

ANOSOV GROUPS: LOCAL MIXING, COUNTING, AND EQUIDISTRIBUTION.

SAM EDWARDS, MINJU LEE, AND HEE OH

ABSTRACT. For a Zariski dense Anosov subgroup Γ of a semisimple real algebraic group G , we describe the asymptotic behavior of matrix coefficients $\Phi(g) = \langle gf_1, f_2 \rangle$ in $L^2(\Gamma \backslash G)$ for any $f_1, f_2 \in C_c(\Gamma \backslash G)$. These asymptotics involve higher rank analogues of Burger-Roblin measures. As an application, for any symmetric subgroup H of G , we obtain a bisector counting result for Γ -orbits with respect to the corresponding generalized Cartan decomposition of G . Moreover, we obtain analogues of the results of Duke-Rudnick-Sarnak and Eskin-McMullen for counting discrete Γ -orbits in affine symmetric spaces $H \backslash G$. The link between mixing and counting is provided by an equidistribution result for the translates $\Gamma \backslash \Gamma H a_i$ as $a_i \in H \backslash G$ tends to ∞ .

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1. INTRODUCTION

Let G be a connected semisimple real algebraic group. We fix a Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$, where K is a maximal compact subgroup and $\exp \mathfrak{a}^+$ is a positive Weyl chamber of a maximal real split torus of G . Let $\Gamma < G$ be a Zariski dense discrete subgroup. Consider a matrix coefficient

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of $L^2(\Gamma \backslash G)$ given by

$$\langle \exp(tu)f_1, f_2 \rangle = \int_{\Gamma \backslash G} f_1(x \exp(tu))f_2(x) dx, \quad (1.1)$$

where $u \in \mathfrak{a}^+ - \{0\}$ and dx denotes the G -invariant measure on $\Gamma \backslash G$. Understanding its asymptotic behavior as $t \rightarrow \infty$ is of basic importance in the study of dynamics of flows in $\Gamma \backslash G$, and has many applications, including to equidistribution and counting problems. A classical result due to Howe-Moore [22] implies that

$$\lim_{t \rightarrow \infty} \langle \exp(tu)f_1, f_2 \rangle = \frac{1}{\text{Vol}(\Gamma \backslash G)} \int f_1 dx \int f_2 dx. \quad (1.2)$$

In particular, if Γ has infinite co-volume in G , then for any non-zero $u \in \mathfrak{a}^+$,

$$\lim_{t \rightarrow \infty} \langle \exp(tu)f_1, f_2 \rangle = 0. \quad (1.3)$$

This leads us to the following local mixing type question: for a given unit vector $u \in \mathfrak{a}^+$, do there exist a normalizing function $\Psi_{\Gamma, u} : (0, \infty) \rightarrow (0, \infty)$ and locally finite Borel measures μ_u, μ_u^* on $\Gamma \backslash G$ such that for any $f_1, f_2 \in C_c(\Gamma \backslash G)$ ¹

$$\lim_{t \rightarrow \infty} \Psi_{\Gamma, u}(t) \langle \exp(tu)f_1, f_2 \rangle = \mu_u(f_1)\mu_u^*(f_2)? \quad (1.4)$$

When G has rank one, this was completely answered by Roblin and Winter ([48], [58]) for geometrically finite subgroups and by Oh and Pan [35] for co-abelian subgroups of convex cocompact subgroups.

When G has rank at least two, the location of the vector u relative to the *limit cone* of Γ turns out to play an important role in Question (1.4). The limit cone of Γ , which we denote by \mathcal{L}_Γ , is defined as the smallest closed cone in \mathfrak{a}^+ containing the Jordan projection of Γ . Benoist showed that \mathcal{L}_Γ is convex and has non-empty interior [2]. Indeed, it is not hard to show that if $u \notin \mathcal{L}_\Gamma$, then for any $f_1, f_2 \in C_c(\Gamma \backslash G)$,

$$\langle \exp(tu)f_1, f_2 \rangle = 0 \quad \text{for all } t \text{ large enough;}$$

see Proposition 2.20.

In this paper, for a large class of discrete subgroups Γ , called Anosov subgroups of G , we prove the local mixing result, giving a positive answer to Question (1.4) for all directions u in the interior of \mathcal{L}_Γ . We also give applications to counting and equidistribution results associated to a symmetric subgroup H of G .

Let P be a minimal parabolic subgroup of G and $\mathcal{F} := G/P$ the Furstenberg boundary. We denote by $\mathcal{F}^{(2)}$ the unique open G -orbit in $\mathcal{F} \times \mathcal{F}$ via the diagonal action. A Zariski dense discrete subgroup $\Gamma < G$ is called *Anosov* if there exists a finitely generated word hyperbolic group Σ such

¹for a topological space X , the notation $C_c(X)$ means the space of all continuous functions on X with compact support, and if M is a compact group acting continuously on X , the notation $C_c(X)^M$ means the subspace of $C_c(X)$ consisting of M -invariant functions.

that $\Gamma = \Phi(\Sigma)$ where $\Phi : \Sigma \rightarrow G$ is a P -Anosov representation, i.e., Φ induces a continuous equivariant map ζ from the Gromov boundary $\partial\Sigma$ to \mathcal{F} such that for all $x \neq y \in \partial\Sigma$, $(\zeta(x), \zeta(y))$ belongs to $\mathcal{F}^{(2)}$. The notion of Anosov representations was first introduced by Labourie for surface groups [27], and then extended by Guichard and Wienhard [19] to general word hyperbolic groups.

Remark 1.5. We remark that there are notions of Anosov subgroups with respect to a general parabolic subgroup, but we only deal with those with respect to a minimal parabolic subgroup in this paper. The notion of a P -Anosov representation for a general discrete subgroup requires a certain contraction property, which is automatic for Zariski dense subgroups (see [19, Theorem 1.5]).

If G has rank one, the class of Anosov groups coincides with the class of Zariski dense convex cocompact subgroups of G [19, Theorem 5.15]. Guichard and Wienhard [19, Theorem 1.2] showed that P -Anosov representations form an open subset of the space $\text{Hom}(\Sigma, G)$. The class of Anosov subgroups includes subgroups of a real-split simple algebraic group which arise as the Zariski dense image of a Hitchin representation [21] of a surface subgroup studied by Labourie and Fock-Goncharov ([27], [14]), as well as Schottky groups (cf. Lemma 7.2). We refer to ([24], [18], [5], etc.) for other equivalent definitions of Anosov subgroups, and to ([25], [57]) for excellent survey articles.

In the rest of the introduction, we let $\Gamma < G$ be an Anosov subgroup. Following Quint [41], the growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(tu) = t\psi_\Gamma(u)$, such that for any unit vector $u \in \mathfrak{a}^+$,

$$\psi_\Gamma(u) := \inf_{\substack{\text{open cones } \mathcal{C} \subset \mathfrak{a}^+ \\ u \in \mathcal{C}}} \limsup_{t \rightarrow \infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \leq t\}, \quad (1.6)$$

where $\mu : G \rightarrow \mathfrak{a}^+$ is the Cartan projection and $\|\cdot\|$ is the norm on \mathfrak{a} induced from the left-invariant Riemannian metric on G . Observe that in the rank one case, ψ_Γ is simply the critical exponent of Γ . Quint [41] showed that ψ_Γ is a concave and upper semi-continuous function which is positive on $\text{int } \mathcal{L}_\Gamma$; here $\text{int } \mathcal{L}_\Gamma$ denotes the interior of \mathcal{L}_Γ . If $2\rho \in \mathfrak{a}^*$ denotes the sum of all positive roots with respect to the choice of \mathfrak{a}^+ , then $\psi_\Gamma \leq 2\rho$. When Γ is a lattice, it follows from [15] that $\psi_\Gamma = 2\rho$. On the other hand, when Γ is of infinite co-volume in a simple Lie group of rank at least 2, Quint deduced from [33] that $\psi_\Gamma \leq 2(\rho - \eta_G)$, where $2\eta_G$ is the sum of the maximal strongly orthogonal subset of the root system of G [46].

Local mixing. Let N^+ and N^- denote the maximal expanding and contracting horospherical subgroups, respectively, associated with \mathfrak{a}^+ (see (2.1), (2.2)), and M the centralizer of $\exp \mathfrak{a}$ in K . For each $u \in \text{int } \mathcal{L}_\Gamma$, Quint [42]

constructed a higher-rank analogue of the Patterson-Sullivan density supported on the limit set Λ_Γ , which is the minimal Γ -invariant subset of \mathcal{F} . Using this, we define the $N^\pm M$ -invariant Burger-Roblin measures m_u^{BR} and $m_u^{\text{BR}*}$, respectively, on $\Gamma \backslash G$ (see (3.8) and (3.10)), which can be considered as the higher rank generalizations of the Burger-Roblin measures in the rank one case ([1], [48], [36]). We denote by i the opposition involution of \mathfrak{a} (Definition 3.1), and set $r := \text{rank}(G) = \dim \mathfrak{a} \geq 1$.

Theorem 1.7. *For any unit vector $u \in \text{int } \mathcal{L}_\Gamma$, there exists $\kappa_u > 0$ such that for all $f_1, f_2 \in C_c(\Gamma \backslash G)^M$,*

$$\lim_{t \rightarrow +\infty} t^{(r-1)/2} e^{t(2\rho - \psi_\Gamma)(u)} \int_{\Gamma \backslash G} f_1(x \exp(tu)) f_2(x) dx = \kappa_u \cdot m_{i(u)}^{\text{BR}}(f_1) m_u^{\text{BR}*}(f_2).$$

We mention that this theorem is not expected to hold for $u \in \partial \mathcal{L}_\Gamma$ in view of [11, Theorem 1.1]. See Theorem 7.11 for a more refined version of this theorem; in fact it is this refined version which is needed in the application to counting problems as stated in Theorems 1.9 and 1.11.

Equidistribution of maximal horospheres. We also obtain the following equidistribution result for translates of maximal horospheres:

Theorem 1.8. *For any unit vector $u \in \text{int } \mathcal{L}_\Gamma$, $f \in C_c(\Gamma \backslash G)^M$, $\phi \in C_c(N^+)$, and $x = [g] \in \Gamma \backslash G$, we have*

$$\lim_{t \rightarrow \infty} t^{(r-1)/2} e^{t(2\rho - \psi_\Gamma)(u)} \int_{N^+} f(xn \exp(tu)) \phi(n) dn = \kappa_u \cdot m_{i(u)}^{\text{BR}}(f) \mu_{gN^+,u}^{\text{PS}}(\phi),$$

where dn and $\mu_{gN^+,u}^{\text{PS}}$ are respectively the Lebesgue and Patterson-Sullivan measures on gN^+ as defined in (4.7) and (4.1).

Bisector counting for a generalized Cartan decomposition. Let H be a symmetric subgroup of G , i.e. H is the identity component of the set of fixed points for an involution σ of G . Up to a conjugation, we may assume that σ commutes with the Cartan involution θ which fixes K . We then have a generalized Cartan decomposition $G = H\mathcal{W}(\exp \mathfrak{b}^+)K$, where $\mathfrak{b}^+ \subset \mathfrak{a}^+$ and \mathcal{W} is a subgroup of the Weyl group (see Section 8 for details). Set $r_0 := \text{rank } H \backslash G = \dim \mathfrak{b}$. Note that $1 \leq r_0 \leq r$.

Theorem 1.9. *For any unit vector $v \in \mathfrak{b}^+ \cap \text{int } \mathcal{L}_\Gamma$, there exist $c > 0$ and a norm $|\cdot|$ on \mathfrak{b} such that for any right $H \cap M$ -invariant bounded subset $\Omega_H \subset H$ with $\mu_{H,v}^{\text{PS}}(\partial \Omega_H) = 0$ and any left M -invariant bounded subset $\Omega_K \subset K$ with $\mu_{K,i(v)}^{\text{PS},*}(\partial \Omega_K^{-1}) = 0$, we have*

$$\lim_{T \rightarrow \infty} \frac{\#(\Gamma \cap \Omega_H(\exp \mathfrak{b}_T^+) \Omega_K)}{e^{\psi_\Gamma(v)T} \cdot T^{(r_0-r)/2}} = c \mu_{H,v}^{\text{PS}}(\Omega_H) \mu_{K,i(v)}^{\text{PS},*}(\Omega_K^{-1}),$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : |w| \leq T\}$ and $\mu_{H,v}^{\text{PS}}$ and $\mu_{K,i(v)}^{\text{PS},*}$ are measures on H and K defined in (8.7) and Lemma 9.19 respectively.

When Γ is a cocompact lattice in a rank one Lie group and H is compact, this goes back to Margulis' thesis from 1970 (see [29] for an English translation published in 2004). In the case when Γ is a geometrically finite subgroup of a rank one Lie group, this was shown in [48] for H compact, and in [36] and [32] for general symmetric subgroups.

When the rank of G is at least 2 and H is compact, this theorem was proved by Quint [45] and Thirion [55] for Schottky groups² and by Sambarino for Anosov subgroups [50] (see also [10]). Hence the main novelty of this paper lies in our treatment of non-compact symmetric subgroups H in a general higher rank case. It is interesting to note the presence of the decaying polynomial term $T^{(r_0-r)/2}$ when $\mathfrak{a} \neq \mathfrak{b}$, as the results in loc. cit. have all purely exponential terms. We mention that a related counting result was obtained for $\mathrm{SO}(p, q-1) \backslash \mathrm{SO}(p, q)$ in a recent paper of Carvajales [9]; in this case, \mathfrak{b} lies in the wall of \mathfrak{a} and hence Theorem 1.9 does not apply, and the asymptotic is again purely exponential.

By the concavity and upper semi-continuity of ψ_Γ , there exists a unique unit vector $u_\Gamma \in \mathfrak{a}^+$ (called the maximal growth direction) such that

$$\psi_\Gamma(u_\Gamma) = \max_{v \in \mathfrak{a}^+, \|v\|=1} \psi_\Gamma(v).$$

It is known that $u_\Gamma \in \mathrm{int} \mathcal{L}_\Gamma$ ([45], [50]). When $u_\Gamma \in \mathfrak{b}^+$, the norm $|\cdot|$ in Theorem 1.9 associated to u_Γ may simply be taken as the Euclidean norm $\|\cdot\|$ as above, i.e. the one obtained from the inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{a} induced by the Killing form. For a general vector $v \in \mathfrak{b}^+ \cap \mathrm{int} \mathcal{L}_\Gamma$, one may take any norm that arises from an inner product for which v and $(\nabla \psi_\Gamma(v))^\perp = \{w \in \mathfrak{b} : \langle \nabla \psi_\Gamma(v), w \rangle = 0\}$ are orthogonal.

Example. When $\mathfrak{a} = \mathfrak{b}$, we automatically have $u_\Gamma \in \mathfrak{b}^+ \cap \mathrm{int} \mathcal{L}_\Gamma$; so Theorem 1.9 applies. For groups G of rank one, this is always the case for any symmetric subgroup H . In general, this case arises as follows: let ι be any involution of G that commutes with the Cartan involution θ and fixes \mathfrak{a} pointwise. Then defining $\sigma := \iota \circ \theta$, we have $\sigma|_{\mathfrak{a}} = -1$, and hence $\mathfrak{a} = \mathfrak{b}$. For example, for any element $m \in K$ of order two which commutes with $\exp \mathfrak{a}$, $\iota(g) := m g m$ satisfies the above conditions. More specifically, the pair $G = \mathrm{PGL}_n(\mathbb{R})$ and $H = \mathrm{PO}(p, n-p)$ may be realized this way by taking $m = \mathrm{diag}(\mathrm{Id}_p, -\mathrm{Id}_{n-p})$.

Counting in affine symmetric spaces. Around 1993, Duke-Rudnick-Sarnak [12] and Eskin-McMullen [13] showed the following (see also [4], [15], [16], [17] etc.):

²Thirion's work applies to the so-called Ping-Pong groups which are slightly more general than Schottky groups.

Theorem 1.10. *Let $\Gamma < G$ be a lattice such that $\mathfrak{v}_0\Gamma \subset H \backslash G$ is discrete for $\mathfrak{v}_0 = [H]$. Suppose that $(H \cap \Gamma) \backslash H$ has finite volume. We have, as $T \rightarrow \infty$,*

$$\lim_{T \rightarrow \infty} \frac{\#(\mathfrak{v}_0\Gamma \cap \mathfrak{v}_0(\exp \mathfrak{b}_T^+)K)}{\mathfrak{m}(\mathfrak{v}_0(\exp \mathfrak{b}_T^+)K)} = \frac{\text{Vol}((\Gamma \cap H) \backslash H)}{\text{Vol}(\Gamma \backslash G)},$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : \|w\| \leq T\}$ and \mathfrak{m} is a suitably normalized G -invariant measure on $H \backslash G$.

In order to state an analogue of Theorem 1.10, we introduce the following condition on an H -orbit: a closed orbit $[e]H \subset \Gamma \backslash G$ is said to be *uniformly proper* if there exists a neighborhood \mathcal{O} of $[e]$ in $\Gamma \backslash G$ such that

$$\{[h] \in (\Gamma \cap H) \backslash H : [h] \exp(\mathfrak{b}^+ \cap \mathcal{L}_\Gamma) \cap \mathcal{O} \neq \emptyset\}$$

is bounded.³ We show in Lemma 9.35 that $[e]H$ is uniformly proper whenever $r = r_0$ and Λ_Γ is contained in the open set $HP/P \subset \mathcal{F}$ (cf. [53, Proposition. 7.1.8]).

When $\mathfrak{v}_0\Gamma$ is discrete, the measure $\mu_{H,v}^{\text{PS}}$ in Theorem 1.9 induces a locally finite Borel measure on $(\Gamma \cap H) \backslash H$, whose total measure will be called the skinning constant $\text{sk}_{\Gamma,v}(H)$ of H with respect to Γ and v .

Theorem 1.11. *Let $\Gamma < G$ be an Anosov subgroup such that $\mathfrak{v}_0\Gamma \subset H \backslash G$ is discrete for $\mathfrak{v}_0 = [H]$. Suppose that $[e]H$ is uniformly proper. Then there exists a constant $c > 0$ such that*

$$\lim_{T \rightarrow \infty} \frac{\#(\mathfrak{v}_0\Gamma \cap \mathfrak{v}_0(\exp \mathfrak{b}_T^+)K)}{e^{\delta_\Gamma T} \cdot T^{(r_0-r)/2}} = c \text{sk}_{\Gamma,u_\Gamma}(H) \quad (1.12)$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : \|w\| \leq T\}$ and $\delta_\Gamma = \psi_\Gamma(u_\Gamma)$.

See Theorem 9.27 for a more refined version where u_Γ is replaced by a more general $v \in \mathfrak{b}^+ \cap \text{int } \mathcal{L}_\Gamma$.

Remark 1.13. (1) When G has rank one, this is proved in [36] and [32] for any geometrically finite group Γ under the finite skinning constant hypothesis. In higher rank case, the finite skinning constant hypothesis seems insufficient for our approach to work.

(2) We mention a recent work of Carvajales [10, Theorem B] where a special case of this theorem was obtained for $G = \text{PSL}_n(\mathbb{R})$, $H = \text{SO}(p, n-p)$, Γ acts properly discontinuously on $H \backslash G$. In this case, $\mathfrak{a} = \mathfrak{b}$ and hence there is no polynomial term in (9.29).

There are many symmetric spaces $H \backslash G$ admitting a proper discontinuous action by a Zariski dense Schottky subgroup $\Gamma < G$ [3]. See also [18, Corollary 1.10, Remark 6.2] for examples of symmetric spaces of real-split simple Lie groups on which Anosov groups arising from Hitchin representations act properly discontinuously. There are also many Anosov subgroups

³As $[e]H$ is closed, the inclusion $(H \cap \Gamma) \backslash H \rightarrow \Gamma \backslash G$ is a proper map and hence the set $S_b := \{[h] \in (\Gamma \cap H) \backslash H : [h] \exp b \cap \mathcal{O} \neq \emptyset\}$ is bounded for each $b \in \mathfrak{b}^+ \cap \mathcal{L}_\Gamma$. Now the uniform properness of $[e]H$ means that the union $\bigcup_{b \in \mathfrak{b}^+ \cap \mathcal{L}_\Gamma} S_b$ is also bounded.

Γ contained in $\mathrm{PSL}_n(\mathbb{Z})$ (i.e., thin matrix groups in the terminology of [52]) arising from Hitchin representations, as constructed in Kac-Vinberg [23] and Long-Reid-Thislethwaite [28] (see also [57, Theorem 24]). When $H < \mathrm{PSL}_n$ is defined over \mathbb{Q} , we have $v_0\Gamma$ is discrete, and hence Theorem 1.11 may be applied to these settings.

On the proofs. The main ingredient of Theorem 1.7 is the following mixing result for the Bowen-Margulis-Sullivan measures proved by Thirion [55] for Schottky groups and by Sambarino [50] for Anosov groups which arise from representations of the fundamental group of a closed negatively curved Riemannian manifold, using thermodynamic formalism. Given the recent work [8] which introduces the geodesic flow for Anosov groups which is shown to be a metric Anosov flow, Sambarino's proof is known to extend to all Anosov groups (see [10, Appendix]):

Theorem 1.14. *Let $u \in \mathrm{int} \mathcal{L}_\Gamma$ be a unit vector. For any $f_1, f_2 \in C_c(\Gamma \backslash G)^M$,*

$$\lim_{t \rightarrow \infty} t^{(r-1)/2} \int_{\Gamma \backslash G} f_1(x) f_2(x \exp(tu)) dm_u^{\mathrm{BMS}}(x) = \kappa_u \cdot m_u^{\mathrm{BMS}}(f_1) m_u^{\mathrm{BMS}}(f_2),$$

where m_u^{BMS} is the BMS-measure associated to u (see (3.5) and (7.10)).

Using the product structures of the Haar measure dx and $dm_u^{\mathrm{BMS}}(x)$, one can deduce mixing for one measure from that of the other via the study of transversal intersections. This observation is originally due to Roblin [48] in the case of the unit tangent bundle of a rank one locally symmetric manifold, and has been extended and utilized in ([32], [36]) to the frame bundle. This study leads us to generalize the definition of the family of Burger-Roblin measures m_u^{BR} and $m_u^{\mathrm{BR}^*}$ for $u \in \mathrm{int} \mathcal{L}_\Gamma$, which turn out to control the asymptotic behavior of matrix coefficients as in Theorem 1.7.

As in [12] and [13] (also as in [36] and [32]), passing from Theorem 1.7 to Theorems 1.9 and 1.11 requires the following equidistribution statement for translates of H -orbits. The idea of using mixing in the equidistribution and counting problem goes back to [29]:

Proposition 1.15. *For any unit vector $v \in \mathfrak{b}^+ \cap \mathrm{int} \mathcal{L}_\Gamma$, there exists $\kappa_v > 0$ such that for any $f \in C_c(\Gamma \backslash G)^M$ and $\phi \in C_c(H)^{H \cap M}$,*

$$\lim_{t \rightarrow \infty} t^{(r-1)/2} e^{(2\rho - \Theta)(tv)} \int_H f([h] \exp(tv)) \phi(h) dh = \kappa_v \cdot \mu_{H,v}^{\mathrm{PS}}(\phi) m_{\mathfrak{i}(v)}^{\mathrm{BR}}(f)$$

where dh denotes the Haar measure on H and $\Theta \in \mathfrak{a}^*$ is given by $\Theta(w) = \langle \nabla \psi_\Gamma(v), w \rangle$ as in (8.3).

Organization: We start by reviewing some basic notions, including higher rank analogues of Patterson-Sullivan measures as defined by Quint [42] in Section 2. Section 3 introduces generalized BMS-measures, in particular higher-rank versions of Burger-Roblin measures are defined. The product structure of these measures is discussed in Section 4. We then deduce

equidistribution of translates of PS-measures on horospheres from local mixing in Section 5. This is then used in Section 6 to show mixing for the Haar measure and equidistribution of translates of Lebesgue measures on maximal horospheres. Properties of the main types of discrete subgroups we study are discussed in Section 7. The remainder of the paper is mainly devoted to proving the claimed counting statements. As a first step towards this, we prove equidistribution of translates of orbits of symmetric subgroups in $\Gamma \backslash G$ in Section 8. These equidistribution statements are combined with the strong wavefront property in Section 9 to give the various counting results.

