THE HOPF-TSUJI-SULLIVAN DICHOTOMY IN HIGHER RANK AND APPLICATIONS TO ANOSOV SUBGROUPS

MARC BURGER, OR LANDESBERG, MINJU LEE, AND HEE OH

Abstract. We establish an extension of the Hopf-Tsuji-Sullivan dichotomy to any Zariski dense discrete subgroup of a semisimple real algebraic group G. We then apply this dichotomy to Anosov subgroups of G, which surprisingly presents a different phenomenon depending on the rank of the ambient group G.

Contents

1. Introduction

Let G be a connected simple real algebraic group of rank one, (X, d) the associated Riemannian symmetric space and ∂X the geometric boundary of X. We fix a base point $o \in X$, and $\pi : \mathrm{T}^1(X) \to X$ denotes the canonical projection of a vector to its basepoint. The Hopf parametrization of the unit tangent bundle $\mathrm{T}^1(X)$ maps a vector $v \in \mathrm{T}^1(X)$ to

$$(v^+, v^-, \beta_{v^+}(o, \pi(v)))$$

where $v^+, v^- \in \partial X$ are respectively the forward and backward endpoints of the geodesic determined by v and for $\xi \in \partial X$ and $x, y \in X$, $\beta_\xi(x, y)$ denotes the Busemann function given by

$$\beta_\xi(x, y) = \lim_{z \to \xi} d(y, z) - d(x, z).$$

This gives a homeomorphism

$$\mathrm{T}^1(X) \simeq (\partial X \times \partial X - \Delta(\partial X)) \times \mathbb{R}$$

Oh is partially supported by the NSF.

where $\Delta(\partial X)$ denotes the diagonal embedding of ∂X into $\partial X \times \partial X$ and the geodesic flow $\mathcal{G}^t$ on $\mathrm{T}^1(X)$ corresponds to the translation flow on $\mathbb{R}$.

The Hopf-Tsuji-Sullivan dichotomy in rank one. Let $\Gamma < G$ be a non-elementary discrete subgroup. A Borel probability measure ν on ∂X is called a Γ-conformal measure of dimension $\delta \geq 0$ if for any $\gamma \in \Gamma$ and $\xi \in \partial X$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{-\delta \beta_\xi (o, \gamma(o))}$$

where $\gamma_* \nu(Q) = \nu(\gamma^{-1}Q)$ for any Borel subset $Q \subset \partial X$.

Each Γ-conformal measure ν on ∂X determines a unique geodesic flow invariant Borel measure m_ν on $\mathrm{T}^1(\Gamma\backslash X)$, which is locally equivalent to $\nu \otimes \nu \otimes ds$ in the Hopf coordinates, where ds denotes the Lebesgue measure on $\mathbb{R}$. The following criterion known as the *Hopf-Tsuji-Sullivan dichotomy* relates dynamical properties of the geodesic flow $\mathcal{G}^t$ with respect to the measure m_ν, the ν-size of the conical[1] limit points of Γ and the divergence property of the Poincare series $\mathcal{P}(s) = \sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)}$ at the dimension of ν: we denote by $\Lambda_{\mathsf{con}} \subset \partial X$ the set of all conical limit points of Γ.

Theorem 1.1. *Let G be a connected simple real algebraic group of rank one and $\Gamma < G$ a non-elementary discrete subgroup. Let ν be a Γ-conformal measure on ∂X of dimension δ. The following are equivalent:*

1. $\nu(\Lambda_{\mathsf{con}}) > 0$ *(resp.* $\nu(\Lambda_{\mathsf{con}}) = 0$*);*
2. $\nu(\Lambda_{\mathsf{con}}) = 1$ *(resp.* $\nu(\Lambda_{\mathsf{con}}) = 0$*);*
3. *the geodesic flow $\mathcal{G}^t$ is conservative (resp. completely dissipative) with respect to m_ν;*
4. *the geodesic flow $\mathcal{G}^t$ is ergodic (resp. non-ergodic) with respect to m_ν;*
5. $\sum_{\gamma \in \Gamma} e^{-\delta d(o, \gamma o)} = \infty$ *(resp.* $\sum_{\gamma \in \Gamma} e^{-\delta d(o, \gamma o)} < \infty$*) where δ is the conformal dimension of ν and $o \in X$ is any point.*

Most equivalences are due to Sullivan for real hyperbolic spaces [30] (see also [31], [3]) and to Burger-Mozes for proper CAT (-1) spaces [7, Sec. 6.3] and its complete form can be found in Nicholl's book [22, Ch. 8] when X is a real hyperbolic space and in Roblin's thesis [26, Thm. 1.7] for a proper CAT (-1) spaces.

We denote by $\Lambda \subset \partial X$ the limit set of Γ, which is the unique Γ-minimal subset of ∂X and by δ_Γ the critical exponent of Γ, that is, the abscissa of the convergence of the Poincare series $\mathcal{P}(s)$ of Γ. The group Γ is called a divergent type if $\mathcal{P}(\delta_\Gamma) = \infty$. Patterson and Sullivan constructed a Γ-conformal measure, say, ν_{PS}, supported on the limit set Λ of dimension δ_Γ, called the Patterson-Sullivan measure. Theorem 1.1 implies that whether Γ is of divergent type or not is completely determined by the positivity of $\nu_{\mathrm{PS}}(\Lambda_{\mathsf{con}})$, and vice versa.

[1] A point $\xi \in \partial X$ is called a conical limit point of Γ if a geodesic ray toward ξ accumulates in $\Gamma\backslash X$.

The case of convex cocompact groups. A discrete group with $\Lambda = \Lambda_{\mathsf{con}}$ is called a *convex cocompact* subgroup. They are also characterized by the property that Γ acts cocompactly on the convex hull of Λ in X. For a convex cocompact subgroup Γ, there exists a unique Γ-conformal measure supported on the limit set Λ, namely the Patterson-Sullivan measure ν_{PS}. The associated geodesic flow invariant measure on $\mathrm{T}^1(\Gamma\backslash X)$, called the Bowen-Margulis-Sullivan measure, is known to be the measure of maximal entropy [30]. An immediate consequence of Theorem 1.1 for convex cocompact groups is as follows:

Theorem 1.2. *Let $\Gamma < G$ be a convex cocompact subgroup. Then*

(1) *the geodesic flow $\mathcal{G}^t$ on $\mathrm{T}^1(\Gamma\backslash X)$ is conservative and ergodic with respect to the Bowen-Margulis-Sullivan measure m^{BMS};*
(2) *Γ is of divergent type, i.e., $\sum_{\gamma\in\Gamma} e^{-\delta_\Gamma d(o,\gamma o)} = \infty$.*

The unit tangent bundle of $\Gamma\backslash X$ is a double quotient space $\Gamma\backslash G/M$ where M is a compact subgroup of G commuting with the one-parameter diagonal subgroup $\{a_t\}$ which induces the geodesic flow. When Γ is Zariski dense in addition, the lifted Bowen-Margulis-Sullivan measure, considered as an M-invariant measure on $\Gamma\backslash G$, is also ergodic for the diagonal flow $\{a_t\}$ whenever M is connected [32, Thm. 1.1]. The only case of M disconnected is when $G \simeq \mathrm{SL}_2(\mathbb{R})$ and $M = \{\pm e\}$, in which case m^{BMS} has at most two ergodic components [20].

The Hopf-Tsuji-Sullivan dichotomy in higher rank. The main aim of this article is to extend the Hopf-Tsuji-Sullivan dichotomy for discrete subgroups of higher rank semisimple real algebraic groups G, while replacing the geodesic flow of the rank one space with any one-parameter subgroup of diagonal elements of G (Theorem 1.4). Each one-parameter subgroup of diagonal elements corresponds to a direction, say, u, in the positive Weyl chamber of G. We introduce the u-*directional* conical limit set and u-*directional* Poincare series, whose properties relative to a given Γ-conformal density is shown to determine ergodic properties of the action of the one-parameter subgroup $\{\exp(t\mathsf{u}) : t \in \mathbb{R}\}$ with respect to an associated measure on $\Gamma\backslash G$. We then apply the dichotomy together with recent local mixing results of Chow and Sarkar [9] to Anosov subgroups Γ. We discover a surprising phenomenon that the rank of the ambient group G dictates a completely opposite behavior for Γ as stated in Theorem 1.6. We also deduce recurrent properties of the Burger-Roblin measures for each interior direction of the limit cone of Γ (Corollary 1.7), which plays an important role in the recent measure classification result of Landesberg, Lee, Lindenstrauss and Oh [18, Thm. 1.1].

In order to state these results precisely, we now let G be a connected, semisimple real algebraic group. Let P be a minimal parabolic subgroup of G with a fixed Langlands decomposition $P = MAN$. Here A is a maximal real split torus of G, M is a compact subgroup commuting with A and

N is a maximal horospherical subgroup. We fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a} = \mathrm{Lie}(A)$ so that $\log N$ consists of positive root subspaces. We fix a maximal compact subgroup $K < G$ so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds, and denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection, i.e., for $g \in G$, $\mu(g) \in \mathfrak{a}^+$ is the unique element such that $g \in K \exp \mu(g) K$.

Let $\Gamma < G$ be a Zariski dense discrete subgroup of G. We denote by $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ the limit cone of Γ, which is the asymptotic cone of $\mu(\Gamma)$. Benoist showed that $\mathcal{L}_\Gamma$ is a convex cone with non-empty interior [4]. Let $\mathcal{F}$ denote the Furstenberg boundary G/P and $\Lambda \subset \mathcal{F}$ the limit set of Γ, which is the unique Γ-minimal subset. For a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν_ψ on $\mathcal{F}$ is called a (Γ, ψ)-conformal measure if for any $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_* \nu_\psi}{d\nu_\psi}(\xi) = e^{\psi(\beta_\xi(e,\gamma))} \tag{1.1}$$

where β denotes the $\mathfrak{a}$-valued Busemann function (see Def. 2.1). Quint showed in [24, Thm. 8.1] that a (Γ, ψ)-conformal measure may exist only when $\psi \geq \psi_\Gamma$ where $\psi_\Gamma : \mathfrak{a} \to \mathbb{R}$ denotes the growth indicator function of Γ (Def. 5.1). Moreover, he constructed a (Γ, ψ)-conformal measure supported on Λ for every linear form $\psi \geq \psi_\Gamma$ satisfying $\psi(\mathsf{v}) = \psi_\Gamma(\mathsf{v})$ for some $\mathsf{v} \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+$ [24, Thm. 8.4].

Let $\mathrm{i} : \mathfrak{a}^+ \to \mathfrak{a}^+$ denote the opposition involution given by $\mathrm{i}(\mathsf{v}) = -\mathrm{Ad}_{w_0}(\mathsf{v})$ where w_0 is the longest Weyl element. In rank one groups, i is the identity map. Letting $\mathcal{F}^{(2)}$ denote the unique open diagonal G-orbit in $\mathcal{F} \times \mathcal{F}$, the quotient space G/M is homeomorphic to $\mathcal{F}^{(2)} \times \mathfrak{a}$ via the Hopf parametrization which maps gM to $(gP, gw_0P, \beta_{gP}(e, g))$ for any $g \in G$.

For a given pair of Γ-conformal measures ν_ψ and $\nu_{\psi \circ \mathrm{i}}$ on $\mathcal{F}$ with respect to ψ and $\psi \circ \mathrm{i}$ respectively, one can use the Hopf parameterization to define a non-zero A-invariant Borel measure $\mathsf{m}(\nu_\psi, \nu_{\psi\circ \mathrm{i}})$ on the quotient space $\Gamma\backslash G/M$, which is locally equivalent to $d\nu_\psi \otimes d\nu_{\psi \circ \mathrm{i}} \otimes db$ in the Hopf coordinates, where db denotes the Lebesgue measure on $\mathfrak{a}$; we will call it the Bowen-Margulis-Sullivan measure (or simply BMS-measure) associated to the pair $(\nu_\psi, \nu_{\psi\circ \mathrm{i}})$ (Section 4). For simplicity, we write m_ψ for $\mathsf{m}(\nu_\psi, \nu_{\psi\circ \mathrm{i}})$, although the measure depends on the choice of conformal measures ν_ψ and $\nu_{\psi\circ \mathrm{i}}$, not only on ψ.

For $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, we will say that m_ψ is u-*balanced* if

$$\limsup_{T \to \infty} \frac{\int_0^T \mathsf{m}_\psi(\mathcal{O}_1 \cap \mathcal{O}_1 \exp(t\mathsf{u}))\, dt}{\int_0^T \mathsf{m}_\psi(\mathcal{O}_2 \cap \mathcal{O}_2 \exp(t\mathsf{u}))\, dt} < \infty \tag{1.2}$$

for any bounded Borel subsets $\mathcal{O}_i \subset \Gamma\backslash G/M$ with $\Omega \cap \operatorname{int} \mathcal{O}_i \neq \emptyset$, where $\Omega = \{[g] \in \Gamma\backslash G/M : gP, gw_0P \in \Lambda\}$.

Each BMS measure m_ψ on $\Gamma\backslash G/M$ can be considered as an AM-invariant measure on $\Gamma\backslash G$, which we will also denote by m_ψ, by abuse of notation. While the set $\mathcal{E} = \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$ is the unique P-minimal subset of $\Gamma\backslash G$, it breaks into finitely many P°-minimal subsets in general where P°

denotes the identity component of P. For each P°-minimal subset $Y \subset \Gamma\backslash G$, the restriction $\mathsf{m}_\psi|_Y$ gives an A-invariant measure.

The conical limit set Λ_{con} of Γ is given by

$$\Lambda_{\mathsf{con}} := \{gP \in \mathcal{F} : \limsup \Gamma g A^+ \neq \emptyset\} \tag{1.3}$$

where $A^+ = \exp \mathfrak{a}^+$ and lim sup denotes the topological limit superior, i.e. all accumulation points of the given family of sets.

Definition 1.3 (Directional conical limit set)**.** For each $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, we define the set of u-directional conical limit points as follows:

$$\Lambda_{\mathsf{u}} := \{gP \in \mathcal{F} : \limsup_{t\to+\infty} \Gamma g \exp(t\mathsf{u}) \neq \emptyset\};$$

this is a dense Borel measurable subset of Λ_{con} if non-empty.

It is easy to see that $\Lambda_{\mathsf{u}} \neq \emptyset$ only when $\mathsf{u} \in \mathcal{L}_\Gamma$.

