CONFORMAL MEASURE RIGIDITY FOR REPRESENTATIONS VIA SELF-JOININGS

DONGRYUL M. KIM AND HEE OH

Abstract. Let Γ be a Zariski dense discrete subgroup of a connected simple real algebraic group G_1. We discuss a rigidity problem for discrete faithful representations $\rho : \Gamma \to G_2$ and a surprising role played by higher rank conformal measures of the associated self-joining group. Our approach recovers rigidity theorems of Sullivan, Tukia and Yue, as well as applies to Anosov representations, including Hitchin representations.

More precisely, for a given representation ρ with a boundary map f defined on the limit set Λ, we ask whether the extendability of ρ to G_1 can be detected by the property that f pushes forward some Γ-conformal measure class $[\nu_\Gamma]$ to a $\rho(\Gamma)$-conformal measure class $[\nu_{\rho(\Gamma)}]$. When Γ is of divergence type in a rank one group or when ρ arises from an Anosov representation, we give an affirmative answer by showing that if the self-joining $\Gamma_\rho = (\mathrm{id} \times \rho)(\Gamma)$ is Zariski dense in $G_1 \times G_2$, then the push-forward measures $(\mathrm{id} \times f)_* \nu_\Gamma$ and $(f^{-1} \times \mathrm{id})_* \nu_{\rho(\Gamma)}$, which are higher rank Γ_ρ-conformal measures, cannot be in the same measure class.

1. Introduction

For $i = 1, 2$, let G_i be a connected simple noncompact real algebraic group, and (X_i, d_i) the Riemannian symmetric space associated to G_i. Let $\Gamma < G_1$ be a Zariski dense discrete subgroup. Let

$$\rho : \Gamma \to G_2$$

be a discrete faithful Zariski dense representation. In this paper, we are interested in the rigidity problem for ρ, that is, when can ρ be extended to a Lie group isomorphism $G_1 \to G_2$? The rigidity of representations of discrete subgroups of a connected simple real algebraic group has been extensively studied in the last four decades. Especially when Γ is a lattice in G_1, which is not locally isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, we have the celebrated Mostow strong rigidity when $G_1 = G_2$ is of rank one and Margulis superrigidity when G_1 is of higher rank; in particular any discrete faithful Zariski dense representation of Γ into G_2 extends to a Lie group isomorphism $G_1 \to G_2$ ([26], [28], [22]).

We will be interested in more general discrete subgroups, which are not necessarily lattices and which do admit non-trivial[1] discrete faithful representations. The main aim of this paper is to present a new viewpoint in studying the rigidity problem for representations of discrete subgroups in

Oh is partially supported by the NSF grant No. DMS-1900101.

[1] We say $\rho : \Gamma \to G_2$ is trivial if it extends to a Lie group isomorphism $G_1 \to G_2$

terms of conformal measure classes of self-joining groups. For a given representation $\rho : \Gamma \to G_2$, the self-joining of Γ via ρ is the following discrete subgroup of $G_1 \times G_2$:

$$\Gamma_\rho := (\mathrm{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\}.$$

A basic observation is that Γ_ρ is not Zariski dense in $G_1 \times G_2$ if and only if ρ is trivial. This led us to search properties of the self-joining group Γ_ρ which would have direct implications on the rigidity of ρ; this was posed as a question in our earlier paper [18]. The main goal of this paper is to explain our findings that by studying higher rank conformal measure classes of Γ_ρ, we can relate the rigidity of ρ with a relation among conformal measure classes of Γ and $\rho(\Gamma)$ via its ρ-boundary map.

What is a ρ-boundary map? It is well-understood since Mostow's original proof of his strong rigidity theorem [26] that investigating the behavior of ρ *on the sphere at infinity* is important in the rigidity problem. For each $i = 1, 2$, let $\mathcal{F}_i$ denote the Furstenberg boundary of G_i, which is the quotient G_i/P_i by a minimal parabolic subgroup P_i of G_i. We denote by $\Lambda = \Lambda_\Gamma$ the limit set of Γ, defined as the set of all accumulation points of $\Gamma(o)$ in $\mathcal{F}_1$ for $o \in X_1$; it is the unique Γ-minimal subset of $\mathcal{F}_1$ (Definition 2.4). By a ρ-boundary map, we mean a ρ-equivariant continuous embedding of Λ to $\mathcal{F}_2$. There can be at most one boundary map (Lemma 4.5). It will be our standing hypothesis that ρ admits a boundary map

$$f : \Lambda \to \mathcal{F}_2.$$

Conformal measures. Our rigidity criterion involves the notion of conformal measures, which was introduced and studied extensively by Patterson and Sullivan for rank one discrete groups ([27], [36]). This notion was generalized by Quint [30] to a discrete subgroup of any connected semisimple real algebraic group G as follows. Let X be the associated symmetric space, A a maximal diagonalizable subgroup with $\mathfrak{a} = \operatorname{Lie} A$ and $\mathcal{F}$ the Furstenberg boundary of G.

Definition 1.1. Fix $o \in X$ and let $\Delta < G$ be a closed subgroup. A Borel probability measure ν on $\mathcal{F}$ is called a Δ-conformal measure (with respect to o) if there exists a linear form $\psi \in \mathfrak{a}^*$ such that for any $g \in \Delta$ and $\eta \in \mathcal{F}$,

$$\frac{dg_*\nu}{d\nu}(\eta) = e^{\psi(\beta_\eta(o, go))} \tag{1.1}$$

where $g_*\nu(B) = \nu(g^{-1}B)$ for any Borel subset $B \subset \mathcal{F}$ and $\beta_\cdot(\cdot,\cdot)$ denotes the $\mathfrak{a}$-valued Busemann map (see Definition 2.1)[2]. We call ψ a linear form associated to ν. By a Δ-conformal measure class $[\nu]$, we mean the set of all Borel measures on $\mathcal{F}$ which are equivalent to ν.

[2]For $x \in X$, $d\nu_x(\eta) = e^{\psi(\beta_\eta(o,x))} d\nu(\eta)$ is a (Δ, ψ)-conformal measure with respect to x. The family $\{\nu_x : x \in X\}$ is referred to as a (Δ, ψ)-conformal density. The uniqueness of a conformal measure is to be understood in terms of its associated conformal density.

Generalizing the construction of Patterson-Sullivan for rank one groups, Quint constructed a Δ-conformal measure supported on the limit set $\Lambda_\Delta \subset \mathcal{F}$ for any Zariski dense discrete subgroup Δ of G [30].

Rigidity of ρ and conformal measures. We denote by

$$\mathfrak{R}_{\rm disc}(\Gamma, G_2)$$

the space of all discrete faithful Zariski dense representations $\rho : \Gamma \to G_2$ admitting boundary maps $f : \Lambda \to \mathcal{F}_2$. If ν_Γ is a Γ-conformal measure supported on the limit set Λ of Γ, then the pushforward $f_*\nu_\Gamma$ is a Borel measure supported on $f(\Lambda) = \Lambda_{\rho(\Gamma)}$. If $\rho : \Gamma \to G_2$ extends to a Lie group isomorphism $G_1 \to G_2$, then $f_*\nu_\Gamma$ is a $\rho(\Gamma)$-conformal measure. By investigating higher rank conformal measures for the associated self-joining group Γ_ρ, we prove the converse holds: if $[f_*\nu_\Gamma]$ is a $\rho(\Gamma)$-conformal measure class, then ρ extends to a Lie group isomorphism in the following two settings:

(1) Γ is of divergence type in a rank one Lie group;
(2) ρ arises from an Anosov representation.

We now describe our main theorems in these two settings.

Divergence-type groups. When rank $G_1 = 1$, a discrete subgroup $\Gamma < G_1$ is *of divergence type* if $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d_1(o_1,\gamma o_1)} = \infty$ where δ_Γ denotes the critical exponent of Γ. The divergence type condition is satisfied for lattices, geometrically finite groups, as well as normal subgroups Γ of convex cocompact subgroups Γ_0 with $\Gamma_0/\Gamma \simeq \mathbb{Z}^d$ for $d = 1, 2$ ([39, Proposition 2], [8, Proposition 3.7], [31, Theorem 4.7]). Another important class of divergence-type groups are finitely generated discrete subgroups of $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{SO}^\circ(3,1)$ whose limit set is $\mathbb{S}^2 = \partial\mathbb{H}^3$; this follows from [6, Corollary 11.2] and the tameness theorem ([1], [5]).

For Γ of divergence type, there exists a unique Γ-conformal measure of dimension δ_Γ, say, ν_Γ, supported on Λ ([36], [32, Corollary 1.8]).

Theorem 1.2. *Let* $\operatorname{rank} G_1 = 1$ *and* $\Gamma < G_1$ *be of divergence type. Let* $\rho \in \mathfrak{R}_{\rm disc}(\Gamma, G_2)$. *Suppose* $\operatorname{rank} G_2 = 1$ *or* $f : \Lambda \to \mathcal{F}_2$ *is a continuous extension of* ρ. *Then the following are equivalent:*

(1) $[f_*\nu_\Gamma]$ *is a* $\rho(\Gamma)$*-conformal measure class;*
(2) ρ *extends to a Lie group isomorphism* $G_1 \to G_2$.

In particular, if $G_1 \not\simeq G_2$ *(e.g.,* $\operatorname{rank} G_2 \geq 2$*), then* $f_*\nu_\Gamma$ *is not equivalent to any* $\rho(\Gamma)$*-conformal measure.*

That f is a *continuous* extension of ρ means that for any sequence $g_\ell \in \Gamma$, the convergence $g_\ell o_1 \to \xi$ implies the convergence $\rho(g_\ell) o_2 \to f(\xi)$ in the sense of Definition 2.3, where $o_1 \in X_1$ and $o_2 \in X_2$. We refer to Example 4.3 of representations admitting boundary maps which are continuous extensions.

On the rigidity results of Mostow and Sullivan. The Mostow-Sullivan quasiconformal rigidity ([25, Theorem 12.1], [37, Theorem V]) can be deduced from Theorem 1.2:

Corollary 1.3. (1) *Let $n \geq 3$. Any discrete subgroup $\Gamma < \mathrm{SO}^\circ(n,1)$ of divergence type with $\Lambda = \mathbb{S}^{n-1}$ is quasiconformally rigid.*
(2) *Any finitely generated discrete subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ with $\Lambda = \mathbb{S}^2$ is quasiconformally rigid.*

For Γ as in the above corollary, ν_Γ is given by the Lebesgue measure $\mathrm{Leb}_{\mathbb{S}^{n-1}}$. On the other hand, for $n \geq 3$, a quasiconformal homeomorphism $f : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ has non-zero Jacobian at $\mathrm{Leb}_{\mathbb{S}^{n-1}}$-almost every point [25, Theorem 9.4]. Therefore $\mathrm{Leb}_{\mathbb{S}^{n-1}} \ll f_* \mathrm{Leb}_{\mathbb{S}^{n-1}}$[3] and hence Corollary 1.3(1) follows from Theorem 1.2 (see also Lemma 3.2). If $\Gamma < \mathrm{SO}^\circ(n,1)$ is a lattice, Mostow [26] (see also Prasad [28, Theorem B] for non-uniform lattices) showed that any discrete faithful representation of Γ into $\mathrm{SO}^\circ(n,1)$ is indeed a quasiconformal deformation, as the first step of his proof of rigidity. Hence Mostow rigidity is a special case of Corollary 1.3(1). Since Γ as in (2) is of divergence type as mentioned before, the case (2) is now a special case of (1).

Remark 1.4. When $X_1 = X_2$ is the real hyperbolic space, Theorem 1.2 is due to Sullivan when $\delta_\Gamma = \delta_{\rho(\Gamma)}$ [38, Theorem 5] and to Tukia in general [41, Theorem 3C]. Yue extended it for a general rank one space [43, Theorem A]. Their proofs use the ergodicity of the geodesic flow with respect to the Bowen-Margulis-Sullivan measure on the unit tangent bundle of $\Gamma\backslash X_1$. The rank one hypothesis on G_2 is essential for their arguments.

Anosov representations. The notion of Anosov representations was introduced by Labourie for surface groups [19] and was extended by Guichard-Wienhard for hyperbolic groups [13] (see also [14]). For a finitely generated hyperbolic group Σ, let $\rho_i : \Sigma \to G_i$ be a discrete faithful Zariski dense Anosov representation with respect to a minimal parabolic subgroup of G_i for $i = 1, 2$. This means that denoting by $\partial\Sigma$ the Gromov boundary, there exists an equivariant homeomorphism $f_i : \partial\Sigma \to \Lambda_{\rho_i(\Sigma)} \subset \mathcal{F}_i$ and the limit set $\Lambda_{\rho_i(\Sigma)}$ is antipodal, that is, two distinct points are in general position.

Setting $\Gamma = \rho_1(\Sigma)$, consider a representation of Γ given by $\rho := \rho_2 \circ \rho_1^{-1} : \Gamma \to G_2$ with boundary map $f = f_2 \circ f_1^{-1} : \Lambda \to \mathcal{F}_2$. The following may be regarded as an Anosov version of the rigidity theorem of Sullivan-Tukia-Yue ([38], [41], [43]).

Theorem 1.5. *Let ν_Γ be any Γ-conformal measure on Λ_Γ. Either*[4]

$$f_*\nu_\Gamma \perp \nu_{\rho(\Gamma)}$$

for all $\rho(\Gamma)$-conformal measures $\nu_{\rho(\Gamma)}$ on $\Lambda_{\rho(\Gamma)}$ or $\rho : \Gamma \to G_2$ extends to a Lie group isomorphism $G_1 \to G_2$.

[3]The notation $\nu_1 \ll \nu_2$ means that ν_1 is absolutely continuous with respect to ν_2.

[4]The notation $\nu_1 \perp \nu_2$ means that ν_1 and ν_2 are mutually singular to each other.

This is equivalent to saying that either $(f_1^{-1})_*\nu_\Gamma \perp (f_2^{-1})_*\nu_{\rho(\Gamma)}$ or there exists a Lie group isomorphism $\Phi : G_1 \to G_2$ such that $\rho_2 = \Phi \circ \rho_1$.

Hitchin representations. Let Γ be a torsion-free uniform lattice of $\mathrm{PSL}_2(\mathbb{R})$, and π_d denote the d-dimensional irreducible representation

$$\pi_d : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_d(\mathbb{R})$$

which is unique up to conjugation. A Hitchin representation $\rho : \Gamma \to \mathrm{PSL}_d(\mathbb{R})$ is a representation which belongs to the same connected component as $\pi_d|_\Gamma$ in the character variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}_d(\mathbb{R}))/\sim$ where the equivalence is given by conjugations. A Hitchin representation is known to be an Anosov representation with respect to a minimal parabolic subgroup by Labourie [19]. Moreover, the Zariski closure of its image is a connected simple real algebraic group (see [12], [33, Corollary 1.5]). Therefore by considering ρ as a representation into the Zariski closure of $\rho(\Gamma)$, we deduce the following corollary from Theorem 1.5. The Furstenberg boundary $\mathcal{F}$ of $\mathrm{PSL}_d(\mathbb{R})$ is the full flag variety of $\mathbb{R}^d$. Let $f : \mathbb{S}^1 \to \mathcal{F}$ be the ρ-boundary map.

Corollary 1.6. *For any Hitchin representation $\rho : \Gamma \to \mathrm{PSL}_d(\mathbb{R})$, $f_*\mathrm{Leb}_{\mathbb{S}^1}$ is mutually singular to any $\rho(\Gamma)$-conformal measure on $f(\mathbb{S}^1)$, except for the case when $d = 2$ and ρ is a conjugation.*

Graph-conformal measures of self-joinings and proofs of main theorems. As discussed before, the main novelty of our approach is the introduction of higher rank conformal measures for self-joinings in this rigidity problem. Let $G = G_1 \times G_2$, and recall the self-joining group:

$$\Gamma_\rho = (\mathrm{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) \in G : \gamma \in \Gamma\}.$$

The existence of a ρ-boundary map f implies that the limit set of Γ_ρ is of the form $\Lambda_\rho = (\mathrm{id} \times f)(\Lambda) = \{(\xi, f(\xi)) : \xi \in \Lambda\}$, where $\mathrm{id} \times f : \Lambda \to \Lambda_\rho$ is the diagonal embedding $\eta \mapsto (\eta, f(\eta))$.