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2. (Γ, ψ) -PATTERSON-SULLIVAN MEASURES

Let G be a connected, semisimple real algebraic group, and $\Gamma < G$ be a Zariski dense discrete subgroup. In this section, we review the notion of (Γ, ψ) -Patterson-Sullivan measures associated to a certain class of linear forms ψ on \mathfrak{a} , as constructed by Quint in [42]. We present these measures as analogously as possible to the Patterson-Sullivan measures on the limit set of Γ in the rank one case.

We fix, once and for all, a Cartan involution θ of the Lie algebra \mathfrak{g} of G , and decompose \mathfrak{g} as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} and \mathfrak{p} are the $+1$ and -1 eigenspaces of θ , respectively. We denote by K the maximal compact subgroup of G with Lie algebra \mathfrak{k} . We also choose a maximal abelian subalgebra \mathfrak{a} of \mathfrak{p} . Fixing a left G -invariant and right K -invariant Riemannian metric on G induces a Weyl-group invariant inner product and corresponding norm on \mathfrak{a} , which we denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Note also that the choice of this Riemannian metric induces a G -invariant metric $d(\cdot, \cdot)$ on G/K . The identity coset $[e]$ in G/K is denoted by o .

Let $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber \mathfrak{a}^+ of \mathfrak{a} , let $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M , and we set

$$N = N^-$$

to be the maximal contracting horospherical subgroup for A : for an element a in the interior of A^+ ,

$$N^- = \{g \in G : a^{-n}ga^n \rightarrow e \text{ as } n \rightarrow +\infty\}. \quad (2.1)$$

Note that $\log(N)$ is the sum of all positive root subspaces for our choice of \mathfrak{a}^+ . Similarly, we will also need to consider the maximal expanding horospherical subgroup

$$N^+ := \{g \in G : a^n ga^{-n} \rightarrow e \text{ as } n \rightarrow +\infty\}. \quad (2.2)$$

We set

$$P^+ = MAN^+, \quad \text{and} \quad P = MAN;$$

they are minimal parabolic subgroups of G . The quotient

$$\mathcal{F} = G/P$$

is known as the Furstenberg boundary of G , and is isomorphic to K/M .

Definition 2.3 (Busemann function). The Iwasawa cocycle $\sigma : G \times \mathcal{F} \rightarrow \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$, $\exp \sigma(g, \xi)$ is the A -component of gk in the KAN decomposition, where $\xi = [k] \in K/M$:

$$gk \in K \exp(\sigma(g, \xi))N.$$

The Busemann function $\beta : \mathcal{F} \times G/K \times G/K \rightarrow \mathfrak{a}$ is now defined as follows: for $\xi \in \mathcal{F}$ and $[g], [h] \in G/K$,

$$\beta_\xi([g], [h]) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi).$$

Observe that the Busemann function is continuous in all three variables. To ease notation, we will write $\beta_\xi(g, h) = \beta_\xi([g], [h])$. The following identities will be used throughout the article:

$$\begin{aligned} \beta_\xi(g, h) + \beta_\xi(h, q) &= \beta_\xi(g, q), \\ \beta_{g\xi}(gh, gq) &= \beta_\xi(h, q), \text{ and} \\ \beta_\xi(e, g) &= -\sigma(g^{-1}, \xi). \end{aligned} \tag{2.4}$$

Geometrically, if $\xi = [k] \in K/M$, then for any unit vector $u \in \mathfrak{a}^+$,

$$\langle \beta_\xi(g, h), u \rangle = \lim_{t \rightarrow +\infty} d([g], \xi_t) - d([h], \xi_t),$$

where $\xi_t = k \exp(tu)o \in G/K$.

Definition 2.5 (Conformal measures and densities). Given $\psi \in \mathfrak{a}^*$ and a closed subgroup $\Delta < G$, a Borel probability measure ν_ψ on \mathcal{F} is called a (Δ, ψ) -conformal measure if, for any $\gamma \in \Delta$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_*\nu_\psi}{d\nu_\psi}(\xi) = e^{\psi(\beta_\xi(e, \gamma))}, \tag{2.6}$$

where $\gamma_*\nu_\psi(Q) = \nu_\psi(\gamma^{-1}Q)$ for any Borel subset $Q \subset \mathcal{F}$. If ν_ψ is a (Δ, ψ) -conformal measure, then the collection $\{g_*\nu_\psi : g \in G\}$ is called a (Δ, ψ) -conformal density.

Definition 2.7 (Lebesgue measure). Let m_o denote the K -invariant probability measure on \mathcal{F} , and ρ denote the half sum of all positive roots with respect to \mathfrak{a}^+ . Then, using the decomposition of the Haar measure in the KAN coordinates, one can check the following (cf. [47, Proposition 3.3]): if m_o is a $(G, 2\rho)$ -conformal measure, i.e. for any $g \in G$ and $\xi \in \mathcal{F}$,

$$\frac{dg_*m_o}{dm_o}(\xi) = e^{2\rho(\beta_\xi(e, g))}. \tag{2.8}$$

Limit set and Limit cone.

Definition 2.9 (Limit set). The limit set Λ_Γ of Γ is defined to be the set of all points $x \in \mathcal{F}$ such that the Dirac measure δ_x is a limit point (in the space of Borel probability measures on \mathcal{F}) of $\{\gamma_* m_o : \gamma \in \Gamma\}$.

Benoist showed that Λ_Γ is the minimal Γ -invariant closed subset of \mathcal{F} . Moreover, Λ_Γ is Zariski dense in \mathcal{F} [2, Section 3.6].

An element of G is called elliptic if it is contained in a compact subgroup, and hyperbolic if it is conjugate to an element of A^+ . Any $g \in G$ can be written as the commuting product

$$g = g_h g_e g_u \tag{2.10}$$

where g_h is hyperbolic, g_e is elliptic and g_u is unipotent. An element $g \in G$ is called *loxodromic* if g_h is conjugate to an element of $\text{int } A^+$.

Lemma 2.11. *For any open subset $U \subset \mathcal{F}$ with $U \cap \Lambda_\Gamma \neq \emptyset$, $U \cap \Lambda_\Gamma$ is contained in any smooth submanifold of \mathcal{F} of smaller dimension.*

Proof. This is proved in [58] when G has rank one, and our proof is similar. Since $\mathfrak{n}^+ := \text{Lie } N^+$ is nilpotent, the map $\mathfrak{n}^+ \rightarrow G/P$ given by $x \mapsto \exp(x)[e]$ is algebraic and its image $N^+[e]$ is Zariski open and dense in G/P for $[e] = P$. Therefore $N^+[e] \cap \Lambda_\Gamma$ is Zariski dense in $N^+[e]$. Suppose that there exists an open subset $U \subset \mathcal{F}$ such that $U \cap \Lambda_\Gamma \neq \emptyset$ is contained in a smooth submanifold S of \mathcal{F} of smaller dimension. Any Zariski dense Γ contains a loxodromic element, say γ . We may assume without loss of generality that $\gamma = am$ where $m \in M$ and $a \in \text{int } A^+$. Choose a basis of \mathfrak{n}^+ consisting of eigenvectors of Ad_a , and for $x \in \mathfrak{n}^+$, we write $x = (x_1, \dots, x_d)$ for the coordinates with respect to this basis. It follows that there exist $0 < c_i < 1$, $i = 1, \dots, d$ such that

$$\text{Ad}_a x = (c_1 x_1, \dots, c_d x_d). \tag{2.12}$$

Choose $\ell \in \mathbb{N}$ such that $c_1 > \max_{1 \leq i \leq d} c_i^{\ell+1}$. By the implicit function theorem, after shrinking U and rearranging the indices if necessary, we may assume that $U \cap S = \{[\exp(x)] \in U : x_1 = f(x_2, \dots, x_d)\}$ for some smooth function f . Let p be the Taylor polynomial of f of degree ℓ . Then, by shrinking U further, there exists $C > 0$ such that for all $x \in U$,

$$|f(x_2, \dots, x_d) - p(x_2, \dots, x_d)| \leq C \|(x_2, \dots, x_d)\|^{\ell+1} \tag{2.13}$$

(here $\|\cdot\|$ denotes the Euclidean norm on \mathfrak{n}^+). Since the action of Ad_a on the polynomial ring $\mathbb{R}[\mathfrak{n}^+]$ is diagonalizable, we can write

$$x_1 - p(x_2, \dots, x_d) = \sum_{i=1}^k p_i(x), \tag{2.14}$$

where $p_i \in \mathbb{R}[\mathfrak{n}^+]$ are non-zero polynomials such that $p_i(\text{Ad}_a x) = \beta_i \cdot p_i(x)$ where $1 > \beta_1 > \dots > \beta_k > 0$. Note that $\beta_1 \geq c_1$, due to the presence of x_1

in (2.14). Since $U \cap \Lambda_\Gamma \subset S$, combining (2.13) and (2.14), we conclude

$$\left| \sum_{i=1}^k p_i(x) \right| \leq C \|(x_2, \dots, x_d)\|^{\ell+1}, \text{ whenever } [\exp x] \in U \cap \Lambda_\Gamma. \quad (2.15)$$

Now let $x \in \mathfrak{n}^+$ be such that $[\exp x] \in \Lambda_\Gamma$. Since $[e] \in U \cap \Lambda_\Gamma$ and $\text{Ad}_{\gamma^n} x \rightarrow 0$ as $n \rightarrow \infty$, we have $[\exp \text{Ad}_{\gamma^n} x] \in U \cap \Lambda_\Gamma$ for all sufficiently large n . Applying (2.12) and (2.15), we obtain

$$\left| \sum_{i=1}^k p_i(\text{Ad}_{\gamma^n} x) \right| = \left| \sum_{i=1}^k \beta_i^n \cdot p_i(\text{Ad}_{m^n} x) \right| \leq C \left(\max_{1 \leq i \leq d} c_i^{\ell+1} \right)^n \|\text{Ad}_{m^n} x\|^{\ell+1}.$$

Therefore, by dividing by β^n ,

$$\left| p_1(\text{Ad}_{m^n} x) + \sum_{i=2}^k \left(\frac{\beta_i}{\beta_1} \right)^n p_i(\text{Ad}_{m^n} x) \right| \leq C \left(\frac{\max_{1 \leq i \leq d} c_i^{\ell+1}}{\beta_1} \right)^n \|\text{Ad}_{m^n} x\|^{\ell+1}.$$

Since M is a compact subgroup, it follows that for some sequence $n_i \rightarrow \infty$, $m^{n_i} \rightarrow e$. Taking the limit along this subsequence yields $p_1(x) = 0$. This shows that $\{x \in \mathfrak{n}^+ : [\exp x] \in \Lambda_\Gamma\} \subset \{x \in \mathfrak{n}^+ : p_1(x) = 0\}$, implying that $N^+[e] \cap \Lambda_\Gamma$ is not Zariski dense in $N^+[e]$, yielding a contradiction. \square

We remark that a weaker version of this lemma was proved for $G = \text{SL}_n(\mathbb{C})$ by Cantat (see [20, Sec.6]).

Definition 2.16 (Cartan projection). The Cartan projection $\mu : G \rightarrow \mathfrak{a}^+$ is defined as follows: for each $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that

$$g \in K \exp(\mu(g))K.$$

The Jordan projection of g is defined as $\lambda(g) \in \mathfrak{a}^+$, where $\exp \lambda(g)$ is the element of A^+ conjugate to g_h where g_h is as in (2.10).

Definition 2.17 (Limit cone). The *limit cone* $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ of Γ is defined as the smallest closed cone containing the *Jordan projection* of Γ .

Quint showed the following:

Theorem 2.18. [41, Theorem IV.2.2] *The growth indicator function ψ_Γ , defined in (1.6), is concave, upper-semicontinuous, and satisfies*

$$\mathcal{L}_\Gamma = \{u \in \mathfrak{a}^+ : \psi_\Gamma(u) > -\infty\}.$$

Moreover, $\psi_\Gamma(u)$ is non-negative on \mathcal{L}_Γ and positive on $\text{int } \mathcal{L}_\Gamma$.

We deduce from Theorem 2.18:

Lemma 2.19. *The cone \mathcal{L}_Γ is the smallest closed cone containing $\mu(\Gamma)$.*

Proof. Observe the following:

$$\begin{aligned} u \in \mathcal{L}_\Gamma &\Leftrightarrow \psi_\Gamma(u) > -\infty \text{ (by Theorem 2.18)} \\ &\Leftrightarrow \mu(\Gamma) \cap \mathcal{C} \neq \emptyset \text{ for all open cones } \mathcal{C} \text{ containing } u \\ &\Leftrightarrow u \in \overline{\mathbb{R}_+\mu(\Gamma)}. \end{aligned}$$

We therefore conclude that $\mathcal{L}_\Gamma = \overline{\mathbb{R}_+\mu(\Gamma)}$, as desired. \square

Proposition 2.20. *Let \mathfrak{m} be a locally finite Borel measure on $\Gamma \backslash G$ and $\mathcal{C} \subset \mathfrak{a}^+$ a closed cone with $\mathcal{L}_\Gamma \subset \text{int } \mathcal{C}$. For any $f_1, f_2 \in C_c(\Gamma \backslash G)$, there exists $t_0 > 0$ such that for all $v \in \mathfrak{a}^+ - \mathcal{C}$ with $\|v\| \geq t_0$,*

$$\int_{\Gamma \backslash G} f_1(x \exp(v)) f_2(x) d\mathfrak{m}(x) = 0.$$

In particular, if $u \in \mathfrak{a}^+ - \mathcal{L}_\Gamma$, then $\int_{\Gamma \backslash G} f_1(x \exp(tu)) f_2(x) d\mathfrak{m}(x) = 0$ for all $t \gg 1$ large enough.

Proof. It suffices to check that for any given compact subset $L \subset G$ such that $L = L^{-1}$, we have

$$L \exp(-v) \cap \Gamma L = \emptyset$$

for all sufficiently large $v \in \mathfrak{a}^+ - \mathcal{C}$.

Suppose that there exist a compact subset $L \subset G$, sequences $\ell_n, \ell'_n \in L$, $\gamma_n \in \Gamma$, and $v_n \in \mathfrak{a} - \mathcal{C}$ with $\|v_n\| = t_n \rightarrow \infty$ such that

$$\ell_n \exp(-v_n) = \gamma_n \ell'_n.$$

We may assume that v_n/t_n converges to some unit vector $v \in (\mathfrak{a}^+ - \text{int } \mathcal{C})$; hence $v \notin \mathcal{L}_\Gamma$. By [2, Lemma 4.6], there exists a compact subset $M = M(L)$ of \mathfrak{a} such that for all $g \in G$,

$$\mu(LgL) \subset \mu(g) + M.$$

Since $v \notin \mathcal{L}_\Gamma$, by Lemma 2.19, we can find an open cone \mathcal{D} containing v such that $\mathcal{D} \cap \mu(\Gamma) = \emptyset$. Then

$$\mu(\gamma_n^{-1}) = \mu(\ell'_n \exp(v_n) \ell_n^{-1}) \in \mu(\exp(v_n)) + M.$$

As \mathcal{D} is open, there exists n_0 such that $\frac{1}{t_n}(v_n + M) \subset \mathcal{D}$ for all $n \geq n_0$. Since $\mu(\exp(v_n)) + M = t_n(\frac{v_n}{t_n} + \frac{1}{t_n}M)$, we conclude $\mu(\gamma_n^{-1}) \in \mathcal{D}$ for all $n \geq n_0$. This yields a contradiction.

The second claim follows from the first one as we can find a closed cone \mathcal{C} such that $u \notin \mathcal{C}$ and $\mathcal{L}_\Gamma \subset \text{int } \mathcal{C}$. \square

Set

$$D_\Gamma := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma \text{ on } \mathfrak{a}^+\},$$

which is a non-empty set [44, Section 4.1]. An element $\psi \in D_\Gamma$ is said to be tangent to ψ_Γ at $u \in \mathfrak{a}$ if $\psi(u) = \psi_\Gamma(u)$. The following collection of linear forms is of particular importance:

$$D_\Gamma^* := \{\psi \in D_\Gamma : \psi \text{ is tangent to } \psi_\Gamma \text{ at some unit vector } u \in \mathcal{L}_\Gamma \cap \text{int } \mathfrak{a}^+\}. \quad (2.21)$$

Definition 2.22 (Patterson-Sullivan measures). For $\psi \in \mathfrak{a}^*$, a (Γ, ψ) -conformal measure supported on Λ_Γ will be called a (Γ, ψ) -PS measure.

Generalizing the work of Patterson-Sullivan ([38], [54]), Quint [42] constructed a (Γ, ψ) -PS measure for every $\psi \in D_\Gamma^*$.

Maximal growth direction. Since ψ_Γ is concave, upper-semicontinuous, and the unit norm ball in \mathfrak{a} is strictly convex, there exists a unique unit vector $u_\Gamma \in \mathcal{L}_\Gamma$ (called the maximal growth direction) such that

$$\delta_\Gamma := \max_{u \in \mathfrak{a}^+, \|u\|=1} \psi_\Gamma(u) = \psi_\Gamma(u_\Gamma). \quad (2.23)$$

Note that u_Γ must be stabilized by the *opposition involution* (see Def. 3.1)

Example 2.24. If $G = \mathrm{PSL}_3(\mathbb{R})$, then $u_\Gamma = \mathrm{diag}(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ for any Zariski dense subgroup Γ .

Uniqueness of tangent forms. The following lemma follows from [44, Sec 4.1] (see also [49, Lemma 4.8]).

Lemma 2.25. (1) For any $u \in \mathrm{int} \mathcal{L}_\Gamma$, there exists a linear form $\psi_u \in D_\Gamma^*$ tangent to ψ_Γ at u .

(2) For any $u \in \mathrm{int} \mathcal{L}_\Gamma$ at which ψ_Γ is differentiable, there exists a unique linear form $\psi_u \in D_\Gamma^*$ tangent to ψ_Γ at u , and it is given by

$$\psi_u(\cdot) = \langle (\nabla \psi_\Gamma)(u), \cdot \rangle = D_u \psi_\Gamma(\cdot);$$

(3) If $u_\Gamma \in \mathrm{int} \mathcal{L}_\Gamma$ and ψ_Γ is differentiable at u_Γ , then ψ_{u_Γ} is given by

$$\psi_{u_\Gamma}(\cdot) = \delta_\Gamma \langle u_\Gamma, \cdot \rangle.$$

Proof. Let $P \subset \mathfrak{a}$ be an affine hyperplane such that $P \cap \mathfrak{a}^+$ is an $(r-1)$ -simplex and $P \cap \mathcal{L}_\Gamma$ is a bounded convex subset of $P \simeq \mathbb{R}^{r-1}$. We now define a set $S \subset P \times \mathbb{R}$ by $S := \{(x, y) \in (P \cap \mathcal{L}_\Gamma) \times \mathbb{R} : 0 \leq y \leq \psi_\Gamma(x)\}$. Since $P \cap \mathcal{L}_\Gamma$ is convex and $\psi_\Gamma : \mathfrak{a} \rightarrow \mathbb{R}$ is concave, S is convex. Since $\mathbb{R}(P \cap \mathrm{int} \mathcal{L}_\Gamma) \supset \mathrm{int} \mathcal{L}_\Gamma$, it suffices to prove (1) for $u \in P \cap \mathrm{int} \mathcal{L}_\Gamma$. Note that $(u, \psi_\Gamma(u)) \in \partial S$, and hence by the supporting hyperplane theorem, we can find a hyperplane $C \subset P \times \mathbb{R}$ passing through $(u, \psi_\Gamma(u))$ such that the interior of S is contained in a connected component of $P \times \mathbb{R} - C$. Note that as $u \in \mathrm{int} \mathcal{L}_\Gamma$, such a hyperplane C must be the graph of a function. We may therefore write $C = \{(x, \varphi(x)) \in P \times \mathbb{R}\}$ for some affine map $\varphi : P \rightarrow \mathbb{R}$ satisfying $\varphi(x) \geq \psi_\Gamma(x)$ for all $x \in P \cap \mathcal{L}_\Gamma$. Consider the unique linear form in \mathfrak{a}^* which extends φ , which we also denote by φ by abuse of notation. Since $\varphi(x) \geq \psi_\Gamma(x)$ for all $x \in P \cap \mathcal{L}_\Gamma$ and \mathcal{L}_Γ has non-empty interior, it follows that $\varphi \geq \psi_\Gamma$. Since $\varphi(u) = \psi_\Gamma(u)$, this proves (1).

To prove (2), define $\psi_u(\cdot) := \langle (\nabla \psi_\Gamma)(u), \cdot \rangle$. By differentiating $\psi_\Gamma(tu) = t\psi_\Gamma(u)$ with respect to t , we get by the chain rule that

$$\langle \nabla \psi_\Gamma(tu), u \rangle = \psi_\Gamma(u). \quad (2.26)$$

Hence $\psi_u(u) = \psi_\Gamma(u)$ by plugging $t = 1$. Next, let \mathfrak{b} be a vector space such that $\mathfrak{a} = \mathbb{R}u \oplus \mathfrak{b}$, and let $v \in \mathfrak{b}$ be arbitrary. Consider the closed interval

$I = \{s \in \mathbb{R} : u + sv \in \mathcal{L}_\Gamma\}$ and let $f(s) := \psi_\Gamma(u + sv)$. Note that $f : I \rightarrow \mathbb{R}$ is concave, differentiable at $s = 0$, and $f'(0) = \langle \nabla \psi_\Gamma(u), v \rangle$. Hence, using (2.26),

$$f(s) \leq \psi_\Gamma(u) + s \langle \nabla \psi_\Gamma(u), v \rangle = \langle \nabla \psi_\Gamma(u), u + sv \rangle = \psi_u(u + sv).$$

As $v \in \mathfrak{b}$ is arbitrary, this implies $\psi_u \geq \psi_\Gamma$. Hence $\psi_u \in D_\Gamma^*$.

To show the uniqueness, suppose that $\psi \in D_\Gamma^*$ is tangent to ψ_Γ at u . Let $v \in \mathfrak{b}$ be arbitrary. Define $g : I \rightarrow \mathbb{R}$ by $g(s) := \psi(u + sv)$. Then $g \geq f$ and $g(0) = f(0)$. Since f is a concave function on an interval I and differentiable at $0 \in S$, it follows that $g(s) = f(0) + sf'(0)$. Since $f(0) = \psi_\Gamma(u)$ and $f'(0) = \langle \nabla \psi_\Gamma(u), v \rangle$, it follows that

$$\psi(u + sv) = \psi_\Gamma(u) + s \langle \nabla \psi_\Gamma(u), v \rangle = \langle \nabla \psi_\Gamma(u), u + sv \rangle;$$

this proves the uniqueness.

Next we claim that $\nabla \psi_\Gamma(u_\Gamma) = cu_\Gamma$ for some $c \neq 0$. Consider a curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow \mathbb{S}^{r-1} \cap \mathfrak{a}^+$ such that $\alpha(0) = u_\Gamma$. By definition of u_Γ , $s \mapsto \psi_\Gamma(\alpha(s))$ achieves its maximum at $s = 0$. Hence, its derivative at $s = 0$ vanishes, and $\langle \nabla \psi_\Gamma(u_\Gamma), \alpha'(0) \rangle = 0$. Since $\alpha'(0) \in \mathbb{T}\mathbb{S}^{r-1}$ can be arbitrary, $\nabla \psi_\Gamma(u_\Gamma)$ is parallel to u_Γ . Combining this with (2.26), the claim follows. Since $\psi_\Gamma(u_\Gamma) = \delta_\Gamma$, we have $c = \delta_\Gamma$. This completes the proof of the lemma. \square

3. GENERALIZED BMS MEASURES

Using the notation introduced in Section 2, given a pair of Γ -conformal measures on \mathcal{F} , we now define an MA -invariant locally finite Borel measure on $\Gamma \backslash G$, which we call a generalized BMS-measure. Haar measures, BR-measures, and BMS measures are all constructed in this way.

