# MATH 310 FALL 2019
# PROBLEM SET 9

Due Thursday, Nov 21th, at 11:35 AM, in class.

(1) Show that for $a > 0$ real parameter (Hint: replace $\sin az$ by $e^{iaz}$ and integrate over an indented domain)
$$\int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2+1)} dx = \pi(1 - e^{-a})$$

(2) By integrating $(e^{\pm 2iz} - 1)/z^2$ show that
$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi$$

(3) Show that for a real parameter
$$\text{PV} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x-a)} dx = -\frac{\pi a}{a^2+1}$$

(4) Show that for $0 < a < b$ real parameters
$$\text{PV} \int_{0}^{\infty} \frac{x^{a-1}}{x^b - 1} dx = -\frac{\pi}{b} \cot\left(\frac{\pi a}{b}\right)$$

(5) Show that the sum of residues of a rational function on the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is equal to 0.

(6) Show that
$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx = \pi(\sqrt{2} - 1)$$

(7) Let $D$ be an exterior domain. Suppose that $f$ is analytic on $D \cup \partial D$ and at $\infty$. Show that
$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw = f(z) - f(\infty)$$

(8) Let $f$ be a meromorphic function on $\mathbb{C}$ with finite many poles, none in the real line. Suppose that there is a constant $K$ so that $|f(z)| \leq K/|z|$ for $z$ sufficiently large. Let $a > 0$. Show that
$$\int_{-\infty}^{\infty} f(x)e^{iax} = 2\pi i \sum \big( \text{ residues of } e^{iaz} f(z) \text{ in the upper half-space } \big)$$
. This formula is useful to find some Fourier Transforms.

”but even complete mastery does not guarantee success” Ahlfors, on using calculus of residues to compute definite integrals