DICHOTOMY AND MEASURES ON LIMIT SETS OF ANOSOV GROUPS.

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Abstract. Let G be a connected semisimple real algebraic group. For a Zariski dense Anosov subgroup $\Gamma < G$, we show that a Γ-conformal measure is supported on the limit set of Γ if and only if its *dimension* is Γ-critical. This implies the uniqueness of a Γ-conformal measure for each critical dimension, answering the question posed in our earlier paper with Edwards [14]. We obtain this by proving a higher rank analogue of the Hopf-Tsuji-Sullivan dichotomy for the *maximal diagonal* action. Other applications include an analogue of the Ahlfors measure conjecture for Anosov subgroups.

1. Introduction

Let G be a connected semisimple real algebraic group. In this paper, we investigate properties of Γ-conformal measures on the Furstenberg boundary of G for a certain class of discrete subgroups Γ of G, called *Anosov* subgroups. Associated to each conformal measure does there exist a linear form on the Cartan subspace of the Lie algebra of G, which may be regarded as the *dimension* of the measure. We show that a Γ-conformal measure is supported on the limit set of Γ if and only if this dimension is Γ-critical. We deduce this result from a higher rank analogue of the Hopf-Tsuji-Sullivan dichotomy for the maximal diagonal action, which relates the supports of conformal measures, critical exponents of Poincare series, and the dynamical properties of the action of a maximal diagonal subgroup on $\Gamma\backslash G$ relative to higher rank generalizations of Bowen-Margulis-Sullivan measures. Applications include an analogue of the Ahlfors measure conjecture for Anosov subgroups of G.

To state our main results precisely, we let $P = MAN$ be a minimal parabolic subgroup of G with a fixed Langlands decomposition, where A is a maximal real split torus of G, M is the maximal compact subgroup centralizing A and N is the unipotent radical of P. Let $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{a} = \operatorname{Lie} A$ and $\mathfrak{a}^+$ denote the positive Weyl chamber so that $\log N$ consists of positive root subspaces. Let K be a maximal compact subgroup so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Let $\mu : G \to \mathfrak{a}^+$ denote the Cartan projection map defined by the condition $\exp \mu(g) \in KgK$ for all $g \in G$.

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A finitely generated discrete subgroup $\Gamma < G$ is called an *Anosov subgroup* (with respect to P) if there exist constants $C, C' > 0$ such that for all $\gamma \in \Gamma$ and all simple root α of $(\mathfrak{g}, \mathfrak{a})$,

$$\alpha(\mu(\gamma)) \geq C|\gamma| - C'$$

where $|\gamma|$ denotes the word length of γ with respect to a fixed finite symmetric set of generators of Γ. The notion of Anosov subgroups was first introduced by Labourie for surface groups [26], and was extended to general word hyperbolic groups by Guichard-Wienhard [18]. Several equivalent characterizations have been established, one of which is the above definition (see [17], [21], [22], [23]). Anosov subgroups are regarded as natural generalizations of convex cocompact subgroups of rank one groups.

Uniqueness of conformal measures. We set $\mathcal{F} := G/P$ which is the Furstenberg boundary of G. Let $\Gamma < G$ be a Zariski dense discrete subgroup. A Borel probability measure ν on $\mathcal{F}$ is called a Γ-conformal measure if, there exists a linear form $\psi \in \mathfrak{a}^*$ such that for any $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(e,\gamma))} \tag{1.1}$$

where β denotes the $\mathfrak{a}$-valued Busemann function defined in Def. (2.2). We call ν a (Γ, ψ)-conformal measure and ψ the dimension of ν. Although ψ is a linear form instead of a number, we find it convenient to treat it as a sort of dimension of the measure ν and hence the name.

If ρ denotes the half sum of all positive roots of $(\mathfrak{g}, \mathfrak{a})$, the K-invariant probability measure on $\mathcal{F}$ (the Lebesgue measure) is the unique G-conformal measure of dimension 2ρ [40].

We let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of Γ (see Def. (2.3)). Let $\mathcal{L} \subset \mathfrak{a}^+$ denote the limit cone of Γ, which is the asymptotic cone of the Cartan projection of Γ.

We mention that the dimension of a Γ-conformal measure is always bounded below by ψ_Γ [36]. We call a linear form $\psi \in \mathfrak{a}^*$ Γ-*critical*, or simply, critical, if it is tangent to ψ_Γ, i.e.,

$$\psi \geq \psi_\Gamma \quad \text{and} \quad \psi(u) = \psi_\Gamma(u) \quad \text{for some } u \in \mathcal{L} \cap \operatorname{int} \mathfrak{a}^+.$$

When G has rank one, ψ_Γ is simply the critical exponent δ of Γ and hence a critical linear form is just given by δ. Note that the dimension ψ of a Γ-conformal measure is either critical or $\psi > \psi_\Gamma$.

We denote by Λ the limit set of Γ, which is the unique Γ-minimal subset of $\mathcal{F}$. For each Γ-critical dimension $\psi \in \mathfrak{a}^*$, Quint constructed a (Γ, ψ)-conformal measure supported on the limit set Λ, following the approach of Patterson and Sullivan ([33], [43], [36]). Moreover, for any Anosov subgroup of *the second kind* (see [15, Def. 5.1]), a (Γ, ψ)-conformal measure exists for any dimension $\psi \geq \max(\psi_\Gamma, \rho)$ by [15, Cor. 5.3].

Our first theorem gives a criterion on the support of a conformal measure in terms of its dimension. This generalizes Sullivan's theorem [43] that for

$\Gamma < \mathrm{SO}(n,1)$ convex cocompact, any Γ-conformal measure of dimension equal to the critical exponent is necessarily supported on the limit set.

Theorem 1.2. *Let $\Gamma < G$ be a Zariski dense Anosov subgroup. For any Γ-conformal measure λ on $\mathcal{F}$, we have*

$$\lambda(\Lambda) = \begin{cases} 1 & \textit{if its dimension is } \Gamma\textit{-critical} \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, for each Γ-critical linear form $\psi \in \mathfrak{a}^*$, there exists a unique Γ-conformal measure on $\mathcal{F}$ with dimension ψ.

The second part follows from the first part together with the result in [28], which showed that there exists a unique Γ supported measure *supported on* Λ for each critical dimension. These results together also imply that the space of all Γ-conformal measures on $\mathcal{F}$ is homeomorphic to the space of directions in the interior of the limit cone of Γ. It also follows from [28, Thm. 10.20] that conformal measures of distinct critical dimensions are mutually singular to each other. The study of Γ-conformal measures is directly related to the study of positive joint eigenfunctions on the associated locally symmetric manifold $\Gamma\backslash G/K$ for the ring of G-invariant differential operators ([45], [15]).

Remark 1.1. When the rank of G is at most 3, it was proved in [14] that any conformal measure of critical dimension is supported on Λ, and the general case was posed as an open problem there (see Remark 1.2).

Analogue of the Ahlfors measure conjecture. The Ahlfors measure conjecture [3] says that the limit set of a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is either $\mathbb{S}^2$ or has Lebesgue measure zero; this is now a theorem following from the works of Agol [2], Calegari-Gabai [8] and Canary [9]. The following theorem is analogous to the case of Ahlfors' conjecture proved by Ahlfors himself for convex cocompact subgroups [3]. We denote by Leb the Lebesgue measure on $\mathcal{F}$.

Theorem 1.3. *For any Zariski dense Anosov subgroup $\Gamma < G$, we have either*

$$\Lambda = \mathcal{F} \quad \textit{or} \quad \mathrm{Leb}(\Lambda) = 0.$$

In the former case, $\mathrm{rank}(G) = 1$ *and Γ is cocompact in G.*

Higher rank analogue of the Hopf-Tsuji-Sullivan dichotomy. Both theorems are deduced from a higher rank analogue of the Hopf-Tsuji-Sullivan dichotomy for the action of the maximal diagonal subgroup A. To state this dichotomy, we need to introduce some notations first. Letting $\mathcal{F}^{(2)}$ denote the unique open diagonal G-orbit in $\mathcal{F}\times\mathcal{F}$, the quotient space G/M is homeomorphic to $\mathcal{F}^{(2)} \times \mathfrak{a}$ via the Hopf parameterization. The notation i denotes the opposition involution of $\mathfrak{a}$, and let db denote the Lebesgue measure on $\mathfrak{a}$. For a given pair of Γ-conformal measures λ_ψ and $\lambda_{\psi\circ\mathrm{i}}$ on $\mathcal{F}$ with respect to ψ and $\psi \circ \mathrm{i}$ respectively, one can use the Hopf parameterization to define a non-zero A-invariant Borel measure $\mathsf{m}_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}$ on the quotient space $\Gamma\backslash G/M$, which is locally equivalent to $d\lambda_\psi \otimes d\lambda_{\psi\circ\mathrm{i}} \otimes db$ in the Hopf coordinates. We will call it the Bowen-Margulis-Sullivan measure (or simply BMS measure) associated to the pair $(\lambda_\psi, \lambda_{\psi\circ\mathrm{i}})$. Each BMS measure $\mathsf{m}_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}$ on $\Gamma\backslash G/M$ can be considered as an AM-invariant measure on $\Gamma\backslash G$, for which we will use the same notation. For example, for $\psi = 2\rho = \psi\circ\mathrm{i}$, the corresponding measure $\mathsf{m}_{\lambda_{2\rho},\lambda_{2\rho}}$ is a G-invariant measure on $\Gamma\backslash G$.

The conical limit set of Γ is defined as

$$\Lambda_c = \{gP \in \mathcal{F} : gA^+ \text{ accumulates on } \Gamma\backslash G\},$$

in other words, $\Lambda_c = \{gP \in \mathcal{F} : \limsup \Gamma g A^+ \neq \emptyset\}$,[1] where $A^+ = \exp \mathfrak{a}^+$. For Anosov subgroups, we have

$$\Lambda = \Lambda_c,$$

as proved in [22] using the Morse property.

For $\psi \in \mathfrak{a}^*$, let $\mathcal{M}_\psi$ denote the collection of all (Γ,ψ)-conformal measures.

