Let $f: \mathbb{D} \to \mathbb{C}$ be a continuous function on a compact domain $\mathbb{D}$, S.t. $f$ is holomorphic inside $\mathbb{D}$, i.e., $|f(z)| \leq M$ for all $z \in \mathbb{D}$, then $|f(z)| \leq M$ for all $z \in \mathbb{D}$.

We prove an important lemma (extreme value theorem for complex functions). Let $g: \mathbb{K} \to \mathbb{C}$ be a continuous function on a compact domain $\mathbb{K} \subseteq \mathbb{C}$, then $|g|$ is a continuous, and hence it maps compact sets to compact sets. Thus 

$$|g(K)| = \text{img}(g) = L \subseteq \mathbb{R} \text{ s.t. } L \text{ is compact},$$

Thus $L$ contains its supremum and infimum, i.e., $\exists L_s, L_i \in L$ s.t.

$$L_s = \sup_{x \in L} x \\
L_i = \inf_{x \in L} x$$

Then $bL = \text{img}(g)$, we get that there exist $p, q \in \mathbb{K}$ s.t.

$$|g(p)| = L_s = \sup_{x \in L} x \quad \text{and} \quad |g(q)| = L_i = \inf_{x \in L} x.$$ 

In other words, $g$ attains its supremum and infimum on $\mathbb{K}$ with respect to the modulus.

We assumed $\mathbb{D}$ was bounded, so by definition $\overline{\mathbb{D}}$ is closed and bounded, and compact. Since $f$ is continuous, we apply the lemma to get that $|f|$ attains its supremum/maximum on $\overline{\mathbb{D}}$, i.e., there $\exists z_0 \in \mathbb{D}$ s.t.

$$|f| \leq S \quad \text{and} \quad \exists z_0 \in \overline{\mathbb{D}} \text{ s.t. } |f(z_0)| = S.$$ 

We have two cases.

1. $z_0 \in \partial \mathbb{D}$. Then since $f$ is holomorphic, we apply the Maximum Principle for holomorphic functions to get that $f$ is constant on $\mathbb{D}$. But $f$ extends continuously to $\overline{\mathbb{D}}$ by assumption, and so $f$ is constant on $\partial \mathbb{D} = \overline{\mathbb{D}}$, i.e., Thus $|f(z)| = S \quad \forall z \in \overline{\mathbb{D}}$.

Since $|f(z)| \leq M$ for all $z \in \overline{\mathbb{D}}$, we must have that $S \leq M$, in which case $|f(z)| \leq M \quad \forall z \in \overline{\mathbb{D}}$.

2. $z_0 \in \mathbb{D}$. Then $|f(z_0)| = S \quad \forall z \in \mathbb{D}$, since $|f(z)| \leq M \quad \forall z \in \overline{\mathbb{D}}$, we get $S \leq M$.

Hence by def. of supremum $|f(z)| \leq S \leq M \quad z \in \overline{\mathbb{D}}$.

D must be bounded so that $\overline{\mathbb{D}}$ is compact for the result to hold, as it relies on results of compactness.
b) Suppose \( p: \mathbb{C} \to \mathbb{C} \) is a polynomial that does not vanish. By part 1, 
\( p \) is holomorphic on \( \mathbb{C} \) and into \( \mathbb{C} \), so \( p: \mathbb{C} \to \{0\} \).

Now, we also proved that \( \frac{1}{z} \) is holomorphic whenever \( z \neq 0 \), i.e.,
\( \frac{1}{z}: \{z \in \mathbb{C} | z \neq 0\} \to \mathbb{C} \) is holomorphic.

By class 9/3, the composition \( (g \circ p(z) = \frac{1}{p(z)}; \mathbb{C} \to \mathbb{C} \) is well-defined and holomorphic, since \( p(z) \neq 0 \) \( \forall z \in \mathbb{C} \).