For $R > 0$ and $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, we define the following tube-like subset of Γ whose Cartan projection lies within distance R from the ray $\mathbb{R}_+\mathsf{u}$:

$$\Gamma_{\mathsf{u},R} := \{\gamma \in \Gamma : \|\mu(\gamma) - t\mathsf{u}\| < R \quad \text{for some } t \geq 0\},$$

where $\|\cdot\|$ is an Euclidean norm on $\mathfrak{a}$. The following theorem extends Theorem 1.1 to all Zariski dense subgroups of higher rank semisimple real algebraic groups:

Theorem 1.4 (The Hopf-Tsuji-Sullivan dichotomy in higher rank)**.** *Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. Fix $\psi \in \mathfrak{a}^*$ and let $\nu_\psi, \nu_{\psi\circ \mathrm{i}}$ be a pair of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-conformal measures respectively, and let $\mathsf{m}_\psi = \mathsf{m}(\nu_\psi, \nu_{\psi\circ\mathrm{i}})$ denote the associated* BMS *measure on $\Gamma\backslash G/M$. For any $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, the following conditions (1)-(5) are equivalent and imply (6). If $\psi(\mathsf{u}) > 0$ and m_ψ is u-balanced, then (6) implies (7). Moreover, the first cases of (1)-(7) can occur only when $\psi(\mathsf{u}) = \psi_\Gamma(\mathsf{u})$.*

(1) $\max(\nu_\psi(\Lambda_{\mathsf{u}}), \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})) > 0$ *(resp. $\nu_\psi(\Lambda_{\mathsf{u}}) = 0 = \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})$);*
(2) $\max(\nu_\psi(\Lambda_{\mathsf{u}}), \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})) = 1$ *(resp. $\nu_\psi(\Lambda_{\mathsf{u}}) = 0 = \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})$);*
(3) $(\Gamma\backslash G/M, \{\exp(t\mathsf{u})\}, \mathsf{m}_\psi)$ *is conservative (resp. totally dissipative);*
(4) $(\Gamma\backslash G/M, \{\exp(t\mathsf{u})\}, \mathsf{m}_\psi)$ *is ergodic (resp. non-ergodic);*
(5) *For some (and hence for all) P°-minimal subset $Y \subset \Gamma\backslash G$, the system $(Y, \{\exp(t\mathsf{u})\}, \mathsf{m}_\psi|_Y)$ is ergodic and conservative (resp. $\mathsf{m}_\psi(Y) = 0$, or non-ergodic and totally dissipative);*
(6) $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty$ *for some $R > 0$ (resp. $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} < \infty$ for all $R > 0$);*
(7) $\nu_\psi(\Lambda_{\mathsf{u}}) = 1 = \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})$ *(resp. $\nu_\psi(\Lambda_{\mathsf{u}}) = 0 = \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})$).*

Remark 1.5. (1) When G has rank one, $\psi \circ \mathrm{i} = \psi$ for any $\psi \in \mathfrak{a}^*$, as the opposition involution i is trivial. Moreover, the m_ψ being u-balanced condition is not needed for the implication (6) $\Rightarrow$ (7). For Γ non-elementary, (1)-(7) are all equivalent to each other, except for (5), and for Γ Zariski dense, these conditions imply (5).

(2) When the rank of G is at least 2, we need Γ to be Zariski dense for the equivalence of (3) and (4). The reason is that, when Γ is not Zariski dense, the Jordan projection of Γ may not generate a dense subgroup of A while in the rank one case, the Jordan projection of any non-elementary subgroup generates a dense subgroup of A [15].

(3) We emphasize here that although the implication (3) $\Rightarrow$ (1) is a direct consequence of the definition of m_ψ, the proof for (3) $\Rightarrow$ (7) under the further u-balanced condition of m_ψ requires the discussion of the directional Poincare series.

For discrete subgroups of a product of two rank one Lie groups whose projection to each factor is convex cocompact, Burger announced that $\nu_\psi(\Lambda_{\mathsf{u}}) = 1$ for all $\psi \in \mathfrak{a}^*$ and $\mathsf{u} \in \operatorname{int} \mathcal{L}_\Gamma$ such that $\psi(\mathsf{u}) = \psi_\Gamma(\mathsf{u})$ [6, Thm. 3]. Indeed, we show that this is a special case of a more general phenomenon which holds for all Anosov subgroups whose ambient group has rank at most 3.

The case of Anosov subgroups. Although there are notions of Anosov subgroups with respect to a general parabolic subgroup [13], we will restrict our attention only to those Anosov subgroups with respect to a minimal parabolic subgroup. Recall that a Zariski dense discrete subgroup $\Gamma < G$ is an *Anosov subgroup* (with respect to a minimal parabolic subgroup P) if it is a finitely generated word hyperbolic group which admits a Γ-equivariant embedding ζ of the Gromov boundary $\partial\Gamma$ into $\mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y$ in $\partial\Gamma$ [13, Prop. 2.7 and Thm. 1.5]. We note that Zariski dense images of representations of a surface subgroup in the Hitchin component [17] as well as Schottky subgroups provide ample examples of Anosov subgroups ([25, Prop. 3.3], see also [11, Lem. 7.2]). Let Γ be an Anosov subgroup for the rest of the introduction. Set

$$D_\Gamma^\star := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma, \psi(\mathsf{v}) = \psi_\Gamma(\mathsf{v}) \text{ for some } \mathsf{v} \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+\}.$$

For each $\psi \in D_\Gamma^\star$, there exists a unique unit vector $\mathsf{v} \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+$ such that $\psi(\mathsf{v}) = \psi_\Gamma(\mathsf{v})$ and v necessarily belongs to $\operatorname{int} \mathcal{L}_\Gamma$ ([21, Prop. 4.11] and [25, Lem. 4.3(i)], see also [29, Lem. 4.3] and [8, Thm. A.2(3)]).

Moreover, for each $\psi \in D_\Gamma^\star$, there exists a unique (Γ, ψ)-conformal probability measure, say ν_ψ, *supported on* Λ and the map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D_\Gamma^\star$ and the space $\mathcal{S}_\Gamma$ of all Γ-conformal probability measures supported on Λ; hence $\mathcal{S}_\Gamma$ is homeomorphic to the set of unit vectors of $\operatorname{int} \mathcal{L}_\Gamma$ (see [19, Thm. 1.3] and references therein). It was also shown in ([19], [20]) that for any $\psi \in D_\Gamma^\star$ and $\mathsf{m}_\psi = \mathsf{m}(\nu_\psi, \nu_{\psi \circ \mathrm{i}})$,

- $\Lambda = \Lambda_{\mathsf{con}}$;
- for any P°-minimal subset $Y \subset \Gamma\backslash G$, $\mathsf{m}_\psi|_Y$ is A-ergodic;
- $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty$.

On the other hand, the divergence of the directional Poincare series (i.e., $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))}$ for some $R > 0$) turns out to depend on the rank of G:

Theorem 1.6. *Let $\Gamma < G$ be an Anosov subgroup. For any $\psi \in D_\Gamma^\star$ and $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, the following conditions are equivalent and the first cases of (1)-(4) can occur only when $\mathsf{u} \in \operatorname{int} \mathcal{L}_\Gamma$:*

1. $\operatorname{rank} G \leq 3$ *and* $\psi(\mathsf{u}) = \psi_\Gamma(\mathsf{u})$ *(resp.* $\operatorname{rank} G > 3$ *or* $\psi(\mathsf{u}) \neq \psi_\Gamma(\mathsf{u})$*);*
2. $\nu_\psi(\Lambda_\mathsf{u}) = 1 = \nu_{\psi\circ \mathrm{i}}(\Lambda_{\mathsf{i}(\mathsf{u})})$ *(resp.* $\nu_\psi(\Lambda_\mathsf{u}) = 0 = \nu_{\psi\circ \mathrm{i}}(\Lambda_{\mathsf{i}(\mathsf{u})})$*);*
3. *For some (and hence for all) P°-minimal subset $Y \subset \Gamma\backslash G$, the system $(Y, \{\exp(t\mathsf{u})\}, \mathsf{m}_\psi|_Y)$ is ergodic and conservative (resp. non-ergodic and totally dissipative);*
4. $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty$ *for some $R > 0$ (resp.* $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} < \infty$ *for all $R > 0$).*

For $\psi \in D_\Gamma^\star$ and $\mathsf{u} \in \operatorname{int} \mathcal{L}_\Gamma$ with $\psi(\mathsf{u}) = \psi_\Gamma(\mathsf{u})$, Chow and Sarkar proved in [9] the following local mixing result that for any $f_1, f_2 \in C_c(\Gamma\backslash G)$,

$$\lim_{t\to+\infty} t^{(\operatorname{rank} G-1)/2} \int_{\Gamma\backslash G} f_1(x \exp t\mathsf{u}) f_2(x) d\mathsf{m}_\psi(x) = \kappa_\mathsf{u}\, \mathsf{m}_\psi(f_1)\mathsf{m}_\psi(f_2) \quad (1.4)$$

for some constant $\kappa_\mathsf{u} > 0$ depending only on u (see [27] where this is proved for M-invariant functions for some special cases).

Using the shadow lemma (Lemma 3.4), we deduce from this local mixing result (1.4) that the u-directional Poincare series $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))}$ diverges if and only if $\operatorname{rank} G \leq 3$. Together with Theorem 1.4, this implies Theorem 1.6.

Let $\mathsf{m}_\psi^{\mathrm{BR}}$ denote the Burger-Roblin measure associated to ν_ψ, that is, the MN-invariant measure on $\Gamma\backslash G$ which is induced from a measure on G/M locally equivalent to $d\nu_\psi \otimes dm_o \otimes db$ where m_o is the K-invariant probability measure on $\mathcal{F}$ (cf. [11, Sec. 3]). Lee and Oh proved that each $\mathsf{m}_\psi^{\mathrm{BR}}$ is MN-ergodic and its restrictions to P°-minimal subsets of $\Gamma\backslash G$ yield all N-ergodic components ([19, Thm. 10.1], [20, Thm. 1.3]). For $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, we consider the following directional recurrent set

$$\mathcal{R}_\mathsf{u} := \{x \in \Gamma\backslash G : \limsup_{t\to+\infty} x \exp(t\mathsf{u}) \neq \emptyset\}.$$

Since $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, this is a P-invariant dense Borel subset of $\mathcal{E}$.

An immediate consequence of Theorem 1.6 is the following:

Corollary 1.7. *For any $\psi \in D_\Gamma^\star$ and $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, we have*

1. *If* $\operatorname{rank} G \leq 3$ *and* $\mathsf{u} \in \operatorname{int} \mathcal{L}_\Gamma$ *with* $\psi(\mathsf{u}) = \psi_\Gamma(\mathsf{u})$*, then*
$$\mathsf{m}_\psi^{\mathrm{BR}}(\Gamma\backslash G - \mathcal{R}_\mathsf{u}) = 0.$$
2. *In all other cases,* $\mathsf{m}_\psi^{\mathrm{BR}}(\mathcal{R}_\mathsf{u}) = 0$.

This corollary is one of the main ingredients of the recent measure classification result [18, Thm. 1.1].

Added after revision: Sambarino posted a preprint (arXiv:2202:02213) showing ergodicity for rank G at most 2 and non-ergodicity for rank G at least 4, with a different approach based on work of Guivarch.

Organization. In section 2, we collect basic definitions. In section 3, we show that the set of directional conical limit points is either null or conull for any (Γ, ψ)-conformal measure. In section 4, we prove that the conservativity of the Bowen-Margulis-Sullivan measure for one parameter diagonal flow implies its ergodicity, extending Hopf's argument. In section 5, we relate the directional Poincare series with respect to ψ and the correlation functions of the BMS measures and provide the proof of Theorem 1.4. In section 6, we specialize to Anosov groups and prove Theorem 1.6.

2. Preliminaries

Let G be a connected, semisimple real algebraic group. We decompose $\mathfrak{g} = \operatorname{Lie} G$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of a Cartan involution θ of $\mathfrak{g}$, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and by $X = G/K$ the associated symmetric space. Choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$ and a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. Set $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M. Consider the following pair of opposite maximal horospherical subgroups:

$$N = N^- := \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\} \text{ and}$$

$$N^+ := \{g \in G : a^n g a^{-n} \to e \text{ as } n \to +\infty\}$$

for any $a \in \operatorname{int} A^+$; this definition is independent of the choice of $a \in \operatorname{int} A^+$.

We set

$$P = MAN, \quad \text{and} \quad P^+ = MAN^+;$$

they are minimal parabolic subgroups of G and $P \cap P^+ = MA$. The quotient space $\mathcal{F} = G/P$ is called the Furstenberg boundary of G, and via the Iwasawa decomposition $G = KP$, $\mathcal{F}$ is isomorphic to K/M.

Let $\mathrm{N}_K(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K, and $\mathcal{W} := \mathrm{N}_K(\mathfrak{a})/M$ denote the Weyl group. Fixing a left G-invariant and right K-invariant Riemannian metric d on G induces a Riemannian metric on the associated symmetric space $X = G/K$, which we also denote by d by abuse of notation. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the associated $\mathcal{W}$-invariant inner product and norm on $\mathfrak{a}$.

For $R > 0$, set $A_R = \{a \in A : \|\log a\| \le R\}$, $A_R^+ = A_R \cap A^+$, and

$$G_R := K A_R^+ K.$$

$\mathfrak{a}$-valued Buseman functions. The product map $K \times A \times N \to G$ is a diffeomorphism, yielding the well-known Iwasawa decomposition $G = KAN$. The Iwasawa cocycle $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$ with $\xi = [k]$ for $k \in K$, $\exp \sigma(g, \xi)$ is the A-component of gk in the KAN decomposition, that is,

$$gk \in K \exp(\sigma(g, \xi)) N.$$

Definition 2.1. The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g,h) := \sigma(g^{-1},\xi) - \sigma(h^{-1},\xi).$$

Denote by $w_0 \in \mathcal{W}$ the unique element of $\mathcal{W}$ such that $\mathrm{Ad}_{w_0}\mathfrak{a}^+ = -\mathfrak{a}^+$.

Definition 2.2 (Visual maps)**.** For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$

Note that for $g \in G$, $g^\pm = g(e^\pm)$.

The opposition involution $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined by

$$\mathrm{i}(\mathsf{v}) = -\mathrm{Ad}_{w_0}(\mathsf{v}). \tag{2.1}$$

When G is a product of rank one groups, i is trivial.

The set $\mathcal{F}^{(2)} = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\} = G.(e^+, e^-)$ is the unique open G-orbit. The $\mathfrak{a}$-valued Gromov product on $\mathcal{F}^{(2)}$ is defined as follows: for $(g^+, g^-) \in \mathcal{F}^{(2)}$,

$$\mathcal{G}(g^+, g^-) := \beta_{g^+}(e,g) + \mathrm{i}\left(\beta_{g^-}(e,g)\right).$$

Lemma 2.3. [5, Prop. 8.12] *There exist $c, c' > 0$ such that for all $g \in G$,*

$$c^{-1}\|\mathcal{G}(g^+, g^-)\| \le d(o, gAo) \le c\,\|\mathcal{G}(g^+, g^-)\| + c'.$$

Definition 2.4 (Cartan projection)**.** For $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, such that

$$g \in K \exp(\mu(g))K.$$

When $\mu(g) \in \operatorname{int}\mathfrak{a}^+$ and $g = k_1 \exp(\mu(g))k_2$, we write $\kappa_1(g) := [k_1] \in K/M$ and $\kappa_2(g) := k_2 \in M\backslash K$, which are well-defined.

In the whole paper, we fix the constant $d = d(G) \ge 2$ as in the following lemma.

Lemma 2.5. [19, Lem. 5.8] *There exists $d \ge 2$ such that for any $R > 1$ and any $g \in G$,*

$$\mu(G_R g G_R) \subset \mu(g) + \mathfrak{a}_{dR}.$$

Definition 2.6. We say that a sequence $g_i \to \infty$ regularly in G if $\alpha(\mu(g_i)) \to \infty$ as $i \to \infty$ for every simple root α corresponding to $\mathfrak{a}^+$.

