MATH 310 FALL 2019
PROBLEM SET 1

Due Thursday, Sep 12th, at 11:35 AM, in class.

(1) Find all possible complex numbers corresponding to the following expressions
  (a) $\sqrt{i}$
  (b) $\sqrt[3]{-1+i}$
  (c) $\left(\frac{1+3i}{2}\right)^6$
  (d) $e^{-2+i\frac{\pi}{3}}$
  (e) $\log(\frac{e^5}{2} + i\frac{e^5\sqrt{3}}{2})$
  (f) $\cos(i)$

(2) Show that the following functions are holomorphic, then find their complex derivative
  (a) $f(z) = \cos(z)$
  (b) $f(z) = \sin(z)$
  (c) $f(z) = \sum_{k=0}^{n} a_k z^k$, where a_k are fixed complex numbers

(3) Define the function

$$f(z) = \begin{cases} e^{-z^{-4}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that f satisfies the Cauchy-Riemann equations in all $\mathbb{C}$. Then show that f is not holomorphic at 0. What went wrong?

(4) Take $f = u + iv$ a holomorphic function. Show that after writting (x, y) in polar coordinates (r, θ) the Cauchy-Riemann equations are equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

(5) Show that if $f : U \to \mathbb{C}$ is holomorphic on a domain U and f has image contained in some (1-dimensional) real affine line inside $\mathbb{C} = \mathbb{R}^2$, then f is constant.

(6) Use the following example to show that holes can be an obstruction to find a harmonic conjugate.
  - Verify that $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ defined as $u(x, y) = \log(\sqrt{x^2 + y^2})$ is harmonic.
  - Take $(-1, -1)$ as a base point. Use the formula of harmonic conjugate for u along the polygonal paths $(-1, -1) \to (-1, 1) \to (1, 1)$ and $(-1, -1) \to (1, -1) \to (1, 1)$. Compare the two values obtained and draw a conclusion about finding a harmonic conjugate for u in $\mathbb{C} \setminus \{0\}$

(7) Denote by $\langle v\,, w\rangle$ the dot product for $v, w \in \mathbb{R}^2$. We say that a function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is conformal if it has continuous first order partial real derivatives and for any $v, w \in \mathbb{R}^2, p \in U$ we have that $\frac{\langle v,w\rangle}{|v|.|w|} = \frac{\langle \mathcal{J}_f(p)v, \mathcal{J}_f(p)w\rangle}{|\mathcal{J}_f(p)v|.|\mathcal{J}_f(p)w|}$. We say that f preserves orientation if $\det(\mathcal{J}_f(p)) > 0$ for all $p \in U$. Show that if a function is conformal and preserves orientation then is holomorphic. Show also that if f is holomorphic and $f'(z) \neq 0$, then f is conformal at z and preserves orientation. (Hint: find a description of 2×2 real matrices A so that the multiplication with vectors $v \to A.v$ is angle preserving)

A cornerstone of our thinking is that in the infinitely small every function becomes linear (from an unknown mathematical physicist, 1915).