Problem 1. (2pt each) Justify with a few words if the following claims are True or False.

1. If \( f : U \to \mathbb{C} \) is holomorphic then it satisfies the Cauchy-Riemann equations.
   \[
   \text{True} \quad \partial_x f = \partial_y f = \frac{1}{i} \partial_y f
   \]

2. If \( f : U \to \mathbb{C} \) satisfies the Cauchy-Riemann equations then \( f \) is holomorphic.
   \[
   \text{False} \quad f \text{ is not necessarily } C^1
   \]

3. If \( f : U \to \mathbb{C} \) is holomorphic then \( \int_\eta f(z)dz = 0 \) for any piecewise smooth closed path \( \eta \) in \( U \).
   \[
   \text{False} \quad \text{only if } \eta = \partial D (\text{Cauchy})
   \]

4. If \( f_n : E \to \mathbb{C} \) is a sequence of continuous functions and \( \lim_{n \to \infty} f_n(z) = f(z) \in \mathbb{C} \), \( \forall z \in E \). Then \( f \) is continuous.
   \[
   \text{False} \quad \text{need uniform convergence}
   \]

5. Given \( a \in \mathbb{C} \setminus \{0\} \), the equation \( e^z = a \) has infinitely many solutions.
   \[
   \text{True} \quad a = re^{i\theta} \Rightarrow z = \log r + i(\theta + 2\pi k) \quad k \in \mathbb{Z}
   \]

Problem 2. (10pt) Let \( a, z \in \mathbb{C} \) such that \( |z| = 1 \) and \( 1 - \overline{a}z \neq 0 \). Show that \( |\frac{z - a}{1 - \overline{a}z}| = 1 \)

\[
|z - a| = |z - a\overline{z} - z\overline{z}| = |z| |1 - a\overline{z}| \Rightarrow |\frac{z - a}{1 - \overline{a}z}| = 1
\]

Since \( |z| = |\overline{z}| = 1 \),

\[
\left| \frac{z - a}{1 - \overline{a}z} \right| = |1 - \overline{a}z|
\]
Problem 3. (15pt) Let $f : U \to \mathbb{C}$ continuous, with continuous partial derivatives $\partial_x f, \partial_y f$. Let $\text{Re} f = u, \text{Im} f = v$ so that $f = u + iv$. Denote by $\nabla u$ the vector in $\mathbb{R}^2$ with coordinates $\nabla u = (\partial_x u, \partial_y u)$ (similarly $\nabla v = (\partial_x v, \partial_y v)$). Show that

$f$ is holomorphic $\iff |\nabla u| = |\nabla v|$, and $\nabla u, \nabla v$ are orthogonal. $(\langle \nabla u, \nabla v \rangle = 0)$

Assuming any of these conditions, also show that $|\nabla u| = |\nabla v| = |f'|$.

$$\Rightarrow \quad \nabla u = (u_x, u_y), \quad \nabla v = (v_x, v_y) = (-u_y, u_x) \quad \Rightarrow \quad i \nabla u = \nabla v$$

(*) then $|\nabla u| = 1|\nabla v|$ and $\langle \nabla u, \nabla v \rangle = 0$ oriented

$\leq \quad \text{from } |\nabla u| = 1|\nabla v|$ & $\langle \nabla u, \nabla v \rangle = 0$ oriented we have $i \nabla u = \nabla v$

$$\Rightarrow \quad u_x = v_y, \quad u_y = -v_x$$

(Cauchy-Riemann equations $\Rightarrow$ $f$ is holomorphic)

$f$ is $C^1$

in (*) $|\nabla u| = 1|\nabla v| = \sqrt{u_x^2 + u_y^2}$

where $f'(z) = u_x + i u_y$

$$\Rightarrow \quad 1|\nabla u| = 1|\nabla v| = |f'|$$
Problem 4. (15pt) Prove the Mean Value Property \( u(z_0) = \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) \frac{d\theta}{2\pi}, z_0 \in D, \) where \( u : D \to \mathbb{R} \) is harmonic and \( D \) contains the disk \( |z - z_0| \leq \rho, \) by following:

1. (6pt) Use Green's theorem to show that

\[
0 = \oint \left( -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) = \iint_D \left( \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right) dx \, dy = 0
\]

Since \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \)

2. (5pt) Write the previous equation in polar coordinates \( z - z_0 = r \cos \theta + ir \sin \theta = re^{i\theta} \) to conclude

\[
0 = \int_0^{2\pi} \frac{\partial u}{\partial r} (z_0 + re^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial u}{\partial r} (r, \theta) d\theta
\]

\[
dx = r \sin \theta \, d\theta, \quad dy = r \cos \theta \, d\theta
\]

\[
0 = \int_0^{2\pi} \frac{\partial u}{\partial y} r \sin \theta + \frac{\partial u}{\partial x} r \cos \theta \, d\theta = r \int_0^{2\pi} \frac{\partial u}{\partial r} (r, \theta) d\theta
\]

\[
\Rightarrow \int_0^{2\pi} \frac{\partial u}{\partial r} (r, \theta) d\theta = 0 \quad (r \neq 0)
\]