Definition 2.7.
(1) A sequence $g_i \in G$ is said to converge to $\xi \in \mathcal{F}$, if $g_i \to \infty$ regularly in G and $\lim_{i\to\infty} \kappa_1(g_i)^+ = \xi$.
(2) A sequence $p_i = g_i(o) \in X$ is said to converge to $\xi \in \mathcal{F}$ if g_i does.

Definition 2.8 (Limit set)**.** For a Zariski dense discrete subgroup $\Gamma < G$, we define the limit set Λ of Γ as follows: fixing $p \in X$,

$$\Lambda := \left\{ \lim_{i\to\infty} \gamma_i p \in \mathcal{F} : \gamma_i \in \Gamma \right\}.$$

By [19, Lem. 2.13], this definition is independent of the choice of $p \in X$ and coincides with one given by Benoist [4, Def. 3.6]; in particular, it is the unique Γ-minimal subset of $\mathcal{F}$.

We later use the fact that Λ is a Zariski dense subset of $\mathcal{F}$ [4, Lem. 3.6].

For any real-valued functions $f(t)$ and $g(t)$, we write $f(t) \ll g(t)$ if there exists $C > 0$ such that $f(t) \le Cg(t)$ for all $t > 1$. We write $f(t) \asymp g(t)$ if $f(t) \ll g(t)$ and $g(t) \ll f(t)$.

3. A zero-one law for $\nu_\psi(\Lambda_{\mathsf{u}})$

Let $\Gamma < G$ be a Zariski dense discrete subgroup of G. Fix $\psi \in \mathfrak{a}^*$, and a (Γ, ψ)-conformal measure ν_ψ on $\mathcal{F}$ as defined in (1.1).

Recalling the notation Λ_{u} from Definition 1.3, the goal of this section is to prove the following dichotomy:

Proposition 3.1. *For any $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, we have*

$$\nu_\psi(\Lambda_{\mathsf{u}}) = 1 \quad \text{or} \quad \nu_\psi(\Lambda_{\mathsf{u}}) = 0.$$

The proof of this proposition is based on the study of shadows.

Shadows. For $p, q \in X = G/K$ and $r > 0$, the shadow of the r-ball around q as seen from p is defined by

$$O_r(p, q) := \{g^+ \in \mathcal{F} : go = p,\ gA^+o \cap B(q, r) \neq \emptyset\},$$

where $B(q, r) = \{x \in X : d(x, q) < r\}$.

Similarly, for $\xi \in \mathcal{F}$, we define the shadow of the r-ball around q as seen from ξ to be

$$O_r(\xi, q) := \{g^+ \in \mathcal{F} : g^- = \xi,\ go \in B(q, r)\}.$$

Note the following G-equivariance property: for any $g \in G$ and $r > 0$,

$$gO_r(p, q) = O_r(gp, gq) \quad \text{and} \quad gO_r(\xi, q) = O_r(g\xi, gq). \tag{3.1}$$

Note that for any $\xi \in \mathcal{F}$, $q \in X$ and $R > 0$,

$$\bigcup_{r > R} O_r(\xi, q) = \{\eta \in \mathcal{F} : (\xi, \eta) \in \mathcal{F}^{(2)}\}. \tag{3.2}$$

Lemma 3.2. [19, Lem. 5.7] *There exists $\kappa > 0$ such that for any $r > 0$ and $g \in G$, we have*

$$\sup_{\xi \in O_r(o, go)} \|\beta_\xi(e, g) - \mu(g)\| \le \kappa r.$$

The following lemma is an immediate consequence of [19, Lem. 5.6]:

Lemma 3.3. *For any $S > 0$ and a sequence $g_i \to \infty$ regularly in G, we have, for all sufficiently large i, the closure of $O_S(o, g_i o) \times O_S(g_i o, o)$ is contained in $\mathcal{F}^{(2)}$.*

The following shadow lemma plays an important role in our paper. It was first presented in [2, Thm. 3.3] and then in [24, Thm. 8.2] in slightly different forms.

Lemma 3.4 (Shadow lemma)**.** [19, Lem. 7.8] *There exists* $S_0 > 0$ *such that* $c_1 := \inf_{\gamma\in\Gamma} \nu_\psi(O_S(\gamma o, o)) > 0$. *Moreover, there exists* $\kappa > 0$ *such that for all* $S > S_0$*, and for all* $\gamma \in \Gamma$*,*

$$c_1 e^{-\kappa\|\psi\| S} e^{-\psi(\mu(\gamma))} \le \nu_\psi(O_S(o, \gamma o)) \le e^{\kappa\|\psi\| S} e^{-\psi(\mu(\gamma))}.$$

For any $R > 0$, set

$$G_{\mathsf{u},R} := \{g \in G : \|\mu(g) - t\mathsf{u}\| < R \quad \text{for some } t \ge 0\}. \tag{3.3}$$

Lemma 3.5. *Let* $R, S > 0$. *If* $g \in G_{\mathsf{u},R}$*, then*

$$O_S(o, go) \subset \{k^+ \in \mathcal{F} : k\exp(t\mathsf{u})o \in B(go, R + 2dS) \text{ for some } t > 0\}.$$

Proof. For $\xi \in O_S(o, go)$, there exist $k \in K$ and $a \in A^+$ such that $kao \in B(go, S)$ and $\xi = k^+$. It follows that $g^{-1}ka \in G_S$, and hence $\|\mu(g) - \log a\| \le dS$ by Lemma 2.5. On the other hand, since $g \in G_{\mathsf{u},R}$, there exists $t \ge 0$ such that $\|\mu(g) - t\mathsf{u}\| < R$, and hence

$$\begin{aligned} d(k\exp(t\mathsf{u})o, go) &\le d(k\exp(t\mathsf{u})o, kao) + d(kao, go) \\ &< \|t\mathsf{u} - \log a\| + S \le \|t\mathsf{u} - \mu(g)\| + \|\mu(g) - \log a\| + S \\ &\le R + (d+1)S. \end{aligned}$$

This proves the lemma. □

The following Vitali-covering type lemma is a key ingredient of the proof of Proposition 3.1.

Lemma 3.6 (Covering lemma)**.** *Fix* $R > 0$ *and consider* $\{O_R(o, \gamma o) : \gamma \in \Gamma'\}$ *for some infinite subset* $\Gamma' \subset \Gamma_{\mathsf{u},R}$. *There exists a subset* $\Gamma'' \subset \Gamma'$ *such that* $\{O_R(o, \gamma o) : \gamma \in \Gamma''\}$ *consists of pairwise disjoint shadows and*

$$\bigcup_{\gamma\in\Gamma'} O_R(o, \gamma o) \subset \bigcup_{\gamma\in\Gamma''} O_{10dR}(o, \gamma o). \tag{3.4}$$

Proof. Enumerate $\Gamma' = \{\gamma_i : i \in \mathbb{N}\}$ so that $\|\mu(\gamma_i)\|$ is nondecreasing. Set $i_0 = 0$ and inductively define

$$i_{n+1} := \min\{i > i_n : O_R(o, \gamma_i o) \bigcap \left(\cup_{j\le n} O_R(o, \gamma_{i_j} o)\right) = \emptyset\}.$$

Set $\Gamma'' := \{\gamma_{i_n} : n \in \mathbb{N}\}$ so that $\{O_R(o, \gamma o) : \gamma \in \Gamma''\}$ consists of pairwise disjoint shadows.

For each $\gamma_i \in \Gamma'$, we claim that $O_R(o, \gamma_i o) \subset O_{10dR}(o, \gamma o)$ for some $\gamma \in \Gamma''$. We may assume that $i_n < i < i_{n+1}$ for some n. By definition of i_{n+1}, there exists $j \le n$ such that $O_R(o, \gamma_i o) \cap O_R(o, \gamma_{i_j} o) \ne \emptyset$. In particular, there exists $k_1 \in K$, $a_i, a_{i_j} \in A^+$ such that $k_1 a_i o \in B(\gamma_i o, R)$ and $k_1 a_{i_j} o \in B(\gamma_{i_j} o, R)$. Since $\gamma_i^{-1} k_1 a_i, \gamma_{i_j}^{-1} k_1 a_{i_j} \in G_R$, we have

$$\|\mu(\gamma_i) - \log a_i\| \le dR \text{ and } \|\mu(\gamma_{i_j}) - \log a_{i_j}\| \le dR$$

by Lemma 2.5. On the other hand, there exists $t_i, t_{i_j} \ge 0$ such that

$$\|\mu(\gamma_i) - t_i\mathsf{u}\| \le R \text{ and } \|\mu(\gamma_{i_j}) - t_{i_j}\mathsf{u}\| \le R,$$

as $\gamma_i, \gamma_{i_j} \in \Gamma_{\mathsf{u},R}$. Observe that

$$\begin{aligned}
&\|\mu(\gamma_i)\| = d(o, \gamma_i o) \le d(o, k_1 a_{t_i} o) + d(k_1 a_{t_i} o, k_1 a_i o) + d(k_1 a_i o, \gamma_i o)\\
&\le d(o, k_1 a_{t_i} o) + dR + 2R = d(o, k_1 a_{t_{i_j}} o) + dR + 2R + (t_i - t_{i_j})\\
&\le d(o, k_1 a_{i_j} o) + 2dR + 3R + (t_i - t_{i_j}) \le d(o, \gamma_{i_j} o) + 2dR + 4R + (t_i - t_{i_j})\\
&= \|\mu(\gamma_{i_j})\| + 2dR + 4R + (t_i - t_{i_j}) \le \|\mu(\gamma_i)\| + 2dR + 4R + (t_i - t_{i_j}),
\end{aligned}$$

and hence $t_i' := t_i + 2dR + 4R \ge t_{i_j}$.

Now let $k_2^+ \in O_R(o, \gamma_i o)$ be arbitrary and $b \in A^+$ be such that $k_2 b o \in B(\gamma_i o, R)$. We have $\|\mu(\gamma_i) - \log b\| \le dR$ by Lemma 2.5. Since $\gamma_i \in \Gamma_{\mathsf{u},R}$, there exists $s \ge 0$ such that $\|\mu(\gamma_i) - s\mathsf{u}\| \le R$. Since

$$\begin{aligned}
&d(k_2 a_s o, k_1 a_{t_i'} o) \le d(k_2 a_s o, k_2 b o)\\
&+ d(k_2 b o, \gamma_i o) + d(\gamma_i o, k_1 a_i o) + d(k_1 a_i o, k_1 a_{t_i} o) + d(k_1 a_{t_i} o, k_1 a_{t_i'} o)\\
&\le (dR + R) + R + R + (dR + R) + (2dR + 4R) = 4dR + 8R,
\end{aligned}$$

there exists $0 \le s' \le s$ such that $d(k_2 a_{s'} o, k_1 a_{t_{i_j}} o) \le 4dR + 8R$ by Lemma 3.7 below. Finally,

$$\begin{aligned}
d(k_2 a_{s'} o, \gamma_{i_j} o) &< d(k_2 a_{s'} o, k_1 a_{t_{i_j}} o) + d(k_1 a_{t_{i_j}} o, k_1 a_{i_j} o) + d(k_1 a_{i_j} o, \gamma_{i_j} o)\\
&\le (4dR + 8R) + (dR + R) + R = 5dR + 10R,
\end{aligned}$$

which implies that $k_2^+ \in O_{5dR+10R}(o, \gamma_{i_j} o) \subset O_{10dR}(o, \gamma_{i_j} o)$, since $d \ge 2$. This finishes the proof. $\square$

Lemma 3.7. *Let $k_1, k_2 \in K$, $t_1, t_2 \ge 0$ be arbitrary. For any $0 \le s_1 \le t_1$, there exists $0 \le s_2 \le t_2$ such that*

$$d(k_1 \exp(s_1 \mathsf{u}) o, k_2 \exp(s_2 \mathsf{u}) o) \le d(k_1 \exp(t_1 \mathsf{u}) o, k_2 \exp(t_2 \mathsf{u}) o).$$

Proof. This follows from the CAT(0) property of G/K (cf. [10]). Consider the geodesic triangle $\triangle(pqr)$ in G/K with vertices $p = o$, $q = k_1 \exp(t_1 \mathsf{u}) o$ and $r = k_1 \exp(t_2 \mathsf{u}) o$. Let $\triangle(p'q'r')$ be the triangle in the Euclidean space which has the same corresponding side length to $\triangle pqr$. Let $0 \le s_2 \le t_2$ be arbitrary and r_1' be a point on the side $p'r'$ such that the segment $p'r_1'$ has length $\ell(p'r_1') = s_2$. By a straightforward computation in Euclidean geometry, we can find a point q_1' on the side $p'q'$ such that

$$\ell(q_1' r_1') \le \ell(q'r') = \ell(qr) = d(k_1 \exp(t_1 \mathsf{u}) o, k_2 \exp(t_2 \mathsf{u}) o).$$

Set $s_1 := \ell(p'q_1')$. Since G/K is a CAT(0) space, we get

$$d(k_1 \exp(s_1 \mathsf{u}) o, k_2 \exp(s_2 \mathsf{u}) o) \le \ell(q_1' r_1'),$$

from which the lemma follows. $\square$

We may write $\Lambda_{\mathsf{u}} = \cup_{R>0} \Lambda_{\mathsf{u},R}$ where

$$\Lambda_{\mathsf{u},R} := \bigcap_{m \ge 1} \bigcup_{\substack{\gamma \in \Gamma_{\mathsf{u},R},\\ \|\mu(\gamma)\| \ge m}} O_R(o, \gamma o), \qquad \text{where } \Gamma_{\mathsf{u},R} := \Gamma \cap G_{\mathsf{u},R}. \tag{3.5}$$

Lemma 3.8. *If $R > 1$ is large enough, for any $f \in L^1(\nu_\psi)$ and for ν_ψ-a.e. $\xi \in \Lambda_{\mathsf{u},R}$, we have*

$$\lim_{i\to\infty} \frac{1}{\nu_\psi(O_R(o,\gamma_i o))} \int_{O_R(o,\gamma_i o)} f\, d\nu_\psi = f(\xi)$$

for any sequence $\gamma_i \to \infty$ in $\Gamma_{\mathsf{u},R}$ such that $\xi \in O_R(o,\gamma_i o)$.

We define a maximal operator M_R on $L^1(\nu_\psi)$ as follows: for all $f \in L^1(\nu_\psi)$ and all $\xi \in \Lambda_{\mathsf{u},R}$, set

$$M_R f(\xi) := \limsup_{\substack{\gamma\in\Gamma_{\mathsf{u},R}, \|\mu(\gamma)\|\to\infty,\\ \xi\in O_R(o,\gamma o)}} \frac{1}{\nu_\psi(O_R(o,\gamma o))} \int_{O_R(o,\gamma o)} f\, d\nu_\psi;$$

this is well-defined by the definition of $\Lambda_{\mathsf{u},R}$.

Note that Lemma 3.8 holds trivially for $f \in C(\Lambda)$. Once the weak type inequality for the maximal functions is established as in Lemma 3.9, Lemma 3.8 follows from a standard argument using the density of $C(\Lambda)$ in $L^1(\nu_\psi)$.