A general higher rank conformal measure seems mysterious, but by the graph structure of the self-joining group Γ_ρ allows very explicit types of conformal measures, which we call graph-conformal measures. We write $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ where $\mathfrak{a}_i = \mathfrak{a} \cap \mathrm{Lie}\, G_i$ for $i = 1, 2$. Let $\pi_i : \mathfrak{a} \to \mathfrak{a}_i$ be the projection. A basic but crucial observation is that for a (Γ, ψ_1)-conformal measure ν_{Γ,ψ_1} on Λ, its pushforward measure by $\mathrm{id} \times f$:

$$(\mathrm{id} \times f)_*\nu_{\Gamma,\psi_1}$$

is a $(\Gamma_\rho, \psi_1 \circ \pi_1)$-conformal measure, and conversely, (Γ_ρ, σ)-conformal measure on Λ_ρ for a linear form $\sigma \in \mathfrak{a}^*$ that factors through $\mathfrak{a}_1$ is of such form (Proposition 4.6). We call them *graph-conformal* measures of Γ_ρ.

For a given Γ_ρ-conformal measure ν, the essential subgroup $\mathsf{E}_\nu(\Gamma_\rho) < \mathfrak{a}$ for ν consists of all $u \in \mathfrak{a}$ such that for any Borel subset $B \subset \mathcal{F}$ with $\nu(B) > 0$ and any $\varepsilon > 0$, there exists $g \in \Gamma_\rho$ such that

$$\nu(B \cap gB \cap \{\xi \in \mathcal{F} : \|\beta_\xi(o, go) - u\| < \varepsilon\}) > 0.$$

Here is our key proposition linking the higher rank conformal measures and our rigidity question (see Proposition 3.6).

Proposition 1.7. *Let ν_1 be a (Γ, ψ_1)-conformal measure on Λ and ν_2 a $(\rho(\Gamma), \psi_2)$-conformal measure on $\Lambda_{\rho(\Gamma)}$. If*

$$\mathsf{E}_{(\mathrm{id} \times f)_* \nu_1}(\Gamma_\rho) \not\subset \{(u_1, u_2) \in \mathfrak{a} : \psi_1(u_1) = \psi_2(u_2)\},$$

then

$$(f^{-1} \times \mathrm{id})_*(\nu_2) \not\ll (\mathrm{id} \times f)_*(\nu_1) \text{ and hence } \nu_2 \not\ll f_* \nu_1.$$

The essential subgroup $\mathsf{E}_\nu(\Gamma_\rho)$ is usually used as a tool to decide the ergodicity of the corresponding Burger-Roblin measure m_ν^{BR} (see Definition 3.4) for the maximal horospherical action on $\Gamma_\rho \backslash G$ ([35], [32], [20, Proposition 9.2]). In this paper, we use it as a tool to determine linear forms associated to conformal measures in the same measure class as ν. In fact, recalling that Γ_ρ is Zariski dense in G if and only if ρ does not extend to a Lie group isomorphism $G_1 \to G_2$, Theorem 1.2 follows from Proposition 1.7 and the following dichotomy Theorem 1.8. For Γ and ν_Γ as in Theorem 1.2, we set

$$\nu_{\mathrm{graph}} = (\mathrm{id} \times f)_* \nu_\Gamma,$$

which is the unique (Γ_ρ, σ_1)-conformal measure where $\sigma_1(u_1, u_2) = \delta_\Gamma u_1$ (Corollary 4.7). We refer to Theorem 8.1 for any undefined terminologies:

Theorem 1.8. *Let Γ, ρ be as in Theorem 1.2. In each of the two complimentary cases, the claims (1) to (4) are equivalent to each other.*

1. *Γ_ρ is Zariski dense in G (resp. not Zariski dense in G);*
2. *$\mathsf{E}_{\nu_{\mathrm{graph}}}(\Gamma_\rho) = \mathfrak{a}$ (resp. $\mathsf{E}_{\nu_{\mathrm{graph}}}(\Gamma_\rho) = \mathbb{R}(\delta_{\rho(\Gamma)}, \delta_\Gamma)$);*
3. *$m^{\mathrm{BR}}_{\nu_{\mathrm{graph}}}$ is NM-ergodic (resp. $m^{\mathrm{BR}}_{\nu_{\mathrm{graph}}}$ is not NM-ergodic);*
4. *For any (Γ_ρ, ψ)-conformal measure ν on Λ_ρ for $\psi \neq \sigma_1$, we have $[\nu_{\mathrm{graph}}] \neq [\nu]$ (resp. for any (Γ_ρ, ψ)-conformal measure ν on Λ_ρ for a tangent linear form ψ, we have $[\nu_{\mathrm{graph}}] = [\nu]$.)*

If ρ is an Anosov representation, the essential subgroup for *any* Γ_ρ-conformal measure on Λ_ρ is equal to $\mathfrak{a}$, as shown in [20, Proposition 10.2]. In our general setting, we do not know the size of a general essential subgroup. However for the essential subgroup corresponding to the graph-conformal measure, we are able to make an extensive use of the graph structures of Γ_ρ and Λ_ρ to prove $\mathsf{E}_{\nu_{\mathrm{graph}}}(\Gamma_\rho) = \mathfrak{a}$.

Cannon-Thurston map. Let $\mathcal{M}$ be a closed hyperbolic 3-manifold that fibers over a circle with fiber a closed orientable surface S. Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a uniform lattice such that $\Gamma \backslash \mathbb{H}^2$ is homeomorphic to S by ϕ. Let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ be the holonomy representation induced by $\phi : \Gamma \backslash \mathbb{H}^2 \to S \subset \mathcal{M}$. Let $F : \mathbb{H}^2 \to \mathbb{H}^3$ be a lift of ϕ. Cannon and Thurston [7, Section 4] proved that F continuously extends to the ρ-boundary map $f : \partial \mathbb{H}^2 \to \partial \mathbb{H}^3$, which is surjective but not injective. The map f is called a Cannon-Thurston map. Mj generalized this result for a general lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ (see [23, Theorem 8.6], [24]). Indeed, Theorem1.8 (see Theorem 8.1) applies to the Cannon-Thurston map, since we do not require f to be injective in our proof of Theorem 8.1. Since $\mathrm{PSL}_2(\mathbb{R})$ is not isomorphic to $\mathrm{PSL}_2(\mathbb{C})$, Γ_ρ is Zariski dense in $G = \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{C})$. Hence Theorem1.8 says that the Burger-Roblin measure $m^{\mathrm{BR}}_{\nu_{\mathrm{graph}}}$ is NM-ergodic where $\nu_{\mathrm{graph}} = (\mathrm{id} \times f)_* \mathrm{Leb}_{\mathbb{S}^1}$, where $N < G$ is the product of strict upper triangular subgroups and $M = \{e\} \times \{\mathrm{diag}(e^{i\theta}, e^{-i\theta})\}$. Therefore we have:

Corollary 1.9. *For Lebesgue almost all $\xi \in \mathbb{S}^1$, the orbit $[g]NM$ with $gP = (\xi, f(\xi))$ is dense in the space $\{[h] \in \Gamma_\rho \backslash (\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{C})) : hP \in \Lambda_\rho\}$.*

Organization. In section 2, we review basic properties of Zariski dense discrete subgroups of semisimple real algebraic groups. In section 3, after recalling notions of conformal measures and essential subgroups, we discuss how the essential subgroup of a given conformal measure influences other conformal measures which are absolutely continuous with respect to ν. We also discuss the relation between essential subgroups and ergodic properties of Burger-Roblin measures. In section 4, we introduce the notion of graph-conformal measures for self-joining groups and explain their important role in the rigidity problem. We also prove the uniqueness of a ρ-boundary map. In section 5, we discuss tangent linear forms and show that self-joining subgroups admit infinitely many such forms. In section 6, we introduce the notion of weak-Myrberg limit set and show that it is of full measure with respect to graph-conformal measures. In section 7, we prove that the essential subgroups for graph-conformal measures are all of $\mathfrak{a}$ under the Zariski dense hypothesis on Γ_ρ. In section 8, we establish the dichotomy for the Zariski density of self-joining groups in terms of essential subgroups and singularity of conformal measures. We then deduce the main theorems stated in the introduction. In section 9, we discuss general Anosov representations where both G_1 and G_2 have no rank restrictions.

2. Preliminaries

Let G be a connected semisimple real algebraic group and $\mathfrak{g} = \mathrm{Lie}\, G$ denote its Lie algebra. We fix a Cartan involution θ of $\mathfrak{g}$ and consider the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. Denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Fixing a left G-invariant and right K-invariant Riemannian metric on G induces a Weyl-group invariant norm on $\mathfrak{a}$, which we denote by $\|\cdot\|$. Note that the choice of this Riemannian metric induces a left G-invariant metric d on the symmetric space $X := G/K$. Let $o \in X$ denote the point corresponding to the coset $[K]$.

Let $A := \exp \mathfrak{a}$ and choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, set $A^+ = \exp \mathfrak{a}^+$. Denote by M the centralizer of A in K. Set N to be the maximal contracting horospherical subgroup for A: for an element a in the interior of A^+, $N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}$. We set $P = MAN$, which is a minimal parabolic subgroup of G. The quotient

$$\mathcal{F} = G/P$$

is called the Furstenberg boundary of G, and is isomorphic to K/M. Two points $\xi, \eta \in \mathcal{F}$ are said to be in general position if the diagonal orbit $G(\xi, \eta)$ is open in $\mathcal{F} \times \mathcal{F}$.

Definition 2.1 (Busemann map)**.** The Iwasawa cocycle $H : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$ with $\xi = [k] \in K/M$, $\exp H(g, \xi)$ is the A-component of gk in the KAN decomposition so that

$$gk \in K \exp(H(g, \xi))N.$$

The Busemann map $\beta : \mathcal{F} \times X \times X \to \mathfrak{a}$ is now defined as follows: for $\xi \in \mathcal{F}$ and $[g], [h] \in X$,

$$\beta_\xi([g], [h]) := H(g^{-1}, \xi) - H(h^{-1}, \xi).$$

Observe that the Busemann map is continuous in all three variables. To ease notation, we will sometimes write $\beta_\xi(g, h) = \beta_\xi([g], [h])$. We have

$$\beta_\xi(g, h) + \beta_\xi(h, q) = \beta_\xi(g, q) \quad \text{and} \quad \beta_{g\xi}(gh, gq) = \beta_\xi(h, q)$$

for any $g, h, q \in G$ and $\xi \in \mathcal{F}$.

Jordan projection. An element $g \in G$ is loxodromic if

$$g = hamh^{-1}$$

for some $a \in \operatorname{int} A^+$, $m \in M$ and $h \in G$. The Jordan projection of g is defined to be

$$\lambda(g) := \log a \in \operatorname{int} \mathfrak{a}^+.$$

The attracting fixed point of g is given by

$$y_g := hP \in \mathcal{F};$$

for any $\xi \in \mathcal{F}$ in general position with $y_{g^{-1}}$, the sequence $g^\ell \xi$ converges to y_g as $\ell \to \infty$.

Let $\Delta < G$ be a discrete subgroup. We write $\lambda(\Delta)$ for the set of all Jordan projections of loxodromic elements of Δ. The following result is due to Benoist [3] (see [20, Lemma 10.3] for the second part).

Theorem 2.2. *If $\Delta < G$ is Zariski dense, then $\lambda(\Delta)$ generates a dense subgroup of $\mathfrak{a}$. Moreover, for any finite subset $S \subset \lambda(\Delta)$, $\lambda(\Delta) - S$ generates a dense subgroup of $\mathfrak{a}$.*

The limit cone $\mathcal{L}_\Delta \subset \mathfrak{a}^+$ is defined as the smallest closed cone containing the Jordan projection of Δ. If Δ is Zariski dense, $\mathcal{L}_\Delta$ is a convex cone with non-empty interior [2].

Limit set. Let Π denote the set of all simple roots of $\mathfrak{g} = \operatorname{Lie} G$ with respect to $\mathfrak{a}^+$. We say that a sequence $g_\ell \to \infty$ regularly in G if $\alpha(\mu(g_\ell)) \to \infty$ as $\ell \to \infty$ for all $\alpha \in \Pi$.

Definition 2.3. A sequence $g_\ell \in G$, or $g_\ell o \in X$, is said to converge to $\xi \in \mathcal{F}$ if

- $g_\ell \to \infty$ regularly in G;
- $\lim_{\ell \to \infty} \kappa_1(g_\ell) P = \xi$ where $\kappa_1(g_\ell) \in K$ is an element such that $g_\ell \in \kappa_1(g_\ell) A^+ K$.

Definition 2.4 (Limit set)**.** The limit set $\Lambda_\Delta \subset \mathcal{F}$ is defined as the set of all accumulation points of $\Delta(o)$ in $\mathcal{F}$, that is,

$$\Lambda_\Delta = \{\lim_{\ell \to \infty} g_\ell o \in \mathcal{F} : g_\ell \in \Delta\}.$$

This may be an empty set for a general discrete subgroup. However, we have the following result of Benoist for Zariski dense subgroups ([2, Section 3.6], [20, Lemma 2.15]):

Theorem 2.5. *If $\Delta < G$ is Zariski dense, then Λ_Δ is the unique Δ-minimal subset of $\mathcal{F}$ and the set of all attracting fixed points of loxodromic elements of Δ is dense in Λ_Δ.*

For each $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, such that

$$g \in K \exp(\mu(g)) K.$$

Definition 2.6 ($\mathfrak{a}^+$-valued distance)**.** We define $\underline{a} : X \times X \to \mathfrak{a}^+$ by

$$\underline{a}(p, q) := \mu(g^{-1} h)$$

where $p = g(o)$ and $q = h(o)$.

Growth indicator function. Following Quint [29], the growth indicator function $\psi_\Delta : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ is defined as follows: for any cone $\mathcal{C} \subset \mathfrak{a}$, let $\tau_{\mathcal{C}}$ denote the abscissa of convergence of the series $\sum_{g \in \Delta, \mu(g) \in \mathcal{C}} e^{-s\|\mu(g)\|}$. For $u \in \mathfrak{a} - \{0\}$, we define

$$\psi_\Delta(u) = \|u\| \inf_{u \in \mathcal{C}} \tau_{\mathcal{C}}$$

where the infimum is taken over all open cones $\mathcal{C}$ containing u. We also set $\psi_\Delta(0) = 0$. It is immediate from the definition that ψ_Δ is homogeneous of degree one. Quint [29, Theorem 1.1.1] showed that ψ_Δ is a concave and upper semi-continuous function satisfying

$$\mathcal{L}_\Delta = \{\psi_\Delta \geq 0\}.$$

Moreover $\psi_\Delta > 0$ on $\operatorname{int} \mathcal{L}_\Delta$ and $\psi_\Delta = -\infty$ outside $\mathcal{L}_\Delta$.

3. Conformal measures and essential subgroups

Let G be a connected semisimple real algebraic group. We continue notations $X, \mathcal{F}, o$, etc from section 2. Let $\Delta < G$ be a discrete subgroup.

Definition 3.1 (Conformal measures)**.** A Borel probability measure ν_o on $\mathcal{F}$ is called a Δ-conformal measure (with respect to o) if there exists a linear form $\psi \in \mathfrak{a}^*$ such that for all $\eta \in \mathcal{F}$ and $g \in \Delta$,

$$\frac{dg_*\nu_o}{d\nu_o}(\eta) = e^{\psi(\beta_\eta(o,go))}. \tag{3.1}$$

In this case, we say ν_o is a (Δ, ψ)-conformal measure. For $x \in X$, $d\nu_x(\eta) = e^{\psi(\beta_\eta(o,x))} d\nu_o(\eta)$ is a (Δ, ψ)-conformal measure with respect to x.

