THE HOPF-TSUJI-SULLIVAN DICHOTOMY IN HIGHER RANK AND APPLICATIONS TO ANOSOV SUBGROUPS

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Abstract. We establish an extension of the Hopf-Tsuji-Sullivan dichotomy to any regular Zariski dense discrete subgroup of a semisimple real algebraic group $G$. We then apply this dichotomy to Anosov subgroups of $G$, which surprisingly presents a different phenomenon depending on the rank of the ambient group $G$.

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1. INTRODUCTION

Let $G$ be a connected simple real algebraic group of rank one, $(X,d)$ the associated Riemannian symmetric space and $\partial X$ the geometric boundary of $X$. We fix a base point $o \in X$, and $\pi : T^1(X) \to X$ denotes the canonical projection of a vector to its basepoint. The Hopf parametrization of the unit tangent bundle $T^1(X)$ maps a vector $v \in T^1(X)$ to

$$(v^+, v^-, \beta_{v^+}(o, \pi(v)))$$

where $v^+, v^- \in \partial X$ are respectively the forward and backward endpoints of the geodesic determined by $v$ and for $\xi \in \partial X$ and $x, y \in X$, $\beta_\xi(x, y)$ denotes the Busemann function given by

$$\beta_\xi(x, y) = \lim_{z \to \xi} d(y, z) - d(x, z).$$

This gives a homeomorphism

$$T^1(X) \simeq (\partial X \times \partial X - \Delta(\partial X)) \times \mathbb{R}$$

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where $\Delta(\partial X)$ denotes the diagonal embedding of $\partial X$ into $\partial X \times \partial X$ and the geodesic flow $\mathcal{G}^t$ on $T^1(X)$ corresponds to the translation flow on $\mathbb{R}$.

**The Hopf-Tsuji-Sullivan dichotomy in rank one.** Let $\Gamma < G$ be a non-elementary discrete subgroup. A Borel probability measure $\nu$ on $\partial X$ is called a $\Gamma$-conformal measure of dimension $\delta \geq 0$ if for any $\gamma \in \Gamma$ and $\xi \in \partial X$,

$$d\gamma_*\nu(\xi) = e^{-\delta d(o,\gamma(o))}$$

where $\gamma_*\nu(Q) = \nu(\gamma^{-1}Q)$ for any Borel subset $Q \subset \partial X$.

Each $\Gamma$-conformal measure $\nu$ on $\partial X$ determines a unique geodesic flow invariant Borel measure $m_\nu$ on $T^1(\Gamma\backslash X)$, which is locally equivalent to $\nu \otimes \nu \otimes ds$ in the Hopf coordinates, where $ds$ denotes the Lebesgue measure on $\mathbb{R}$. The following criterion known as the Hopf-Tsuji-Sullivan dichotomy relates dynamical properties of the geodesic flow $\mathcal{G}^t$ with respect to the measure $m_\nu$, the $\nu$-size of the conical\(^1\) limit points of $\Gamma$ and the divergence property of the Poincare series $P(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma(o))}$ at the dimension of $\nu$: we denote by $\Lambda_{\text{con}} \subset \partial X$ the set of all conical limit points of $\Gamma$.

**Theorem 1.1.** Let $G$ be a connected simple real algebraic group of rank one and $\Gamma < G$ a non-elementary discrete subgroup. Let $\nu$ be a $\Gamma$-conformal measure on $\partial X$ of dimension $\delta$. The following are equivalent:

1. $\nu(\Lambda_{\text{con}}) > 0$ (resp. $\nu(\Lambda_{\text{con}}) = 0$);
2. $\nu(\Lambda_{\text{con}}) = 1$ (resp. $\nu(\Lambda_{\text{con}}) = 0$);
3. the geodesic flow $\mathcal{G}^t$ is conservative (resp. completely dissipative) with respect to $m_\nu$;
4. the geodesic flow $\mathcal{G}^t$ is ergodic (resp. non-ergodic) with respect to $m_\nu$;
5. $\sum_{\gamma \in \Gamma} e^{-\delta d(o,\gamma(o))} = \infty$ (resp. $\sum_{\gamma \in \Gamma} e^{-\delta d(o,\gamma(o))} < \infty$) where $\delta$ is the conformal dimension of $\nu$ and $o \in X$ is any point.

Most equivalences are due to Sullivan for real hyperbolic spaces [29] (see also [30], [2]) and to Burger-Mozes for proper CAT (-1) spaces [6, Sec. 6.3] and its complete form can be found in Nicholl’s book [21, Ch. 8] when $X$ is a real hyperbolic space and in Roblin’s thesis [25, Thm. 1.7] for a proper CAT (-1) spaces.

We denote by $\Lambda \subset \partial X$ the limit set of $\Gamma$, which is the unique $\Gamma$-minimal subset of $\partial X$ and by $\delta_\Gamma$ the critical exponent of $\Gamma$, that is, the abscissa of the convergence of the Poincare series $P(s)$ of $\Gamma$. The group $\Gamma$ is called a divergent type if $P(\delta_\Gamma) = \infty$. Patterson and Sullivan constructed a $\Gamma$-conformal measure, say, $\nu_{\text{PS}}$, supported on the limit set $\Lambda$ of dimension $\delta_\Gamma$, called the Patterson-Sullivan measure. Theorem 1.1 implies that whether $\Gamma$ is of divergent type or not is completely determined by the positivity of $\nu_{\text{PS}}(\Lambda_{\text{con}})$, and vice versa.

\(^1\)A point $\xi \in \partial X$ is called a conical limit point of $\Gamma$ if a geodesic ray toward $\xi$ accumulates in $\Gamma\backslash X$. 

The case of convex cocompact groups. A discrete group with $\Lambda = \Lambda_{\text{con}}$ is called a convex cocompact subgroup. They are also characterized by the property that $\Gamma$ acts cocompactly on the convex hull of $\Lambda$ in $X$. For a convex cocompact subgroup $\Gamma$, there exists a unique $\Gamma$-conformal measure supported on the limit set $\Lambda$, namely the Patterson-Sullivan measure $\nu_{\text{PS}}$. The associated geodesic flow invariant measure on $T^1(\Gamma \backslash X)$, called the Bowen-Margulis-Sullivan measure, is known to be the measure of maximal entropy [29]. An immediate consequence of Theorem 1.1 for convex cocompact groups is as follows:

**Theorem 1.2.** Let $\Gamma < G$ be a convex cocompact subgroup. Then

1. the geodesic flow $G^t$ on $T^1(\Gamma \backslash X)$ is conservative and ergodic with respect to the Bowen-Margulis-Sullivan measure $m_{\text{BMS}}$;
2. $\Gamma$ is of divergent type, i.e., $\sum_{\gamma \in \Gamma} e^{-\delta \gamma d(o,\gamma o)} = \infty$.

The unit tangent bundle of $\Gamma \backslash X$ is a double quotient space $\Gamma \backslash G/M$ where $M$ is a compact subgroup of $G$ commuting with the one-parameter diagonal subgroup $\{a_t\}$ which induces the geodesic flow. When $\Gamma$ is Zariski dense in addition, the lifted Bowen-Margulis-Sullivan measure, considered as an $M$-invariant measure on $\Gamma \backslash G$, is also ergodic for the diagonal flow $\{a_t\}$ whenever $M$ is connected [31, Thm. 1.1]. The only case of $M$ disconnected is when $G \cong \text{SL}_2(\mathbb{R})$ and $M = \{\pm e\}$, in which case $m_{\text{BMS}}$ has at most two ergodic components [19].

The Hopf-Tsuji-Sullivan dichotomy in higher rank. The main aim of this article is to extend the Hopf-Tsuji-Sullivan dichotomy for discrete subgroups of higher rank semisimple real algebraic groups $G$, while replacing the geodesic flow of the rank one space with any one-parameter subgroup of diagonal elements of $G$ (Theorem 1.5). Each one-parameter subgroup of diagonal elements corresponds to a direction, say, $u$, in the positive Weyl chamber of $G$. We introduce the $u$-directional conical limit set and $u$-directional Poincare series, whose properties relative to a given $\Gamma$-conformal density is shown to determine ergodic properties of the action of the one-parameter subgroup $\{\exp(tu) : t \in \mathbb{R}\}$ with respect to an associated measure on $\Gamma \backslash G$.