Definition 3.1 (Opposition involution). Denote by $w_0 \in K$ a representative of the unique element of the Weyl group $N_K(A)/M$ such that $\text{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $i : \mathfrak{a} \rightarrow \mathfrak{a}$ is defined by

$$i(u) = -\text{Ad}_{w_0}(u).$$

Note that for all $g \in G$, we have

$$\begin{aligned} \lambda(g^{-1}) &= i(\lambda(g)), & \mu(g^{-1}) &= i(\mu(g)), & \text{and} \\ i(\mathfrak{a}^+) &= \mathfrak{a}^+ & \text{and} & & \psi_\Gamma \circ i &= \psi_\Gamma. \end{aligned}$$

In particular, i preserves $\text{int } \mathcal{L}_\Gamma$.

Note that for all rank one groups, i is the identity map.

Example. When $G = \text{PSL}_d(\mathbb{R})$, with the Riemannian metric given by the inner product $\langle X, Y \rangle = \text{tr}(XY^t)$, we have

$$\begin{aligned} \mathfrak{a} &= \{\text{diag}(t_1, \dots, t_d) : t_1 + \dots + t_d = 0\} \\ \mathfrak{a}^+ &= \{\text{diag}(t_1, \dots, t_d) \in \mathfrak{a} : t_1 \geq \dots \geq t_d\} \end{aligned} \tag{3.2}$$

and $\langle \cdot, \cdot \rangle : \mathfrak{a} \rightarrow \mathbb{R}$ is given by $\langle X, Y \rangle = \text{tr}(XY)$. The opposition involution is given by

$$i(\text{diag}(t_1, \dots, t_d)) = \text{diag}(-t_d, \dots, -t_1).$$

For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$

Observe that $(gm)^\pm = g^\pm$ for all $g \in G$, $m \in M$; we may thus also view the above as maps from G/M to \mathcal{F} . Hence, for the identity element $e \in G$, $e^+ = [P]$, $e^- = [w_0P]$ and $g^\pm = g(e^\pm)$ for any $g \in G$. Let $\mathcal{F}^{(2)}$ denote the unique open G -orbit in $\mathcal{F} \times \mathcal{F}$:

$$\mathcal{F}^{(2)} = G.(e^+, e^-) = \{(gP, gw_0P) \in G/P \times G/P : g \in G\}.$$

Example. If $G = \text{PGL}_d(\mathbb{R})$, \mathcal{F} may be identified with the space of complete flags $\{\{V_1 \subset \dots \subset V_{d-1}\} : \dim V_i = i\}$ in \mathbb{R}^d ; $\mathcal{F}^{(2)}$ is then identified with the set of pairs of flags $(\{V_1 \subset \dots \subset V_{d-1}\}, \{W_1 \subset \dots \subset W_{d-1}\})$ in general position, i.e., $V_i \oplus W_{d-i} = \mathbb{R}^d$ for all $1 \leq i \leq d-1$.

Definition 3.3 (Hopf parameterization). The homeomorphism $G/M \rightarrow \mathcal{F}^{(2)} \times \mathfrak{a}$ given by $gM \mapsto (g^+, g^-, b = \beta_{g^-}(e, g))$ is called the Hopf parameterization of G/M .

Example. Using the linear fractional transformation action of $G = \text{PSL}_2(\mathbb{R})$ on $\mathbb{H}^2 \cup \hat{\mathbb{R}}$, we have $P^- = \text{Stab}(\infty)$ and $P^+ = \text{Stab}(0)$, where $P = P^-$ and P^+ are the upper and lower triangular subgroups of G respectively. Hence

$$g^+ = gP = g(\infty), \quad g^- = gw_0P = g(0) \in \partial\mathbb{H}^2 = G/P$$

where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We will make use of the following identities, which are all straightforward:

$$\begin{aligned} \beta_{g^-}(e, g) &= -\sigma(g^{-1}, kw_0P) = -i(\log a) \text{ if } g = kan \in KAN^+; \\ \beta_{g^+}(e, g) &= -\sigma(g^{-1}, kP) = \log a \text{ if } g = kah \in KAN^-. \end{aligned} \quad (3.4)$$

In particular, for $a \in A$,

$$\beta_{e^+}(e, a) + i(\beta_{e^-}(e, a)) = 0.$$

The generalized BMS-measure: m_{ν_1, ν_2} . Fix a pair of linear forms $\psi_1, \psi_2 \in \mathfrak{a}^*$. Let $\nu_1 = \nu_{\psi_1}$ and $\nu_2 = \nu_{\psi_2}$ be respectively (Γ, ψ_1) and (Γ, ψ_2) conformal measures on \mathcal{F} . Using the Hopf parametrization, define the following locally finite Borel measure \tilde{m}_{ν_1, ν_2} on G/M as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{m}_{\nu_1, \nu_2}(g) = e^{\psi_1(\beta_{g^+}(e, g)) + \psi_2(\beta_{g^-}(e, g))} d\nu_1(g^+) d\nu_2(g^-) db, \quad (3.5)$$

where $db = d\ell(b)$ is the Lebesgue measure on \mathfrak{a} .

Lemma 3.6. *The measure \tilde{m}_{ν_1, ν_2} is left Γ -invariant and right A -quasi-invariant: for all $a \in A$,*

$$a_*\tilde{m}_{\nu_1, \nu_2} = e^{(-\psi_1 + \psi_2 \circ i)(\log a)} \tilde{m}_{\nu_1, \nu_2}.$$

Proof. Let $\gamma \in \Gamma$ and $g \in G$ be arbitrary. Note

$$\beta_{\gamma g^\pm}(e, \gamma g) = \beta_{\gamma g^\pm}(e, \gamma) + \beta_{\gamma g^\pm}(\gamma, \gamma g) = \beta_{\gamma g^\pm}(e, \gamma) + \beta_{g^\pm}(e, g).$$

Recall the conformality of the measures ν_1 and ν_2 :

$$d\nu_1(\gamma g^+) = e^{\psi_1(\beta_{g^+}(e, \gamma^{-1}))} d\nu_1(g^+) \text{ and } d\nu_2(\gamma g^-) = e^{\psi_2(\beta_{g^-}(e, \gamma^{-1}))} d\nu_2(g^-).$$

Combining these, we have

$$\begin{aligned} d\tilde{m}_{\nu_1, \nu_2}(\gamma g) &= e^{\psi_1(\beta_{\gamma g^+}(e, \gamma g)) + \psi_2(\beta_{\gamma g^-}(e, \gamma g))} d\nu_1(\gamma g^+) d\nu_2(\gamma g^-) dl(b + \beta_{\gamma g^-}(e, \gamma)) \\ &= e^{\psi_1(\beta_{g^+}(e, g)) + \psi_2(\beta_{g^-}(e, g))} d\nu_1(g^+) d\nu_2(g^-) dl(b) = d\tilde{m}_{\nu_1, \nu_2}(g). \end{aligned}$$

Let $a \in A$. By the identities (2.4) and (3.4),

$$\begin{aligned} \beta_{g^\pm}(e, ga) &= \beta_{g^\pm}(e, g) + \beta_{g^\pm}(g, ga) = \beta_{g^\pm}(e, g) + \beta_{e^\pm}(e, a); \\ ga^\pm &= g^\pm, \text{ and } \beta_{e^+}(e, a) = -i\beta_{e^-}(e, a) = \log a. \end{aligned}$$

Combining with definition (3.5), we have

$$\begin{aligned} d\tilde{m}_{\nu_1, \nu_2}(ga) &= e^{\psi_1(\beta_{g^+}(e, ga)) + \psi_2(\beta_{g^-}(e, ga))} d\nu_1(g^+) d\nu_2(g^-) dl(b + \beta_{e^-}(e, a)) \\ &= e^{\psi_1(\beta_{g^+}(e, g) + \beta_{e^+}(e, a)) + \psi_2(\beta_{g^-}(e, g) + \beta_{e^-}(e, a))} d\nu_1(g^+) d\nu_2(g^-) dl(b) \\ &= e^{(\psi_1 - \psi_2 \circ i)(\log a)} d\tilde{m}_{\nu_1, \nu_2}(g). \end{aligned}$$

This proves the claim. \square

The measure \tilde{m}_{ν_1, ν_2} gives rise to a left Γ -invariant and right M -invariant measure on G , by integrating along the fibers of $G \rightarrow G/M$ with respect to the Haar measure on M . By abuse of notation, we will also denote this measure by \tilde{m}_{ν_1, ν_2} . We denote by m_{ν_1, ν_2} the measure on $\Gamma \backslash G$ induced by \tilde{m}_{ν_1, ν_2} , and call it the generalized BMS-measure associated to the pair (ν_1, ν_2) .

BMS-measure: $m_{\nu_\psi, \nu_{\psi \circ i}}^{\text{BMS}}$. Let $\psi \in \mathfrak{a}^*$ and let ν_ψ and $\nu_{\psi \circ i}$ be respectively (Γ, ψ) and $(\Gamma, \psi \circ i)$ -PS measures. We set

$$m_{\nu_\psi, \nu_{\psi \circ i}}^{\text{BMS}} := m_{\nu_\psi, \nu_{\psi \circ i}} \quad (3.7)$$

and call it the Bowen-Margulis-Sullivan measure associated to $(\nu_\psi, \nu_{\psi \circ i})$. By Lemma 3.6, $m_{\nu_\psi, \nu_{\psi \circ i}}^{\text{BMS}}$ is an A -invariant measure, whose support is given by

$$\text{supp}(m_{\nu_\psi, \nu_{\psi \circ i}}^{\text{BMS}}) = \{x \in \Gamma \backslash G : x^\pm \in \Lambda_\Gamma\};$$

since Λ_Γ is Γ -invariant, the condition $x^\pm \in \Lambda_\Gamma$ is a well-defined condition.

BR-measures: $m_{\nu_\psi}^{\text{BR}}$. We set

$$m_{\nu_\psi}^{\text{BR}} = m_{m_o, \nu_\psi}; \quad (3.8)$$

and call it the Burger-Roblin measure associated to ν_ψ . Note that the support of $m_{\nu_\psi}^{\text{BR}}$ is given by

$$\text{supp}(m_{\nu_\psi}^{\text{BR}}) = \{x \in \Gamma \backslash G : x^- \in \Lambda_\Gamma\}.$$

Lemma 3.9. *The Burger-Roblin measure $m_{\nu_\psi}^{\text{BR}}$ is right N^+ -invariant.*

Proof. Let $g \in G$ and $n \in N^+$. By the identities (2.4) and (3.4), we have $gn^- = g^-$, $\beta_{n^-}(e, n) = 0$ and

$$\beta_{gn^-}(e, gn) = \beta_{gn^-}(e, g) + \beta_{gn^-}(g, gn) = \beta_{gn^-}(e, g) + \beta_{n^-}(e, n) = \beta_{g^-}(e, g).$$

On the other hand, by the conformality (2.6),

$$\begin{aligned} dm_o(gn^+) &= e^{2\rho(\beta_{gn^+}(gn^{-1}, e))} dm_o(g^+) \\ &= e^{2\rho(\beta_{gn^+}(gn^{-1}, gn) + \beta_{gn^+}(gn, e))} dm_o(g^+) \\ &= e^{2\rho(\beta_{g^+}(e, g) - \beta_{gn^+}(e, gn))} dm_o(g^+). \end{aligned}$$

Combining these, we have

$$\begin{aligned} d\tilde{m}_{m_o, \nu_\psi}(gn) &= e^{2\rho(\beta_{gn^+}(e, gn)) + \psi(\beta_{g^-}(e, gn))} dm_o(gn^+) d\nu_\psi(g^-) d\ell(b) \\ &= e^{2\rho(\beta_{g^+}(e, g)) + \psi(\beta_{g^-}(e, g))} dm_o(g^+) d\nu_\psi(g^-) d\ell(b) \\ &= d\tilde{m}_{m_o, \nu_\psi}(g). \end{aligned}$$

□

Similarly, but with a different parametrization $g = (g^+, g^-, b = \beta_{g^+}(e, g))$, we define the following N^- -invariant locally finite measure:

$$d\tilde{m}_{\nu_\psi}^{\text{BR}*}(g) = e^{\psi(\beta_{g^+}(e, g)) + 2\rho(\beta_{g^-}(e, g))} d\nu_\psi(g^+) dm_o(g^-) db. \quad (3.10)$$

Haar measure m^{Haar} . Recall that the K -invariant probability measure m_o is a conformal density for the linear form 2ρ . We denote by $dx = dm^{\text{Haar}}(x)$ the generalized BMS measure associated to the pair (m_o, m_o) :

$$dx = dm^{\text{Haar}} := dm_{m_o, m_o}. \quad (3.11)$$

Since m_o is a $(G, 2\rho)$ -conformal measure, dm^{Haar} is G -invariant; the proofs of Lemma 3.6 and Lemma 3.9 show that m_o is invariant under AM and N^\pm . As these subgroups generate G , the G -invariance follows).

4. DISINTEGRATION OF THE BMS AND BR-MEASURES ALONG N -ORBITS

In this section, we fix a linear form $\psi \in D_\Gamma^*$, a (Γ, ψ) -PS measure ν_ψ and a $(\Gamma, \psi \circ i)$ -PS measure $\nu_{\psi \circ i}$ on \mathcal{F} . To simplify the notations, we write

$$\nu := \nu_\psi, \nu_i := \nu_{\psi \circ i}, \tilde{m}^{\text{BMS}} := \tilde{m}_{\nu_\psi, \nu_{\psi \circ i}}^{\text{BMS}}, \tilde{m}^{\text{BR}} := \tilde{m}_{\nu_\psi \circ i}^{\text{BR}}, \tilde{m}^{\text{BR}*} := \tilde{m}_{\nu_\psi}^{\text{BR}*}.$$

4.1. PS-measures on gN^\pm . We start by defining measures on N^\pm . Firstly, for $g \in G$, define $\mu_{gN^+}^{\text{PS}} := \mu_{gN^+, \nu}^{\text{PS}}$ and $\mu_{gN^-}^{\text{PS}} := \mu_{gN^-, \nu_i}^{\text{PS}}$ on N^\pm by the formulas: for $n \in N^+$ and $h \in N^-$,

$$d\mu_{gN^+}^{\text{PS}}(n) := e^{\psi(\beta_{(gn)^+}(e, gn))} d\nu((gn)^+) \quad (4.1)$$

and

$$d\mu_{gN^-}^{\text{PS}}(h) := e^{\psi \circ i(\beta_{(gh)^-}(e, gh))} d\nu_i((gh)^-).$$

These are left Γ -invariant; for any $\gamma \in \Gamma$ and $g \in G$,

$$\mu_{\gamma g N^\pm}^{\text{PS}} = \mu_{g N^\pm}^{\text{PS}}.$$

When xN^\pm is closed in $\Gamma \backslash G$ for $x = [g] \in \Gamma \backslash G$, $\mu_{g N^\pm}^{\text{PS}}$ induces a locally finite Borel measure on $\text{Stab}_{N^\pm}(x) \backslash N^\pm \simeq xN^\pm$ which we will denote by $d\mu_{xN^\pm}^{\text{PS}}$.

Recalling that A normalizes N^\pm , we will use the following lemma:

Lemma 4.2. *For any $g \in G$, $a \in A$, $n_0, n \in N^+$, we have*

$$d\mu_{gN^+}^{\text{PS}}(an_0na^{-1}) = e^{-\psi(\log a)} d\mu_{gan_0N^+}^{\text{PS}}(n).$$

Proof. By (2.4) and (3.4), we have $(gan_0na^{-1})^+ = gan_0n^+$, and

$$\begin{aligned} \beta_{gan_0n^+}(e, gan_0na^{-1}) &= \beta_{gan_0n^+}(e, gan_0n) + \beta_{gan_0n^+}(gan_0n, gan_0na^{-1}) \\ &= \beta_{gan_0n^+}(e, gan_0n) + \beta_{e^+}(e, a^{-1}). \end{aligned}$$

Also note that $\beta_{e^+}(e, a^{-1}) = -\log a$. Consequently,

$$\begin{aligned} d\mu_{gN^+}^{\text{PS}}(an_0na^{-1}) &= e^{\psi(\beta_{gan_0n^+}(e, gan_0na^{-1}))} d\nu(gan_0n^+) \\ &= e^{-\psi(\log a)} e^{\psi(\beta_{gan_0n^+}(e, gan_0n))} d\nu(gan_0n^+) \\ &= e^{-\psi(\log a)} d\mu_{gan_0N^+}^{\text{PS}}(n). \end{aligned}$$

□

The measures $\mu_{gN^\pm}^{\text{PS}}$ allow us to decompose the BMS-measure as follows: The product map $N^+ \times P^- \rightarrow G$ is a diffeomorphism onto a Zariski open neighborhood of e .

4.2. Product structure of BMS measures. Given $g \in G$, the BMS measure \tilde{m}^{BMS} can be disintegrated in gN^+P^- as follows.

Lemma 4.3. *For $g \in G$, $f \in C_c(gN^+P^-)$, and $nham \in N^+N^-AM$,*

$$\tilde{m}^{\text{BMS}}(f) = \int_{N^+} \left(\int_{N^-AM} f(gnham) dm da d\mu_{gnN^-}^{\text{PS}}(h) \right) d\mu_{gN^+}^{\text{PS}}(n).$$

Proof. By the identities (2.4) and (3.4), we have $gnha^- = gnh^-$, $gnha^+ = gn^+$, and

$$\begin{aligned} \beta_{gnh^\pm}(e, gnha) &= \beta_{gnh^\pm}(e, gnh) + \beta_{gnh^\pm}(gnh, gnha) \\ &= \beta_{gnh^\pm}(e, gnh) + \beta_{e^\pm}(e, a). \end{aligned}$$

Note that $\beta_{e^-}(e, a) = -i \log a$ and $\beta_{e^+}(e, a) = \log a$. Hence

$$\begin{aligned} &d\tilde{m}^{\text{BMS}}(gnha) \\ &= e^{\psi(i(\beta_{gnh^-}(e, gnha) + \beta_{gnh^+}(e, gnha)))} d\nu_i(gnh^-) d\nu(gn^+) d\ell(\beta_{gnh^-}(e, gnha)) \\ &= e^{\psi(i\beta_{gnh^-}(e, gnh) - \log a + \beta_{gnh^+}(e, gnh) + \log a)} \\ &\quad \times d\nu_i(gnh^-) d\nu(gn^+) d\ell(\beta_{gnh^-}(e, gnh) + \log a) \\ &= da d\mu_{gnN^-}^{\text{PS}}(h) d\mu_{gN^+}^{\text{PS}}(n). \end{aligned}$$

Hence for $nham \in N^+N^-AM$,

$$d\tilde{m}^{\text{BMS}}(gnham) = dm d\tilde{m}^{\text{BMS}}(gnha) = dm da d\mu_{gnN^-}^{\text{PS}}(h) d\mu_{gN^+}^{\text{PS}}(n),$$

proving the claim. \square

In a similar manner, one can decompose the BMS measure according to gP^-N^+ :

Lemma 4.4. *For $g \in G$, $f \in C_c(gP^-N^+)$, and $hamn \in N^-AMN^+$,*

$$\tilde{m}^{\text{BMS}}(f) = \int_{N^-AM} \left(\int_{N^+} f(ghamn) d\mu_{ghamN^+}^{\text{PS}}(n) \right) e^{-\psi(\log a)} dm da d\mu_{gN^-}^{\text{PS}}(h).$$

Proof. For each $m \in M$, consider the change of variable $n_0 = mnm^{-1}$. Then for $hamn \in N^-AMN^+$, we have

$$d\tilde{m}^{\text{BMS}}(ghamn) = d\tilde{m}^{\text{BMS}}(ghan_0m) = dm d\tilde{m}^{\text{BMS}}(ghan_0).$$

By the identity (3.4), we have $\beta_{gh^-}(e, ghan_0) = \beta_{gh^-}(e, gh) - i \log a$ and

$$i(\beta_{gh^-}(e, ghan_0)) + \beta_{ghan_0^+}(e, ghan_0) = i(\beta_{gh^-}(e, gh) - i \log a) + \beta_{ghan_0^+}(e, ghan_0).$$

It follows that

$$\begin{aligned} d\tilde{m}^{\text{BMS}}(ghan_0) &= e^{\psi(\beta_{ghan_0^+}(e, ghan_0))} d\nu(ghan_0^+) e^{-\psi(\log a)} da e^{\psi \circ i(\beta_{gh^-}(e, gh))} d\nu_1(gh^-) \\ &= d\mu_{ghamN^+}^{\text{PS}}(n) e^{-\psi(\log a)} da d\mu_{gN^-}^{\text{PS}}(h). \end{aligned}$$

Hence

$$d\tilde{m}^{\text{BMS}}(ghamn) = d\mu_{ghamN^+}^{\text{PS}}(n) e^{-\psi(\log a)} dm da d\mu_{gN^-}^{\text{PS}}(h),$$

finishing the proof. \square

Define, for $ham \in N^-AM$,

$$d\mu_{gP^-}^{\text{PS}}(ham) = e^{-\psi(\log a)} dm da d\mu_{gN^-}^{\text{PS}}(h).$$

This also allows us to succinctly rewrite the decomposition in Lemma 4.4 as follows: for any $f \in C_c(gP^-N^+)$,

$$\tilde{m}^{\text{BMS}}(f) = \int_{P^-} \int_{N^+} f(gpn) d\mu_{gpN^+}^{\text{PS}}(n) d\mu_{gP^-}^{\text{PS}}(p). \quad (4.5)$$

Lebesgue measures on gN^\pm . For $g \in G$, we note that the Haar measure on gN^\pm can be given as follows: for $n \in N^\pm$,

$$d\mu_{gN^-}^{\text{Leb}}(n) = e^{2\rho(\beta_{(gn)^-}(e, gn))} dm_o((gn)^-), \quad (4.6)$$

and

$$d\mu_{gN^+}^{\text{Leb}}(n) = e^{2\rho(\beta_{(gn)^+}(e, gn))} dm_o((gn)^+). \quad (4.7)$$

Using (2.6), it can be checked that these are N^- and N^+ invariant measures respectively. Moreover, $d\mu_{gN^\pm}^{\text{Leb}}$ does not depend on $g \in G$, so we simply write dn .

4.3. Decomposition of m^{BR} . Similarly to Lemma 4.4, the BR and BR_* measures can be decomposed in terms of the gP^-N^+ decomposition of G :

For all $f \in C_c(gP^-N^+)$,

$$\tilde{m}^{\text{BR}}(f) = \int_{P^-} \int_{N^+} f(gpn) \, dn \, d\mu_{gP^-}^{\text{PS}}(p); \quad (4.8)$$

$$\tilde{m}^{\text{BR}_*}(f) = \int_{P^-} \int_{N^+} f(ghamn) \, d\mu_{ghamN^+}^{\text{PS}}(n) e^{-2\rho(\log a)} \, dm \, da \, dh.$$

We also have the following description of the BR-measures $\tilde{m}_{\nu_\psi}^{\text{BR}}$ on G/M :

Lemma 4.9. *For $g = k \exp(b)n \in KAN^+$ and $[g] = gM$, we have*

$$d\tilde{m}_{\nu_\psi}^{\text{BR}}([g]) = e^{-(\psi \circ i)(b)} \, dn \, d\ell(b) \, d\tilde{\nu}_\psi(k)$$

where $\tilde{\nu}_\psi$ is given by

$$d\tilde{\nu}_\psi(k) := d\nu_\psi(k^-) \, dm. \quad (4.10)$$

Proof. For $g = k \exp(b)n \in KAN^+$, we have $\beta_{g^-}(e, g) = -i(b)$. Since m_o is a $(G, 2\rho)$ -conformal measure, we have

$$\begin{aligned} d\tilde{m}_{\nu_\psi}^{\text{BR}}([g]) &= e^{2\rho(\beta_{k \exp(b)n^+}(e, k \exp(b)n))} e^{-(\psi \circ i)(b)} \, dm_o(k \exp(b)n^+) \, db \, d\nu_\psi(k^-) \, dm \\ &= e^{-(\psi \circ i)(b)} \, dn \, db \, d\tilde{\nu}_\psi(k). \end{aligned}$$

□

5. BMS-MIXING AND TRANSLATES OF PS-MEASURES

In this section, we fix

- (1) an element $u \in \mathcal{L}_\Gamma \cap \text{int } \mathfrak{a}^+$,
- (2) a linear form $\psi \in D_\Gamma^*$ tangent to ψ_Γ at u ,
- (3) a (Γ, ψ) -PS measure $\nu = \nu_\psi$ on \mathcal{F} , and
- (4) a $(\Gamma, \psi \circ i)$ -PS measure $\nu_i = \nu_{\psi \circ i}$ on \mathcal{F} .