Now suppose \( p \) is not constant. Hence \( p(z) = \sum a_n z^n \) \( \forall \) \( z \), the exists at
least one \( i \) \( \in \{1, \ldots, n\} \) such that \( a_i \neq 0 \). So, let \( \alpha \) be the largest such \( i \), and we have
that \( p \) is a complex polynomial of degree \( n \); we then from analysis that its
leading term dominates:

\[
\lim_{z \to 0} |p(z)| = \lim_{z \to 0} \left| \sum a_n z^n \right| = \lim_{z \to 0} \left| z^n \sum a_n z^{-n} \right|
\]

since \( \sum a_n z^{-n} = a_n z^{-n} + \cdots + a_n z^{-n} = \lim_{z \to \infty} |\sum a_n z^{-n}| \)

\[
= 0.
\]

Moreover, note that \( |\sum a_n \theta^n| = |\sum a_n \theta^n| \]

Puting these observations together, we have that \( p(Re^{i\theta}) \to \infty \) as \( R \to \infty \). Hence, as \( R \) gets large, the values of \( \frac{1}{p(Re^{i\theta})} \) get
arbitrarily small for fixed \( \theta \).

c) We want to show that if \( p \) is not constant then \( \exists z_0 \) s.t. \( p(z_0) = 0 \).

We proceed by contradiction. Suppose \( \forall z \in \mathbb{C} \), \( p(z) \neq 0 \) i.e., \( p \) does not vanish.
We want to show that \( p \) is not constant. Toward a contradiction, suppose that \( p \) is not constant.

Then we have by (b) that, in fixed \( \Theta \),

\[
\lim_{R \to \infty} \frac{|1|}{|p(Re^{i\Theta})|} = 0.
\]

Fix some \( z_0 \in \mathbb{C} \). Since \( p \) does not vanish, \( p(z_0) \neq 0 \) \( \neq 1/z_0 \).

Then there is some large \( R > 0 \) s.t. \( R > |z_0| \) and
\[
\left| \frac{1}{p(z)} \right| \leq \left| \frac{1}{p(z_0)} \right| \quad \text{for all } \Theta. \quad \text{In other words,}
\]

Let \( \text{Re} e = \theta \in [0, \pi] \) be any point on the boundary of \( B_R(0) \). Then the modulus of \( \frac{1}{p} \) along the boundary of \( B_R(0) \) is bounded above by \( \left| \frac{1}{p(z_0)} \right| \).

Considering \( B_R(0) \), a compact domain, we apply (a), setting \( \left| \frac{1}{p(z_0)} \right| = M \), we see that \( \left| \frac{1}{p(z)} \right| \leq M \) for all \( z \in B_R(0) \), and hence \( \left| \frac{1}{p(z)} \right| \leq M \) for all \( z \in B_R(0) \).

In particular, this tells us that \( \frac{1}{p(z)} \) achieves its supremum in \( B_R(0) \), at the point \( z_0 \in B_R(0) \). By the maximum modulus principle, we get that \( \frac{1}{p(z)} \) is constant on \( B_R(0) \).

This holds for all \( R \geq 1 \), to which the initial inequality held. But since \( \frac{1}{p(z_0)} = 0 \), take \( R \) to infinity to get that \( \frac{1}{p(z)} \) is constant on \( \bigcup_{R \geq 1} B_R(0) \).

Hence \( \frac{1}{p(z)} \) is constant, implying \( p(z) \) is constant in all \( C \). This contradicts our assumption, which completes the proof by contradiction.

Hence if \( p \) is not constant there exist \( z_0 \) so that \( p(z_0) = 0 \).
(2) Let $f : D \to \mathbb{C}$ be holomorphic with no zeros.

a) Take $g = \frac{1}{f}$, which exists everywhere since $f$ does not vanish and is holomorphic since $f$ and the function $z \mapsto \frac{1}{z}$ are both holomorphic (see class, 9.3).

Suppose $|f(z)|$ attains its supremum in $D$. Then by real analysis,

$$\frac{1}{|f(z)|} = \left| \frac{1}{f(z)} \right| \text{ attains its supremum in } D,$$

$g = \frac{1}{f}$ is a constant in $D$, as desired.

b) Suppose $D$ is bounded and $f$ extends continuously to $\bar{D}$. By (a), we know that $D$ is compact, so $f$ attains a supremum in $D$.

Thus by 2.1, $f$ is constant on $D$.

If $|f|$ attains its minimum in $D$, then by (a), $f$ is constant in $\mathbb{C}$.