We then apply the dichotomy together with recent local mixing results of Chow and Sarkar [8] to Anosov subgroups $\Gamma$. We discover a surprising phenomenon that the rank of the ambient group $G$ dictates a completely opposite behavior for $\Gamma$ as stated in Theorem 1.7. We also deduce recurrent properties of the Burger-Roblin measures for each interior direction of the limit cone of $\Gamma$ (Corollary 1.8), which plays an important role in the recent measure classification result of Landesberg, Lee, Lindenstrauss and Oh [17, Thm. 1.1].

In order to state these results precisely, we now let $G$ be a connected, semisimple real algebraic group. Let $P$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition $P = MAN$. Here $A$ is a maximal real split torus of $G$, $M$ is a compact subgroup commuting with $A$ and
N is a maximal horospherical subgroup. We fix a positive Weyl chamber $a^+ \subset a = \text{Lie}(A)$ so that $\log N$ consists of positive root subspaces. We fix a maximal compact subgroup $K < G$ so that the Cartan decomposition $G = K(\exp a^+)K$ holds, and denote by $\mu : G \to a^+$ the Cartan projection, i.e., for $g \in G$, $\mu(g) \in a^+$ is the unique element such that $g \in K \exp \mu(g)K$.

Let $\Gamma < G$ be a Zariski dense discrete subgroup of $G$. We denote by $L_\Gamma \subset a^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$. Benoist showed that $L_\Gamma$ is a convex cone with non-empty interior [3]. Let $\mathcal{F}$ denote the Furstenberg boundary $G/P$ and $\Lambda \subset \mathcal{F}$ the limit set of $\Gamma$, which is the unique $\Gamma$-minimal subset. For a linear form $\psi \in a^*$, a Borel probability measure $\nu_\psi$ on $\mathcal{F}$ is called a $(\Gamma, \psi)$-conformal measure if for any $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$ \frac{d\gamma_*\nu_\psi}{d\nu_\psi}(\xi) = e^{\psi(\beta_\gamma(e, \gamma))} \quad (1.1) $$

where $\beta$ denotes the $a$-valued Busemann function (see Def. 2.1). Quint showed in [23, Thm. 8.1] that a $(\Gamma, \psi)$-conformal measure may exist only when $\psi \geq \psi_\Gamma$ where $\psi_\Gamma : a \to \mathbb{R}$ denotes the growth indicator function of $\Gamma$ (Def. 5.1). Moreover, he constructed a $(\Gamma, \psi)$-conformal measure supported on $\Lambda$ for every linear form $\psi \geq \psi_\Gamma$ satisfying $\psi(v) = \psi_\Gamma(v)$ for some $v \in L_\Gamma \cap \text{int } a^+$ [23, Thm. 8.4].

Let $i : a^+ \to a^+$ denote the opposition involution given by $i(v) = -\text{Ad}_{w_0}(v)$ where $w_0$ is the longest Weyl element. In rank one groups, $i$ is the identity map. Letting $\mathcal{F}^{(2)}$ denote the unique open diagonal $G$-orbit in $\mathcal{F} \times \mathcal{F}$, the quotient space $G/M$ is homeomorphic to $\mathcal{F}^{(2)} \times a$ via the Hopf parametrization which maps $gM$ to $(gP, gw_0P, \beta_gP(e, g))$ for any $g \in G$.

For a given pair of $\Gamma$-conformal measures $\nu_\psi$ and $\nu_{\psi_0i}$ on $\mathcal{F}$ with respect to $\psi$ and $\psi \circ i$ respectively, one can use the Hopf parameterization to define a non-zero $A$-invariant Borel measure $m(\nu_\psi, \nu_{\psi_0i})$ on the quotient space $\Gamma \backslash G/M$, which is locally equivalent to $d\nu_\psi \otimes d\nu_{\psi_0i} \otimes db$ in the Hopf coordinates, where $db$ denotes the Lebesgue measure on $a$; we will call it the Bowen-Margulis-Sullivan measure (or simply BMS-measure) associated to the pair $(\nu_\psi, \nu_{\psi_0i})$ (Section 4). For simplicity, we write $m_\psi$ for $m(\nu_\psi, \nu_{\psi_0i})$, although the measure depends on the choice of conformal measures $\nu_\psi$ and $\nu_{\psi_0i}$, not only on $\psi$.

For $u \in \text{int } a^+$, we will say that $m_\psi$ is $u$-balanced if

$$ \lim \sup_{T \to \infty} \int_0^T \frac{m_\psi(O_1 \cap O_1 \exp(tu))}{m_\psi(O_2 \cap O_2 \exp(tu))} dt < \infty \quad (1.2) $$

for any bounded Borel subsets $O_i \subset \Gamma \backslash G/M$ with $m_\psi(\text{int } O_i) > 0$.

Each BMS measure $m_\psi$ on $\Gamma \backslash G/M$ can be considered as an $AM$-invariant measure on $\Gamma \backslash G$, which we will also denote by $m_\psi$, by abuse of notation. While the set $\mathcal{E} = \{ [g] \in \Gamma \backslash G : gP \in \Lambda \}$ is the unique $P$-minimal subset of $\Gamma \backslash G$, it breaks into finitely many $P^o$-minimal subsets in general where $P^o$
denotes the identity component of $P$. For each $P$-minimal subset $Y \subset \Gamma \setminus G$, the restriction $m_\psi|_Y$ gives an $A$-invariant measure.

The conical limit set $\Lambda_{\text{con}}$ of $\Gamma$ is given by

$$\Lambda_{\text{con}} := \{gP \in F : \limsup_{gA^+} \neq \emptyset\}$$

where $A^+ = \exp a^+$ and $\limsup$ denotes the topological limit superior, i.e. all accumulation points of the given family of sets.

**Definition 1.3** (Directional conical limit set). For each $u \in \text{int } a^+$, we define the set of $u$-directional conical limit points as follows:

$$\Lambda_u := \{gP \in F : \limsup_{t \to +\infty} \Gamma g \exp(tu) \neq \emptyset\};$$

this is a dense Borel measurable subset of $\Lambda_{\text{con}}$.

It is easy to see that $\Lambda_u \neq \emptyset$ only when $u \in L$.

For $R > 0$ and $u \in \text{int } a^+$, we define the following tube-like subset of $\Gamma$ whose Cartan projection lies within distance $R$ from the ray $\mathbb{R}_+u$:

$$\Gamma_{u,R} := \{\gamma \in \Gamma : \|\mu(\gamma) - tu\| < R \text{ for some } t \geq 0\},$$

where $\| \cdot \|$ is an Euclidean norm on $a$.

**Definition 1.4.** We say that $\Gamma$ is regular if for any simple root $\alpha$ of $a$, $\alpha(\mu(\gamma)) \to \infty$ whenever $\gamma \to \infty$ in $\Gamma$.

Note that if $L - \{0\} \subset \text{int } a^+$, then $\Gamma$ is regular. In particular, in rank one Lie groups, all subgroups are regular. In the higher rank case, the regularity assumption on $\Gamma$ is restrictive (for examples, lattices are never regular by the Poincare recurrence theorem) but important for us in several aspects. For regular subgroups, each $\Gamma$-orbit in the symmetric space $X = G/K$ accumulates on the Furstenberg boundary $\mathcal{F}$ in the sense of Definition 2.7 and we have the shadow lemma ([18, Lem. 7.8], see Lemma 3.4), which is a basic tool in the proof of the following theorem, which extends Theorem 1.1 to all higher rank groups:

**Theorem 1.5** (The Hopf-Tsuji-Sullivan dichotomy in higher rank). Let $G$ be a connected semisimple real algebraic group and $\Gamma < G$ be a regular Zariski dense discrete subgroup. Fix $\psi \in a^*$ and let $\nu_\psi, \nu_{\psi\circ i}$ be a pair of $(\Gamma, \psi)$ and $(\Gamma, \psi \circ i)$-conformal measures respectively, and let $m_\psi = m(\nu_\psi, \nu_{\psi\circ i})$ denote the associated BMS measure on $\Gamma \setminus G/M$. For any $u \in \text{int } a^+$, the following conditions (1)-(5) are equivalent and imply (6). If $\psi(u) > 0$ and $m_\psi$ is $u$-balanced, then (6) implies (7). Moreover, the first cases of (1)-(7) can occur only when $\psi(u) = \psi_T(u)$.

