MATH 310 FALL 2019
PROBLEM SET 7

Due Thursday, Nov 7th, at 11:35 AM, in class.

(1) Show that if z_0 is not a removable singularity for $f(z)$, then z_0 is an essential singularity for $e^{f(z)}$. (Hint: in the case z_0 is a pole for f, try to use that if U is open and g is holomorphic in U, then $g(U)$ is open.)

(2) Show that for z_0 isolated singularity of f, the Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ classifies the type of singularity at z_0 by
   (a) Removable, if $a_k = 0$ for all $k \leq -1$
   (b) Pole, if $a_k = 0$ for all $k \leq n \leq -1$, and $a_n \neq 0$
   (c) Essential, if that are infinitely many negative indices k with $a_k \neq 0$

(3) Show that if f is holomorphic in $\mathbb{C}$ has two periods $w_1, w_2 \in \mathbb{C}$ that are $\mathbb{Z}$-linearly independent (i.e. if $nw_1 + mw_2 = 0, n, m \in \mathbb{Z}$ then $n, m = 0$). Show that f is constant. Show that if a meromorphic function g in $\mathbb{C}$ has 3 $\mathbb{Z}$-linearly independent periods, then g is constant.

(4) Evaluate the following residues
   (a) $\text{Res}\left[\frac{1}{z^2+4}, 2i\right]$
   (b) $\text{Res}\left[\frac{\sin z}{z^2}, 0\right]$
   (c) $\text{Res}\left[\frac{z}{\log z}, 1\right], \text{Re}(z) > 0$
   (d) $\text{Res}\left[\frac{e^z}{z^5}, 0\right]$

(5) Evaluate the following counter-clockwise integrals, using the residue theorem
   (a) $\int_{|z|=1} \frac{\sin z}{z^2} dz$
   (b) $\int_{|z|=2} \frac{z}{\cos z} dz$
   (c) $\int_{|z-1|=1} \frac{1}{z^8-1} dz$
   (d) $\int_{|z|=2} \frac{e^z}{z^2-1} dz$

(6) Suppose $P(z), Q(z)$ are polynomials such that Q only has simple zeros $z_1, \ldots, z_m$ and $\deg P < \deg Q$. Show that the partial fractions decomposition of P/Q is given by

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^{m} \frac{P(z_j)}{Q'(z_j)} \frac{1}{z - z_j}$$

(7) Using calculus of residue, show that
   (a) $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$
   (b) $\int_0^{\infty} \frac{x^2}{x^4+1} dx = \frac{\pi}{2\sqrt{2}}$

The method of evaluating real integrals by passing to the complex numbers (passage du réel á l'imaginaire) goes as early as the 18th century