Since $f$ extends continuously to $\bar{D}$ (go to the boundary), $f$ must be constant in $\partial D$ as well. Hence $|f|$ attains its minimum in $\partial D$. That is, since $|f(z)| = S$ for $z \in \partial D$ and $f \to \bar{D}$ continuously, $f(z) = S e^{i \theta}$ constantly in $\partial D$ and $|f(z)| = S$ for $z \in \partial D$.

Thus $|f|$ attains its minimum in $\partial D$.

If $|f|$ attains its minimum in $\partial D$, then we are done.

In either case, assuming $D$ is bounded and $f$ extends continuously to $\bar{D}$, $|f|$ attains its minimum in $\partial D$. 
3) Let \( f: \mathbb{D} \to \mathbb{C} \) be holomorphic and 1-1. Assume \( f \) extends continuously to \( \partial \mathbb{D} \setminus \{ z \} \) such that \( |f(z)| \leq M \) for \( z \in \mathbb{D} \setminus \{ z \} \). Show that if \( f(z) \) is any \( z \in \mathbb{D} \), then there exists a unique point \( z_0 \in \mathbb{D} \) such that \( f(z_0) = z \).

Suppose first that there exists only one discontinuity point \( z \), \( x \) extends continuously to \( \partial \mathbb{D} \setminus \{ z \} \) and \( f(z) = e^{\log(z - x)} \) is well-defined on the disk, i.e., why \( \log(z - x) \) has unique outputs on \( \mathbb{D} \).

Wlog, assume that \( z_1 = 1 \in \mathbb{D} \) (same up to rotation). Then \( \log(z - x) \) is a branch of \( z \). \( f(z) = e^{\log(z - x)} \) on \( \mathbb{D} \). However, the angles possible for the vector \( z - z_1 = z - 1 \) are in the open interval \((- \pi/2, \pi/2\)\), which means that \( \arg(z - z_1) \in (- \pi/2, \pi/2) \) and hence \( \log(z - z_1) \) can be chosen as the canonical branch to be well-defined on \( \mathbb{D} \). Moreover, the function \( e^{\log(z - z_1)} \) is holomorphic on \( \mathbb{D} \), since \( \log(z - z_1) \) is holomorphic on \( \mathbb{C} \setminus \{z_1\} \).

We prove that the "replacement function" \( f(z) \) extends continuously to \( z \in \partial \mathbb{D} \). First observe that \( (z - z_1)^2 = 0 \) for \( z - z_1 = 0 \), but for small \( \varepsilon > 0 \), \( (z - z_1)^{\varepsilon} = 0 \) whenever \( z - z_1 \). To prove the former statement, we have

\[
\lim_{z \to z_1} e^{\log(z - z_1)^{\varepsilon}} = \lim_{z \to z_1} (z - z_1)^{\varepsilon} = 0.
\]

Now observe that

\[
\lim_{z \to z_1} (z - z_1)^{\varepsilon} = M \lim_{z \to z_1} |(z - z_1)^{\varepsilon}| = 0
\]

(by supposition).

Hence \( (z - z_1)^{\varepsilon} = f(z) \) extends continuously to \( \partial \mathbb{D} \setminus \{z_1\} \) (by virtue of the continuity of \( (z - z_1)^{\varepsilon} \) and of the cont. extension of \( f \)), and we just showed that it extends cont. to \( z_1 \in \mathbb{D} \). Hence the function extends continuously to \( \partial \mathbb{D} \) as a whole set.

Thus \( |(z - z_1)^{\varepsilon} f(z)| \leq M |z - z_1|^{\varepsilon} \) and hence for all \( z \in \mathbb{D} \). This holds for all small \( \varepsilon \in \mathbb{C} \) (\( \varepsilon > 0 \)), so \( f(z) \) can take another limit to get that

\[
\lim_{\varepsilon \to 0} (z - z_1)^{\varepsilon} f(z) = f(z), \text{ and hence, taking } \varepsilon \to 0 \text{ with}
\]

1)}
the above result for each $\varepsilon$ shows that $|f(z)| \le M$ for all $z \in D$.

Now assume that there exist a many discontinuity points $z_1, \ldots, z_\lambda$ s.t. $f$ extends cont to 00 \{z_1, \ldots, z_\lambda\}$ and $|f(z)| \le M$ at $z=0 (z_1, \ldots, z_\lambda)$.

