RIGIDITY OF KLEINIAN GROUPS VIA SELF-JOININGS

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Abstract. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$ be a finitely generated non-Fuchsian Kleinian group whose ordinary set $\Omega = \mathbb{S}^2 - \Lambda$ has at least two components. Let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be a faithful discrete non-Fuchsian representation with boundary map $f : \Lambda \to \mathbb{S}^2$ on the limit set.

In this paper, we obtain a new rigidity theorem: if f is *conformal on* Λ, in the sense that f maps every circular slice of Λ into a circle, then f extends to a Möbius transformation g on $\mathbb{S}^2$ and ρ is the conjugation by g. Moreover, unless ρ is a conjugation, the set of circles C such that $f(C \cap \Lambda)$ is contained in a circle has empty interior in the space of all circles meeting Λ. This answers a question asked by McMullen on the rigidity of maps $\Lambda \to \mathbb{S}^2$ sending vertices of every tetrahedron of zero-volume to vertices of a tetrahedron of zero-volume.

The novelty of our proof is a new viewpoint of relating the rigidity of Γ with the higher rank dynamics of the self-joining $(\mathrm{id} \times \rho)(\Gamma) < \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$.

1. Introduction

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ be a finitely generated torsion-free Kleinian group. Consider the following discreteness locus of Γ in the space of representations of Γ into $\mathrm{PSL}_2(\mathbb{C})$:

$$\mathfrak{R}_{\mathrm{disc}}(\Gamma) = \{\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C}) : \text{discrete, faithful}\};$$

each $\rho \in \mathfrak{R}_{\mathrm{disc}}(\Gamma)$ gives rise to a hyperbolic manifold $\rho(\Gamma)\backslash\mathbb{H}^3$ which is homotopy equivalent to $\Gamma\backslash\mathbb{H}^3$. Another commonly used notation for $\mathfrak{R}_{\mathrm{disc}}(\Gamma)$ is $\mathcal{AH}(\Gamma)$ where $\mathcal{H}$ stands for hyperbolic and $\mathcal{A}$ for the topology on this space given by the algebraic convergence (cf. [27]).

We denote by $\mathrm{Möb}(\mathbb{S}^2)$ the group of all Möbius transformations on $\mathbb{S}^2$, by which we mean the group generated by inversions with respect to circles in $\mathbb{S}^2$. As well-known, $\mathrm{Möb}(\mathbb{S}^2)$ is equal to the group of conformal automorphisms of $\mathbb{S}^2$. The group $\mathrm{PSL}_2(\mathbb{C})$ can be identified with the subgroup consisting of compositions of even number of inversions with respect to circles in $\mathbb{S}^2$; in particular, it is a normal subgroup of $\mathrm{Möb}(\mathbb{S}^2)$ of index two. This means that conjugations by elements of $\mathrm{Möb}(\mathbb{S}^2)$ are contained in $\mathfrak{R}_{\mathrm{disc}}(\Gamma)$; we call them *trivial* elements of $\mathfrak{R}_{\mathrm{disc}}(\Gamma)$. Note that $\rho \in \mathfrak{R}_{\mathrm{disc}}(\Gamma)$ is trivial if and only if $\Gamma\backslash\mathbb{H}^3$ and $\rho(\Gamma)\backslash\mathbb{H}^3$ are isometric to each other.

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The rigidity question on Γ concerns a criterion on when a given representation

$$\rho \in \mathfrak{R}_{\text{disc}}(\Gamma)$$

is trivial. Denote by $\Lambda \subset \mathbb{S}^2$ the limit set of Γ, that is, the set of all accumulation points of $\Gamma(o)$, $o \in \mathbb{H}^3$. A ρ-equivariant continuous embedding

$$f : \Lambda \to \mathbb{S}^2$$

is called a ρ-boundary map. There can be at most one ρ-boundary map. Two important class of representations admitting boundary maps are as follows. Firstly, if both Γ and $\rho(\Gamma)$ are geometrically finite, and ρ is type-preserving, then the ρ-boundary map always exists by Tukia [29]. Secondly, if ρ is a quasiconformal deformation of Γ, i.e., there exists a quasiconformal homeomorphism $F : \mathbb{S}^2 \to \mathbb{S}^2$ such that for all $\gamma \in \Gamma$, $\rho(\gamma) = F \circ \gamma \circ F^{-1}$, then the restriction of F to Λ is the ρ-boundary map.