1. $\max(\nu_\psi(\Lambda_u), \nu_{\psi\circ i}(\Lambda_{i(u)}) > 0$ (resp. $\nu_\psi(\Lambda_u) = 0 = \nu_{\psi\circ i}(\Lambda_{i(u)})$);  
2. $\max(\nu_\psi(\Lambda_u), \nu_{\psi\circ i}(\Lambda_{i(u)}) = 1$ (resp. $\nu_\psi(\Lambda_u) = 0 = \nu_{\psi\circ i}(\Lambda_{i(u)})$);  
3. $(\Gamma \setminus G/M, \{\exp(tu)\}, m_\psi)$ is conservative (resp. totally dissipative);  
4. $(\Gamma \setminus G/M, \{\exp(tu)\}, m_\psi)$ is ergodic (resp. non-ergodic).
(5) For some (and hence for all) \( P^\circ \)-minimal subset \( Y \subset \Gamma \setminus G \), the system \((Y, \{\exp(tu)\}, m_\psi \mid Y)\) is ergodic and conservative (resp. non-ergodic and totally dissipative);

(6) \( \sum_{\gamma \in \Gamma \setminus \mathcal{R}} e^{-\psi(\mu(\gamma))} = \infty \) for some \( R > 0 \) (resp. \( \sum_{\gamma \in \Gamma \setminus \mathcal{R}} e^{-\psi(\mu(\gamma))} < \infty \) for all \( R > 0 \));

(7) \( \nu_\psi(\Lambda_u) = 1 = \nu_\psi \circ i(\Lambda_{i(u)}) \) (resp. \( \nu_\psi(\Lambda_u) = 0 = \nu_\psi \circ i(\Lambda_{i(u)}) \)).

Remark 1.6. (1) When \( G \) has rank one, \( \psi \circ i = \psi \) for any \( \psi \in a^* \), as the opposition involution \( i \) is trivial. Moreover, the \( m_\psi \) being \( u \)-balanced condition is not needed for the implication \( (6) \Rightarrow (7) \). For \( \Gamma \) non-elementary, (1)-(7) are all equivalent to each other, except for (5), and for \( \Gamma \) Zariski dense, these conditions imply (5).

(2) When the rank of \( G \) is at least 2, we need \( \Gamma \) to be Zariski dense for the equivalence of (3) and (4). The reason is that, when \( \Gamma \) is not Zariski dense, the Jordan projection of \( \Gamma \) may not generate a dense subgroup of \( A \) while in the rank one case, the Jordan projection of any non-elementary subgroup generates a dense subgroup of \( A \) [14].

(3) We mention that the equivalences among (3), (4), and (5) do not require the regularity assumption on \( \Gamma \).

(4) We emphasize here that although the implication \( (3) \Rightarrow (1) \) is a direct consequence of the definition of \( m_\psi \), the proof for \( (3) \Rightarrow (7) \) under the further \( u \)-balanced condition of \( m_\psi \) requires the discussion of the directional Poincare series.

For discrete subgroups of a product of two rank one Lie groups whose projection to each factor is convex cocompact, Burger announced that \( \nu_\psi(\Lambda_u) = 1 \) for all \( \psi \in a^* \) and \( u \in \text{int} \mathcal{L}_\Gamma \) such that \( \psi(u) = \psi_{\Gamma}(u) \) [5, Thm. 3]. Indeed, we show that this is a special case of a more general phenomenon which holds for all Anosov subgroups whose ambient group has rank at most 3.

The case of Anosov subgroups. Although there are notions of Anosov subgroups with respect to a general parabolic subgroup [12], we will restrict our attention only to those Anosov subgroups with respect to a minimal parabolic subgroup. Recall that a Zariski dense discrete subgroup \( \Gamma < G \) is an Anosov subgroup (with respect to a minimal parabolic subgroup \( P \)) if it is a finitely generated word hyperbolic group which admits a \( \Gamma \)-equivariant embedding \( \zeta \) of the Gromov boundary \( \partial \Gamma \) into \( \mathcal{F} \) such that \((\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)} \) for all \( x \neq y \) in \( \partial \Gamma \) [12, Prop. 2.7 and Thm. 1.5]. We note that Zariski dense images of representations of a surface subgroup in the Hitchin component [16] as well as Schottky subgroups provide ample examples of Anosov subgroups ([24, Prop. 3.3], see also [10, Lem. 7.2]). Let \( \Gamma \) be an Anosov subgroup for the rest of the introduction. First of all, \( \Gamma \) is regular, as \( \mathcal{L}_\Gamma - \{0\} \) is contained in \( \text{int} a^+ \) [20, Prop. 4.6]. Set

\[ D_\Gamma^* := \{ \psi \in a^*: \psi \geq \psi_{\Gamma}, \psi(v) = \psi_{\Gamma}(v) \text{ for some } v \in \mathcal{L}_\Gamma \cap \text{int} a^+ \}. \]
For each $\psi \in D^*_\Gamma$, there exists a unique unit vector $v \in L_\Gamma \cap \text{int } a^+$ such that $\psi(v) = \psi_T(v)$ and $v$ necessarily belongs to $\text{int } L_\Gamma$ ([20, Prop. 4.11] and [24, Lem. 4.3(i)], see also [28, Lem. 4.3] and [7, Thm. A.2(3)]).

Moreover, for each $\psi \in D^*_\Gamma$, there exists a unique $(\Gamma, \psi)$-conformal probability measure, say $\nu_\psi$, supported on $\Lambda$ and the map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D^*_\Gamma$ and the space $S_\Gamma$ of all $\Gamma$-conformal probability measures supported on $\Lambda$; hence $S_\Gamma$ is homeomorphic to the set of unit vectors of $\text{int } L_\Gamma$ (see [18, Thm. 1.3] and references therein). It was also shown in ([18], [19]) that for any $\psi \in D^*_\Gamma$ and $m_\psi = m(\nu_\psi, \nu_{\psi, 01})$,

- $\Lambda = \Lambda_{\text{con}}$;
- for any $P^\infty$-minimal subset $Y \subset \Gamma \setminus G$, $m_\psi|_Y$ is $A$-ergodic;
- $\sum_{G} e^{-\psi(\mu(\gamma))} = \infty$.

On the other hand, the divergence of the directional Poincaré series (i.e., $\sum_{G} e^{-\psi(\mu(\gamma))}$) for some $R > 0$) turns out to depend on the rank of $G$:

**Theorem 1.7.** Let $\Gamma < G$ be an Anosov subgroup. For any $\psi \in D^*_\Gamma$ and $u \in \text{int } a^+$, the following conditions are equivalent and the first cases of (1)-(4) can occur only when $u \in \text{int } L_\Gamma$:

1. $\text{rank } G \leq 3$ and $\psi(u) = \psi_T(u)$ (resp. $\text{rank } G > 3$ or $\psi(u) \neq \psi_T(u)$);
2. $\nu_\psi(\Lambda_u) = 1 = \nu_{\psi, 01}(\Lambda_i(u))$ (resp. $\nu_\psi(\Lambda_u) = 0 = \nu_{\psi, 01}(\Lambda_i(u))$);
3. For some (and hence for all) $P^\infty$-minimal subset $Y \subset \Gamma \setminus G$, the system $(\Gamma, \{\exp(tu)\}, m_\psi|_Y)$ is ergodic and conservative (resp. non-ergodic and totally dissipative);
4. $\sum_{G} e^{-\psi(\mu(\gamma))} = \infty$ for some $R > 0$ (resp. $\sum_{G} e^{-\psi(\mu(\gamma))} < \infty$ for all $R > 0$).

For $\psi \in D^*_\Gamma$ and $u \in \text{int } L_\Gamma$ with $\psi(u) = \psi_T(u)$, Chow and Sarkar proved in [8] the following local mixing result that for any $f_1, f_2 \in C_c(\Gamma \setminus G)$,

$$\lim_{t \to +\infty} (\text{rank } G - 1)^{1/2} \int_{\Gamma \setminus G} f_1(x \exp tu)f_2(x)dm_\psi(x) = \kappa_u m_\psi(f_1)m_\psi(f_2)$$

for some constant $\kappa_u > 0$ depending only on $u$ (see [26] where this is proved for $M$-invariant functions for some special cases).