As before, we set

$$m^{\text{BMS}} = m_{\nu, \nu_i}^{\text{BMS}}, \quad \mu_{gN^+}^{\text{PS}} = \mu_{gN^+, \nu}^{\text{PS}}.$$

For all $t \geq 0$ and $v \in \ker \psi$, define

$$a(t, v) := \exp(tu + \sqrt{t}v) \in A.$$

Definition 5.1. We say that m^{BMS} satisfies the *local mixing property* if there exist functions $\Psi : (0, \infty) \rightarrow (0, \infty)$ and $J : \ker \psi \rightarrow (0, \infty)$ such that

- (1) For all $v \in \ker \psi$ and $f_1, f_2 \in C_c(\Gamma \backslash G)^M$,

$$\lim_{t \rightarrow \infty} \Psi(t) \int_{\Gamma \backslash G} f_1(x a(t, v)) f_2(x) \, dm^{\text{BMS}}(x) = J(v) m^{\text{BMS}}(f_1) m^{\text{BMS}}(f_2); \quad (5.2)$$

- (2) There exists $C = C(f_1, f_2) > 0$ such that for all $(t, v) \in (0, \infty) \times \ker \psi$ with $a(t, v) \in A^+$,

$$\left| \Psi(t) \int_{\Gamma \backslash G} f_1(x a(t, v)) f_2(x) dm^{\text{BMS}}(x) \right| < C.$$

The main goal of this section is to establish the following:

Proposition 5.3. *Suppose that m^{BMS} satisfies the local mixing property for the pair (Ψ, J) . Then for any $x = [g] \in \Gamma \backslash G$, $v \in \ker \psi$, $f \in C_c(\Gamma \backslash G)^M$, and $\phi \in C_c(N^+)$,*

$$\lim_{t \rightarrow \infty} \Psi(t) \int_{N^+} f(xn a(t, v)) \phi(n) d\mu_{gN^+}^{\text{PS}}(n) = J(v) m^{\text{BMS}}(f) \mu_{gN^+}^{\text{PS}}(\phi), \quad (5.4)$$

and there exists $C' = C'(f, \phi) > 0$ such that

$$\left| \Psi(t) \int_{N^+} f(xn a(t, v)) \phi(n) d\mu_{gN^+}^{\text{PS}}(n) \right| < C'$$

for all $(t, v) \in (0, \infty) \times \ker \psi$ with $a(t, v) \in A^+$.

For $\varepsilon > 0$, let G_ε denote the open ball of radius ε around e in G . For a subgroup $S < G$, we define $S_\varepsilon := S \cap G_\varepsilon$. The choices $S = P^\pm, N^\pm, A$ are the only subgroups we will require. We will carry out a thickening argument using PS measures as in e.g. [37]; the following lemma is needed:

Lemma 5.5. *For any $g \in G$,*

$$\nu(gN^+(e^+)) > 0 \quad \text{and} \quad \nu_1(gN^-(e^-)) > 0.$$

Proof. The Zariski density of Λ_Γ in \mathcal{F} is proved in [2]. This also follows from Lemma 2.11. Since each $gN^\pm(e^\pm)$ is a Zariski open subset of \mathcal{F} and the support of ν is equal to Λ_Γ , the conclusion follows. \square

We will also need the following continuity property of the PS-measures [36, Proposition 2.15]:

Lemma 5.6. *For any fixed $\rho \in C_c(N^\pm)$ and $g \in G$, the map $N^\mp \rightarrow \mathbb{R}$ given by $h \mapsto \mu_{ghN^\pm}^{\text{PS}}(\rho)$ is continuous.*

Proof. We will only prove the case when $\rho \in C_c(N^+)$; the other case can be proved similarly. Define a function $\tilde{\rho}_g : N^- \times G/P \rightarrow \mathbb{R}$ by

$$\tilde{\rho}_g(h, \xi) := \begin{cases} \rho(n) e^{\psi(\beta_{ghn^+}(e, ghn))} & \text{if } \xi = ghn^+ \text{ for some } n \in N^+, \\ 0 & \text{otherwise.} \end{cases}$$

Since $N^+ \cap P = \{e\}$, $\tilde{\rho}_g$ is well-defined. By continuity of the Busemann function, $\tilde{\rho}_g$ is continuous in $h \in N^-$. This gives

$$\mu_{ghN^+}^{\text{PS}}(\rho) = \int_{N^+} \rho(n) e^{\psi(\beta_{ghn^+}(e, ghn))} d\nu(ghn^+) = \int_{G/P} \tilde{\rho}_g(h, \xi) d\nu(\xi),$$

hence $|\mu_{gh_1N^+}^{\text{PS}}(\rho) - \mu_{gh_2N^+}^{\text{PS}}(\rho)| \leq \max_{\xi \in G/P} |\tilde{\rho}_g(h_1, \xi) - \tilde{\rho}_g(h_2, \xi)|$. The continuity of $\tilde{\rho}_g$ then implies the claimed statement. \square

A function on N^\pm is said to be *radial* if it is invariant under conjugation by elements of M i.e. $f(mnm^{-1}) = f(n)$ for all $m \in M$, and $n \in N^\pm$.

Corollary 5.7. *Given $\varepsilon > 0$ and $g \in G$, there exist $R > 1$ and a non-negative radial function $\rho_{g,\varepsilon} \in C_c(N_R^-)$ such that for all $n \in N_\varepsilon^+$,*

$$\mu_{gnN^-}^{\text{PS}}(\rho_{g,\varepsilon}) > 0.$$

Proof. For each $j \in \mathbb{N}$, let $\phi_j \in C_c(N_{j+1}^-)$ be a nonnegative radial function such that $\phi_j|_{N_j^-} = 1$. By Lemma 5.5, for each $n \in N^+$, there exists some $j_n \in \mathbb{N}$ such that $\mu_{gnN^-}^{\text{PS}}(N_{j_n}^-) > 0$. By Lemma 5.6, for each $n \in N^+$, there exists $r_n > 0$ such that

$$\mu_{gn_0N^-}^{\text{PS}}(N_{j_n}^-) > 0 \quad \text{for all } n_0 \in \mathcal{B}(n) := \{n_0 \in N^+ : \text{dist}(n, n_0) < r_n\}.$$

Using the relative compactness of N_ε^+ , we choose $n_1, \dots, n_k \in N^+$ such that $N_\varepsilon^+ \subset \bigcup_{i=1}^k \mathcal{B}(n_i)$. Choosing $R := \max(j_{n_1}, \dots, j_{n_k}) + 1$ and $\rho_{g,\varepsilon} := \phi_{R-1}$ completes the proof. \square

Proof of Proposition 5.3. Fixing $v \in \ker(\psi)$, for simplicity, we denote $a_t = a(t, v)$. Let $x = [g]$, and $\varepsilon_0 > 0$ be such that $\phi \in C_c(N_{\varepsilon_0}^+)$. By Corollary 5.7, there exist $R > 0$ and a nonnegative $\rho_{g,\varepsilon_0} \in C_c(N_R^-)$ such that

$$\mu_{gnN^-}^{\text{PS}}(\rho_{g,\varepsilon_0}) > 0 \quad \text{for all } n \in N_{\varepsilon_0}^+.$$

Given arbitrary $\varepsilon > 0$, choose a non-negative function $q_\varepsilon \in C_c(A_\varepsilon)$ satisfying $\int_A q_\varepsilon(a) da = 1$. Then

$$\begin{aligned} & \int_{N^+} f(xna_t) \phi(n) d\mu_{gN^+}^{\text{PS}}(n) \\ &= \int_{N^+} f(xna_t) \phi(n) \left(\frac{1}{\mu_{gnN^-}^{\text{PS}}(\rho_{g,\varepsilon_0})} \int_{N^-A} \rho_{g,\varepsilon_0}(h) q_\varepsilon(a) da d\mu_{gnN^-}^{\text{PS}}(h) \right) d\mu_{gN^+}^{\text{PS}}(n) \\ &= \int_{N^+} \left(\int_{N^-A} f(xna_t) \frac{\phi(n) \rho_{g,\varepsilon_0}(h) q_\varepsilon(a)}{\mu_{gnN^-}^{\text{PS}}(\rho_{g,\varepsilon_0})} da d\mu_{gnN^-}^{\text{PS}}(h) \right) d\mu_{gN^+}^{\text{PS}}(n). \end{aligned} \tag{5.8}$$

We now define a right M -invariant function $\tilde{\Phi}_\varepsilon \in C_c(gN_{\varepsilon_0}^+N_R^-A_\varepsilon M) \subset C_c(G)$ by

$$\tilde{\Phi}_\varepsilon(g_0) := \begin{cases} \frac{\phi(n) \rho_{g,\varepsilon_0}(h) q_\varepsilon(a)}{\mu_{gnN^-}^{\text{PS}}(\rho_{g,\varepsilon_0})} & \text{if } g_0 = gnham, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the continuity of $\tilde{\Phi}_\varepsilon$ is a consequence of Lemma 5.6. Also observe that $\tilde{\Phi}_\varepsilon$ depends on our choice of representative for $x = [g]$.

We now assume without loss of generality $f \geq 0$ and define, for all $\varepsilon > 0$, functions f_ε^\pm as follows: for all $z \in \Gamma \backslash G$,

$$f_\varepsilon^+(z) := \sup_{b \in N_\varepsilon^+ N_\varepsilon^- A_\varepsilon} f(zb) \quad \text{and} \quad f_\varepsilon^-(z) := \inf_{b \in N_\varepsilon^+ N_\varepsilon^- A_\varepsilon} f(zb).$$

Since $u \in \text{int } \mathfrak{a}^+$, for every $\varepsilon > 0$, there exists $t_0(R, \varepsilon) > 0$ such that

$$a_t^{-1} N_R^- a_t \subset N_\varepsilon^- \quad \text{for all } t \geq t_0(R, \varepsilon).$$

Then, as $\text{supp}(\tilde{\Phi}_\varepsilon) \subset gN_{\varepsilon_0}^+ N_R^- A_\varepsilon M$, we have

$$f(xna_t) \tilde{\Phi}_\varepsilon(gnha) \leq f_{2\varepsilon}^+(xnhaa_t) \tilde{\Phi}_\varepsilon(gnha) \quad (5.9)$$

for all $nha \in N^+ N^- A$ and $t \geq t_0(R, \varepsilon)$. Similarly,

$$f_{2\varepsilon}^-(xnhaa_t) \tilde{\Phi}_\varepsilon(gnha) \leq f(xna_t) \tilde{\Phi}_\varepsilon(gnha).$$

We now use $f_{2\varepsilon}^+$ to give an upper bound on the limit we are interested in; $f_{2\varepsilon}^-$ is used in an analogous way to provide a lower bound. Entering the definition of $\tilde{\Phi}_\varepsilon$ and the above inequality (5.9) into (5.8) gives

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \Psi(t) \int_{N^+} f(xna_t) \phi(n) d\mu_{gN^+}^{\text{PS}}(n) \\ & \leq \limsup_{t \rightarrow \infty} \Psi(t) \int_{N^+} \left(\int_{N^- AM} f_{2\varepsilon}^+(xnhaa_t) \tilde{\Phi}_\varepsilon(gnha) dm da d\mu_{gnN^-}^{\text{PS}}(h) \right) d\mu_{gN^+}^{\text{PS}}(n) \\ & = \limsup_{t \rightarrow \infty} \Psi(t) \int_G f_{2\varepsilon}^+([g_0]a_t) \tilde{\Phi}_\varepsilon(g_0) d\tilde{m}^{\text{BMS}}(g_0) \\ & = \limsup_{t \rightarrow \infty} \Psi(t) \int_{\Gamma \backslash G} f_{2\varepsilon}^+([g_0]a_t) \Phi_\varepsilon([g_0]) dm^{\text{BMS}}([g_0]), \end{aligned}$$

where

$$\Phi_\varepsilon([g_0]) := \sum_{\gamma \in \Gamma} \Phi_\varepsilon(\gamma g_0),$$

and Lemma 4.3 was used in the second to last line of the above calculation. By the standing assumption (5.2), we then have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \Psi(t) \int_{N^-} f(xna_t) \phi(n) d\mu_{gN^-}^{\text{PS}}(n) \\ & \leq J(v) m^{\text{BMS}}(f_{2\varepsilon}^+) m^{\text{BMS}}(\tilde{\Phi}_\varepsilon) = J(v) m^{\text{BMS}}(f_{2\varepsilon}^+) \tilde{m}^{\text{BMS}}(\tilde{\Phi}_\varepsilon). \end{aligned}$$

Using Lemma 4.3 and the M -invariance of $\tilde{\Phi}_\varepsilon$, we have

$$\begin{aligned} \tilde{m}^{\text{BMS}}(\tilde{\Phi}_\varepsilon) &= \int_{N^+} \left(\int_{N^- A} \tilde{\Phi}_\varepsilon(gnha) da d\mu_{gnN^-}^{\text{PS}}(h) \right) d\mu_{gN^+}^{\text{PS}}(n) \\ &= \int_{N^+} \frac{\phi(n)}{\mu_{gnN^-}^{\text{PS}}(\rho_{g, \varepsilon_0})} \left(\int_{N^- A} \rho_{g, \varepsilon_0}(h) q_\varepsilon(a) da d\mu_{gnN^-}^{\text{PS}}(h) \right) d\mu_{gN^+}^{\text{PS}}(n) \\ &= \mu_{gN^+}^{\text{PS}}(\phi). \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, taking $\varepsilon \rightarrow 0$ gives

$$\limsup_{t \rightarrow \infty} \Psi(t) \int_{N^-} f(xna_t) \phi(n) d\mu_{gN^-}^{\text{PS}}(n) \leq J(v) m^{\text{BMS}}(f) \mu_{gN^+}^{\text{PS}}(\phi).$$

The lower bound given by replacing $f_{2\varepsilon}^+$ with $f_{2\varepsilon}^-$ in the above calculations proves the first statement.

For the second claim of the proposition, observe that if $tu + \sqrt{t}v \in \mathfrak{a}^+$, then

$$f(xna_t)\tilde{\Phi}_\varepsilon(gnha) \leq f_{R+\varepsilon}^+(xnhaa_t)\tilde{\Phi}_\varepsilon(gnha),$$

as in (5.9). Hence

$$\Psi(t) \int_{N^+} f(xna_t)\phi(n) d\mu_{gN^+}^{\text{PS}}(n) \leq C(f_{R+\varepsilon}^+, \Phi_\varepsilon).$$

Choosing $C'(f, \phi) := C(f_{R+\varepsilon}^+, \Phi_\varepsilon)$ finishes the proof. \square

6. TRANSLATES OF LEBESGUE MEASURES AND HAAR MIXING

We continue with the setup of Section 5: recall that we have fixed $u \in \mathcal{L}_\Gamma \cap \text{int } \mathfrak{a}^+$, a linear form $\psi \in D_\Gamma^*$ such that $\psi(u) = \psi_\Gamma(u)$, a (Γ, ψ) -PS measure $\nu = \nu_\psi$ and a $(\Gamma, \psi \circ \mathfrak{i})$ -PS measure $\nu_1 = \nu_{\psi \circ \mathfrak{i}}$ on \mathcal{F} . We set

$$m^{\text{BMS}} = m_{\nu, \nu_1}^{\text{BMS}}, m^{\text{BR}} = m_{\nu_1}^{\text{BR}}, m^{\text{BR}*} = m_\nu^{\text{BR}*}, \text{ and } \mu_{gN^+}^{\text{PS}} = \mu_{gN^+, \nu}^{\text{PS}}.$$

The main goal in this section is to prove a local mixing statement for the Haar measure on $\Gamma \backslash G$. In order to do this, we first convert equidistribution of translates of $\mu_{gN^+}^{\text{PS}}$ (Proposition 5.3) into equidistribution of translates of the Lebesgue measure on xN^+ :

Proposition 6.1. *Suppose that m^{BMS} satisfies the local mixing property for the pair (Ψ, J) . Then for any $x = [g] \in \Gamma \backslash G$, $v \in \ker \psi$, $f \in C_c(\Gamma \backslash G)^M$, and $\phi \in C_c(N^+)$,*

$$\lim_{t \rightarrow \infty} \Psi(t) e^{(2\rho - \psi)(tu + \sqrt{t}v)} \int_{N^+} f(xna(t, v))\phi(n) dn = J(v) m^{\text{BR}}(f) \mu_{gN^+}^{\text{PS}}(\phi),$$

and there exists $C'' = C''(f, \phi) > 0$ such that

$$\left| \Psi(t) e^{(2\rho - \psi)(tu + \sqrt{t}v)} \int_{N^+} f(xna(t, v))\phi(n) dn \right| < C''$$

for all $(t, v) \in (0, \infty) \times \ker \psi$ with $a(t, v) \in A^+$.

Proof. For $\varepsilon_0 > 0$, set $\mathcal{B}_{\varepsilon_0} = N_{\varepsilon_0}^- A_{\varepsilon_0} M N_{\varepsilon_0}^+$. Note that $MN_{\varepsilon_0}^+ = N_{\varepsilon_0}^+ M$ by the choice of the invariant metric on G . Given $x_0 \in \Gamma \backslash G$, let $\varepsilon_0(x_0)$ denote the maximum number r such that the map $G \rightarrow \Gamma \backslash G$ given by $h \mapsto x_0 h$ for $h \in G$ is injective on \mathcal{B}_r . Note that $\psi(u) = \psi_\Gamma(u) = \psi_\Gamma(\mathfrak{i}(u))$. Fixing $v \in \ker \psi$, we set, for all $t \in \mathbb{R}$,

$$a_t := a(t, v).$$

By using a partition of unity if necessary, it suffices to prove that for any $x_0 \in \Gamma \backslash G$ and $\varepsilon_0 = \varepsilon_0(x_0)$, the claims of the proposition hold for any non-negative $f \in C(x_0 \mathcal{B}_{\varepsilon_0})^M$, non-negative $\phi \in C(N_{\varepsilon_0}^+)$, and $x = [g] \in x_0 \mathcal{B}_{\varepsilon_0}$, that is,

$$\lim_{t \rightarrow \infty} \Psi(t) e^{(2\rho - \psi)(\log a_t)} \int_{N^+} f(xna_t)\phi(n) dn = J(v) m^{\text{BR}}(f) \mu_{gN^+}^{\text{PS}}(\phi), \quad (6.2)$$

and for $\log a_t \in \mathfrak{a}^+$, we have

$$\Psi(t)e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f(xna_t)\phi(n) dn < C'', \quad \text{for some } C'' = C''(f, \phi).$$

Moreover, we may assume that f is given as

$$f([g]) = \sum_{\gamma \in \Gamma} \tilde{f}(\gamma g) \quad \text{for all } g \in G,$$

for some non-negative $\tilde{f} \in C_c(g_0\mathcal{B}_{\varepsilon_0}) \subset C_c(G)$. Note that for $x = [g] \in [g_0]\mathcal{B}_{\varepsilon_0}$,

$$\int_{N^+} f([g]na_t)\phi(n) dn = \sum_{\gamma \in \Gamma} \int_{N^+} \tilde{f}(\gamma gna_t)\phi(n) dn. \quad (6.3)$$

Note that $\tilde{f}(\gamma gna_t) = 0$ unless $\gamma gna_t \in g_0\mathcal{B}_{\varepsilon_0}$. Together with the fact that $\text{supp}(\phi) \subset N_{\varepsilon_0}^+$, it follows that the summands in (6.3) are non-zero for only finitely many elements $\gamma \in \Gamma \cap g_0\mathcal{B}_{\varepsilon_0}a_t^{-1}N_{\varepsilon_0}^+g^{-1}$.