The set of values $\{\beta_\eta(o, go) \in \mathfrak{a} : g \in \Delta, \eta \in \operatorname{supp}(\nu_o)\}$ may not be large enough to determine the linear form to which ν_o is associated; in general, there may be multiple linear forms associated to the same conformal measure. This phenomenon occurs when $\dim \mathfrak{a} > 1$ and hence definitely a higher rank feature which does not arise in rank one situation.

Lemma 3.2. *Let ν_1 and ν_2 be Δ-conformal measures on $\mathcal{F}$ such that $\nu_1 \ll \nu_2$. If ν_2 is Δ-ergodic, then*

$$[\nu_1] = [\nu_2].$$

Proof. Suppose $\nu_1 \ll \nu_2$. We need to prove $\nu_2 \ll \nu_1$. Let $B \subset \mathcal{F}$ be a ν_1-null Borel subset. Since $g_*\nu_1 \ll \nu_1$ for all $g \in \Delta$ by the Δ-conformality of ν_1, we have $\nu_1(\Delta B) = 0$. Since ν_2 is Δ-ergodic, ΔB is either ν_2-null or ν_2-conull. The latter case implies that ΔB should also be ν_1-conull since $\nu_1 \ll \nu_2$, which contradicts the hypothesis $\nu_1(B) = 0$. Hence ΔB is ν_2-null; in particular, $\nu_2(B) = 0$. This proves the claim. ☐

Essential subgroups.

Definition 3.3 (Essential subgroup for ν)**.** For a Δ-conformal measure ν with respect to o, we define the subset $\mathsf{E}_\nu(\Delta) \subset \mathfrak{a}$ as follows: $u \in \mathsf{E}_\nu(\Delta)$ if for any Borel subset $B \subset \mathcal{F}$ with $\nu(B) > 0$ and any $\varepsilon > 0$, there exists $g \in \Delta$ such that

$$\nu(B \cap gB \cap \{\xi \in \mathcal{F} : \|\beta_\xi(o, go) - u\| < \varepsilon\}) > 0. \tag{3.2}$$

It is easy to see that $\mathsf{E}_\nu(\Delta)$ is a closed subgroup of $\mathfrak{a}$. We call $\mathsf{E}_\nu(\Delta)$ the essential subgroup for ν. In rank one case, this subgroup was defined in ([35], see also [32]) in order to study the ergodic properties of horospherical actions. The higher rank analogue given as above was studied in [20] in relation with ergodicity of the maximal horospherical action with respect to the higher rank Burger-Roblin measures which we now recall.

Burger-Roblin measures in higher rank. For $p = m(\exp a)n \in MAN$, set $H(p) = a \in \mathfrak{a}$; in the notation of Definition 2.1, $H(p) = H(p, [P]) = \beta_{[P]}(p^{-1}, e)$.

Definition 3.4 (Burger-Roblin measures)**.** Given a (Δ, ψ)-conformal measure ν on $\mathcal{F}$ (with respect to o), we define $\tilde{m}_\nu^{\mathrm{BR}}$ on G as follows: for $\Phi \in C_c(G)$,

$$\tilde{m}_\nu^{\mathrm{BR}}(\Phi) = \int_{kp \in KP} \Phi(kp) e^{\psi(H(p))}\, d\nu(kP) dp$$

where dp denotes the left Haar measure on P. By the Δ-conformal property of ν (3.1), $\tilde{m}_\nu^{\mathrm{BR}}$ is left Δ-invariant and hence induces a locally finite measure on $\Delta \backslash G$ [10], which we denote by m_ν^{BR}. We call this measure the Burger-Roblin (or BR) measure associated to ν.

The BR measures on $\Delta\backslash G$ are NM-invariant and A-quasi-invariant measures ([10, Lemma 3.9]). The action of NM on $\Gamma_\rho \backslash G$ will be referred to as a maximal horospherical action. The NM-ergodicity of the BR-measure m_ν^{BR} is directly related to the size of the essential subgroup $\mathsf{E}_\nu(\Delta)$ by the following higher rank version of a theorem of Schmidt ([35], see also [32, Proposition 2.1]).

Proposition 3.5 ([20, Proposition 9.2])**.** *For any Δ-conformal ergodic measure ν on $\mathcal{F}$, we have*

$$\mathsf{E}_\nu(\Delta) = \mathfrak{a} \text{ if and only if } (\Delta\backslash G, m_\nu^{\mathrm{BR}}) \text{ is } NM\text{-ergodic.}$$

Singularity of conformal measures by essential subgroups. The following proposition is one of key ingredients of this paper:

Proposition 3.6 ([20, Proof of Lemma 10.21])**.** *For $i = 1, 2$, let ν_i be a (Δ, ψ_i)-conformal measure for some $\psi_i \in \mathfrak{a}^*$. If $\nu_2 \ll \nu_1$, then*

$$\psi_1(w) = \psi_2(w) \quad \text{for all } w \in \mathsf{E}_{\nu_1}(\Delta).$$

In particular, if $\mathsf{E}_{\nu_1}(\Delta) = \mathfrak{a}$, then $\nu_2 \ll \nu_1$ implies $\psi_1 = \psi_2$.

Proof. Suppose that $\nu_2 \ll \nu_1$. Consider the Radon-Nikodym derivative $\phi := \frac{d\nu_2}{d\nu_1} \in L^1(\mathcal{F}, \nu_1)$. Note that there exists a ν_1-conull set $F \subset \mathcal{F}$ such that for all $\xi \in F$ and $\gamma \in \Delta$, we have

$$\phi(\gamma^{-1}\xi) = e^{(\psi_2 - \psi_1)(\beta_\xi(o, \gamma o))} \phi(\xi). \tag{3.3}$$

Choose $0 < r_1 < r_2$ such that

$$B := \{\xi \in \mathcal{F} : r_1 < \phi(\xi) < r_2\}$$

has a positive ν_1-measure.

Suppose that there exists a vector $w \in \mathsf{E}_{\nu_1}(\Delta)$ such that $\psi_1(w) \neq \psi_2(w)$. Since $\mathsf{E}_{\nu_1}(\Delta)$ is a subgroup of $\mathfrak{a}$, we have $\mathbb{Z}w \subset \mathsf{E}_{\nu_1}(\Delta) \cap \{\psi_1 \neq \psi_2\}$. Hence we may assume by replacing w with some element of $\mathbb{Z}w$ if necessary that

$$e^{(\psi_2 - \psi_1)(w)} > \frac{2r_2}{r_1}. \tag{3.4}$$

Choose $\varepsilon > 0$ such that $e^{\|\psi_2 - \psi_1\| \varepsilon} < 2$. Since $\nu_1(B) > 0$ and $w \in \mathsf{E}_{\nu_1}(\Delta)$, there exists $\gamma \in \Delta$ such that

$$B' := B \cap \gamma B \cap \{\xi \in \mathcal{F} : \|\beta_\xi(o, \gamma o) - w\| < \varepsilon\}$$

has a positive ν_1-measure. Now note that

$$\begin{aligned} \int_{B'} \phi(\gamma^{-1}\xi)\, d\nu_1(\xi) &> e^{(\psi_2 - \psi_1)(w) - \|\psi_2 - \psi_1\|\varepsilon} \int_{B'} \phi(\xi)\, d\nu_1(\xi) \\ &> \frac{r_2}{r_1} \int_{B'} \phi(\xi)\, d\nu_1(\xi) \end{aligned}$$

by (3.3), (3.4), and the choice of ε. In particular,

$$\nu_1 \left(\left\{ \xi \in B' : \phi(\gamma^{-1}\xi) > \frac{r_2}{r_1} \phi(\xi) \right\} \right) > 0.$$

It follows that there exists $\xi \in B' \cap F$ such that

$$\phi(\gamma^{-1}\xi) > \frac{r_2}{r_1} \phi(\xi). \tag{3.5}$$

On the other hand, for $\xi \in B'$, both ξ and $\gamma^{-1}\xi$ belong to B. By the definition of B, we have $\phi(\xi) > r_1$ and $\phi(\gamma^{-1}\xi) < r_2$. Therefore, for all $\xi \in B'$, we get

$$\phi(\gamma^{-1}\xi) < r_2 = \frac{r_2}{r_1} r_1 < \frac{r_2}{r_1} \phi(\xi).$$

This is a contradiction to (3.5). ☐

4. Graph-conformal measures of self-joinings

For $i = 1, 2$, let G_i be a connected semisimple real algebraic group and let (X_i, d_i) be the associated Riemannian symmetric space. Let (X, d) be the Riemannian product $(X_1 \times X_2, \sqrt{d_1^2 + d_2^2})$. Set

$$G = G_1 \times G_2.$$

Then G acts as isometries on X. For $\square \in \{A, M, N, P, K\}$, we consider the corresponding subgroups of G by setting

$$\square = \square_1 \times \square_2.$$

In particular, $A = A_1 \times A_2$. Let $A^+ = A_1^+ \times A_2^+$. Let $\mathfrak{a}$ denote the Lie algebra of A, and $\mathfrak{a}^+ = \log A^+$. We note that

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \quad \text{and} \quad \mathfrak{a}^+ = \mathfrak{a}_1^+ \oplus \mathfrak{a}_2^+,$$

where $\mathfrak{a}_i = \operatorname{Lie} A_i$ and $\mathfrak{a}_i^+ = \operatorname{Lie} A_i^+$ for $i = 1, 2$.

Let $\Gamma < G_1$ be a Zariski dense discrete subgroup with limit set Λ. Let $\rho : \Gamma \to G_2$ be a discrete faithful Zariski dense representation.

Definition 4.1 (Self-joining)**.** We define the self-joining of Γ via ρ as

$$\Gamma_\rho := (\operatorname{id} \times \rho)(\Gamma) = \{(g, \rho(g)) \in G : g \in \Gamma\},$$

which is a discrete subgroup of G.

We begin by recalling the following:

Lemma 4.2 (cf. [18, Lemma 4.1])**.** *If G_1 and G_2 are simple, the self-joining $\Gamma_\rho < G$ is not Zariski dense if and only if ρ extends to a Lie group isomorphism $G_1 \to G_2$.*

In the rest of the paper, we assume that there exists a ρ-equivariant continuous map

$$f : \Lambda \to \mathcal{F}_2.$$

We will not assume that f is injective, mentioned otherwise. When it is injective, we call it a ρ-boundary map.

Example 4.3. Cases where a ρ-boundary map exists include the following:

(1) If Γ and $\rho(\Gamma)$ are geometrically finite and ρ is type-preserving, the ρ-boundary map exists and is a continuous extension of ρ; this was first shown by Tukia for $G_1 = G_2 = \mathrm{SO}^\circ(n,1)$ [40, Theorem 3.3] and generalized to all rank one groups by [42, Theorem 0.1] and [9, Theorem A.4].

(2) If $\rho : \Gamma \to \mathrm{SO}^\circ(n,1)$ is a quasiconformal deformation of $\Gamma < \mathrm{SO}^\circ(n,1)$, i.e., there exists a quasiconformal homeomorphism $F : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ such that $\rho(\gamma) = F \circ \gamma \circ F^{-1}$ for all $\gamma \in \Gamma$, then $f = F|_\Lambda$ is the ρ-boundary map.

(3) Let $\Gamma < G_1$ be convex cocompact. A Zariski dense representation $\rho : \Gamma \to G_2$ is an *Anosov* representation (with respect to P_2) if there is a ρ-boundary map

$$f : \Lambda \to \mathcal{F}_2$$

which maps two distinct points of Λ to points of $\mathcal{F}_2$ in general position. Moreover, by the work of Kapovich-Leeb-Porti [16, Theorem 1.4], f is a continuous extension of ρ.

Uniqueness of the boundary map. We will prove that a ρ-boundary map exists uniquely. First observe that, since Λ (resp. $\Lambda_{\rho(\Gamma)}$) is the unique Γ (resp. $\rho(\Gamma)$) minimal subset of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$), it follows from the equivariance of f that

$$f(\Lambda) = \Lambda_{\rho(\Gamma)}.$$

Moreover,

$$\Lambda_\rho = (\mathrm{id} \times f)(\Lambda) = \{(\xi, f(\xi)) \in \mathcal{F}_1 \times \mathcal{F}_2 : \xi \in \Lambda\}$$

is the unique Γ_ρ-minimal subset of $\mathcal{F}$; therefore it is the limit set of Γ_ρ by Theorem 2.5.

We will use the following nowhere smoothness property of the limit set:

Lemma 4.4 ([10, Lemma 2.11])**.** *Let Δ be a Zariski dense subgroup of a connected semisimple real algebraic group G. For any open subset U of the Furstenberg boundary $\mathcal{F}$ of G, $U \cap \Lambda_\Delta$ is not contained in any smooth submanifold of positive co-dimension.*

Lemma 4.5 (Uniqueness)**.** *If $g \in \Gamma$ and $\rho(g)$ are loxodromic, then*

$$f(y_g) = y_{\rho(g)}.$$

In particular, when G_1 and G_2 are simple, f is the unique ρ-equivariant continuous map $\Lambda \to \mathcal{F}_2$.

Proof. Let $g \in \Gamma$ be a loxodromic element. Note that if $\xi \in \Lambda$ is in general position with $y_{g^{-1}}$, then, by the ρ-equivariance and continuity of f,

$$\rho(g)^\ell f(\xi) = f(g^\ell \xi) \to f(y_g) \quad \text{as } \ell \to +\infty. \tag{4.1}$$

On the other hand, if $\rho(g)$ is loxodromic and $f(\xi)$ is in general position with $y_{\rho(g)^{-1}}$, then $\rho(g)^\ell f(\xi) \to y_{\rho(g)}$ as $\ell \to +\infty$.

Let $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) denote the set of points in $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) which are in general position with $y_{g^{-1}}$ (resp. $y_{\rho(g)^{-1}}$). Since $\mathcal{F}_1 - \mathcal{O}_1$ is contained in a proper smooth submanifold of $\mathcal{F}_1$ (as a consequence of Bruhat decomposition of G_1), we deduce from Lemma 4.4 that $\mathcal{F}_1 - \mathcal{O}_1$ cannot contain any non-empty open subset of Λ. In other words, $\Lambda \cap \mathcal{O}_1$ is dense in Λ.

By the Zariski density of $\rho(\Gamma)$ in G_2, the limit set $f(\Lambda) = \Lambda_{\rho(\Gamma)}$ is Zariski dense in $\mathcal{F}_2$, and hence $\mathcal{O}_2 \cap f(\Lambda) \neq \emptyset$. As f, regarded as a map $\Lambda \to f(\Lambda)$, is continuous, $f^{-1}(\mathcal{O}_2 \cap f(\Lambda))$ is a non-empty open subset of Λ. Therefore there exists $\xi_0 \in \Lambda \cap \mathcal{O}_1 \cap f^{-1}(\mathcal{O}_2 \cap f(\Lambda))$. Since $f(\xi_0) \in \mathcal{O}_2$, we have $\rho(g)^\ell f(\xi_0) \to y_{\rho(g)}$ as $\ell \to +\infty$. On the other hand, since $\xi_0 \in \mathcal{O}_1$, $\rho(g)^\ell f(\xi_0) \to f(y_g)$ as $\ell \to +\infty$ by (4.1). Therefore, $f(y_g) = y_{\rho(g)}$. This implies the first claim.

