\beta)}\tilde{\phi}\tilde{\gamma}_j]$. Since each component of $p^{-1}(\beta)$, which is a lift of β, is separating, $T_{p^{-1}(\beta)}$ is Torelli. As a result, $[\tilde{f}\tilde{\phi}\tilde{\gamma}_j] = [\tilde{\phi}\tilde{\gamma}_j]$. Similarly, we have $[\tilde{f}\tilde{\phi}\tilde{\delta}_j] = [\tilde{\phi}\tilde{\delta}_j]$.

Now we consider $[\tilde{f}\tilde{\phi}\tilde{\alpha}]$. Since $\tilde{\phi}\tilde{\alpha}$ is a lift of $\phi\alpha$, $T^{-1}_{p^{-1}(\phi\alpha)}\tilde{\phi}\tilde{\alpha} = \tilde{\phi}\tilde{\alpha}$. Furthermore, since $\alpha \cap \beta = \emptyset$, $\tilde{\phi}\tilde{\alpha}$, a lift of $\phi\alpha$, does not intersect $p^{-1}(\phi\beta)$. It implies that $T^{-1}_{p^{-1}(\phi\beta)}\tilde{\phi}\tilde{\alpha} = \tilde{\phi}\tilde{\alpha}$. Finally, since $T_{p^{-1}(\beta)}$ is Torelli again, we conclude $[\tilde{f}\tilde{\phi}\tilde{\alpha}] = [\tilde{\phi}\tilde{\alpha}]$.

So far, we have proved $m(\tilde{f}) \ge 2g+1$. Suppose to the contrary that $m(\tilde{f}) = 2g+2$, which means that $\tilde{f}$ is Torelli. Then $\tilde{f}\tilde{\phi}\tilde{\eta}$ should be homologous to $\tilde{\phi}\tilde{\eta}$. It implies $[T^{-1}_{p^{-1}(\phi\beta)} T^{-1}_{p^{-1}(\phi\alpha)}\tilde{\phi}\tilde{\eta}] = [\tilde{\phi}\tilde{\eta}]$ since $T_{p^{-1}(\beta)}$ is Torelli. Because any two components of $p^{-1}(\phi\alpha)$ bound a subsurface, they are homologous. In particular, since $\tilde{\phi}\tilde{\alpha}$ is a component of $p^{-1}(\phi\alpha)$, each of its components is homologous to $\tilde{\phi}\tilde{\alpha}$. Hence, $[T^{-1}_{p^{-1}(\phi\beta)}T^{-1}_{p^{-1}(\phi\alpha)}\tilde{\phi}\tilde{\eta}] = [T^{-1}_{p^{-1}(\phi\beta)}T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta}]$. Noting that $T^{-g}_{\tilde{\alpha}}\tilde{\eta}$ can be isotoped into arbitrary neighborhood of $\tilde{\alpha}\cup\tilde{\eta}$, $T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta}$ can also be isotoped into arbitrary neighborhood of $\tilde{\phi}\tilde{\alpha}\cup\tilde{\phi}\tilde{\eta}$. Since $\tilde{\phi}\tilde{\alpha}\cup\tilde{\phi}\tilde{\eta}$ and $p^{-1}(\phi\beta)$ are disjoint compact sets, we have $T^{-1}_{p^{-1}(\phi\beta)}T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta} = T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta}$. Summing up the above argument, we obtain

$$[\tilde{\phi}\tilde{\eta}] = [\tilde{f}\tilde{\phi}\tilde{\eta}] = [T^{-1}_{p^{-1}(\phi\beta)}T^{-1}_{p^{-1}(\phi\alpha)}\tilde{\phi}\tilde{\eta}] = [T^{-1}_{p^{-1}(\phi\beta)}T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta}] = [T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta}],$$

where the first equality is the assumption. However,

$$[T^{-g}_{\tilde{\phi}\tilde{\alpha}}\tilde{\phi}\tilde{\eta}] = [\tilde{\phi}\tilde{\eta}] - g\cdot\hat{i}(\tilde{\phi}\tilde{\eta},\tilde{\phi}\tilde{\alpha})[\tilde{\phi}\tilde{\alpha}],$$

which implies that $\hat{i}(\tilde{\phi}\tilde{\eta},\tilde{\phi}\tilde{\alpha}) = 0$. It contradicts our choice of η that $i(\tilde{\eta},\tilde{\alpha}) = 1$. Therefore, $m(\tilde{f}) = 2g+1$. Setting $f_{g+1} = \tilde{f}$ completes the proof of Theorem 1.5. □