Suppose $\gamma gN_{\varepsilon_0}^+a_t \cap g_0\mathcal{B}_{\varepsilon_0} \neq \emptyset$. Then $\gamma ga_t \in g_0N_{\varepsilon_0}^-A_{\varepsilon_0}MN^+$, and there are unique elements $p_{t,\gamma} \in N_{\varepsilon_0}^-A_{\varepsilon_0}M$ and $n_{t,\gamma} \in N^+$ such that

$$\gamma ga_t = g_0p_{t,\gamma}n_{t,\gamma} \in g_0P_{\varepsilon_0}^-N^+.$$

Let $\Gamma_{t,v}$ denote the subset $\Gamma \cap g_0(N_{\varepsilon_0}^-A_{\varepsilon_0}MN^+)a_t^{-1}g^{-1}$. Note that although $\Gamma_{t,v}$ may possibly be infinite, only finitely many of the terms in the sums we consider will be non-zero. This gives

$$\begin{aligned} \int_{N^+} f([g]na_t)\phi(n) dn &= \sum_{\gamma \in \Gamma} \int_{N^+} \tilde{f}(\gamma gna_t)\phi(n) dn \\ &= \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{f}(\gamma ga_t(a_t^{-1}na_t))\phi(n) dn \\ &= e^{-2\rho(\log a_t)} \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{f}(\gamma ga_t n)\phi(a_t n a_t^{-1}) dn \\ &= e^{-2\rho(\log a_t)} \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{f}(g_0p_{t,\gamma}n_{t,\gamma}n)\phi(a_t n a_t^{-1}) dn \\ &= e^{-2\rho(\log a_t)} \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{f}(g_0p_{t,\gamma}n)\phi(a_t n_{t,\gamma}^{-1}n a_t^{-1}) dn. \end{aligned}$$

Since $\text{supp}(\tilde{f}) \subset g_0\mathcal{B}_{\varepsilon_0}$, we have

$$\begin{aligned} &\sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{f}(g_0p_{t,\gamma}n)\phi(a_t n_{t,\gamma}^{-1}n a_t^{-1}) dn \\ &\leq \sum_{\gamma \in \Gamma_{t,v}} \left(\sup_{n \in N_{\varepsilon_0}^+} \phi(a_t n_{t,\gamma}^{-1} a_t^{-1}(a_t n a_t^{-1})) \right) \cdot \int_{N^+} \tilde{f}(g_0p_{t,\gamma}n) dn, \end{aligned}$$

and

$$\begin{aligned} & \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{f}(g_0 p_{t,\gamma} n) \phi(a_t n_{t,\gamma}^{-1} n a_t^{-1}) dn \\ & \geq \sum_{\gamma \in \Gamma_{t,v}} \left(\inf_{n \in N_{\varepsilon_0}^+} \phi(a_t n_{t,\gamma}^{-1} a_t^{-1} (a_t n a_t^{-1})) \right) \cdot \int_{N^+} \tilde{f}(g_0 p_{t,\gamma} n) dn. \end{aligned}$$

Since u belongs to $\text{int } \mathcal{L}_\Gamma$, there exist $t_0(v) > 0$ and $\alpha > 0$ such that

$$a_t N_r^+ a_t^{-1} \subset N_{re^{-\alpha t}}^+ \quad \text{for all } r > 0 \text{ and } t > t_0(v).$$

Therefore, for all $n \in N_{\varepsilon_0}^+$ and $t > t_0(v)$, we have

$$\phi_{\varepsilon_0 e^{-\alpha t}}^-(a_t n_{t,\gamma}^{-1} a_t^{-1}) \leq \phi(a_t n_{t,\gamma}^{-1} a_t^{-1} (a_t n a_t^{-1})) \leq \phi_{\varepsilon_0 e^{-\alpha t}}^+(a_t n_{t,\gamma}^{-1} a_t^{-1}), \quad (6.4)$$

where

$$\phi_\varepsilon^+(n) := \sup_{b \in N_\varepsilon^+} \phi(nb), \text{ and } \phi_\varepsilon^-(n) := \inf_{b \in N_\varepsilon^+} \phi(nb) \quad \text{for all } n \in N^+, \varepsilon > 0.$$

We now have the following chain of inequalities (for $t > t_0(v)$):

$$\sum_{\gamma \in \Gamma_{t,v}} \phi_{\varepsilon_0 e^{-\alpha t}}^-(a_t n_{t,\gamma}^{-1} a_t^{-1}) \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 p_{t,\gamma} n) dn \quad (6.5)$$

$$\leq e^{2\rho(\log a_t)} \int_{N^+} f([g]na_t) \phi(n) dn$$

$$\leq \sum_{\gamma \in \Gamma_{t,v}} \phi_{\varepsilon_0 e^{-\alpha t}}^+(a_t n_{t,\gamma}^{-1} a_t^{-1}) \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 p_{t,\gamma} n) dn. \quad (6.6)$$

By Lemmas 5.5 and 5.6, there exist $R > 0$ and a radial function $\rho \in C_c(N_R^+)$ such that $\rho(n) \geq 0$ for all $n \in N^+$, and $\mu_{g_0 p N^+}^{\text{PS}}(\rho) > 0$ for all $p \in N_{\varepsilon_0}^- A_{\varepsilon_0} M$. Define $\tilde{F} \in C_c(g_0 N_{\varepsilon_0}^- A_{\varepsilon_0} M N_R^+)$ by

$$\tilde{F}(g) := \begin{cases} \frac{\rho(n)}{\mu_{g_0 p N^+}^{\text{PS}}(\rho)} \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 p v) dv & \text{if } g = g_0 p n \in g_0 N_{\varepsilon_0}^- A_{\varepsilon_0} M N_R^+, \\ 0 & \text{otherwise.} \end{cases}$$

Since ρ is radial, \tilde{F} is right M -invariant. The key property of \tilde{F} we will use is the following: for all $p \in P_{\varepsilon_0}^-$,

$$\int_{N^+} \tilde{F}(g_0 p n) d\mu_{g_0 p N^+}^{\text{PS}}(n) = \int_{N_R^+} \tilde{F}(g_0 p n) d\mu_{g_0 p N^+}^{\text{PS}}(n) = \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 p n) dn.$$

Returning to (6.5), we now give an upper bound for $\int_N f([g]na_t)\phi(n) dn$; the lower bound can be dealt with in a similar fashion. We observe:

$$\begin{aligned}
& e^{2\rho(\log a_t)} \int_{N^+} f([g]na_t)\phi(n) dn \\
& \leq \sum_{\gamma \in \Gamma_{t,v}} \phi_{\varepsilon_0}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} a_t^{-1}) \int_{N_{\varepsilon_0}^+} \tilde{f}(g_0 p_{t,\gamma} n) dn \\
& = \sum_{\gamma \in \Gamma_{t,v}} \phi_{\varepsilon_0}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} a_t^{-1}) \int_{N_R^+} \tilde{F}(g_0 p_{t,\gamma} n) d\mu_{g_0 p_{t,\gamma} N^+}^{\text{PS}}(n) \\
& = \sum_{\gamma \in \Gamma_{t,v}} \int_{N_R^+} \tilde{F}(g_0 p_{t,\gamma} n) \phi_{\varepsilon_0}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} a_t^{-1}) d\mu_{g_0 p_{t,\gamma} N^+}^{\text{PS}}(n).
\end{aligned}$$

Similarly as before, we have, for all $t > t_0(v)$ and $n \in N_R^+$,

$$\begin{aligned}
\phi_{\varepsilon_0}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} a_t^{-1}) & = \phi_{\varepsilon_0}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} n(n)^{-1} a_t^{-1}) \\
& \leq \phi_{(R+\varepsilon_0)}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} n a_t^{-1}). \tag{6.7}
\end{aligned}$$

Hence (6.5) is bounded above by

$$\begin{aligned}
& \leq \sum_{\gamma \in \Gamma_{t,v}} \int_{N_R^+} \tilde{F}(g_0 p_{t,\gamma} n) \phi_{(R+\varepsilon_0)}^+ e^{-\alpha t} (a_t n_{t,\gamma}^{-1} n a_t^{-1}) d\mu_{g_0 p_{t,\gamma} N^+}^{\text{PS}}(n) \\
& = \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{F}(g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} n a_t) \phi_{(R+\varepsilon_0)}^+ e^{-\alpha t} (n) d\mu_{g_0 p_{t,\gamma} N^+}^{\text{PS}}(n_{t,\gamma} a_t^{-1} n a_t).
\end{aligned}$$

By Lemma 4.2,

$$d\mu_{g_0 p_{t,\gamma} N^+}^{\text{PS}}(n_{t,\gamma} a_t^{-1} n a_t) = e^{-\psi(\log a_t^{-1})} d\mu_{g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} N^+}^{\text{PS}}(n).$$

Since $g_0 p_{t,\gamma} n_{t,\gamma} a_t^{-1} = \gamma g$, it follows that for all $t > t_0(v)$,

$$\begin{aligned}
& e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n) dn \\
& \leq \sum_{\gamma \in \Gamma_{t,v}} \int_{N^+} \tilde{F}(\gamma g n a_t) \phi_{(R+\varepsilon_0)}^+ e^{-\alpha t} (n) d\mu_{\gamma g N^+}^{\text{PS}}(n) \\
& \leq \int_{N^+} \left(\sum_{\gamma \in \Gamma} \tilde{F}(\gamma g n a_t) \right) \phi_{(R+\varepsilon_0)}^+ e^{-\alpha t} (n) d\mu_{g N^+}^{\text{PS}}(n).
\end{aligned}$$

Define a function F on $\Gamma \backslash G$ by

$$F([g]) := \sum_{\gamma \in \Gamma} \tilde{F}(\gamma g).$$

Then for any $\varepsilon > 0$ and for all $t > t_0(v)$ such that $(R + \varepsilon_0)e^{-\alpha t} \leq \varepsilon$,

$$\begin{aligned} \Psi(t) \int_{N^+} F([g]na_t)\phi_\varepsilon^-(n) d\mu_{gN^+}^{\text{PS}}(n) &\leq \Psi(t)e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n) dn \\ &\leq \Psi(t) \int_{N^+} F([g]na_t)\phi_\varepsilon^+(n) d\mu_{gN^+}^{\text{PS}}(n). \end{aligned}$$

Since F is right M -invariant, by Proposition 5.3, letting $\varepsilon \rightarrow 0$ gives

$$\lim_{t \rightarrow \infty} \Psi(t)e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n) dn = J(v) m^{\text{BMS}}(F) \mu_{gN^+}^{\text{PS}}(\phi).$$

From the definition of F , together with Lemma 4.4 in the form (4.5), and (4.8), we have

$$\begin{aligned} m^{\text{BMS}}(F) &= \tilde{m}^{\text{BMS}}(\tilde{F}) \\ &= \int_{P^-} \left(\int_{N^+} \tilde{F}(g_0pn) d\mu_{g_0pN^+}^{\text{PS}}(n) \right) d\mu_{g_0P^-}^{\text{PS}}(p) \\ &= \int_{P^-} \left(\int_{N^+} \tilde{f}(g_0pn) dn \right) d\mu_{g_0P^-}^{\text{PS}}(p) \\ &= \tilde{m}^{\text{BR}}(\tilde{f}) = m^{\text{BR}}(f). \end{aligned}$$

This finishes the proof of the first statement. For the second statement, note that the following inequalities corresponding to (6.4), and (6.7) hold with the weaker assumption $tu + \sqrt{t}v \in \mathfrak{a}^+$, rather than $t > t_0(v)$: for all $n \in N_{\varepsilon_0}^+$,

$$\phi(a_t n_{t,\gamma}^{-1} a_t^{-1}(a_t n a_t^{-1})) \leq \phi_{\varepsilon_0}^+(a_t n_{t,\gamma}^{-1} a_t^{-1}),$$

and for all $n \in N_R^+$,

$$\phi_{\varepsilon_0}^+(a_t n_{t,\gamma}^{-1} a_t^{-1}) = \phi_{\varepsilon_0}^+(a_t n_{t,\gamma}^{-1} n(n)^{-1} a_t^{-1}) \leq \phi_{R+\varepsilon_0}^+(a_t n_{t,\gamma}^{-1} n a_t^{-1}).$$

Now proceeding similarly as in the proof of the first statement, we have

$$\begin{aligned} &\int_{N^+} f([g]na_t)\phi(n) dn \\ &\leq e^{(\psi-2\rho)(\log a_t)} \int_{N^+} \left(\sum_{\gamma \in \Gamma} \tilde{F}(\gamma gna_t) \right) \phi_{R+\varepsilon_0}^+(n) d\mu_{gN^+}^{\text{PS}}(n), \end{aligned}$$

and hence

$$\begin{aligned} &\Psi(t)e^{(2\rho-\psi)(\log a_t)} \int_{N^+} f([g]na_t)\phi(n) dn \\ &\leq \Psi(t) \int_{N^+} F([g]na_t)\phi_{R+\varepsilon_0}^+(n) d\mu_{gN^+}^{\text{PS}}(n) \leq C'(F, \phi_{R+\varepsilon_0}^+), \end{aligned}$$

provided $\log a_t \in \mathfrak{a}^+$. By setting $C''(f, \phi) := C'(F, \phi_{R+\varepsilon_0}^+)$, this finishes the proof of the proposition. \square

With the help of Proposition 5.3, we are now ready to prove:

Proposition 6.8. *Suppose that m^{BMS} satisfies the local mixing property for the pair (Ψ, J) . Then for any $f_1, f_2 \in C_c(\Gamma \backslash G)^M$ and $v \in \ker \psi$, we have*

$$\lim_{t \rightarrow \infty} \Psi(t) e^{(2\rho - \psi)(tu + \sqrt{t}v)} \int_{\Gamma \backslash G} f_1(xa(t, v)) f_2(x) dx = J(v) m^{\text{BR}}(f_1) m^{\text{BR}^*}(f_2),$$

and there exists $C_0 = C_0(f_1, f_2) > 0$ such that if $a(t, v) \in \mathfrak{a}^+$,

$$\left| \Psi(t) e^{(2\rho - \psi)(tu + \sqrt{t}v)} \int_{\Gamma \backslash G} f_1(xa(t, v)) f_2(x) dx \right| < C_0.$$

Proof. Note that the hypotheses above coincide with those of Propositions 5.3 and 6.1; this allows us to apply Proposition 6.1 in the following argument.

By compactness, we can find $\varepsilon_0 > 0$ and $x_i \in \Gamma \backslash G$, $i = 1, \dots, \ell$ such that the map $G \rightarrow \Gamma \backslash G$ given by $g \rightarrow x_i g$ is injective on $R_{\varepsilon_0} = N_{\varepsilon_0}^- A_{\varepsilon_0} N_{\varepsilon_0}^+ M$, and $\bigcup_{i=1}^{\ell} x_i R_{\varepsilon_0/2}$ contains both $\text{supp } f_1$ and $\text{supp } f_2$. As before, set $a_t = \exp(tu + \sqrt{t}v)$. We use continuous partitions of unity to write f_1 and f_2 as finite sums $f_1 = \sum_{i=1}^{\ell} f_{1,i}$ and $f_2 = \sum_{j=1}^{\ell} f_{2,j}$ with $\text{supp } f_{1,i} \subset x_i R_{\varepsilon_0/2}$ and $\text{supp } f_{2,j} \subset x_j R_{\varepsilon_0/2}$. Writing $p = \text{ham} \in N^- AM$ and using the following decomposition of the Haar measure on G :

$$d(\text{ham}n) = e^{-2\rho(\log a)} dn dm da dh$$

(cf. [26, Prop 8.45]), we have

$$\begin{aligned} & \int_{\Gamma \backslash G} f_1(xa_t) f_2(x) dx & (6.9) \\ &= \sum_{i,j} \int_{R_{\varepsilon_0}} f_{1,i}(x_i p n a_t) f_{2,j}(x_j p n) e^{-2\rho(\log a)} dn dm da dh \\ &= \sum_{i,j} \int_{N_{\varepsilon_0}^- A_{\varepsilon_0} M} \left(\int_{N_{\varepsilon_0}^+} f_{1,i}(x_i p n a_t) f_{2,j}(x_j p n) dn \right) e^{-2\rho(\log a)} dm da dh. \end{aligned}$$

Applying Proposition 6.1, it follows (cf. also (4.8))

$$\begin{aligned} & \lim_{t \rightarrow \infty} \Psi(t) e^{(2\rho - \psi)(\log a_t)} \int_{\Gamma \backslash G} f_1(xa_t) f_2(x) dx \\ &= J(v) \sum_i m^{\text{BR}}(f_{1,i}) \sum_j \int_{N_{\varepsilon_0}^- A_{\varepsilon_0} M} \mu_{x_i p N^+}^{\text{PS}}(f_{2,j}(x_i p \cdot)) e^{-2\rho(\log a)} dm da dh \\ &= J(v) \sum_i m^{\text{BR}}(f_{1,i}) \sum_j m^{\text{BR}^*}(f_{2,j}) \\ &= J(v) m^{\text{BR}}(f_1) m^{\text{BR}^*}(f_2) \end{aligned}$$

where the second last equality is valid by (4.8). This justifies the first statement. For the second statement, note that if $tu + \sqrt{t}v \in \mathfrak{a}^+$, (6.9) together

with Proposition 6.1 gives

$$\left| \Psi(t) e^{(2\rho - \psi)(\log at)} \int_{\Gamma \backslash G} f_1(xa_t) f_2(x) dx \right| \leq C_0(f_1, f_2),$$

where $C_0(f_1, f_2) := \sum_{i,j} C'''(f_{1,i}, f_{2,j}) \int_{N_{\varepsilon_0}^- A_{\varepsilon_0} M} e^{-2\rho(\log(a(t,v)))} dm da dh$. This completes the proof. \square

We make the following observation, which will be used in the proof of Theorem 7.9.

Corollary 6.10. *Fix $u \in \mathcal{L}_\Gamma \cap \text{int } \mathfrak{a}^+$, and let $\psi \in D_\Gamma^*$ be tangent to ψ_Γ at u . For $k = 1, 2$, let ν_k and $\bar{\nu}_k$ be respectively (Γ, ψ) and $(\Gamma, \psi \circ i)$ -PS measures on Λ_Γ . Setting $m_k^{\text{BMS}} := m_{\nu_k, \bar{\nu}_k}^{\text{BMS}}$, suppose that there exist functions $\Psi_k : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that for all $f_1, f_2 \in C_c(\Gamma \backslash G)^M$,*

$$\lim_{t \rightarrow \infty} \Psi_k(t) \int_{\Gamma \backslash G} f_1(x \exp(tu)) f_2(x) dm_k^{\text{BMS}}(x) = m_k^{\text{BMS}}(f_1) m_k^{\text{BMS}}(f_2).$$

Then $\nu_1 = \nu_2$ and $\bar{\nu}_1 = \bar{\nu}_2$.

Proof. By an argument similar to the proof of Proposition 6.8,

$$\lim_{t \rightarrow \infty} \Psi_k(t) e^{t(2\rho - \psi_\Gamma)(u)} \int_{\Gamma \backslash G} f_1(x \exp(tu)) f_2(x) dx = m_{\bar{\nu}_k}^{\text{BR}}(f_1) m_{\nu_k}^{\text{BR}*}(f_2).$$

Fix $f_1 \in C_c(\Gamma \backslash G)^M$ with $m_{\bar{\nu}_k}^{\text{BR}}(f_1) > 0$ for each $k = 1, 2$. By considering $f_2 \in C_c(\Gamma \backslash G)^M$ with $m_{\nu_k}^{\text{BR}*}(f_2) > 0$, it follows from the hypothesis that $c_0 := \lim_{t \rightarrow \infty} \frac{\Psi_1(t)}{\Psi_2(t)} > 0$. Set $c_1 := c_0 \cdot \frac{m_{\bar{\nu}_2}^{\text{BR}}(f_1)}{m_{\bar{\nu}_1}^{\text{BR}}(f_1)}$. Then for any $f_2 \in C_c(\Gamma \backslash G)^M$, we have

$$m_{\nu_1}^{\text{BR}*}(f_2) = c_1 \cdot m_{\nu_2}^{\text{BR}*}(f_2). \quad (6.11)$$

Recall from Lemma 3.6 that for all $a \in A$, $a_* m_{\nu_k}^{\text{BR}*} = e^{(-\psi + 2\rho)(\log a)} m_{\nu_k}^{\text{BR}*}$. We claim that $\nu_1 = \nu_2$. Let $g_0 \in G$ be arbitrary. Fix $\varepsilon > 0$ and consider $F \in C(G/P)$ supported in $(g_0 N_\varepsilon^+)^-$. Choose $q_1 \in C_c(A)$, and a radial function $q_2 \in C_c(N)$ such that $\int_A q_1 da = \int_{N^-} q_2 dn = 1$. Define $\tilde{f}_2 \in C_c(G)$ by

$$\tilde{f}_2(g) := \begin{cases} F(g_0 h^+) q_1(a) q_2(n) e^{-\psi(\log a)} & \text{if } g = g_0 h m a n \in g_0 N^+ P^-, \\ 0 & \text{otherwise.} \end{cases}$$

Note that \tilde{f}_2 is M -invariant, as q_2 is radial. Defining $f_2 \in C_c(\Gamma \backslash G)$ by $f_2([g]) := \sum_{\gamma \in \Gamma} \tilde{f}_2(\gamma g)$, a direct computation shows

$$\begin{aligned} m_{\nu_k}^{\text{BR}*}(f_2) &= \int_{N^+} \int_{P^-} \tilde{f}_2(g_0 h m a n) e^{\psi(\log a)} dm da dn d\nu_k(g_0 h^+) \\ &= \int_{N^+} F(g_0 h^+) \left(\int_{P^-} q_1(a) q_2(n) dm da dn \right) d\nu_k(g_0 h^+) \\ &= \nu_k(F). \end{aligned}$$

Hence for any $g_0^- \in G/P$, $\nu_1(F) = c_1 \cdot \nu_2(F)$ for all $F \in C(G/P)$ supported in $(g_0\mathcal{O}_\varepsilon)^-$. By using a partition of unity, we get $\nu_1(F) = c_1 \cdot \nu_2(F)$ for all $F \in C(G/P)$. Since $|\nu_1| = |\nu_2| = 1$, we have $c_1 = 1$ and hence $\nu_1 = \nu_2$. Repeating the same argument for $i(v)$, we also get $\bar{\nu}_1 = \bar{\nu}_2$ (this implies $c_0 = 1$). \square

7. ANOSOV GROUPS

7.1. Anosov subgroups. Let Σ be a finitely generated word hyperbolic group and let $\partial\Sigma$ denote the Gromov boundary of Σ . We call a Zariski dense discrete subgroup $\Gamma < G$ *Anosov* if it arises as the image of a P -Anosov representation of Σ . A representation $\Phi : \Sigma \rightarrow G$ is P -Anosov if Φ induces a continuous equivariant map $\zeta : \partial\Sigma \rightarrow \mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y \in \partial\Sigma$.

Let $\tau_d : \mathrm{PSL}_2(\mathbb{R}) \rightarrow \mathrm{PSL}_d(\mathbb{R})$ be the d -dimensional irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$. For any torsion-free uniform lattice Σ in $\mathrm{PSL}_2(\mathbb{R})$, the connected component of $\tau_d|_\Sigma$ in the space $\mathrm{Hom}(\Sigma, \mathrm{PSL}_d(\mathbb{R}))$ is called the Hitchin component. Representations $\Sigma \rightarrow \mathrm{PSL}_d(\mathbb{R})$ in the Hitchin component are known to be P -Anosov [27]. In fact, Hitchin components are defined for representations of Σ into any split real simple Lie group G , and all representations $\Sigma \rightarrow G$ in the Hitchin component are known to be P -Anosov ([14], [19]).

We mention that if $\rho_i : \Sigma \rightarrow G_i$ are P_i -Anosov where P_i is a minimal parabolic subgroup of G_i , then $\rho_1 \times \rho_2 : \Sigma \rightarrow G_1 \times G_2$ is $P_1 \times P_2$ -Anosov whenever its image is Zariski dense. Indeed, if $\zeta_i : \partial\Sigma \rightarrow G_i/P_i$ denotes the limit map of ρ_i , the map $\zeta(x) = (\zeta_1(x), \zeta_2(x))$ provides the desired limit map for $\rho_1 \times \rho_2$, and hence $\{(\rho_1(g), \rho_2(g)) : g \in \Sigma\}$ is an Anosov subgroup of $G_1 \times G_2$.

One subclass of Anosov groups consists of Schottky groups, which generalize the Schottky subgroups of rank one Lie groups.

For a loxodromic element $g \in G$, we denote by $y_g \in \mathcal{F}$ the unique attracting fixed point of g . Let $\gamma_1, \dots, \gamma_p$ be loxodromic elements of G ($p \geq 2$). For each $1 \leq i \leq p$, set $\xi_i^{+1} = y_{\gamma_i}$ and $\xi_i^{-1} = y_{\gamma_i^{-1}}$.

Definition 7.1. The subgroup Γ generated by $\{\gamma_1, \dots, \gamma_p\}$ is called Schottky if there exist open subsets $b_i^\pm, B_i^\pm \subset \mathcal{F}$, $1 \leq i \leq p$ and $0 < \varepsilon < 1$ such that

- (1) for all $i \neq j$ and $\omega, \varpi \in \{-1, 1\}$, $b_i^\omega \subset B_j^\varpi$ and $\bar{b}_i^\omega \times \bar{b}_j^\varpi \subset \mathcal{F}^{(2)}$;
- (2) for all i and $\omega \in \{-1, 1\}$, $\xi_i^\omega \in \mathrm{int} b_i^\omega$, $\gamma_i^\omega B_i^\omega \subset b_i^\omega$ and the restriction of γ_i to B_i^ω is ε -Lipschitz;
- (3) the intersection $\bigcap_{1 \leq i \leq p, \omega \in \{1, -1\}} B_i^\omega$ is non-empty.

