MATH 310 FALL 2019
PROBLEM SET 2

Due Thursday, Sep 19th, at 11:35 AM, in class.

(1) Show that the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in polar coordinates $x = r\cos\theta, y = r\sin\theta$ is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

(2) Check that for $u(z) = \mathrm{Im}\left(\frac{1}{z^2}\right)$, $z \neq 0$, $u(0) = 0$ the partial derivatives $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist and satisfy $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Then show that u is <u>not</u> harmonic. What went wrong?

(3) A *fractional linear transformation* (also known as *Möbius tranformation*) is a function of the form

$$f(z) = \frac{az+b}{cz+d}, \quad cz+d \neq 0$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$

(a) Show that $f'(z) = \frac{ad-bc}{(cz+d)^2}$

(b) Show that for $f(z) = \frac{az+b}{cz+d}, g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ the composition $f \circ g$ is a Möbius transformation. Moreover, show that the coefficients for the fraction expression of $f \circ g$ correspond to the matrix multiplication $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

(c) Show that the inverse function f^{-1} is a Möbius transformation defined by $f^{-1}(w) = \frac{-dw+b}{cw-a}$

(d) Show that the image of a real line under a fractional linear transformation is a line or a circle. Show also that the image of a circle under a fractional linear transformation is a line or a circle. (Hint: Show first that every linear transformation is the composition of a dilations $z \to \lambda z$, translations $z \to z + \eta$ and inversions $z \to \frac{1}{z}$. Then prove that each of those transformations send lines and circles to lines and circles). Extra challenge: Why we have to consider both lines and circles at the same time?

(4) Use Green's theorem to show that for a domain D with piecewise smooth boundary ∂D, $\mathrm{Area}(D) = \int_{\partial D} x dy$. Use this to calculate the area of a unit square and a unit ball in $\mathbb{C}$.

(5) Let $P, Q : D \to \mathbb{C}$ be continuous functions and let $\gamma : [a, b] \to D$ be piecewise smooth path.

(a) We say that a *reparametrization* of γ is a path $\tilde{\gamma} = \gamma \circ \phi$ where $\phi : [\alpha, \beta] \to [a, b]$ is a smooth function with $\phi'(s) > 0$ for all $s \in [\alpha, \beta]$ and $\phi(\alpha) = a$, $\phi(\beta) = b$. Show that $\int_{\tilde{\gamma}} Pdx + Qdy = \int_{\gamma} Pdx + Qdy$. Notice that this says that the line integral is always *independent from the parametrization.*

(b) Define γ^- as $\gamma^- : [-b, -a] \to D$ as $\gamma^-(t) = \gamma(-t)$. Show that $\int_{\gamma^-} Pdx+Qdy = -\int_{\gamma} Pdx+Qdy$. Notice that this says that the integral distinguishes the orientation of the path.

(c) Show that the following statements are equivalent

- $\int_{\gamma} Pdx + Qdy$ depends only on the endpoints of γ, for all paths γ .
- $\int_{\eta} Pdx + Qdy = 0$ for all closed paths η.

(6) Determine whether the following path integrals are independent of the path (i.e. depend only on the endpoints).

(a) $xdx + ydy$

(b) $x^2dx + y^5dy$

(c) $ydx + xdy$

(d) $ydx - xdy$

Calculus integralis est methodus, ex data differentialium relation inveniendi relationem ipsarum quantitatum (Integral calculus is the method for finding, from a given relation of differentials, the relation of the quantities themselves). L. EULER