MATH 310 FALL 2019
PROBLEM SET 5

(1) Show that $\sum_{k=1}^{\infty}\frac{1}{k^p}$ diverges for $p \leq 1$.
(2) Show that $\sum_{k=1}^{\infty}\frac{1}{k\log k}$ diverges and $\sum_{k=1}^{\infty}\frac{1}{k(\log k)^2}$ converges.
(3) Show that $f_k(z) = \frac{z^k}{k}$ converges uniformly for $|z| < 1$. Show that $f_k'(z)$ does not converge uniformly for $|z| < 1$. Does it converge uniformly for $|z| \leq r$ for $r < 1$?
(4) Show that if $f_k(z)$ converges uniformly in the sets $E_1, E_2, \ldots E_n$ then it converges uniformly in their union $\cup_{j=1}^{n} E_j$.
(5) Find the radius of convergence of the following series.
   (a) $\sum_{k=0}^{\infty} 2^k z^k$
   (b) $\sum_{k=1}^{\infty} \frac{k^k}{1_2^k k^k} z^k$
   (c) $\sum_{k=3}^{\infty} (\log k)^{k/2} z^k$
(6) Show that the function $f(z) = \sum_{k=1}^{\infty} z^{k!}$ is analytic on the unit disk $|z| < 1$. Show that if $\lambda^n = 1$ for some integer n, then $|f(r\lambda)| \to \infty$ as $r \to 1$.
(7) Show that the series $\sum a_k z^k$, its derivative series $\sum k.a_k z^k$ and its integral series $\sum \frac{a_k}{k+1} z^k$ have all the same radius of convergence.
(8) Find the radius of convergence of the following functions at the indicated points
   (a) $\frac{1}{z-1}$, at $z = i$
   (b) $\log(z)$, at $z = 1 + 2i$
   (c) $z^{3/2}$, at $z = 3$
   (d) $\frac{z^2-1}{z^3-1}$, at $z = 2$

Die Potenzreihen sind deshalb besonders bequem, weil man mit ihnen fast wie mit Polynomen rechnen kann (Power series are therefore especially convenient because one can compute with them almost as with polynomials).-C. CARATHÉODORY