Lemma 3.9. *If $R > 1$ is large enough, then M_R is of weak type $(1,1)$; for all $f \in L^1(\nu_\psi)$ and $\lambda > 0$, we have*

$$\nu_\psi(\{\xi \in \Lambda_{\mathsf{u},R} : |M_R f(\xi)| > \lambda\}) \ll \frac{1}{\lambda}\|f\|_{L^1(\nu_\psi)}$$

where the implied constant is independent of f.

Proof. Let $R > 1$ be large enough to satisfy Lemma 3.4. Let $\lambda > 0$ be arbitrary. By definition of M_R, there exists an infinite subset $\Gamma' \subset \Gamma_{\mathsf{u},R}$ such that

$$\{\xi \in \Lambda_{\mathsf{u},R} : |M_R f(\xi)| > \lambda\} \subset \bigcup_{\gamma\in\Gamma'} O_R(o,\gamma o), \text{ and}$$

$$\frac{1}{\nu_\psi(O_R(o,\gamma o))} \int_{O_R(o,\gamma o)} f\, d\nu_\psi > \lambda \text{ for all } \gamma \in \Gamma'.$$

By Lemma 3.6, there exists $\Gamma'' \subset \Gamma'$ so that $\{O_R(o,\gamma o) : \gamma \in \Gamma''\}$ consists of pairwise disjoint shadows and

$$\bigcup_{\gamma\in\Gamma'} O_R(o,\gamma o) \subset \bigcup_{\gamma\in\Gamma''} O_{10dR}(o,\gamma o). \tag{3.6}$$

Hence, by Lemma 3.4,

$$\begin{aligned} &\nu_\psi(\{\xi \in \Lambda_{\mathsf{u},R} : |M_R f(\xi)| > \lambda\}) \le \nu_\psi(\bigcup_{\gamma\in\Gamma'} O_R(o,\gamma o)) \\ &\le \nu_\psi(\bigcup_{\gamma\in\Gamma''} O_{10dR}(o,\gamma o)) \le \sum_{\gamma\in\Gamma''} \nu_\psi(O_{10dR}(o,\gamma o)) \\ &\asymp \sum_{\gamma\in\Gamma''} \nu_\psi(O_R(o,\gamma o)) \le \frac{1}{\lambda}\int_{\cup_{\gamma\in\Gamma''} O_R(o,\gamma o)} f\, d\nu_\psi \le \frac{1}{\lambda}\|f\|_{L^1(\nu_\psi)}. \end{aligned}$$

$\square$

Proof of Proposition 3.1. Let $R > 1$ be large enough to satisfy Lemma 3.9. Suppose that $\nu_\psi(\Lambda_{\mathsf{u}}) > 0$. Then for all sufficiently large $R > 1$, we have $\nu_\psi(\Lambda_{\mathsf{u},R}) > 0$. By applying Lemma 3.8 with $f = \mathbb{1}_{\Lambda_{\mathsf{u}}^c}$, there exists $\xi \in \Lambda_{\mathsf{u},R}$, we obtain a sequence $\gamma_i \in \Gamma$ such that $\xi \in O_R(o, \gamma_i o)$ and

$$\lim_{i\to\infty} \frac{\nu_\psi(O_R(o,\gamma_i o) \cap \Lambda_{\mathsf{u}}^c)}{\nu_\psi(O_R(o,\gamma_i o))} = 0.$$

Since $\nu_\psi(O_R(o,\gamma_i o)) \asymp e^{-\psi(\mu(\gamma_i))}$ by Lemma 3.4,

$$\lim_{i\to\infty} e^{\psi(\mu(\gamma_i))}\nu_\psi(O_R(o,\gamma_i o) \cap \Lambda_{\mathsf{u}}^c) = 0. \tag{3.7}$$

By Lemma 3.2,

$$\begin{aligned}\nu_\psi(O_R(o,\gamma_i o) \cap \Lambda_{\mathsf{u}}^c) &= \int \mathbb{1}_{O_R(o,\gamma_i o)\cap\Lambda_{\mathsf{u}}^c}(\xi)\, d\nu_\psi(\xi) \\ &= \int \mathbb{1}_{O_R(\gamma_i^{-1}o,o)\cap\Lambda_{\mathsf{u}}^c}(\xi) e^{\psi(\beta_\xi(e,\gamma_i^{-1}))}\, d\nu_\psi(\xi) \\ &\asymp e^{-\psi(\mu(\gamma_i))}\nu_\psi(O_R(\gamma_i^{-1}o,o)\cap\Lambda_{\mathsf{u}}^c).\end{aligned}$$

Hence as $i \to \infty$,

$$\nu_\psi(O_R(\gamma_i^{-1}o,o)\cap\Lambda_{\mathsf{u}}^c) \asymp e^{\psi(\mu(\gamma_i))}\nu_\psi(O_R(o,\gamma_i o)\cap\Lambda_{\mathsf{u}}^c) \to 0.$$

Passing to a subsequence, we may assume that $\gamma_i^{-1}o$ converges to some $\eta_0 \in \Lambda$. By [19, Lem. 5.6], for all sufficiently large i,

$$\nu_\psi(O_{R/2}(\eta_0,o)\cap\Lambda_{\mathsf{u}}^c) \le \nu_\psi(O_R(\gamma_i^{-1}o,o)\cap\Lambda_{\mathsf{u}}^c).$$

Therefore

$$\nu_\psi(O_{R/2}(\eta_0,o)\cap\Lambda_{\mathsf{u}}^c) = 0.$$

Since $R > 1$ is an arbitrary large number, varying R, we get from (3.2) that

$$\nu_\psi(\Lambda_{\mathsf{u}}^c \cap \{\eta \in \mathcal{F} : (\eta,\eta_0) \in \mathcal{F}^{(2)}\}) = 0. \tag{3.8}$$

We now claim that for any $\eta \in \Lambda_{\mathsf{u}}^c$, there exists a neighborhood U_η of η such that $\nu_\psi(\Lambda_{\mathsf{u}}^c \cap U_\eta) = 0$. If $(\eta,\eta_0) \in \mathcal{F}^{(2)}$, this is immediate from (3.8). Otherwise, by the Zariski density of Γ and the fact that Λ is the unique Γ-minimal subset of $\mathcal{F}$, we can find $\gamma \in \Gamma$ such that $(\gamma\eta,\eta_0) \in \mathcal{F}^{(2)}$. The claim follows again from (3.8), since ν_ψ is Γ-quasi-invariant. This finishes the proof. $\square$

4. Hopf's argument for higher rank cases

Let $\Gamma < G$ be a Zariski dense discrete subgroup. We fix $\psi \in \mathfrak{a}^*$ and a pair $(\nu_\psi, \nu_{\psi\circ \mathrm{i}})$ of (Γ,ψ) and $(\Gamma,\psi\circ\mathrm{i})$-conformal measures on $\mathcal{F}$ respectively.

Definition 4.1 (Hopf parametrization of G/M)**.** The map

$$gM \mapsto (g^+, g^-, b = \beta_{g^+}(e,g))$$

gives a homeomorphism between G/M and $\mathcal{F}^{(2)} \times \mathfrak{a}$.

Bowen-Margulis-Sullivan measures. Define the following A-invariant Radon measure $\tilde{\mathsf{m}} = \tilde{\mathsf{m}}(\nu_\psi, \nu_{\psi\circ \mathrm{i}})$ on G/M as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{\mathsf{m}}(g) = e^{\psi(\mathcal{G}(g^+, g^-))}\, d\nu_\psi(g^+) d\nu_{\psi\circ\mathrm{i}}(g^-) db$$

where db is the Lebesgue measure on $\mathfrak{a}$. We note that this is a non-zero measure; otherwise, ν_ψ is supported on a proper Zariski subvariety of $\mathcal{F}$ by Fubini's theorem, but since Γ is Zariski dense and ν_ψ is Γ-conformal, that is not possible. The measure $\tilde{\mathsf{m}}$ is left Γ-invariant, and hence induces a measure on $\Gamma\backslash G/M$, which we denote by m.

We fix $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$ and set for all $t \in \mathbb{R}$,

$$a_t := \exp t\mathsf{u}.$$

Recall the following definitions:

(1) A Borel subset $B \subset \Gamma\backslash G/M$ is called a *wandering set* for m if for m-a.e. $x \in B$, we have $\int_{-\infty}^{\infty} \mathbb{1}_B(xa_t)\, dt < \infty$.
(2) We say that $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is *conservative* if there is no wandering set $B \subset \Gamma\backslash G/M$ with $\mathsf{m}(B) > 0$.
(3) We say that $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is *completely dissipative* if $\Gamma\backslash G/M$ is a countable union of wandering sets modulo m.

Proposition 4.2. *The flow $(\Gamma\backslash G/M, \mathsf{m}, \{a_t = \exp(t\mathsf{u})\})$ is conservative (resp. completely dissipative) if and only if $\max(\nu_\psi(\Lambda_\mathsf{u}), \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})) > 0$ (resp. $\nu_\psi(\Lambda_\mathsf{u}) = 0 = \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})$).*

Proof. Suppose that $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is conservative. Let B be a compact subset of $\Gamma\backslash G/M$ with $\mathsf{m}(B) > 0$. If we set $B_0^\pm := \{x \in B : \limsup_{t\to\pm\infty} xa_t \cap B \neq \emptyset\}$, then $\mathsf{m}(B_0^+ \cup B_0^-) > 0$. Since $\tilde{\mathsf{m}}$ is equivalent to $\nu_\psi \otimes \nu_{\psi\circ\mathrm{i}} \otimes db$, it follows that $\mathsf{m}(B_0^+) > 0$ (resp. $\mathsf{m}(B_0^-) > 0$) if and only if $\nu_\psi(\Lambda_\mathsf{u}) > 0$ (resp. $\nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})}) > 0$). Hence $\max(\nu_\psi(\Lambda_\mathsf{u}), \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})})) > 0$.

Now suppose that $\nu_\psi(\Lambda_\mathsf{u}) > 0$ (resp. $\nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})}) > 0$). Then by Proposition 3.1, $\nu_\psi(\Lambda_\mathsf{u}) = 1$ (resp. $\nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathrm{i}(\mathsf{u})}) = 1$.) Hence for m a.e. $[g]$, we have $g^+ \in \Lambda_\mathsf{u}$ (resp. $g^- \in \Lambda_{\mathrm{i}(\mathsf{u})}$), and hence $[g]a_{t_i}$ is convergent for some sequence $t_i \to \pm\infty$. It follows that for m a.e. x, there exists a compact subset B such that $\int_\mathbb{R} \mathbb{1}_B(xa_t) dt = \infty$. We claim that this implies that $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is conservative. Assume in contradiction that there exists a wandering set $W \subset \Gamma\backslash G/M$ with $0 < \mathsf{m}(W) < \infty$. By the σ-compactness of $\Gamma\backslash G/M$, there exists a compact subset B such that

$$\mathsf{m}\{x \in W : \int_\mathbb{R} \mathbb{1}_B(xa_t) dt = \infty\} \geq \mathsf{m}(W)/2. \tag{4.1}$$

On the other hand, there exists an integer $n \geq 1$ for which the set

$$W_n := \left\{ w \in W : \int_{-\infty}^{\infty} \mathbb{1}_W(wa_t)\, dt \leq n \right\}$$

has m-measure strictly bigger than $\mathsf{m}(W)/2$. Note that the set $E := W_n \exp(\mathbb{R}\mathsf{v})$ is $\{a_t\}$-invariant and any $w \in E$ satisfies $\int_{-\infty}^{\infty} \mathbb{1}_W(wa_t)\, dt \leq n$. Hence, for any $R > 0$, we get

$$\begin{aligned}
&\int_{W_n}\int_{-R}^{R} \mathbb{1}_B(wa_t)dtd\mathsf{m} = \int_{-R}^{R}\int_{W_n} \mathbb{1}_B(wa_t)d\mathsf{m}dt \\
&= \int_{-R}^{R} \mathsf{m}(Ba_{-t}\cap W_n)dt = \int_{-R}^{R} \mathsf{m}(B\cap W_n a_t)dt \\
&= \int_{-R}^{R} \mathsf{m}((B\cap E)\cap W_n a_t)dt = \int_{-R}^{R}\int_{B\cap E} \mathbb{1}_{W_n}(xa_{-t})d\mathsf{m}dt \\
&= \int_{B\cap E}\int_{-R}^{R} \mathbb{1}_{W_n}(xa_{-t})dtd\mathsf{m} \le \int_{B\cap E}\int_{\mathbb{R}} \mathbb{1}_{W_n}(xa_{-t})dtd\mathsf{m} \\
&\le \int_{B\cap E} n\ dt = n\cdot \mathsf{m}(B\cap E) < \infty
\end{aligned}$$

where finiteness follows from the fact that B is compact and m is Radon. Hence $\int_{W_n}\int_{\mathbb{R}} \mathbb{1}_B(wa_t)dtd\mathsf{m} < \infty$; so

$$\mathsf{m}\{x\in W : \int_{\mathbb{R}} \mathbb{1}_B(wa_t)dt < \infty\} \ge \mathsf{m}(W_n) > \mathsf{m}(W)/2.$$

contradicting (4.1). The rest of the claims can be proven similarly. □

Let $\tilde{\mathsf{m}}'$ denote the M-invariant lift of $\tilde{\mathsf{m}}$ to G and m' the measure on $\Gamma\backslash G$ induced by $\tilde{\mathsf{m}}'$. Since Γ is Zariski dense, there exists a normal subgroup $M_\Gamma < M$ of finite index such that each P°-minimal subset of $\Gamma\backslash G$ is M_Γ-invariant and the collection of all P°-minimal subsets is parameterized by M/M_Γ ([12, Thm. 1.9 and 2], see also [20, Sec. 3]).

We will need the following notion:

Definition 4.3 (Transitivity group)**.** For $g\in G$ with $g^\pm\in\Lambda$, define the subset $\mathcal{H}^s_\Gamma(g) < AM$ as follows: $am\in\mathcal{H}^s_\Gamma(g)$ if and only if there exist $\gamma\in\Gamma$ and a sequence $h_i\in N^-\cup N^+$, $i=1,\dots,k$ such that

$$(gh_1h_2\dots h_r)^\pm\in\Lambda \text{ for all } 1\le r\le k \quad\text{and}\quad \gamma gh_1h_2\dots h_k = gam.$$

It is not hard to check that $\mathcal{H}^s_\Gamma(g)$ is a subgroup (cf. [32, Lem. 3.1]); it is called the strong transitivity subgroup.

The following was obtained in [20] using the work of Guivarch-Raugi [12, Thm. 1.9].

Lemma 4.4. [20, Coro. 3.8] *For any $g\in G$ with $g^\pm\in\Lambda$, the closure of $\mathcal{H}^s_\Gamma(g)$ contains AM_Γ.*

We now prove the following higher rank version of the Hopf-dichotomy, using Lemma 4.4.