The fundamental role played by the boundary map in the study of rigidity of Γ is well-understood, going back to the proofs of Mostow's and Sullivan's rigidity theorems ([19], [20], [25]). By the Ahlfors measure conjecture ([2], [3]) now confirmed by the works of Canary [7], Agol [1] and Calegari-Gabai [6], the limit set Λ is either all of $\mathbb{S}^2$ or of Lebesgue measure zero. Mostow rigidity theorem ([19], [20], [21]) says that if Γ is a lattice, that is, if $\Gamma\backslash\mathbb{H}^3$ has finite volume, then any $\rho \in \mathfrak{R}_{\text{disc}}(\Gamma)$ is trivial; he obtained this by showing that the ρ-boundary map has to be conformal on $\mathbb{S}^2$. More generally, for any finitely generated Kleinian group Γ with $\Lambda = \mathbb{S}^2$, Sullivan showed that any quasiconformal deformation of Γ is trivial [25]. In fact, Sullivan's original theorem says that any ρ-equivariant quasiconformal homeomorphism of $\mathbb{S}^2$ which is conformal on the ordinary set $\Omega = \mathbb{S}^2 - \Lambda$ is a Möbius transformation. However Ahlfors measure conjecture implies that this is meaningful only when $\Lambda = \mathbb{S}^2$ (cf. [14, Section 3.13]).

In this paper, we concern the case when $\Lambda \neq \mathbb{S}^2$. For example, any geometrically finite Kleinian group which is not a lattice satisfies $\Lambda \neq \mathbb{S}^2$ [26]. We prove that if the ρ-boundary map is *conformal on* Λ, then ρ is trivial, provided the ordinary set $\Omega = \mathbb{S}^2 - \Lambda$ has at least two connected components. By the "conformality of f on Λ", we mean that f maps *circles in* Λ into circles.

Circular slices. The main result of this paper is the following rigidity theorem in terms of the behavior of f on circular slices of Λ: a circular slice of Λ is a subset of the form $C \cap \Lambda$ for some circle $C \subset \mathbb{S}^2$. We denote by $\mathcal{C}_\Lambda$ the space of all circles in $\mathbb{S}^2$ meeting Λ.

Theorem 1.1. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a finitely generated Zariski dense Kleinian group whose ordinary set Ω has at least two components. Let $\rho \in \mathfrak{R}_{\text{disc}}(\Gamma)$ be a Zariski dense representation with boundary map $f : \Lambda \to \mathbb{S}^2$.*

If f maps every circular slice of Λ into a circle, then ρ is a conjugation by some $g \in \text{Möb}(\mathbb{S}^2)$ and $f = g|_\Lambda$.

Moreover, unless ρ is a conjugation, the following subset of $\mathcal{C}_\Lambda$

$$\{C \in \mathcal{C}_\Lambda : f(C \cap \Lambda) \text{ is contained in a circle}\} \tag{1.1}$$

has empty interior.

We call Λ *doubly stable* if for any $\xi \in \Lambda$, there exists a circle $C \ni \xi$ such that for any sequence of circles C_i converging to C, $\# \limsup(C_i \cap \Lambda) \geq 2$. The assumption that Γ is finitely generated with Ω disconnected was used only to ensure that Λ is doubly stable (Lemma 3.2, Theorem 4.3).

Remark 1.2.
(1) This theorem holds rather trivially when $\Lambda = \mathbb{S}^2$, in which case all circular slices of Λ are circles.
(2) If $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is geometrically finite with connected limit set, then Ω is disconnected (cf. [16, Chapter IX]); hence Theorem 1.1 applies.

Tetrahedra of zero-volume. A quadruple of points in $\mathbb{S}^2$ determines an ideal tetrahedron of the hyperbolic 3-space $\mathbb{H}^3$. Gromov-Thurston's proof of Mostow rigidity theorem for closed hyperbolic 3-manifolds uses the fact that a homeomorphism of $\mathbb{S}^2$ mapping vertices of a maximal volume tetrahedron to vertices of a maximal volume tetrahedron is a Möbius transformation ([10] [28, Chapter 6]). In view of this, Curtis McMullen asked us whether one can consider the other extreme type of tetrahedra, namely, those of zero-volume in the study of rigidity of Γ.

Noting that $f : \Lambda \to \mathbb{S}^2$ maps every circular slice of Λ into a circle if and only if f maps any quadruple of points in Λ lying in a circle into a circle, the following is a reformulation of Theorem 1.1, which answers McMullen's question in the affirmative:

Theorem 1.3. *Let Γ, ρ be as in Theorem 1.1. If the ρ-boundary map $f : \Lambda \to \mathbb{S}^2$ maps vertices of every tetrahedron of zero-volume to vertices of a tetrahedron of zero-volume, then f is the restriction of a Möbius transformation g and ρ is the conjugation by g.*

Cross ratios. Theorem 1.3 can also be stated in terms of cross ratios: note that for four distinct points $\xi_1, \xi_2, \xi_3, \xi_4 \in \hat{\mathbb{C}}$, the cross ratio $[\xi_1 : \xi_2 : \xi_3 : \xi_4]$ is a real number if and only if all $\xi_1, \xi_2, \xi_3, \xi_4$ lie in a circle.