Using the shadow lemma (Lemma 3.4), we deduce from this local mixing result (1.4) that the $u$-directional Poincaré series $\sum_{G} e^{-\psi(\mu(\gamma))}$ diverges if and only if $\text{rank } G \leq 3$. Together with Theorem 1.5, this implies Theorem 1.7.

Let $m^{\text{BR}}_\psi$ denote the Burger-Roblin measure associated to $\nu_\psi$, that is, the $MN$-invariant measure on $\Gamma \setminus G$ which is induced from a measure on $G/M$ locally equivalent to $d\nu_\psi \otimes dm_o \otimes db$ where $m_o$ is the $K$-invariant probability measure on $\mathcal{F}$ (cf. [10, Sec. 3]). Lee and Oh proved that each $m^{\text{BR}}_\psi$ is $MN$-ergodic and its restrictions to $P^\infty$-minimal subsets of $\Gamma \setminus G$ yield all $N$-ergodic components ([18, Thm. 10.1], [19, Thm. 1.3]). For $u \in \text{int } a^+$, we consider
the following directional recurrent set
\[ \mathcal{R}_u := \{ x \in \Gamma \backslash G : \limsup_{t \to +\infty} x \exp(tu) \neq \emptyset \}. \]

Since \( u \in \text{int} a^+ \), this is a \( P \)-invariant dense Borel subset of \( \mathcal{E} \).

An immediate consequence of Theorem 1.7 is the following:

**Corollary 1.8.** For any \( \psi \in D^\star \Gamma \) and \( u \in \text{int} a^+ \), we have

1. If \( \text{rank } G \leq 3 \) and \( u \in \text{int} L^\Gamma \) with \( \psi(u) = \psi_T(u) \), then
   \[ m^\text{BR}_\psi(\Gamma \backslash G - \mathcal{R}_u) = 0. \]
2. In all other cases, \( m^\text{BR}_\psi(\mathcal{R}_u) = 0. \)

This corollary is one of the main ingredients of the recent measure classification result [17, Thm. 1.1].

*Added:* After we sent this paper to Andres Sambarino, he sent us a note giving a rough outline of a proof of the statement that for Anosov groups and for \( \psi(u) = \psi_T(u) \), the ergodicity of \( (\Gamma \backslash G/M, \exp(tu), m_\psi) \) holds if and only if \( \text{rank } G \leq 3 \), with a different approach based on the work of Guivarch.

**Organization.** In section 2, we collect basic definitions. In section 3, we show that the set of directional conical limit points is either null or conull for any \( (\Gamma, \psi) \)-conformal measure. In section 4, we prove that the conservativity of the Bowen-Margulis-Sullivan measure for one parameter diagonal flow implies its ergodicity, extending Hopf’s argument. In section 5, we relate the directional Poincare series with respect to \( \psi \) and the correlation functions of the BMS measures and provide the proof of Theorem 1.5. In section 6, we specialize to Anosov groups and prove Theorem 1.7.

## 2. Preliminaries

Let \( G \) be a connected, semisimple real algebraic group. We decompose \( \mathfrak{g} = \text{Lie } G \) as \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \), where \( \mathfrak{k} \) and \( \mathfrak{p} \) are the +1 and −1 eigenspaces of a Cartan involution \( \theta \) of \( \mathfrak{g} \), respectively. We denote by \( K \) the maximal compact subgroup of \( G \) with Lie algebra \( \mathfrak{k} \), and by \( X = G/K \) the associated symmetric space. Choose a maximal abelian subalgebra \( \mathfrak{a} \) of \( \mathfrak{p} \) and a closed positive Weyl chamber \( \mathfrak{a}^+ \) of \( \mathfrak{a} \). Set \( A := \exp \mathfrak{a} \) and \( A^+ = \exp \mathfrak{a}^+ \). The centralizer of \( A \) in \( K \) is denoted by \( M \). Consider the following pair of opposite maximal horospherical subgroups:

\[ N = N^- := \{ g \in G : a^{-n}ga^n \to e \text{ as } n \to +\infty \} \text{ and } \]
\[ N^+ := \{ g \in G : a^n ga^{-n} \to e \text{ as } n \to +\infty \} \]

for any \( a \in \text{int } A^+ \); this definition is independent of the choice of \( a \in \text{int } A^+ \).

We set
\[ P = MAN, \quad \text{and } P^+ = MAN^+; \]
they are minimal parabolic subgroups of \( G \) and \( P \cap P^+ = MA \). The quotient space \( \mathcal{F} = G/P \) is called the Furstenberg boundary of \( G \), and via the Iwasawa decomposition \( G = KP, \mathcal{F} \) is isomorphic to \( K/M \).
Let $N_K(a)$ be the normalizer of $a$ in $K$, and $W := N_K(a)/M$ denote the Weyl group. Fixing a left $G$-invariant and right $K$-invariant Riemannian metric $d$ on $G$ induces a Riemannian metric on the associated symmetric space $X = G/K$, which we also denote by $d$ by abuse of notation. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the associated $W$-invariant inner product and norm on $a$.

For $R > 0$, set $A_R = \{ a \in A : \| \log a \| \leq R \}$, $A_R^+ = A_R \cap A^+$, and

$$G_R := KA_R^+K.$$

**$\mathfrak{a}$-valued Busemann functions.** The product map $K \times A \times N \to G$ is a diffeomorphism, yielding the well-known Iwasawa decomposition $G = KAN$. The Iwasawa cocycle $\sigma : G \times F \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times F$ with $\xi = [k]$ for $k \in K$, $\exp \sigma(g, \xi)$ is the $A$-component of $gk$ in the $KAN$ decomposition, that is,

$$gk \in K \exp(\sigma(g, \xi))N.$$

**Definition 2.1.** The $\mathfrak{a}$-valued Busemann function $\beta : F \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in F$ and $g, h \in G$,

$$\beta_{\xi}(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi).$$

Denote by $w_0 \in W$ the unique element of $W$ such that $\text{Ad}_{w_0} a^+ = -a^+$.

**Definition 2.2 (Visual maps).** For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$

Note that for $g \in G$, $g^\pm = g(e^\pm)$.

The opposition involution $i : \mathfrak{a} \to \mathfrak{a}$ is defined by

$$i(v) = -\text{Ad}_{w_0}(v). \quad (2.1)$$

When $G$ is a product of rank one groups, $i$ is trivial.

The set $\mathcal{F}^{(2)} = \{(g^+, g^-) \in F \times F : g \in G\} = G(e^+, e^-)$ is the unique open $G$-orbit. The $\mathfrak{a}$-valued Gromov product on $\mathcal{F}^{(2)}$ is defined as follows: for $(g^+, g^-) \in \mathcal{F}^{(2)}$,

$$\mathcal{G}(g^+, g^-) := \beta_{g^+}(e, g) + i(\beta_{g^-}(e, g)).$$

**Lemma 2.3.** [4, Prop. 8.12] There exist $c, c' > 0$ such that for all $g \in G$,

$$c^{-1}\|\mathcal{G}(g^+, g^-)\| \leq d(o, gAo) \leq c\|\mathcal{G}(g^+, g^-)\| + c'.$$

**Definition 2.4 (Cartan projection).** For $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of $g$, such that

$$g \in K \exp(\mu(g))K.$$

When $\mu(g) \in \text{int } \mathfrak{a}^+$ and $g = k_1 \exp(\mu(g))k_2$, we write $\kappa_1(g) := [k_1] \in K/M$ and $\kappa_2(g) := k_2 \in M \setminus K$, which are well-defined.

In the whole paper, we fix the constant $d = d(G) \geq 2$ as in the following lemma.
Lemma 2.5. [18, Lem. 5.8] There exists $d \geq 2$ such that for any $R > 1$ and any $g \in G$,
$$\mu(G_R g G_R) \subset \mu(g) + a_d R.$$ 

Definition 2.6. We say that a sequence $g_i \to \infty$ regularly in $G$ if $\alpha(\mu(g_i)) \to \infty$ as $i \to \infty$ for every simple root $\alpha$ corresponding to $a^+$. 