Proposition 4.5. *Let Y be a P°-minimal subset of $\Gamma\backslash G$ such that $\mathsf{m}'(Y) > 0$. Then $(\mathsf{m}'|_Y,\{a_t\})$ is conservative if and only if $(\mathsf{m}'|_Y,\{a_t\})$ is ergodic.*

Proof. Suppose that $(\mathsf{m}'|_Y, \{a_t\})$ is conservative. Fix $x_0 \in \operatorname{supp}(\mathsf{m}'|_Y)$ and let $B_n \subset \Gamma\backslash G$ denote the ball of radius n centered at x_0. Let r be a positive function on $[0,\infty)$ which is affine on each $[n, n+1]$ and $r(n) = 1/(2^{n+1}\mathsf{m}'(B_{n+1}))$. Then the function $\rho(x) := r(d(x_0, x))$ is a positive Lipschitz function on $\Gamma\backslash G$ with a uniform Lipschitz constant. In particular, it is uniformly continuous and $\rho \in L^1(\mathsf{m}')$, since

$$\|\rho\|_{L^1(\mathsf{m}')} = \sum_{n=1}^{\infty} \int_{B_n - B_{n-1}} \rho \, d\mathsf{m}' \le \sum_{n=1}^{\infty} \frac{1}{2^n \mathsf{m}'(B_n)} \mathsf{m}'(B_n) < \infty.$$

By the definition of ρ, for all $[g] \in \Gamma\backslash G$ such that $g^+ \in \Lambda_{\mathsf{u}}$ and $g^- \in \Lambda_{\mathsf{i}(\mathsf{u})}$, we have

$$\int_0^\infty \rho([g]a_t)\, dt = \int_0^\infty \rho([g]a_{-t})\, dt = \infty. \tag{4.2}$$

Now let $f \in C_c(\Gamma\backslash G)$ be arbitrary. By the Hopf ratio ergodic theorem, the following f_+ and f_- are well-defined and equal m'-a.e.:

$$f_+(x) := \lim_{T\to\infty} \frac{\int_0^T f(xa_t)\, dt}{\int_0^T \rho(xa_t)\, dt} \text{ and } f_-(x) := \lim_{T\to\infty} \frac{\int_0^T f(xa_{-t})\, dt}{\int_0^T \rho(xa_{-t})\, dt}. \tag{4.3}$$

By the uniform continuity of f and ρ, (4.2) and the assumption that $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$, $f_\pm$ are $N^\pm$-invariant respectively. Let $\tilde{f}_\pm : G \to \mathbb{R}$ be a left Γ-invariant lift of $f_\pm$. Let $\mathcal{B}$ denote the Borel σ-algebra of G and set

$$\Sigma := \{B \in \mathcal{B} : \tilde{\mathsf{m}}'(B \triangle B^\pm) = 0 \text{ for some } B^\pm \in \mathcal{B} \text{ such that } \Gamma B^\pm = B^\pm N^\pm \}.$$

Let $F : G \to \mathbb{R}$ be a Σ-measurable and left Γ-invariant function such that $F(g) = \tilde{f}_+(g) = \tilde{f}_-(g)$ for $\tilde{\mathsf{m}}'$ a.e $g \in G$. Set

$$E := \left\{ gAM : \begin{array}{c} F|_{gAM} \text{ is measurable and} \\ F(gam) = \tilde{f}_+(gam) = \tilde{f}_-(gam) \\ \text{for Haar a.e. } am \in AM \end{array} \right\} \subset G/AM.$$

By Fubini's theorem, E has full measure in $G/AM \simeq \mathcal{F}^{(2)}$ with respect to the measure $d\nu_\psi \otimes d\nu_{\psi\circ\mathrm{i}}$. For all small $\varepsilon > 0$, define functions $F^\varepsilon, \tilde{f}^\varepsilon_\pm : G \to \mathbb{R}$ by

$$F^\varepsilon(g) := \tfrac{1}{\operatorname{Vol}(AM)_\varepsilon} \int_{(AM)_\varepsilon} F(g\ell)\, d\ell, \ \ \tilde{f}^\varepsilon_\pm(g) := \tfrac{1}{\operatorname{Vol}(AM)_\varepsilon} \int_{(AM)_\varepsilon} \tilde{f}_\pm(g\ell)\, d\ell$$

where $(AM)_\varepsilon$ denotes the ε-ball around e in AM and $d\ell$ is the Haar measure on AM. Note that if $gAM \in E$, then F^ε and $\tilde{f}^\varepsilon_\pm$ are continuous and identical on gAM. Moreover, F^ε is left Γ-invariant and $\tilde{f}^\varepsilon_\pm$ are $N^\pm$-invariant, as AM normalizes $N^\pm$. Using the isomorphism between G/AM and $\mathcal{F}^{(2)}$ given by $gAM \mapsto (g^+, g^-)$, we may consider E as a subset of $\mathcal{F}^{(2)}$. We then define

$$E^+ := \{\xi \in \Lambda : (\xi, \eta') \in E \quad \text{for } \nu_{\psi\circ\mathrm{i}}\text{-a.e. } \eta' \in \Lambda\};$$
$$E^- := \{\eta \in \Lambda : (\xi', \eta) \in E \quad \text{for } \nu_\psi\text{-a.e. } \xi' \in \Lambda\}.$$

Then E^+ is ν_ψ-conull and E^- is $\nu_{\psi\circ\mathrm{i}}$-conull by Fubini's theorem. By a similar argument as in [20, Lem. 4.6], we can show that for any $gAM \in E$ with $g^\pm \in E^\pm$, and any $\varepsilon > 0$, $F^\epsilon|_{gAM}$ is AM_Γ-invariant, using the fact that the closure of $\mathcal{H}^s_\Gamma(g)$ contains AM_Γ (Lemma 4.4). It follows that F is Σ_0-measurable where

$$\Sigma_0 := \{B \in \mathcal{B} : B = \Gamma BAM_\Gamma\}.$$

We claim that if f is M-invariant, then F is constant on the m'-conull set $E^\dagger := \{g \in G : g^\pm \in E^\pm\}$. Using Hopf's ratio ergodic theorem once more, this would in turn imply that m' is $M\{a_t\}$-ergodic. Assume f is M-invariant. Since $F = \lim_{\varepsilon\to 0} F^\varepsilon$ m'-a.e. by the Lebesgue differentiation theorem, it suffices to show that F^ε is constant on $E^\dagger$. Since F^ε is AM-invariant on $E^\dagger$ and $F^\varepsilon(gh) = F^\varepsilon(g)$ for all $g \in E^\dagger$ and $h \in N^\pm$ with $gh \in E^\dagger$, it is again enough to show that for any $g_1, g_2 \in E^\dagger$, there exist $h_1, h_2, h_3 \in N^+ \cup N^-$ such that $g_1h_1h_2h_3 \in g_2AM$ and $g_1h_1, g_1h_1h_2 \in E^\dagger$.

We note that if $(\xi, \eta_1), (\xi, \eta_2) \in \mathcal{F}^{(2)}$, then there exist $g \in G, h \in N$ such that $(\xi, \eta_1) = (g^+, g^-)$ and $(\xi, \eta_2) = ((gh)^+, (gh)^-)$. Similarly, if $(\xi_1, \eta), (\xi_2, \eta) \in \mathcal{F}^{(2)}$, then there exist $g \in G, h \in N^+$ such that $(\xi_1, \eta) = (g^+, g^-)$ and $(\xi_2, \eta) = ((gh)^+, (gh)^-)$.

Note that E^+ is Γ-invariant. Since the limit set Λ is the unique Γ-minimal subset of $\mathcal{F}$, the closure of E^+ contains Λ, and in particular it is Zariski dense. Therefore we can choose $\xi \in E^+$ such that $(\xi, g_1^-), (\xi, g_2^-) \in \mathcal{F}^{(2)}$. Let $h_1, h_2, h_3 \in N^+ \cup N^-$ be such that

$$\begin{aligned}(\xi, g_1^-) &= (g_1h_1^+, g_1h_1^-)\\ (\xi, g_2^-) &= (g_1h_1h_2^+, g_1h_1h_2^-)\\ (g_2^+, g_2^-) &= (g_1h_1h_2h_3^+, g_1h_1h_2h_3^-).\end{aligned}$$

Hence the claim is proved. In particular, m' is AM-ergodic.

Let $\tilde{Y} \subset G$ be the Γ-invariant lift of Y. In order to show that $\mathsf{m}'|_Y$ is $\{a_t\}$-ergodic, it suffices to show that F, associated to an arbitrary function $f \in C_c(\Gamma\backslash G)$, is constant on $\tilde{Y}$. It follows from the AM-ergodicity of m' that Σ_0 is $\tilde{\mathsf{m}}'$-equivalent to a finite σ-algebra generated by $\{B.s : s \in M_\Gamma\backslash M\}$ for some $B \in \Sigma_0$. Since $\{\tilde{Y}.s : s \in M_\Gamma\backslash M\} \subset \Sigma_0$ and the $\tilde{Y}.s$'s are mutually disjoint, it follows that $\tilde{Y} = B.s \bmod \tilde{\mathsf{m}}'$ for some $s \in M_\Gamma\backslash M$.

Since F is constant on $B.s$, being Σ_0-measurable, it implies that F is constant on $\tilde{Y}$, concluding that $\mathsf{m}'|_Y$ is $\{a_t\}$-ergodic.

Now to show the converse, assume that $(\mathsf{m}'|_Y, \{a_t\})$ is ergodic. Since the quotient map $\Gamma\backslash G \to \Gamma\backslash G/M$ is a proper map, it suffices to show that $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is conservative when it is ergodic. Assume that $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is ergodic. Then it is either conservative or completely dissipative by the Hopf decomposition theorem [16]. Suppose it is completely dissipative. Then it is isomorphic to a translation on $\mathbb{R}$ with respect to the Lebesgue measure. This implies that the dimension of $\mathfrak{a}$ must be one, since $\tilde{\mathsf{m}} = \tilde{\mathsf{m}}(\nu_\psi, \nu_{\psi\circ\mathrm{i}})$ gives measure zero on any one dimensional flow otherwise. It also implies that $\nu_\psi \otimes \nu_{\psi\circ \mathrm{i}}$ is supported on a single Γ-orbit, say, $\Gamma(\xi_0, \eta_0)$ in $\mathcal{F}^{(2)}$. Since ν_ψ (resp. $\nu_{\psi\circ\mathrm{i}}$) must be an atomic measure supported on $\Gamma\xi_0$ (resp. $\Gamma\eta_0$), it follows that $(\Gamma\xi_0 \times \Gamma\eta_0) \cap \mathcal{F}^{(2)} = \Gamma(\xi_0, \eta_0)$. This implies that $\Gamma\eta_0 \subset \Gamma_{\xi_0}\eta_0$ where Γ_{ξ_0} denotes the stabilizer of ξ_0 in Γ. Since the limit set of Γ_{ξ_0} is finite (as we are in the rank one situation), this is a contradiction as Γ is non-elementary. This proves that m is conservative for the $\{a_t\}$-action. $\square$

5. Directional Poincare series

Let $\Gamma < G$ be a Zariski dense discrete subgroup. We define the limit cone $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ as the asymptotic cone of $\mu(\Gamma)$. Then $\mathcal{L}_\Gamma$ coincides with the smallest cone containing the Jordan projection of Γ and is a convex cone with non-empty interior [4].

Quint [23] introduced the following:

Definition 5.1. The growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(t\mathsf{u}) = t\psi_\Gamma(\mathsf{u})$ for all $t > 0$, such that for any unit vector $\mathsf{u} \in \mathfrak{a}^+$,

$$\psi_\Gamma(\mathsf{u}) := \inf_{\substack{\text{open cones } \mathcal{C}\subset\mathfrak{a}^+ \\ \mathsf{u}\in\mathcal{C}}} \tau_{\mathcal{C}}$$

where $\tau_{\mathcal{C}}$ is the abscissa of convergence of the series $\sum_{\gamma\in\Gamma,\mu(\gamma)\in\mathcal{C}} e^{-t\|\mu(\gamma)\|}$.

We consider ψ_Γ as a function on $\mathfrak{a}$ by setting $\psi_\Gamma = -\infty$ outside $\mathfrak{a}^+$.

Quint showed that ψ_Γ is upper semi-continuous, $\psi_\Gamma > 0$ on $\operatorname{int}\mathcal{L}_\Gamma$, $\psi_\Gamma \geq 0$ on $\mathcal{L}_\Gamma$ and $\psi_\Gamma = -\infty$ outside $\mathcal{L}_\Gamma$ [23, Thm. IV.2.2].

Lemma 5.2. *Let $\psi \in \mathfrak{a}^*$ and $\mathsf{u} \in \operatorname{int}\mathfrak{a}^+$ be such that $\psi(\mathsf{u}) > \psi_\Gamma(\mathsf{u})$. Then for any $R > 0$,*

$$\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} < \infty.$$

Proof. Since $\psi(\mathsf{u}) > \psi_\Gamma(\mathsf{u})$, the upper-semi continuity of ψ_Γ implies that there exists a small open convex cone $\mathcal{C}$ containing u such that $\overline{\mathcal{C}} \subset \operatorname{int}\mathfrak{a}^+$ and $\psi > \psi_\Gamma$ on $\overline{\mathcal{C}}$. Since $\psi > \psi_\Gamma$ on some open convex cone $\mathcal{C}'$ containing $\overline{\mathcal{C}}$, we can choose a continuous homogeneous function $\theta : \mathfrak{a} \to \mathbb{R}$ such that $\psi \geq \theta > \psi_\Gamma$ on $\mathcal{C}$ and $\theta > \psi_\Gamma$ on $\mathfrak{a}^+$. Since $\psi_\Gamma = -\infty$ outside $\mathfrak{a}^+$, we have $\theta > \psi_\Gamma$ on $\mathfrak{a} - \{0\}$. Applying [23, Lem. III.1.3] to the measure $\sum_{\gamma\in\Gamma}\delta_{\mu(\gamma)}$ on $\mathfrak{a}^+$, we get

$$\sum_{\gamma\in\Gamma,\mu(\gamma)\in\mathcal{C}} e^{-\psi(\mu(\gamma))} \leq \sum_{\gamma\in\Gamma} e^{-\theta(\mu(\gamma))} < \infty,$$

Since $\#\{\gamma \in \Gamma_{\mathsf{u},R} : \mu(\gamma) \notin \mathcal{C}\} < \infty$ for any $R > 0$, the lemma follows. $\square$

Let $\psi \in \mathfrak{a}^*$ and fix a pair of (Γ, ψ) and $(\Gamma, \psi\circ \mathrm{i})$-conformal measures $(\nu_\psi, \nu_{\psi\circ\mathrm{i}})$ on $\mathcal{F}$ respectively. We let m denote the BMS measure on $\Gamma\backslash G/M$ associated to $(\nu_\psi, \nu_{\psi\circ\mathrm{i}})$.