Theorem 1.4 (Dichotomy for the maximal diagonal action)**.** *Let Γ be a Zariski dense Anosov subgroup of G. Let $\psi \in \mathfrak{a}^*$ be such that $\mathcal{M}_\psi \neq \emptyset$. Then the following are all equivalent to each other:*

1. $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty$ *(resp.* $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} < \infty$*);*
2. ψ *is Γ-critical (resp. $\psi > \psi_\Gamma$);*
3. *for any $\lambda_\psi \in \mathcal{M}_\psi$, $\lambda_\psi(\Lambda_c) > 0$ (resp. $\lambda_\psi(\Lambda_c) = 0$);*
4. *for any $\lambda_\psi \in \mathcal{M}_\psi$, $\lambda_\psi(\Lambda_c) = 1$ (resp. $\lambda_\psi(\Lambda_c) = 0$);*
5. *for any $(\lambda_\psi, \lambda_{\psi\circ\mathrm{i}}) \in \mathcal{M}_\psi \times \mathcal{M}_{\psi\circ\mathrm{i}}$, the diagonal Γ-action on $(\mathcal{F}^{(2)}, \lambda_\psi \otimes \lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}})$ is ergodic and completely conservative (resp. non-ergodic and completely dissipative);*
6. *for any $(\lambda_\psi, \lambda_{\psi\circ\mathrm{i}}) \in \mathcal{M}_\psi \times \mathcal{M}_{\psi\circ\mathrm{i}}$, the A-action on $(\Gamma\backslash G/M, \mathsf{m}_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}})$ is ergodic and completely conservative (resp. non-ergodic and completely dissipative);*
7. *for any $(\lambda_\psi, \lambda_{\psi\circ\mathrm{i}}) \in \mathcal{M}_\psi \times \mathcal{M}_{\psi\circ\mathrm{i}}$ and any P°-minimal subset $\mathcal{E}_0$ of $\Gamma\backslash G$, the A-action on $(\mathcal{E}_0, \mathsf{m}_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}|_{\mathcal{E}_0})$ is ergodic and completely conservative (resp. either $\mathsf{m}_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}(\mathcal{E}_0) = 0$, or non-ergodic and completely dissipative).*

In the rank one case, the A-action on $\Gamma\backslash G/M$ corresponds to the geodesic flow on the unit tangent bundle of the locally symmetric manifold $\Gamma\backslash G/K$. Therefore this theorem generalizes the Hopf-Tsuji-Sullivan dichotomy for the geodesic flow in the rank one case ([46], [43], [44], [19], [1], [11], [31]); we refer to Roblin's article [41] for the most comprehensive exposition.

Theorem 1.3 is deduced from Theorem 1.4 and Theorem 7.1 proved by Quint [38], using the matrix coefficient bounds for higher rank simple algebraic groups in [32]. This in turn implies that, unless $\Gamma\backslash G$ is compact, 2ρ

[1] For a sequence S_n of subsets of a topological space X, $\limsup S_n$ is defined as the set of all possible limits $s = \lim_{i\to\infty} s_{n_i}$ in X where $s_{n_i} \in S_{n_i}$ for some infinite sequence n_i.

is not Γ-critical and hence the Haar measure on $\Gamma\backslash G$ is non-ergodic for the AM-action.

Since there exists a Γ-conformal measure supported on Λ for each critical dimension, Theorem 1.2 immediately follows from Theorem 1.4 together with the uniqueness of Γ-conformal measures supported on Λ [13, Thm. 7.9].

Remark 1.2. When $\operatorname{rank} G$ is at most 3, it was shown in [14] that any (Γ, ψ)-conformal measure is supported on the u-directional conical limit set Λ_u where u is the unique unit vector $\psi(u) = \psi_\Gamma(u)$; this implies Theorem 1.2. The proof of this result was based on the Hopf-Tsuji-Sullivan Dichotomy for *one dimensional* diagonal flows $\{\exp(tu) : t \in \mathbb{R}\}$ as established in [7]. When the rank of G exceeds 3, *directional* conical limit sets have negligible conformal measures, and hence this result of [14] did not prove Theorem 1.2. We note that while the dichotomy for one dimensional diagonal flows was obtained for any Zariski dense discrete subgroup, our proof of Theorem 1.4 is heavily based on the hypothesis that Γ is Anosov.

While some of the implications of Theorem 1.4 were previously obtained in ([28], [29]), the implication (1) $\Rightarrow$ (3) is the main new result of this paper, which is needed for the application to Theorem 1.2. Fixing a (Γ, ψ)-conformal measure λ_ψ for a critical $\psi \in \mathfrak{a}^*$, we consider the generalized Bowen-Margulis-Sullivan measure $\mathsf{m} = m^{\mathrm{BMS}}_{\lambda_\psi, \lambda_{\psi\circ \mathrm{i}}}$ on $\Gamma\backslash G$ for some conformal measure $\lambda_{\psi\circ\mathrm{i}}$ of dimension $\psi\circ\mathrm{i}$ (see (2.4) for the definition). We use a variant of the Borel-Cantelli lemma for the A^+ action (Lemma 5.3) by relating the correlations functions of m with the Poincare series $\sum_{\gamma\in\Gamma, \|\mu(\gamma)\|\le T} e^{-\psi(\mu(\gamma))}$. This requires a control on the multiplicity of certain shadows (Lemma 3.1), the proof of which uses the following property of Anosov subgroups that for any $x \in \Gamma\backslash G$, accumulations of an orbit xA in $\Gamma\backslash G$ can occur only via sequences in $A^+ \cup w_0 A^+ w_0^{-1}$ where w_0 is the longest Weyl element. In other words, for any other Weyl element $w \neq e, w_0$, the subset xwA^+w^{-1} is a proper embedding of wA^+w^{-1}, as was first observed in [28, Lem. 8.13]. See Lemmas 2.8 and 3.4. This phenomenon makes this higher rank situation a bit more like a rank one situation where the one dimensional subgroup A is simply the union $A^+ \cup w_0 A^+ w_0^{-1}$. Based on this and other properties of Anosov subgroups, we are able to extend the rank one argument in [41] to this higher rank Anosov setting.

In a higher rank simple algebraic group, the conical limit has Lebesgue measure zero for a discrete subgroup of infinite co-volume (see Proposition 7.6). We end the introduction by the following question:

Question 1.5. *Let G be a connected simple real algebraic group with rank at least 2 and $\Gamma < G$ be a Zariski dense discrete subgroup. Is the following true?:*

$$\Lambda = \mathcal{F} \quad \textit{if and only if} \quad \Gamma \textit{ is a a lattice in } G.$$

We remark that $\Lambda = \mathcal{F}$ is equivalent to the minimality of the P-action on $\Gamma\backslash G$, which means that every P-orbit is dense in $\Gamma\backslash G$. Hence a weaker (still unknown) question than the above is whether Γ is necessarily a lattice if the NM-action is minimal on $\Gamma\backslash G$, or equivalently if $\mathcal{F}$ is equal to the set of horospherical limit points of Γ, in the sense of [27]. In view of a theorem of Fraczyk and Gelander [16], one can also ask whether the infinite injectivity radius of $\Gamma\backslash G$ implies that Λ cannot be all of $\mathcal{F}$ in higher rank setting.

Organization. In section 2, basic definitions and properties of Anosov subgroups will be recalled. In section 3, we prove a uniform bound on the multiplicity of certain shadows, which is a main technical ingredient. In section 4, we show that if $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty$, then for a large compact subset $Q \subset \Gamma\backslash G$, the events $P_a = Q \cap Qa^{-1}$, $a \in A^+$ do not have a strong correlation with respect to the BMS measures of the form $m^{\mathrm{BMS}}_{\lambda_\psi, \lambda_{\psi\circ \mathrm{i}}}$; this will be used as a main input for the Borel-Cantelli lemma in section 5 to show that any (Γ, ψ)-conformal measure is necessarily supported on the conical limit set Λ_c. In section 6, we establish all the equivalences of Theorem 1.4. In section 7, we prove Theorem 1.3.

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2. Preliminaries

Let G be a connected semsimple real algebraic group. We let $P = MAN$, $\mathfrak{g}, \mathfrak{a}, \mathfrak{a}^+$, etc, be as defined in the introduction. We fix a maximal compact subgroup $K < G$ so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection, i.e., for $g \in G$, its Cartan projection $\mu(g) \in \mathfrak{a}^+$ is the unique element such that

$$g \in K \exp \mu(g) K. \tag{2.1}$$

We fix a norm $\|\cdot\|$ on $\mathfrak{a}$ which is induced from the Killing form on $\mathfrak{g}$. The quotient space $X = G/K$ is the associated Riemannian symmetric space. We denote by d the Riemannian distance on X induced by $\|\cdot\|$. We also set $o = [K] \in X$.

Denote by $w_0 \in K$ a representative of the unique element of the Weyl group $\mathcal{W} = N_K(A)/M$ such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined by

$$\mathrm{i}(u) = -\mathrm{Ad}_{w_0}(u) \quad \text{for } u \in \mathfrak{a}.$$

We have $\mathrm{i}(\mu(g)) = \mu(g^{-1})$ for all $g \in G$.

The Furstenberg boundary $\mathcal{F} = G/P$ is isomorphic to K/M as K acts on $\mathcal{F}$ transitively with $K \cap P = M$. The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi) \tag{2.2}$$

where the Iwasawa cocycle $\sigma(g^{-1},\xi)\in\mathfrak{a}$ is defined by the relation $g^{-1}k\in K\exp(\sigma(g^{-1},\xi))N$ for $\xi=kP$, $k\in K$.

Let $\Gamma<G$ be a Zariski dense discrete subgroup of G. Denote by $\mathcal{L}\subset\mathfrak{a}^+$ the limit cone of Γ, which is the asymptotic cone of $\mu(\Gamma)$, i.e.,

$$\mathcal{L}=\{v\in\mathfrak{a}^+ : v=\lim_{i\to\infty} t_i\mu(\gamma_i) \text{ for some } t_i\to 0 \text{ and } \gamma_i\to\infty \text{ in } \Gamma\}.$$

It is a convex cone with non-empty interior [4].

