**Problem 1.** Consider the initial value

$$\frac{dy}{dt} = \frac{1+y^4}{4(t+2)y^3}, y(0) = 1$$

(a). Solve for $y$. Express your solution in the form $y = F(t)$.

(b). What is the interval of existence?

**Solution.** Separating the variables

$$\frac{4y^3 dy}{1+y^4} = \frac{dt}{t+2}$$

and integrate:

$$\ln|1+y^4| = \ln|t+2| + C$$

Exponentiate:

$$(1+y^4) = C(t+2)$$

Solve for $y$:

$$y = \pm\sqrt[4]{C(t+2)-1}$$

The initial values says $y(0) = 1$, so $y$ is positive:

$$1 = \sqrt[4]{C(t+2)-1}$$

Solve for $C$, one gets

$$C = 1$$

and thus the solution to the IVP is

$$y = \sqrt[4]{t+1}$$

...answer to Part (a)

This solution makes sense when

$$t+1 \geq 0$$

So the interval of existence is

$$[-1, \infty)$$

...answer to Part (b)

**Grading Break-up.** 15 points for Part (a). 5 points for Part (b). If Part (a) solution is incorrect but Part (b) is correct based on the incorrect answer in Part (b), no point will be deducted.

**Common Errors.**

- (SERIOUS) Many people wrote $\ln|1+y^4| = \ln|t+2| + C \Rightarrow 1+y^4 = t+2+C$ and thus unfortunately commit the crime of abusing algebra. Currently I am nice enough to deduct only 7 points if you are lucky enough to get the correct answer. In the future no partial credits will be given.
- For nonlinear ODE, you should NEVER use the existence and uniqueness theorem to determine the interval of existence. The theorem is not so strong as that! The only way to do it is through the explicit expression of solutions. cf. Rec. Notes #3 Page 23. I'll give 1 point as partial credit in this case.
- $\sqrt[n]{0}$ makes sense and it is simply 0 as long as $n$ is a positive integer! Only negative power of 0 will blow up!. So there is no point to exclude $-1$ in the interval of existence.

**Problem 2.** Consider the initial value problem:

$$3x^2 + 2xy^2 + 2x^2y\frac{dy}{dx} = 0, \quad y(2) = -3$$

(a) Show that the equation is exact.

(b) Solve the equation and find an explicit expression for $y$.

(c) Find the interval of existence for the solution.

**Solution.** Set

$$M = 3x^2 + 2xy^2, N = 2x^2y.$$

Check the exactness:

$$M_y = 4xy, N_x = 4xy.$$

So we have exactness.

...answer to Part (a)

Recover $\psi$ by integrating $M$ with the variable $x$:

$$\psi(x, y) = \int M dx = \int (3x^2 + 2xy^2)dx = x^3 + x^2y^2 + \phi(y).$$

Take partial derivative of $\psi$ with the variable $y$:

$$\frac{\partial \psi}{\partial y} = 2x^2y + \phi'(y),$$

and equate it to $N$:

$$2x^2y + \phi'(y) = 2x^2y$$

Therefore

$$\phi'(y) = 0 \Rightarrow \phi(y) = 0$$

So

$$\psi(x, y) = x^3 + x^2y^2$$

and the general (implicit) solution to the ODE is

$$x^3 + x^2y^2 = C$$

Put in the initial value $y(2) = -3$ to get $C$:

$$2^3 + 2^2(-3)^2 = 8 + 36 = 44 = C$$

And solve $y$ to get the explicit expression:

$$y = \pm\sqrt{\frac{44 - x^3}{x^2}}.$$

The initial value $y(2) = -3$ says $y$ should be negative, so we omit the positive sign. Therefore

$$y = -\sqrt{\frac{44 - x^3}{x^2}}.$$

...answer to Part (b)

This solution makes sense when

$$\frac{44 - x^3}{x^2} \geq 0, x \neq 0$$

Solve it:

$$x \neq 0, x \leq \sqrt[3]{44}$$

Since the initial value is given at $x = 2$, the interval of existence is

$$(0, \sqrt[3]{44}]$$

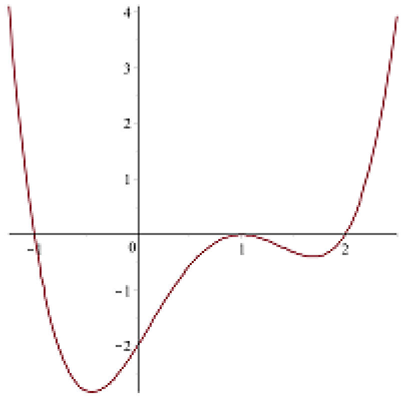
...answer to Part (c)

**Grading Break-up.** 5 points for Part (a). 10 points for Part (b). 5 points for Part (c). If Part (b) solution is incorrect but Part (c) is correct based on the incorrect answer in Part (b), no point will be deducted.