Just as above, for small $\varepsilon$, define

$$g(z) = \left( \prod_{i=1}^{\lambda} \frac{1}{z-z_i} \right) \varepsilon f(z).$$

By exactly the same proof as above, $g$ is holomorphic in the interior of $D$ and extends continuously to all of $\partial_D$. Since

$$\lim_{z \to z_i} \left| g(z) \right| = \lim_{z \to z_i} \left| \prod_{i=1}^{\lambda} \frac{1}{z-z_i} \varepsilon |f(z)| \right|$$

$$= \left( \prod_{i=1}^{\lambda} \lim_{z \to z_i} \left| z-z_i \right| \right) \varepsilon \lim_{z \to z_i} |f(z)|$$

$$\le M \lim_{z \to z_i} |z-z_i|^\varepsilon = 0, \quad \text{since } \varepsilon \text{ is bounded.}$$

Hence by la., $\left| g(z) \right| \le M$ for all $z \in D$, $z$ for all $z \in \partial D$.

Now taking

$$\lim_{\varepsilon \to 0} g(z) = f(z) \text{ gets us that since the result holds for each } \varepsilon \text{ and }$$

$$f, g \text{ are continuous, } |f(z)| \le M \text{ for all } z \in D.$$
1) Show that for the following cases of domain U a holomorphic \( f \) is not \( f(U) \) that extends continuously to the set \( U \). If \( f(z) = z \) for \( z \in U \) then \( |f(z)| \leq |z| \) for \( z \in U \).

\( a) \ U = \{ z \mid \text{Im}(z) > 0 \} \)

We want to find a holomorphic map \( \sigma(z) : D \to \{ z \mid \text{Im}(z) > 0 \} \) that preserves boundaries, where \( D \) is the unit disk.

Let \( T : D \to \mathbb{H} = \{ z \mid \text{Im}(z) > 0 \} \) by \( T(z) = \frac{1 + i z}{1 - z} \). By part 2, the Mobius transform is a bijection and function from \( D \to \mathbb{H} \) and holomorphic and continuous on \( D \) (since \( 1 \notin D \)).

We show that \( \text{Im}(T(z)) \geq 1 \) for all \( z \in D \), \( \text{Im}(T(z)) > 0 \).

Let \( z \in D \) and then

\[ T(z) = \frac{i}{1 - z} \frac{z(1 - i) - i(1 + i)}{1 - z} = \frac{1 - \overline{z}}{1 - z} \frac{1 - i|z|^2}{1 - z} = \frac{(1 - |z|^2) - (1 - |z|^2)i}{1 - z} \]

Since \( z \in D \), \( |z| < 1 \) and hence \( 1 - |z|^2 > 0 \). Similarly \( 2 \text{Re}(z) \leq |z|^2 < 1 + |z|^2 \) since \( |z| < 1 \), so \( 1 - 2|z|^2 + |z|^2 > 0 \). Thus \( \text{Im}(T(z)) > 0 \), so \( T(z) \in \mathbb{H} \), as desired.

We claim that \( T \) is surjective. Let \( \omega \in \mathbb{H} \). \( \omega = x + iy \) for \( y > 0 \).

Set \( z = \frac{w - i}{w + i} \), and we claim that \( T(z) = \omega \). We have

\[ T(z) = \frac{i + \frac{w - i}{w + i}}{1 - \frac{w - i}{w + i}} = \frac{2i}{2w} = \omega, \text{ as desired.} \]

We must show that \( z \in D \), \( |z| < 1 \):

\[ |\frac{w - i}{w + i}|^2 = \frac{(w - i)(\overline{w} + i)}{(w + i)(\overline{w} - i)} = \frac{1 - |w|^2}{2|1 - w|^2} \leq 1 \text{ since } \text{Im}(z) = \frac{\text{Im}(w)}{2} \]

since \( \text{Im}(z) > 0 \) and hence \( 1 - |w|^2 - 2\text{Im}(z) - 1 \leq 1 - |w|^2 - 2\text{Im}(z) - 1 \).