Definition 2.7. (1) A sequence $g_i \in G$ is said to converge to $\xi \in \mathcal{F}$ regularly in $G$ if $\alpha(\mu(g_i)) \to \infty$ as $i \to \infty$ for every simple root $\alpha$ corresponding to $a^+$. 

(2) A sequence $g_i = g_i(o) \in X$ is said to converge to $\xi \in \mathcal{F}$ if $g_i$ does. 

Definition 2.8 (Limit set). For a Zariski dense discrete subgroup $\Gamma < G$, we define the limit set $\Lambda$ of $\Gamma$ as follows: fixing $p \in X$,
$$\Lambda := \{ \lim_{i \to \infty} \gamma_i p \in \mathcal{F} : \gamma_i \in \Gamma \}.$$ 
By [18, Lem. 2.13], this definition is independent of the choice of $p \in X$ and coincides with one given by Benoist [3, Def. 3.6]; in particular, it is the unique $\Gamma$-minimal subset of $\mathcal{F}$. 

We later use the fact that $\Lambda$ is a Zariski dense subset of $\mathcal{F}$ [3, Lem. 3.6]. 

Definition 2.9. We say $\Gamma$ is regular if any sequence $\gamma_i \to \infty$ in $\Gamma$ tends to $\infty$ regularly. 

We note that for $\Gamma$ regular, any infinite sequence of $\Gamma$ has an accumulation point in $\Lambda$. 

For any real-valued functions $f(t)$ and $g(t)$, we write $f(t) \ll g(t)$ if there exists $C > 0$ such that $f(t) \leq C g(t)$ for all $t > 1$. We write $f(t) \asymp g(t)$ if $f(t) \ll g(t)$ and $g(t) \ll f(t)$. 

3. A zero-one law for $\nu_\psi(\Lambda_u)$ 

Let $\Gamma < G$ be a Zariski dense regular discrete subgroup of $G$. Fix $\psi \in a^*$, and a $(\Gamma, \psi)$-conformal measure $\nu_\psi$ on $\mathcal{F}$ as defined in (1.1).

Recalling the notation $\Lambda_u$ from Definition 1.3, the goal of this section is to prove the following dichotomy: 

Proposition 3.1. For any $u \in \text{int } a^+$, we have 
$$\nu_\psi(\Lambda_u) = 1 \quad \text{or} \quad \nu_\psi(\Lambda_u) = 0.$$ 

The proof of this proposition is based on the study of shadows. 

Shadows. For $p, q \in X = G/K$ and $r > 0$, the shadow of the $r$-ball around $q$ as seen from $p$ is defined by 
$$O_r(p, q) := \{ g^+ \in \mathcal{F} : go = p, g A^+ o \cap B(q, r) \neq \emptyset \},$$ 
where $B(q, r) = \{ x \in X : d(x, q) < r \}$.

Similarly, for $\xi \in \mathcal{F}$, we define the shadow of the $r$-ball around $q$ as seen from $\xi$ to be 
$$O_r(\xi, q) := \{ g^+ \in \mathcal{F} : g^- = \xi, \ go \in B(q, r) \}.$$
Note the following $G$-equivariance property: for any $g \in G$ and $r > 0$,
\[ gO_r(p,q) = O_r(gp,gq) \quad \text{and} \quad gO_r(\xi,q) = O_r(g\xi,gq). \] (3.1)

Note that for any $\xi \in \mathcal{F}$, $q \in X$ and $R > 0$,
\[ \bigcup_{r > R} O_r(\xi,q) = \{ \eta \in \mathcal{F} : (\xi, \eta) \in \mathcal{F}^{(2)} \}. \] (3.2)

**Lemma 3.2.** [18, Lem. 5.7] There exists $\kappa > 0$ such that for any $r > 0$ and $g \in G$, we have
\[ \sup_{\xi \in O_r(o,go)} \| \beta_\xi(e,g) - \mu(g) \| \leq \kappa r. \]

The following lemma is an immediate consequence of [18, Lem. 5.6]:

**Lemma 3.3.** For any $S > 0$ and a sequence $g_i \to \infty$ regularly in $G$, we have, for all sufficiently large $i$, the closure of $O_S(o,g_i o) \times O_S(g_i o,o)$ is contained in $\mathcal{F}^{(2)}$.

The regularity assumption on $\Gamma$ is crucial for the following shadow lemma:

**Lemma 3.4** (Shadow lemma). [18, Lem. 7.8] There exist $c_1, \kappa > 0$ such that for all sufficiently large $R > 1$, and for all $\gamma \in \Gamma$,
\[ c_1 e^{-\kappa \| \psi \| R} e^{-\psi(\mu(\gamma))} \leq \nu_\psi(O_R(o,\gamma o)) \leq c_1 e^{\kappa \| \psi \| R} e^{-\psi(\mu(\gamma))}. \]
Moreover,
\[ \inf_{\gamma \in \Gamma} \nu_\psi(O_R(\gamma o,o)) > 0. \]

For any $R > 0$, set
\[ G_{u,R} := \{ g \in G : \| \mu(g) - tu \| < R \ \text{for some} \ t \geq 0 \}. \] (3.3)

**Lemma 3.5.** Let $R, S > 0$. If $g \in G_{u,R}$, then
\[ O_S(o,go) \subset \{ k^+ \in \mathcal{F} : k \exp(tu)o \in B(go,R + 2dS) \ \text{for some} \ t > 0 \}. \]

**Proof.** For $\xi \in O_S(o,go)$, there exist $k \in K$ and $a \in A^+$ such that $kao \in B(go,S)$ and $\xi = k^+$. It follows that $g^{-1}ka \in G_S$, and hence $\| \mu(g) - \log a \| \leq dS$ by Lemma 2.5. On the other hand, since $g \in G_{u,R}$, there exists $t \geq 0$ such that $\| \mu(g) - tu \| < R$, and hence
\[ d(k \exp(tu)o,go) \leq d(k \exp(tu)o,kao) + d(kao,go) \leq \| tu - \log a \| + S \leq \| tu - \mu(g) \| + \| \mu(g) - \log a \| + S \leq R + (d + 1)S. \]

This proves the lemma. \[\square\]

The following Vitali-covering type lemma is a key ingredient of the proof of Proposition 3.1.
Lemma 3.6 (Covering lemma). Fix $R > 0$ and consider $\{O_R(o, \gamma o) : \gamma \in \Gamma'\}$ for some infinite subset $\Gamma' \subset \Gamma_{u,R}$. There exists a subset $\Gamma'' \subset \Gamma'$ such that $\{O_R(o, \gamma o) : \gamma \in \Gamma''\}$ consists of pairwise disjoint shadows and
\[
\bigcup_{\gamma \in \Gamma'} O_R(o, \gamma o) \subset \bigcup_{\gamma \in \Gamma''} O_{10dR}(o, \gamma o). \quad (3.4)
\]

Proof. Enumerate $\Gamma' = \{\gamma_i : i \in \mathbb{N}\}$ so that $\|\mu(\gamma_i)\|$ is nondecreasing. Set $i_0 = 0$ and inductively define
\[
i_{n+1} := \min\{i > i_n : O_R(o, \gamma_i o) \bigcap \bigcup_{j \leq n} O_R(o, \gamma_j o) = \emptyset\}.
\]
Set $\Gamma'' := \{\gamma_{i_n} : n \in \mathbb{N}\}$ so that $\{O_R(o, \gamma o) : \gamma \in \Gamma''\}$ consists of pairwise disjoint shadows.