To prove the uniqueness, suppose that $f_1, f_2 : \Lambda \to \mathcal{F}_2$ are ρ-equivariant continuous maps. First consider the case when Γ_ρ is Zariski dense. By Theorem 2.5, projecting $\Lambda_\rho \subset \mathcal{F}_1 \times \mathcal{F}_2$ to its first factor Λ, the set $\Lambda' = \{y_g : g \in \Gamma \text{ loxodromic}, \rho(g) \text{ loxodromic}\}$ is dense in Λ. By the first part of this lemma, $f_1 = f_2$ on Λ' and hence by the continuity of f_1, f_2, we get $f_1 = f_2$ on Λ. In the case when Γ_ρ is not Zariski dense, by Lemma 4.2, ρ extends to a Lie group isomorphism $G_1 \to G_2$ which must be an algebraic group isomorphism. Hence ρ maps loxodromic element to a loxodromic element. Again, the first part of this lemma implies that $f_1 = f_2$ on Λ. ☐

Graph-conformal measures. In the rest of the paper, for each $i = 1, 2$, we denote by

$$\pi_i : \mathfrak{a} \to \mathfrak{a}_i \tag{4.2}$$

the canonical projection. For a linear form ψ_i on $\mathfrak{a}_i^*$, we define a linear form $\sigma_{\psi_i} \in \mathfrak{a}^*$ by

$$\sigma_{\psi_i} := \psi_i \circ \pi_i;$$

so, $\sigma_{\psi_i}(u_1, u_2) = \psi_i(u_i)$ for all $(u_1, u_2) \in \mathfrak{a}_1 \oplus \mathfrak{a}_2$. We fix $o = (o_1, o_2) \in X$.

Proposition 4.6. *Let $\psi_1 \in \mathfrak{a}_1^*$.*

1. *If ν_{Γ,ψ_1} is a (Γ, ψ_1)-conformal measure on Λ with respect to o_1, then $(\mathrm{id} \times f)_* \nu_{\Gamma,\psi_1}$ is a $(\Gamma_\rho, \sigma_{\psi_1})$-conformal measure with respect to o.*
2. *Any $(\Gamma_\rho, \sigma_{\psi_1})$-conformal measure on Λ_ρ with respect to o is of the form $(\mathrm{id} \times f)_* \nu_{\Gamma,\psi_1}$ for some (Γ, ψ_1)-conformal measure ν_{Γ,ψ_1} on Λ.*

Proof. For simplicity, we write $\nu_\Gamma = \nu_{\Gamma,\psi_1}$. Clearly, the measure $(\mathrm{id} \times f)_* \nu_\Gamma$ is supported on Λ_ρ. Since $\Lambda_\rho = (\mathrm{id} \times f)(\Lambda)$, any Borel subset $\tilde{E} \subset \Lambda_\rho$ is of the form $\tilde{E} = (\mathrm{id} \times f)(E)$ for some Borel subset $E \subset \Lambda$. Let $\gamma = (g, \rho(g)) \in \Gamma_\rho$. Then

$$\begin{aligned} \gamma_*(\mathrm{id} \times f)_* \nu_\Gamma(\tilde{E}) &= (\mathrm{id} \times f)_* \nu_\Gamma(\gamma^{-1}\tilde{E}) \\ &= (\mathrm{id} \times f)_* \nu_\Gamma((\mathrm{id} \times f)(g^{-1}E)) \\ &= \nu_\Gamma(g^{-1}E) = g_* \nu_\Gamma(E). \end{aligned} \tag{4.3}$$

Since ν_Γ is a (Γ, ψ_1)-conformal measure with respect to o_1, we have

$$\begin{aligned} g_*\nu_\Gamma(E) &= \int_E e^{\psi_1(\beta_\eta(o_1, go_1))} d\nu_\Gamma(\eta) \\ &= \int_{(\mathrm{id} \times f)(E)} e^{\psi_1(\beta_\eta(o_1, go_1))} d(\mathrm{id} \times f)_* \nu_\Gamma((\mathrm{id} \times f)(\eta)). \end{aligned}$$

Since the first component of $(\beta_{(\mathrm{id} \times f)(\eta)}(o, \gamma o))$ is given by $\beta_\eta(o_1, go_1)$, by performing a change of variable $\xi = (\mathrm{id} \times f)(\eta)$, we get

$$g_*\nu_\Gamma(E) = \int_{\tilde{E}} e^{\sigma_{\psi_1}(\beta_\xi(o, \gamma o))} d(\mathrm{id} \times f)_* \nu_\Gamma(\xi).$$

Together with (4.3), we obtain

$$\gamma_*(\mathrm{id} \times f)_* \nu_\Gamma(\tilde{E}) = \int_{\tilde{E}} e^{\sigma_{\psi_1}(\beta_\xi(o, \gamma o))} d(\mathrm{id} \times f)_* \nu_\Gamma(\xi).$$

Since $\gamma \in \Gamma_\rho$ is an arbitrary element, this proves $(\mathrm{id} \times f)_* \nu_\Gamma$ is a $(\Gamma_\rho, \sigma_{\psi_1})$-conformal measure with respect to o. This proves (1).

To prove (2), let ν be a $(\Gamma_\rho, \sigma_{\psi_1})$-conformal measure on Λ_ρ with respect to o. We denote by $\pi : \mathcal{F} \to \mathcal{F}_1$ the canonical projection and claim that the pushforward $\pi_*\nu$ is a (Γ, ψ_1)-conformal measure on Λ with respect to o_1. To see this, consider any $g \in \Gamma$ and any Borel subset $E \subset \Lambda$. Then

$$g_*\pi_*\nu(E) = \pi_*\nu(g^{-1}E) = \nu(g^{-1}E \times \mathcal{F}_2).$$

Since ν is supported on $\Lambda_\rho = (\mathrm{id} \times f)(\Lambda)$ and f is ρ-equivariant, we have

$$g_*\pi_*\nu(E) = \nu((\mathrm{id} \times f)(g^{-1}E)) = (g, \rho(g))_* \nu((\mathrm{id} \times f)(E)).$$

By the $(\Gamma_\rho, \sigma_{\psi_1})$-conformality of ν, we have

$$g_*\pi_*\nu(E) = \int_{(\mathrm{id} \times f)(E)} e^{\psi_1(\beta_{\pi(\xi)}(o_1, go_1))} d\nu(\xi) = \int_E e^{\psi_1(\beta_\eta(o_1, go_1))} d\pi_*\nu(\eta).$$

Hence we have

$$\frac{dg_*\pi_*\nu}{d\pi_*\nu}(\eta) = e^{\psi_1(\beta_\eta(o_1, go_1))},$$

proving that $\pi_*\nu$ is a (Γ, ψ_1)-conformal measure on Λ with respect to o_1. Set $\nu_{\Gamma,\psi_1} = \pi_*\nu$. Now for any Borel subset $E \subset \Lambda$,

$$\nu((\mathrm{id} \times f)(E)) = \pi_*\nu(E) = \nu_{\Gamma,\psi_1}(E) = (\mathrm{id} \times f)_* \nu_{\Gamma,\psi_1}((\mathrm{id} \times f)(E)).$$

Since both ν and $(\mathrm{id} \times f)_* \nu_{\Gamma,\psi_1}$ are supported on $\Lambda_\rho = (\mathrm{id} \times f)(\Lambda)$, this proves that $\nu = (\mathrm{id} \times f)_* \nu_{\Gamma,\psi_1}$, finishing the proof. $\square$

We have the following corollary:

Corollary 4.7. *If $\nu_{\Gamma,\psi}$ is a unique (Γ,ψ)-conformal measure on Λ, then $(\mathrm{id}\times f)_*\nu_{\Gamma,\psi}$ is the unique $(\Gamma_\rho,\sigma_{1,\psi})$-conformal measure on Λ_ρ with respect to o; in particular, $(\mathrm{id}\times f)_*\nu_{\Gamma,\psi}$ is Γ_ρ-ergodic.*

Proof. The first claim is clear from Proposition 4.6. Ergodicity of $(\mathrm{id}\times f)_*\nu_{\Gamma,\psi}$ follows immediately from the uniqueness. □

Definition 4.8 (Graph-conformal measures)**.** By a graph-conformal measure of Γ_ρ, we mean a (conformal) measure of the form $(\mathrm{id}\times f)_*\nu_\Gamma$ for some Γ-conformal measure ν_Γ on Λ.

Using this terminology, Proposition 4.6 can be reformulated as follows:

Proposition 4.9. *Let $\sigma\in\mathfrak{a}^*$ be a linear form which factors through $\mathfrak{a}_1$. A (Γ_ρ,σ)-conformal measure on Λ_ρ is a graph-conformal measure of Γ_ρ, and conversely, any graph-conformal measure of Γ_ρ is of such a form.*

Lemma 4.10. *Suppose f is injective. Let ν_{Γ,ψ_1} and $\nu_{\rho(\Gamma),\psi_2}$ be Γ-conformal and $\rho(\Gamma)$-conformal measures respectively.*

Then $(f^{-1}\times\mathrm{id})_*\nu_{\rho(\Gamma),\psi_2}$ is a $(\Gamma_\rho,\sigma_{\psi_2})$-conformal measure, and we have

$$(f^{-1}\times\mathrm{id})_*\nu_{\rho(\Gamma),\psi_2}\ll(\mathrm{id}\times f)_*\nu_{\Gamma,\psi_1}\ \text{if and only if}\ \nu_{\rho(\Gamma),\psi_2}\ll f_*\nu_{\Gamma,\psi_1}.$$

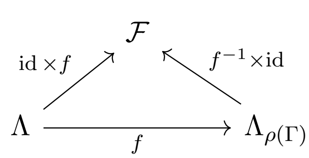
and

$$[(f^{-1}\times\mathrm{id})_*\nu_{\rho(\Gamma),\psi_2}]=[(\mathrm{id}\times f)_*\nu_{\Gamma,\psi_1}]\ \text{if and only if}\ [\nu_{\rho(\Gamma),\psi_2}]=[f_*\nu_{\Gamma,\psi_1}].$$

Proof. The first claim can be proved similarly as (1) of Proposition 4.6. The second claim follows since any Borel subset $\tilde{E}$ of Λ_ρ is of the form $(\mathrm{id}\times f)(E)=(f^{-1}\times\mathrm{id})(f(E))$ for a Borel subset $E\subset\Lambda$. Moreover

$$(\mathrm{id}\times f)_*\nu_{\Gamma,\psi_1}(\tilde{E})=\nu_{\Gamma,\psi_1}(E)\quad\text{and}\quad(f^{-1}\times\mathrm{id})_*\nu_{\rho(\Gamma),\psi_2}(\tilde{E})=\nu_{\rho(\Gamma),\psi_2}(f(E)),$$

as indicated in the following commutative diagram:



□

5. Tangent forms for self-joinings

We use the same notations as in section 4. For a discrete subgroup Δ of a semisimple real algebraic group G, the growth indicator function ψ_Δ defined in section 2 plays an important role in studying Δ-conformal measures. For instance, if there exists a (Δ,ψ)-conformal measure, then $\psi\ge\psi_\Delta$ [30, Theorem 8.1].

Definition 5.1. A linear form $\psi \in \mathfrak{a}^*$ is said to be tangent to ψ_Δ at $u \in \mathfrak{a}^+ - \{0\}$ if $\psi \geq \psi_\Delta$ and $\psi(u) = \psi_\Delta(u)$; note that u must belong to $\mathcal{L}_\Delta$, as $\psi_\Delta = -\infty$ outside $\mathcal{L}_\Delta$. We simply say that ψ is a tangent (linear) form.

For any linear form ψ tangent to ψ_Δ at some point of $\operatorname{int} \mathfrak{a}^+ \cap \mathcal{L}_\Delta$, Quint [30, Theorem 8.4] constructed a (Δ, ψ)-conformal measure supported on the limit set Λ_Δ, generalizing Patterson-Sullivan's construction.

For instance, if Δ is a lattice in G, ψ_Δ is equal to the sum of all positive roots (counted with multiplicity) on $\mathfrak{a}^+$ and hence there is only one tangent form at $\operatorname{int} \mathfrak{a}^+$, which is ψ_Δ itself and the corresponding conformal measure is simply the K-invariant probability measure on $\mathcal{F}$. In contrast, we show in this section that there are infinitely many tangent forms for self-joinings.

Let G_1 and G_2 be connected semisimple real algebraic groups. Let $\Gamma < G_1$ be a Zariski dense discrete subgroup and $\rho : \Gamma \to G_2$ a discrete faithful Zariski dense representation. Let $\mathcal{L}_\rho$ and ψ_ρ denote the limit cone and the growth indicator function of the self-joining $\Gamma_\rho < G = G_1 \times G_2$ respectively.

The main goal of this section is to prove the following:

Theorem 5.2. *If Γ_ρ is Zariski dense in G, there are infinitely many linear forms which are tangent to ψ_ρ at* $\operatorname{int} \mathfrak{a}^+$.

Corollary 5.3. *For any finitely collection of linear forms $\varphi_1, \cdots, \varphi_n \in \mathfrak{a}^*$, there exists a (Γ_ρ, ψ)-conformal measure on Λ_ρ for some linear form $\psi \in \mathfrak{a}^* - \{\varphi_i : i = 1, \cdots, n\}$.*

Proof. By Theorem 5.2, we have a linear form $\psi \notin \{\varphi_1, \cdots, \varphi_n\}$ tangent to ψ_ρ at $\operatorname{int} \mathfrak{a}^+$. By [30, Theorem 8.4], there exists a (Γ_ρ, ψ)-conformal measure supported on Λ_ρ. □

For a linear form $\psi \in \mathfrak{a}^*$, we let $-\infty \leq \delta_\psi \leq \infty$ denote the abscissa of convergence of the series $\sum_{g \in \Delta} e^{-s\psi(\mu(g))}$. We will use the following lemma in the proof of Theorem 5.2.

Lemma 5.4 ([17, Theorem 2.5])**.** *Let $\Delta < G$ be a Zariski dense discrete subgroup. For any linear form $\psi \in \mathfrak{a}^*$ such that $\psi|_{\mathcal{L}_\Delta} \geq 0$ and $\delta_\psi < \infty$, the linear form $\delta_\psi \psi \in \mathfrak{a}^*$ is tangent to ψ_Δ.*

Proof of Theorem 5.2. For convenience, we let C_ρ be the space of all linear forms tangent to ψ_ρ at $\operatorname{int} \mathfrak{a}^+$. Since ψ_ρ is concave, the subset $S := \{(u, t) : u \in \mathcal{L}_\rho, 0 \leq t \leq \psi_\rho(u)\}$ is a convex subset. Hence for each $v \in \operatorname{int} \mathcal{L}_\rho$, we may apply the supporting hyperplane theorem to S to get a linear form $\psi_v \in \mathfrak{a}^*$ tangent to ψ_ρ at v. Hence $\psi_v \in \mathsf{C}_\rho$ and

$$\psi_\rho(v) = \min_{\psi \in \mathsf{C}_\rho} \psi(v).$$

Since Γ_ρ is Zariski dense, we have $\operatorname{int} \mathcal{L}_\rho \neq \emptyset$. Choose unit vectors $\mathsf{y} \in \mathfrak{a}_1^+$ and $\mathsf{x} \in \mathfrak{a}_2^+$ so that the line segment $\mathsf{A} = [\mathsf{x}, \mathsf{y}] = \{(1-t)\mathsf{x} + t\mathsf{y} : 0 \leq t \leq 1\}$ intersects $\operatorname{int} \mathcal{L}_\rho$. Since $\mathcal{L}_\rho$ is convex, the intersection $\mathsf{A} \cap \operatorname{int} \mathcal{L}_\rho$ is an open line segment. Let J be the closure of $\mathsf{A} \cap \operatorname{int} \mathcal{L}_\rho$. We write $J = [v_1, v_2]$ where

$v_1 = (1-t_1)\mathsf{x} + t_1\mathsf{y}$ and $v_2 = (1-t_2)\mathsf{x} + t_2\mathsf{y} \in \mathsf{A}$ are two endpoints of J and $t_1 < t_2$.