We have shown that \( T : D \to \mathbb{H} \) is surjective. Now we want to show that

\( T \) preserves boundaries as well, i.e., that \( T(0) = \mathbb{R} = \{ z \mid \text{Im}(z) = 0 \} = \mathbb{C} \).

Suppose \( z \in D \), \( r = |z| \in \mathbb{R} \). Then \( T(z) = \frac{i + e^{it}}{1 - e^{it}} \) (assuming \( \theta = 0 \), \( \theta \) applies \#3 since \( T \) extends continuously).

\[ \frac{i + e^{it}}{1 - e^{it}} = \frac{\cos \theta + i \sin \theta}{1 - \cos \theta - i \sin \theta} = \frac{(1 - \cos \theta + i \sin \theta)}{2 - 2 \cos \theta} \]

\[ = \frac{-\sin \theta + i \cos \theta}{2 - 2 \cos \theta} \]
\[
\begin{align*}
\frac{1}{2-\cos \Theta}
&= (\sin^2 \Theta - (1-\cos^2 \Theta)) (\sin^2 \Theta) + (\sin \Theta \sin \Theta + (\cos \Theta + 1)(1-\cos \Theta)') \\
\Rightarrow \Im(T(z)) &= -\sin^2 \Theta - \cos^2 \Theta + 1 = (\sin^2 \Theta - \cos^2 \Theta) + 1 = -1 + 1 = 0, \text{ i.e.} \\
\Im(T(z)) &\in \mathbb{D}.
\end{align*}
\]

To recap, we have $T : \mathbb{D} \to \mathbb{H}$ holomorphic and surjective and preserves boundaries. Then for $T : \mathbb{D} \to \mathbb{C}$ is holomorphic on $\mathbb{D}$ with continuous extension to the boundary $\partial \mathbb{D}$ (with one point $0$ without limit), we have $y$ since composition of holomorphic not.

For $z \in \mathbb{D}$, so

\[
|f(z)| \leq M, \quad \forall z \in \partial \mathbb{D}.
\]

Now we apply Liouville's since $f(z)$ is holomorphic on $\mathbb{D}$ and extends continuously to $\partial \mathbb{D}$ which is compact, so get that $|f(z)| \leq M, \forall z \in \partial \mathbb{D}$, and in particular, $\forall z \in \partial \mathbb{D}$.

Now we apply surjectivity: since $T(\mathbb{D}) = \mathbb{H}$, i.e. $T$ "covers" all of $\mathbb{H}$, we can replace this with

\[
f(w) \leq M, \quad \forall w \in \mathbb{H}, \text{ i.e. } w = T(z). \text{ In complete, the proof.}
\]

b) Now suppose $U = \{z \mid \Re \Im(z) > 0\}$, the strip with boundary $\{z \mid \Im(z) = \{0, \pi\}\}$. Again, we produce a holomorphic map with the desired properties, but have we use only part (a), let

\[
T(z) = \log : H \to U = \{z \mid 0 < \Im(z) < \pi\}, \text{ fixing the branch}
\]

\[
\log(z) = \log r + i \arg(z), \text{ per class on } 8\pi \mathbb{Z}.
\]

For $z = re^{i\theta}$, we get

\[
\log(z) = \log r + i \theta, \text{ as desired.}
\]

First, we know that $\log(z)$ is holomorphic on the cut plane $\{z \mid \Re(z) > 0 \& \Im(z) > 0\}$. Since $H \leq \{z \mid \Re(z) > 0 \& \Im(z) > 0\}$, and for all $z \in H$, $\Im(z) > 0$ by def. we have that $T : H \to U$ is holomorphic.

Now we show that $\log(u)$. Take some $z \in H$, so $z = re^{i\theta}$ s.t $r > 0, \theta \in (0, \pi)$ strictly. Then $\log(z) = \log r + i \theta$ s.t. $\Im(\log(z)) = \theta > 0$, as desired.

