Qualifying Exam Transcript

Note: the following transcript is only an approximate version of the questions and answers of the exam. I have omitted the questions that asked for details of something that I said when I was answering a main question. Most of the answers (if not all) just indicate the path that I took to solve what I was asked.

Algebraic Topology

- (Agol) Definition of an Eileenberg-McLane space.
  Wrote down the definition

- (Agol) Example of $K(\mathbb{Z}, 1)$.
  Said S^1 and showed that has the right homotopy groups using its universal cover.

- (Agol) Example of $K(\mathbb{Z}_n, 1)$.
  Instead of talking about lens spaces I ended talking about the general construction of a CW complex example. Got to say why the space constructed has the right homotopy groups.

- (Agol) Example of $K(\mathbb{Z}, 2)$.
  Said $\mathbb{C}P^\infty$. I used the fact that the complex projective spaces are the skeletons of the CW structure, so I had to compute some of their homotopy groups. Since I used the long exact sequence for the circle bundle defined by the quotient $S^{2n+1} \to \mathbb{C}P^n$, got asked to describe the homomorphisms involved in a long exact sequence of a fiber bundle.

- (Agol) Why are they relevant?
  Mentioned that for the case of $\mathbb{C}P^\infty$ helps to define characteristic classes, but I couldn't say much more. At the end, he was trying to make me mention that is a way of defining homology for groups.

- (Agol) Define the first Chern class / Euler class of a complex line bundle.
  Mentioned that in this case both are the same cohomology class, defined with the orientation given by the complex structure (the fundamental class defined for oriented vector bundles in Thom's isomorphism).

- (Agol) Now define it as the pullback of a fundamental class.
  Defined the canonical vector bundle over $\mathbb{C}P^\infty$ and said how every complex

line bundle is obtained as a pullback of it. Went back to what I had of $\mathbb{C}P^{\infty}$ to say that its cohomology is a polynomial ring generated by a cohomology class of degree 2, concluding that the class that I wanted is the pullback of this class. At the end, had to show that this class does not depend on the map used to take the bundle pullback. Did that by showing that any two choices are homotopic (argument from Milnor-Stasheff book).

- (Lott) Define a connection on vector bundles.
  Gave the standard definition.

- (Lott) Give examples of singular cochains representing generators of $H^1(S^1, \mathbb{R})$ and $H^1(S^1, \mathbb{Z})$.
  For real coefficients I assigned to each path its angle variation over 2π. Got stuck with integer coefficients, but solved at the end thanks to the advice of looking at the pullback under the covering map $\mathbb{R} \to S^1$. Since $\mathbb{R}$ is contractible, it will be the differential of a 0-cochain, which can be represented by a function $\theta : \mathbb{R} \to \mathbb{Z}$ such that $\theta(x + 2\pi) = \theta(x) + 1$. Then for a path in S^1, take a lifting to $\mathbb{R}$ and assign the difference of the values of θ at the end points.

- (Gronsky) What does the hairy-ball theorem says?
  Stated that the even dimensional spheres can not have non-vanishing vector fields and mentioned how this says that one can not comb a hairy ball.

Riemannian Geometry

- (Lott) Define the index form.
  Gave the definition and mentioned that comes from calculating the second variation of the energy along a proper variation of a geodesic. Got to define some of this words.

- (Lott) Use it to prove Cartan-Hadamard theorem.
  More precisely, to show that the exponential map is a local diffeomorphism. By contradiction, we will have a proper non-trivial Jacobi field. The Jacobi field equation implies that the index form will be 0, but it is positive by the non-positivity of the curvature.
  Taking the pullback metric, it remains to show that the exponential map is a covering map. This is obtained because satisfies the path lifting property thanks to be a local isometry onto a complete space.

- (Lott-Agol) Statement of Hopf-Rinow for one step in the proof of Cartan-Hadamard.
  Stated all the equivalent properties of a complete space, mentioning how appears in the last step of the previous question.

- (Zworski) How this is related to have a minimizing geodesic between every pair of points?
  Stated that this property is satisfied by complete spaces. As a counterexample for the converse I gave the upper-half plane.

- (Zworski) What does Weinstein-Synge's theorem says?
  Just wrote down the statement.

Partial Differential Equations

- (Zworski) Define the fundamental solution of the heat equation.
  Wrote down the function.

- (Zworski) Say $\partial_t u - \Delta u = f \in C^\infty$. Is u smooth?
  Tried to argue like it was the initial-value problem, but he warm me to try another path. At the end I said that given that the fundamental solution has singular support equal to $\{0\}$, u is as amooth as f.

- (Zworski) Define singular support and the result that you are using.
  Gave the definition and that if $P(E) = \delta_0$, $singsupp(E) = \{0\}$ and $P(u) = f$ then $singsupp(u) = singsupp(f)$.

- (Zworski) Solve $u_{xx}.u_x + u_{xy}.u_y = 1$.
  Got scared at the beginning, but he reminded me that my syllabus only covered the first-order case of non-linear equations, hence it must be reduced to $u_x^2 + u_y^2 = 2x + g(y)$. Then the key-word that he was looking for me to say was characteristics.

- (Zworski) Speak about regularity. Which one is done by the book and under which conditions?
  Said that the book (Evans-PDE) covers H^2 and (after some struggle) that the key condition is ellipticity.

- (Agol) Talk about minimal surfaces.
  Said that the functional to minimize is the area, but got in a lot of trouble trying to deduce the equations. For the case of the graph of a function got confused trying to state the Euler-Lagrange equation as the area integrand. Got corrected and final got asked why the previous discussed theory does not apply here (lack of ellipticity).