We now suppose that

$$\#\mathsf{C}_\rho < \infty. \tag{5.1}$$

Since ψ_ρ is concave and upper semi-continuous, it is continuous on the interval J. Hence the finiteness assumption on C_ρ implies that there exists a finite partition of $J = [v_1, v_2]$ into $v_1 = w_0 < w_1 < \cdots < w_n = v_2$ and pairwise distinct

$$\psi_1, \cdots, \psi_n \in \mathsf{C}_\rho$$

such that $\psi_\rho = \psi_\ell$ on each $[w_{\ell-1}, w_\ell]$ for $1 \le \ell \le n$.

Claim (1): $n = 1$. If $n \ge 2$, then $w_1 \in \operatorname{int} J$, and hence we find infinitely many elements of C_ρ by considering convex linear combinations

$$(1-c)\psi_1 + c\psi_2$$

for all $0 < c < 1$; hence $n = 1$.

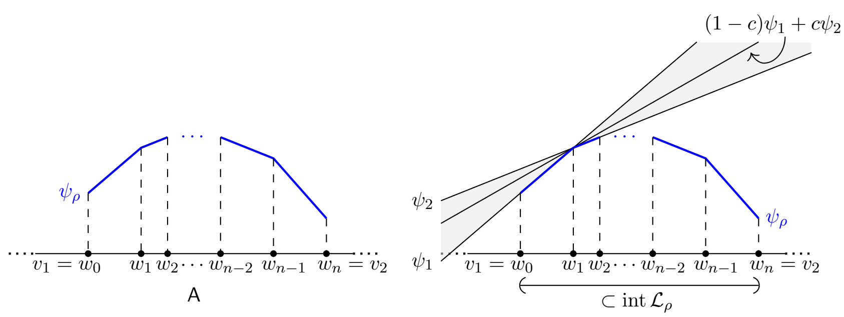


FIGURE 1. Partition of J (left) and when $n \ge 2$ (right)

Claim (2): $J = \mathsf{A}$. This claim is same as $v_1, v_2 \notin \operatorname{int} \mathfrak{a}^+$. Suppose that $v_i \in \operatorname{int} \mathfrak{a}^+$. Since $v_i \in \partial \mathcal{L}_\rho$, it follows from the supporting hyperplane theorem applied to $\mathcal{L}_\rho$ that there exists a hyperplane P tangent to $\mathcal{L}_\rho$ at v_i. Since $\mathcal{L}_\rho$ is a closed cone in $\mathfrak{a}^+$, P is a linear subspace. Hence there exists a linear form $\varphi \in \mathfrak{a}^*$ such that $\ker \varphi = \mathsf{P}$ and $\varphi \ge 0$ on $\mathcal{L}_\rho$. In particular, $\varphi > 0$ on $\operatorname{int} \mathcal{L}_\rho$. Then for any $c > 0$, the linear combination $\psi_1 + c\varphi$ belongs to C_ρ, yielding a contradiction to (5.1).

Claim (3): $\psi_\rho = 0$ on $J = \mathsf{A}$. By Claims (1) and (2), $\psi_\rho = \psi_1$ on A. Since the growth indicator function ψ_Γ is upper semi-continuous, it attains a maximum on the unit sphere of $\mathfrak{a}_1$. Hence we may choose $\Phi \in \mathfrak{a}_1^*$ such that $\Phi > \psi_\Gamma$ on $\mathfrak{a}_1 - \{0\}$. By [29, Lemma 3.1.3], we have $\delta_\Phi < \infty$. We now set $\tilde{\varphi}_1 := \Phi \circ \pi_1$. Since $\tilde{\varphi}_1(\mu((g, \rho(g)))) = \Phi(\mu(g))$ for $g \in \Gamma$, we have

$$\delta_{\tilde{\varphi}_1} = \delta_\Phi < \infty.$$

Since $\pi_1(\mathcal{L}_\rho) \subset \mathfrak{a}_1^+$ and $\Phi > 0$ on $\mathfrak{a}_1^+ - \{0\}$, we have $\tilde{\varphi}_1 \geq 0$ on $\mathcal{L}_\rho$. Hence by Lemma 5.4, $\varphi_1 := \delta_{\tilde{\varphi}_1}\tilde{\varphi}_1$ is tangent to ψ_ρ; in particular,

$$\psi_\rho \leq \varphi_1 \quad \text{and} \quad \varphi_1|_{\mathfrak{a}_2} = 0. \tag{5.2}$$

Similarly, we have a linear form $\varphi_2 \in \mathfrak{a}^*$ which is tangent to ψ_ρ and factors through $\pi_2 : \mathfrak{a} \to \mathfrak{a}_2$. It implies

$$\psi_\rho \leq \varphi_2 \quad \text{and} \quad \varphi_2|_{\mathfrak{a}_1} = 0. \tag{5.3}$$

Since $\mathsf{x} \in \mathcal{L}_\rho \cap \mathsf{A} \cap \mathfrak{a}_2$ and $\mathsf{y} \in \mathcal{L}_\rho \cap \mathsf{A} \cap \mathfrak{a}_1$, we deduce from (5.2) and (5.3):

$$0 \leq \psi_1(\mathsf{x}) = \psi_\rho(\mathsf{x}) \leq \varphi_1(\mathsf{x}) = 0 \quad \text{and} \quad 0 \leq \psi_1(\mathsf{y}) = \psi_\rho(\mathsf{y}) \leq \varphi_2(\mathsf{y}) = 0.$$

Since ψ_1 is a linear form and $\mathsf{A} = [\mathsf{x}, \mathsf{y}]$, we have

$$\psi_\rho = \psi_1 = 0 \text{ on } \mathsf{A}.$$

Finishing the proof. Since A intersects $\operatorname{int}\mathcal{L}_\rho$ and $\psi_\rho > 0$ on $\operatorname{int}\mathcal{L}_\rho$ by [29, Theorem 1.1.1], Claim (3) yields a contradiction. Therefore, $\#\mathsf{C}_\rho = \infty$, finishing the proof.

6. Weak-Myrberg limit sets of self-joinings

Let G_1 be a connected simple real algebraic group of rank one and (X_1, d_1) the associated rank one symmetric space. Let $\Gamma < G_1$ be a non-elementary discrete subgroup with limit set $\Lambda = \Lambda_\Gamma \subset \mathcal{F}_1$.

Divergence-type groups. Fix $o_1 \in X_1$. If the Poincaré series of Γ diverges at the critical exponent $s = \delta_\Gamma$, i.e., $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d_1(o_1,\gamma o_1)} = \infty$, then Γ is said to be of divergence type.

Theorem 6.1 ([36], see also [32, Corollary 1.8])**.** *If Γ is of divergence type, there exists a unique Γ-conformal measure of dimension δ_Γ with respect to $o_1 \in X_1$. In particular, it is Γ-ergodic.*

We denote by $\Lambda_{\Gamma,c}$ the set of conical limit points of Γ. That is, $\xi \in \Lambda_{\Gamma,c}$ if and only if any geodesic ray toward ξ accumulates on a compact subset of $\Gamma\backslash X_1$. Let

$$\Lambda_{\Gamma,M} \subset \Lambda_{\Gamma,c}$$

denote the set of Myrberg limit points for Γ, that is, $\xi \in \Lambda_{\Gamma,M}$ if and only if any geodesic ray toward ξ is dense in the union of all geodesics connecting limit points, modulo Γ. Equivalently, $\xi \in \Lambda_{\Gamma,M}$ if and only if for any $\eta \neq \eta' \in \Lambda$, there exists a sequence $\gamma_\ell \in \Gamma$ such that $\gamma_\ell\xi \to \eta$ and $\gamma_\ell o_1 \to \eta'$ as $\ell \to \infty$.

The Hopf-Tsuji-Sullivan dichotomy [36] (see also [32, Theorem 1.7]) states:

Theorem 6.2. *The following are equivalent to each other for any Γ-conformal measure ν_Γ of dimension δ_Γ:*

1. *Γ is of divergence type;*
2. *$\nu_\Gamma(\Lambda_{\Gamma,c}) = 1$;*
3. *$\nu_\Gamma(\Lambda_{\Gamma,M}) = 1$.*

A non-elementary discrete subgroup $\Gamma < G_1$ is called *geometrically finite* (resp. convex cocompact) if the unit neighborhood of the convex core of $\Gamma\backslash X_1$ has finite volume (resp. compact). As remarked before, geometrically finite groups are of divergence type ([39, Proposition 2], [8, Proposition 3.7]). Moreover, if Γ is a convex cocompact subgroup of $\mathrm{SO}^\circ(n,1) = \mathrm{Isom}^+(\mathbb{H}^n_{\mathbb{R}})$, then ν_Γ is the δ_Γ-dimensional Hausdorff measure on Λ with respect to the spherical metric on $\mathbb{S}^{n-1}$ [36, Theorem 8].

Weak-Myrberg limit set of Γ_ρ. In the rest of the section, we assume that

$$\Gamma < G_1 \text{ is Zariski dense and of divergence type.}$$

Let $\rho : \Gamma \to G_2$ be a discrete faithful Zariski dense representation where G_2 is a connected semisimple real algebraic group. Let X_2 be the Riemannian symmetric space associated to G_2. Let $G = G_1 \times G_2$. We assume that there exists a ρ-equivariant continuous map $f : \Lambda \to \mathcal{F}_2$. As before, we denote by Γ_ρ the associated self-joining subgroup of G. We denote by $\mathcal{F}^{(2)}$ the subset of $\mathcal{F} \times \mathcal{F}$ consisting of all pairs in general position. We set

$$\Lambda_\rho^{(2)} = \mathcal{F}^{(2)} \cap (\Lambda_\rho \times \Lambda_\rho).$$

Note that if f is injective in addition, then $\Lambda_\rho^{(2)} = (\Lambda_\rho \times \Lambda_\rho) - \text{Diagonal}$. However we are not assuming the injectivity of f.

We recall from [20] that the *Myrberg limit set* of Γ_ρ is defined as

$$\Lambda_{\rho,M} := \left\{ \xi \in \Lambda_\rho : \begin{array}{c} \forall (\xi_0, \eta_0) \in \Lambda_\rho^{(2)}, \ \exists \text{ a sequence } \gamma_\ell \in \Gamma_\rho \text{ s.t.} \\ \lim \gamma_\ell \xi = \xi_0 \quad \text{and} \quad \lim \gamma_\ell o = \eta_0 \end{array} \right\}.$$

Definition 6.3. We introduce the *weak-Myrberg limit set* as follows:

$$\Lambda_{\rho,wM} := \bigcap_{\gamma_0 \in \Gamma_\rho : \text{loxodromic}} \Lambda_\rho(\gamma_0)$$

where

$$\Lambda_\rho(\gamma_0) = \left\{ \xi \in \Lambda_\rho : \begin{array}{c} \forall \eta \in \Lambda_\rho \text{ with } (y_{\gamma_0}, \eta) \in \Lambda_\rho^{(2)}, \ \exists \text{ a sequence } \gamma_\ell \in \Gamma_\rho \\ \text{s.t.} \lim \gamma_\ell \xi = y_{\gamma_0} \quad \text{and} \quad \lim \gamma_\ell o = \eta \end{array} \right\}.$$

Note that the definition does not depend on the choice of the basepoint o.

Definition 6.4. We say f is a *continuous* extension of ρ if for any sequence $g_\ell \in \Gamma$, the convergence $g_\ell o_1 \to \xi$ implies the convergence $\rho(g_\ell) o_2 \to f(\xi)$ in the sense of Definition 2.3.

The rest of this section is devoted to proving the following:

Proposition 6.5. *Suppose either that* $\operatorname{rank} G_2 = 1$ *or that* f *is a continuous extension of* ρ*. Then*

$$(\mathrm{id} \times f)_* \nu_\Gamma (\Lambda_{\rho,wM}) = 1.$$

We begin by recalling some basic hyperbolic features of X_1. The Gromov product of $x, y \in X_1$ at $p \in X_1$ is defined as

$$\langle x, y\rangle_p = \frac{1}{2}\left(d_1(x,p) + d_1(p,y) - d_1(x,y)\right).$$

For $\xi \neq \eta \in \mathcal{F}_1$, the Gromov product $\langle \xi, \eta\rangle_p$ is defined as

$$\langle \xi, \eta\rangle_p = \lim_{\ell\to\infty} \langle x_\ell, y_\ell\rangle_p$$

for any sequences $x_\ell, y_\ell \in X_1$ converging to ξ, η respectively. We also set $\langle \xi, \xi\rangle_p = \infty$. The Gromov product $\langle \xi, \eta\rangle_p$ is roughly a distance from p to the geodesic connecting ξ and η; more precisely, there exists a constant $c > 0$ depending only on the hyperbolicity of X_1 such that for any $p \in X_1$ and $\xi, \eta, \zeta \in X_1 \cup \mathcal{F}_1$,

$$|d(p, [\xi,\eta]) - \langle \xi, \eta\rangle_p| < c \quad \text{and} \quad \langle \xi, \eta\rangle_p \geq \min\{\langle \xi, \zeta\rangle_p, \langle \zeta, \eta\rangle_p\} - c \tag{6.1}$$

where $[\xi, \eta]$ denotes the unique geodesic in X_1 connecting ξ and η. For any $q \in [\xi, \eta]$, we have

$$\langle \xi, \eta\rangle_p = \frac{1}{2}\left(\beta_\xi(p,q) + \beta_\eta(p,q)\right).$$

The visual metric on $\mathcal{F}_1$ at p is defined by

$$d_p(\xi,\eta) = e^{-\langle \xi,\eta\rangle_p} \text{ if } \xi \neq \eta \text{ and } d_p(\xi,\xi) = 0.$$

If we normalize the metric so that X_1 has the sectional curvature at most -1, then d_p is indeed a *metric*; this was proved by Bourdon [4, Section 1.1].