For each $\gamma_i \in \Gamma'$, we claim that $O_R(o, \gamma_i o) \subset O_{10dR}(o, \gamma o)$ for some $\gamma \in \Gamma''$. We may assume that $i_{n-1} < i < i_{n+1}$ for some $n$. By definition of $i_{n+1}$, there exists $j \leq n$ such that $O_R(o, \gamma_j o) \cap O_R(o, \gamma_i o) \neq \emptyset$. In particular, there exists $k_1 \in K$, $a_i, a_{ij} \in A^+$ such that $k_1 a_i o \in B(\gamma_i o, R)$ and $k_1 a_{ij} o \in B(\gamma_{ij} o, R)$. Since $\gamma_i^{-1} k_1 a_i, \gamma_{ij}^{-1} k_1 a_{ij} \in \Gamma_{u,R}$, we have
\[
\|\mu(\gamma_i) - \log a_i\| \leq dR \text{ and } \|\mu(\gamma_{ij}) - \log a_{ij}\| \leq dR
\]
by Lemma 2.5. On the other hand, there exists $t_i, t_{ij} \geq 0$ such that
\[
\|\mu(\gamma_{ij}) - t_i u\| \leq R \text{ and } \|\mu(\gamma_{ij}) - t_{ij} u\| \leq R,
\]
as $\gamma_i, \gamma_{ij} \in \Gamma_{u,R}$. Observe that
\[
\|\mu(\gamma_i)\| = d(o, \gamma_i o) \leq d(o, k_1 a_i o) + d(k_1 a_i o, k_1 a_{ij} o) + d(k_1 a_{ij} o, \gamma_i o)
\]
\[
\leq d(o, k_1 a_{ij} o) + dR + 2R = d(o, k_1 a_{ij} o) + dR + 2R + (t_i - t_{ij})
\]
\[
\leq d(o, k_1 a_{ij} o) + 2dR + 3R + (t_i - t_{ij}) \leq d(o, \gamma_{ij} o) + 2dR + 4R + (t_i - t_{ij})
\]
\[
= \|\mu(\gamma_{ij})\| + 2dR + 4R + (t_i - t_{ij}) \leq \|\mu(\gamma_i)\| + 2dR + 4R + (t_i - t_{ij}),
\]
and hence $t_i' := t_i + 2dR + 4R \geq t_{ij}$.

Now let $k_2^+ \in O_R(o, \gamma_i o)$ be arbitrary and $b \in A^+$ be such that $k_2 b o \in B(\gamma_i o, R)$. We have $\|\mu(\gamma_i) - \log b\| \leq dR$ by Lemma 2.5. Since $\gamma_i \in \Gamma_{u,R}$, there exists $s \geq 0$ such that $\|\mu(\gamma_i) - su\| \leq R$. Since
\[
d(k_2 a_{ij} o, k_1 a_{ij} o) \leq d(k_2 a_{ij} o, k_2 b o)
\]
\[
+ d(k_2 b o, \gamma_i o) + d(\gamma_i o, k_1 a_{ij} o) + d(k_1 a_{ij} o, k_1 a_{ij} o) + d(k_1 a_{ij} o, \gamma_{ij} o)
\]
\[
\leq (dR + R) + R + (dR + R) + (2dR + 4R) = 4dR + 8R,
\]
there exists $0 \leq s' \leq s$ such that $d(k_2 a_{ij} o, k_1 a_{ij} o) \leq 4dR + 8R$ by Lemma 3.7 below. Finally,
\[
d(k_2 a_{ij} o, \gamma_{ij} o) < d(k_2 a_{ij} o, k_1 a_{ij} o) + d(k_1 a_{ij} o, k_1 a_{ij} o) + d(k_1 a_{ij} o, \gamma_{ij} o)
\]
\[
\leq (4dR + 8R) + (dR + R) + R = 5dR + 10R,
\]
which implies that $k_2^+ \in O_{5dR + 10R}(o, \gamma_{ij} o) \subset O_{10dR}(o, \gamma_{ij} o)$, since $d \geq 2$. This finishes the proof. $\square$
Lemma 3.7. Let $k_1, k_2 \in K$, $t_1, t_2 \geq 0$ be arbitrary. For any $0 \leq s_1 \leq t_1$, there exists $0 \leq s_2 \leq t_2$ such that
\[
d(k_1 \exp(s_1u)o, k_2 \exp(s_2u)o) \leq d(k_1 \exp(t_1u)o, k_2 \exp(t_2u)o).
\]

Proof. This follows from the CAT(0) property of $G/K$ (cf. [9]). Consider the geodesic triangle $\triangle(pqr)$ in $G/K$ with vertices $p = o$, $q = k_1 \exp(t_1u)o$ and $r = k_1 \exp(t_2u)o$. Let $\triangle(p'q'r')$ be the triangle in the Euclidean space which has the same corresponding side length to $\triangle pqr$. Let $0 \leq s_2 \leq t_2$ be arbitrary and $r'_1$ be a point on the side $p'r'$ such that the segment $p'r'_1$ has length $\ell(p'r'_1) = s_2$. By a straightforward computation in Euclidean geometry, we can find a point $q'_1$ on the side $p'q'$ such that
\[
\ell(q'_1r'_1) \leq \ell(q'r') = \ell(qr) = d(k_1 \exp(t_1u)o, k_2 \exp(t_2u)o).
\]
Set $s_1 := \ell(p'q'_1)$. Since $G/K$ is a CAT(0) space, we get
\[
d(k_1 \exp(s_1u)o, k_2 \exp(s_2u)o) \leq \ell(q'_1r'_1),
\]
from which the lemma follows. $\square$

We may write $\Lambda_u = \bigcup_{R > 0} \Lambda_{u,R}$ where
\[
\Lambda_{u,R} := \bigcap_{m \geq 1} \bigcup_{\gamma \in \Gamma_{u,R}, \|\mu(\gamma)\| \geq m} O_R(o, \gamma o), \quad \text{where } \Gamma_{u,R} := \Gamma \cap G_{u,R}. \tag{3.5}
\]

Lemma 3.8. If $R > 1$ is large enough, for any $f \in L^1(\nu_\psi)$ and for $\nu_\psi$-a.e. $\xi \in \Lambda_{u,R}$, we have
\[
\lim_{i \to \infty} \frac{1}{\nu_\psi(O_R(o, \gamma_i o))} \int_{O_R(o, \gamma_i o)} f \, d\nu_\psi = f(\xi)
\]
for any sequence $\gamma_i \to \infty$ in $\Gamma_{u,R}$ such that $\xi \in O_R(o, \gamma_i o)$.

We define a maximal operator $M_R$ on $L^1(\nu_\psi)$ as follows: for all $f \in L^1(\nu_\psi)$ and all $\xi \in \Lambda_{u,R}$, set
\[
M_Rf(\xi) := \limsup_{\gamma \in \Gamma_{u,R}, \|\mu(\gamma)\| \to \infty} \frac{1}{\nu_\psi(O_R(o, \gamma o))} \int_{O_R(o, \gamma o)} f \, d\nu_\psi;
\]
this is well-defined by the definition of $\Lambda_{u,R}$.

Note that Lemma 3.8 holds trivially for $f \in C(\Lambda)$. Once the weak type inequality for the maximal functions is established as in Lemma 3.9, Lemma 3.8 follows from a standard argument using the density of $C(\Lambda)$ in $L^1(\nu_\psi)$.

Lemma 3.9. If $R > 1$ is large enough, then $M_R$ is of weak type $(1,1)$; for all $f \in L^1(\nu_\psi)$ and $\lambda > 0$, we have
\[
\nu_\psi(\{\xi \in \Lambda_{u,R} : |M_Rf(\xi)| > \lambda\}) \ll \frac{1}{\lambda} \|f\|_{L^1(\nu_\psi)}
\]
where the implied constant is independent of $f$. 
Proof. Let $R > 1$ be large enough to satisfy Lemma 3.4. Let $\lambda > 0$ be arbitrary. By definition of $M_R$, there exists an infinite subset $\Gamma' \subset \Gamma_{u,R}$ such that
\[
\{\xi \in \Lambda_{u,R} : |M_R f(\xi)| > \lambda\} \subset \bigcup_{\gamma \in \Gamma'} O_R(o, \gamma o), \text{ and}
\]
\[
\frac{1}{\nu_\psi(O_R(o, \gamma o))} \int_{O_R(o, \gamma o)} f d\nu_\psi > \lambda \text{ for all } \gamma \in \Gamma'.
\]
By Lemma 3.6, there exists $\Gamma'' \subset \Gamma'$ so that $\{O_R(o, \gamma o) : \gamma \in \Gamma''\}$ consists of pairwise disjoint shadows and
\[
\bigcup_{\gamma \in \Gamma'} O_R(o, \gamma o) \subset \bigcup_{\gamma \in \Gamma''} O_{10dR}(o, \gamma o). \tag{3.6}
\]
Hence, by Lemma 3.4,
\[
\nu_\psi(\{\xi \in \Lambda_{u,R} : |M_R f(\xi)| > \lambda\}) \leq \nu_\psi\left( \bigcup_{\gamma \in \Gamma'} O_R(o, \gamma o) \right)
\]
\[
\leq \nu_\psi\left( \bigcup_{\gamma \in \Gamma''} O_{10dR}(o, \gamma o) \right) \leq \sum_{\gamma \in \Gamma''} \nu_\psi(O_{10dR}(o, \gamma o))
\]
\[
\leq \sum_{\gamma \in \Gamma''} \nu_\psi(O_R(o, \gamma o)) \leq \frac{1}{\lambda} \int_{\bigcup_{\gamma \in \Gamma''} O_R(o, \gamma o)} f d\nu_\psi \leq \frac{1}{\lambda} \|f\|_{L^1(\nu_\psi)}.
\]
\[
\square
\]

