MATH 310 FALL 2019
PROBLEM SET 1

Due Thursday, Sep 12th, at 11:35 AM, in class.

(1) Find all possible complex numbers corresponding to the following expressions
   (a) $\sqrt{\imath}$
   (b) $\sqrt[3]{-1 + \imath}$
   (c) $(\frac{1+3\imath}{2})^6$
   (d) $e^{-2+\frac{\pi}{3} \imath}$
   (e) $\log(\frac{e^{\sqrt{3}}}{2} + i e^{\sqrt{3}})$
   (f) $\cos(i)$

(2) Show that the following functions are holomorphic, then find their complex derivative
   (a) $f(z) = \cos(z)$
   (b) $f(z) = \sin(z)$
   (c) $f(z) = \sum_{k=0}^{n} a_k z^k$, where $a_k$ are fixed complex numbers

(3) Define the function
   $$f(z) = \begin{cases} e^{-z} - 4 & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
   Show that $f$ satisfies the Cauchy-Riemann equations in all $\mathbb{C}$. Then show that $f$ is not holomorphic at 0. What went wrong?

(4) Take $f = u + iv$ a holomorphic function. Show that after writting $(x, y)$ in polar coordinates $(r, \theta)$
the Cauchy-Riemann equations are equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

(5) Show that if $f : U \to \mathbb{C}$ is holomorphic on a domain $U$ and $f$ has image contained in some (1-dimensional) real affine line inside $\mathbb{C} = \mathbb{R}^2$, then $f$ is constant.

(6) Use the following example to show that holes can be an obstruction to find a harmonic conjugate.
   - Verify that $u : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ defined as $u(x, y) = \log(\sqrt{x^2 + y^2})$ is harmonic.
   - Take $(-1, -1)$ as a base point. Use the formula of harmonic conjugate for $u$ along the polygonal paths $(-1, -1) \to (-1, 1) \to (1, 1)$ and $(-1, -1) \to (1, -1) \to (1, 1)$. Compare the two values obtained and draw a conclusion about finding a harmonic conjugate for $u$ in $\mathbb{C} \setminus \{0\}$.

(7) Denote by $\langle v, w \rangle$ the dot product for $v, w \in \mathbb{R}^2$. We say that a function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is
conformal if it has continuous first order partial real derivatives and for any $v, w \in \mathbb{R}^2, p \in U$ we have that
$$\frac{\langle \frac{\partial f}{\partial r}, v \rangle}{\|v\|} = \frac{\langle \frac{\partial f}{\partial \theta}, v \rangle}{\|v\|}$$. We say that $f$ preserves orientation if $\det(\frac{\partial f}{\partial \theta}) > 0$ for all $p \in U$. Show that if a function is conformal and preserves orientation then is holomorphic. Show also that if $f$ is holomorphic and $f'(z) \neq 0$, then $f$ is conformal at $z$ and preserves orientation. (Hint: find a description of $2 \times 2$ real matrices $A$ so that the multiplication with vectors $v \to A.v$ is angle preserving)

A cornerstone of our thinking is that in the infinitely small every function becomes linear (from an unknown mathematical physicist, 1915).