Corollary 1.4. *Let Γ, f be as in Theorem 1.1. If $[f(\xi_1) : f(\xi_2) : f(\xi_3) : f(\xi_4)] \in \mathbb{R}$ for any distinct $\xi_1, \xi_2, \xi_3, \xi_4 \in \Lambda$ with $[\xi_1 : \xi_2 : \xi_3 : \xi_4] \in \mathbb{R}$, then f extends to a Möbius transformation on $\hat{\mathbb{C}}$.*

On the proof of Theorem 1.1. The novelty of our approach is to relate the rigidity question for a Kleinian group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ with the dynamics of one parameter diagonal subgroups on the quotient of a higher rank semisimple *real* algebraic group $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ by a self-joining discrete subgroup.

For a given $\rho \in \mathfrak{R}_{\mathrm{disc}}(\Gamma)$, we consider the following self-joining of Γ via ρ:

$$\Gamma_\rho = (\mathrm{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\},$$

which is a discrete subgroup of G. A basic but crucial observation is that ρ is trivial if and only if Γ_ρ is not Zariski dense in G (Lemma 4.1). Our strategy is then to prove that if f maps *too many* circular slices of Λ into circles, then Γ_ρ cannot be Zariski dense in G. We achieve this by considering the action of Γ_ρ on the space $\mathcal{T}_\rho$ of all tori in the Furstenberg boundary $\mathbb{S}^2 \times \mathbb{S}^2$ intersecting the limit set $\Lambda_\rho = \{(\xi, f(\xi)) \in \mathbb{S}^2 \times \mathbb{S}^2 : \xi \in \Lambda\}$. Here a torus means an ordered pair of circles in $\mathbb{S}^2$.

(1) On one hand, using the Koebe-Maskit theorem ([15], [23], see Theorem 3.4) and the hypothesis that the ordinary set Ω has at least 2 components, we show the existence of a torus $T \in \mathcal{T}_\rho$ such that
$$T \notin \overline{\Gamma_\rho T_0}$$
for any torus $T_0 = (C_0, D_0)$ with $f(C_0 \cap \Lambda) \subset D_0$; in particular $\overline{\Gamma_\rho T_0} \neq \mathcal{T}_\rho$.

(2) On the other hand, we prove in Theorem 2.1 that the Zariski density of Γ_ρ implies the existence of a dense subset $\tilde{\Lambda}_\rho$ of Λ_ρ such that $\overline{\Gamma_\rho T_0} = \mathcal{T}_\rho$ for any torus T_0 meeting $\tilde{\Lambda}_\rho$. Denoting by A the two dimensional diagonal subgroup of G, the main ingredients for this step are the existence of a dense orbit of some *regular* one-parameter diagonal semigroup in the non-wandering set of the A-action on $\Gamma_\rho\backslash G$ (Theorem 2.2) as well as a theorem of Prasad-Rapinchuk [22] on the existence of $\mathbb{R}$-regular elements (Theorem 2.4). Therefore, if the subset (1.1) has non-empty interior, we can find a torus $T_0 = (C_0, D_0)$ satisfying that $f(C_0 \cap \Lambda) \subset D_0$ and $\overline{\Gamma_\rho T_0} = \mathcal{T}_\rho$.

The incompatibility of (1) and (2) implies that either the subset (1.1) has empty interior or Γ_ρ is not Zariski dense in G, as desired.

Question. There are several different proofs of Mostow rigidity theorem ([19], [20], [21]). By the viewpoint suggested in this paper, it will be interesting to find yet another proof, which directly shows the following reformulation: for any lattice $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ and $\rho \in \mathfrak{R}_{\mathrm{disc}}(\Gamma)$, the self-joining Γ_ρ is not Zariski dense in $\mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$.

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2. Dense orbits in the space of Tori

Let $G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$ and let $X = \mathbb{H}^3 \times \mathbb{H}^3$ be the Riemannian product of two hyperbolic 3-spaces. It follows from $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$ that $G \simeq \mathrm{Isom}^\circ(X)$. In the whole paper, we regard G as a *real* algebraic group and the Zariski density of a subset of G is to be understood accordingly. The action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{H}^3$ extends continuously to the compactification $\mathbb{H}^3 \cup \partial\mathbb{H}^3$ and its action on $\partial\mathbb{H}^3 \simeq \mathbb{S}^2$ is given by the Möbius transformation action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{S}^2$. We set $\mathcal{F} = \mathbb{S}^2 \times \mathbb{S}^2$, which coincides with the so-called Furstenberg boundary of G. Note that $\mathcal{F}$ is not the geometric boundary of X. Clearly, the action of G extends continuously to the compact space $X \cup \mathcal{F}$.

For a Zariski dense subgroup Δ of G, its limit set $\Lambda_\Delta \subset \mathcal{F}$ is defined as all possible accumulation points of $\Delta(o)$, $o \in X$, *on* $\mathcal{F}$. It is a *non-empty* Δ-minimal subset of $\mathcal{F}$ ([4, Section 3.6], [13, Lemma 2.13]).

By a torus T, we mean an ordered pair $T = (C_1, C_2) \subset \mathcal{F}$ of circles in $\mathbb{S}^2$. The group G acts on the space of tori by extending the action of $\mathrm{PSL}_2(\mathbb{C})$ on the space of circles componentwise. The main goal of this section is to prove the following: denote by $\mathcal{T}_\Delta$ the space of all tori in $\mathcal{F}$ intersecting Λ_Δ.