This is the same definition as given in [45, Section 4.2], except for the extra second condition in (1), which we added to ensure the following lemma:

Lemma 7.2. *Any Zariski dense Schottky subgroup $\Gamma < G$ is Anosov.*

Proof. The Gromov boundary $\partial\Gamma$ can be identified with the set of infinite words of the form $\underline{a} = (a_0 a_1 a_2 \cdots)$ where $a_i \in \{\gamma_1^{\pm 1}, \dots, \gamma_p^{\pm 1}\}$ and $a_i \neq a_{i+1}^{-1}$. Fix an element $\xi \in \bigcap_{1 \leq i \leq p, \omega \in \{1, -1\}} B_i^\omega$, which exists by (3). Under the above definition of a Schottky group, the proof of [44, Proposition 3.3] gives that the map $\underline{a} \rightarrow \lim_{n \rightarrow \infty} (a_0 a_1 \cdots a_n) \xi$ induces a Γ -equivariant homeomorphism $\zeta : \partial\Gamma \rightarrow \Lambda_\Gamma$ (see also [45, Proposition 4.5]). If $\xi \neq \eta$ in Λ_Γ , there exists $\gamma \in \Gamma$ such that $\gamma\xi = \lim_{n \rightarrow \infty} (a_0 a_1 \cdots a_n) \xi$ and $\gamma\eta = \lim_{n \rightarrow \infty} (a'_0 a'_1 \cdots a'_n) \xi$ with $a_0 = \gamma_i^\omega$ and $a'_0 = \gamma_j^\varpi$ for some $i \neq j$ and $\omega, \varpi \in \{\pm 1\}$. It follows from (1) and (2) that $\gamma\xi \in \overline{b_j^\varpi}$ and $\gamma\eta \in \overline{b_j^\omega}$. Hence $(\gamma\xi, \gamma\eta) \in \mathcal{F}^{(2)}$ by (1); consequently, $(\xi, \eta) \in \mathcal{F}^{(2)}$. This shows that Γ is Anosov. \square

Schottky groups are found everywhere, in the following sense:

Lemma 7.3. *Any Zariski dense discrete subgroup Γ contains a Zariski dense Schottky subgroup.*

Proof. This follows from the proof of a more general theorem [2, Proposition 4.3]. We give a sketch of the proof for the sake of completeness. Since the set of loxodromic elements of Γ is Zariski dense, we may choose a loxodromic element $\gamma_1 \in \Gamma$. There exists a proper Zariski closed subset $F_{\gamma_1} \subset G$ which contains all Zariski connected and Zariski closed proper subgroups of G containing γ_1 [56, Proposition 4.4]. We may choose a second loxodromic element $\gamma_2 \in \Gamma - F_{\gamma_1}$ such that $(y_{\gamma_2^{\pm 1}}, y_{\gamma_1}), (y_{\gamma_2^{\pm 1}}, y_{\gamma_1^{-1}}) \subset \mathcal{F}^{(2)}$. Moreover we can assume that γ_2^k generates a Zariski connected subgroup, and hence $\gamma_2^k \notin F_{\gamma_1}$, for any $k \in \mathbb{N}$. Let $\varphi_i \in G$ be so that $\gamma_i \in \varphi_i(\text{int } A^+) \varphi_i^{-1}$. Then $y_{\gamma_i^{\pm 1}} = \varphi_i e^\pm$. For $\varepsilon > 0$, let $b_i^{\pm 1}(\varepsilon)$ be the ε -neighborhood of $\varphi_i e^\pm$ and set $B_i^{\pm 1}(\varepsilon) = \varphi_i N_{\varepsilon^{-1}}^\pm e^\pm$ where $N_{\varepsilon^{-1}}^\pm$ denotes the ε^{-1} -neighborhood of e in N^\pm . For any $\varepsilon_1, \varepsilon_2 > 0$, $\gamma_i^{\pm k} B_i^{\pm 1}(\varepsilon_2) \subset b_i^{\pm 1}(\varepsilon_1)$ for all sufficiently large k . It follows that we can find $k, \varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that γ_1^k and γ_2^k satisfy the conditions in 7.1 with $b_i^{\pm 1} := b_i^{\pm 1}(\varepsilon_1)$ and $B_i^{\pm 1} := B_i^{\pm 1}(\varepsilon_2)$. \square

7.2. BMS-mixing. The following theorem was first proved by Sambarino [50] when Σ is the fundamental group of a closed negatively curved manifold \mathcal{M} . The fact that the geodesic flow on $T^1(\mathcal{M})$ is a transitive Anosov flow is a key ingredient which gives a Markov coding of the flow and hence makes it possible to translate the mixing of the BMS measures to a statement in thermodynamic formalism as formulated by Thirion ([50, Theorem 3.8], [55]). In [8], the authors defined the geodesic flow associated to ρ which is Hölder equivalent to the Gromov geodesic flow of Σ and showed that this flow is a transitive metric Anosov flow which admits a Markov coding ([6], [7], [39], see also [8, Theorem 3.4]). Given this, the proof of ([50, Theorem 3.8]) extends to general Anosov groups (see [10, Appendix]):

Theorem 7.4. *Let $u \in \text{int } \mathcal{L}_\Gamma$ be a unit vector. Let $\psi \in D_\Gamma^*$ be tangent to ψ_Γ at u , and ν and ν_i be respectively (Γ, ψ) and $(\Gamma, \psi \circ i)$ -PS measures on Λ_Γ .*

Set $m^{\text{BMS}} := m_{\nu, \nu_i}^{\text{BMS}}$. Then there exists $\kappa_u > 0$ such that for any $v \in \ker \psi$ and any $f_1, f_2 \in C_c(\Gamma \backslash G)^M$,

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{(r-1)/2} \int_{\Gamma \backslash G} f_1(x) f_2(x \exp(tu + \sqrt{t}v)) dm^{\text{BMS}}(x) \\ = \kappa_u e^{-I(v)/2} m^{\text{BMS}}(f_1) m^{\text{BMS}}(f_2), \end{aligned}$$

where $I : \ker(\psi) \rightarrow \mathbb{R}$ is given by

$$I(v) := c \cdot \frac{\|v\|_*^2 \|u\|_*^2 - \langle v, u \rangle_*^2}{\|u\|_*^2} \quad (7.5)$$

for some inner product $\langle \cdot, \cdot \rangle_*$ and some $c > 0$. Moreover, the left-hand side is uniformly bounded over all $(t, v) \in (0, \infty) \times \ker \psi$ with $tu + \sqrt{t}v \in \mathfrak{a}^+$.

Although the second statement of Theorem 7.4 is not stated in [50, Theorem 3.8], its proof uses the same technique as [55, Theorem 1.1], where the corresponding statement can be found.

Remark 7.6. Theorem 7.4 is the main reason for the assumption that Γ is a Anosov subgroup. In fact, all our results stated in the introduction hold whenever Γ satisfies Theorems 7.4 and 7.7.

In the rest of the paper, let Γ be an Anosov subgroup of G . The following theorem was proved by Sambarino [49] for a special case and by Potrie-Sambarino [40, Propositions 4.6 and 4.11] in general:

Theorem 7.7. (1) $\mathcal{L}_\Gamma \subset \text{int } \mathfrak{a}^+$.
(2) ψ_Γ is strictly concave and analytic on $\text{int } \mathcal{L}_\Gamma$.

By Lemma 2.25, and Theorem 7.7, we get the following corollary:

Corollary 7.8. For each unit vector $u \in \text{int } \mathcal{L}_\Gamma$, there exists a unique $\psi_u \in D_\Gamma^*$ tangent to ψ_Γ at u , which is given by

$$\psi_u(\cdot) = \langle \nabla \psi_\Gamma(u), \cdot \rangle.$$

Theorem 7.9. For any $\psi \in D_\Gamma^*$, there exists a unique (Γ, ψ) -PS measure on Λ_Γ . In particular, this measure is Γ -ergodic.

Proof. We only need to address the uniqueness. This is proved in [51, Theorem 3.1, 7.8] when Γ is the fundamental group of a negatively curved manifold. The general case follows from Theorem 7.4 by Corollary 6.10. \square

For each unit vector $u \in \text{int } \mathcal{L}_\Gamma$, we denote by ν_u the unique (Γ, ψ_u) -PS measure on Λ_Γ . We now set

$$m_u^{\text{BMS}} := m_{\nu_u, \nu_i(u)}^{\text{BMS}}, \quad m_{i(u)}^{\text{BR}} := m_{\nu_i(u)}^{\text{BR}}, \quad m_u^{\text{BR}*} := m_{\nu_u}^{\text{BR}*}. \quad (7.10)$$

We mention that all three measures are infinite measures when G has rank at least 2 (cf. [31, Corollary 4.9]).

We therefore deduce the following from Proposition 5.3, Proposition 6.8, and Theorem 7.4.

Theorem 7.11. *Let $u \in \text{int } \mathcal{L}_\Gamma$ be a unit vector.*

(1) *For any $f_1, f_2 \in C_c(\Gamma \backslash G)^M$ and $v \in \ker \psi_u$,*

$$\begin{aligned} \lim_{t \rightarrow +\infty} t^{(r-1)/2} e^{(2\rho - \psi_u)(tu + \sqrt{t}v)} \int_{\Gamma \backslash G} f_1(x \exp(tu + \sqrt{t}v)) f_2(x) dx \\ = \kappa_u e^{-I(v)/2} m_{i(u)}^{\text{BR}}(f_1) m_u^{\text{BR}*}(f_2). \end{aligned}$$

(2) *For any $f \in C_c(\Gamma \backslash G)^M$, $\phi \in C_c(N^+)$, $x = [g] \in \Gamma \backslash G$, and $v \in \ker \psi_u$,*

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{(r-1)/2} e^{(2\rho - \psi_u)(tu + \sqrt{t}v)} \int_{N^+} f(xn \exp(tu + \sqrt{t}v)) \phi(n) dn \\ = \kappa_u e^{-I(v)/2} m_{i(u)}^{\text{BR}}(f) \mu_{gN^+, \nu_u}^{\text{PS}}(\phi). \end{aligned}$$

Moreover the left-hand sides of the above equalities are uniformly bounded for all $(t, v) \in (0, \infty) \times \ker \psi_u$ with $tu + \sqrt{t}v \in \mathfrak{a}^+$.

Recalling that $\psi_u(u) = \psi_\Gamma(u)$, the special case of Theorem 7.11 when $v = 0$ now implies Theorem 1.7 and Theorem 1.8.

8. EQUIDISTRIBUTION OF TRANSLATES OF $\Gamma \backslash \Gamma H$

Symmetric subgroups of G . Let $H < G$ be a symmetric subgroup; that is to say, H is the identity component of the set of fixed points of an involution σ of G . We start by reviewing some general structure theory regarding symmetric subgroups; see Chapter 6 of [53] for more details on this. The involution σ induces a Lie algebra involution on \mathfrak{g} , which (using a slight abuse of notation) we also denote by σ . There exists a Cartan involution of G that commutes with σ ; without loss of generality, we may assume that θ from Section 2 commutes with σ . These involutions give rise to the decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ into the $+1$ and -1 eigenspace decompositions of θ and σ , respectively. Let \mathfrak{a} be a maximal abelian subalgebra of \mathfrak{p} such that $\mathfrak{b} := \mathfrak{a} \cap \mathfrak{q}$ is a maximal abelian subalgebra of $\mathfrak{p} \cap \mathfrak{q}$. Denote the dimension of \mathfrak{a} by r and the dimension of \mathfrak{b} by r_0 .

Let $\Sigma_\sigma \subset \mathfrak{b}^*$ be the root system of \mathfrak{b} , i.e.

$$\Sigma_\sigma = \{\lambda \in \mathfrak{b}^* - \{0\} : \text{there exists } X \in \mathfrak{g} \text{ with } \text{ad}_Y X = \lambda(Y)X \text{ for all } Y \in \mathfrak{b}\}.$$

From now on, we fix a closed positive Weyl chamber $\mathfrak{b}^+ \subset \mathfrak{b}$ for Σ_σ and assume that it has been chosen compatibly with \mathfrak{a}^+ as follows: denoting the positive roots of \mathfrak{a} by Σ^+ , we assume that there exists a collection of positive roots Σ_σ^+ of \mathfrak{b} such that the elements of Σ_σ^+ are all obtained by restricting elements of Σ^+ to \mathfrak{b} , hence $\mathfrak{b}^+ \subset \mathfrak{a}^+$. We will denote $B = \exp(\mathfrak{b})$ and $B^+ = \exp(\mathfrak{b}^+)$.

Let $\mathcal{W}_\sigma := N_K(\mathfrak{b})/Z_K(\mathfrak{b})$ and $\mathcal{W}_{\sigma, \theta} := N_{K \cap H}(\mathfrak{b})/Z_{K \cap H}(\mathfrak{b})$. There then exists a finite set of representatives $\mathcal{W} \subset N_K(\mathfrak{a}) \cap N_K(\mathfrak{b})$ for $\mathcal{W}_{\sigma, \theta} \backslash \mathcal{W}_\sigma$, and we have the following generalized Cartan decomposition:

$$G = H \exp(\mathfrak{b}) K = H \mathcal{W} \exp(\mathfrak{b}^+) K, \quad (8.1)$$

in the sense that for any $g \in G$, there exist unique elements $b \in B^+$ and $\omega \in \mathcal{W}$ such that

$$g \in H\omega bK.$$

Directions in $\mathfrak{b}^+ \cap \text{int } \mathcal{L}_\Gamma$. Let Γ be an Anosov subgroup of G . In the rest of this section, we assume that

$$\mathfrak{b}^+ \cap \text{int}(\mathcal{L}_\Gamma) \neq \emptyset.$$

Since $\mathcal{L}_\Gamma \subset \text{int}(\mathfrak{a}^+)$ by Theorem 7.7, it follows that $\mathfrak{b}^+ \cap \text{int}(\mathfrak{a}^+) \neq \emptyset$.

We now fix a unit vector (with respect to the norm $\|\cdot\|$ on \mathfrak{a})

$$v \in \mathfrak{b}^+ \cap \text{int } \mathcal{L}_\Gamma$$

and set

$$\delta := \psi_\Gamma(v) > 0. \quad (8.2)$$

By Corollary 7.8, the linear form $\Theta \in \mathfrak{a}^*$ defined as

$$\Theta(w) = \langle \nabla \psi_\Gamma(v), w \rangle \quad (8.3)$$

gives the unique linear form in D_Γ^* such that $\Theta(v) = \psi_\Gamma(v) = \delta$.

Lemma 8.4. *We have $\mathcal{L}_\Gamma \cap \ker \Theta = \{0\}$.*

Proof. We use the fact that ψ_Γ is strictly concave (Theorem 7.7). Since $\Theta \geq \psi_\Gamma$ on \mathfrak{a} , and $\Theta(v) = \psi_\Gamma(v)$, it follows that $\Theta(w) > \psi_\Gamma(w)$ for all vectors $w \in \mathfrak{a}^+ - \mathbb{R}v$. Since $\psi_\Gamma \geq 0$ on \mathcal{L}_Γ , we have $\Theta > 0$ on $\mathcal{L}_\Gamma - \{0\}$, i.e., $\mathcal{L}_\Gamma \cap \ker \Theta = \{0\}$. \square

We use the following notation: for $t > 0$ and $w \in \ker \Theta$,

$$a(t, w) := \exp(tv + \sqrt{t}w), \quad \text{and} \quad \underline{a}(t, w) := tv + \sqrt{t}w.$$

We set

$$m^{\text{BR}} = m_{\nu_1(v)}^{\text{BR}} \quad \text{and} \quad m^{\text{BR}^*} = m_{\nu_v}^{\text{BR}^*}.$$

Patterson-Sullivan measures on H . Let $P = MAN$ be the minimal parabolic subgroup. Since $\mathfrak{b}^+ \cap \text{int}(\mathfrak{a}^+) \neq \emptyset$, it follows that $M = Z_K(\mathfrak{b})$, and the unipotent subgroup whose Lie algebra is the sum of positive root spaces corresponding to Σ_σ coincides with N .

Lemma 8.5. *We have $H \cap N = \{e\}$*

Proof. Fix $\alpha \in \Sigma_\sigma$. Since $v \in \mathfrak{b}^+ \cap \text{int } \mathfrak{a}^+$, we have $\alpha(v) > 0$. Letting $X \in \mathfrak{n}$ be such that $[v, X] = \alpha(v)X$, we have $\sigma(X) = \alpha(v)^{-1}\sigma([v, X]) = \alpha(v)^{-1}[\sigma(v), \sigma(X)] = \alpha(v)^{-1}[-v, \sigma(X)]$, i.e. $[v, \sigma(X)] = -\alpha(v)\sigma(X)$. Therefore $\sigma(X) \in \mathfrak{n}^+$. Since σ fixes H point-wise and swaps N and N^+ , we get $H \cap N = \{e\}$. \square

By [30, Theorem 3-(iii)],

$$H \cap P = (H \cap M)(H \cap A)(H \cap N) = (H \cap M)(H \cap A). \quad (8.6)$$

Together with the fact $H \cap B = H \cap N = \{e\}$, it then follows that $H \cap MBN = H \cap M$.

Definition 8.7. Define a measure $\mu_H^{\text{PS}} = \mu_{H,v}^{\text{PS}}$ on H as follows: for $\phi \in C_c(H)$, let

$$\mu_H^{\text{PS}}(\phi) = \int_{h_0 \in H/(H \cap P)} \int_{p \in H \cap P} \phi(h_0 p) e^{\Theta(\beta_{h_0^+}(e, h_0 p))} dp dv(h_0^+),$$

where dp is a right-Haar measure on $H \cap P$. The measure defined above is $\Gamma \cap H$ -invariant: for any $\gamma \in \Gamma \cap H$, $\gamma_* \mu_H^{\text{PS}} = \mu_H^{\text{PS}}$. Therefore, if $\Gamma \backslash \Gamma H$ is closed in $\Gamma \backslash G$, $d\mu_{H,v}^{\text{PS}}$ induces a locally finite Borel measure on $\Gamma \backslash \Gamma H \simeq (\Gamma \cap H) \backslash H$, which we denote by $\mu_{[e]H}^{\text{PS}} = \mu_{[e]H,v}^{\text{PS}}$.

For a subset $S \subset G$ and $\varepsilon > 0$, set $S_\varepsilon := \{s \in S : d(e, s) \leq \varepsilon\}$. Let $M' \subset M$ be a Borel section for the map $m \mapsto (H \cap M)m$, and $P' = M'BN$ be the subset of the minimal parabolic subgroup $P = MAN$. Note that the map $H \times P' \rightarrow G$ given by $(h, p') \mapsto hp'$ is injective. For $\varepsilon > 0$, let $\rho_\varepsilon \in C((NB)_\varepsilon)$ be a non-negative function such that

$$\int_{NB} \rho_\varepsilon(nb) dn db = 1,$$

and $\rho_\varepsilon(mnbm^{-1}) = \rho_\varepsilon(nb)$ for all $m \in M$ and $nb \in NB$. Fixing $\phi \in C_c(H)^{H \cap M}$ and $\varepsilon > 0$ smaller than the injectivity radius of $\text{supp}(\phi)$, define $\tilde{\Phi}_\varepsilon \in C_c(G)$ by

$$\tilde{\Phi}_\varepsilon(g) := \begin{cases} \phi(h)\rho_\varepsilon(nb) & \text{if } g = hm'nb \in HP', \\ 0 & \text{otherwise.} \end{cases} \quad (8.8)$$

Observe that $\tilde{\Phi}_\varepsilon$ is right M -invariant. Define now $\Phi_\varepsilon \in C_c(\Gamma \backslash G)$ by $\Phi_\varepsilon([g]) = \sum_{\gamma \in \Gamma} \tilde{\Phi}_\varepsilon(\gamma g)$. Since $\tilde{\Phi}_\varepsilon$ is right M -invariant, so is Φ_ε .

Lemma 8.9. For $\phi \in C_c(H)^{H \cap M}$ and Φ_ε as above, we have

$$m^{\text{BR}*}(\Phi_\varepsilon) = \mu_H^{\text{PS}}(\phi)(1 + O(\varepsilon)).$$

Proof. Note that $\text{supp}(\tilde{\Phi}_\varepsilon) \subset HM(NB)_\varepsilon$. The proof of the lemma now relies on the following three observations.

Firstly, for each $hm \in HM'$ and $n \in N$,

$$d(hmn) = e^{2\rho(\beta_{(hmn)^-}(e, hmn))} dm_o(hmn^-) = dn$$

is a Lebesgue measure on hmN .

Secondly, for $g = hm'nb \in HP'$, the decomposition

$$\beta_g(e, g) = \beta_{h^+}(e, h) + \beta_{e^+}(e, nb) = \beta_{h^+}(e, h) + i(\log b)$$

induces an isomorphism $A \cong (A \cap H) \times B$. This implies that

$$d(\beta_g(e, g)) = d(\beta_{h^+}(e, h)) d(\beta_{e^+}(e, p)) = d(\beta_{h^+}(e, h)) d(b),$$

and for all $hm' \in HM'$ and $nb \in (NB)_\varepsilon$,

$$e^{\Theta(\beta_{h^+}(e, hm'nb))} = e^{\Theta(\beta_{h^+}(e, h))}(1 + O(\varepsilon))$$

by continuity of the Busemann function.

Finally, we also have:

$$\begin{aligned}\beta_{(hm'nb)^-}(e, hm'nb) &= \beta_{(hm'n)^-}(e, hm'n) + \beta_{(hm'n)^-}(hm'n, hm'nb) \\ &= \beta_{(hm'n)^-}(e, hm'n) + \beta_{e^-}(e, b) \\ &= \beta_{(hm'n)^-}(e, hm'n) - i(\log b).\end{aligned}$$

Consequently, $e^{2\rho(\beta_{(hm'nb)^-}(e, hm'nb))} = e^{2\rho(\beta_{(hm'n)^-}(e, hm'n))}(1 + O(\varepsilon))$. Using the definition of $m^{\text{BR}*}(\Phi_\varepsilon)$ and the second and third observations above give

$$\begin{aligned}m^{\text{BR}*}(\Phi_\varepsilon) &= \int_{H \times M' \times (NB)_\varepsilon} \phi(h) \rho_\varepsilon(nb) e^{\Theta(\beta_{h^+}(e, hm'nb)) + 2\rho(\beta_{(hm'n)^-}(e, hm'nb))} \\ &\quad \times dm_o((hm'n)^-) d(\beta_{h^+}(e, hm'nb)) d\nu(h^+) \\ &= (1 + O(\varepsilon)) \int_{H \times M' \times (NB)_\varepsilon} \phi(h) e^{\Theta(\beta_{h^+}(e, h))} \rho_\varepsilon(nb) e^{2\rho(\beta_{(hm'n)^-}(e, hm'n))} \\ &\quad \times dm_o((hm'n)^-) db d(\beta_{h^+}(e, h)) d\nu(h^+).\end{aligned}$$

We now choose a section $H_0 \subset H$ for the map $h \mapsto h(H \cap P)$, and write $h = h_0 m a_h \in H_0(M \cap H)(A \cap H) = H_0(H \cap P)$. Using the first observation above, we then have

$$\begin{aligned}&= (1 + O(\varepsilon)) \int_{H_0(H \cap P)} \phi(h_0 m a_h) e^{\Theta(\beta_{h_0^+}(e, h))} \left(\int_{(NB)_\varepsilon} \rho_\varepsilon(nb) dn db \right) d(a_h) d\nu(h_0^+) \\ &= (1 + O(\varepsilon)) \int_{h_0 \in H/(H \cap P)} \int_{p \in H \cap P} \phi(h_0 p) e^{\Theta(\beta_{h_0^+}(e, h_0 p))} dp d\nu(h_0^+) \\ &= (1 + O(\varepsilon)) \mu_H^{\text{PS}}(\phi).\end{aligned}$$

□

Equidistribution of translates of $\Gamma \backslash \Gamma H$.

Proposition 8.10. *For any $f \in C_c(\Gamma \backslash G)^M$, $\phi \in C_c(H)^{H \cap M}$, and $w \in \mathfrak{b} \cap \ker \Theta$,*

$$\lim_{t \rightarrow \infty} t^{(r-1)/2} e^{(2\rho - \Theta)(\underline{a}(t, w))} \int_H f([h]a(t, w)) \phi(h) dh = \kappa_v e^{-I(w)/2} m^{\text{BR}}(f) \mu_H^{\text{PS}}(\phi),$$

and there exists $C' = C'(f, \phi) > 0$ such that for all (t, w) with $\underline{a}(t, w) \in \mathfrak{b}^+$,

$$\left| t^{(r-1)/2} e^{(2\rho - \Theta)(\underline{a}(t, w))} \int_H f([h]a(t, w)) \phi(h) dh \right| < C'. \quad (8.11)$$

Proof. For $\varepsilon > 0$, let $R_\varepsilon := N_\varepsilon A_\varepsilon N_\varepsilon^+ M$ and define $f_\varepsilon^\pm \in C_c(\Gamma \backslash G)$ by

$$f_\varepsilon^+(y) = \sup_{g \in R_\varepsilon} f(yg), \text{ and } f_\varepsilon^-(y) = \inf_{g \in R_\varepsilon} f(yg). \quad (8.12)$$

Since R_ε is right M -invariant, it follows that $f_\varepsilon^\pm \in C_c(\Gamma \backslash G)^M$. Let $C_0 \subset H$ denote the support of ϕ ; we may assume that C_0 injects to its image under the map $G \mapsto \Gamma \backslash G$. Choosing $\rho_\varepsilon \in C_c((NB)_\varepsilon)$ and defining Φ_ε as above, we let $d\lambda(m')$ denote the density on M' of total mass one such that

$$d(hm'nb) = dh d\lambda(m') dn db \quad (8.13)$$

(where $h \in H$, $m' \in M'$, $n \in N$, and $b \in B$) is a Haar measure on G .

