QUIZ 7 FOR CALC 4

(1) Find the largest open interval for the following differential equation where you are guaranteed to have a solution

$$\sin 2t\ y^{(4)}(t) + \tan t\ y(t) = t, y(\frac{\pi}{4}) = 0, y'(\frac{\pi}{4}) = 1, y''(\frac{\pi}{4}) = 0, y'''(\frac{\pi}{4}) = -1$$

Although many students got this correct, very few have really noticed that $\tan t$ also blows up at $x = k\pi + \pi/2, k = 0, \pm 1, \pm 2, \cdots$. You should always keep in mind what the existence and uniqueness theorems says: the equation

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_{n-1}(t)y'(t) + p_n(t)y(t) = g(t), y(t_0) = y_0, y'(t_0) = z_0$$

have a unique solution if ALL $p(t), q(t)$ AND $g(t)$ ARE CONTINUOUS!
So here is the argument: $\sin 2t = 0$ for $t = 0, \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi, \cdots$. And $\tan t$ blows up at $t = \pm\pi/2, \pm 3\pi/2, \cdots$. So the maximal interval that both contains the initial $\pi/4$ and makes the theorem work is $(0, \pi/2)$.
For those who got full points, think about the example: what if the equation is changed to $\sin 2t\ y^{(4)}(t) + \tan 3t\ y(t) = \sec t$ and you are asked to find the maximal interval?

(2) Find the general solution of the equation

$$y''(t) - 4y'(t) + 4y(t) = \frac{e^{2t}}{1+t^2}$$

This is so similar to the homework problem and I have already written on my blackboard a harder problem. I'll just post the procedure (with typo possibly) below

First we have <u>two solutions of the homogeneous version</u>:

$$y_1(t) = e^{2t}, y_2(t) = te^{2t}.$$

Now <u>formulate the linear equation</u> for $u_1'(t), u_2'(t)$

$$\begin{aligned} e^{2t}u_1'(t) + te^{2t}u_2'(t) &= 0 \\ 2e^{2t}u_1'(t) + (2t+1)e^{2t}u_2'(t) &= \frac{e^{2t}}{1+t^2} \end{aligned}$$

Let's <u>solve it</u>. The first equation implies

$$u_1'(t) = -tu_2'(t).$$

Substituting into the second,

$$-2te^{2t}u_2'(t) + (2t+1)e^{2t}u_2'(t) = \frac{e^{2t}}{1+t^2}.$$

This shows immediately

$$u_2'(t) = \frac{1}{1+t^2}$$

and then

$$u_1'(t) = -tu_2'(t) = -\frac{t}{1+t^2}.$$

Now perform the integration to obtain

$$u_1(t) = -\frac{1}{2}\ln(1+t^2)$$

(we ignored the constant and also the absolute value here) and

$$u_2(t) = \arctan t$$

So the particular solution we are looking for is

$$P(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = -\frac{1}{2}\ln(1+t^2)e^{2t} + (\arctan t)te^{2t}$$

and the general solution is

$$y(t) = c_1e^{2t} + c_2te^{2t} - \frac{1}{2}\ln(1+t^2)e^{2t} + (\arctan t)te^{2t}.$$

(3) What is the integrating factor of the first order differential equation

$$y'(t) + p(t)y(t) = g(t)?$$

and what is the general solution?

Don't make any mistakes: the integrating factor is

$$I(t) = \exp(\int p(t)dt)$$

and the general solution is

$$y(t) = \frac{\int I(t)g(t)}{I(t)}$$

I would like to add an important remark explaining why I gave this problem in this quiz. In fact this formula is also obtained by the method of variation of parameters. More precisely, notice that

$$y_1(t) = \frac{C}{\exp(\int p(t)dt)} = \frac{C}{I(t)}$$

is the general solution of the homogeneous equation

$$y'(t) + p(t)y(t) = 0$$

(by separating variables). Now we change the constant C into some $u(t)$. Then by the linear equation

$$u'(t)y_1(t) = g(t),$$

we have

$$u'(t) = I(t)g(t),$$

so

$$u(t) = \int I(t)g(t)dt$$

and the general solution is

$$y(t) = Cy_1(t) + u(t)y_1(t) = \frac{C + \int I(t)g(t)}{I(t)} = \frac{\int I(t)g(t)}{I(t)}$$

(the constant C is now cooperated into the integral).

In fact the technique also works for n-th order equation. You can google "variation of perimeter" to find the general version. Since that's not required in the course, I am not going to write them here.