MATH 310 FALL 2019
PROBLEM SET 4

Due Thursday, Oct 2nd, at 11:35 AM, in class.

(1) Calculate the following integrals, using the Cauchy Integral Formula or Cauchy's Theorem. All circles are oriented counter-clockwise
 (a) $\int_{|z|=2} \frac{z^n}{z-1} dz, \quad n \geq 0$
 (b) $\int_{|z|=1} \frac{z^n}{z-2} dz, \quad n \geq 0$
 (c) $\int_{|z|=1} \frac{\sin z}{z} dz$
 (d) $\int_{|z|=1} \frac{dz}{z^2(z^2-4)e^z}$

(2) Use the Cauchy Integral Formula to derive the Mean Value Property of harmonic functions
$$u(z_0) = \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi}, z_0 \in D$$
where $u : D \to \mathbb{R}$ is harmonic and D contains the disk $|z - z_0| \leq \rho$. (Hint: Recall that a disk is a nice domain to find a harmonic conjugate).

(3) Show that if $u : \mathbb{C} \to \mathbb{R}$ is harmonic and bounded, then u is constant. (Hint: what happens with $|e^f|$ for $f = u + iv$, when u, v are real valued?).

(4) Show that if $f : \mathbb{C} \to \mathbb{C}$ is entire and there is a disk $D \subset \mathbb{C}$ so that $f(\mathbb{C}) \cap D = \emptyset$, then f is constant.

(5) Let $L \subset \mathbb{C}$ be a real line and $f : D \to \mathbb{C}$ continuous so that f is holomorphic in $D \setminus L$. Prove that f is holomorphic in D.

(6) Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic in the unit disk $\mathbb{D}$. Denote by d the *diameter* of the image of D,
$$d = \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|.$$
Show that $2|f'(0)| \leq d$. (Hint: Verify first that $f'(0) = \frac{1}{2\pi i} \int_{|w|=r} \frac{-f(-w)}{w^2} dw$. Then add the two integral formulas for $f'(0)$ at our disposal).

(7) Weierstarss's Theorem say that every continuous real function on the interval $[0, 1]$ can be approximated by polynomials. This means that for any $g : [0, 1] \to \mathbb{R}$ continuous and $\epsilon > 0$ there exists a polynomial P such that $|g(x) - P(x)| < \epsilon$, $0 \leq x \leq 1$. Show that not all continuous functions in the unit disk $\mathbb{D}$ can be approximated by polynomials in the complex variable z.

(8) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that for some $k \geq 0$ there are real constant $A, B > 0$ such that
$$|f(z)| \leq A + B|z|^k, \quad z \in \mathbb{C}$$
Show that f is a polynomial of degree $\leq k$.

Integralsatz and Integralformel sind zusammen von solcher Tragweite, dass man ohne Uebertreibung sagen kann, in diesen beiden Integralen liege die ganze jetzige Functionentheorie conzentrirt vor (The integral theorem and the integral formula together are of such scope that one can say without exaggeration: the whole of contemporary function theory is concentrated in these two integrals) - L. KRONECKER.