Theorem 2.1. *Let Δ be a Zariski dense subgroup of G. There exists a dense subset $\tilde{\Lambda}_\Delta$ of Λ_Δ such that for any torus T with $T \cap \tilde{\Lambda}_\Delta \neq \emptyset$, the orbit ΔT is dense in $\mathcal{T}_\Delta$.*

This theorem may be viewed as a higher rank analogue of [18, Theorem 4.1]. The rest of this section is devoted to its proof. It is convenient to use the upper half-space model of $\mathbb{H}^3$ so that $\partial\mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$. The visual maps $G \to \mathcal{F}$, $g \mapsto g^\pm$, are defined as follows: for $g = (g_1, g_2) \in G$ with $g_i \in \mathrm{PSL}_2(\mathbb{C})$,

$$g^+ = (g_1(\infty), g_2(\infty)) \quad \text{and} \quad g^- = (g_1(0), g_2(0)).$$

For $t \in \mathbb{C}$, we set $a_t = \mathrm{diag}(e^{t/2}, e^{-t/2})$ and define the following subgroups of G:

$$A = \{(a_{t_1}, a_{t_2}) : t_1, t_2 \in \mathbb{R}\} \text{ and } M = \{(a_{t_1}, a_{t_2}) : t_1, t_2 \in i\mathbb{R}\}.$$

For $u = (u_1, u_2) \in \mathbb{R}^2$, we write $a_u = (a_{u_1}, a_{u_2})$ and consider the following one-parameter semisubgroup

$$A_u^+ = \{a_{tu} : t \geq 0\}.$$

A loxodromic element $h \in \mathrm{PSL}_2(\mathbb{C})$ is of the form $h = \varphi a_{t_h} m_h \varphi^{-1}$ where $t_h > 0$ and $m_h \in \mathrm{PSO}(2)$ are uniquely determined and $\varphi \in \mathrm{PSL}_2(\mathbb{C})$. We call $t_h > 0$ the Jordan projection of h and m_h the rotational component of h. The attracting and repelling fixed points of h on $\mathbb{S}^2$ are given by $y_h = \varphi(\infty)$ and $y_{h^{-1}} = \varphi(0)$, respectively.

For a loxodromic element $g = (g_1, g_2) \in G$, that is, each g_i is loxodromic, its Jordan projection $\lambda(g)$ and the rotational component $\tau(g)$ are defined componentwise: $\lambda(g) = (t_{g_1}, t_{g_2}) \in \mathbb{R}^2_{>0}$ and $\tau(g) = (m_{g_1}, m_{g_2}) \in M$.

Dense A_u^+-orbit. For a Zariski dense subgroup Δ of G, we consider the following AM-invariant subset

$$\mathcal{R}_\Delta = \{[g] \in \Delta \backslash G : g^+, g^- \in \Lambda_\Delta\}.$$

Let $\mathcal{L} = \mathcal{L}_\Delta \subset \mathbb{R}^2_{\geq 0}$ denote the limit cone of Δ, which is the smallest closed cone containing the Jordan projection $\lambda(\Delta) = \{\lambda(\delta) : \delta \in \Delta\}$. The Zariski density of Δ implies that $\mathcal{L}$ has non-empty interior [4, Section 1.2].

We use the following theorem which is an immediate consequence of the result of Dang [9] (this also follows from [8] and [5]):

Theorem 2.2. *For any Zariski dense subgroup $\Delta < G$ and any $u \in \operatorname{int} \mathcal{L}_\Delta$, there exists a dense A_u^+-orbit in $\mathcal{R}_\Delta$.*

Proof. As shown in [9, Theorem 7.1 and its proof], the semigroup $S^+ := \{a_u^n : n \in \mathbb{N} \cup \{0\}\}$ acts on $\mathcal{R}_\Delta$ topologically transitively: for any non-empty open subsets $\mathcal{O}_1, \mathcal{O}_2$ of $\mathcal{R}_\Delta$, $\mathcal{O}_1 a_u^n \cap \mathcal{O}_2 \neq \emptyset$ for some $n \in \mathbb{N}$. This implies the existence of a dense S^+-orbit on $\mathcal{R}_\Delta$ (cf. [24, Proposition 1.1]). Since $S^+ \subset A_u^+$, this proves the claim. □

In the following, we fix $u \in \operatorname{int} \mathcal{L}_\Delta$ and a dense A_u^+-orbit, say $[g_0]A_u^+$, in $\mathcal{R}_\Delta$, provided by Theorem 2.2. Set

$$\tilde{\Lambda}_\Delta = \Delta g_0^+ = \left\{ \delta g_0^+ \in \Lambda_\Delta : \delta \in \Delta \right\}; \tag{2.1}$$

note that this is a dense subset of Λ_Δ, as Λ_Δ is a Δ-minimal subset.