We fix a unit vector $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$ such that $\psi(\mathsf{u}) > 0$, and set

$$a_t := \exp(t\mathsf{u}) \quad \text{and} \quad \delta := \psi(\mathsf{u}).$$

For an interval $I \subset \mathbb{R}$, we sometimes write $a_I = \{a_t : t \in I\}$. We make the following simple observation: for any $R > 0$,

$$\sum_{\gamma \in \Gamma_{\mathsf{i}(\mathsf{u}),R}} e^{-\psi(\mathrm{i}(\mu(\gamma)))} = \sum_{\gamma^{-1} \in \Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma^{-1}))} = \sum_{\gamma \in \Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))}. \tag{5.1}$$

Lemma 5.3. *If* $\max(\nu_\psi(\Lambda_{\mathsf{u}}), \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathsf{i}(\mathsf{u})})) > 0$, *then there exists* $R > 0$ *such that*

$$\sum_{\gamma \in \Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty = \sum_{\gamma \in \Gamma_{\mathsf{i}(\mathsf{u}),R}} e^{-\psi(\mathrm{i}(\mu(\gamma)))}.$$

Proof. Without loss of generality, we may assume that $\nu_\psi(\Lambda_{\mathsf{u}}) > 0$. Recall that $\Lambda_{\mathsf{u}} = \cup_{n\in\mathbb{N}} \Lambda_{\mathsf{u},n}$ where

$$\Lambda_{\mathsf{u},n} = \bigcap_{m=1}^{\infty} \bigcup_{\|\mu(\gamma)\| \geq m, \gamma \in \Gamma_{\mathsf{u},n}} O_n(o, \gamma o).$$

Hence $\nu_\psi(\Lambda_{\mathsf{u},n}) > 0$ for some n. Now by Lemma 3.4, we have for all $m \geq 1$,

$$0 < \nu_\psi(\Lambda_{\mathsf{u},n}) \leq \sum_{\substack{\|\mu(\gamma)\| \geq m, \\ \gamma \in \Gamma_{\mathsf{u},n}}} \nu_\psi(O_n(o, \gamma o)) \ll \sum_{\substack{\|\mu(\gamma)\| \geq m, \\ \gamma \in \Gamma_{\mathsf{u},n}}} e^{-\psi(\mu(\gamma))}.$$

Since the implicit constant above is independent of m, it follows that the series $\sum_{\gamma \in \Gamma_{\mathsf{u},n}} e^{-\psi(\mu(\gamma))}$ diverges, which implies the claim by (5.1). $\square$

The rest of this section is devoted to the proof of the following:

Proposition 5.4. *Suppose that* m *is* u*-balanced as defined in* (1.2). *If* $\sum_{\gamma \in \Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty$ *for some* $R > 0$, *then*

$$\nu_\psi(\Lambda_{\mathsf{u}}) = 1 = \nu_{\psi\circ\mathrm{i}}(\Lambda_{\mathsf{i}(\mathsf{u})}).$$

Proof of this proposition involves investigating the relation between the u-directional Poincare series and the correlation function of m for the a_t-action.

Multiplicity of shadows.

Lemma 5.5. *For any* $R > 0$ *and* $D > 0$, *we have*

$$\sup_{T>0} \sum_{\substack{\gamma \in \Gamma_{\mathsf{u},R}, \\ T \leq \psi(\mu(\gamma)) \leq T+D}} \mathbb{1}_{O_R(o,\gamma o)} < \infty.$$

Proof. Suppose that there exist $\gamma_1, \cdots, \gamma_m \in \Gamma_{\mathsf{u},R}$ and $k \in K$ such that $k^+ \in \cap_{i=1}^m O_R(o, \gamma_i o)$ and $T \le \psi(\mu(\gamma_i)) \le T + D$. By Lemma 3.5, for all $1 \le i \le m$, there exists $t_i \ge 0$ such that $ka_{t_i}o \in B(\gamma_i o, (2d+1)R)$. Since $\gamma_i^{-1}ka_{t_i} \in G_{(2d+1)R}$, we have $\|\mu(\gamma_i) - t_i u\| \le d(2d+1)R$ by Lemma 2.5. In particular,

$$t_i\psi(\mathsf{u}) \le \psi(\mu(\gamma_i)) + \|\psi\|d(2d+1)R \le T + D + \|\psi\|d(2d+1)R,$$

and similarly

$$t_i\psi(\mathsf{u}) \ge T - \|\psi\|d(2d+1)R.$$

Hence $|\psi(\mathsf{u})(t_i - t_1)| < 2\|\psi\|d(2d+1)R + D$. Note that as $\psi(\mathsf{u}) > 0$, for all $1 \le i \le m$,

$$\begin{aligned} d(\gamma_i o, \gamma_1 o) &\le d(\gamma_i o, ka_{t_i}o) + d(ka_{t_i}o, ka_{t_1}o) + d(ka_{t_1}o, \gamma_1 o) \\ &\le 2d(2d+1)R + |t_i - t_1| \\ &\le S := 2d(2d+1)R + (\psi(\mathsf{u}))^{-1}(2\|\psi\|d(2d+1)R + D). \end{aligned}$$

Since there are only finitely many $\gamma_i o$ in a bounded ball of radius S, it follows that m is bounded above by a constant depending only on S. This proves the claim. ∎

Corollary 5.6. *For any large enough $R > 0$, we have, for any $D > 0$,*

$$\sup_{T>0} \sum_{\substack{\gamma \in \Gamma_{\mathsf{u},R}, \\ T \le \psi(\mu(\gamma)) \le T+D}} e^{-\psi(\mu(\gamma))} < \infty.$$

Proof. By Lemmas 3.4 and 3.5, there exists $C = C(\psi) > 0$ such that for all R large enough, and any $T > 0$,

$$\sum_{\substack{\gamma \in \Gamma_{\mathsf{u},R}, \\ T \le \psi(\mu(\gamma)) \le T+D}} e^{-\psi(\mu(\gamma))} \le \sum_{\substack{\gamma \in \Gamma_{\mathsf{u},R}, \\ T \le \psi(\mu(\gamma)) \le T+D}} C \cdot \nu_\psi(O_R(o, \gamma o)) < \infty$$

by Lemma 5.5. ∎

Directional Poincare series. For $r > 0$ and $g \in G$, we set

$$Q_r := G_r A_r = K A_r K A_r, \text{ and} \tag{5.2}$$

$$\mathcal{L}_r(o, g(o)) := \{(h^+, h^-) \in \mathcal{F}^{(2)} : h \in G_r \cap gG_r \exp(\mathbb{R}_- \mathsf{u})\}.$$

Lemma 5.7. *For any $r > 0$, we have $Q_r \subset G_{2r}$.*

Proof. Let $g \in Q_r$ be arbitrary. By definition, $g = k_1 a_1 k_2 a_2$ for some $k_1, k_2 \in K$ and $a_1, a_2 \in A_r$. Since

$$\begin{aligned} d(go, o) = d(a_1 k_2 a_2 o, o) &\le d(a_1 k_2 a_2 o, a_1 k_2 o) + d(a_1 k_2 o, o) \\ &= d(a_2 o, o) + d(o, a_1^{-1} o) < 2r, \end{aligned}$$

the lemma follows. ∎

The following is the main ingredient of the proof of Proposition 5.4:

Proposition 5.8. *Suppose that* $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))}=\infty$ *for some* $R>0$. *If* r *is large enough, we have the following for any* $T>1$:

$$\int_0^T\int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q_r\cap\gamma Q_r a_{-t}\cap\gamma' Q_r a_{-t-s})\,dt\,ds \ll \left(\sum_{\substack{\gamma\in\Gamma_{\mathsf{u},4dr}\\ \psi(\mu(\gamma))\le\delta T}} e^{-\psi(\mu(\gamma))}\right)^2; \tag{5.3}$$

$$\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q_{4r}\cap\gamma Q_{4r}a_{-t})\,dt \gg \sum_{\substack{\gamma\in\Gamma_{\mathsf{u},r}\\ \psi(\mu(\gamma))<\delta T}} e^{-\psi(\mu(\gamma))} \tag{5.4}$$

where the implied constants are independent of T.

Lemma 5.9. *If* $Q_r\cap\gamma Q_r a_{-t}\neq\emptyset$ *for* $\gamma\in\Gamma$ *and* $t,r>0$, *then*

$$\|\mu(\gamma)-t\mathsf{u}\|\le 4dr.$$

Proof. If $Q_r\cap\gamma Q_r a_{-t}\neq\emptyset$, there exist $q_0,q_0'\in Q_r$ such that $q_0=\gamma q_0' a_{-t}$. By Lemma 2.5 and Lemma 5.7,

$$t\mathsf{u}=\mu(a_t)=\mu(q_0^{-1}\gamma q_0')\in\mu(\gamma)+\mathfrak{a}_{4dr}.$$

$\square$

In order to prove Proposition 5.8, we will bound the integrals appearing in the lemma from below and above using shadows, and then apply the shadow lemma (Lemma 3.4). For this purpose, we observe several relations between the sets defined in (5.2) and shadows.

Lemma 5.10. *If* $g\in Q_r\cap\gamma Q_r a_{-t}$ *for* $\gamma\in\Gamma$ *and* $t,r>0$, *then*
(1) $(g^+,g^-)\in\mathcal{L}_r(o,\gamma o)$;
(2) $|\psi(\mathcal{G}(g^+,g^-))|<2\|\psi\|cr$ *where* c *is from Lemma 2.3;*
(3) $[g]A\cap Q_r\cap\gamma Q_r a_{-t}\subset[g]A_{2dr}$.

Proof. (1) is immediate from the definition of $\mathcal{L}_r(o,\gamma o)$. Since $g\in Q_r$, $go\in B(o,2r)$ and hence $\|\mathcal{G}(g^+,g^-)\|<2cr$ by Lemma 2.3 and (2) follows. (3) follows from the stronger inclusion $gA\cap Q_r\subset gA_{2dr}$ which follows from Lemma 2.5 and Lemma 5.7. $\square$

Lemma 5.11. *For any* $g\in G$ *and* $r>0$, *we have*

$$\mathcal{L}_r(o,g(o))\subset O_{4r}(o,g(o))\times O_{4r}(g(o),o).$$

Proof. Let $(h^+,h^-)\in\mathcal{L}_r(o,g(o))$; so $h\in B(o,2r)$ such that $ha_to\in B(g(o),2r)$ for some $t\ge 0$. Write $o=ha_0n_0o$ for some $a_0n_0\in AN$. Since the Hausdorff distance between $a_0n_0A^+o$ and A^+o is $d(a_0n_0o,o)$ [10, 1.6.6 (4)], we can find $q'\in ha_0n_0A^+o$ such that $d(q',ha_to)<d(ha_0n_0o,ho)<2r$. Hence, $d(g(o),q')<d(g(o),ha_to)+d(ha_to,q')<4r$ and it follows that $h^+\in O_{4r}(o,g(o))$. A similar argument shows that $h^-\in O_{4r}(g(o),o)$. $\square$

Lemma 5.12. *For all large enough $r > 1$, we have for any $t > 1$,*

$$\tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a_{-t}) \ll e^{-\psi(\mu(\gamma))}$$

where the implied constant is independent of $t > 1$.

Proof. If r is large enough, we get by Lemma 5.11, Lemma 3.4 and Lemma 5.10:

$$\begin{aligned}
&\tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a_{-t}) \\
&= \int \mathbb{1}_{Q_r \cap \gamma Q_r a_{-t}}([gb]) e^{\psi(\mathcal{G}(g^+, g^-))} d\nu_\psi(g^+) d\nu_{\psi \circ \mathrm{i}}(g^-)\, db \\
&= \int_{\mathcal{L}_r(o, \gamma o)} \left(\int_A \mathbb{1}_{Q_r \cap \gamma Q_r a_{-t}}([gb])\, db \right) e^{\psi(\mathcal{G}(g^+, g^-))} d\nu_\psi(g^+) d\nu_{\psi \circ \mathrm{i}}(g^-) \\
&\le \nu_\psi(O_{4r}(o, \gamma o)) \operatorname{Vol}(A_{2dr}) e^{2\|\psi\| cr} \\
&\ll e^{-\psi(\mu(\gamma))}.
\end{aligned}$$

□

Lemma 5.13. *If $Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s} \neq \emptyset$ for $\gamma, \gamma' \in \Gamma$ and $r, t, s > 0$, then*

(1) $\|\mu(\gamma) - t\mathsf{u}\|$, $\|\mu(\gamma^{-1}\gamma') - s\mathsf{u}\|$, $\|\mu(\gamma') - (t+s)\mathsf{u}\| \le 4dr$;

(2) $\psi(\mu(\gamma)) + \psi(\mu(\gamma^{-1}\gamma')) \le \psi(\mu(\gamma')) + 12dr\|\psi\|$.

Proof. Note that from the hypothesis, the intersections

$$Q_r \cap \gamma Q_r a_{-t},\ Q_r \cap \gamma^{-1}\gamma' Q_r a_{-s},\ Q_r \cap \gamma' Q_r a_{-t-s}$$

are all nonempty. By Lemma 5.9, we obtain (1).

(2) follows since

$$\begin{aligned}
&|\psi(\mu(\gamma)) + \psi(\mu(\gamma^{-1}\gamma')) - \psi(\mu(\gamma'))| \\
&= |\psi(\mu(\gamma) - t\mathsf{u}) + \psi(\mu(\gamma^{-1}\gamma') - s\mathsf{u}) - \psi(\mu(\gamma') - (t+s)\mathsf{u})| \\
&\le 4dr\|\psi\| + 4dr\|\psi\| + 4dr\|\psi\| = 12dr\|\psi\|.
\end{aligned}$$

□

Proof of (5.3) **in Proposition 5.8.** Fix $s, t > 0$. Let r be large enough so that $\sum_{\gamma \in \Gamma_{\mathsf{u},4dr}} e^{-\psi(\mu(\gamma))} = \infty$. In the following proof, the notation $\sum''$ means the sum over all $(\gamma, \gamma') \in \Gamma_{\mathsf{u},4dr} \times \Gamma$ satisfying:

$$\begin{aligned}
\gamma^{-1}\gamma' &\in \Gamma_{\mathsf{u},4dr}; \\
\psi(\mu(\gamma)) &\in (\delta t - 4dr\|\psi\|, \delta t + 4dr\|\psi\|);\ \textit{and} \\
\psi(\mu(\gamma^{-1}\gamma')) &\in (\delta s - 4dr\|\psi\|, \delta s + 4dr\|\psi\|).
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q_r\cap\gamma Q_r a_{-t}\cap\gamma' Q_r a_{-t-s})\\
&=\sum{}'' \tilde{\mathsf{m}}(Q_r\cap\gamma Q_r a_{-t}\cap\gamma' Q_r a_{-t-s}) \text{ by Lemma 5.13(1)}\\
&\ll\sum{}'' e^{-\psi(\mu(\gamma'))} \qquad \text{by Lemma 5.12}\\
&\ll\sum{}'' e^{-\psi(\mu(\gamma))}e^{-\psi(\mu(\gamma^{-1}\gamma'))} \qquad \text{by Lemma 5.13(2)}\\
&\ll\left(\sum_{\substack{\gamma\in\Gamma_{\mathsf{u},4dr},\\ \psi(\mu(\gamma))\in(\delta t-c_0,\delta t+c_0)}} e^{-\psi(\mu(\gamma))}\right)\left(\sum_{\substack{\gamma'\in\Gamma_{\mathsf{u},4dr},\\ \psi(\mu(\gamma'))\in(\delta s-c_0,\delta s+c_0)}} e^{-\psi(\mu(\gamma'))}\right)
\end{aligned}$$

where $c_0 = 4dr\|\psi\|$.