**Common Errors.**

- If $\phi(y)$ involves $x$ and you dare to continue, at least 5 points will be gone. $\phi(y)$ SHOULD ONLY DEPEND ON $y$! cf. Rec. Notes #5 Page 11, the remarks on the right in blue.
- Quite a few regarded the solution of exact ODE as $y = \psi(x, y) = x^3 + x^2y^2 + C$, which is absolute nonsense. Please READ Rec. Notes #4 Page 9 to 16! At least 5 points will be deducted.
- Similarly the existence and uniqueness theorem for nonlinear ODE DOES NOT APPLY here. 1 point will be given as partial credit.
- In Part (b) you are asked to solve this equation with an INITIAL VALUE and you are asked to find the EXPLICIT expression. 2 point will be deducted if you did not solve for $C$ and 2 point will be deducted if you did not find the explicit solution. If the $\pm$ sign is left in your answer, 1 point will be deducted.
- Similarly, $\sqrt{0}$ makes sense and it is simply 0. So there is no point of excluding $\sqrt[3]{44}$ in the final solution.

**Problem 3.** Consider the differential equation $\frac{dy}{dx} = f(y)$ where the graph of $f(y)$ given below.



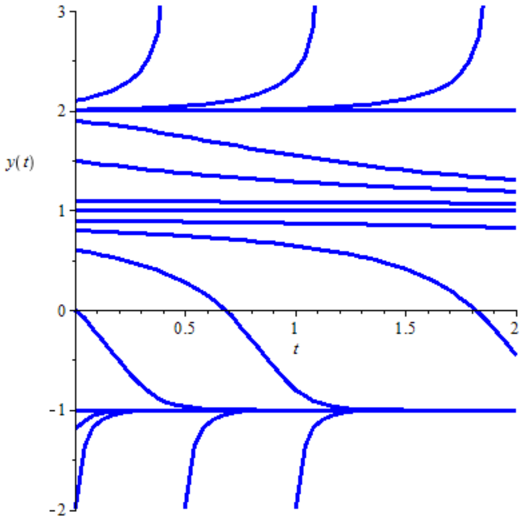

(a). Draw the phase line (use the space above)

(b). Sketch a few solutions showing the behavior near equilibrium points and asymptotic behavior

(c). Determine the equilibrium points and classify them as asymptotically stable, unstable or semi-stable.

**Solution.** Phase line looks like


And the sketch for Part (b) looks like




The equilibrium $y = 2$ is unstable.
The equilibrium $y = 1$ is semistable.
The equilibrium $y = -1$ is stable.

**Grading Break-up.** 4 points for Part (a). 3 points for Part (b) 3 points for Part (c). Due to the difficulty of the problem, no partial credit will be given for wrong answers. For example, if two arrows are wrong in Part (a), then 2 points deducted in Part (a), 1 point deducted in Part (b) and 2 point deducted in Part (c).

**Common Errors.**

- There has been confusions on the three objects: direction field, integral curves and phase line. Although for the sake of stability they are the same, in essence they are very different. Please check Rec. Notes #5 Page 6 and 7. If the tendency is correct, then only 1 point will be deducted each part.
- The integral curves in your drawing of Part (b) should not intersect the equilibrium solution. It is even wrong to connect the integral curve above 1 and below 1. 1 point will be deducted for the former and 2 points for the latter.

**Problem 4.** Consider the differential equation

$$y'' - 2y' - 8y = 0\,, \quad y(0) = \alpha, y'(0) = 2$$

(a). Solve the initial value problem.

(b). The solution can exhibit 3 possible long term behaviors depending on the value of $\alpha$. Determine them (along with the corresponding $\alpha$ values).

**Solution.** The characteristic equation is

$$r^2 - 2r - 8 = 0 \Rightarrow r_1 = 4, r_2 = -2$$

So the general solution of the ODE is

$$y(t) = C_1 e^{4t} + C_2 e^{-2t}$$

The initial values imply

$$\alpha = C_1 + C_2, 2 = 4C_1 - 2C_2$$

Solve:

$$C_1 = \frac{1}{3}(\alpha + 1), C_2 = \frac{1}{3}(2\alpha - 1)$$

So the solution to the IVP is

$$y(t) = \frac{1}{3}(\alpha + 1)e^{4t} + \frac{1}{3}(2\alpha - 1)e^{-2t}$$

...answer to Part (a)

When $t \to \infty$, the second term goes to zero. So the coefficient of the first term determines the long term behavior. In summary, as $t \to infty$

- $C_1 > 0$, i.e., $\alpha > -1$, then $y(t) \to \infty$.
- $C_1 = 0$, i.e., $\alpha = -1$, then $y(t) \to 0$.
- $C_1 < 0$, i.e., $\alpha < -1$, then $y(t) \to -\infty$.

