SOLUTIONS TO QUIZ 5

(1) (4 pt) Verify that

$$y_1(t) = e^{-t}$$

is a solution of the

$$-(t+1)y''(t) - ty'(t) + y(t) = 0, t > 0$$

Do it yourself

(2) (3 pt) Find another solution of the above equation.

A Cheating Solution: The simplest solution that do not require any thinking is: $y_2(t) = 2e^{-t}$. Although this does not help to obtain the general solution, it did answer the question. You will get 3 points if you do that. But don't expect the same thing in the exams.

The appropriate way to ask shall be:

"Find another solution of the above equation that is linearly independent to $y_1(t) = e^{-t}$."

Really Quick Solution: If you have really done your homework 11, you will see that actually this problem is very similar to Problem 3d, where you should have found that $y_2(t) = t$ is a solution. In fact, it can be seen almost immediately that $y_2(t) = t$ IS ALSO A PARTICULAR SOLUTION HERE. So THIS IS THE ANSWER. All you need to do is to verify as in Problem (1) and also verify that $W(y_1, y_2) \neq 0$ to see the linear independency.

If you don't know what do I mean by linear independency, please read Section 3.2 in the book carefully as Dr. Z asked in his emails.

Standard Solution: If you did not think of the above argument, then probably you have got no choice but to proceed the standard procedure. Putting $y_1(t) = e^{-t}$ into the equation of $v(t)$, you will obtain

$$e^{-t}v''(t) + (-2e^{-t} + \frac{t}{t+1}e^{-t})v'(t) = 0.$$

By my method, you will have

$$(\ln(v'(t)))' = \frac{v''(t)}{v'(t)} = 2 - \frac{t}{t+1} = 2 - \frac{t+1-1}{t+1} = 1 + \frac{1}{t+1}.$$

Watch the signs here!! Many students messed up here and obtained something that cannot be integrated. Then we have

$$\ln(v'(t)) = t + \ln(t+1).$$

Taking exponential, you have

$$v'(t) = e^t(t+1).$$

Perform the integration by parts:

$$v(t) = \int e^t(t+1)dt = \int (t+1)de^t = (t+1)e^t - \int e^t d(t+1) = (t+1)e^t - \int e^t dt = te^t$$

So the solution is

$$y_2(t) = v(t)y_1(t) = te^t e^{-t} = t.$$

By Dr. Z's method, assuming $u(t) = v'(t)$, you will have a first-order linear homogeneous equation

$$u'(t) + (-2 + \frac{t}{t+1})u(t) = 0.$$

Recall that the general solution of a general first-order linear equation

$$u'(t) + p(t)u(t) = g(t)$$

is

$$y(t) = \frac{\int I(t)g(t)dt}{I(t)},$$

where $I(t) = e^{\int p(t)dt}$. Now for the equation in our hand, you have $g(t) = 0$, so the numerator becomes a constant which nobody cares about. So taking the constant to be 1 you will have

$$u(t) = \frac{1}{I(t)}.$$

It suffices to compute $I(t)$. You should have seen the integration of $t/(t+1)$ in the previous quizzes. If you did forget how to do this, my board work has also showed the trick of "plus 1 minus 1". In more detail:

$$\int \left(-2 + \frac{t}{t+1}\right) dt = \int \left(-2 + \frac{t+1-1}{t+1}\right) dt = \int \left(-1 - \frac{1}{t+1}\right) dt = -t - \ln|t+1| + C.$$

Since the absolute value and the constant here does not interfere with the final solution, you will have

$$v'(t) = u(t) = \frac{1}{e^{-t-\ln(t+1)}} = e^t(t+1).$$

The rest is totally the same as above.

(3) (1 pt) Write down the general solution.

Solution: The Wronskian theorem tells you that

$$y(t) = C_1 e^{-t} + C_2 t.$$

Or you can just use the standard procedure to conclude that

$$y(t) = C_1 y_1(t) + C_2 v(t) y_1(t) = C_1 e^{-t} + C_2 te^t e^{-t} = C_1 e^{-t} + C_2 t.$$