In fact, $\log(z)$ surjective: take $z \in U$, i.e. $z = x + iy$ s.t. $0 < \Im(z) < \pi$

Define $e^x e^{iy} \in H$ by definition; then $\log(z) = \log(\sqrt{x^2 + y^2}) + iy$

\[
= x + iy = z, \text{ as desired.}
\]

So $\log : H \to U$ is holomorphic and surjective. It also preserves boundaries, let $z = \partial H$ s.t. $z = x + iy$ for some $x \in \mathbb{R}$. Then

\[
\Im(z) = \frac{1}{2} \pi, \text{ as desired.}
\]

Use $\sqrt{z}$ to allow continuity.
\[ \log(z) = \begin{cases} \log x + 10 = \log x & \text{if } x > 0 \\ \log|x| + 1 \pi & \text{if } x < 0 \end{cases} \]

Hence \( \im (\log(z)) \in \{0, \pi\} \), that is, \( \log(z) \in \mathbb{U} \).

We have proven that \( \log: \mathbb{H} \to \mathbb{U} \) is holomorphic, surjective, and boundary-preserving. Take \((f \circ z): \mathbb{H} \to \mathbb{C}\) holomorphic, we have

\[
|f(z)| \leq M \quad \forall z \in \mathbb{H}, \quad \text{so} \quad |(f \circ z)(u)| = |f(z)| \leq M \quad \forall z \in \mathbb{H}
\]

Applying \( \text{H}(a) \), we get that \( |f(z)| \leq M \quad \forall z \in \mathbb{H} \).

Noting finally, finally, that \( \text{H}(u) = \mathbb{U} \) since \( z \) is surjective, we have \( |f(z)| \leq M \quad \forall w \in \mathbb{U} \), that \( f(z) \equiv w \).
(5) Show that for $D$ a bounded domain with piecewise smooth boundary
\[
S_{DD} \overline{z} dz = 2i \text{Area}(D).
\]
By Gamelin p. 102, \( d\overline{z} = dx + idy \), so we get that
\[
S_{DD} \overline{z} dz = S_{DD} (x-iy)(dx + idy)
\]
\[
= S_{DD} x dx + y dy + i \int_{\partial D} y dx + x dy \quad (\text{see Gamelin})
\]
Now we apply Green's Theorem since $D$ is piecewise smooth and $D$ bounded.
and $x, y, -y$ are continuously differentiable functions in $C$ ($p. 73$).
We get, setting $P_1(x,y) = x, Q_1(x,y) = y, P_2(x,y) = y, Q_2(x,y) = -x$
\[
\begin{align*}
&= \iint_D \left( \frac{\partial Q_2}{\partial x} - \frac{\partial P_2}{\partial y} \right) dx dy + \int_{\partial D} (Q_2(y,x) - P_2(x,y)) dx \\
&= \iint_D (1-1) dx dy + i \int_{\partial D} (1-1) dx dy \\
&= 0 + 2i \text{Area}(D) \\
&= 2i \text{Area}(D)
\end{align*}
\]
(6) Suppose $f: D \to \mathbb{C}$ has a primitive $F$, i.e. $\frac{dF}{dz} = f$, let $\gamma$ be a plane curve smooth path, let $A$ be a point along $\gamma$ arbitrarily. Since $\gamma$ is closed and piecewise smooth, there exists a parametrization of $\gamma$: $[0, 1] \to D$ such $\gamma(0) = \gamma(1) = A$.
Since we proved last week that line integrals are invariant under parametrization,
\[
\text{so } f(\gamma) dz \text{ can be described as a line integral from } A \text{ to } A.
\]
But by class, 9/2, since $f$ has a primitive $F$, $\gamma$ is a path from $A$ to $A$,
\[
\text{so } f(\gamma) dz = F(A) - F(A) = 0, \text{ since } F \text{ is well-defined. Thus for all closed piecewise smooth paths } \gamma, \text{ so } f(\gamma) dz = 0 \text{ whenever } f \text{ has a primitive.}
\]
Now assume $f(\gamma) dz = 0$ for any closed piecewise smooth path $\gamma$. We want to show that $f$ has a primitive, i.e. there is a function $F(z)$, $\text{holomorphic in } D$, $F'(z) = f(z)$.
By Pett 2, $\text{ if } f$ is $C^1$, we get $f(\gamma) dz$ depends only on the endpoints of $\gamma$.
For all paths $\gamma$, we have $f(\gamma) dz$ is independent of path [since $f$ is continuous on $D$].
Fix some point $z_0 \in D$, we define $F(\gamma) = f(\gamma) dz$ for $\gamma \in D$, where $\gamma$ is any piecewise smooth path from $z_0$ to $\gamma$. We prove that $F(\gamma)$ satisfies the Cauchy-Riemann equations with continuous portions in the first order.