The growth indicator function $\psi_\Gamma : \mathfrak{a}^+\to\mathbb{R}\cup\{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(tu)=t\psi_\Gamma(u)$ for all $t\in\mathbb{R}$, such that for any unit vector $u\in\mathfrak{a}^+$,

$$\psi_\Gamma(u):=\inf_{u\in\mathcal{C},\text{open cones } \mathcal{C}\subset\mathfrak{a}^+}\tau_{\mathcal{C}} \tag{2.3}$$

where $\tau_{\mathcal{C}}$ is the abscissa of convergence of the series $\sum_{\gamma\in\Gamma,\mu(\gamma)\in\mathcal{C}} e^{-t\|\mu(\gamma)\|}$. We have $\psi_\Gamma\geq 0$ on $\mathcal{L}$ and $\psi_\Gamma=-\infty$ outside $\mathcal{L}$.

The generalized BMS-measures m_{ν_1,ν_2}. For $g\in G$, we consider the following visual images:

$$g^+:=gP\in\mathcal{F} \quad \text{and} \quad g^-:=gw_0P\in\mathcal{F}.$$

Let $\mathcal{F}^{(2)}$ denote the unique open G-orbit in $\mathcal{F}\times\mathcal{F}$ under the diagonal action. In fact,

$$\mathcal{F}^{(2)}=\{(g^+,g^-):g\in G\}.$$

Then the map

$$gM\mapsto(g^+,g^-,b=\beta_{g^-}(e,g))$$

gives a homeomorphism $G/M\simeq\mathcal{F}^{(2)}\times\mathfrak{a}$, called the Hopf parametrization of G/M.

For a pair of linear forms $\psi_1,\psi_2\in\mathfrak{a}^*$ and a pair of (Γ,ψ_1) and (Γ,ψ_2) conformal measures ν_1 and ν_2 respectively, define a locally finite Borel measure $\tilde{m}_{\nu_1,\nu_2}$ on G/M as follows: for $g=(g^+,g^-,b)\in\mathcal{F}^{(2)}\times\mathfrak{a}$,

$$d\tilde{m}_{\nu_1,\nu_2}(g)=e^{\psi_1(\beta_{g^+}(e,g))+\psi_2(\beta_{g^-}(e,g))}\,d\nu_1(g^+)d\nu_2(g^-)db, \tag{2.4}$$

where $db=d\ell(b)$ is the Lebesgue measure on $\mathfrak{a}$. By abuse of notation, we also denote by $\tilde{m}_{\nu_1,\nu_2}$ the M-invariant measure on G induced by $\tilde{m}_{\nu_1,\nu_2}$. This is always left Γ-invariant and right A quasi-invariant: for all $a\in A$,

$$a_*\tilde{m}_{\nu_1,\nu_2}=e^{(-\psi_1+\psi_2\circ\mathrm{i})(\log a)}\,\tilde{m}_{\nu_1,\nu_2};$$

we refer to [13] for more details on these measures. We denote by m_{ν_1,ν_2} the M-invariant measure on $\Gamma\backslash G$ induced by $\tilde{m}_{\nu_1,\nu_2}$.

We will need the following notion:

Definition 2.5. *Let $g_i\in G$ be a sequence whose Cartan decomposition is given by $g_i=k_ia_i\ell_i\in KA^+K$. As $i\to\infty$,*

(1) we say that $g_i\to\infty$ regularly if $\alpha(\log a_i)\to\infty$ for all simple root α of $(\mathfrak{g},\mathfrak{a})$;

(2) we say that g_i converges to $\xi \in \mathcal{F}$, if $g_i \to \infty$ regularly and $\lim_{i\to\infty} k_i^+ = \xi$;
(3) we say that $p_i = g_i(o) \in X$ converges to $\xi \in \mathcal{F}$ if g_i does.

We then define the limit set Λ of Γ as the set of all accumulation points of $\Gamma(o)$ in $\mathcal{F}$; this is the unique Γ-minimal subset ([28, Lem. 2.13], [4]). As in the introduction, we also define the conical limit set:

$$\Lambda_c = \left\{ gP \in \mathcal{F} : \begin{array}{c} \text{there exist } \gamma_i \in \Gamma \text{ and } a_i \to \infty \text{ in } A^+ \\ \text{such that } \gamma_i g a_i \text{ is bounded} \end{array} \right\}.$$

In the rest of this section, we assume that $\Gamma < G$ is a Zariski dense Anosov subgroup (with respect to P) as defined in the introduction. We collect some important properties of Anosov subgroups that we will be using.

Lemma 2.6. ([22], [18]) *If $\Gamma < G$ is Anosov, then we have:*

(1) *(Regularity) If $\gamma_i \to \infty$ in Γ, then $\gamma_i \to \infty$ regularly as $i \to \infty$.*
(2) *(Antipodality) If $\xi, \eta \in \Lambda$ are distinct, then $(\xi, \eta) \in \mathcal{F}^{(2)}$.*
(3) *(Conicality) $\Lambda = \Lambda_c$.*

Indeed, these three properties characterize Anosov subgroups [22, Thm. 1.1]. Note that the regularity of (1) implies that $\Gamma(o) \cup \Lambda$ is compact. Moreover, by [28, Lem. 2.10], we have:

Lemma 2.7. *For any compact subset $Q \subset G$, the union $\Gamma(Q) \cup \Lambda$ is compact.*

The following is a consequence of the antipodal property of Anosov subgroups, and plays a key role in this paper.

Lemma 2.8. [28, Lem. 8.13] *Let $\Gamma < G$ be Anosov. For $x = [g] \in \Gamma\backslash G$, the following are equivalent:*

(1) $\limsup xA \neq \emptyset$;
(2) $\limsup xA^+ \cup \limsup xw_0A^+ \neq \emptyset$;
(3) $\{gP, gw_0P\} \cap \Lambda \neq \emptyset$.

Theorem 2.9. [34] *For Γ Anosov, we have*

$$\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}.$$

Corollary 2.10. *If $\Gamma < G$ is Anosov and rank $G \geq 2$, there exists no finite A-invariant Borel measure on $\Gamma\backslash G$.*

Proof. Suppose there exists a finite A-invariant Borel measure m on $\Gamma\backslash G$. Let $v \in \operatorname{int} \mathfrak{a}^+$. By the Poincare recurrence theorem, m-almost all points are recurrent for the action of $\exp \mathbb{R}v$. In particular, there exist $g \in G$, $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that $\gamma_i g \exp(t_i v)$ is bounded. Then the sequence $\mu(\gamma_i^{-1})$ stays in a bounded distance from the ray $\mathbb{R}_+ v$ by [4, Lem. 4.6]; it follows that $v \in \mathcal{L}$. Therefore $\mathcal{L} = \mathfrak{a}^+ \cup \{0\}$. If rank $G \geq 2$, then $\mathfrak{a}^+ - \{0\} \neq \operatorname{int} \mathfrak{a}^+$. Hence the claim follows from Theorem 2.9. $\square$

3. Uniform bound on the multiplicity of shadows

For $p \in X = G/K$ and $S > 0$, we set $B(p, S) := \{x \in X : d(x, p) < S\}$.

Recall the notation $o = [K] \in X = G/K$. For $p \in X$ and $S > 0$, the shadow of the ball $B(p, S)$ as seen from o is defined by

$$O_S(o, p) := \{\xi \in \mathcal{F} : \text{ for some } k \in K \text{ with } \xi = kP,\ kA^+o \cap B(p, S) \neq \emptyset\}.$$

Lemma 3.1. *Let $\Gamma < G$ be a Zariski dense Anosov subgroup of G. For any $S, D > 0$, there exists $q = q(S, D) > 0$ such that for any $T > 0$, the shadows*

$$\{O_S(o, \gamma o) : T < \|\mu(\gamma)\| < T + D\}$$

have multiplicity at most q.

The rest of this section is devoted to the proof of Lemma 3.1.

Throughout the section, we fix a compact subset Q of G. The notation $x \approx_Q y$ means that $x - y$ is contained in a bounded set that depends only on Q. We will simply write $x \approx y$ if the implicit bounded set depends only on Γ and G.

Lemma 3.2. [4, Lem. 4.6] *For all $g \in G$ and $q_1, q_2 \in Q$, we have*

$$\mu(q_1 g q_2) \approx_Q \mu(g).$$

Lemma 3.3. *Let $a \in A$ and $w \in \mathcal{W}$ be such that $waw^{-1} \in A^+$. If $Q \cap \gamma Q a^{-1} \neq \emptyset$, then $\mu(\gamma) \approx_Q \mathrm{Ad}_w \log a$.*

Proof. If $Q \cap \gamma Q a^{-1} \neq \emptyset$, then there exists $q_0, q_0' \in Q$ such that $q_0 a = \gamma q_0'$. The conclusion follows from Lemma 3.2. □

We set $A^- = w_0 A^+ w_0^{-1}$, and for any $C > 0$, set $A_C := \{a \in A : \|\log a\| \leq C\}$. The following lemma is a key ingredient in the proof of Lemma 3.1; we use the regularity and antipodality of Anosov subgroups.

Lemma 3.4. *Let $\Gamma < G$ be Anosov. There exists $C_0 > 1$ depending only on Q such that whenever $Q \cap \gamma_1 Q a_1^{-1} \cap \gamma_2 Q a_2^{-1} \neq \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma$ and $a_1, a_2 \in A^+$, we have*

1. *$a_1^{-1} a_2 \in (A^+ \cup A^-) A_{C_0}$;*
2. *$\mu(\gamma_2) \approx_Q \mu(\gamma_1) + \mu(\gamma_1^{-1}\gamma_2)$ or $\mu(\gamma_1) \approx_Q \mu(\gamma_2) + \mu(\gamma_2^{-1}\gamma_1)$.*

Proof. We first prove (1). Suppose not. Then there exists a compact set $Q \subset G$ and sequences $q_{0,i}, q_{1,i}, q_{2,i} \in Q$, $a_{1,i}, a_{2,i} \in A^+$ and $\gamma_{1,i}, \gamma_{2,i} \in \Gamma$ such that

$$a_{1,i}^{-1} a_{2,i} \notin (A^+ \cup A^-) A_i, \tag{3.5}$$

$$q_{0,i}\, a_{1,i} = \gamma_{1,i}\, q_{1,i}, \quad q_{0,i}\, a_{2,i} = \gamma_{2,i}\, q_{2,i} \tag{3.6}$$

where $A_i = \{a \in A : \|\log a\| \leq i\}$.