Denote by $\mathcal{T}_\Delta^\spadesuit$ the space of all tori T with $\# T \cap \Lambda_\Delta \geq 2$.

Corollary 2.3. *For any torus T meeting $\tilde{\Lambda}_\Delta$, the closure of ΔT contains $\mathcal{T}_\Delta^\spadesuit$.*

Proof. Note that $H = \mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{R})$ is a subgroup of G, as $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$. The space $\mathcal{T}$ of all tori in $\mathcal{F}$ can be identified with the quotient space G/H. Let T be a torus containing $\delta_0 g_0^+ \in \tilde{\Lambda}_\Delta$ for some $\delta_0 \in \Delta$. By the identification of $\mathcal{T} = G/H$, we may write $T = gH$ for some $g \in G$. Then for some $h \in H$, $(gh)^+ = \delta_0 g_0^+$. If we denote by P the stabilizer subgroup of (∞, ∞) in G, which is equal to the product of two upper triangular subgroups of $\mathrm{PSL}_2(\mathbb{C})$, this implies that for some $p \in P$, $gh = \delta_0 g_0 p$. Write $p = nam$ where n belongs to the strict upper triangular subgroup, $a \in A$ and $m \in M$. We claim that $\overline{[g]hA_u^+} \supset (\mathcal{R}_\Delta - [g_0]A_u^+)ma$. Let $x \in \mathcal{R}_\Delta - [g_0]A_u^+$. Since $\overline{[g_0]A_u^+} = \mathcal{R}_\Delta$, there exists a sequence $t_i \to +\infty$ such that $x = \lim_{i \to \infty} [g_0] a_{t_i u}$. Since $u = (u_1, u_2) \in \operatorname{int} \mathcal{L}_\Delta$, we have $u_1 > 0, u_2 > 0$, and hence $a_{-t_i u} n a_{t_i u} \to e$ as $i \to \infty$.

Therefore

$$\lim_{i \to \infty} [g] h a_{t_i u} = \lim_{i \to \infty} [g_0] n a m a_{t_i u} = \lim_{i \to \infty} [g_0] a_{t_i u} (a_{-t_i u} n a_{t_i u}) a m = x a m;$$

so $xam \in \overline{[g]hA_u^+}$. This proves the claim. Since $\mathcal{R}_\Delta$ is AM-invariant, and $\mathcal{R}_\Delta - [g_0]AM$ is dense in $\mathcal{R}_\Delta$ (as Λ_Δ is a perfect set), it follows that

$$\overline{[g]hA_u^+} \supset \mathcal{R}_\Delta.$$

Since $A_u^+ \subset H$, this implies that $\overline{[g]H} \supset \mathcal{R}_\Delta H$. Since $\mathcal{R}_\Delta H = \Delta \backslash \mathcal{T}_\Delta^\spadesuit$ and $T = gH$, we get $\overline{\Delta T} \supset \mathcal{T}_\Delta^\spadesuit$, as desired. □

Loxodromic element $\delta \in \Delta$ with $\tau(\delta)$ generating M. We use the following special case of a theorem of Prasad and Rapinchuk [22]:

Theorem 2.4. [22, Theorem 1, Remark 1] *Any Zariski dense subgroup $\Delta < G$ contains a loxodromic element δ such that $\tau(\delta)$ generates a dense subgroup of M.*

Corollary 2.5. *If Δ is Zariski dense in G, then $\mathcal{T}_\Delta^\spadesuit$ is dense in $\mathcal{T}_\Delta$.*

Proof. Let $\delta = (\delta_1, \delta_2) \in \Delta$ be as given by Theorem 2.4. Since M has no isolated point, there exists a sequence m_j, which we may assume tends to $+\infty$, by replacing δ by δ^{-1} if necessary, that $\tau(\delta)^{m_j}$ converges to e. It follows that the semigroup generated by $\tau(\delta)$ is also dense in M. Let $T = (C_1, C_2) \in \mathcal{T}_\Delta$ be any torus. It suffices to construct a sequence $T_n = (C_{1,n}, C_{2,n}) \in \mathcal{T}_\Delta^\spadesuit$ which converges to T. We begin by fixing a point $\xi = (\xi_1, \xi_2) \in T \cap \Lambda_\Delta$. Since Δ acts minimally on Λ_Δ, there exists a sequence $\delta_n = (\delta_{1,n}, \delta_{2,n}) \in \Delta$ such that that $\delta_n y_\delta$ converges to ξ as $n \to \infty$; recall that $y_\delta \in \mathcal{F}$ denotes the attracting fixed point of δ. Fix a point $\eta = (\eta_1, \eta_2) \in \Lambda_\Delta - \{y_\delta, y_{\delta^{-1}}\}$.