3. (2pt) Use the previous equation and differentiation under the integral sign to show that \( f(r) = \int_0^{2\pi} u(r, \theta) d\theta \) is constant.

\[
f'(r) = \frac{d}{dr} \left( \int_0^{2\pi} u(r, \theta) d\theta \right) = \int_0^{2\pi} \frac{\partial u}{\partial r} (r, \theta) d\theta = 0
\]

\[
\text{Derivative under integral sign}
\]

Page 4
4. (2pt) Conclude the Mean Value property by replacing \( r = \rho \) and sending \( r \) to 0 in

\[
\int_0^{2\pi} u(r, \theta) d\theta
\]

\[
f(\rho) = \int_0^{2\pi} u(2\rho e^{i\theta}) d\theta
\]

\[
\lim_{r \to 0} f(r) = \int_0^{2\pi} u(2\rho) d\theta = 2\pi 2\rho
\]
Problem 5. (15pt) Prove the Fundamental Theorem of Algebra by following these steps

1. (6pt) Let \( f: \overline{D} \rightarrow \mathbb{C} \) be a continuous function on a compact domain \( \overline{D} \), so that \( f \) is holomorphic in the interior \( D \). Show that if \( |f(z)| \leq M \) for all \( z \in \partial D \), then \( |f(z)| \leq M \) for all \( z \in \overline{D} \). Why is it important that \( \overline{D} \) is compact?

   \( \overline{D} \) attains its maximum in \( \overline{D} = \partial D \cup D \) (needs \( \overline{D} \) compact)

   If happens in \( \partial D \): then \( |f(z)| \leq M \) in \( \partial D \) implies

   \( |f(\zeta)| \leq M \) in \( \overline{D} \), since biggest value is in \( \partial D \)

   If happens in \( D \); By maxmimum principle, it is constant \( = C \)

   \[ \Rightarrow |C| \leq M \Rightarrow |f(z)| \leq M \quad \forall z \in \overline{D} \]

2. (4pt) If a polynomial \( p: \mathbb{C} \rightarrow \mathbb{C} \) does not vanish then \( 1/p: \mathbb{C} \rightarrow \mathbb{C} \) is a well defined holomorphic function. See that for \( p \) not constant and for large \( R \) the values \( 1/p(Re^{i\theta}) \) are arbitrarily small.

   \[ p(z) = a_0 + a_1 z + \ldots + a_n \]

   \[ \Rightarrow \frac{1}{p(z)} = \frac{1}{a_0} \cdot \frac{1}{\frac{z}{a_0} + \frac{a_1}{a_0} z + \ldots + \frac{a_n}{a_0} z^n} \]

   As \( |z| \rightarrow +\infty \)

   \[ \text{then } \left| \frac{1}{z} \right| \rightarrow 0 \]

   \[ \Rightarrow \frac{1}{p(z)} \rightarrow 0 \cdot \frac{1}{a_0} = 0. \]

3. (5pt) Use the Maximum Principle for \( 1/p \) for larger and larger balls to show that if \( p \) is not constant then exists \( z_0 \) so that \( p(z_0) = 0 \).

   If \( 1/p \) is entire then \( |1/p(z)| \leq \max_{\theta \in \mathbb{R}} 1/p(Re^{i\theta}) \rightarrow 0 \)

   \[ \Rightarrow \left( \forall \zeta \in D \right) \]

   \[ \text{since } \frac{1}{p(z)} \neq 0. \]
Problem 6. (10pt) Calculate the following integrals, using the Cauchy Integral Formula or Cauchy's Theorem. All circles are oriented counter-clockwise.

1. (3pt)
\[
\int_{|z|=1} \frac{e^z}{z^n} \, dz, \quad n \in \mathbb{Z}
\]
\[
\because \quad \frac{d^n e^z}{dz^n} = e^z
\]
\[
= e^0 = \frac{n!}{2\pi i} \int_{|z|=1} \frac{e^z}{z^{n+1}} \, dz
\]
\[
\Rightarrow \int_{|z|=1} \frac{e^z}{z^n} \, dz = \frac{2\pi i}{(n-1)!}
\]

2. (3pt)
\[
\int_{|z|=1} \frac{\sin z}{z-2} \, dz
\]
\[
\frac{\sin z}{z-2} \text{ is holomorphic in } |z| \leq 1
\]
\[
\Rightarrow \int_{|z|=1} \frac{\sin z}{z-2} \, dz = 0 \quad \text{(Cauchy)}
\]