Observe that (3.5) implies $a_{1,i}^{-1} a_{2,i} \to \infty$ in A and $a_{1,i}, a_{2,i} \to \infty$ in A^+. Observe that $a_{1,i}, a_{2,i} \to \infty$ regularly, by (3.6) and Lemmas 2.6 and 3.2.

Passing to a subsequence, we may assume that for each $m = 1, 2$, $q_{m,i}$ converges to some $q_m \in Q$, and $\gamma_{m,i}^{-1} q_{0,i} o$ converges to some element $\xi \in \Lambda$ as $i \to \infty$. Since $\gamma_{m,i}^{-1} q_{0,i} o = q_{m,i} a_{m,i}^{-1} o$, it follows that $\xi = q_m^-$ by [28, Lem. 2.11] for each $m = 1, 2$. Therefore $q_m^- \in \Lambda$. On the other hand, we have

$$\gamma_{1,i}^{-1} \gamma_{2,i}\, q_{2,i} = q_{1,i}\, a_{1,i}^{-1} a_{2,i}. \tag{3.7}$$

Note that $\gamma_{1,i}^{-1} \gamma_{2,i} \to \infty$ and there exists $w_i \in \mathcal{W} - \{e, w_0\}$ such that $w_i^{-1} a_{1,i}^{-1} a_{2,i} w_i \in A^+$. Passing to a subsequence, we may assume that $w_i = w$ is constant and $\gamma_{1,i}^{-1} \gamma_{2,i}\, q_{2,i}\, o$ converges to an element of Λ by Lemma 2.7. By (3.7) and [28, Lem. 2.11], it follows that $q_1 w^+ \in \Lambda$. This contradicts Lemma 2.6, as neither $q_1 w^+ = q_1^-$ nor $(q_1 w^+, q_1^-) \in \mathcal{F}^{(2)}$, proving (1).

To prove (2), observe that we have $\mu(\gamma_1) \approx_Q \log a_1$, $\mu(\gamma_2) \approx_Q \log a_2$ by Lemma 3.3, since $Q \cap \gamma_1 Q a_1^{-1} \cap \gamma_2 Q a_2^{-1} \neq \emptyset$. On the other hand, it follows from (1) that

$$\mu(\gamma_2^{-1}\gamma_1) \approx_Q \log a_1^{-1} a_2 \quad \text{or} \quad \mu(\gamma_1^{-1}\gamma_2) \approx_Q \log a_2^{-1} a_1.$$

Hence (2) is proved. $\square$

The following lemma follows from Theorem 2.9 and the fact that the angle between two walls of a Weyl chamber is at most $\pi/2$.

Lemma 3.8. *There exist constants $\beta_1, \beta_2 > 0$ depending only on Γ such that for all $x, y \in \mu(\Gamma)$, we have*

$$\|x + y\|^2 \geq \|x\|^2 + \|y\|^2 + \beta_1 \|x\| \|y\| - \beta_2.$$

Proof of Lemma 3.1. Suppose that there exist $\xi \in \bigcap_{i=1}^n O_S(o, \gamma_i o)$ and $T < \|\mu(\gamma_i)\| < T + D$ for some γ_i $(i = 1, \cdots, n)$. Set $Q := K A_S^+ K$. Choose $k \in K$ such that $\xi = kP$. Then $d(kA^+ o, \gamma_i o) \leq S$. It follows that there exists a sequence $a_1, \cdots, a_n \in A^+$ such that $k \in Q \cap \gamma_1 Q a_1^{-1} \cap \cdots \cap \gamma_n Q a_n^{-1}$.

We claim that there exists $D' = D'(Q, D) > 0$ such that

$$\max_{i,j} \|\mu(\gamma_i^{-1} \gamma_j)\| < D'. \tag{3.9}$$

This implies that $n \leq \#\{\gamma \in \Gamma : \|\mu(\gamma)\| \leq D'\}$.

To prove (3.9), we apply Lemma 3.4(2) to each pair (γ_i, γ_j); suppose first that $\mu(\gamma_j) \approx_Q \mu(\gamma_i) + \mu(\gamma_i^{-1}\gamma_j)$. Since $\|\mu(\gamma_j)\| \leq T + D$, there exists $D_1 = D_1(Q) > 0$ such that

$$\|\mu(\gamma_i) + \mu(\gamma_i^{-1}\gamma_j)\|^2 \leq (\|\mu(\gamma_j)\| + D_1)^2 \leq (T + D + D_1)^2. \tag{3.10}$$

Set $D_2 = D + D_1$. By Lemma 3.8 and (3.10), we deduce that

$$\beta_1 \|\mu(\gamma_i^{-1}\gamma_j)\| T + \|\mu(\gamma_i^{-1}\gamma_j)\|^2 < 2 D_2 T + D_2^2 + \beta_2,$$

in particular, $\|\mu(\gamma_i^{-1}\gamma_j)\| < \max(\sqrt{D_2^2 + \beta_2}, 2 D_2 \beta_1^{-1})$. The other case of Lemma 3.4(2) also yields the same conclusion by a symmetric argument.

This proves the claim (3.9). $\square$

We remark that the boundedness of the multiplicity of the *intersection* of shadows and the limit set for projective Anosov representations, with respect to the *word length* $|\gamma|$ is given [35, Prop. 3.5].

4. Poincare series and the average of correlations

Let $\Gamma < G$ be a Zariski dense Anosov subgroup. We fix $\psi \in \mathfrak{a}^*$ and a (Γ, ψ)-conformal measure λ_ψ on $\mathcal{F}$ (not necessarily supported on Λ). We assume that

$$\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty.$$

This implies that ψ is Γ-critical by [39, Lem. III.1.3]. Therefore, there exists a $(\Gamma, \psi \circ \mathrm{i})$-conformal measure, say $\lambda_{\psi\circ\mathrm{i}}$, e.g., as constructed by Quint.

Let

$$\tilde{\mathsf{m}} = \tilde{m}^{\mathrm{BMS}}_{\lambda_\psi, \lambda_{\psi\circ\mathrm{i}}}$$

denote the generalized BMS measure on G, which is left Γ-invariant and right AM-invariant.

The notations $x \lesssim_z y$ (resp. $x \ll_z y$) are to be understood that $x \le y + C$ (resp. $x \le Cy$) for some constant $C > 0$ that depends on z.

The main aim of this section is to prove the following proposition. For $r > 0$ and any subset $S \subset A$, we set $S_r = \{a \in S : \|\log a\| \le r\}$.

Proposition 4.1. *Let $Q_r = KA_r^+KA_r$ for $r > 0$. For any sufficiently large $r > 1$, the following holds: for any $T \ge 1$,*

$$\int_{A_T^+}\int_{A_T^+} \sum_{\gamma_1,\gamma_2\in\Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma_1 Q_r a_1^{-1} \cap \gamma_2 Q_r a_2^{-1})\, da_1\, da_2 \ll \Bigg(\sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\le T}} e^{-\psi(\mu(\gamma))} \Bigg)^2$$

and

$$\int_{A_T^+} \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a^{-1})\, da \gg \sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\le T}} e^{-\psi(\mu(\gamma))}$$

where the implied constants depend only on r.

The rest of this section is devoted to the proof of this proposition, given as the proofs of Propositions 4.11 and 4.14.

The $\mathfrak{a}$-valued Gromov product on $\mathcal{F}^{(2)}$ is defined as follows: for $(g^+, g^-) \in \mathcal{F}^{(2)}$,

$$\mathcal{G}(g^+, g^-) := \beta_{g^+}(e, g) + \mathrm{i}\left(\beta_{g^-}(e, g)\right);$$

this is well-defined independent of the choice of $g \in G$.

Lemma 4.2. [5, Prop. 8.12] *There exist c, $c' > 0$ such that for all $g \in G$,*

$$c^{-1}\|\mathcal{G}(g^+, g^-)\| \le d(o, gAo) \le c\|\mathcal{G}(g^+, g^-)\| + c'.$$

For $r > 0$, let
$$G_r = KA_r^+K$$
and
$$\mathcal{L}_r(o, go) := \{(h^+, h^-) \in \mathcal{F}^{(2)} : h \in G_r,\ ha \in gG_r \text{ for some } a \in A^+\}.$$

Lemma 4.3. *For any $g \in G$ and $r > 0$, we have*
$$\mathcal{L}_r(o, go) \subset O_{2r}(o, go) \times O_{2r}(go, o).$$

Proof. Let $(\xi, \eta) \in \mathcal{L}_r(o, go)$. Then there exists $h \in G_r$ such that $(h^+, h^-) = (\xi, \eta)$ and $d(hA^+o, go) < r$. Let $k \in K$ be such that $k^+ = h^+$. Then the Hausdorff distance between kA^+o and hA^+o is given by $d(o, ho) < r$ [12, 1.6.6(4)] and hence $d(kA^+o, go) < 2r$. It follows that $\xi = k^+ \in O_{2r}(o, go)$. A similar computation shows that $\eta \in O_{2r}(go, o)$. □

Lemma 4.4. *Let $r > 0$. If $g \in Q_r \cap \gamma Q_r a^{-1}$ for $\gamma \in \Gamma$ and $a \in A^+$, then*
(1) *$(g^+, g^-) \in \mathcal{L}_{2r}(o, \gamma o)$.*
(2) *$|\psi(\mathcal{G}(g^+, g^-))| < 2\|\psi\| cr$ where c is from Lemma 4.2.*
(3) *$gA \cap Q_r \cap \gamma Q_r a^{-1} \subset gA_{4r}$.*

Proof. (1) follows from the definition since $Q_r \subset G_{2r}$. (2) follows from Lemma 4.2 and the fact that $d(gAo, o) \le d(go, o) < 2r$. (3) follows from the stronger inclusion $gA \cap Q_r \subset gA_{4r}$; if g, $gb \in Q_r$ for some $b \in A$, then $b \in Q_r \cdot Q_r \subset G_{4r}$ since $Q_r \subset G_{2r}$. Note that $G_{4r} \cap A = A_{4r}$. □

We will need the following shadow lemma:

Lemma 4.5. [28, Lem. 7.8]*: There exists $S_0 > 0$ such that for all $S \ge S_0$ and all $\gamma \in \Gamma$, we have*
$$e^{-\psi(\mu(\gamma))} \ll \lambda_\psi(O_S(o, \gamma o)) \ll e^{-\psi(\mu(\gamma))}.$$
with implied constants independent of γ.