For each fixed $n \in \mathbb{N}$, note that, as $k \to \infty$, the sequence $\delta_n \delta^k \eta$ converges to $\delta_n y_\delta$, while rotating around $\delta_n y_\delta$ by the amount given by $\tau(\delta)^k$. Since $\tau(\delta)$ generates a dense semigroup of M, we can find a sequence $k_n \to \infty$ such that for each $i = 1, 2$,

$$d(\delta_{i,n} y_{\delta_i}, \delta_{i,n} \delta_i^{k_n} \eta_i) < \tfrac{1}{n} \quad \text{and} \quad \tfrac{\pi}{2} - \tfrac{1}{n} < \theta_{i,n} < \tfrac{\pi}{2} + \tfrac{1}{n}$$

where $\theta_{i,n}$ is the angle at $\delta_{i,n} y_{\delta_i}$ of the triangle determined by the center of C_i, $\delta_{i,n} y_{\delta_i}$ and $\delta_{i,n} \delta_i^{k_n} \eta_i$. For each $i = 1, 2$, we now choose $p_i \in C_i - \bigcup_n \{\delta_{i,n} y_{\delta_i}, \delta_{i,n} \delta_i^{k_n} \eta_i\}$ and set $C_{i,n}$ to be the circle passing through $\delta_{i,n} y_{\delta_i}, \delta_{i,n} \delta_i^{k_n} \eta_i$ and p_i.

From the construction, each sequence $C_{i,n}$ converges to the circle tangent to C_i at ξ_i and passing through $p_i \in C_i$, which must be equal to C_i itself; therefore if we set $T_n = (C_{1,n}, C_{2,n})$,

$$T_n \to T \quad \text{as} \quad n \to \infty.$$

Since $T_n \cap \Lambda_\Delta$ contains both $\delta_n y_\delta$ and $\delta_n \delta^{k_n} \eta$, we have $T_n \in \mathcal{T}_\Delta^\spadesuit$. This completes the proof. □

Proof of Theorem 2.1. It suffices to consider the set $\tilde{\Lambda}_\Delta$ defined in (2.1) by Corollary 2.3 and Corollary 2.5.

3. Limits of circular slices and Koebe-Maskit theorem

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary Kleinian group and $\Omega = \mathbb{S}^2 - \Lambda$ its ordinary set, i.e., $\Lambda \subset \mathbb{S}^2$ denotes the limit set of Γ. We refer to [14] and [17] for general facts on the theory of Kleinian groups.

Definition 3.1. (1) We call a circle C doubly stable for Λ if for any sequence of circles C_i converging to C, $\# \limsup(C_i \cap \Lambda) \geq 2$.

(2) We call Λ doubly stable if for any $\xi \in \Lambda$, there exists a circle $C \ni \xi$, which is doubly stable for Λ.

The main goal of this section is to prove the following lemma:

Lemma 3.2. *If Γ is finitely generated and Ω is not connected, then Λ is doubly stable.*

In the rest of this section, we assume Γ is finitely generated. Lemma 3.2 is an immediate consequence of the following lemma, since, if $\xi_1, \xi_2 \in \Omega$ belong to different components of Ω, then for any $\xi \in \Lambda$, the circle C passing through ξ, ξ_1, ξ_2 is not contained in the closure of any component of Ω.

Lemma 3.3. *Let $C \subset \mathbb{S}^2$ be a circle such that $C \not\subset \overline{\Omega_0}$ for any component Ω_0 of Ω. If C_n is a sequence of circles converging to C, then*[1]

$$\# \limsup (C_n \cap \Lambda) \geq 2.$$

The main ingredient is the following formulation of the Koebe-Maskit theorem ([15, Theorem 6], [23, Theorem 1]):

Theorem 3.4. *Let $\{\Omega_i\}$ be the collection of all components of the ordinary set Ω. Then for any $\alpha > 2$, $\sum_i \mathrm{Diam}(\Omega_i)^\alpha < \infty$ where $\mathrm{Diam}(\Omega_i)$ is the diameter of Ω_i in the spherical metric on $\mathbb{S}^2$.*

We will only need the following immediate corollary of Theorem 3.4:

Corollary 3.5. *For any $\varepsilon > 0$, there are only finitely many components of the ordinary set of Γ with diameter bigger than ε.*

Proof of Lemma 3.3. Given Corollary 3.5, the proof is similar to the proof of [12, Lemma 8.1], which deals with the case when all components of Ω are round disks.

Let C and $C_n \to C$ be as in the statement of the lemma. It suffices to show that there exists $\varepsilon_0 > 0$ such that $C_{n_i} \cap \Lambda$ contains two points of distance at least ε_0 for some infinite sequence $n_i \to \infty$. Suppose not. Then, letting I_n be the minimal connected subset of C_n containing $C_n \cap \Lambda$, we have $\mathrm{Diam}(I_n) \to 0$ as $n \to \infty$.