Compute partials of $F(\gamma)$ by taking rectangular paths. Suppose $\gamma = (x_0, y_0) \to D$, fix some $(x, y) \in D$ and take $\gamma$ a path from $z_0$ to $(x, y)$.
for $x \in \text{closed } x$, $F(x, y)$ can be obtained by following $\gamma$ from $z_0$ to $(x, y)$ and then the straight path $\gamma_1(t) = x + t \varepsilon$, $0 \leq t \leq 1$

$\gamma_2(t) = u$

$F(x + \varepsilon, y) = \int_0^1 f(x + t \varepsilon, y) \, dt$

Then $\lim_{\varepsilon \to 0} \frac{F(x + \varepsilon, y) - F(x, y)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 f(x + t \varepsilon, y) \, dt = \int_0^1 f(x, y) \, dt - f(x, y)$ by the above worked since $f$ is continuous. Similarly, $F(x, y + \varepsilon) = \int_0^1 f(x, y + t \varepsilon) \, dt$

So if $f(x, y) = f(x, y)$ and $y(t) = y + t \varepsilon$, $y'(t) = \varepsilon$.

Hence $\frac{dF}{dx} = f(x, y) \, dx + \varepsilon f(x, y) \, dy$

$= f(x, y) \, dx + \varepsilon \frac{dy}{dx} f(x, y) \, \frac{dx}{\varepsilon} = \frac{df}{dz}$, as desired.

Moreover, we get that $\frac{dF}{dx} = \frac{df}{dz}$ from the above statements by class on 9/18.

The question is the Cauchy-Riemann equations. Moreover, since $f(z)$ is continuous, $\frac{dF}{dz}$ and $\frac{dF}{dx}$ are continuous.

Putting these pieces together, we get that $EF$ holomorphic at $-\frac{dF}{dz} = f$; hence $f: D \to C$ is a primitive.

We have shown that a holomorphic $f: D \to C$ has a primitive if $\int f(z) \, dz = 0$ for any closed path in $D$.
Suppose \( h: C \to \mathbb{C} \) is s.t. \( |h(z)| \leq M \) \( \forall z \in C \). We want to show \( \left| \oint_{\partial C} h(z) \, dz \right| \leq M \cdot \text{Length} (\partial C) \)

We denote the infinitesimal arc-length by \( |dz| = \sqrt{(dx)^2 + (dy)^2} \). Hence, \( \oint_{\partial C} h(z) \, dz \)

denotes \( \int_0^1 h(x(t)) \cdot |x'(t)| \, dt \).

We complete the proof by approximating the line integral using Riemann sums. Parametrize \( \gamma \) by \( \mathbf{r}(t) = x(t) + iy(t) \) for \( a \leq t \leq b \) and take some partition \( a = t_0 < t_1 < \ldots < t_n = b \). Then, per Corollary \( p.102 \),

\[
\oint_{\partial C} h(z) \, dz = \oint_{t_0}^{t_n} h(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \, dt
\]

can be approximated in the \( n \)th dissection by

\[
\sum_{j=0}^{n-1} h(\mathbf{r}(t_j)) \cdot \frac{\mathbf{r}(t_{j+1}) - \mathbf{r}(t_j)}{t_{j+1} - t_j}
\]

where \( 0 \leq j \leq n \).

Simplifying, we get

\[
\sum_{j=0}^{n-1} h(\mathbf{r}(t_j)) \cdot \left[ (x_{j+1} - x_j) + i(y_{j+1} - y_j) \right]
\]

\[
= \sum_{j=0}^{n-1} h(\mathbf{r}(t_j)) \cdot [Z_{j+1} - Z_j]
\]

Then for each \( n \in \mathbb{N} \), we have

1. \( \left| \sum_{j=0}^{n-1} h(\mathbf{r}(t_j)) \cdot [Z_{j+1} - Z_j] \right| \leq \sum_{j=0}^{n-1} |h(\mathbf{r}(t_j))| |Z_{j+1} - Z_j| \) by the triangle inequality

\[
\leq \sum_{j=0}^{n-1} M |Z_{j+1} - Z_j| \quad \text{since each} \ Z_j \in C
\]

\[
= M \sum_{j=0}^{n-1} \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2}
\]

Note that \( \sum_{j=0}^{n-1} \sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2} \) is an approximation of \( \text{Length}(\partial C) \) for this partition.