Lemma 4.6. *Let $r > 0$. For any $a \in A^+$, we have*
$$\tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a^{-1}) \ll_r e^{-\psi(\mu(\gamma))}.$$

Proof. By Lemmas 4.3, 4.4 and 4.5, we have
$$\begin{aligned}
&\tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a^{-1}) \\
&= \int_{\mathcal{L}_{2r}(o,\gamma o)} \left(\int_A \mathbb{1}_{Q_r \cap \gamma Q_r a^{-1}}(gb) e^{\psi(\mathcal{G}(g^+, g^-))}\, db \right) d\lambda_\psi(g^+)\, d\lambda_{\psi\circ \mathrm{i}}(g^-) \\
&\le \int_{O_{4r}(o,\gamma o) \times O_{4r}(\gamma o, o) \cap \mathcal{F}^{(2)}} \mathrm{Vol}(A_{4r}) e^{2\|\psi\| cr}\, d\lambda_\psi(g^+)\, d\lambda_{\psi\circ \mathrm{i}}(g^-) \\
&\ll_r e^{-\psi(\mu(\gamma))},
\end{aligned}$$
which proves the lemma. □

The following is easy to prove (cf. [7, Lem. 5.14]).

Lemma 4.7. *There exists $\ell_0 > 0$ such that any $\gamma \in \Gamma$ with $\|\mu(\gamma)\| > \ell_0$ and any $(\xi, \eta) \in O_{S_0}(o, \gamma o) \times O_{S_0}(\gamma o, o)$ satisfies $\|\mathcal{G}(\xi, \eta)\| < \ell_0$.*

In the rest of this section, we fix constants S_0, ℓ_0 c, c' from Lemmas 4.2, 4.5 and 4.7. We set

$$r_0 := S_0 + c\ell_0 + c' + 1. \tag{4.8}$$

Lemma 4.9. *For all $r > r_0$, there exists $C_2 = C_2(r) > 0$ such that for any $T \geq C_2$ and any $g \in G$ with*

$$(g^+, g^-) \in \bigcup \{O_{S_0}(o, \gamma o) \times O_{S_0}(\gamma o, o) : \gamma \in \Gamma, \ell_0 < \|\mu(\gamma)\| < T - C_2\},$$

we have

$$\int_{A_T^+} \int_A \mathbb{1}_{Q_r \cap \gamma Q_r a^{-1}}(gb)\, db\, da \geq \operatorname{Vol}(A_r) \operatorname{Vol}(A_1^+).$$

Proof. Let $C_2' = C_2'(r)$ be the implied constant in Lemma 3.3 associated to $Q = Q_r$. Set $C_2 := C_2' + 1$. Let $T > C_2$. Let $g \in G$ and $\gamma \in \Gamma$ be such that $\ell_0 < \|\mu(\gamma)\| < T - C_2$ and $(g^+, g^-) \in O_{S_0}(o, \gamma o) \times O_{S_0}(\gamma o, o)$. By Lemmas 4.2 and 4.7, we have $d(o, gAo) \leq c\|\mathcal{G}(g^+, g^-)\| + c' \leq c\ell_0 + c'$. Therefore, we may assume without loss of generality that $d(o, go) \leq c\ell_0 + c'$ by replacing g by an element of gA.

Since $g^+ \in O_{S_0}(o, \gamma o)$, there exists $k \in K$ such that $k^+ = g^+$ and $d(kao, \gamma o) < S_0$ for some $a \in A^+$.

Since $d(kao, gao) \leq d(o, go)$ by [12, 1.6.6(4)], we get

$$\begin{aligned} d(\gamma o, gA^+ o) &\leq d(\gamma o, kao) + d(kao, gao) \\ &\leq d(\gamma o, kao) + d(ko, go) \\ &\leq S_0 + c\ell_0 + c' = r_0 - 1. \end{aligned}$$

Since $r \geq r_0$, we have $g \in G_{r-1}$ and $ga_0 \in \gamma G_{r-1}$ for some $a_0 \in A^+$.

Therefore $g \in G_{r-1} \cap \gamma G_{r-1} a_0^{-1}$. By Lemma 3.3, this implies that $\|\mu(\gamma) - \log a_0\| \leq C_2'$. Since $\|\mu(\gamma)\| \leq T - C_2' - 1$, we have $a_0 \in A_{T-1}^+$, and hence $a_0 A_1^+ \subset A_T^+$. Since $Q_r = G_r A_r$, we have $gA_r \subset Q_r \cap \gamma Q_r a^{-1}$ for all $a \in a_0 A_1$. Therefore

$$\begin{aligned} &\int_{A_T^+} \int_A \mathbb{1}_{Q_r \cap \gamma Q_r a^{-1}}(gb)\, db\, da \geq \int_{a_0 A_1^+} \int_A \mathbb{1}_{Q_r \cap \gamma Q_r a^{-1}}(gb)\, db\, da \\ &\geq \int_{a_0 A_1^+} \int_A \mathbb{1}_{gA_r}(gb)\, db\, da \geq \operatorname{Vol}(A_1^+) \operatorname{Vol}(A_r). \end{aligned}$$

This finishes the proof. □

We now deduce the following from Lemma 3.1 and the shadow lemma 4.5.

Lemma 4.10. *For any $D > 0$, we have:*

$$\sup_{T>0} \sum_{\substack{\gamma \in \Gamma, \\ T < \|\mu(\gamma)\| < T + D}} e^{-\psi(\mu(\gamma))} < \infty.$$

Proof. For any $T > 0$,

$$\sum_{T<\|\mu(\gamma)\|<T+D} e^{-\psi(\mu(\gamma))} \ll \sum_{T<\|\mu(\gamma)\|<T+D} \lambda_\psi(O_{S_0}(o,\gamma o)) \le q(S_0, D)$$

where $q(S_0, D)$ is given by Lemma 3.1. This proves Lemma 4.10. ☐

We are now ready to give estimates for correlation functions in terms of Poincaré series, which was the main goal of the section.

Proposition 4.11. *For all $r > r_0$, we have, for all $T \ge 1$,*

$$\int_{A_T^+}\int_{A_T^+} \sum_{\gamma_1,\gamma_2\in\Gamma} \tilde{\mathsf{m}}(Q_r\cap\gamma_1 Q_r a_1^{-1}\cap\gamma_2 Q_r a_2^{-1})\, da_1\, da_2 \ll_r \Bigg(\sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\le T}} e^{-\psi(\mu(\gamma))}\Bigg)^2.$$

Proof. Let $C_0 > 0$ be as in Lemma 3.4(1) associated to $Q = Q_r$. Set

$$E_{\gamma_1,\gamma_2} := \left\{(a_1,a_2)\in A_T^+\times A_T^+ : \begin{array}{c} Q_r\cap\gamma_1 Q_r a_1^{-1}\cap\gamma_2 Q_r a_2^{-1}\neq\emptyset \\ \mu(\gamma_2)\approx_{Q_r}\mu(\gamma_1)+\mu(\gamma_1^{-1}\gamma_2)\end{array}\right\}, \quad (4.12)$$

where the implied constant for $\approx_{Q_r}$ is chosen to be the one in Lemma 3.4(2) with $Q = Q_r$. Note that by Lemma 3.3, the subset E_{γ_1,γ_2} is contained in some bounded ball around $(\mu(\gamma_1),\mu(\gamma_2))$ whose radius depends only on r. Hence the volume of E_{γ_1,γ_2} has a uniform upper bound depending only on r. Observe that if there exists $(a_1,a_2)\in E_{\gamma_1,\gamma_2}$, then $\|\mu(\gamma_i)\|\approx\|\log a\|\le T$. Since the angle between any two walls of $\mathfrak{a}^+$ is at most $\pi/2$, we deuce $\|\mu(\gamma_1^{-1}\gamma_2)\|\le\|\mu(\gamma_1)+\mu(\gamma_1^{-1}\gamma_2)\|\lesssim_r\|\mu(\gamma_2)\|\lesssim T$.

Therefore we get

$$\begin{aligned}
&\int_{A_T^+}\int_{A_T^+}\sum_{\gamma_1,\gamma_2\in\Gamma}\tilde{\mathsf{m}}(Q_r\cap\gamma_1 Q_r a_1^{-1}\cap\gamma_2 Q_r a_2^{-1})\,da_1\,da_2\\
&\le 2\int_{A_T^+}\int_{A_T^+}\sum_{\gamma_1,\gamma_2\in\Gamma}\tilde{\mathsf{m}}(Q_r\cap\gamma_1 Q_r a_1^{-1}\cap\gamma_2 Q_r a_2^{-1})\mathbb{1}_{E_{\gamma_1,\gamma_2}}(a_1,a_2)\,da_1\,da_2\\
&\ll_r \int_{A_T^+}\int_{A_T^+}\sum_{\gamma_1,\gamma_2\in\Gamma} e^{-\psi(\mu(\gamma_2))}\mathbb{1}_{E_{\gamma_1,\gamma_2}}(a_1,a_2)\,da_1\,da_2\\
&\ll_r \sum_{\substack{\gamma_1,\gamma_2\in\Gamma,\\ \|\mu(\gamma_1)\|\lesssim_r T,\\ \|\mu(\gamma_1^{-1}\gamma_2)\|\lesssim_r T}} e^{-\psi(\mu(\gamma_1))}e^{-\psi(\mu(\gamma_1^{-1}\gamma_2))}\int_{A_T^+}\int_{A_T^+}\mathbb{1}_{E_{\gamma_1,\gamma_2}}(a_1,a_2)\,da_1\,da_2\\
&\ll_r \Bigg(\sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\lesssim_r T}} e^{-\psi(\mu(\gamma))}\Bigg)^2; \qquad (4.13)
\end{aligned}$$

note here that the first inequality follows from Lemma 3.4(2) and the symmetricity of the expression with respect to γ_1,γ_2. The second inequality is due to Lemma 4.6. The third inequality is valid again by Lemma 3.4(2).