Setting $\eta = \mathrm{Diam}(C)/2$, we have $\mathrm{Diam}(C_n) > \eta$ for all sufficiently large n. Let $0 < \varepsilon < \eta/4$ be arbitrary. Since $\mathrm{Diam}(I_n) \to 0$, we have $\mathrm{Diam}(I_n) < \varepsilon$ for all large n. Noting that $C_n - I_n$ is a connected subset of Ω, let Ω_n be the connected component of Ω containing $C_n - I_n$. Then C_n is contained in the ε-neighborhood of Ω_n, which implies

$$\mathrm{Diam}(\Omega_n) \geq \mathrm{Diam}(C_n) - 2\varepsilon > \eta/2.$$

By Corollary 3.5, the collection $\{\Omega_n : \mathrm{Diam}(\Omega_n) > \eta/2\}$ must be a finite set, say, $\{\Omega_1, \cdots, \Omega_N\}$. Therefore, for some $1 \leq j \leq N$, there exists an infinite sequence C_{n_i} contained in the ε-neighborhood of Ω_j. Hence C is contained in the 2ε-neighborhood of Ω_j. Since the collection $\{\Omega_1, \cdots, \Omega_N\}$ does not

[1] For a sequence of subsets S_n in a topological space, we define $\limsup S_n = \bigcap_n \overline{\bigcup_{i \geq n} S_i}$.

depend on ε, we can find a sequence $\varepsilon_k \to 0$ and a fixed $1 \le j \le N$ such that C is contained in the $2\varepsilon_k$-neighborhood of Ω_j. It follows that $C \subset \overline{\Omega_j}$, contradicting the hypothesis on C. This finishes the proof.

4. Self-joinings of Kleinian groups and Proof of Theorem 1.1.

Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a Zariski dense discrete subgroup with limit set Λ. As before, we denote by $\Omega = \mathbb{S}^2 - \Lambda$ its ordinary set.

We fix a discrete faithful representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ such that $\rho(\Gamma)$ is Zariski dense.

We now define the self-joining of Γ via ρ as

$$\Gamma_\rho := (\mathrm{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\},$$

which is a discrete subgroup of G.

We begin by recalling two standard facts:

Lemma 4.1. *The subgroup Γ_ρ is Zariski dense in G if and only if ρ is not a conjugation by an element of* $\mathrm{M\ddot{o}b}(\mathbb{S}^2)$.

Proof. It is clear that if ρ is a conjugation by an element of $\mathrm{M\ddot{o}b}(\mathbb{S}^2)$, then Γ_ρ is not Zariski dense in G. To see the converse, let $G_0 < G$ be the Zariski closure of Γ_ρ and suppose that $G_0 \neq G$. Denote by $\pi_i : G = \mathrm{PSL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ the projection onto the i-th component.

We now claim that $\pi_1|_{G_0}$ is an isomorphism. Since Γ is Zariski dense, $\pi_1|_{G_0}$ is surjective. Hence, it suffices to show that $\pi_1|_{G_0}$ is injective. Note that $G_0 \cap \ker \pi_1 = G_0 \cap (\{e\} \times \mathrm{PSL}_2(\mathbb{C}))$ is a normal subgroup of G_0. Hence, $G_0 \cap \ker \pi_1$ is normalized by $\{e\} \times \mathrm{PSL}_2(\mathbb{C})$ since $\rho(\Gamma)$ is Zariski dense $\mathrm{PSL}_2(\mathbb{C})$. Thus, $G_0 \cap \ker \pi_1$ is a normal subgroup of $\ker \pi_1$. As $\ker \pi_1 \cong \mathrm{PSL}_2(\mathbb{C})$ is simple, $G_0 \cap \ker \pi_1$ is either trivial or $\{e\} \times \mathrm{PSL}_2(\mathbb{C})$. In the latter case, note that $\{e\} \times \mathrm{PSL}_2(\mathbb{C}) < G_0$. Since $\pi_1|_{G_0}$ is surjective, it follows that $G_0 = G$, yielding contradiction. Therefore $\pi_1|_{G_0}$ is injective, and hence an isomorphism. Similarly, $\pi_2|_{G_0}$ is an isomorphism. Hence, $\pi_2|_{G_0} \circ \pi_1|_{G_0}^{-1}$ is a Lie group automorphism of $\mathrm{PSL}_2(\mathbb{C})$. Hence it is a conjugation by a Möbius transformation (cf. [11]). Since this map restricts to ρ on Γ, it finishes the proof. $\square$

Since ρ gives an isomorphism from Γ to $\rho(\Gamma)$ and f is an equivariant embedding, it follows that ρ maps every loxodromic element γ to a loxodromic element $\rho(\gamma)$ and f sends the attracting fixed point of $\gamma \in \Gamma$ to the attracting fixed point of $\rho(\gamma)$. Since the set of attracting fixed points of loxodromic elements of Γ is dense in Λ, this implies the following.

