MATH 310 FALL 2019
PROBLEM SET 10

Due Tuesday, Dec 3rd, at 11:35 AM, in class.

(1) Show that $z^4 + 2z^2 - z + 1$ has exactly one root in each quadrant.

(2) For a fixed real number α, find the number of zeros of $z^4 + z^3 + 4z^2 + \alpha z + 3$ satisfying Re$z > 0$ (Answer depends on α).

(3) Show that if m, n are positive integers, then the polynomial

$$p(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^m}{m!} + 3z^n$$

has exactly n zeros in the unit disk.

(4) Suppose D is a bounded domain with piecewise smooth boundary. Let $f(z)$ be meromorphic and $g(z)$ holomorphic on $D \cup \partial D$, so that $f(z) \neq 0$ in ∂D. Show that

$$\frac{1}{2\pi i}\int_{\partial D} g(z)\frac{f'(z)}{f(z)}dz = \sum_{j=1}^{n} m_j g(z_j)$$

Where $\{z_j\}$ are the zeros and poles of f, and m_j is the order of $f(z)$ at z_j

(5) Let D be a bounded domain, and let $f(z)$ be a continuous function on $D \cup \partial D$ that is holomorphic in D. Show that $\partial f(D) \subseteq f(\partial D)$. (Recall that $f(D)$ is open). Can you give an example when the inclusion is strict?

(6) Let γ be a closed path in a domain D such that $W(\gamma, \zeta) = 0$ for all $\zeta \notin D$. Let f be a holomorphic function on D except at finitely many points $z_1, \ldots, z_m$ not in the image of γ. Show that

$$\int_\gamma f(z)dz = 2\pi i \sum_k W(\gamma, z_k)\mathrm{Res}[f, z_k]$$

"Die größte Schwierigkeit bei der Planung eines Lehrbuches der Funktionentheorie liegt in der Auswahl des Stoffes. Man muB sich von vornherein entschliefen, alle Fragen wegzulassen, deren Darstellung zu grofle Vorbereitungen verlangt. (The greatest difficulty in planning a textbook on function theory lies in the choice of material. You have to decide beforehand to leave aside all questions whose treatment requires too much preparatory development.)" CARATHEODORY