The last inequality is obtained by reindexing $\gamma_1^{-1}\gamma_2 \in \Gamma$ with a new variable. Finally note that (4.13) together with Lemma 4.10 finishes the proof. $\square$

Proposition 4.14. *For all $r > r_0$, we have, for all $T > 0$,*

$$\int_{A_T^+} \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a^{-1})\, da \gg_r \sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\le T}} e^{-\psi(\mu(\gamma))}.$$

Proof. By Lemma 4.7, for all $a \in A^+$ and $\gamma \in \Gamma$ with $\|\mu(\gamma)\| \ge \ell_0$,

$$\begin{aligned}&\tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a^{-1})\\ &\gg \int_{O_{S_0}(o,\gamma o)\times O_{S_0}(\gamma o, o)\cap \mathcal{F}^{(2)}} \int_A \mathbb{1}_{Q_r\cap\gamma Q_r a^{-1}}(gb)\, db\, d\lambda_\psi(g^+)\, d\lambda_{\psi\circ \mathrm{i}}(g^-).\end{aligned}$$

Hence by Lemmas 4.5 and 4.9,

$$\int_{A_T^+} \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a^{-1})\, da \gg_r \sum_{\substack{\gamma\in\Gamma\\ \|\mu(\gamma)\|\le T-C_2}} e^{-\psi(\mu(\gamma))}.$$

This finishes the proof in view of Lemma 4.10. $\square$

5. Conical limit points and Poincare series

We begin by recalling:

Lemma 5.1. [28, Lem. 7.11]. *Let $\Gamma < G$ be a Zariski dense discrete subgroup and $\psi \in \mathfrak{a}^*$. If there exists a (Γ,ψ)-conformal measure λ_ψ with $\lambda_\psi(\Lambda_c) > 0$, then*

$$\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty.$$

The goal of this section is to establish the converse for Anosov subgroups:

Proposition 5.2. *Let $\Gamma < G$ be a Zariski dense Anosov subgroup of G. Let $\psi \in \mathfrak{a}^*$. If $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty$, then for any (Γ,ψ)-conformal measure λ_ψ, we have*

$$\lambda_\psi(\Lambda_c) > 0.$$

We will need the following version of the Borel-Cantelli lemma:

Lemma 5.3. *Let (Ω, M) be a Borel probability measure space. Let $\mathbb{1}_{P_a}(\omega)$ be a jointly measurable function of $(a,\omega) \in A^+\times\Omega$. Suppose that $\int_{A^+} \mathsf{M}(P_a)\, da = \infty$ and*

$$\liminf_{T\to\infty} \frac{\int_{A_T^+}\int_{A_T^+} \mathsf{M}(P_a\cap P_b)\, db\, da}{\left(\int_{A_T^+} \mathsf{M}(P_a)\, da\right)^2} \le C \tag{5.4}$$

for some $C < \infty$. Then we have

$$\mathsf{M}(\{\omega : \int_{A^+} \mathbb{1}_{P_a}(\omega)\, da = \infty\}) > \frac{1}{C}.$$

Proof. The proof is an easy adaption of [1]. Set

$$b(t,\omega)=\frac{\int_{A_t^+}\mathbb{1}_{P_a}(\omega)\,da}{\int_{A_t^+}\mathsf{M}(P_a)\,da}\quad\text{and}\quad \Omega_0=\{\omega:\int_{A^+}\mathbb{1}_{P_a}(\omega)\,da=\infty\}.$$

Since $\int_{A^+}\mathsf{M}(P_a)\,da=\infty$, we have $b(t,\omega)\to 0$ as $t\to\infty$ for all $\omega\in\Omega_0^c$. On the other hand, by (5.4), there exists $t_n\to\infty$ such that

$$C_n:=\int_\Omega b(t_n,\omega)^2\,d\mathsf{M}(\omega)\to C_\infty$$

for some $C_\infty\le C$, as $n\to\infty$. Since the family of functions $\omega\mapsto b(t_n,\omega)$ is uniformly bounded in their L^2-norms, they are uniformly integrable. Hence $\int_{\Omega_0^c} b(t_n,\omega)\,d\mathsf{M}(\omega)\to 0$ as $n\to\infty$. Since $\int_\Omega b(t,\omega)\,d\mathsf{M}(\omega)=1$, it follows that

$$\lim_{n\to\infty}\int_{\Omega_0} b(t_n,\omega)\,d\mathsf{M}(\omega)=1.$$

By the Cauchy-Schwartz inequality, for all $n\ge 1$,

$$\mathsf{M}(\Omega_0)C_n=\int_{\Omega_0}1^2\,d\mathsf{M}(\omega)\int_{\Omega_0}b(t_n,\omega)^2\,d\mathsf{M}(\omega)\ge\Big(\int_{\Omega_0}b(t_n,\omega)\,d\mathsf{M}(\omega)\Big)^2.$$

Therefore

$$\mathsf{M}(\Omega_0)\ge\lim_{n\to\infty}\frac{1}{C_n}=\frac{1}{C_\infty}\ge\frac{1}{C}$$

and the conclusion follows. $\square$

Let m denote the measure on $\Gamma\backslash G$ induced from $\tilde{\mathsf{m}}=\tilde{m}^{\mathrm{BMS}}_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}$. For $r>0$, set $G_r:=KA_r^+K$, $Q_r:=G_rA_r$ and $\mathsf{M}:=\mathsf{m}|_{\Gamma Q_r}$. For all $a\in A^+$, set

$$P_a:=\Gamma(Q_r\cap\Gamma Q_r a^{-1})\subset\Gamma\backslash G.$$

We will prove:

Proposition 5.5. *For all sufficiently large $r>1$, we have, for all $T\ge 1$,*

$$\int_{A_T^+}\mathsf{M}(P_a)\,da\gg_r\sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\le T}}e^{-\psi(\mu(\gamma))}\ \textit{and}$$

$$\int_{A_T^+}\int_{A_T^+}\mathsf{M}(P_{a_1}\cap P_{a_2})\,da_1\,da_2\ll_r\Big(\sum_{\substack{\gamma\in\Gamma,\\ \|\mu(\gamma)\|\le T}}e^{-\psi(\mu(\gamma))}\Big)^2.$$

Proof. Note that for all $a,a_1,a_2\in A^+$,

$$\mathsf{M}(P_a)\gg_r\sum_{\gamma\in\Gamma}\tilde{\mathsf{m}}(Q_r\cap\gamma Q_r a^{-1})\ \text{and}$$

$$\mathsf{M}(P_{a_1}\cap P_{a_2})\ll_r\sum_{\gamma_1,\gamma_2\in\Gamma}\tilde{\mathsf{m}}(Q_r\cap\gamma_1Q_ra_1^{-1}\cap\gamma_2Q_ra_2^{-1});$$

Indeed, if Q_r is small enough so that it injects to $\Gamma\backslash G$, then we have equalities in the above by the definition of measures. In general, the above inequalities follow by covering Q_r by finitely many subsets which inject to $\Gamma\backslash G$ and the implied constants depend only on the multiplicity of the covering. With this observation, the proposition follows from Propositions 4.11 and 4.14. $\square$

Proof of Proposition 5.2. By Proposition 5.5 and Lemma 5.3, the following set has positive m-measure:

$$W_r := \{[g] \in \Gamma Q_r : \int_{A^+} \mathbb{1}_{\Gamma Q_r}(ga)\, da = \infty\} \tag{5.6}$$

for all r large enough. On the other hand, note that for all $[g] \in W_r$, there exists $a_i \to \infty$ in A^+ such that $[g]a_i$ is bounded; and hence $g^+ \in \Lambda_c$. Therefore $\lambda_\psi(\Lambda_c) > 0$. This finishes the proof. $\square$

6. Dichotomy theorem for the A-action

We begin by recalling the notion of complete conservativity and dissipativity. Let H be either a countable group or a connected closed subgroup of A. We denote by dh the Haar measure on H. Consider the dynamical system (Ω, μ, H) where Ω is a separable, locally compact and σ-compact topological space on which H acts continuously and μ is a Radon measure which is quasi-invariant by H.

A Borel subset $B \subset \Omega$ is called wandering if $\int_H \mathbb{1}_B(h.w)dh < \infty$ for μ-almost all $w \in B$. The Hopf decomposition theorem says that Ω can be written as the disjoint union $\Omega_C \cup \Omega_D$ of H-invariant subsets where Ω_D is a countable union of wandering subsets which is maximal in the sense that Ω_C does not contain any wandering subset of positive measure. If $\mu(\Omega_D) = 0$ (resp. $\mu(\Omega_C) = 0$), the system is called completely conservative (resp. dissipative). When (Ω, μ, H) is ergodic, H is countable and μ is atom-less, then it is completely conservative (cf. [20, Thm. 14]).