We then obtain

$$\begin{aligned} \int_H f([h]a(t, w))\phi(h) dh &= \int_{C_0} f([h]a(t, w))\phi(h) \left(\int_{(NB)_\varepsilon} \rho_\varepsilon(nb) dn db \right) dh \\ &= \int_{C_0(NB)_\varepsilon} f([h]a(t, w))\Phi_\varepsilon([h]nb) dh dn db. \end{aligned}$$

Using the M -invariance of f and the definitions of Φ_ε and $d\lambda$ gives

$$f([h]a(t, w))\Phi_\varepsilon([h]nb) = \int_{M'} f([h]m'a(t, w))\Phi_\varepsilon([h]m'nb) d\lambda(m'),$$

and so

$$\int_H f([h]a(t, w))\phi(h) dh = \int_{C_0 M' (NB)_\varepsilon} f([h]m'a(t, w))\Phi_\varepsilon([h]m'nb) dh d\lambda(m') dn db.$$

Since $v \in \text{int } \mathfrak{b}^+$, for all (t, w) such that $\underline{a}(t, w) \in \mathfrak{b}^+$, and for all $nb \in (NB)_\varepsilon$, we have

$$\begin{aligned} f(xhm'a(t, w)) &= f(xhm'nba(t, w) \cdot (a(t, w)^{-1}(nb)^{-1}a(t, w))) \\ &\leq f_\varepsilon^+(xhm'nba(t, w)). \end{aligned} \tag{8.14}$$

Consequently,

$$\begin{aligned} &\int_{C_0} f([h]a(t, w))\phi(h) dh \\ &\leq \int_{C_0 M' (NB)_\varepsilon} f_\varepsilon^+([h]m'nba(t, w))\Phi_\varepsilon([h]m'nb) dh d\lambda(m') dn db \\ &= \int_{\Gamma \setminus G} f_\varepsilon^+(ya(t, w))\Phi_\varepsilon(y) dy. \end{aligned}$$

A similar computation shows that

$$\int_{C_0} f([h]a(t, w))\phi(h) dh \geq \int_{\Gamma \setminus G} f_\varepsilon^-(ya(t, w))\Phi_\varepsilon(y) dy.$$

On the other hand, we have

$$\begin{aligned} &\lim_{t \rightarrow \infty} t^{(r-1)/2} e^{(2\rho - \Theta)(\underline{a}(t, w))} \int_{\Gamma \setminus G} f_\varepsilon^\pm(ya(t, w))\Phi_\varepsilon(y) dy \\ &= \kappa_v e^{-I(w)/2} m^{\text{BR}}(f_\varepsilon^\pm) m^{\text{BR}^*}(\Phi_\varepsilon) \\ &= \kappa_v e^{-I(w)/2} m^{\text{BR}}(f_\varepsilon^\pm) \mu_H^{\text{PS}}(\phi)(1 + O(\varepsilon)). \end{aligned}$$

Taking $\varepsilon \rightarrow 0$ in the last equality proves the first statement. The second statement is clear with the choice of $C' = C(f_\varepsilon^+, \Phi_\varepsilon)$, finishing the proof. \square

9. COUNTING IN AFFINE SYMMETRIC SPACES

Let Γ be a Zariski dense Anosov subgroup of G , and let H be a symmetric subgroup of G . We continue to use the notation for $v, \nu, \delta, r, r_0, \Theta$, and $G = H\mathcal{W}\exp(\mathfrak{b}^+)K$, etc. from Section 8; hence $v \in \mathfrak{b}^+ \cap \text{int } \mathcal{L}_\Gamma$ is a unit vector (with respect to the norm $\|\cdot\|$ on \mathfrak{a}). We denote by $|\cdot|$ the norm on \mathfrak{a} induced by an inner product (\cdot, \cdot) with respect to which v and $\ker \Theta$ are orthogonal to each other, and such that $|v| = 1$.

In the following we fix a convex cone $\mathcal{C} \subset \mathfrak{b}^+ \cap (\text{int}(\mathfrak{a}^+) \cup \{0\})$ such that

$$v \in \text{int}_{\mathfrak{b}} \mathcal{C} \quad \text{and} \quad \overline{\mathcal{C}} \cap \ker \Theta = \{0\} \quad (9.1)$$

where $\text{int}_{\mathfrak{b}} \mathcal{C}$ means the interior of \mathcal{C} in the relative topology of \mathfrak{b} . Note that there are convex cones which contain $\mathcal{L}_\Gamma \cap \mathfrak{b}^+$ and satisfy (9.1) by Theorem 7.7 (1) and Lemma 8.4.

Remark 9.2. Note that if $v = u_\Gamma \in \mathfrak{b}^+$, then by Lemma 2.25, $\Theta(w) = \delta_\Gamma \langle u_\Gamma, w \rangle$, hence the cone $\mathfrak{b}^+ \cap (\text{int}(\mathfrak{a}^+) \cup \{0\})$ satisfies the conditions placed on \mathcal{C} above.

By the condition (9.1), we have

$$\mathcal{C} \subset \{tv + \sqrt{t}w : t \geq 0, w \in \ker \Theta\}. \quad (9.3)$$

Denote

$$\mathcal{C}_T := \{w \in \mathcal{C} : |w| < T\} \quad \text{for } T > 1.$$

For $w \in \ker \Theta$, set

$$R_T(w) := \{t \in \mathbb{R} : tv + \sqrt{t}w \in \mathcal{C}_T\}.$$

for all sufficiently large T , $R_T(w)$ is an interval of the form

$$R_T(w) = [t_w, \frac{1}{2} \left(-|w|^2 + \sqrt{|w|^4 + 4T^2} \right)].$$

Lemma 9.4. (1) *There exists $c > 0$ such that for all $w \in \mathfrak{b} \cap \ker \Theta$ and $T > 1$,*

$$e^{-\delta T} T^{(r-r_0)/2} \int_{R_T(w)} t^{\frac{r_0-r}{2}} e^{\delta t} dt \leq \delta^{-1} e^{-c\delta|w|^2}.$$

(2) *We have*

$$\lim_{T \rightarrow \infty} e^{-\delta T} T^{(r-r_0)/2} \int_{R_T(w)} t^{\frac{r_0-r}{2}} e^{\delta t} dt = \delta^{-1} e^{-\delta|w|^2/2}. \quad (9.5)$$

Proof. Note that for any non-zero vector $x \in \mathcal{C}$, $(v, x) > 0$. Since \mathcal{C} is a convex cone with $\overline{\mathcal{C}} \cap \ker \Theta = \{0\}$, it follows that there exists $0 < \theta_0 < \pi/2$ such that the angle between any vector in \mathcal{C} and v is at most θ_0 . Now, as v is perpendicular to $\ker \Theta$ with respect to (\cdot, \cdot) , we have that for any t such that $\underline{a}(t, w) \in \mathcal{C}$,

$$\frac{\sqrt{t}|w|}{t|v|} \leq \tan \theta_0, \quad \text{or, equivalently, } |w|^2 \leq \tan^2 \theta_0 \cdot t.$$

In particular, for $t \in R_T(w)$, we have

$$T^2 \geq t^2 + t|w|^2 \geq \left(\frac{1}{\tan^2 \theta_0} + \frac{1}{\tan^4 \theta_0} \right) |w|^4.$$

This gives the upper bound

$$\frac{-|w|^2 + \sqrt{|w|^4 + 4T^2}}{2} - T = -\frac{|w|^2}{2} + \frac{|w|^4}{2(\sqrt{|w|^4 + 4T^2} + 2T)} \leq -c|w|^2,$$

with

$$c := \frac{1}{2} \left(1 - \left(1 + 4 \left(\frac{1}{\tan^2 \theta_0} + \frac{1}{\tan^4 \theta_0} \right) \right)^{-1/2} \right) > 0. \quad (9.6)$$

Now, by changing variables,

$$\int_{t_w}^{\frac{-|w|^2 + \sqrt{|w|^4 + 4T^2}}{2}} t^{\frac{r_0-r}{2}} e^{\delta t} dt = \int_{t_w-T}^{\frac{-|w|^2 + \sqrt{|w|^4 + 4T^2}}{2} - T} (t+T)^{\frac{r_0-r}{2}} e^{\delta(t+T)} dt,$$

and hence

$$\begin{aligned} & e^{-\delta T} T^{\frac{r-r_0}{2}} \int_{t_w}^{\frac{-|w|^2 + \sqrt{|w|^4 + 4T^2}}{2}} t^{\frac{r_0-r}{2}} e^{\delta t} dt \\ &= \int_{t_w-T}^{\frac{-|w|^2 + \sqrt{|w|^4 + 4T^2}}{2} - T} \left(\frac{t}{T} + 1 \right)^{\frac{r_0-r}{2}} e^{\delta t} dt \\ &\leq \int_{-\infty}^{-c|w|^2} e^{\delta t} dt = \frac{1}{\delta} e^{-c\delta|w|^2}, \end{aligned} \quad (9.7)$$

which proves (1).

The second claim (2) follows as well, because by the dominated convergence theorem, (9.7) converges to

$$\int_{-\infty}^{-\frac{|w|^2}{2}} e^{\delta t} dt = \frac{1}{\delta} e^{-\frac{\delta|w|^2}{2}}.$$

□

We fix a left $(H \cap M)$ -invariant function $\tau_H \in C_c(H)$ with its support injecting to $\Gamma \backslash G$, and a right M -invariant function $\tau_K \in C(K)$. Define a function $Z_T : G \rightarrow \mathbb{R}$ as follows:

$$Z_T(g) := \begin{cases} \tau_H(h) \mathbf{1}_{\mathcal{C}_T}(\log b) \tau_K(k) & g = hbk \in H \exp(\mathcal{C})K, \\ 0 & g \notin H \exp(\mathcal{C})K. \end{cases}$$

Since $\mathcal{C} \subset \text{int}(\mathfrak{a}^+) \cup \{0\}$, hM and Mk are uniquely determined and hence Z_T is well-defined.

Definition 9.8 (Bi-sector counting function). For $T > 0$, define $F_T = F_{T, \tau_H, \tau_K} : \Gamma \backslash G \rightarrow \mathbb{R}$ by

$$F_T([g]) = \sum_{\gamma \in \Gamma} Z_T(\gamma g). \quad (9.9)$$

For $\Phi \in C_c(\Gamma \backslash G)$ and a left M -invariant Borel function f on K , we define the following M -invariant function on $\Gamma \backslash G$: for $x \in \Gamma \backslash G$,

$$\Phi * f(x) := \int_{k \in K} \Phi(xk) f(k) dk.$$

In the definition (9.11) below, the integral over the trivial subspace $\{0\}$ should be interpreted as $\int_0 f(w) dw = f(0)$. In particular, if $\mathfrak{b} \cap \ker \Theta = \{0\}$, then $c_v = \frac{s_v \kappa_v}{\delta}$.

Proposition 9.10. *Let $\Phi \in C_c(\Gamma \backslash G)$. As $T \rightarrow \infty$, we have*

$$\langle F_T, \Phi \rangle \sim c_v e^{\delta T} T^{(r_0 - r)/2} \mu_H^{\text{PS}}(\tau_H) m^{\text{BR}}(\Phi * \tau_K)$$

where c_v is given by

$$c_v := \frac{s_v \kappa_v}{\delta} \int_{\mathfrak{b} \cap \ker \Theta} e^{-(I(w) + \delta|w|^2)/2} dw; \quad (9.11)$$

here κ_v and $I(w)$ are as in Theorem 7.4 and in (7.5) respectively, and $s_v = \frac{1}{|\det S_v|}$, where $S_v : \mathfrak{a} \rightarrow \mathfrak{a}$ is any linear map such that $S_v|_{\ker \Theta} = \text{Id}$ and $S_v v$ is a unit vector orthogonal to $\ker \Theta$ with respect to the inner product on \mathfrak{a} induced by the Killing form.

Proof. In view of the decomposition (8.1), we will need the following formula for the Haar measure dg on G ; for all $\phi \in C_c(G)$,

$$\int_G \phi(g) dg = \sum_{w \in \mathcal{W}} \int_H \int_K \left(\int_{\mathfrak{b}^+} \phi(hw(\exp b)k) \xi(b) db \right) dk dh, \quad (9.12)$$

where $\xi : \mathfrak{b} \rightarrow \mathbb{R}$ is given by

$$\xi(b) = \prod_{\alpha \in \Sigma_{\sigma}^{\pm}} (\sinh \alpha(b))^{\ell_{\alpha}^+} (\cosh \alpha(b))^{\ell_{\alpha}^-};$$

here $\ell_{\alpha}^{\pm} := \dim(\mathfrak{g}_{\alpha}^{\pm})$, where each $\mathfrak{g}_{\alpha}^{\pm}$ is the ± 1 eigenspace of the root space \mathfrak{g}_{α} with respect to the involution $\theta\sigma$ (cf. [53], [16, p.18]).

Substituting $b = \underline{a}(t, w)$ for $t \geq 0$ and $w \in \mathfrak{b} \cap \ker \Theta$ gives $db = s_v t^{\frac{r_0 - 1}{2}} dt dw$. Now,

$$\langle F_T, \Phi \rangle = \int_K \tau_K(k) \int_{b \in \mathcal{C}_T} \left(\int_{[e]H} \Phi([h](\exp b)k) \tau_H(h) d[h] \right) \xi(b) db dk. \quad (9.13)$$

Hence

$$\begin{aligned} & \langle F_T, \Phi \rangle \\ &= \int_{\mathfrak{b} \cap \ker \Theta} \int_{t \in R_T(w)} t^{\frac{r_0-1}{2}} \xi(\underline{a}(t, w)) \left(\int_{[e]H} (\Phi * \tau_K)([h]a(t, w)) \tau_H(h) d[h] \right) s_v dt dw \\ &= s_v e^{\delta T} T^{(r_0-r)/2} \int_{\mathfrak{b} \cap \ker \Theta} p_T(w) dw, \end{aligned}$$

where we define $p_T(w)$ to be

$$e^{-\delta T} T^{(r-r_0)/2} \int_{R_T(w)} t^{\frac{r_0-1}{2}} \xi(\underline{a}(t, w)) \left(\int_{[e]H} (\Phi * \tau_K)([h]a(t, w)) \tau_H(h) d[h] \right) dt. \quad (9.14)$$

We next look for an integrable function on $\mathfrak{b} \cap \ker \Theta$ that bounds the family of functions $p_T(w)$ from above, in order to apply the dominated convergence theorem. Note that there exists a constant $c_1 > 0$ such that for all (t, w) with $\underline{a}(t, w) \in \mathcal{C}$,

$$e^{-2\rho(\underline{a}(t, w))} \xi(\underline{a}(t, w)) \leq c_1, \quad (9.15)$$

and by Proposition 8.10, we may assume

$$\left| t^{\frac{r-1}{2}} e^{2\rho(\underline{a}(t, w)) - \delta t} \int_{[e]H} (\Phi * \tau_K)([h]a(t, w)) \tau_H(h) d[h] \right| \leq c_1$$

as well. Hence

$$p_T(w) \leq c_1^2 \psi(w),$$

where $\psi(w) = \delta^{-1} e^{-c\delta|w|^2}$ is as given in Lemma 9.4. Since ψ is integrable over $\mathfrak{b} \cap \ker \Theta$, we may apply Proposition 8.10 and the dominated convergence theorem to deduce that

$$\begin{aligned} & \int_K \tau_K(k) \int_{b \in \mathcal{C}_T} \left(\int_{[e]H} \Phi([h](\exp b)k) \tau_H(h) d[h] \right) \xi(b) db dk \\ & \sim s_v \kappa_v e^{\delta T} T^{(r_0-r)/2} \cdot \mu_H^{\text{PS}}(\tau_H) \int_{w \in \mathfrak{b} \cap \ker \Theta} \delta^{-1} e^{-\delta|w|^2/2} e^{-I(w)/2} m^{\text{BR}}(\Phi * \tau_K) dw \\ & = c_v e^{\delta T} T^{(r_0-r)/2} \mu_H^{\text{PS}}(\tau_H) m^{\text{BR}}(\Phi * \tau_K), \end{aligned}$$

with c_v as given in the statement of the proposition. \square

We now fix a left $(H \cap M)$ -invariant compact subset $\Omega_H \subset H$ and a right M -invariant compact subset $\Omega_K \subset K$. Let $\varepsilon > 0$ be a number smaller than the injectivity radius at $[e]$ in $\Gamma \backslash G$. There exists a symmetric neighborhood \mathcal{O}_ε of e in G such that for all $T \geq 1$,

$$\begin{aligned} \Omega_{H, \varepsilon}^- \exp(\mathcal{C}_{T, \varepsilon}^-) \Omega_{K, \varepsilon}^- & \subset \bigcap_{g \in \mathcal{O}_\varepsilon} \Omega_H \exp(\mathcal{C}_T) \Omega_K g, \text{ and} \\ \Omega_H \exp(\mathcal{C}_T) \Omega_K \mathcal{O}_\varepsilon & \subset \Omega_{H, \varepsilon}^+ \exp(\mathcal{C}_{T, \varepsilon}^+) \Omega_{K, \varepsilon}^+. \end{aligned} \quad (9.16)$$

where $\mathcal{C}_{T,\varepsilon}^+ = \mathcal{C}_T + \mathfrak{b}_\varepsilon$, $\mathcal{C}_{T,\varepsilon}^- = \cap_{b \in \mathfrak{b}_\varepsilon} (\mathcal{C}_T + b)$, $\Omega_{K,\varepsilon}^+ = \Omega_K K_\varepsilon$, $\Omega_{K,\varepsilon}^- = \cap_{k \in K_\varepsilon} \Omega_K k$, and $\Omega_{H,\varepsilon}^\pm$ are defined similarly (see [16], [17]). We will additionally fix convex cones $\mathcal{C}^\flat, \mathcal{C}^\sharp \subset \mathfrak{b}^+ \cap (\text{int}(\mathfrak{a}^+) \cup \{0\})$ such that $\mathcal{C}^\flat \subset \text{int}_\mathfrak{b} \mathcal{C}$, $\mathcal{C} \subset \text{int}_\mathfrak{b} \mathcal{C}^\sharp$, and which satisfy the condition (9.1). Note that $\mathcal{C}_{T,\varepsilon}^+$ is no longer contained in \mathcal{C} , but there exists $T_0 > 0$ (independent of $0 < \varepsilon < 1$) such that

$$\mathcal{C}_{T,\varepsilon}^+ - \mathcal{C}_{T_0,\varepsilon}^+ \subset \mathcal{C}_{T+\varepsilon}^\sharp \quad \text{and} \quad \mathcal{C}_{T-\varepsilon}^\flat - \mathcal{C}_{T_0}^\flat \subset \mathcal{C}_{T,\varepsilon}^- \quad (9.17)$$

Choose a nonnegative function $\phi_\varepsilon \in C_c(G)$ such that $\int_G \phi_\varepsilon(g) dg = 1$ and $\text{supp}(\phi_\varepsilon) \subset \mathcal{O}_\varepsilon$. Define $\Phi_\varepsilon : \Gamma \backslash G \rightarrow \mathbb{R}$ by

$$\Phi_\varepsilon([g]) = \sum_{\gamma \in \Gamma} \phi_\varepsilon(\gamma g). \quad (9.18)$$

Lemma 9.19. *Let $\tau \in C(K)$ be left M -invariant. Then*

$$\lim_{\varepsilon \rightarrow 0} m^{\text{BR}}(\Phi_\varepsilon * \tau) = \int_K \tau(k^{-1}) \mu_{K,i(v)}^{\text{PS},*}(k).$$

where $\mu_{K,i(v)}^{\text{PS},*} := \tilde{\nu}_{\Theta \circ i}$ is given in (4.10).

Proof. Set $\tilde{\nu} := \tilde{\nu}_{\Theta \circ i}$. We use Lemma 4.9 and write

$$\begin{aligned} m^{\text{BR}}(\Phi_\varepsilon * \tau) &= \int_K \int_G \phi_\varepsilon(gk) \tau(k) d\tilde{m}^{\text{BR}}(g) dk \\ &= \int_K \int_G \phi_\varepsilon(k' \exp(q)nk) \tau(k) e^{-\Theta(q)} dn dq d\tilde{\nu}(k') dk. \end{aligned}$$

Substituting $g = \exp(q)nk \in AN^+K$, the density of the Haar measure is given by

$$dg = e^{-2\rho(q)} dn dq dk.$$

For $g \in G$, let $\kappa(g)$ denote the K -component of g , and a_g denote the logarithm of A -component of g , in the decomposition $G = AN^+K$. Then

$$\begin{aligned} m^{\text{BR}}(\Phi_\varepsilon * \tau) &= \int_K \int_G \phi_\varepsilon(k'g) \tau(\kappa(g)) e^{(2\rho-\Theta)(a_g)} dg d\tilde{\nu}(k') \\ &= \int_K \int_G \phi_\varepsilon(g) \tau(\kappa(k^{-1}g)) e^{(2\rho-\Theta)(a_{k^{-1}g})} dg d\tilde{\nu}(k). \end{aligned}$$

By shrinking \mathcal{O}_ε if necessary, we can assume that for all $k \in K$,

$$\kappa(k^{-1}\mathcal{O}_\varepsilon) \subset k^{-1}K_\varepsilon.$$

By the uniform continuity of τ , there exist positive $\eta = \eta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that for all $g \in \mathcal{O}_\varepsilon$ and $k \in K$,

$$\tau(k^{-1}) - \eta \leq \tau(\kappa(k^{-1}g)) \leq \tau(k^{-1}) + \eta.$$

It follows from the fact that the multiplication map $A \times N^+ \times K \rightarrow G$ is a diffeomorphism that for some $C > 1$, we have that for all $g \in \mathcal{O}_\varepsilon$ and $k \in K$,

$$1 - C\varepsilon \leq e^{(2\rho-\Theta)(a_{k^{-1}g})} \leq 1 + C\varepsilon.$$

Since $\int_G \phi_\varepsilon dg = 1$, we get

$$(1-C\varepsilon) \int_K (\tau(k^{-1})-\eta) d\tilde{\nu}(k) \leq m^{\text{BR}}(\Phi_\varepsilon * \tau) \leq (1+C\varepsilon) \int_K (\tau(k^{-1})+\eta) d\tilde{\nu}(k).$$

The claim now follows from letting $\varepsilon \rightarrow 0$. \square

Corollary 9.20. *Let $\mathcal{C} \subset \mathfrak{b}^+ \cap (\text{int } \mathfrak{a}^+ \cup \{0\})$ be a convex cone satisfying (9.1). If $\mu_{K,i(v)}^{\text{PS},*}(\partial\Omega_K^{-1}) = \mu_H^{\text{PS}}(\partial\Omega_H) = 0$, then*

$$\lim_{T \rightarrow \infty} \frac{\#(\Gamma \cap \Omega_H \exp(\mathcal{C}_T) \Omega_K)}{e^{\delta T} T^{(r_0-r)/2}} = c_v \mu_H^{\text{PS}}(\Omega_H) \mu_{K,i(v)}^{\text{PS},*}(\Omega_K^{-1}).$$

If $v = u_\Gamma \in \mathfrak{b}^+$, then we may take \mathcal{C} to be \mathfrak{b}^+ .

Proof. Write $\tilde{\nu} := \mu_{K,i(v)}^{\text{PS},*}$ for simplicity. For $g \in H(B^+ \cap \text{int } A^+)K$, let $g = h_g(\exp b_g)k_g$ denote the HB^+K decomposition of $g \in G$; note that $h_g(H \cap M)$, b_g and Mk_g are uniquely defined.