Proof of Proposition 6.5. If Γ is of divergence type, then $\nu_\Gamma(\Lambda_{\Gamma,M}) = 1$ by Theorem 6.2. Therefore Proposition 6.5 is an immediate consequence of the following:

Lemma 6.6. *We have*

$$(\mathrm{id} \times f)(\Lambda_{\Gamma,M}) \subset \Lambda_{\rho,wM}.$$

Proof. Let $\xi_1 \in \Lambda_{\Gamma,M}$ be an arbitrary element and let $\xi = (\xi_1, f(\xi_1))$. Letting $\gamma_0 = (g_0, \rho(g_0)) \in \Gamma_\rho$ be an arbitrary loxodromic element, we need to show that

$$\xi \in \Lambda_\rho(\gamma_0). \tag{6.2}$$

Noting that $y_{\gamma_0} = (y_{g_0}, f(y_{g_0})) \in \Lambda_\rho$, let $\eta = (\eta_1, f(\eta_1)) \in \Lambda_\rho$ be any element which is in a general position with y_{γ_0}. As $\xi_1 \in \Lambda_{\Gamma,M}$, there exists a sequence $g_\ell \in \Gamma$ such that

$$g_\ell \xi_1 \to y_{g_0} \quad \text{and} \quad g_\ell o_1 \to \eta_1 \quad \text{as } \ell \to \infty. \tag{6.3}$$

Take any $\xi_{-1} \in \Lambda - f^{-1}(f(\xi_1))$; this is possible since $\rho(\Gamma)$ is Zariski dense and hence $f(\Lambda)$ is not singleton. Since for all $\ell \geq 1$,

$$\langle g_\ell \xi_1, g_\ell \xi_{-1}\rangle_{g_\ell o_1} = \langle \xi_1, \xi_{-1}\rangle_{o_1} < \infty,$$

(6.3) and (6.1) imply that

$$\lim_{\ell\to\infty} g_\ell \xi_{-1} = \lim_{\ell\to\infty} g_\ell o_1 = \eta_1. \tag{6.4}$$

Since $\lim_{\ell\to\infty} g_\ell o_1 = \eta_1 \neq y_{g_0}$ and $\lim_{\ell\to\infty} g_\ell \xi_1 = y_{g_0}$, we have

$$\langle \xi_1, g_\ell^{-1} y_{g_0}\rangle_{o_1} = \langle g_\ell \xi_1, y_{g_0}\rangle_{g_\ell o_1} \to \infty \quad \text{as } \ell \to \infty.$$

It follows that

$$\lim_{\ell\to\infty} g_\ell^{-1} y_{g_0} = \xi_1. \tag{6.5}$$

As f is ρ-equivariant and continuous, we get from (6.3), (6.4) and Lemma 4.5 that as $\ell \to \infty$,

$$\rho(g_\ell) f(\xi_1) = f(g_\ell \xi_1) \to f(y_{g_0}) = y_{\rho(g_0)} \text{ and}$$
$$\rho(g_\ell) f(\xi_{-1}) = f(g_\ell \xi_{-1}) \to f(\eta_1).$$

Hence (6.2) follows once we verify that

$$\lim_{\ell\to\infty} \rho(g_\ell) o_2 = f(\eta_1). \tag{6.6}$$

If f is a continuous extension of ρ, this is automatic from (6.3). Hence we now assume that $\operatorname{rank} G_2 = 1$, and so X_2 is a rank one symmetric space, in the rest of the proof. We use the same notation $\langle \cdot, \cdot \rangle$. for the Gromov product in $X_2 \cup \mathcal{F}_2$. By the hyperbolicity of X_2, we have a constant $c > 0$ depending only on X_2 as in (6.1) so that for all $\ell \geq 1$,

$$\begin{aligned} \langle f(\xi_1), f(\xi_{-1})\rangle_{o_2} &= \langle f(g_\ell\xi_1), f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2} \\ &\geq \min\{\langle f(g_\ell\xi_1), y_{\rho(g_0)}\rangle_{\rho(g_\ell)o_2}, \langle y_{\rho(g_0)}, f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2}\} - c \\ &= \min\{\langle f(\xi_1), \rho(g_\ell)^{-1} y_{\rho(g_0)}\rangle_{o_2}, \langle y_{\rho(g_0)}, f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2}\} - c. \end{aligned}$$

On the other hand, by (6.5) and the continuity of f, we have, as $\ell \to \infty$,

$$\rho(g_\ell)^{-1} y_{\rho(g_0)} = f(g_\ell^{-1} y_{g_0}) \to f(\xi_1)$$

which implies $\langle f(\xi_1), \rho(g_\ell)^{-1} y_{\rho(g_0)}\rangle_{o_2} \to \infty$ as $\ell \to \infty$. Hence for all large enough $\ell \geq 1$, we have

$$\langle f(\xi_1), f(\xi_{-1})\rangle_{o_2} \geq \langle y_{\rho(g_0)}, f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2} - c.$$

Again, it follows from the hyperbolicity of X_2 that

$$\langle f(\xi_1), f(\xi_{-1})\rangle_{o_2} \geq \min\{\langle y_{\rho(g_0)}, f(\eta_1)\rangle_{\rho(g_\ell)o_2}, \langle f(\eta_1), f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2}\} - 2c. \tag{6.7}$$

Now suppose to the contrary that (6.6) does not hold. From the choice of ξ_{-1}, we have $f(\xi_1) \neq f(\xi_{-1})$ and hence

$$\langle f(g_\ell\xi_1), f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2} = \langle f(\xi_1), f(\xi_{-1})\rangle_{o_2} < \infty.$$

Since $f(g_\ell\xi_1) \to y_{\rho(g_0)}$ and $f(g_\ell\xi_{-1}) \to f(\eta_1)$ as $\ell \to \infty$, by passing to a subsequence, we may assume that $\rho(g_\ell)o_2$ converges to either $y_{\rho(g_0)}$ or $f(\eta_1)$. Since we are assuming that (6.6) does not hold, we must have

$$\lim_{\ell\to\infty} \rho(g_\ell) o_2 = y_{\rho(g_0)}.$$

Since $\lim_{\ell\to\infty} f(g_\ell\xi_{-1}) = f(\eta_1) \neq y_{\rho(g_0)}$, it implies that

$$\langle f(\eta_1), f(g_\ell\xi_{-1})\rangle_{\rho(g_\ell)o_2} \to \infty \quad \text{as } \ell \to \infty.$$

Then, for all large enough $\ell \geq 1$, we have from (6.7) that

$$\langle f(\xi_1), f(\xi_{-1})\rangle_{o_2} \geq \langle y_{\rho(g_0)}, f(\eta_1)\rangle_{\rho(g_\ell)o_2} - 2c.$$

It follows from (6.1) that the distance between $\rho(g_\ell)o_2$ and the geodesic $[y_{\rho(g_0)}, f(\eta_1)]$ connecting $y_{\rho(g_0)}$ and $f(\eta_1)$ is uniformly bounded. Hence for some $B > 0$, $\rho(g_\ell)o_2$ converges to $y_{\rho(g_0)}$ within the B-neighborhood of the translation axis $\mathcal{L}_{\rho(g_0)}$ of $\rho(g_0)$. Since $\rho(g_0)$ acts cocompactly on any fixed neighborhood of $\mathcal{L}_{\rho(g_0)}$, there exists a sequence $n_\ell \to \infty$ and a compact subset $\mathcal{C} \subset X_2$ such that

$$\rho(g_0)^{-n_\ell}\rho(g_\ell)o_2 \in \mathcal{C} \cap \rho(\Gamma)o_2.$$

Since $\mathcal{C} \cap \rho(\Gamma)o_2$ is a finite subset, by passing to a subsequence, we may assume that $\rho(g_0)^{-n_\ell}\rho(g_\ell)$ is a constant sequence in $\rho(\Gamma)$, say, $\rho(g)$, for some $g \in \Gamma$. As ρ is faithful, it follows that

$$g_\ell = g_0^{n_\ell} g$$

for all $\ell \geq 1$, and hence $g_\ell o_1$ converges to y_{g_0} as $\ell \to \infty$; this contradicts the second condition in (6.3) as $\eta_1 \neq y_{g_0}$. Therefore we have proved (6.6), completing the proof. $\square$

7. Essential subgroups for graph-conformal measures

Let G_1 be a connected simple real algebraic group of rank one, and $\Gamma < G_1$ be a Zariski dense discrete subgroup of divergence type. Let $\rho : \Gamma \to G_2$ and $f : \Lambda \to \mathcal{F}_2$ be as in section 6. Let ν_Γ be the unique Γ-conformal measure of dimension δ_Γ on Λ and set

$$\nu_{\text{graph}} = (\text{id} \times f)_* \nu_\Gamma.$$

In this case, $\mathfrak{a}_1 = \mathbb{R}$ and ν_{graph} is a (Γ_ρ, σ_1)-conformal measure where

$$\sigma_1(u_1, u_2) = \delta_\Gamma u_1.$$

The main goal of this section is to prove:

Theorem 7.1. *Let $\Gamma < G_1$ be of divergence type. Suppose either that* $\operatorname{rank} G_2 = 1$ *or that f is a continuous extension of ρ. If Γ_ρ is Zariski dense, then*

$$\mathsf{E}_{\nu_{\text{graph}}}(\Gamma_\rho) = \mathfrak{a}.$$

Remark 7.2. (1) In fact, our proof shows a slightly stronger statement that for any non-trivial normal subgroup $\Gamma' < \Gamma$,

$$\mathsf{E}_{\nu_{\text{graph}}}(\Gamma'_\rho) = \mathfrak{a}.$$

(2) If Γ_ρ is an Anosov subgroup with respect to P, Theorem 7.1 is a special case of [20, Proposition 10.2]. The proof there uses the Anosov property in a crucial way.

Covering lemma. Let $p = (p_1, p_2) \in X$, $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \Lambda_\rho$. As Λ_ρ is the graph of f, we have $\xi \neq \eta \in \Lambda_\rho$ implies that $\xi_1 \neq \eta_1$. We define d_p on Λ_ρ as follows: for any $\xi \neq \eta \in \Lambda_\rho$, set

$$d_p(\xi, \eta) := e^{-\delta_\Gamma \cdot \langle \xi_1, \eta_1 \rangle_{p_1}} := d_{p_1}(\xi_1, \eta_1)^{\delta_\Gamma},$$

and $d_p(\xi, \eta) = 0$ if $\xi = \eta$. Let $n_0 > 0$ be a constant so that the normalized metric space $(X_1, n_0 d_1)$ has sectional curvature at most -1. Then, as remarked before, $d_{p_1}^{n_0} = d_p^{n_0/\delta_\Gamma}$ is a genuine metric by [4, Section 1.1]. We fix $N_0 \geq 1$ such that for all $a, b \geq 0$,

$$\left(a^{\frac{n_0}{\delta_\Gamma}} + b^{\frac{n_0}{\delta_\Gamma}} \right)^{\frac{\delta_\Gamma}{n_0}} \leq N_0(a + b).$$

Then we have the following pseudo-triangle inequality: for all $\xi, \eta, \zeta \in \Lambda_\rho$,

$$\begin{aligned} d_p(\xi, \eta) &= \left(d_p(\xi, \eta)^{\frac{n_0}{\delta_\Gamma}} \right)^{\frac{\delta_\Gamma}{n_0}} \\ &\leq \left(d_p(\xi, \zeta)^{\frac{n_0}{\delta_\Gamma}} + d_p(\zeta, \eta)^{\frac{n_0}{\delta_\Gamma}} \right)^{\frac{\delta_\Gamma}{n_0}} \\ &\leq N_0(d_p(\xi, \zeta) + d_p(\zeta, \eta)). \end{aligned} \tag{7.1}$$

For $\xi \in \Lambda_\rho$ and $r > 0$, set

$$B_p(\xi, r) := \{\eta \in \Lambda_\rho : d_p(\xi, \eta) < r\}.$$

It is standard to deduce the following from the triangle inequality (7.1):

Lemma 7.3 (Covering lemma)**.** *For any finite collection of open d_p-balls $B_p(\xi_1, r_1), \cdots, B_p(\xi_n, r_n)$ for $\xi_j \in \Lambda_\rho$ and $r_j > 0$, there is a subcollection of disjoint balls $B_p(\xi_{i_1}, r_{i_1}), \cdots, B_p(\xi_{i_\ell}, r_{i_\ell})$ such that*

$$\bigcup_{j=1}^{n} B_p(\xi_j, r_j) \subset \bigcup_{j=1}^{\ell} B_p(\xi_{i_j}, 3N_0 r_{i_j}).$$

where $N_0 \geq 1$ is as in (7.1)*.*

Shadow lemma. For $g \in G$, its visual images are defined by

$$g^+ = gP \in \mathcal{F} \quad \text{and} \quad g^- = gw_0P \in \mathcal{F}$$

where $w_0 \in G$ is an element of the normalizer of A such that $w_0 P w_0^{-1} \cap P = AM$.

Let $x, y \in X$ and $r > 0$. The shadow of the ball

$$B(y, r) = \{z \in X : d(z, y) < r\}$$

viewed from x is defined as follows:

$$O_r(x, y) := \{gk^+ \in \mathcal{F} : k \in K, gk(\text{int}A^+)o \cap B(y, r) \neq \emptyset\}$$

where $g \in G$ satisfies $x = go$. The shadow of $B(y, r)$ viewed from $\xi \in \mathcal{F}$ can also be defined:

$$O_r(\xi, y) := \{h^+ \in \mathcal{F} : h^- = \xi, ho \in B(y, r)\}.$$

The following is an analogue of Sullivan's shadow lemma:

Lemma 7.4 ([20, Lemma 5.7]). *There exists $\kappa > 0$ such that for any $x, y \in X$ and $r > 0$, we have*

$$\sup_{\xi \in O_r(x,y)} \|\beta_\xi(x,y) - \underline{a}(x,y)\| \leq \kappa r.$$

Jordan projections of self-joinings. We will need the following lemma which we deduce from Theorem 2.2:

Lemma 7.5. *Suppose that Γ_ρ is Zariski dense in G. For any $Q > 0$, the subset*

$$\{\lambda(\gamma) \in \mathfrak{a} : \gamma \in \Gamma_\rho, \sigma_1(\lambda(\gamma)) \geq Q\}$$

generates a dense subgroup of $\mathfrak{a}$.

Proof. Note that for a given $Q > 0$, $\{\lambda(\gamma) \in \mathfrak{a} : \gamma \in \Gamma_\rho, \sigma_1(\lambda(g)) \leq Q\}$ is not a finite subset in general. Hence this is not a direct consequence of Theorem 2.2. On the other hand, as Γ_ρ is Zariski dense, we can find a Zariski dense Schottky subgroup $\Gamma' < \Gamma$ such that Γ'_ρ is Zariski dense in G (see for example [10, Lemma 7.3]). Since Γ' is convex cocompact, there are only finitely many closed geodesics in $\Gamma' \backslash X_1$ of length bounded by a fixed number. Since the set of closed geodesics in $\Gamma' \backslash X_1$ is in one to one correspondence with the set $[\Gamma']$ of conjugacy classes of loxodromic elements while the Jordan projection of a loxodromic element is the length of the corresponding closed geodesic, we have

$$\#\{[g'] \in [\Gamma'] : \lambda(g') < \delta_\Gamma^{-1} Q\} < \infty.$$

Therefore $\#\{[\gamma'] \in [\Gamma'_\rho] : \sigma_1(\lambda(\gamma')) < Q\} < \infty$. Hence Theorem 2.2 implies that $\{\lambda(\gamma') \in \mathfrak{a} : \gamma \in \Gamma'_\rho, \sigma_1(\lambda(\gamma')) \geq Q\}$ generates a dense subgroup of $\mathfrak{a}$. This implies the claim. ☐

Main proposition. Since $\mathsf{E}_{\nu_{\text{graph}}}(\Gamma_\rho)$ is a closed subgroup of $\mathfrak{a}$, Theorem 7.1 follows from the following proposition by Lemma 7.5:

Proposition 7.6. *Let Γ, ρ, f be as in Theorem 7.1. Let $\gamma_0 \in \Gamma_\rho$ be any loxodromic element such that $\sigma_1(\lambda(\gamma_0)) > 1 + \log 3N_0$. For any $\varepsilon > 0$ and Borel subset $B \subset \mathcal{F}$ with $\nu_{\text{graph}}(B) > 0$, there exists $\gamma \in \Gamma_\rho$ such that*

$$B \cap \gamma\gamma_0\gamma^{-1}B \cap \left\{\xi \in \Lambda_\rho : \|\beta_\xi(o, \gamma\gamma_0\gamma^{-1}o) - \lambda(\gamma_0)\| < \varepsilon\right\}$$

has a positive ν_{graph}-measure. In particular,

$$\lambda(\gamma_0) \in \mathsf{E}_{\nu_{\text{graph}}}(\Gamma_\rho).$$

In the rest of this section, we fix a loxodromic element $\gamma_0 \in \Gamma_\rho$ such that $\sigma_1(\lambda(\gamma_0)) > 1 + \log 3N_0$. Set

$$\xi_0 := y_{\gamma_0}$$

and let $\eta \in \Lambda_\rho$ be any element which is in general position with ξ_0. Since $(\xi_0, \eta) \in \Lambda_\rho^{(2)}$, we can choose $p = go \in X$ where $g \in G$ such that $g^+ = \xi_0$ and $g^- = \eta$.

