MATH 310 FALL 2019
PROBLEM SET 3

Due Thursday, Sep 26th, at 11:35 AM, in class.

(1) Prove the Fundamental Theorem of Algebra by following these steps
  (a) Let $f : \overline{D} \to \mathbb{C}$ be a continuous function on a compact domain $\overline{D}$ so that f is holomorphic in the interior D. If $|f(z)| \leq M$ for all $z \in \partial D$, then $|f(z)| \leq M$ for all $z \in \overline{D}$. Why is it important that D is bounded?
  (b) If a polynomial $p : \mathbb{C} \to \mathbb{C}$ does not vanish then $1/p : \mathbb{C} \to \mathbb{C}$ is a well defined holomorphic function. See that for p not constant and for large R the values $1/p(Re^{i\theta})$ are arbitrarily small.
  (c) Use the Maximum Principle for $1/p$ for larger and larger balls to show that if p is not constant then exists z_0 so that $p(z_0) = 0$.

(2) Let $f : D \to \mathbb{C}$ be a holomorphic function with no zeros (i.e. does not vanish).
  (a) Show that if $|f(z)|$ attains its infimum in D then f is constant
  (b) Show that if D is bounded and f extends continuously to $\overline{D}$ then $|f|$ attains its minimum in ∂D.

(3) Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and bounded, where $\mathbb{D}$ is the unit disk. Assume that f extends continuously to $\partial\mathbb{D} \setminus \{z_1, z_2, \ldots, z_n\}$ so that $|f(z)| \leq M$ for $z \in \partial\mathbb{D} \setminus \{z_1, z_2, \ldots, z_n\}$. Show that $|f(z)| \leq M$ for any $z \in \mathbb{D}$. (Hint: work with $(z - z_1)^\epsilon f(z)$ for small epsilon. Why is $(z - z_1)^\epsilon := e^{\epsilon \log(z - z_1)}$ well-defined on the disk?)

(4) Show that for the following cases of domain U a holomorphic and bounded function $f : U \to \mathbb{C}$ that extends continuously to ∂U so that $|f(z)| \leq M$ for $z \in \partial U$ then $|f(z)| \leq M$ for $z \in U$. (Hint: find a map that identifies U with $\mathbb{D}$. What happens to the boundary ∂U?)
  (a) U the half-space $\{z \,|\, \mathrm{Im}(z) > 0\}$
  (b) U the strip $\{z \,|\, \pi > \mathrm{Im}(z) > 0\}$ (Think about the exponential map)

(5) Show that for D a bounded domain with piecewise smooth boundary $2i\mathrm{Area}(D) = \int_{\partial D} \overline{z} dz$.

(6) Show that a holomorphic function $f : D \to \mathbb{C}$ has a primitive (or antiderivative) F (meaning $\frac{dF}{dz} = f$) if and only $\int_\gamma f(z)dz = 0$ for any closed piecewise smooth path γ.

(7) It is customary to denote the infinitesimal arc-length by $|dz| = \sqrt{(dx)^2 + (dy^2)}$. Hence $\int_\gamma h(z)|dz|$ denotes $\int_a^b h(\gamma(t))\sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}dt$. Show that if h is so that $|h(z)| \leq M$ then $|\int_\gamma h(z)dz| \leq M. \int_\gamma |dz| = M.\mathrm{Length}(\gamma)$

(8) Take γ a path between 0 and i. Find the following integrals (and justify why you were given enough information to do so!)
  (a) $\int_\gamma e^{\pi z} dz$
  (b) $\int_\gamma (z - 1)^m dz$, $m \neq -1$ (Assume that γ does not pass through $z = 1$).
  (c) $\int_\gamma \cos(z) dz$

On a letter that Gauss wrote to Bessel on December 18, 1811: "What should we make of $\int \phi x.dx$ for $x = a + bi$?"