Since this inequality holds for each partition of the \( \gamma \) path, we can take \( n \) to infinity (i.e., smaller and smaller refinements) to get that

\[
\left| \oint_{\partial C} h(z) \, dz \right| \leq M \cdot \text{Length}(\partial C)
\]

since the left hand side of the inequality \( 0 \) was an approximation of \( \left| \oint_{\partial C} h(z) \, dz \right| \) and the RHS is an approximation, by definition, of \( M \cdot \sqrt{\text{Length}(\partial C)^2} \)

\[
= M \cdot \text{Length}(\partial C)
\]

By the above argument, this proves \( M \cdot \text{Length}(\partial C) \), which completes the proof.
Take $\gamma$ a path from 0 to 1. Find the following integrals (and justify why you need given some information to do so.)

a) $\int_{\gamma} e^{iz} \, dz$

We know that $e^{iz}$ is continuous in all $z$, and furthermore that the complex derivative of $e^{iz}$ is $e^{iz}$ (part 1). By the chain rule, the derivative of $e^{iz}$ is $i \pi e^{iz}$; hence a primitive for $e^{iz}$ is

$$\frac{e^{iz}}{i},$$

which is a scalar multiple of a holomorphic function and hence holomorphic (by class 3/8). Similarly, the derivative is $(\frac{1}{i} e^{iz})' = \frac{1}{i} (e^{iz})'$

as desired.

By p. 107 in Gamelin, $\int_{\gamma} e^{iz} \, dz = F(1) - F(0)$ where $F$ is a primitive of $e^{iz}$ and $F(1)$, $F(0)$ are the endpoints of $\gamma$. Thus, we get

$$\int_{\gamma} e^{iz} \, dz = \left[ \frac{e^{iz}}{i} \right]_0^1 = e^{i1}/i - e^{i0}/i = -1/i - 1/i = -2/i.$$

b) $\int_{\gamma} (z-1)^m \, dz$, $m \neq -1$. Assume that $\gamma$ does not pass through $z=1$.

By part 1, we know that $(z-1)^m$ is continuous on $\gamma$; hence $\gamma$ is a path within a domain where $(z-1)^m$ is continuous, since $\gamma$ does not pass through $z=1$. Also by part 1, $(z-1)^m$ is holomorphic with primitive

$$F(z) = \frac{(z-1)^{m+1}}{m+1}.$$ Since $m \neq 1$, this is a non-constant and we proved a

Now, we have

$$F'(z) = \frac{1}{m+1} \cdot (m+1) (z-1)^{m+1-1} = (z-1)^m,$$

as desired.

So

$$\int_{\gamma} (z-1)^m \, dz = \int_{0}^{1} \frac{(z-1)^m}{m+1} \, dz = (1)^{m+1} - (0)^{m+1} = (2)^m - (1)^m,$$

as desired.
c) \( \int_0^1 \cos(z) \, dz \).

Observe that \( \cos(z) \) is continuous in \( \mathbb{C} \); it is also holomorphic (part 1)

with primitive

\[ F(z) = \sin(z), \text{ since we proved in part 1 that } F \text{ is holomorphic} \]

\[ F'(z) = \cos(z), \text{ as desired} \]

Hence by Thm. 6.6 on p.107

\[ \int_0^1 \cos(z) = \sin(1) - \sin(0) \]

Note that \( \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \) (class 8/29), so

\[ \sin(i) = \frac{e^i - e^{-i}}{2i} = \frac{e^{-1} - e}{2i} = \frac{e - \frac{1}{e}}{2i} \]

\[ = \frac{e}{2i} - \frac{1}{2i} \]

\[ \sin(0) = \frac{e^0 - e^0}{2i} = 0 \]

So \( \int_0^1 \cos(z) = \frac{e^{-1} - e}{2i} \).