3. (4pt)
\[
\int_{|z|=1} \frac{dz}{z^2(z^2 - 4)e^z}
\]
\[
f(z) = \frac{1}{(z^2+4)e^z} \text{ is holomorphic in } |z| \leq 1
\]
\[
f'(z) = - \frac{2ze^z - (z^2+4)e^z}{(z^2+4)^2(e^z)^2} \Rightarrow f'(0) = \frac{4}{(-4)^2} = \frac{1}{4}
\]
By CIF:
\[
I = f'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z^2(z^2+4)e^z} \Rightarrow \int_{|z|=1} \frac{dz}{z^2(z^2+4)e^z} = \frac{\pi i}{2}
\]
Problem 7. (15pt) Recall that a function $f$ is bounded if there exists $M > 0$ so that $|f| \leq M$

1. (5pt) Use Cauchy estimates to prove that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then $f$ is constant.

   By Cauchy estimates, since $f$ is holomorphic in $B_R(2)$
   
   $\Rightarrow |f(z)| \leq \frac{M}{R}$
   
   $\forall R > 0$
   
   $\Rightarrow f$ is constant (if is connected)

2. (5pt) If $u : \mathbb{C} \to \mathbb{R}$ is bounded and harmonic, then $u$ is constant.

   Define $f = u + iv$ holomorphic ($C$ is convex)
   
   $g = e^f$ is also holomorphic, $|g| = e^{u}$ is bounded
   
   $\Rightarrow g = e^{u}e^{iv}$ is constant $\Rightarrow e^{u}$ is constant
   
   $\Rightarrow u$ is constant.

3. (5pt) If there is an interval $I = [a, b] \subset \mathbb{R}$ so that $u(C) \cap I = \emptyset$, then $u$ is constant.

   Since $C$ is connected, then $u(a) \leq a$ or $u(a) \geq b$

   From 2) $\Rightarrow u$ is constant.
Problem 8. (10pt) Take the series $\sum_{k=1}^{\infty} z^k$

1. (4pt) Find its radius of convergence $R$

$$\mathcal{I} = \{ r \in \mathbb{C} : |z^k| \text{ is bounded} \} \quad \text{since } \exists k = 0 \text{ or } 1 \text{ (and infinite)}$$

$$\Rightarrow r \in \mathcal{I} \iff q(r) \leq 1$$

$$\Rightarrow r = \sup \mathcal{I} = 1$$

2. (2pt) Define the function $f(z) = \sum_{k=1}^{\infty} z^k$, $|z| < R$. Why is $f$ holomorphic?

Because for $|2| < R$ $f$ is uniform limit in compact subsets of polynomials, which are holomorphic. Uniform convergence preserves holomorphicity.

3. (4pt) Let $n$ be integer and $\lambda \in \mathbb{C}$ so that $\lambda^n = 1$. Show that $|f(\lambda^n)| \to \infty$ as $r \to 1$.

$$f(r\lambda) = \sum_{k=1}^{n} (r\lambda)^k + \sum_{k=n+1}^{\infty} r^k$$

$$\Rightarrow |f(r\lambda)| > \left| \sum_{k=n+1}^{\infty} r^k \right|$$

$$\Rightarrow \lim_{r \to 1} \left( \text{finite terms} \right)$$

$$\Rightarrow \sum_{k=1}^{n} \lambda^k$$

$$\Rightarrow N - \sum_{k=1}^{n} \lambda^k \ni N > 0$$

and $r$ sufficiently large

$$\Rightarrow |f(r\lambda)| \to \infty \text{ as } r \to 1$$

Page 9
Bonus problem: ATTEMPT ONLY IF YOU ARE DONE WITH THE REST (10pt)

Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be an entire function (i.e. holomorphic). Assume that \( f \) is not a polynomial. Show that there exists \( a \in \mathbb{C} \) so that the power series expansion at \( a \)

\[
f(z) = \sum_{k=0}^{\infty} c_k(z - a)^k, z \in \mathbb{C}
\]

satisfies \( c_k \neq 0, \forall k \).

\[
I_k = \{ a \in \mathbb{C} / c_k = 0 \} \quad \text{and} \quad f(z) = \sum_{k=0}^{\infty} c_k (z - a)^k
\]

\[
= \{ a \in \mathbb{C} / f^{(k)}(a) = 0 \} \quad \text{since} \quad c_k = \frac{f^{(k)}(a)}{k!}
\]

Since \( f \) is not a polynomial, \( I_k \) is a set of \( \mathbb{C} \) without accumulation points \( \Rightarrow \) \( I_k \) is enumerable, \( \forall k \geq 0 \)

\[
= \bigcup_{k=1}^{\infty} I_k \text{ is enumerable.}
\]

Take \( a \in \mathbb{C} \setminus \bigcup_{k=1}^{\infty} I_k \). This satisfies \( c_k \neq 0 \) \( \forall k \)

and exists since \( \mathbb{C} \) is not enumerable.