Let I_γ denote the interval $\delta^{-1}[\psi(\mu(\gamma))-c_0,\psi(\mu(\gamma))+c_0]$. Note that $I_\gamma\cap[0,T]\neq\emptyset$ implies that $\psi(\mu(\gamma))\le\delta T+c_0$. Hence

$$\begin{aligned}
&\int_0^T\left(\sum_{\substack{\gamma\in\Gamma_{\mathsf{u},4dr},\\ \psi(\mu(\gamma))\in(\delta t-c_0,\delta t+c_0)}} e^{-\psi(\mu(\gamma))}\right)dt\\
&=\sum_{\gamma\in\Gamma_{\mathsf{u},4dr}} e^{-\psi(\mu(\gamma))}\int_0^T \mathbb{1}_{I_\gamma}(t)dt \ll \sum_{\substack{\gamma\in\Gamma_{\mathsf{u},4dr},\\ \psi(\mu(\gamma))\le\delta T+c_0}} e^{-\psi(\mu(\gamma))}.
\end{aligned}$$

Putting these two together along with Corollary 5.6, used in order to remove c_0, concludes the proof of (5.3). □

Lemma 5.14. *For any $S>0$ and $r>0$, there exists $0<\ell(S,r)<\infty$ such that for any $\gamma\in\Gamma$ with $\|\mu(\gamma)\|>\ell(S,r)$, any point $(\xi,\eta)\in O_S(o,\gamma o)\times O_S(\gamma o,o)$ satisfies $\|\mathcal{G}(\xi,\eta)\|<\ell(S,r)$.*

Proof. Suppose not. Then there exists a sequence $\gamma_i\to\infty$ in $\Gamma_{\mathsf{u},r}$ and $(\xi_i,\eta_i)\in O_S(o,\gamma_i o)\times O_S(\gamma_i o,o)$ such that $\|\mathcal{G}(\xi_i,\eta_i)\|\to\infty$.

We may write $\gamma_i = k_i a_i \ell_i$ in KA^+K decomposition, and assume that $k_i\to k_0$ after passing to a subsequence. It follows that $\xi_i\to k_0^+$ and $\eta_i\to\eta_0$ for some $\eta_0\in\mathcal{F}$ such that $(k_0^+,\eta_0)\in\mathcal{F}^{(2)}$ as $\gamma_i\to\infty$ regularly, by Lemma 3.3. Hence $\lim_{i\to\infty}\|\mathcal{G}(\xi_i,\eta_i)\| = \|\mathcal{G}(k_0^+,\eta_0)\|<\infty$, which is a contradiction. □

In the following, we fix a large number S_0 which satisfies Lemma 3.4. For each $r>1$, let $\ell_r := \ell(S_0,r)>0$ be as provided by Lemma 5.14 so that for any $(\xi,\eta)\in\bigcup_{\gamma\in\Gamma_{\mathsf{u},r},\|\mu(\gamma)\|\ge\ell_r} O_{S_0}(o,\gamma o)\times O_{S_0}(\gamma o,o)$, we have $\|\mathcal{G}(\xi,\eta)\|<\ell_r$.

Lemma 5.15. *If $r > 1$ is large enough, the following holds: for any $(\xi, \eta) \in O_{S_0}(o, \gamma o) \times O_{S_0}(\gamma o, o)$ for some $\gamma \in \Gamma_{\mathsf{u},r}$ with $\|\mu(\gamma)\| \geq \ell_r$, there exist $t \in \mathbb{R}$ and $g \in Q_{2r}$ such that*

$$ga_{[t-1,t+1]} \subset \gamma Q_{2r} \quad \text{and} \quad (g^+, g^-) = (\xi, \eta).$$

Proof. Let (ξ, η) be as in the statement. Then by Lemma 3.5, there exists $t \geq 0$ and $k \in K$ such that $\xi = k^+$, $ka_t o \in B(\gamma o, r + (d+1)S_0)$. Let $g \in G$ be such that $(g^+, g^-) = (\xi, \eta)$. Since $\|\mu(\gamma)\| > \ell_r$, by replacing $g \in G$ by an element of gA, we may assume that $d(go, o) < c\ell_r + c'$ where c and c' are as in Lemma 2.3. As $g^+ = k^+$ and hence $k^{-1}g \in P$, it follows by [10, 1.6.6 (4)] that $d(ga_t o, ka_t o) \leq d(go, o)$ for all $t \geq 0$.

Hence for all $s \in [t-1, t+1]$,

$$d(ga_s o, ka_t o) < d(ga_s o, ga_t o) + d(ga_t o, ka_t o) \leq 1 + d(go, o) < 1 + c\ell_r + c'.$$

It follows that $ga_{[t-1,t+1]} \in \gamma G_{r+(d+1)S_0+c\ell_r+c'+1}$. Now if r is large enough,

$$ga_{[t-1,t+1]} \subset \gamma Q_{2r}.$$

Similarly, since $go \in G_{c\ell_r+c'}$, we have $g \in Q_{2r}$, which was to be shown. $\square$

Lemma 5.16. *If $r > 1$ is large enough, the following holds: for any $g \in G$ such that $(g^+, g^-) \in O_{S_0}(o, \gamma o) \times O_{S_0}(\gamma o, o)$ for some $\gamma \in \Gamma_{\mathsf{u},r}$ and $T > 0$ satisfying*

$$\|\mu(\gamma)\| > \ell_r \quad \text{and} \quad 8dr\|\psi\| + \delta < \psi(\mu(\gamma)) < \delta T - 8dr\|\psi\| - \delta,$$

we have

$$\int_0^T \int_A \mathbb{1}_{Q_{4r} \cap \gamma Q_{4r} a_{-t}}([gb])\, db\, dt \geq 2\,\mathrm{Vol}(A_{2r}). \tag{5.5}$$

Proof. Note that replacing g with an element of gA does not affect the validity of (5.5). Hence by Lemma 5.15, we may assume that $g \in Q_{2r}$ and $ga_{[t_0-1,t_0+1]} \subset \gamma Q_{2r}$ for some $t_0 \in \mathbb{R}$.

It follows that $Q_{2r} \cap \gamma Q_{2r} a_{-t} \neq \emptyset$ for all $t \in [t_0 - 1, t_0 + 1]$. Note that $|\psi(\mu(\gamma)) - t_0\delta| \leq 8dr\|\psi\|$ by Lemma 5.9, and hence $[t_0 - 1, t_0 + 1] \subset [0, T]$ by the hypothesis. Since $g \in Q_{2r}$ and hence $g \in G_{4r}$ by Lemma 5.7, we have $gA \cap Q_{4r} \supset gA_{4r}$. Consequently,

$$\int_A \mathbb{1}_{Q_{4r} \cap \gamma Q_{4r} a_{-t}}([gb])\, db \geq \int_{A_{4r}} \mathbb{1}_{\gamma Q_{4r}}([gba_t])\, db. \tag{5.6}$$

By definition of Q_{4r}, there is a uniform lower bound for (5.6), say $\mathrm{Vol}(A_{2r})$, whenever $[ga_t] \cap \gamma Q_{4r} \neq \emptyset$, in particular for all $t \in [t_0 - 1, t_0 + 1]$ by Lemma 5.15. Hence,

$$\begin{aligned}
&\int_0^T \int_A \mathbb{1}_{Q_{4r} \cap \gamma Q_{4r} a_{-t}}([gb])\, db\, dt \\
&\geq \int_{t_0-1}^{t_0+1} \int_A \mathbb{1}_{Q_{4r} \cap \gamma Q_{4r} a_{-t}}([gb])\, db\, dt \geq 2\,\mathrm{Vol}(A_{2r}).
\end{aligned}$$

This proves the lemma. $\square$

Proof of (5.4) **in Proposition 5.8.** By definition of $\tilde{\mathsf{m}}$, we have for any $\gamma \in \Gamma$ and $r, t > 0$,

$$\begin{aligned}
&\tilde{\mathsf{m}}(Q_{4r} \cap \gamma Q_{4r} a_{-t}) \\
&= \int_{\mathcal{F}^{(2)}} \left(\int_A \mathbb{1}_{Q_{4r} \cap \gamma Q_{4r} a_{-t}}([gb])\, db \right) e^{\psi(\mathcal{G}(g^+, g^-))} d\nu_\psi(g^+) d\nu_{\psi \circ \mathrm{i}}(g^-) \\
&\geq \int_{O_{S_0}(o,\gamma o) \times O_{S_0}(\gamma o, o)} \left(\int_A \mathbb{1}_{Q_{4r} \cap \gamma Q_{4r} a_{-t}}([gb])\, db \right) e^{\psi(\mathcal{G}(g^+, g^-))} d\nu_\psi(g^+) d\nu_{\psi \circ \mathrm{i}}(g^-).
\end{aligned}$$

Now Lemma 5.16 implies that if $\gamma \in \Gamma_{\mathsf{u},r}$, $\|\mu(\gamma)\| > \ell_r$ and $(8dr\|\psi\| + \delta) < \psi(\mu(\gamma)) < \delta T - (8dr\|\psi\| + \delta)$, then

$$\begin{aligned}
&\int_0^T \tilde{\mathsf{m}}(Q_{4r} \cap \gamma Q_{4r} a_{-t})\, dt \\
&\geq 2 \operatorname{Vol}(A_{2r}) \int_{O_{S_0}(o,\gamma o) \times O_{S_0}(\gamma o, o)} e^{\psi(\mathcal{G}(g^+, g^-))} d\nu_\psi(g^+) d\nu_{\psi \circ \mathrm{i}}(g^-) \\
&\geq 2 \operatorname{Vol}(A_{2r}) e^{-\|\psi\|\ell_r} \nu_\psi(O_{S_0}(o, \gamma o)) \nu_{\psi \circ \mathrm{i}}(O_{S_0}(\gamma o, o)) \\
&\geq 2 \operatorname{Vol}(A_{2r}) e^{-\|\psi\|\ell_r} \beta(\nu_{\psi \circ \mathrm{i}}) c_1 e^{-\kappa \|\psi\| S_0} e^{-\psi(\mu(\gamma))},
\end{aligned}$$

where the second inequality follows from the lower bound $e^{\psi(\mathcal{G}(g^+, g^-))} \geq e^{-\|\psi\|\ell_r}$ and the last inequality follows from Lemma 3.4. Therefore,

$$\begin{aligned}
\int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q_{4r} \cap \gamma Q_{4r} a_{-t})\, dt &\geq \int_0^T \sum_{\gamma \in \Gamma_{\mathsf{u},r},\, \|\mu(\gamma)\| > \ell_r} \tilde{\mathsf{m}}(Q_{4r} \cap \gamma Q_{4r} a_{-t})\, dt \\
&\gg \sum_{\substack{\gamma \in \Gamma_{\mathsf{u},r}\ \|\mu(\gamma)\| > \ell_r, \\ \psi(\mu(\gamma)) < \delta T - (8dr\|\psi\| + \delta)}} e^{-\psi(\mu(\gamma))}.
\end{aligned}$$

Since $\#\{\gamma \in \Gamma : \|\mu(\gamma)\| \leq \ell_r\}$ is a finite set, this proves the lemma by Corollary 5.6. $\square$

Proposition 5.8 yields:

Corollary 5.17. *Suppose that for any large $r, s \gg 1$, and $T > 1$,*

$$\int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a_{-t})\, dt \asymp \int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q_s \cap \gamma Q_s a_{-t})\, dt \tag{5.7}$$

with the implied constant independent of T. If $\sum_{\gamma \in \Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty$ for some $R > 0$, then for all sufficiently large r, we have for any $T > 1$:

$$\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s})\, dt\, ds \ll \left(\int_0^T \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a_{-t})\, dt \right)^2. \tag{5.8}$$

Proof. By Proposition 5.8, we get

$$\int_0^T \int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s})\, dt\, ds \ll \left(\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}(Q_{16dr} \cap \gamma Q_{16dr} a_{-t})\, dt \right)^2,$$

which implies the claim in view of the hypothesis 5.7. ☐

Proof of Proposition 5.4. We will apply the following version of the Borel-Cantelli lemma:

Lemma 5.18. [3, Lem. 2] *Let* (Ω, M) *be a finite Borel measure space and* $\{P_t : t \geq 0\} \subset \Omega$ *be such that* $(t,\omega) \mapsto \mathbb{1}_{P_t}(\omega)$ *is measurable. Suppose that*

(1) $\int_0^\infty \mathsf{M}(P_t)\, dt = \infty$, *and*
(2) *there is a constant* $C > 0$ *such that*

$$\int_0^T \int_0^T \mathsf{M}(P_t \cap P_s)\, dt\, ds \leq C \left(\int_0^T \mathsf{M}(P_t)\, dt \right)^2 \text{ for all } T \gg 1.$$

Then we have

$$\mathsf{M}\left\{ \omega \in \Omega : \int_0^\infty \mathbb{1}_{P_t}(\omega)\, dt = \infty \right\} > \frac{1}{C}.$$

Suppose that $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty$ for some $R > 0$. Let $r > R$ be large enough to satisfy Proposition 5.8, and consider $Q_r = G_r A_r$. As M commutes with A and $Q_r = KA_r^+ KA_r$, Q_r is an M-invariant subset. Let $[Q_r] = \Gamma\backslash Q_r/M \subset \Gamma\backslash G/M$. Set

$$\mathsf{M} := \mathsf{m}|_{[Q_r]} \text{ and } P_t := \Gamma\backslash\Gamma(Q_r \cap \Gamma Q_r a_{-t}) \subset \Gamma\backslash G/M.$$

We claim that

$$\int_0^T \int_0^T \mathsf{M}(P_t \cap P_s)\, ds\, dt \ll \left(\int_0^T \mathsf{M}(P_t)\, dt \right)^2. \tag{5.9}$$

Since m is assumed to be u-balanced, Corollary 5.17 applies, and hence

$$\int_0^T \int_0^T \mathsf{M}(P_t \cap P_{t+s})\, ds\, dt \ll \left(\int_0^T \mathsf{M}(P_t)\, dt \right)^2. \tag{5.10}$$

Therefore

$$\begin{aligned}\int_0^T \int_0^T \mathsf{M}(P_t \cap P_s)\, ds\, dt &= 2\int_0^T \int_t^T \mathsf{M}(P_t \cap P_s)\, ds\, dt \\ &\leq 2\int_0^T \int_0^T \mathsf{M}(P_t \cap P_{t+s})\, ds\, dt \ll \left(\int_0^T \mathsf{M}(P_t)\, dt \right)^2,\end{aligned}$$

proving the claim. Applying Lemma 5.18 with M and P_t, we conclude that

$$\mathsf{m}\left\{[g] \in [Q_r] : \int_0^\infty \mathbb{1}_{[Q_r]}([g]a_t)dt = \infty\right\} > 0.$$

It follows that $\nu_\psi(\{g^+ \in \mathcal{F} : \limsup [g]a_t \neq \emptyset\}) > 0$ and hence $\nu_\psi(\Lambda_{\mathsf{u}}) > 0$. On the other hand, by (5.1), we have $\sum_{\gamma\in\Gamma_{\mathsf{i}(\mathsf{u}),R}} e^{-\psi\circ \mathsf{i}(\mu(\gamma))} = \infty$. By the same argument as above, this implies that

$$\nu_{\psi\circ\mathsf{i}}(\{g^+ \in \mathcal{F} : \limsup [g]\exp(t\,\mathsf{i}(\mathsf{u})) \neq \emptyset\}) > 0$$

and hence $\nu_{\psi\circ\mathsf{i}}(\Lambda_{\mathsf{i}(\mathsf{u})}) > 0$. This finishes the proof by Proposition 3.1.