...answer to Part (b)

**Grading Break-up.** 15 points for Part (a): 5 points for the general solution; 5 points for setting up the equation of $C_1$ and $C_2$; 5 points for getting the correct solution. 5 points for Part (b)

**Problem 5.** According to Newton's law of cooling the rate of change of the temperature $T$ of an object with respect to time $t$ is given by

$$\frac{dT}{dt} = -k(T - T_a)$$

where $T_a$ is the ambient (or room) temperature and $k$ is a positive constant.
A pot of liquid is put on the stove to boil. The temperature of the liquid reaches 170 °F and then the pot is taken off the burner and placed on a counter in the kitchen. The temperature of the air in the kitchen is 76 °F.

(a). Obtain an expression for $T(t)$, the temperature of the liquid at time $t$. *(Your expression will include $k$)*

(b). After two minutes the temperature of the liquid in the pot is 123 °F. Find an expression for $k$.

**Solution.** Separate the variables

$$\frac{dT}{T - T_a} = -kdt$$

and integrate:

$$\ln|T - T_a| = -kt + C$$

Exponentiate

$$T - T_a = Ce^{-kt}$$

So

$$T = T_a + Ce^{-kt}$$

...answer to Part (a)

Now that $T_a = 76, T(0) = 170, T(2) = 123$, we have

$$170 = 76 + C, 123 = 76 + Ce^{-2k}$$

Therefore $C = 94$ and $e^{-2k} = (123 - 76)/94 = 47/94 = 1/2$. Hence

$$-2k = \ln(1/2) = -\ln 2 \Rightarrow k = \frac{\ln 2}{2}$$

...answer to Part (b)

0.1. **Grading Break-up.** 10 Points for the general solution, 5 points for getting the right constant $C$ and 5 point for getting $k$.

**Problem 6.** Suppose you are trying to solve $y' = \sqrt{t+y}$, $y(0) = 1$ numerically.

(a) If you use the Euler method what will be the local truncation error? Give your answer in terms of $t$, step size $h$ and the solution $\Phi$.

(b) Suppose you tried the Runge-Kutta method with step size $h = 0.1$ and got an error of $2 \times 10^{-1}$ . What do you expect the error to be if you change the step size to 0.01?

**Solution.** Let $y = \phi(t)$ be the solution. Then we have

$$\phi'(t) = \sqrt{t + \phi(t)}.$$

Differentiate:

$$\begin{aligned}\phi''(t) &= (\sqrt{t+\phi(t)})' = \frac{1}{2\sqrt{t+\phi(t)}}(t+\phi(t))' \\ &= \frac{1}{2\sqrt{t+\phi(t)}}(1+\phi'(t)) = \frac{1}{2\sqrt{t+\phi(t)}}(1+\sqrt{t+\phi(t)}) \\ &= \frac{1}{2\sqrt{t+\phi(t)}} + \frac{1}{2}\end{aligned}$$

Therefore the local truncation error can be expressed as

$$e_{n+1} = \frac{1}{2}|\phi''(\overline{t_n})|h^2 = \left|\frac{1}{4\sqrt{\overline{t_n} + \phi(\overline{t_n})}} + \frac{1}{4}\right| h^2$$

...answer to Part (a)

As the local truncation error of Runge-Kutta method is proportion to $h^5$, in general if $h' = ah$ then the error $e' = a^5 e$. Now $e = 2 \times 10^{-1}, h' = 0.01 = 0.1h$, then

$$e' = (0.1)^5 e = 10^{-5} \times 2 \times 10^{-1} = 2 \times 10^{-6}.$$

...answer to Part (b)

**Grading Break-up.** 5 points for Part (a). Wrong $\phi''(t)$ costs 2 points. 5 points for Part (b). Wrong order of $h$ costs 3 points.

**TA's Comment.** I did not grade this problem so I can't say what error is common. But for Part (b), I happened to notice that many people wrote $e_{n+1} = |\phi''(\overline{t_n})|h^5/2$, which is VERY WRONG! Although no points would be deducted for such writing, it is very important to know that ONLY the error of Euler method can be estimated using $\phi''(t)$. The error for RK method is MUCH MORE COMPLICATED than that!