Due Thursday, Nov 7th, at 11:35 AM, in class.

(1) Show that if \( z_0 \) is not a removable singularity for \( f(z) \), then \( z_0 \) is an essential singularity for \( e^{f(z)} \).

(Hint: in the case \( z_0 \) is a pole for \( f \), try to use that if \( U \) is open and \( g \) is holomorphic in \( U \), then \( g(U) \) is open.)

(2) Show that for \( z_0 \) isolated singularity of \( f \), the Laurent series \( f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \) classifies the type of singularity at \( z_0 \) by

(a) Removable, if \( a_k = 0 \) for all \( k \leq -1 \)

(b) Pole, if \( a_k = 0 \) for all \( k \leq n \leq -1 \), and \( a_n \neq 0 \)

(c) Essential, if that are infinitely many negative indices \( k \) with \( a_k \neq 0 \)

(3) Show that if \( f \) is holomorphic in \( \mathbb{C} \) has two periods \( w_1, w_2 \in \mathbb{C} \) that are \( \mathbb{Z} \)-linearly independent (i.e. if \( nw_1 + mw_2 = 0, n, m \in \mathbb{Z} \) then \( n, m = 0 \)). Show that \( f \) is constant. Show that if a meromorphic function \( g \) in \( \mathbb{C} \) has 3 \( \mathbb{Z} \)-linearly independent periods, then \( g \) is constant.

(4) Evaluate the following residues

(a) \( \text{Res} \left[ \frac{1}{z^2 + 4}, 2i \right] \)

(b) \( \text{Res} \left[ \frac{\sin z}{z}, 0 \right] \)

(c) \( \text{Res} \left[ \frac{z}{\log z}, 1 \right], \text{Re}(z) > 0 \)

(d) \( \text{Res} \left[ \frac{z}{z^2}, 0 \right] \)

(5) Evaluate the following counter-clockwise integrals, using the residue theorem

(a) \( \int_{|z|=1} \frac{\sin z}{z} \, dz \)

(b) \( \int_{|z|=2} \frac{z}{\cos z} \, dz \)

(c) \( \int_{|z|=1} \frac{1}{z} \, dz \)

(d) \( \int_{|z|=2} \frac{z^2}{z^2 + 1} \, dz \)

(6) Suppose \( P(z), Q(z) \) are polynomials such that \( Q \) only has simple zeros \( z_1, \ldots, z_m \) and \( \deg P < \deg Q \). Show that the partial fractions decomposition of \( \frac{P(z)}{Q(z)} \) is given by

\[
\frac{P(z)}{Q(z)} = \sum_{j=1}^{m} \frac{P(z_j)}{Q'(z_j)} \frac{1}{z - z_j}
\]

(7) Using calculus of residue, show that

(a) \( \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{\sqrt{2}} \)

(b) \( \int_{0}^{\infty} \frac{x^2}{x^4 + 1} \, dx = \frac{\pi}{2\sqrt{2}} \)