The following is standard (cf. [7, Proof of Thm . 4.2])

Lemma 6.1. *Suppose that μ is H-invariant. Then (Ω, H, μ) is completely conservative if and only if for μ a.e. $x \in \Omega$, there exists a compact subset $B_x \subset \Omega$ such that $\int_{h\in H} \mathbb{1}_{B_x}(h.x)\, dh = \infty$.*

Proof. Suppose (Ω, H, μ) is completely conservative. Suppose that there exists a μ-positive measurable subset $E \subset \Omega$ such that for all $x \in E$, $\int_{h\in H} \mathbb{1}_B(h.x)\, dh < \infty$ for any compact subset B. Then any compact subset of E with positive measure is a wandering set. This proves the only if direction. Now suppose that for μ a.e. $x \in \Omega$, there exists a compact subset $B_x \subset \Omega$ such that $\int_{h\in H} \mathbb{1}_{B_x}(h.x)\, dh = \infty$. Assume that there exists a wandering set $W \subset \Omega$ with $0 < \mu(W) < \infty$. By the σ-compactness of Ω, there exists a compact subset $B \subset \Omega$ such that

$$\mu\{x \in W : \int_H \mathbb{1}_B(h.x)dh = \infty\} \geq \mu(W)/2. \tag{6.2}$$

For any $n \in \mathbb{N}$, set $W_n := \left\{w \in W : \int_H \mathbb{1}_W(h.w)\, dh \leq n\right\}$. Fix n such that $\mu(W_n) > \mu(W)/2$. For any compact subset $C \subset H$, we get, using the

H-invariance of μ,

$$\begin{aligned}
&\int_{W_n}\int_C \mathbb{1}_B(h.w)\,dh\,d\mu = \int_C\int_{W_n} \mathbb{1}_B(h.w)\,d\mu\,dh \\
&= \int_C \mu(B\cap hW_n)\,dh = \int_C\int_{B\cap HW_n} \mathbb{1}_{W_n}(h^{-1}.x)d\mu dh \\
&= \int_{B\cap HW_n}\int_C \mathbb{1}_{W_n}(h^{-1}.x)\,dh\,d\mu \le \int_{B\cap HW_n}\int_H \mathbb{1}_{W_n}(h^{-1}.x)\,dh\,d\mu \\
&\le \int_{B\cap HW_n} n\,d\mu \le n\cdot\mu(B) < \infty.
\end{aligned}$$

Hence $\int_{W_n}\int_H \mathbb{1}_B(h.w)\,dh\,d\mu<\infty$; so

$$\mu\{x\in W : \int_H \mathbb{1}_B(h.w)\,dh<\infty\} \ge \mu(W_n) > \mu(W)/2,$$

contradicting (6.2). □

Proof of Theorem 1.4. The equivalence (1) $\Leftrightarrow$ (2) follows from [37, Prop. 2.8, Lem. 4.4] and [34, Prop. 2.10] (see also [28, Cor. 7.12]).

The equivalence (3) $\Leftrightarrow$ (4) follows because the restriction of λ_ψ to any Γ-invariant measurable subset is again a (Γ,ψ)-conformal measure, up to a positive constant multiple, if not-trivial.

The equivalence (5) $\Leftrightarrow$ (6) follows from the Γ-equivariant homeomorphism $\mathcal{F}^{(2)}\simeq G/AM$ and Lemma 6.1. More precisely, for any Γ-invariant subset $Z\subset\mathcal{F}^{(2)}$, define a Γ-invariant subset $\tilde Z\subset G/M$ by

$$\tilde Z := Z\times A\subset \mathcal{F}^{(2)}\times A$$

using the Hopf parametrization $\mathcal{F}^{(2)}\times A\simeq G/M$. We may view $\tilde Z$ as an A-invariant subset of $\Gamma\backslash G/M$ as well. It follows from Lemma 6.1 that the assignment $Z\mapsto\tilde Z$ preserves the conservativity (and complete dissipativity) of the action of Γ on $(\mathcal{F}^{(2)},\lambda_\psi\otimes\lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}})$ and the action of A on $(\Gamma\backslash G/M, m_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}})$. The equivalence (5) $\Leftrightarrow$ (6) now follows in view of the Hopf decompositions (see the beginning of Section 6) for the systems $(\mathcal{F}^{(2)},\lambda_\psi\otimes\lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}},\Gamma)$ and $(\Gamma\backslash G/M, m_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}, A)$.

The direction (3) $\Rightarrow$ (1) is proved in [28, Lem. 7.11].

The direction (1) $\Rightarrow$ (3) was shown in Proposition 5.2.

For the implication (4) $\Rightarrow$ (5), we will use that all (1) $-$ (4) are equivalent. Suppose that $\lambda_\psi(\Lambda_c)=1$, and hence ψ is Γ-critical. In this case, see ([42], [28, Cor. 4.9]) for the AM-ergodicity of $m_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}$. Hence $\lambda_\psi\otimes\lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}}$ is ergodic. To prove it is conservative, observe that since $\lambda_\psi(\Lambda_c)=1$, and no point in Λ_c can be an atom by Lemma 4.5, λ_ψ is atom-less. Therefore $\lambda_\psi\otimes\lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}}$ has no atom. This implies $\lambda_\psi\otimes\lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}}$ is conservative by [20, Thm. 14]. Therefore (5) follows. To show (5) $\Rightarrow$ (4), suppose that $m_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}$ is completely conservative and ergodic. Fix any compact subset

$B \subset \Gamma\backslash G/M$. Then by Lemma 2.8, we have for $g \in G$,

$\limsup \Gamma gAM \cap B \neq \emptyset$ if and only if $\limsup \Gamma g(A^+ \cup w_0 A^+ w_0^{-1})M \cap B \neq \emptyset$.

Therefore, it follows that $\max(\lambda_\psi(\Lambda_c), \lambda_{\psi\circ\mathrm{i}}(\Lambda_c)) > 0$. On the other hand, Since $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \sum_{\gamma\in\Gamma} e^{-\psi\circ\mathrm{i}(\mu(\gamma))}$, the equivalence (4) $\Leftrightarrow$ (1) implies that $\min(\lambda_\psi(\Lambda_c), \lambda_{\psi\circ\mathrm{i}}(\Lambda_c)) > 0$. This proves (4).

If $\lambda_\psi(\Lambda_c) = 0$, the measure $\lambda_\psi \otimes \lambda_{\psi\circ\mathrm{i}}|_{\mathcal{F}^{(2)}}$ must be non-ergodic by the previous argument, which shows that any ergodic measure would be conservative, which would then imply $\max(\lambda_\psi(\Lambda_c), \lambda_{\psi\circ\mathrm{i}}(\Lambda_c)) > 0$ and hence $\lambda_\psi(\Lambda_c) > 0$ by the equivalence of (1) and (3). This completes the proof of the equivalence (4) $\Leftrightarrow$ (5).

These establish the equivalence of all (1)-(6). To see that these are all equivalent to (7), we recall that for any Γ-critical ψ, the ergodicity of the A-action on $(\mathcal{E}_0, m_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}|_{\mathcal{E}_0})$ is proved in [29, Thm. 1.1]. The conservativity (resp. dissipativity) in (6) and the conservativity in (7) (resp. dissipativity) are equivalent as the projection $\mathcal{E}_0 \to \Gamma\backslash G/M$ has the compact fiber, which is a closed subgroup between M° and M. Hence $(\mathcal{E}_0, A, m_{\lambda_\psi,\lambda_{\psi\circ\mathrm{i}}}|_{\mathcal{E}_0})$ is conservative only when ψ is Γ-critical by (2) $\Leftrightarrow$ (6). This completes the proof of Theorem 1.4.

We also show the following:

Proposition 6.3. *If ψ is Γ-critical, then for any $(\lambda_\psi, \lambda_{\psi\circ\mathrm{i}}) \in \mathcal{M}_\psi \times \mathcal{M}_{\psi\circ\mathrm{i}}$, the diagonal Γ-action on $(\mathcal{F}\times\mathcal{F}, \lambda_\psi \otimes \lambda_{\psi\circ\mathrm{i}})$ is ergodic and completely conservative.*

Proof. By Theorem 1.4, it suffices to show that

$$(\lambda_\psi \times \lambda_{\psi\circ\mathrm{i}})((\Lambda\times\Lambda) - \Lambda^{(2)}) = 0.$$

Set $Q := \Lambda\times\Lambda - \Lambda^{(2)}$ and $Q(x) := \{y \in \Lambda : (x, y) \in Q\}$ for each $x \in \Lambda$. By Lemma 2.6(2), we have

$$Q(x) = \{x\} \quad \text{for all } x \in \Lambda.$$

On the other hand, the conical property of an Anosov subgroup (Lemma 2.6(3)) implies that λ_ψ is not atomic (Prop. 7.4 and Lem. 7.8 of [28]), and hence $\lambda_\psi(Q(x)) = 0$ for all $x \in \Lambda$. Therefore

$$(\lambda_\psi \times \lambda_{\psi\circ\mathrm{i}})(Q) = \int_{x\in\Lambda} \lambda_\psi(Q(x))\, d\lambda_{\psi\circ\mathrm{i}}(x) = 0, \tag{6.4}$$

proving the proposition. $\square$

7. Growth indicator function and Lebesgue measure of Λ

We denote by ρ the half sum of all positive roots of $(\mathfrak{g}, \mathfrak{a})$. A subset $\mathcal{S}$ of positive roots is called strongly orthogonal if any any two distinct roots α, β in $\mathcal{S}$ are strongly orthogonal to each other, i.e., neither of $\alpha\pm\beta$ is a root. Let Θ denote the half sum of all roots in a maximal strongly orthogonal system of $(\mathfrak{g}, \mathfrak{a})$; this does not depend on the choice of a maximal strongly orthogonal system (see [32] where Θ is explicitly given for each simple algebraic group).

Theorem 7.1. *Let G be a connected semisimple real algebraic group with no rank one simple factors. Let $\Gamma < G$ be a discrete subgroup of infinite co-volume. Then*

$$\psi_\Gamma \le 2\rho - \Theta.$$

Proof. This is proved by Quint [38], but the above explicit bound was not formulated, although his proof certainly gives that. We give a slightly different and more direct proof for the sake of completeness.

