MATH 310 FALL 2019
PROBLEM SET 6

Due Thursday, Oct 31st, at 11:35 AM, in class.

(1) Find all possible Laurent expansion centered at 0 of the following functions (indicating domains!)
  (a) $\frac{1}{z^2-z}$
  (b) $\frac{z-1}{z+1}$
  (c) $\frac{1}{(z^2-1)(z^2-4)}$

(2) Let f be a holomorphic function on the annulus $A < |z| < B$. We say that f is even if $f(-z) = f(z)$, and odd if $f(-z) = -f(z)$. Let $f = f_0 + f_1$ the Laurent decomposition of f. Show that if f is even (resp. odd) then f_0, f_1 are even (resp. odd). What does each case say about powers z^k with zero coefficients?

(3) Suppose f is holomorphic on the annulus $\{\sigma < |z| < \rho\}$. Show that there is a unique constant c such that $f(z) - \frac{c}{z}$ has an antiderivative in $\{\sigma < |z| < \rho\}$.

(4) Find the isolated singularities of the following functions. Then determine if they are removable, essential or poles. For each pole, find its order and principal part.
  (a) $\frac{z}{(z-1)^2}$
  (b) $\frac{e^{2z}-1}{z}$
  (c) $e^{1/(z^2+1)}$

(5) Suppose $f(z)$ is meromorphic on the disk $|z| < s$ with only finitely many poles. Show that the Laurent decomposition $f = f_0 + f_1$ with respect to the annulus $s - \epsilon < |z| < s$ is such that f_1 is the sum of the principal parts of $f(z)$ at its poles.

(6) Show that if $f(z)$ is continuous on a domain D, and if f^8 is holomorphic on D, then f is holomorphic on D.

(7) Suppose $f(z) = \sum a_k z^k$ is holomorphic for $|z| < R$, and f extends as a meromorphic function on $|z| < R + \epsilon$ with only one pole z_0 in the circle $|z| = R$. Show that $\lim_{k\to\infty} \frac{a_k}{a_{k+1}} = z_0$.

(8) Suppose u is harmonic on the punctured disk $\{0 < |z| < \rho\}$. Show that if

$$\frac{u(z)}{\log(1/|z|)} \to 0$$

as $z \to 0$, then u extends to be harmonic in 0. What can you say if instead all that you know is that $|u(z)| \leq C \log(1/|z|)$ for some fixed constant C and any $\{0 < |z| < \rho\}$?

At quantopere doctrine de seriebus infinitis Analysin aublimiorem amplificaveret, nemo est, qui ignoret (There is nobody who does not know the extent to which the theory of infinite series has enriched higher analysis). - L. EULER 1748, Introductio.