We also fix

$$0 < \varepsilon < \min(1/2, \delta_\Gamma^{-1}).$$

Covering $\Lambda_{\rho,wM}$ by a certain collection of d_p-balls. We proceed in the same way as in [20, Section 10]. For each $\gamma \in \Gamma_\rho$, let $r_p(\gamma) > 0$ be the supremum of $r \geq 0$ such that

$$\max_{\xi \in B_p(\gamma\xi_0, 3N_0 r)} \|\beta_\xi(p, \gamma\gamma_0^{\pm 1}\gamma^{-1}p) \mp \lambda(\gamma_0)\| < \varepsilon$$

where N_0 is as in (7.1). Consider the following collection of d_p-balls for each $R > 0$:

$$\mathcal{B}_R(\gamma_0, \varepsilon) = \{B_p(\gamma\xi_0, r) : \gamma \in \Gamma_\rho, 0 < r < \min(R, r_p(\gamma))\}.$$

Lemma 7.7. *There exists $s(\gamma_0) > 0$ such that for any $R > 0$, the following holds: if $\xi \in \Lambda_\rho$ and $\gamma_\ell \in \Gamma_\rho$ is a sequence such that $\gamma_\ell^{-1}p \to \eta$ and $\gamma_\ell^{-1}\xi \to y_{\gamma_0}$, then for any $0 < r \leq \min(s(\gamma_0), R)$, there exists $\ell_0 = \ell_0(r) > 0$ such that for all $\ell \geq \ell_0$,*

$$D(\gamma_\ell\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon) \quad \textit{and} \quad \xi \in D(\gamma_\ell\xi_0, r)$$

where

$$D(\gamma_\ell\xi_0, r) := B_p\left(\gamma_\ell\xi_0, \frac{1}{3N_0} e^{-\sigma_1(2\underline{a}(\gamma_\ell^{-1}p, p))} r\right).$$

In particular, for any $R > 0$,

$$\Lambda_{\rho,wM} \subset \bigcup_{D \in \mathcal{B}_R(\gamma_0,\varepsilon)} D.$$

Proof. The second assertion is an immediate consequence of the first by the definition of $\Lambda_{\rho,wM}$. The first claim of the lemma is proved as [20, Lemma 10.12] for the case when Γ_ρ is Anosov. Since $|\beta_{\xi_1}(p_1, q_1)| \leq d_1(p_1, q_1)$ for any $p_1, q_1 \in X_1$ and $\xi_1 \in \mathcal{F}_1$, we have that for all $\gamma \in \Gamma_\rho$ and $\xi \in \Lambda_\rho$,

$$-\sigma_1(\underline{a}(p, \gamma p)) \leq \sigma_1(\beta_\xi(p, \gamma p)) \leq \sigma_1(\underline{a}(p, \gamma p)).$$

Substituting [20, Theorem 5.3] with this special property of the linear form σ_1 and using Lemma 7.4, the proof of [20, Lemma 10.12] can be repeated verbatim. We only explain how to define $s(\gamma_0)$: by the choice of $p \in X$, we have $\xi_0 \in O_{\frac{\varepsilon}{8\kappa}}(\eta, p)$ where $\kappa > 0$ is the constant in Lemma 7.4. Therefore we can choose $s = s(\gamma_0) > 0$ so that

$$B_p(\xi_0, e^{\sigma_1(2\lambda(\gamma_0)) + \frac{1}{2}\delta_\Gamma\varepsilon} s) \subset O_{\frac{\varepsilon}{8\kappa}}(\eta, p);$$

$$\sup_{x \in B_p(\xi_0, s)} \|\beta_x(p, \gamma_0^{\pm 1}p) \mp \lambda(\gamma_0)\| < \frac{\varepsilon}{4}.$$

∎

Approximating Borel subsets by d_p-balls in $\mathcal{B}_R(\gamma_0, \varepsilon)$. It is more convenient to use the following conformal measure ν_p (with respect to the basepoint p):

$$d\nu_p(\xi) = e^{\sigma_1(\beta_\xi(o,p))} d\nu_{\text{graph}}(\xi) \tag{7.2}$$

We now use

$$\nu_p(\Lambda_\rho - \Lambda_{\rho,wM}) = 0$$

(Proposition 6.5) to show the following:

Proposition 7.8. *Let $B \subset \mathcal{F}$ be a Borel subset with $\nu_p(B) > 0$. Then for ν_p-a.e. $\xi \in B$,*

$$\lim_{R \to 0} \sup_{\xi \in D \in \mathcal{B}_R(\gamma_0,\varepsilon)} \frac{\nu_p(B \cap D)}{\nu_p(D)} = 1.$$

Proof. Associated to a measurable function $h : \mathcal{F} \to \mathbb{R}$, we define

$$h^*(\xi) = \lim_{R \to 0} \sup_{D \in \mathcal{B}_R(\gamma_0,\varepsilon), \xi \in D} \frac{1}{\nu_p(D)} \int_D h d\nu_p.$$

By Lemma 7.7, h^* is well-defined on $\Lambda_{\rho,wM}$. Hence by Proposition 6.5, h^* is well-defined ν_p-almost everywhere.

The desired statement is obtained by showing that $h^* = h$ and then by taking $h = \mathbf{1}_B$. Again this is proved in [20, Proposition 10.17], and the key ingredient is the covering lemma for the Anosov setting as stated in [20, Lemma 6.12]. Substituting this with our Lemma 7.3, together with the choice $\varepsilon < \delta_\Gamma^{-1}$ and $\sigma_1(\lambda(\gamma_0)) > 1 + \log 3N_0$, we can repeat the proof of [20, Proposition 10.17] verbatim. $\square$

Remark 7.9. In [20], the Anosov property was used to ensure that the Myrberg limit set $\Lambda_{\rho,M}$ has full ν_p-measure, and hence h^* is well-defined ν_p-almost everywhere. However, in our setting where Γ_ρ is not necessarily Anosov, it does not seem that $\nu_p(\Lambda_{\rho,M}) = 1$ in general. Hence we have replaced $\Lambda_{\rho,M}$ by the weak-Myrberg limit set $\Lambda_{\rho,wM}$ that we could prove to have full ν_p-measure. This was sufficient to prove Proposition 7.8, without Anosov property.

Proof of Proposition 7.6. Anosov version of this proposition is [20, Proposition 10.7]; the key ingredients of its proof are [20, Lemma 10.12, Propostion 10.17]. Substituting these respectively by Lemma 7.7 and Proposition 7.8, the proof works in the same way as [20]. We give a brief sketch. Since $\mathsf{E}_{\nu_{\text{graph}}}(\Gamma_\rho) = \mathsf{E}_{\nu_p}(\Gamma_\rho)$, it suffices to show that

$$\lambda(\gamma_0) \in \mathsf{E}_{\nu_p}(\Gamma_\rho).$$

Let $B \subset \mathcal{F}$ be any Borel subset with $\nu_p(B) > 0$ and $\varepsilon > 0$ be any sufficiently small number. By Proposition 7.8, there exist $\gamma \in \Gamma_\rho$ and $0 < r < r_p(\gamma)$ such that

$$\nu_p(B \cap D) > (1 + e^{-\sigma_1(\lambda(\gamma_0)) - \delta_\Gamma \varepsilon})^{-1} \nu_p(D) \tag{7.3}$$

where $D = B_p(\gamma\xi_0, r)$.

Since $r < r_p(\gamma)$, we have by the definition of $D = B_p(\gamma\xi_0, r)$ and $r_p(\gamma)$,

$$D \subset \{\xi \in \Lambda_\rho : \|\beta_\xi(p, \gamma\gamma_0^{\pm 1}\gamma^{-1}p) \mp \lambda(\gamma_0)\| < \varepsilon\}. \tag{7.4}$$

This implies that

$$(B\cap D)\cap\gamma\gamma_0\gamma^{-1}(B\cap D) \subset B\cap\gamma\gamma_0\gamma^{-1}B\cap\{\xi : \|\beta_\xi(p,\gamma\gamma_0\gamma^{-1}p)-\lambda(\gamma_0)\| < \varepsilon\}. \tag{7.5}$$

By the conformality of ν_p and (7.4), we have

$$\nu_p(B\cap D) + \nu_p(\gamma\gamma_0\gamma^{-1}(B\cap D)) \geq (1 + e^{-\sigma_1(\lambda(\gamma_0))-\delta_\Gamma\varepsilon})\nu_p(B\cap D).$$

It follows from (7.3) that

$$\nu_p(B\cap D) + \nu_p(\gamma\gamma_0\gamma^{-1}(B\cap D)) > \nu_p(D). \tag{7.6}$$

Using that $\varepsilon < \delta_\Gamma^{-1}$ and $\sigma(\lambda(\gamma_0)) > 1 + \log 3N_0$, we can check that

$$\gamma\gamma_0\gamma^{-1}D \subset D.$$

Hence the left hand side of (7.6) is at most $\nu_p(D)$. It follows that

$$\nu_p((B\cap D)\cap\gamma\gamma_0\gamma^{-1}(B\cap D)) > 0.$$

By (7.5), this implies that

$$B\cap\gamma\gamma_0\gamma^{-1}B\cap\{\xi : \|\beta_\xi(p,\gamma\gamma_0\gamma^{-1}p)-\lambda(\gamma_0)\| < \varepsilon\}$$

has positive ν_p-measure. Since B and $\varepsilon > 0$ are arbitrary, this proves that $\lambda(\gamma_0) \in \mathsf{E}_{\nu_p}(\Gamma_\rho)$, as desired.

8. Dichotomy theorems for Zariski density of Γ_ρ

We are finally ready to prove our main theorems. Let $\Gamma < G_1$ be of divergence type. Let G_2 be a connected semisimple real algebraic group. Let $\rho : \Gamma \to G_2$ be a discrete faithful Zariski dense representation admitting the continuous equivariant map $f : \Lambda \to \mathcal{F}_2$.

We suppose in the entire section that

either $\operatorname{rank} G_2 = 1$ or that f is a continuous extension of ρ.

Recall that $\nu_{\text{graph}} = (\text{id}\times f)_*\nu_\Gamma$ is the unique (Γ_ρ, σ_1)-conformal measure on Λ_ρ. We sometimes write $\nu_{\text{graph}} = \nu_{\sigma_1}$.

We establish the following dichotomy of which Theorem 1.8 is a special case:

Theorem 8.1. *In each of the two complementary cases, the claims (1)-(4) are equivalent to each other. We assume G_2 is simple only for the implications* (4) $\Rightarrow$ (1) *in the first case and* (1) $\Rightarrow$ (2), (4) *in the second case.*

The first case is as follows:

(1) Γ_ρ *is Zariski dense in G;*
(2) $\mathsf{E}_{\nu_{\text{graph}}}(\Gamma_\rho) = \mathfrak{a}$*;*
(3) $m^{\text{BR}}_{\nu_{\text{graph}}}$ *is NM-ergodic;*

(4) *For any* (Γ_ρ, ψ)*-conformal measure* ν *on* Λ_ρ *for* $\psi \neq \sigma_1$*, we have* $[\nu_{\text{graph}}] \neq [\nu]$*.*

The second case is as follows (in this case $\operatorname{rank} G_2 = 1$ *as a consequence):*

(1) Γ_ρ *is not Zariski dense in* G*;*
(2) $\mathsf{E}_{\nu_{\text{graph}}}(\Gamma_\rho) = \mathbb{R}(\delta_{\rho(\Gamma)}, \delta_\Gamma)$*;*
(3) $m^{\text{BR}}_{\nu_{\text{graph}}}$ *is not* NM*-ergodic;*
(4) *For any* (Γ_ρ, ψ)*-conformal measure* ν *on* Λ_ρ *for a tangent linear form* ψ*, we have* $[\nu_{\text{graph}}] = [\nu]$*.*

Proof. We prove the equivalences in each of two cases.

First case: The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 6.2, Corollary 4.7 and Proposition 3.5. The implication (1) $\Rightarrow$ (2) was proved in Theorem 7.1. By Proposition 3.6, we have that (2) $\Rightarrow$ (4). In order to prove (4) $\Rightarrow$ (1), we assume that Γ_ρ is not Zariski dense in G. By Lemma 4.2, $\rho : \Gamma \to G_2$ extends to a Lie group isomorphism $G_1 \to G_2$, which we also denote by ρ by abuse of notation. In particular, $\operatorname{rank} G_2 = 1$.

For each $i = 1, 2$, let $\alpha_i \in \mathfrak{a}_i^*$ be the simple root of $(\mathfrak{g}_i, \mathfrak{a}_i^+)$ and define $\sigma_i \in \mathfrak{a}^*$ by

$$\sigma_1(u_1, u_2) = \delta_\Gamma \alpha_1(u_1) \text{ and } \sigma_2(u_1, u_2) = \delta_{\rho(\Gamma)} \alpha_2(u_2). \tag{8.1}$$

Since there is a unique left G_i-invariant Riemannian metric on X_i up to a constant multiple, we may assume that the metric d_i is the one induced from the Killing form of $\mathfrak{g}_i$ for each $i = 1, 2$. Note that for each $\xi \in \mathcal{F}_1$ (resp. $\mathcal{F}_2$) and $x, y \in X_1$ (resp. X_2), the product $\delta_\Gamma \beta_\xi(x, y)$ (resp. $\delta_{\rho(\Gamma)} \beta_\xi(x, y)$) does not change after scaling the metric d_1 (resp. d_2) by the definition of the critical exponent.

Since the differential of ρ must send the Killing form of $\mathfrak{g}_1$ to that of $\mathfrak{g}_2$, ρ induces an isometry $X_1 = G_1/K_1 \to X_2 = G_2/\rho(K_1)$, which we again denote by ρ, as well as the equivariant diffeomorphism $F : \mathcal{F}_1 = G_1/P_1 \to \mathcal{F}_2 = G_2/\rho(P_1)$, so that $f = F|_\Lambda$. It follows that

$$\delta_\Gamma = \delta_{\rho(\Gamma)}$$

and for all $\xi \in \mathcal{F}_1$ and $x, y \in X_1$,

$$\beta_\xi(x, y) = \beta_{F(\xi)}(\rho(x), \rho(y)).$$

For simplicity, we set $\delta := \delta_\Gamma = \delta_{\rho(\Gamma)}$ below.

We claim that $\nu_{\sigma_1} = (\operatorname{id} \times f)_* \nu_\Gamma$ is (Γ_ρ, σ_2)-conformal. First note that $f_* \nu_\Gamma$ is a $(\rho(\Gamma), \delta)$-conformal measure on $\Lambda_{\rho(\Gamma)}$ with respect to $o_2 := \rho(o_1)$: for any $g \in \Gamma$ and $\xi \in \mathcal{F}_1$,

$$\begin{aligned}\frac{d\rho(g)_* f_* \nu_\Gamma}{df_* \nu_\Gamma}(f(\xi)) = \frac{df_* g_* \nu_\Gamma}{df_* \nu_\Gamma}(f(\xi)) &= \frac{dg_* \nu_\Gamma}{d\nu_\Gamma}(\xi) \\ &= e^{\delta \beta_\xi(o_1, g o_1)} = e^{\delta \beta_{f(\xi)}(o_2, \rho(g) o_2)}.\end{aligned}$$

Therefore, by Lemma 4.10, the pushforward $(f^{-1}\times \mathrm{id})_* f_*\nu_\Gamma$ is a (Γ_ρ,σ_2)-conformal measure on Λ_ρ. Since

$$\nu_{\sigma_1} = (\mathrm{id}\times f)_*\nu_\Gamma = (f^{-1}\times \mathrm{id})_* f_*\nu_\Gamma,$$

the claim follows. As $\sigma_2\neq\sigma_1$, it shows the implication (4) $\Rightarrow$ (1).