Proof of Proposition 3.1. Let $R > 1$ be large enough to satisfy Lemma 3.9. Suppose that $\nu_\psi(\Lambda_u) > 0$. Then for all sufficiently large $R > 1$, we have $\nu_\psi(\Lambda_{u,R}) > 0$. By applying Lemma 3.8 with $f = 1_{\Lambda_u}$, there exists $\xi \in \Lambda_{u,R}$, we obtain a sequence $\gamma_i \in \Gamma$ such that $\xi \in O_R(o, \gamma_i o)$ and
\[
\lim_{i \to \infty} \frac{\nu_\psi(O_R(o, \gamma_i o) \cap \Lambda_u^c)}{\nu_\psi(O_R(o, \gamma_i o))} = 0.
\]
Since $\nu_\psi(O_R(o, \gamma_i o)) \asymp e^{-\psi(\mu(\gamma_i))}$ by Lemma 3.4,
\[
\lim_{i \to \infty} e^{\psi(\mu(\gamma_i))} \nu_\psi(O_R(o, \gamma_i o) \cap \Lambda_u^c) = 0. \tag{3.7}
\]
By Lemma 3.2,
\[
\nu_\psi(O_R(o, \gamma_i o) \cap \Lambda_u^c) = \int 1_{O_R(o, \gamma_i o) \cap \Lambda_u^c}(\xi) d\nu_\psi(\xi)
\]
\[
= \int 1_{O_R(\gamma_i^{-1} o, \gamma_i o) \cap \Lambda_u^c}(\xi) e^{\psi(\beta(\gamma_i^{-1}))} d\nu_\psi(\xi)
\]
\[
= e^{-\psi(\mu(\gamma_i))} \nu_\psi(O_R(\gamma_i^{-1} o, \gamma_i o) \cap \Lambda_u^c).
\]
Hence as $i \to \infty$,
\[
\nu_\psi(O_R(\gamma_i^{-1} o, \gamma_i o) \cap \Lambda_u^c) \asymp e^{\psi(\mu(\gamma_i))} \nu_\psi(O_R(o, \gamma_i o) \cap \Lambda_u^c) \to 0.
\]
Passing to a subsequence, we may assume that $\gamma_i^{-1}o$ converges to some $\eta_0 \in \Lambda$. By [18, Lem. 5.6], for all sufficiently large $i$,

$$\nu_\psi(O_{R/2}(\eta_0, o) \cap \Lambda^c_\eta) \leq \nu_\psi(O_R(\gamma_i^{-1}o, o) \cap \Lambda^c_\eta).$$

Therefore

$$\nu_\psi(O_{R/2}(\eta_0, o) \cap \Lambda^c_\eta) = 0.$$ 

Since $R > 1$ is an arbitrary large number, varying $R$, we get from (3.2) that

$$\nu_\psi(\Lambda^c_\eta \cap \{ \eta \in F : (\eta, \eta_0) \in F(2) \}) = 0.$$ 

(3.8)

We now claim that for any $\eta \in \Lambda^c_\eta$, there exists a neighborhood $U_\eta$ of $\eta$ such that $\nu_\psi(\Lambda^c_\eta \cap U_\eta) = 0$. If $(\eta, \eta_0) \in F(2)$, this is immediate from (3.8). Otherwise, by the Zariski density of $\Gamma$ and the fact that $\Lambda$ is the unique $\Gamma$-minimal subset of $F$, we can find $\gamma \in \Gamma$ such that $(\gamma\eta, \eta_0) \in F(2)$. The claim follows again from (3.8), since $\nu_\psi$ is $\Gamma$-quasi-invariant. This finishes the proof. \(\square\)

4. Hopf’s argument for higher rank cases

Let $\Gamma < G$ be a regular Zariski dense discrete subgroup. We fix $\psi \in a^*$ and a pair $(\nu_\psi, \nu_\psi \circ i)$ of $(\Gamma, \psi)$ and $(\Gamma, \psi \circ i)$-conformal measures on $F$ respectively.

**Definition 4.1** (Hopf parametrization of $G/M$). The map 

$$gM \mapsto (g^+, g^-, b = \beta_g(e, g))$$

gives a homeomorphism between $G/M$ and $F(2) \times a$.

**Bowen-Margulis-Sullivan measures.** Define the following $A$-invariant Radon measure $\tilde{m} = \tilde{m}(\nu_\psi, \nu_\psi \circ i)$ on $G/M$ as follows: for $g = (g^+, g^-, b) \in F(2) \times a$,

$$d\tilde{m}(g) = e^{\psi(G(g^+, g^-))} \, d\nu_\psi(g^+)d\nu_\psi \circ i(g^-) \, db$$

where $db$ is the Lebesgue measure on $a$. We note that this is a non-zero measure; otherwise, $\nu_\psi$ is supported on a proper Zariski subvariety of $F$ by Fubini’s theorem, but since $\Gamma$ is Zariski dense and $\nu_\psi$ is $\Gamma$-conformal, that is not possible. The measure $\tilde{m}$ is left $\Gamma$-invariant, and hence induces a measure on $\Gamma \backslash G/M$, which we denote by $m$.

We fix $u \in \text{int } a^+$ and set for all $t \in \mathbb{R}$,

$$a_t := \exp tu.$$

Recall the following definitions:

1. A Borel subset $B \subset \Gamma \backslash G/M$ is called a **wandering set** for $m$ if for $m$-a.e. $x \in B$, we have $\int_{-\infty}^\infty 1_B(xa_t) \, dt < \infty$.
2. We say that $(\Gamma \backslash G/M, m, \{a_t\})$ is **conservative** if there is no wandering set $B \subset \Gamma \backslash G/M$ with $m(B) > 0$.
3. We say that $(\Gamma \backslash G/M, m, \{a_t\})$ is **completely dissipative** if $\Gamma \backslash G/M$ is a countable union of wandering sets modulo $m$. 
Proposition 4.2. The flow \( (\Gamma \setminus G/M, m, \{a_t = \exp(tu)\}) \) is conservative (resp. completely dissipative) if and only if \( \max(\nu_\psi(\Lambda_u), \nu_{\psi\circ i}(\Lambda_{i(u)})) > 0 \) (resp. \( \nu_\psi(\Lambda_u) = 0 = \nu_{\psi\circ i}(\Lambda_{i(u)}) \)).

Proof. Suppose that \( (\Gamma \setminus G/M, m, \{a_t\}) \) is conservative. Let \( B \) be a compact subset of \( \Gamma \setminus G/M \) with \( m(B) > 0 \). If we set \( B_0^+ := \{ x \in B : \limsup_{t \to \pm \infty} xa_t \cap B \neq \emptyset \} \), then \( m(B_0^+) > 0 \). Since \( m \) is equivalent to \( \nu_\psi \otimes \nu_{\psi\circ i} \otimes db \), it follows that \( m(B_0^+) > 0 \) (resp. \( m(B_0^-) > 0 \)) if and only if \( \nu_\psi(\Lambda_u) > 0 \) (resp. \( \nu_{\psi\circ i}(\Lambda_{i(u)}) > 0 \)). Hence \( \max(\nu_\psi(\Lambda_u), \nu_{\psi\circ i}(\Lambda_{i(u)})) > 0 \).