Note that the right translation action of G on $\Gamma\backslash G$ gives a unitary representation $L^2(\Gamma\backslash G)$ with no non-zero fixed vector as $\Gamma\backslash G$ has infinite volume. We then use [32, Thm. 1.2] which gives that for any K-invariant functions $f \in L^2(\Gamma\backslash G)$, any $v \in \mathfrak{a}^+$, and any $\varepsilon > 0$,

$$\langle (\exp v) f, f\rangle \le d_\varepsilon e^{-(1-\varepsilon)\Theta(v)} \|f\|_2^2 \tag{7.2}$$

where $d_\varepsilon > 0$ depends only on ε. Therefore this theorem follows from Proposition 7.3. □

Proposition 7.3. *Suppose that there exists a function $\theta : \mathfrak{a}^+ \to \mathbb{R}$ such that for any K-invariant functions $f \in L^2(\Gamma\backslash G)$, any $v \in \mathfrak{a}^+$, and any $\varepsilon > 0$,*

$$\langle (\exp v) f, f\rangle \le d_\varepsilon e^{-(1-\varepsilon)\theta(v)} \|f\|_2^2 \tag{7.4}$$

where $d_\varepsilon > 0$ depends only on ε. Then

$$\psi_\Gamma \le 2\rho - \theta.$$

Proof. Fix $u \in \mathfrak{a}^+$ be a unit vector such that $\psi_\Gamma(u) > 0$. Fix an open cone $\mathcal{C} \subset \mathfrak{a}^+$ containing u, and set $\mathcal{C}_T = \{v \in \mathcal{C} : \|v\| \le T\}$ and $B_T = K \exp(\mathcal{C}_T)K$ for each $T > 1$.

Define

$$F_T(g,h) := \sum_{\gamma\in\Gamma} \mathbb{1}_{B_T}(g^{-1}\gamma h)$$

which we regard as a function on $\Gamma\backslash G \times \Gamma\backslash G$. Let $\varepsilon > 0$. Let $U_\varepsilon = KU_\varepsilon K$ be a symmetric open neighborhood of e which injects to $\Gamma\backslash G$ such that $U_\varepsilon B_T U_\varepsilon \subset B_{T+\varepsilon}$ for all $T > 1$. Let Φ_ε be a non-negative K-invariant continuous function supported in $\Gamma\backslash\Gamma U$ with $\int_{\Gamma\backslash G} \Phi_\varepsilon dx = 1$.

Let

$$\eta = \eta_{\mathcal{C}} := \sup\{|2\rho(v) - 2\rho(u)| : v \in \mathcal{C}, \|v\| = 1\}.$$

Using that for $g = k_1(\exp v)k_2$, $dg = \Xi(v)dk_1dvdk_2$ with $\Xi(v) \asymp e^{2\rho(v)}$, and (7.4), we compute

$$\begin{aligned}
&\#\Gamma \cap B_T = F_T(e,e)\\
&\leq \int_{\Gamma\backslash G\times\Gamma\backslash G} F_{T+\varepsilon}([g],[h])\Phi_\varepsilon([g])\Phi_\varepsilon([h])dgdh\\
&= \int_{\Gamma\backslash G\times\Gamma\backslash G} \sum_{\gamma\in\Gamma} \mathbb{1}_{B_{T+\varepsilon}}(g^{-1}\gamma h)\Phi_\varepsilon([g])\Phi_\varepsilon([h])\,dg\,dh\\
&= \int_{\Gamma\backslash G}\int_G \mathbb{1}_{B_{T+\varepsilon}}(g^{-1}h)\Phi_\varepsilon([g])\Phi_\varepsilon([h])\,dg\,dh\\
&= \int_{\Gamma\backslash G}\int_G \mathbb{1}_{B_{T+\varepsilon}}(g^{-1})\Phi_\varepsilon([h]g)\Phi_\varepsilon([h])\,dg\,dh\\
&= \int_{K\exp(\mathcal{C}_{T+\varepsilon})K}\left(\int_{\Gamma\backslash G}\Phi_\varepsilon([h]g)\Phi_\varepsilon([h])\,dh\right)dg\\
&\asymp \int_{v\in\mathcal{C}_{T+\varepsilon}} \langle \exp v.\Phi_\varepsilon, \Phi_\varepsilon\rangle e^{2\rho(v)}dv\\
&\leq d_\varepsilon \int_{v\in\mathcal{C}_{T+\varepsilon}} e^{(2\rho-(1-\varepsilon)\theta)(v)}dv\cdot\|\Phi_\varepsilon\|_2^2\\
&\leq d_\varepsilon \int_0^{T+\varepsilon}\int_{v\in\mathcal{C},\|v\|=1} e^{(2\rho-(1-\varepsilon)\theta)(tv)}dvdt\cdot\|\Phi_\varepsilon\|_2^2\\
&\ll e^{(2\rho-(1-\varepsilon)\theta)((T+\varepsilon)u)+2(T+\varepsilon)\eta}
\end{aligned}$$

where the implied constants are independent of $T > 1$. Therefore

$$\limsup_{T\to\infty}\frac{\log\#(\Gamma\cap B_T)}{T} \leq (2\rho-\theta)(u) + \varepsilon\theta(u) + 2\eta.$$

On the other hand, when $\psi_\Gamma(u) > 0$,

$$\psi_\Gamma(u) = \inf_{u\in\mathcal{C}}\limsup_{T\to\infty}\frac{\log\#(\Gamma\cap K\exp(\mathcal{C}_T)K)}{T}$$

where the infimum is taken over all open cones $\mathcal{C}$ containing u. Since $\eta = \eta_{\mathcal{C}} \to 0$ as $\mathcal{C}$ shrinks to the ray $\mathbb{R}_+u$, we get

$$\psi_\Gamma(u) \leq (2\rho-\theta)(u) + \varepsilon\theta(u).$$

Since $\varepsilon > 0$ was arbitrary, this implies

$$\psi_\Gamma(u) \leq (2\rho-\theta)(u)$$

as desired. □

Remark 7.1. (1) Corlette's theorem [10] shows a uniform gap theorem as above for rank one groups with property (T).

(2) We remark that in a recent work [24], a stronger bound $\psi_\Gamma \leq \rho$ was conjectured for Γ Anosov.

A connected simple real algebraic group is isomorphic to one of the following groups: $\mathrm{SO}(n,1), \mathrm{SU}(n,1), \mathrm{Sp}(n,1), F_4$, which are groups of isometries of real, complex, quarternionic hyperbolic spaces and the Cayley plane respectively. If X denotes the corresponding Riemannian symmetric space as listed above, the Hausdorff dimension of ∂X with respect to the Riemannian metric is given by $k(n+1)-2$ where $k = 1, 2, 4$, and 22 respectively([10], [30]) ; they are equal to the volume entropy D_X of X with respect to a properly normalized Riemannian metric on X.

The following theorem is well-known due to the works of Sullivan ([43], [45]), Corlette [10] and Corlette-Iozzi [11].

Theorem 7.5. *Let G be a connected simple algebraic group of rank one. Let $\Gamma < G$ be a convex cocompact subgroup such that $\Gamma\backslash G$ is not compact. Then*

$$\dim_H(\Lambda) < \dim_H(\partial X).$$

where $\dim_H$ denotes the Hausdorff dimension with respect to the Riemannian metric on ∂X.

Proof. Let δ denotes the critical exponent of Γ. By [11, Thm. 6.1, Cor. 6.2], δ is equal to $\dim_H(\Lambda)$ and the bottom, say, λ_0 of the L^2-spectrum of the negative Laplacian is given by $\delta(D_X - \delta)$. Now suppose that $\delta = D_X$. By ([10, Thm. 5.5], [45]), there exists a unique harmonic function on $\Gamma\backslash X$, and it is square-integrable. Since the constant function is a harmonic function, it follows that $\Gamma\backslash X$ has finite volume, and hence compact, as Γ is assumed to be convex cocompact. This proves the claim. □

We now deduce Theorem 1.3 from Theorems 1.4 and 7.1.

Proof of Theorem 1.3. Let $\Gamma < G$ be Zariski dense and Anosov. If $\mathrm{rank} G = 1$ and $\Gamma < G$ is cocompact, then it is immediate that $\Lambda = \mathcal{F}$. We now suppose that Γ is not a cocompact lattice in a rank one group G. We claim that the Lebesgue measure of Λ is zero. We write $G = G_1 G_2$ where G_1 is a product of all simple factors of rank one, and G_2 is a product of all simple factors of rank at least 2. Consider first the case when G_2 is trivial. Then Γ is of the form: $\Gamma = \left(\prod_{i=1}^k \pi_i\right)(\Sigma)$ where Σ is a Gromov hyperbolic group and π_i is a convex cocompact representation of Σ into a rank one simple factor of G. If $k = 1$, it follows from Theorem 7.5. If $k \geq 2$, then the Hausdorff dimension of Λ is at most the maximum of the Hausdorff dimension of the boundary of a rank one factor of G (cf. proof of [25, Theorem 3.1]); therefore it is strictly smaller than the Hausdorff dimension of G/P. Hence the Lebesgue measure of Λ is zero. Now suppose that G_2 is not trivial. Let $p : G \to G_2$ denote the canonical projection. By the Anosov property of Γ, the projection $p(\Gamma) < G_2$ is again an Anosov subgroup. It suffices to prove that the limit set of $p(\Gamma)$ has Lebesgue measure zero. Therefore, we may assume without loss of generality that $G = G_2$. Since Γ has infinite co-volume in G, as $\pi(\Gamma)$ is Gromov hyperbolic, it follows from Theorem 7.1 that the growth indicator function ψ_Γ of Γ satisfies $\psi_\Gamma < 2\rho$, i.e., 2ρ is not Γ-critical. Since the Lebesgue measure on $\mathcal{F}$ is the $(G, 2\rho)$-conformal measure, Theorem 1.4 implies the claim.

Remark 7.2. Note that it is the consequence of Theorem 1.3 that $\psi_\Gamma < 2\rho$ for all Anosov subgroups of G which is not cocompact in G.

For a general discrete subgroup $\Gamma < G$, we record the following:

Proposition 7.6. *If $\Gamma < G$ is a discrete subgroup with $\psi_\Gamma < 2\rho$, then the Lebesgue measure of the conical limit set Λ_c is zero. In particular, if Γ and G are as in Theorem 7.1,* $\mathrm{Leb}(\Lambda_c) = 0$.

Proof. If $\psi_\Gamma < 2\rho$, then $\sum_{\gamma \in \Gamma} e^{-2\rho(\mu(\gamma))} < \infty$ by [39, Lem. III 1.3]. By [28, Lem. 7.11] (Lemma 5.1), this implies that $\mathrm{Leb}(\Lambda_c) = 0$. □

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