The lower bound on $\ell_{\mathcal{C}}(f_g)$ for f_g constructed in the proof can also be calculated in a concrete way by Aougab, Patel, and Taylor [APT22] as follows:

$$\frac{\ell_{\mathcal{C}}(f)}{(g-1)\cdot 80\cdot 2^{13}e^{54}\pi} \le \ell_{\mathcal{C}}(f_g).$$

REMARK 4.1. In the proof, all figures describe one specific example. Any choice of α, β, γ, δ, and η works if it satisfies the condition we provide. That is,

- α and β are nonseparating and separating curves on S_2, respectively, and are disjoint;
- γ and δ are nonseparating simple closed curves that form a basis for the first homology group of the component of $S_2 \setminus \beta$ disjoint from α;
- η is a nonseparating curve on $S_2 \setminus \beta$ with $i(\eta,\alpha) = 1$.

Furthermore, if we modify the map on S_2 to be $f = T_\beta T^{-1}_{\phi\beta}$, then its lift via p_{g+1} is Torelli, which gives another proof of $L_{\mathcal{C}}(2g, g) \asymp \frac{1}{g}$.

5. Small Translation Length and Normal Generation

In this section, we discuss pseudo-Anosov mapping classes with small asymptotic translation lengths and normal generation of mapping class groups. For a general group G and $g \in G$, the *normal closure* $\langle\langle g\rangle\rangle$ of g is the smallest normal subgroup of G containing g. The normal closure can be described in various ways:

$$\langle\langle g\rangle\rangle = \bigcap_{g\in N\trianglelefteq G} N = \langle hgh^{-1} : h\in G\rangle.$$

In a particular case that $\langle\langle g\rangle\rangle = G$, we say g *normally generates* G, and g is said to be a *normal generator* of G.

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Normal generators of mapping class groups of surfaces have been studied by various authors. In [Lon86], Long asked whether there is a pseudo-Anosov normal generator of a mapping class group. This question was recently answered affirmatively by Lanier and Margalit in [LM22]. Indeed, they showed that there is a universal constant so that pseudo-Anosovs with stretch factors less than the constant should be normal generators. Then the asymptote $L_{\mathcal{T}}(0, g) \asymp 1/g$ by Penner [Pen91] deduces the answer. Precisely, Lanier and Margalit proved the following.

THEOREM 5.1 (Lanier–Margalit [LM22]). *If a pseudo-Anosov $\phi \in \mathrm{Mod}(S_g)$ has the stretch factor less than $\sqrt{2}$, then ϕ normally generates* $\mathrm{Mod}(S_g)$.

Since the logarithm of stretch factor of a pseudo-Anosov equals to the translation length of the pseudo-Anosov on the Teichmüller space, Lanier and Margalit's result also means that the small translation length on the Teichmüller space implies the normal generation of the mapping class group. One natural question in this philosophy is whether the same holds in the circumstance of curve complexes. There are several ways to formalize this question:

(1) Is there a universal constant $C > 0$ so that if a pseudo-Anosov $\phi \in \mathrm{Mod}(S_g)$ has $\ell_{\mathcal{C}}(\phi) < C/g$, then $\langle\langle \phi \rangle\rangle = \mathrm{Mod}(S_g)$?
(2) Is there a universal constant $C > 0$ so that if a non-Torelli pseudo-Anosov $\phi \in \mathrm{Mod}(S_g)$ has $\ell_{\mathcal{C}}(\phi) < C/g$, then $\langle\langle \phi \rangle\rangle = \mathrm{Mod}(S_g)$?

Indeed, the first and the third authors of current paper, Kin and Shin, stated (1) in [B+23, Question 1.2].

REMARK 5.2. In the above questions, the factor $1/g$ is necessary since $L_{\mathcal{C}}(2g, g) \asymp 1/g$ [BS20] and due to Theorem 1.6. Furthermore, we separately state above two questions in order to forbid the trivial (Torelli) case in (2) and deal with the same problem.

Proof of Theorem 1.6. The family of pseudo-Anosovs constructed in Theorem 1.5 actually consists of non-normal generators, that is, $\langle\langle f_g \rangle\rangle \neq \mathrm{Mod}(S_g)$. To see this, recall that $f_g = T_{p_g^{-1}(\beta)} T^{-1}_{p_g^{-1}(\phi\beta)} T^{-1}_{p_g^{-1}(\phi\alpha)}$. It can be rewritten as

$$f_g = T_{p_g^{-1}(\beta)} (\tilde{\phi} T^{-1}_{p_g^{-1}(\beta)} \tilde{\phi}^{-1}) (\tilde{\phi} T^{-1}_{p_g^{-1}(\alpha)} \tilde{\phi}^{-1}).$$

Hence, it follows that $\langle\langle f_g \rangle\rangle \leq \langle\langle T_{p_g^{-1}(\beta)}, T_{p_g^{-1}(\alpha)} \rangle\rangle$, where the right-hand side means the smallest normal subgroup containing $T_{p_g^{-1}(\beta)}$ and $T_{p_g^{-1}(\alpha)}$.