Lemma 4.2. *There can be at most one ρ-boundary map $f : \Lambda \to \mathbb{S}^2$. In particular, if ρ is a conjugation by $g \in \mathrm{M\ddot{o}b}(\mathbb{S}^2)$, then $f = g|_\Lambda$.*

Proof of Theorem 1.1. By Lemma 3.2, Theorem 1.1 follows from the following:

Theorem 4.3. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a Zariski dense Kleinian group such that Λ is doubly stable. Let $\rho \in \mathfrak{R}_{\mathrm{disc}}(\Gamma)$ be a Zariski dense representation with boundary map $f : \Lambda \to \mathbb{S}^2$. Unless ρ is a conjugation, the subset*

$$\Lambda_f := \bigcup \{C \cap \Lambda : f(C \cap \Lambda) \text{ is contained in a circle}\} \tag{4.1}$$

has empty interior in Λ; hence

$$\{C \in \mathcal{C}_\Lambda : f(C \cap \Lambda) \text{ is contained in a circle}\}$$

has empty interior in $\mathcal{C}_\Lambda$.

Proof. If $\Lambda = \mathbb{S}^2$, it is easy to prove this. So we assume below that $\Lambda \neq \mathbb{S}^2$. Suppose that ρ is not a conjugation, so that Γ_ρ is Zariski dense by Lemma 4.1. It follows easily from the minimality of the limit set Λ_ρ of Γ_ρ that

$$\Lambda_\rho = \{(\xi, f(\xi)) \in \mathbb{S}^2 \times \mathbb{S}^2 : \xi \in \Lambda\}. \tag{4.2}$$

Let $\tilde{\Lambda}_{\Gamma_\rho}$ be as in Theorem 2.1, which must be of the form $\{(\xi, f(\xi)) : \xi \in \tilde{\Lambda}\}$ for some dense subset $\tilde{\Lambda}$ of Λ.

Suppose on the contrary that Λ_f has non-empty interior. Then $\Lambda_f \cap \tilde{\Lambda} \neq \emptyset$. It follows that there exists $C_0 \in \mathcal{C}_\Lambda$ such that $C_0 \cap \tilde{\Lambda} \neq \emptyset$ and $f(C_0 \cap \Lambda)$ is contained in some circle, say, D_0. Set $T_0 = (C_0, D_0)$. Since $C_0 \cap \tilde{\Lambda} \neq \emptyset$, it follows from Theorem 2.1 that

$$\overline{\Gamma_\rho T_0} = \mathcal{T}_\rho \tag{4.3}$$

where $\mathcal{T}_\rho = \mathcal{T}_{\Gamma_\rho}$ is the space of all tori intersecting Λ_ρ. On the other hand, we now show that the condition $f(C_0 \cap \Lambda) \subset D_0$ implies that $\Gamma_\rho T_0$ cannot be dense in $\mathcal{T}_\rho$, using Lemma 3.3.

Step 1: There exists a circle D which intersects $\Lambda_{\rho(\Gamma)}$ precisely at one point, say $f(\xi_0)$. To show this, fix any $f(\xi) \in \Lambda_{\rho(\Gamma)}$ and let D' be the boundary of the minimal disk B' centered at $f(\xi)$ which contains all of $\Lambda_{\rho(\Gamma)}$. By the minimality of B', $D' \cap \Lambda_{\rho(\Gamma)} \neq \emptyset$. Choose $f(\xi_0) \in D' \cap \Lambda_{\rho(\Gamma)}$, and let D be a circle tangent to D' at $f(\xi_0)$ which does not intersect the interior of B'.

Step 2: By the hypothesis that Λ is doubly stable, we can find a circle C containing ξ_0 which is doubly stable for Λ.

Step 3: Setting $T = (C, D)$, we have $T \notin \overline{\Gamma_\rho T_1}$ for any torus $T_1 = (C_1, D_1)$ with $f(C_1 \cap \Lambda) \subset D_1$. In particular, $T \notin \overline{\Gamma_\rho T_0}$.

Suppose on the contrary that there exists a sequence $\gamma_n \in \Gamma$ such that $\gamma_n C_1$ converges to C and $\rho(\gamma_n) D_1$ converges to D. Since C is doubly stable for Λ, we have

$$\# \limsup (\gamma_n C_1 \cap \Lambda) \geq 2. \tag{4.4}$$

By the ρ-equivariance of f, we have

$$f(\gamma_n C_1 \cap \Lambda) = f(\gamma_n (C_1 \cap \Lambda)) = \rho(\gamma_n) f(C_1 \cap \Lambda) \subset \rho(\gamma_n) D_1 \cap \Lambda_{\rho(\Gamma)}.$$

Hence

$$\limsup f(\gamma_n C_1 \cap \Lambda) \subset \limsup (\rho(\gamma_n) D_1 \cap \Lambda_{\rho(\Gamma)}) \subset D \cap \Lambda_{\rho(\Gamma)}.$$

It now follows from (4.4) and the injectivity of f that

$$\#D \cap \Lambda_{\rho(\Gamma)} \geq 2.$$

This contradicts the choice of D made in Step (1), hence proving Step (3).

Since $(\xi_0, f(\xi_0)) \in T \cap \Lambda_\rho$, we have $T \in \mathcal{T}_\rho$. Hence we obtained a contradiction to (4.3). Therefore Λ_f has empty interior, completing the proof. □

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