Since $\Omega_H(H \cap M) = \Omega_H$ and $M\Omega_K = \Omega_K$, we may define $X_T : G \rightarrow \mathbb{R}$ by

$$X_T(g) = \mathbf{1}_{\Omega_H}(h_g) \mathbf{1}_{\mathcal{C}_T}(b_g) \mathbf{1}_{\Omega_K}(k_g)$$

for $g \in H \exp \mathcal{C}^\sharp K$, and $X_T(g) = 0$ otherwise. Let $\phi_{K,\varepsilon} \in C(K_\varepsilon)^M$ and $\phi_{H,\varepsilon} \in C(H_\varepsilon)^{H \cap M}$ be non-negative functions with integral one. Set

$$\begin{aligned} \tau_{K,\varepsilon}^\pm &:= \mathbf{1}_{\Omega_{K,2\varepsilon}^\pm} * \phi_{K,\varepsilon}, \quad \tau_{H,\varepsilon}^\pm := \mathbf{1}_{\Omega_{H,2\varepsilon}^\pm} * \phi_{H,\varepsilon}, \quad \text{and} \\ F_{T+\varepsilon}^+ &:= F_{\mathcal{C}_{T+\varepsilon}^\sharp, \tau_{H,3\varepsilon}^\pm, \tau_{K,3\varepsilon}^\pm}, \quad F_{T-\varepsilon}^- := F_{\mathcal{C}_{T-\varepsilon}^\flat, \tau_{H,3\varepsilon}^-, \tau_{K,3\varepsilon}^-}. \end{aligned}$$

Note that by definition,

$$\mathbf{1}_{\Omega_{K,\varepsilon}^+} \leq \tau_{K,\varepsilon}^+ \leq \mathbf{1}_{\Omega_{K,3\varepsilon}^+}, \quad \text{and} \quad \mathbf{1}_{\Omega_{K,3\varepsilon}^-} \leq \tau_{K,\varepsilon}^- \leq \mathbf{1}_{\Omega_{K,\varepsilon}^-}. \quad (9.21)$$

By (9.16), (9.17) and (9.21) there exists a uniform constant $C > 0$ such that for all $g \in \mathcal{O}_\varepsilon$,

$$F_{T-\varepsilon}^-([g]) - C \leq \sum_{\gamma \in \Gamma} X_T(\gamma) \leq F_{T+\varepsilon}^+([g]) + C.$$

Integrating this against Φ_ε given in (9.18), we get

$$\langle F_{T-\varepsilon}^-, \Phi_\varepsilon \rangle - C \leq \sum_{\gamma \in \Gamma} X_T(\gamma) \leq \langle F_{T+\varepsilon}^+, \Phi_\varepsilon \rangle + C.$$

For simplicity, we set $x_T := \sum_{\gamma \in \Gamma} X_T(\gamma) = \#\Gamma \cap \Omega_H \exp(\mathcal{C}_T) \Omega_K$. Hence by Proposition 9.10, fixing $\varepsilon_0 > 0$, for all $0 < \varepsilon < \varepsilon_0$, we have

$$\begin{aligned} \limsup_{T \rightarrow \infty} \frac{x_T}{e^{\delta T} T^{(r_0-r)/2}} &\leq c_v \mu_H^{\text{PS}}(\tau_{H,3\varepsilon}^+) m^{\text{BR}}(\Phi_\varepsilon * \tau_{K,3\varepsilon}^+) \\ &\leq c_v \mu_H^{\text{PS}}(\tau_{H,3\varepsilon}^+) m^{\text{BR}}(\Phi_\varepsilon * \tau_{K,9\varepsilon_0}^+). \end{aligned} \quad (9.22)$$

On the other hand, since $\tilde{\nu}(\partial\Omega_K^{-1}) = \mu_H^{\text{PS}}(\partial\Omega_H) = 0$, we get from (9.21) that

$$\lim_{\varepsilon \rightarrow 0} \mu_H^{\text{PS}}(\tau_{H,3\varepsilon}^\pm) = \mu_H^{\text{PS}}(\Omega_H) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \int_K \tau_{K,\varepsilon}^\pm(k^{-1}) d\tilde{\nu}(k) = \tilde{\nu}(\Omega_K^{-1}). \quad (9.23)$$

Now letting $\varepsilon \rightarrow 0$ first using Lemma 9.19 and (9.23), and then letting $\varepsilon_0 \rightarrow 0$, we get

$$\limsup_{T \rightarrow \infty} \frac{x_T}{e^{\delta T} T^{(r_0-r)/2}} \leq c_v \mu_H^{\text{PS}}(\Omega_H) \tilde{\nu}(\Omega_K^{-1}).$$

Similarly, we can show that

$$\liminf_{T \rightarrow \infty} \frac{x_T}{e^{\delta T} T^{(r_0-r)/2}} \geq c_v \mu_H^{\text{PS}}(\Omega_H) \tilde{\nu}(\Omega_K^{-1}),$$

which proves the corollary. \square

Remark 9.24. Note that this corollary implies that the asymptotic of $\#(\Gamma \cap \Omega_H \exp(\mathcal{C}_T) \Omega_K)$ is independent of \mathcal{C} .

Proof of Theorem 1.9. Theorem 1.9 now follows directly from applying Corollary 9.20 and the following observation to a convex cone \mathcal{C} such that $\mathfrak{b} \cap \mathcal{L}_\Gamma \subset \text{int}_\mathfrak{b} \mathcal{C}$; by Lemma 8.4, such a cone always exists.

Lemma 9.25. *Suppose that $\mathfrak{b}^+ \cap \mathcal{L}_\Gamma \subset \text{int}_\mathfrak{b} \mathcal{C}$. Then*

$$\#(\Gamma \cap \Omega_H \exp(\mathfrak{b}^+ - \mathcal{C}) \Omega_K) < \infty.$$

Proof. We can find a smaller closed convex cone $\mathcal{C}' \subset \text{int}_\mathfrak{b} \mathcal{C}$ such that $\mathfrak{b}^+ \cap \mathcal{L}_\Gamma \subset \text{int}_\mathfrak{b} \mathcal{C}'$. Set $\mathcal{Q} := \mathfrak{b}^+ - \mathcal{C}$, $\mathcal{Q}' := \mathfrak{b}^+ - \mathcal{C}'$, and $\Omega'_H := \Omega_H H_1$ where H_1 means the unit neighborhood of e in H . We can find a bi- K -invariant neighborhood $\mathcal{O} \subset G$ of e such that for all $g \in \mathcal{O}$,

$$\Omega_H \exp(\mathcal{Q}_T) K \subset \Omega'_H \exp(\mathcal{Q}'_{T+1}) K g^{-1}.$$

Define $G_T([g]) = \sum_{\gamma \in \Gamma} \mathbf{1}_{\Omega'_H \exp(\mathcal{Q}'_T) K}(\gamma g)$. Now,

$$\#(\Gamma \cap \Omega_H \exp(\mathfrak{b}^+ - \mathcal{C}) \Omega_K) = \sum_{\gamma \in \Gamma} \mathbf{1}_{\Omega_H \exp(\mathcal{Q}_T) K}(\gamma) \leq \langle G_{T+1}, \Phi \rangle,$$

where Φ is a non-negative K -invariant continuous function supported in $[e]\mathcal{O}$ and $\int_{\Gamma \backslash G} \Phi dg = 1$. Now note that

$$\langle G_T, \Phi \rangle = \int_{b \in \mathcal{Q}'_T} \left(\int_{[e]\Omega'_H} \Phi([h](\exp b)) d[h] \right) \xi(b) db.$$

Recalling Ω'_H is $H \cap M$ -invariant, let $0 \leq \psi \leq 1$ be an element of $C_c(H)^{H \cap M}$ which is one on Ω'_H . For $\varepsilon_0 > 0$ smaller than the injectivity radius of $\text{supp}(\psi)$, let $\Psi = \psi \otimes \rho_{\varepsilon_0}$ be given as (8.8). Let $\Phi_{\varepsilon_0}^+(y) = \sup_{g \in N_{\varepsilon_0} A_{\varepsilon_0} N_{\varepsilon_0}^+ M} \Phi(yg)$ be as in (8.12). Since the closure of \mathcal{Q}' is disjoint from $\mathfrak{b}^+ \cap \mathcal{L}_\Gamma$, by Proposition 2.20, there exists $T_0 > 0$ such that $\langle (\exp b) \Phi_{\varepsilon_0}^+, \Psi \rangle = 0$ for all $b \in \mathcal{Q}' - \mathcal{Q}'_{T_0}$.

Then by the same argument as in the proof of Proposition 8.10, we get that for all $b \in \mathcal{Q}' - \mathcal{Q}'_{T_0}$,

$$\begin{aligned} \int_{[e]H} \Phi([h](\exp b))\psi[h]dh &= \int_{[e]H(NB)_{\varepsilon_0}} \Phi([h](\exp b))\Psi([h]nb)dndbdh \\ &\leq \langle (\exp b)\Phi_{\varepsilon_0}^+, \Psi \rangle = 0. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathcal{Q}'} \int_{[e]H} \Phi([h](\exp b))\psi[h]dh\xi(b) db &= \int_{\mathcal{Q}'_{T_0}} \langle (\exp b)\Phi_{\varepsilon_0}^+, \Psi \rangle \xi(b) db \quad (9.26) \\ &\leq \|\Phi_{\varepsilon_0}^+\|_2 \|\Psi\|_2 \text{Vol}(\mathcal{Q}'_{T_0}). \end{aligned}$$

Hence for all $T > T_0$,

$$\#(\Gamma \cap \Omega_H \exp(\mathfrak{b}_T^+ - \mathcal{C})\Omega_K) \leq \langle G_{T+1}, \Phi \rangle \leq \|\Phi_{\varepsilon_0}^+\|_2 \|\Psi\|_2 \text{Vol}(\mathcal{Q}'_{T_0}).$$

This implies the claim in view of Corollary 9.20. \square

For $\omega \in \mathcal{W}$, set $\Gamma^\omega = \omega^{-1}\Gamma\omega$, $H^\omega = \omega^{-1}H\omega$, and $\Omega_{H^\omega} = \omega^{-1}\Omega_H\omega \subset H^\omega$. Then

$$\#(\Gamma \cap \Omega_H \omega \exp(\mathcal{C}_T)\Omega_K) = \#(\Gamma^\omega \cap \Omega_{H^\omega} \exp(\mathcal{C}_T)\Omega_K).$$

Since $\omega \in K$, it follows that $\psi_\Gamma = \psi_{\Gamma^\omega}$, and that the involution which stabilizes H^ω commutes with θ . Hence

$$\delta = \psi_{\Gamma^\omega}(v) = \max_{b \in \mathfrak{b}, \|b\|=1} \psi_{\Gamma^\omega}(b).$$

By applying Corollary 9.20 to Γ^ω and H^ω for each $\omega \in \mathcal{W}$, we can also deduce the asymptotic of $\#(\Gamma \cap \Omega_H \mathcal{W} \exp(\mathcal{C}_T)\Omega_K)$.

Theorem 1.11 follows from the following: we set $\text{sk}_{\Gamma,v}(H) = |\mu_{[e]H,v}^{\text{PS}}|$

Theorem 9.27. *Suppose that $\mathfrak{v}_0\Gamma \subset H \backslash G$ is discrete for $\mathfrak{v}_0 = [H]$ and that $[e]H$ is uniformly proper. Then $\text{sk}_{\Gamma,v}(H) < \infty$ and there exists $c_v > 0$ such that*

$$\lim_{T \rightarrow \infty} \frac{\#(\mathfrak{v}_0\Gamma \cap \mathfrak{v}_0 \exp(\mathfrak{b}^+ \cap \mathcal{L}_\Gamma)_T K)}{e^{\psi_\Gamma(v)T} T^{(r_0-r)/2}} = c_v \text{sk}_{\Gamma,v}(H) \quad (9.28)$$

Moreover, for $v = u_\Gamma$, we have

$$\lim_{T \rightarrow \infty} \frac{\#(\mathfrak{v}_0\Gamma \cap \mathfrak{v}_0(\exp \mathfrak{b}_T^+)K)}{e^{\psi_\Gamma(u_\Gamma)T} T^{(r_0-r)/2}} = c_{u_\Gamma} \text{sk}_{\Gamma,u_\Gamma}(H) \quad (9.29)$$

where $\mathfrak{b}_T^+ = \{w \in \mathfrak{b}^+ : \|w\| \leq T\}$.

Proof. Set $\mathcal{C} := \mathfrak{b}^+ \cap \mathcal{L}_\Gamma$. By Theorem 7.7 (1) and Lemma 8.4, \mathcal{C} satisfies the conditions in (9.1). By hypothesis, there exists an open neighborhood $\mathcal{O} \subset \Gamma \backslash G$ of $[e]$ such that $Y_0 := \{[e]h \in \Gamma \backslash \Gamma H : [e]h \exp \mathcal{C} \cap \mathcal{O}K \neq \emptyset\}$ is bounded. Set

$$F_T([g]) := \sum_{\gamma \in (\Gamma \cap H) \backslash \Gamma} \mathbf{1}_{\mathfrak{v}_0 \exp \mathcal{C}_T K}(\mathfrak{v}_0 \gamma g).$$

Let $\mathcal{O}_\varepsilon \subset G$ and Φ_ε be as in (9.18). We may assume Φ_ε is K -invariant as our functions F_T are K -invariant in deducing the following:

$$\langle F_{T-\varepsilon}, \Phi_\varepsilon \rangle \leq F_T([e]) \leq \langle F_{T+\varepsilon}, \Phi_\varepsilon \rangle. \quad (9.30)$$

Observe that

$$\langle F_{T\pm\varepsilon}, \Phi_\varepsilon \rangle = \int_K \int_{b \in \mathcal{C}_{T\pm\varepsilon}} \left(\int_{[e]H} \Phi_\varepsilon([h](\exp bk)) d[h] \right) \xi(b) db dk.$$

For $S > 0$, let $\tau_S \in C_c([e]H)$ be a function satisfying $0 \leq \tau \leq 1$ and $\tau_S = 1$ on the S -neighborhood of Y_0 . By the definition of Y_0 , it follows that

$$\langle F_{T\pm\varepsilon}, \Phi_\varepsilon \rangle = \int_K \int_{b \in \mathcal{C}_{T\pm\varepsilon}} \left(\int_{[e]H} \Phi_\varepsilon([h](\exp bk)) \tau_S([h]) d[h] \right) \xi(b) db dk.$$

Note that this integral is same as the one in (9.13), as τ_S is compactly supported. Hence we get

$$\lim_{T \rightarrow \infty} \frac{\langle F_{T\pm\varepsilon}, \Phi_\varepsilon \rangle}{e^{\delta(T\pm\varepsilon)T(r_0-r)/2}} = c_v \mu_{[e]H}^{\text{PS}}(\tau_S) m^{\text{BR}}(\Phi_\varepsilon).$$

By sending $\varepsilon \rightarrow 0$ and applying Lemma 9.19, we get from (9.30)

$$\lim_{T \rightarrow \infty} \frac{F_T([e])}{e^{\delta T T(r_0-r)/2}} = c_v \mu_{[e]H}^{\text{PS}}(\tau_S).$$

It follows that $\mu_{[e]H}^{\text{PS}}(\tau_S)$ is a constant function of $S > 0$, and hence

$$|\mu_{[e]H}^{\text{PS}}| = \mu_{[e]H}^{\text{PS}}(\tau_S) < \infty.$$

This proves the first claim. The second claim follows from the first one in view of the following lemma by taking $\mathcal{C} = \mathfrak{b}^+ \cap \mathcal{L}_\Gamma$. \square

Lemma 9.31. *Let $\mathcal{C} \subset \mathfrak{b}^+$ be a convex cone with $v = u_\Gamma \in \text{int}_\mathfrak{b} \mathcal{C}$. Set $\mathcal{Q} := \mathfrak{b}^+ - \mathcal{C}$. Then there exist $0 < \delta' < \delta_\Gamma = \psi_\Gamma(u_\Gamma)$ and $C > 0$ such that for all $T \geq 1$,*

$$\#(\Gamma \cap \Omega_H \exp(\mathcal{Q}_T) \Omega_K) \leq C e^{\delta' T}.$$

Proof. Choose a closed convex cone $\mathcal{C}' \subset \text{int}_\mathfrak{b} \mathcal{C}$ such that $u_\Gamma \in \text{int}_\mathfrak{b} \mathcal{C}'$ and set $\mathcal{Q}' := \mathfrak{b}^+ - \mathcal{C}'$, and $\Omega'_H := \Omega_H H_1$ where H_1 means the unit neighborhood of e in H . We can find a bi- K -invariant neighborhood $\mathcal{O} \subset G$ of e such that for all $g \in \mathcal{O}$,

$$\Omega_H \exp(\mathcal{Q}_T) K \subset \Omega'_H \exp(\mathcal{Q}'_{T+1}) K g^{-1}.$$

Define $G_T([g]) = \sum_{\gamma \in \Gamma} \mathbf{1}_{\Omega'_H \exp(\mathcal{Q}'_T) K}(\gamma g)$. Now,

$$\#(\Gamma \cap \Omega_H \exp(\mathcal{Q}_T) \Omega_K) = \sum_{\gamma \in \Gamma} \mathbf{1}_{\Omega_H \exp(\mathcal{Q}_T) K}(\gamma) \leq \langle G_{T+1}, \Phi \rangle, \quad (9.32)$$

where Φ is a non-negative K -invariant continuous function supported in $[e]\mathcal{O}$ and $\int_{\Gamma \backslash G} \Phi dg = 1$. Now note that

$$\langle G_T, \Phi \rangle = \int_{b \in \mathcal{Q}'_T} \left(\int_{[e]\Omega'_H} \Phi([h](\exp b)) d[h] \right) \xi(b) db.$$

If we write $b = tu_\Gamma + \sqrt{t}w \in \mathcal{Q}'_T$ with $w \in \ker \Theta$, then $t^2 + t\|w\|^2 \leq T^2$ and $\|w\|^2 \geq \tan^2 \theta_0 \cdot t$ for some $0 < \theta_0 < \pi/2$ depending on the distance between u_Γ and \mathcal{Q}' (cf. proof of Lemma 9.4). Hence if $b = tu_\Gamma + \sqrt{t}w \in \mathcal{Q}'_T$, then

$$0 \leq t \leq T \cdot \cos \theta_0. \quad (9.33)$$

Fix a non-negative function $\phi \in C_c(H)$ which is 1 on Ω'_H . Since (8.11) gives that

$$\left| t^{(r-1)/2} e^{(2\rho-\Theta)(\underline{a}(t,w))} \int_H \Phi([h]a(t,w)) \phi(h) dh \right| < C',$$

it follows that for all $t \geq 1$,

$$\left| \int_H \Phi([h]a(t,w)) \phi(h) dh \right| < C' e^{(\Theta-2\rho)(\underline{a}(t,w))}. \quad (9.34)$$

Using (9.15), (9.33), and $\Theta(u_\Gamma) = \delta_\Gamma$, we deduce that

$$\begin{aligned} & \int_{b \in \mathcal{Q}'_T} \left(\int_H \Phi([h]a(t,w)) \phi(h) dh \right) \xi(b) db \\ & \ll \int_{b \in \mathcal{Q}'_T} e^{\delta_\Gamma t} db \leq e^{(\delta_\Gamma \cos \theta_0)T} \text{Vol}(\mathfrak{b}_T). \end{aligned}$$

As $\text{Vol}(\mathfrak{b}_T) = O(T^{r_0})$, for any δ' satisfying $\delta_\Gamma \cos \theta_0 < \delta' < \delta_\Gamma$, we get

$$\int_{b \in \mathcal{Q}'_T} \left(\int_H \Phi([h]a(t,w)) \phi(h) dh \right) \xi(b) db \ll e^{\delta' T}.$$

This proves the lemma by (9.32). \square

Finally we give examples satisfying the hypothesis of Theorem 9.27:

Lemma 9.35. *Suppose that $\mathfrak{a} = \mathfrak{b}$ and $\Lambda_\Gamma \subset HP/P$.*

- (1) *The orbit $[e]H$ is uniformly proper.*
- (2) *The support of $\mu_{[e]H}^{\text{PS}}$ is a compact subset of $(H \cap \Gamma) \backslash H$.*

Proof. The condition $\mathfrak{a} = \mathfrak{b}$ means $H \cap A = \{e\}$, and hence $H \cap P$ is compact by (8.6). Since Λ_Γ is a compact subset of $HP/P \simeq H/(H \cap P)$, it follows that $\Lambda_\Gamma \subset H_0P/P$ for some compact subset $H_0 \subset H$.

To show (1), let \mathcal{C} be a closed cone contained in $\text{int } \mathfrak{a}^+ \cup \{0\}$. It suffices to show that for any given compact subset $Z \subset \Gamma \backslash G$,

$$\{[e]h \in \Gamma \backslash \Gamma H : [e]h \exp \mathcal{C} \cap Z \neq \emptyset\}$$

is bounded. Suppose not; then there exist $h_i \in H$ with $[e]h_i \rightarrow \infty$ in $\Gamma \backslash \Gamma H$, $\gamma_i \in \Gamma$ and $c_i \in \exp \mathcal{C}$ such that $\gamma_i h_i c_i$ converges to some $g \in G$. Since $[e]H \subset \Gamma \backslash G$ is a closed subset, it follows that $c_i \rightarrow \infty$. We may write $h_i c_i = \gamma_i^{-1} g g_i$ where $g_i \rightarrow e$. Set $o = [K] \in G/K$. By passing to a subsequence, we may assume that $\gamma_i^{-1} g g_i o$ converges to some $\xi \in \mathcal{F}$ in the sense of [31, Def.2.7]; note here that $\mu(\gamma_i^{-1} g g_i) \rightarrow \infty$ regularly in \mathfrak{a}^+ as $\mathcal{L}_\Gamma \subset \text{int } \mathfrak{a}^+$. Since $g g_i$ is bounded, it follows that $\xi \in \Lambda_\Gamma$ by [31, Lemma 2.12]. Since $h_i c_i \eta = \gamma_i^{-1} g g_i \eta \rightarrow \xi$ for all $\eta \in \mathcal{F}$ except for points on a proper submanifold

of \mathcal{F} , we may choose $\eta = ne^+ \in N^+e^+$ so that $\lim_{i \rightarrow \infty} h_i c_i n e^+ = \xi$. Writing $\xi = h_0 e^+$ for some $h_0 \in H_0$, we have $\lim_{i \rightarrow \infty} h_i c_i n c_i^{-1} P = h_0 P$. Since $c_i \in \infty$ in $\exp \mathcal{C}$ and $\mathcal{C} \subset \text{int } \mathfrak{a}^+ \cup \{0\}$, we have $\lim_{i \rightarrow \infty} c_i n c_i^{-1} = e$. As HP is open, we may write $c_i n c_i^{-1} = h'_i p_i \in HP$ with $h'_i \in H, p_i \in P$ both tending to e . It follows that $h_i h'_i P \rightarrow h_0 P$ as $i \rightarrow \infty$. Since $H \cap P$ is compact, and the sequence h'_i is bounded, it follows that the sequence h_i is bounded, yielding a contradiction. This proves Claim (1).

Since $\text{supp}(\mu_{[e]H}^{\text{PS}}) = \{[e]h \in [e]H : he^+ \in \Lambda_\Gamma\} \subset \{[e]h_0 : h_0 \in H_0(H \cap P)\}$, Claim (2) follows. \square

We remark that this lemma holds when H is replaced by N^+ by the same proof. Moreover, by replacing Proposition 8.10 by Proposition 7.11 and considering the Iwasawa decomposition $G = N(\exp \mathfrak{a})K$, the proofs of Theorems 1.9 and 9.27 apply for $H = N^+$ and $\mathfrak{b}^+ = \mathfrak{a}^+$.

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MATHEMATICS DEPARTMENT, YALE UNIVERSITY, NEW HAVEN, CT 06520

E-mail address: samuel.edwards@yale.edu

E-mail address: minju.lee@yale.edu

E-mail address: hee.oh@yale.edu