Proof of Theorem 1.4. . The equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 3.1. The equivalence (2) $\Leftrightarrow$ (3) follows from Proposition 4.2. The equivalence (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Proposition 4.5. The implication (1) $\Rightarrow$ (6) is proved in Lemma 5.3, and the implication (6) $\Rightarrow$ (7) follows from Lemma 5.3 and Proposition 5.4.

Remark 5.19. The asymptotic inequality (5.9) shows that if m is u-balanced and $\sum\limits_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi(\mu(\gamma))} = \infty$ for some $R > 0$, then the measure preserving flow $(\Gamma\backslash G/M, \mathsf{m}, \{a_t\})$ is rationally ergodic and the following

$$\mathcal{A}_T = \frac{1}{\mathsf{m}([Q_r])^2}\int_{\Gamma\backslash G/M}\int_0^T \mathbb{1}_{[Q_r]}(xa_t)dtd\mathsf{m}(x)$$

is the asymptotic type of the flow in the sense of [1] and [3, 5].

6. Dichotomy for Anosov groups

Let $\Gamma < G$ be an Anosov subgroup defined as in the introduction. For each $\mathsf{v} \in \operatorname{int}\mathcal{L}_\Gamma$, there exists a unique $\psi_{\mathsf{v}} \in \mathfrak{a}^*$ such that $\psi_{\mathsf{v}} \geq \psi_\Gamma$ and $\psi_{\mathsf{v}}(\mathsf{v}) = \psi_\Gamma(\mathsf{v})$, and a unique $(\Gamma, \psi_{\mathsf{v}})$-conformal measure ν_{v} supported on Λ ([28], [11]). Moreover, $\{u \in \operatorname{int}\mathfrak{a}^+ : \psi_{\mathsf{v}}(\mathsf{u}) = \psi_\Gamma(\mathsf{u})\} = \mathbb{R}_+\mathsf{v}$ ([25], [29]). The assignments $\mathsf{v} \mapsto \psi_{\mathsf{v}}$ and $\mathsf{v} \mapsto \nu_{\mathsf{v}}$ give bijections among $\operatorname{int}\mathcal{L}_\Gamma$, $D_\Gamma^\star$ and the space of all Γ-conformal measures supported on Λ [19, Prop. 4.4 and Thm. 7.7].

For each $\mathsf{v} \in \operatorname{int}\mathcal{L}_\Gamma$, we denote by m_{v} the BMS measure on $\Gamma\backslash G/M$ associated to $(\nu_{\mathsf{v}}, \nu_{\mathsf{i}(\mathsf{v})})$. Chow and Sarkar proved the following theorem for $f_1, f_2 \in C_c(\Gamma\backslash G/M)$.

Theorem 6.1. [9] *Let $\Gamma < G$ be an Anosov subgroup and let $\mathsf{v} \in \operatorname{int}\mathcal{L}_\Gamma$. There exists $\kappa_{\mathsf{v}} > 0$ such that for any $f_1, f_2 \in C_c(\Gamma\backslash G/M)$,*

$$\lim_{t\to+\infty} t^{\frac{\operatorname{rank}(G)-1}{2}}\int_{\Gamma\backslash G/M} f_1(x)f_2(x\exp(t\mathsf{v}))\,d\mathsf{m}_{\mathsf{v}}(x) = \kappa_{\mathsf{v}}\cdot\mathsf{m}_{\mathsf{v}}(f_1)\mathsf{m}_{\mathsf{v}}(f_2).$$

Since m_{v} is A-invariant, the above is equivalent to:

$$\lim_{t\to+\infty} t^{\frac{\operatorname{rank}(G)-1}{2}}\int_{\Gamma\backslash G/M} f_1(x)f_2(x\exp(-t\mathsf{v}))\,d\mathsf{m}_{\mathsf{v}}(x) = \kappa_{\mathsf{v}}\cdot\mathsf{m}_{\mathsf{v}}(f_1)\mathsf{m}_{\mathsf{v}}(f_2). \tag{6.1}$$

In particular, for any $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$, the measure m_v is v-balanced.

Corollary 6.2. *For any $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$ and any bounded Borel subset $Q \subset G/M$ with $\tilde{\mathsf{m}}_\mathsf{v}(\operatorname{int} Q) > 0$, we have*

$$\int_0^\infty \sum_{\gamma \in \Gamma} \tilde{\mathsf{m}}_\mathsf{v}(Q \cap \gamma Q \exp(-t\mathsf{v}))\, dt = \infty \text{ if and only if } \operatorname{rank}(G) \le 3.$$

Proof. Choose $\tilde{f}_1, \tilde{f}_2 \in C_c(G/M)$ so that $0 \le \tilde{f}_1 \le \mathbb{1}_Q \le \tilde{f}_2$ and $\tilde{\mathsf{m}}_\mathsf{v}(\tilde{f}_1) > 0$. For each $i = 1, 2$, let $f_i \in C_c(\Gamma\backslash G/M)$ be defined by $f_i([g]) = \sum_{\gamma\in\Gamma} \tilde{f}_i(\gamma g)$. By (6.1), we get

$$\begin{aligned}
&\int_{\Gamma\backslash G/M} f_i([g]\exp(t\mathsf{v})) f_i([g]) d\mathsf{m}_\mathsf{v}[g] \\
&= \int_{G/M} \sum_{\gamma\in\Gamma} \tilde{f}_i(g\exp(t\mathsf{v}))\tilde{f}_i(g) d\tilde{\mathsf{m}}_\mathsf{v}(g) \asymp t^{(-\operatorname{rank}(G)+1)/2}.
\end{aligned}$$

The claim follows since $\int_1^\infty t^{(-\operatorname{rank}(G)+1)/2} dt = \infty$ if and only if $\operatorname{rank}(G) \le 3$. ∎

By Theorem 1.4, the following theorem implies Theorem 1.6:

Theorem 6.3. *Let $\mathsf{v} \in \operatorname{int} \mathcal{L}_\Gamma$ and $\mathsf{u} \in \operatorname{int} \mathfrak{a}^+$. The following are equivalent:*

1. $\operatorname{rank}(G) \le 3$ *and* $\mathbb{R}\mathsf{u} = \mathbb{R}\mathsf{v}$;
2. $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi_\mathsf{v}(\mu(\gamma))} = \infty$ *for some* $R > 0$.

Proof. Suppose that $\operatorname{rank}(G) \le 3$ and $\mathsf{u} = \mathsf{v}$. Let $a_t = \exp(t\mathsf{v})$. Let $Q_r \subset G/M$ be as in (5.3) of Proposition 5.8. Then for $\delta = \psi_\mathsf{v}(\mathsf{v}) > 0$, we have

$$\int_0^T \int_0^T \sum_{\gamma,\gamma'\in\Gamma} \tilde{\mathsf{m}}_\mathsf{v}(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s})\, dt\, ds \ll \left(\sum_{\substack{\gamma\in\Gamma_{\mathsf{v},4dr} \\ \psi(\mu(\gamma))\le\delta T}} e^{-\psi_\mathsf{v}(\mu(\gamma))} \right)^2. \tag{6.2}$$

Set $Q_r^- := \cap_{0\le s\le r/10} Q_r a_{-s}$. We may assume that $\mathsf{m}_\mathsf{v}(\operatorname{int} Q_r^-) > 0$ by increasing r. Note that

$$\begin{aligned}
&\frac{r}{10}\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}_\mathsf{v}(Q_r^- \cap \gamma Q_r^- a_{-t})\, dt \le \\
&\qquad \int_0^T \int_{0\le s\le r/10} \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}_\mathsf{v}(Q_r \cap \gamma(Q_r \cap Q_r a_{-s})a_{-t})\, ds\, dt.
\end{aligned}$$

By (6.2), we get

$$\int_0^T \sum_{\gamma\in\Gamma} \tilde{\mathsf{m}}_\mathsf{v}(Q_r^- \cap \gamma Q_r^- a_{-t})\, dt \ll \left(\sum_{\substack{\gamma\in\Gamma_{\mathsf{v},4dr}, \\ \psi_\mathsf{v}(\mu(\gamma))<\delta T}} e^{-\psi_\mathsf{v}(\mu(\gamma))} \right)^2.$$

Hence by Corollary 6.2, we get $\sum_{\gamma\in\Gamma_{\mathsf{v},R}} e^{-\psi_{\mathsf{v}}(\mu(\gamma))}=\infty$.

Now suppose that $\sum_{\gamma\in\Gamma_{\mathsf{u},R}} e^{-\psi_{\mathsf{v}}(\mu(\gamma))}=\infty$ for some $R>0$. By Lemma 5.2, $\psi_{\mathsf{v}}(\mathsf{u})=\psi_\Gamma(\mathsf{u})$. This implies $\mathbb{R}\mathsf{v}=\mathbb{R}\mathsf{u}$, as $\mathbb{R}\mathsf{v}$ is the unique line where ψ_{v} and ψ_Γ are equal to each other. This also implies $\mathsf{u}\in\operatorname{int}\mathcal{L}_\Gamma$. By Proposition 5.8, it follows that $\int_0^\infty\sum_{\gamma\in\Gamma}\tilde{\mathsf{m}}_{\mathsf{v}}(Q_r\cap\gamma Q_r a_{-t})\,dt=\infty$. Hence $\operatorname{rank}(G)\le 3$ by Corollary 6.2. $\square$

Remark 6.4. It follows from Theorem 6.3 that when $\operatorname{rank} G\le 3$ and $\mathsf{v}\in\operatorname{int}\mathcal{L}_\Gamma$, the flow $(\Gamma\backslash G/M,\mathsf{m}_{\mathsf{v}},\exp(t\mathsf{v}))$ is rationally ergodic by Remark 5.19.

References

[1] J. Aaronson. *Rational ergodicity and a metric invariant under Markov shifts.* Israel J. Math, 27 (1977), 93-123.

[2] P. Albuquerque. *Patterson-Sullivan theory in higher rank symmetric spaces.* Geometric and Functional Analysis. 9 (1999), 1-28.

[3] J. Aaronson and D. Sullivan. *Rational ergodicity of geodesic flows.* Erg. The. Dynam. Sys. 4 (1984), 165-178.

[4] Y. Benoist. *Proprietes asymptotiques des groupes lineaires.* Geom. Funct. Anal. (1997), 1-47.

[5] J. Bochi, R. Potrie and A. Sambarino. *Anosov representations and dominated splittings.* J. Eur. Math. Soc. 21 (2019), 3343-3414.

[6] M. Burger. *Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2.* Int. Math. Res. Not. 1993, no. 7, 217-225.

[7] M. Burger and S. Mozes. *CAT (-1) spaces, Divergence groups and their commensurators.* Journal of the AMS, 9 (1996), 57-92

[8] León Carvajales. Growth of Quadratic Forms Under Anosov Subgroups. *International Mathematics Research Notices*, 10 2021. rnab181.

[9] M. Chow and P. Sarkar. *Local mixing of one parameter diagonal flows on Anosov homogeneous spaces.* Preprint, arXiv:2105.11377, to appear in IMRN

[10] P. Eberlein. *Geometry of nonpositively curved manifolds.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.

[11] S. Edwards, M. Lee and H. Oh. *Anosov groups: local mixing, counting, and equidistribution* Preprint, arXiv:2003.14277, To appear in Geometry & Topology.

[12] Y. Guivarc'h and A. Raugi. *Actions of large semigroups and random walks on isometric extensions of boundaries.* Ann. Sci. École Norm. Sup. 40 (2007), 209-249.

[13] O. Guichard and A. Wienhard. *Anosov representations: Domains of discontinuity and applications.* Invent. Math. 190, Issue 2 (2012), 357-438.

[14] E. Hopf. *Ergodic theory and the geodesic flow on surfaces of constant negative curvature* BAMS., 77 (1971), 863-877.

[15] I. Kim. Length spectrum in rank one symmetric space is not arithmetic. *Proc. Amer. Math. Soc.*, 134(12):3691–3696, 2006.

[16] U. Krengel. *Ergodic theorems.* De Gruyter Studies in Mathematics, 6. Walter de Gruyter & Co., Berlin, 1985. viii+357 pp.

[17] F. Labourie. *Anosov flows, surface groups and curves in projective space.* Invent. Math. 165 (2006), no. 1, 51–114.

[18] O. Landesberg, M. Lee, E. Lindenstrauss and H. Oh. *Horospherical invariant measures and a rank dichotomy for Anosov groups.* J. Mod. Dyn., 19 (2023), 299–330..

[19] M. Lee and H. Oh. *Invariant measures for horospherical actions and Anosov groups.* Preprint, arXiv:2008.05296, To appear in IMRN.

[20] M. Lee and H. Oh. *Ergodic decompositions of geometric measures on Anosov homogeneous spaces.* Preprint, arXiv:2010.11337, To appear in Israel J. Math.

[21] R. Potrie and A. Sambarino.*Eigenvalues and entropy of a Hitchin representation.* Invent. Math. 209 (2017), no. 3, 885-925.
[22] P. Nicholls. *The ergodic theory of discrete groups.* London Math. Soc. Lecture Notes, vol. 143, Cambridge Univ. Press, Cambridge and New York, 1989.
[23] J.-F. Quint. *Divergence exponentielle des sous-groupes discrets en rang supérieur.* Comment. Math. Helv. 77 (2002), no. 3, 563-608.
[24] J.-F. Quint. *Mesures de Patterson-Sullivan en rang supérieur.* Geom. Funct. Anal. 12 (2002), no. 4, 776-809.
[25] J.-F. Quint. *L'indicateur de croissance des groupes de Schottky.* Ergodic Theory Dynamical Systems. 23 (2003), 249-272.
[26] T. Roblin. *Ergodicité et équidistribution en courbure négative. Mém. Soc. Math. Fr.* No. 95 (2003), vi+96.
[27] A. Sambarino. *The orbital counting problem for hyperconvex representations.* Ann. Inst. Fourier (Grenoble) 65 (2015), no. 4, 1755-1797.
[28] A. Sambarino. *Quantitative properties of convex representations.* Comment. Math. Helv. 89 (2014), no. 2, 443-488.
[29] A. Sambarino. *Hyperconvex representations and exponential growth.* Ergodic Theory Dynam. Systems 34 (2014), no. 3, 986 -1010.
[30] D. Sullivan. *The density at infinity of a discrete subgroup of hyperbolic motions.* Publ. IHES (1979), 171-202
[31] M. Tsuji. *Potential theory in modern function theory.* Maruzen Co. Ltd: Tokyo, 1959.
[32] D. Winter. *Mixing of frame flow for rank one locally symmetric spaces and measure classification.* Israel J. Math. 210 (2015), no. 1, 467-507.

Department of Mathematics, ETH, Zurich, Switzerland
Email address: burger@ethz.ch

Department of Mathematics, Yale University, New Haven, CT 06520
Email address: or.landesberg@yale.edu

Mathematics department, Yale university, New Haven, CT 06520, and Department of Mathematics, University of Chicago, Chicago, IL 60637 (current address)
Email address: minju1@uchicago.edu

Department of Mathematics, Yale University, New Haven, CT 06520
Email address: hee.oh@yale.edu