Since each component of $p_g^{-1}(\beta)$ is separating, $T_{p_g^{-1}(\beta)}$ is Torelli, namely, contained in the kernel of the symplectic representation $\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$. Moreover, any two components of $p_g^{-1}(\alpha)$ bound an essential subsurface, so they are homologous, which means that $T_{p_g^{-1}(\alpha)}$ acts the same as $T_{\tilde{\alpha}}^{g-1}$ on $H_1(S_g; \mathbb{Z})$. As such, $T_{p_g^{-1}(\alpha)}$ acts trivially on the mod $(g-1)$ homology $H_1(S_g, \mathbb{Z}/(g-1)\mathbb{Z})$. Hence, we have that the symplectic representation of $T_{p_g^{-1}(\alpha)}$ is contained

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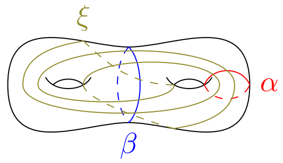

Figure 5 β and ξ fill the surface.

in the kernel of $\mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/(g-1)\mathbb{Z})$. Consequently, the normal closure $\langle\langle T_{p_g^{-1}(\beta)}, T_{p_g^{-1}(\alpha)}\rangle\rangle$ is contained in the kernel of the composition

$$\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z}) \to \mathrm{Sp}(2g,\mathbb{Z}/(g-1)\mathbb{Z}),$$

which is surjective. It follows that

$$\langle\langle f_g\rangle\rangle \le \langle\langle T_{p_g^{-1}(\beta)}, T_{p_g^{-1}(\alpha)}\rangle\rangle \ne \mathrm{Mod}(S_g),$$

so f_g is not a normal generator as desired.

Note that we have a concrete upper bound for $\ell_{\mathcal{C}}(f_g)$ in (4.2):

$$\ell_{\mathcal{C}}(f_g) \le \frac{2}{\lfloor \frac{g-3}{i(\phi\beta,\alpha)+i(\phi\alpha,\alpha)} \rfloor} \le \frac{2(i(\phi\beta,\alpha)+i(\phi\alpha,\alpha))}{g-3-(i(\phi\beta,\alpha)+i(\phi\alpha,\alpha))}.$$

Hence, once we fix α, β, and ϕ, we get a quantitative restriction on the constant C in the above questions. For instance, we can consider the configuration as in Figure 5.

Let $\lambda = T_\xi\beta$. As β and ξ fill the surface S_2, β and $\lambda = T_\xi\beta$ also fill the surface. Since β is separating, $\lambda = T_\xi\beta$ is also separating. Hence, due to Penner [Pen88] or Thurston [Thu88], $\phi = T_\lambda T_\beta^{-1}$ is a Torelli pseudo-Anosov. Furthermore, it follows that β and $\phi\beta$ also fill the surface. Therefore, we can construct f_g as in Theorem 1.5 starting with α, β, and ϕ depicted above.

Since $i(\xi,\beta) = 6$, $i(\lambda,\beta) = i(T_\xi\beta,\beta) = i(\xi,\beta)^2 = 36$ by [FM11, Proposition 3.2]. Now, from $\phi\alpha = T_\lambda\alpha$ and $\phi\beta = T_\lambda\beta$, we have

$$i(\phi\alpha,\alpha) = i(T_\lambda\alpha,\alpha) = i(\lambda,\alpha)^2 = 144,$$
$$i(\phi\beta,\alpha) = i(T_\lambda\beta,\alpha) = i(\lambda,\beta)i(\lambda,\alpha) = 432.$$

Hence, for the resulting f_g,

$$\ell_{\mathcal{C}}(f_g) \le \frac{1152}{g-579}$$

for $g > 579$. Consequently, we conclude Theorem 1.6. □

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H. Baik
Department of Mathematical Sciences
KAIST
291 Daehak-ro Yuseong-gu
Daejeon, 34141
South Korea

hrbaik@kaist.ac.kr

D. M. Kim
Department of Mathematics
Yale University
219 Prospect Street
New Haven, CT 06511
USA

dongryul.kim@yale.edu

C. Wu
Department of Mathematics
University of Wisconsin–Madison
480 Lincoln Drive
Madison, WI 53706
USA

cwu367@math.wisc.edu

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