Second case: The first case we just considered implies

$$(2)\Rightarrow(1)\Leftrightarrow(3).$$

We now claim (1) $\Rightarrow$ (2). Suppose that Γ_ρ is not Zariski dense. We use the same notation used in the proof of the implication (4) $\Rightarrow$ (1) of the first case. In particular, $\delta_\Gamma = \delta_{\rho(\Gamma)}$. Recall that we have in this situation that $\nu_{\sigma_1} = (f^{-1}\times \mathrm{id})_* f_*\nu_\Gamma$ is a (Γ_ρ,σ_2)-conformal measure on Λ_ρ. Therefore, Proposition 3.6 implies that

$$\mathsf{E}_{\nu_{\sigma_1}}(\Gamma_\rho)\subset\{\sigma_1=\sigma_2\} = \mathbb{R}(1,1).$$

It follows from Proposition 7.6 (note that we have not assumed that Γ_ρ is Zariski dense in that proposition) that $\mathsf{E}_{\nu_{\sigma_1}}(\Gamma_\rho)$ contains $\lambda(\Delta_\rho)-E$ for some Zariski dense Schottky subgroup $\Delta<\Gamma$ and a finite subset E. Since ρ induces an isometry $X_1\to X_2$, we have $\lambda(\Delta_\rho)\subset\mathbb{R}(1,1)$, and hence $\lambda(\Delta_\rho)-E$ generates a dense subgroup of $\mathbb{R}(1,1)$ by Theorem 2.2. Consequently, $\mathsf{E}_{\nu_{\sigma_1}}(\Gamma_\rho)=\mathbb{R}(1,1)$, as desired.

It remains to prove (1) $\Leftrightarrow$ (4). To prove (4) $\Rightarrow$ (1), suppose that Γ_ρ is Zariski dense. By Corollary 5.3, there exists a (Γ_ρ,ψ)-conformal measure ν on Λ_ρ for a tangent linear form $\psi\neq\sigma_1$. By the first case, $[\nu]\neq[\nu_{\sigma_1}]$. Therefore (4) does not hold. This proves the claim.

We now show that (1) implies (4). Again, suppose that Γ_ρ is not Zariski dense in G, and hence ρ extends to a Lie group isomorphism $G_1\to G_2$. As before, since ρ induces an isometry $X_1\to X_2$, we get the limit cone $\mathcal{L}_\rho=\mathbb{R}_{\geq 0}(1,1)$, $\delta_\Gamma=\delta_{\rho(\Gamma)}$, which we denote by δ, and the growth indicator function $\psi_\rho(t,t)=\delta t$.

Let $\psi\in\mathfrak{a}^*$ be a linear form tangent to ψ_ρ with a (Γ_ρ,ψ)-conformal measure ν_ψ on Λ_ρ. We need to show that $[\nu_\psi]=[\nu_{\sigma_1}]$. Let $\pi:\mathcal{F}\to\mathcal{F}_1$ denote the canonical projection to the first factor. We first claim that $\pi_*\nu_\psi$ is a Γ-conformal measure of dimension δ on Λ. Since $\psi(1,1)=\delta=\sigma_1(1,1)=\sigma_2(1,1)$, we have $\psi=(1-c)\sigma_1+c\sigma_2$ for some $c\in\mathbb{R}$. As ν_ψ is (Γ_ρ,ψ)-conformal, we have for any $\xi\in\Lambda$ and $\gamma=(g,\rho(g))\in\Gamma_\rho$,

$$\frac{d\gamma_*\nu_\psi}{d\nu_\psi}(\xi,f(\xi)) = e^{(1-c)\delta_\Gamma\beta_\xi(o_1,go_1)+c\delta_{\rho(\Gamma)}\beta_{f(\xi)}(o_2,\rho(g)o_2)} = e^{\delta\beta_\xi(o_1,go_1)}$$

where $o_2=\rho(o_1)$. As in the proof of Proposition 4.6, this implies that for any $g\in\Gamma$ and $\xi\in\Lambda$,

$$\frac{dg_*\pi_*\nu_\psi}{d\pi_*\nu_\psi}(\xi) = e^{\delta\beta_\xi(o_1,go_1)},$$

proving the claim. Since Γ is of divergence type, it follows that $\pi_*\nu_\psi$ is equivalent to ν_Γ by Theorem 6.1. As ν_ψ is supported on $\Lambda_\rho = (\mathrm{id}\times f)(\Lambda)$ and π is injective on Λ_ρ with $\pi|_{\Lambda_\rho}^{-1} = \mathrm{id}\times f$, we have

$$\nu_\psi = (\mathrm{id}\times f)_*\pi_*\nu_\psi.$$

Therefore, $[\nu_\psi] = [(\mathrm{id}\times f)_*\nu_\Gamma] = [\nu_{\sigma_1}]$. This finishes the proof. □

Proof of Theorem 1.2. Let $\nu_{\rho(\Gamma)}$ be a $(\rho(\Gamma),\psi)$-conformal measure on $\Lambda_{\rho(\Gamma)}$. Then by Lemma 4.10, $\nu_{\sigma_2} := (f^{-1}\times \mathrm{id})_*\nu_{\rho(\Gamma)}$ is (Γ_ρ,σ_2)-conformal for $\sigma_2 = \psi\circ\pi_2 \in \mathfrak{a}^*$, and

$$[\nu_{\sigma_2}] = [\nu_{\sigma_1}] \text{ if and only if } [\nu_{\rho(\Gamma)}] = [f_*\nu_\Gamma].$$

Hence if $[f_*\nu_\Gamma] = [\nu_{\rho(\Gamma)}]$, then $[\nu_{\sigma_1}] = [\nu_{\sigma_2}]$. From the first case of Theorem 8.1, Γ_ρ is not Zariski dense, which implies that ρ extends to a Lie group isomorphism $G_1 \to G_2$ by Lemma 4.2. Hence we get (1) ⇒ (2).

Conversely, suppose that ρ extends to a Lie group isomorphism $G_1 \to G_2$. Then Γ_ρ is not Zariski dense and $\operatorname{rank} G_2 = 1$. We can choose $\nu_{\rho(\Gamma)}$ to be a $\rho(\Gamma)$-conformal measure on $\Lambda_{\rho(\Gamma)}$ of dimension $\delta_{\rho(\Gamma)}$, and consider σ_2 and ν_{σ_2} defined same as above. Since σ_2 is tangent to ψ_ρ by Lemma 5.4, it follows from the second case of Theorem 8.1 that $[\nu_{\sigma_2}] = [\nu_{\sigma_1}]$; so $[\nu_{\rho(\Gamma)}] = [f_*\nu_\Gamma]$. This proves (2) ⇒ (1).

Remark 8.2. When Γ is of divergence type, ν_Γ is Γ-ergodic and hence the first condition in Theorem 1.2 is same as saying that some $\rho(\Gamma)$-conformal measure is absolutely continuous with respect to $f_*\nu_\Gamma$ (see Lemma 3.2).

9. Measure class rigidity for Anosov representations

One class of Zariski dense discrete subgroups of G where the space of conformal measures on Λ_Δ is well-understood is the class of Anosov subgroups with respect to P. There are several equivalent definitions of Anosov subgroups (one is given in Example 4.3(3)), and the following is due to Kapovich-Leeb-Porti [15] (see also [19], [13]). Recall that Π denotes the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$:

Definition 9.1 (Anosov representation)**.** For a finitely generated group Σ and a non-empty subset $\Pi_0 \subset \Pi$, a representation $\rho : \Sigma \to G$ is said to be Anosov with respect to Π_0 if for all $g \in \Sigma$ and for all $\alpha \in \Pi_0$,

$$\alpha(\mu(\rho(g))) \geq C_1|g| - C_2$$

where $C_1, C_2 > 0$ are uniform constants and $|\cdot|$ is a word metric. Its image $\Delta = \rho(\Sigma)$ is called an Anosov subgroup with respect to Π_0. Anosov subgroups with respect to P mean Anosov subgroups with respect to Π.

We recall the following theorem which follows from [20, Theorem 1.3], together with [11, Theorem 1.4] and [21, Theorem 1.2].

Theorem 9.2 ([20, Theorem 1.3]). *Let $\Delta < G$ be a Zariski dense Anosov subgroup with respect to P. For each linear form ψ tangent to ψ_Δ, there exists a unique (Δ, ψ)-conformal measure ν_ψ on $\mathcal{F}$ with respect to o. Moreover, we have*

1. *the map $\psi \mapsto \nu_\psi$ gives a bijection between the space of all tangent linear forms and the space of all Δ-conformal measures supported on Λ_Δ;*
2. *if $\psi_1 \neq \psi_2$, then ν_{ψ_1} and ν_{ψ_2} are mutually singular to each other.*

We mention that the space of all tangent linear forms in this case is homeomorphic to $\mathbb{R}^{\operatorname{rank} G-1}$. See also [34] for some partial extension for Anosov subgroups with respect to general Π_0.

We now begin the proof of Theorem 1.5 using notations introduced there. Let $G = G_1 \times G_2$ and $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$. Let $\Psi_i = \psi_i \circ \pi_i$ for each $i = 1, 2$. Then Ψ_i is a linear form on $\mathfrak{a}$ and it follows from Proposition 4.6 and Lemma 4.10 that

$$(\mathrm{id} \times f)_* \nu_{\psi_1} \quad \text{and} \quad (f^{-1} \times \mathrm{id})_* \nu_{\psi_2}$$

are (Γ_ρ, Ψ_1)-conformal and (Γ_ρ, Ψ_2)-conformal measures on Λ_ρ respectively.

Suppose that $\rho : \Gamma \to G_2$ does not extend to a Lie group isomorphism $G_1 \to G_2$. Since G_1 and G_2 are simple, Lemma 4.2 implies that Γ_ρ is a Zariski dense Anosov subgroup of $G_1 \times G_2$. Hence by Theorem 9.2, we have

$$(\mathrm{id} \times f)_* \nu_{\psi_1} \perp (f^{-1} \times \mathrm{id})_* \nu_{\psi_2}.$$

As in the proof of Lemma 4.10, this implies that

$$f_* \nu_{\psi_1} \perp \nu_{\psi_2}.$$

This proves Theorem 1.5.

Remark 9.3. We finally mention that replacing [20, Theorem 10.20] by an analogous result in [34] for Anosov representations with respect to a general parabolic subgroup, we can also prove an analogous statement to Theorem 1.5 in those cases provided that the Furstenberg boundaries and limit sets are appropriately replaced as well.

References

[1] I. Agol. Tameness of hyperbolic 3-manifolds. *arXiv preprint math/0405568*, 2004.
[2] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997.
[3] Y. Benoist. Propriétés asymptotiques des groupes linéaires. II. In *Analysis on homogeneous spaces and representation theory of Lie groups, Okayama–Kyoto (1997)*, volume 26 of *Adv. Stud. Pure Math.*, pages 33–48. Math. Soc. Japan, Tokyo, 2000.
[4] M. Bourdon. Structure conforme au bord et flot géodésique d'un CAT(−1)-espace. *Enseign. Math. (2)*, 41(1-2):63–102, 1995.
[5] D. Calegari and D. Gabai. Shrinkwrapping and the taming of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 19(2):385–446, 2006.
[6] R. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 6(1):1–35, 1993.
[7] J. Cannon and W. Thurston. Group invariant Peano curves. *Geom. Topol.*, 11:1315–1355, 2007.

[8] K. Corlette and A. Iozzi. Limit sets of discrete groups of isometries of exotic hyperbolic spaces. *Trans. Amer. Math. Soc.*, 351(4):1507–1530, 1999.

[9] T. Das, D. Simmons, and M. Urbański. Tukia's isomorphism theorem in CAT(−1) spaces. *Ann. Acad. Sci. Fenn. Math.*, 41(2):659–680, 2016.

[10] S. Edwards, M. Lee, and H. Oh. Anosov groups: local mixing, counting, and equidistribution. *arXiv:2003.14277. To appear in Geom. Topol.*

[11] S. Edwards, M. Lee, and H. Oh. Uniqueness of conformal measures and local mixng for anosov groups. *Michigan Math. J. 72 (Prasad volume), 243-259.*

[12] O. Guichard. In preparation.

[13] O. Guichard and A. Wienhard. Anosov representations: domains of discontinuity and applications. *Invent. Math.*, 190(2):357–438, 2012.

[14] M. Kapovich and B. Leeb. Relativizing characterizations of Anosov subgroups, I. *arXiv preprint arXiv:1807.00160*, 2018.

[15] M. Kapovich, B. Leeb, and J. Porti. Anosov subgroups: dynamical and geometric characterizations. *Eur. J. Math.*, 3(4):808–898, 2017.

[16] M. Kapovich, B. Leeb, and J. Porti. A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings. *Geom. Topol.*, 22(7):3827–3923, 2018.

[17] D. M. Kim, Y. Minsky, and H. Oh. Tent property of the growth indicator functions and applications. *arXiv preprint arXiv:2112.00877*, 2021.

[18] D. M. Kim and H. Oh. Rigidity of Kleinian groups via self-joinings. *arXiv preprint arXiv:2208.05806*, 2022.

[19] F. Labourie. Anosov flows, surface groups and curves in projective space. *Invent. Math.*, 165(1):51–114, 2006.

[20] M. Lee and H. Oh. Invariant measures for horospherical actions and Anosov groups. *arXiv:2008.05296. To appear in Int. Math. Res. Not. IMRN.*

[21] M. Lee and H. Oh. Dichotomy and measures on limit sets of anosov groups. *arXiv preprint arXiv:2203.06794*, 2022.

[22] G. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1991.

[23] M. Mj. Cannon-Thurston maps for surface groups. *Ann. of Math. (2)*, 179(1):1–80, 2014.

[24] M. Mj. Cannon-Thurston maps. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. II. Invited lectures*, pages 885–917. World Sci. Publ., Hackensack, NJ, 2018.

[25] G. Mostow. Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms. *Inst. Hautes Études Sci. Publ. Math.*, (34):53–104, 1968.

[26] G. Mostow. *Strong rigidity of locally symmetric spaces.* Annals of Mathematics Studies, No. 78. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1973.

[27] S. Patterson. The limit set of a Fuchsian group. *Acta Math.*, 136(3-4):241–273, 1976.

[28] G. Prasad. Strong rigidity of $\mathbf{Q}$-rank 1 lattices. *Invent. Math.*, 21:255–286, 1973.

[29] J.-F. Quint. Divergence exponentielle des sous-groupes discrets en rang supérieur. *Comment. Math. Helv.*, 77(3):563–608, 2002.

[30] J.-F. Quint. Mesures de Patterson-Sullivan en rang supérieur. *Geom. Funct. Anal.*, 12(4):776–809, 2002.

[31] M. Rees. Checking ergodicity of some geodesic flows with infinite Gibbs measure. *Ergodic Theory Dynam. Systems*, 1(1):107–133, 1981.

[32] T. Roblin. Ergodicité et équidistribution en courbure négative. *Mém. Soc. Math. Fr. (N.S.)*, (95):vi+96, 2003.

[33] A. Sambarino. Infinitesimal Zariski closures of positive representations. *Preprint, arXiv:2012.10276.*

[34] A. Sambarino. A report on an ergodic dichotomy. *arXiv preprint arXiv:2202.02213*, 2022.

[35] K. Schmidt. *Cocycles on ergodic transformation groups.* Macmillan Lectures in Mathematics, Vol. 1. Macmillan Co. of India, Ltd., Delhi, 1977.

[36] D. Sullivan. The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.*, (50):171–202, 1979.

[37] D. Sullivan. On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions. volume 97 of *Ann. of Math. Stud.* Princeton Univ. Press, 1981.

[38] D. Sullivan. Discrete conformal groups and measurable dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 6(1):57–73, 1982.

[39] D. Sullivan. Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups. *Acta Math.*, 153(3-4):259–277, 1984.

[40] P. Tukia. On isomorphisms of geometrically finite Möbius groups. *Inst. Hautes Études Sci. Publ. Math.*, (61):171–214, 1985.

[41] P. Tukia. A rigidity theorem for Möbius groups. *Invent. Math.*, 97(2):405–431, 1989.

[42] A. Yaman. A topological characterisation of relatively hyperbolic groups. *J. Reine Angew. Math.*, 566:41–89, 2004.

[43] C. Yue. Mostow rigidity of rank 1 discrete groups with ergodic Bowen-Margulis measure. *Invent. Math.*, 125(1):75–102, 1996.

Department of Mathematics, Yale University, New Haven, CT 06511
Email address: dongryul.kim@yale.edu

Department of Mathematics, Yale University, New Haven, CT 06511
Email address: hee.oh@yale.edu