Now suppose that \( \nu_\psi(\Lambda_u) > 0 \) (resp. \( \nu_{\psi\circ i}(\Lambda_{i(u)}) > 0 \)). Then by Proposition 3.1, \( \nu_\psi(\Lambda_u) = 1 \) (resp. \( \nu_{\psi\circ i}(\Lambda_{i(u)}) = 1 \)). Hence for \( \mu \) a.e. \( [g] \), we have \( g^+ \in \Lambda_u \) (resp. \( g^- \in \Lambda_{i(u)} \)), and hence \( [g]a_t \) is convergent for some sequence \( t_i \to \pm \infty \). It follows that for \( \mu \) a.e. \( x \), there exists a compact subset \( B \) such that \( \int_{\mathbb{R}} 1_B(xa_t)dt = \infty \). We claim that this implies that \( (\Gamma \setminus G/M, m, \{a_t\}) \) is conservative. Assume in contradiction that there exists a wandering set \( W \subset \Gamma \setminus G/M \) with \( 0 < m(W) < \infty \). By the \( \sigma \)-compactness of \( \Gamma \setminus G/M \), there exists a compact subset \( B \) such that

\[
m\{x \in W : \int_{\mathbb{R}} 1_B(xa_t)dt = \infty\} \geq m(W)/2. \tag{4.1}
\]

On the other hand, there exists an integer \( n \geq 1 \) for which the set

\[
W_n := \left\{ w \in W : \int_{-\infty}^{\infty} 1_W(wa_t)dt \leq n \right\}
\]

has \( m \)-measure strictly bigger than \( m(W)/2 \). Note that the set \( E := W_n \exp(\mathbb{R}v) \) is \( \{a_t\} \)-invariant and any \( w \in E \) satisfies \( \int_{-\infty}^{\infty} 1_W(wa_t)dt \leq n \). Hence, for any \( R > 0 \), we get

\[
\int_{W_n} \int_{-R}^{R} 1_B(wa_t)dt dm = \int_{-R}^{R} \int_{W_n} 1_B(wa_t)dt dm
\]

\[
= \int_{-R}^{R} m(Ba_{-t} \cap W_n)dt = \int_{-R}^{R} m(B \cap W_n a_t)dt
\]

\[
= \int_{-R}^{R} m((B \cap E) \cap W_n a_t)dt = \int_{-R}^{R} \int_{B \cap E} 1_{W_n}(xa_{-t})dt dm
\]

\[
= \int_{B \cap E} \int_{-R}^{R} 1_{W_n}(xa_{-t})dt dm \leq \int_{B \cap E} \int_{\mathbb{R}} 1_{W_n}(xa_{-t})dt dm
\]

\[
\leq \int_{B \cap E} n dt = n \cdot m(B \cap E) < \infty
\]

where finiteness follows from the fact that \( B \) is compact and \( m \) is Radon. Hence \( \int_{W_n} \int_{\mathbb{R}} 1_B(wa_t)dt dm < \infty \); so

\[
m\{x \in W : \int_{\mathbb{R}} 1_B(wa_t)dt < \infty\} \geq m(W_n) > m(W)/2.
\]

contradicting (4.1). The rest of the claims can be proven similarly. \( \square \)
Let $\tilde{m}'$ denote the $M$-invariant lift of $\tilde{m}$ to $G$ and $\tilde{m}'$ the measure on $\Gamma \backslash G$ induced by $\tilde{m}'$. Since $\Gamma$ is Zariski dense, there exists a normal subgroup $M_\Gamma < M$ of finite index such that each $P^\circ$-minimal subset of $\Gamma \backslash G$ is $M_\Gamma$-invariant and the collection of all $P^\circ$-minimal subsets is parameterized by $M/M_\Gamma$ ([11, Thm. 1.9 and 2], see also [19, Sec. 3]).

We will need the following notion:

**Definition 4.3** (Transitivity group). For $g \in G$ with $g^\pm \in \Lambda$, define the subset $\mathcal{H}^\circ_\Gamma(g) < AM$ as follows: $am \in \mathcal{H}^\circ_\Gamma(g)$ if and only if there exist $\gamma \in \Gamma$ and a sequence $h_i \in N^- \cup N^+$, $i = 1, \ldots, k$ such that $(gh_1h_2\ldots h_r)^\pm \in \Lambda$ for all $1 \leq r \leq k$ and $\gamma gh_1h_2\ldots h_k = gam$.

It is not hard to check that $\mathcal{H}^\circ_\Gamma(g)$ is a subgroup (cf. [31, Lem. 3.1]); it is called the strong transitivity subgroup.

The following was obtained in [19] using the work of Guivarch-Raugi [11, Thm. 1.9].

**Lemma 4.4.** [19, Coro. 3.8] For any $g \in G$ with $g^\pm \in \Lambda$, the closure of $\mathcal{H}^\circ_\Gamma(g)$ contains $AM\Gamma$.

We now prove the following higher rank version of the Hopf-dichotomy, using Lemma 4.4. We remark that the following proposition holds for any Zariski dense subgroup $\Gamma$ which may not be necessarily regular.

**Proposition 4.5.** Let $Y$ be a $P^\circ$-minimal subset of $\Gamma \backslash G$. Then $(m'|_Y, \{a_t\})$ is conservative if and only if $(m'|_Y, \{a_t\})$ is ergodic.

**Proof.** Suppose that $(m'|_Y, \{a_t\})$ is conservative. Fix $x_0 \in \text{supp}(m'|_Y)$ and let $B_n \subset \Gamma \backslash G$ denote the ball of radius $n$ centered at $x_0$. Let $r$ be a positive function on $[0, \infty)$ which is affine on each $[n, n+1]$ and $r(n) = 1/(2^{n+1}m'(B_{n+1}))$. Then the function $\rho(x) := r(d(x_0, x))$ is a positive Lipschitz function on $\Gamma \backslash G$ with a uniform Lipschitz constant. In particular, it is uniformly continuous and $\rho \in L^1(m')$, since

$$
\|\rho\|_{L^1(m')} = \sum_{n=1}^{\infty} \int_{B_n - B_{n-1}} \rho \, dm' \leq \sum_{n=1}^{\infty} \frac{1}{2^n m'(B_n)} m'(B_n) < \infty.
$$

By the definition of $\rho$, for all $[g] \in \Gamma \backslash G$ such that $g^+ \in \Lambda_u$ and $g^- \in \Lambda_{i(u)}$, we have

$$
\int_{0}^{\infty} \rho([g]a_t) \, dt = \int_{0}^{\infty} \rho([g]a_{-t}) \, dt = \infty. \quad (4.2)
$$

Now let $f \in C_\circ(\Gamma \backslash G)$ be arbitrary. By the Hopf ratio ergodic theorem, the following $f_+$ and $f_-$ are well-defined and equal $m'$-a.e.:

$$
f_+(x) := \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(xa_t) \, dt \quad \text{and} \quad f_-(x) := \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(xa_{-t}) \, dt. \quad (4.3)
$$
By the uniform continuity of $f$ and $\rho$, (4.2) and the assumption that $u \in \text{int} a^+$, $f_\pm$ are $N^\pm$-invariant respectively. Let $\hat{f}_\pm : G \to \mathbb{R}$ be a left $\Gamma$-invariant lift of $f_\pm$. Let $\mathcal{B}$ denote the Borel $\sigma$-algebra of $G$ and set

$\Sigma := \{ B \in \mathcal{B} : \tilde{m}'(B \triangle B^\pm) = 0 \text{ for some } B^\pm \in \mathcal{B} \text{ such that } \Gamma B^\pm = B^\pm N^\pm \}.$

Let $F : G \to \mathbb{R}$ be a $\Sigma$-measurable and left $\Gamma$-invariant function such that $F(g) = \hat{f}_+(g) = \hat{f}_-(g)$ for $\tilde{m}'$ a.e. $g \in G$. Set

$$E := \left\{ gAM : \text{ $F(gAM)$ is measurable and $F(gAM) = \hat{f}_+(gAM) = \hat{f}_-(gAM)$ for Haar a.e. } am \in AM \right\} \subset G/AM.$$  

By Fubini’s theorem, $E$ has full measure in $G/AM \simeq \mathcal{F}(2)$ with respect to the measure $d\nu_{\psi} \otimes d\nu_{\psi_1}$. For all small $\varepsilon > 0$, define functions $F^\varepsilon, \tilde{f}_\varepsilon : G \to \mathbb{R}$ by

$$F^\varepsilon(g) := \frac{1}{\text{vol}(AM)_\varepsilon} \int_{(AM)_\varepsilon} F(g\ell) d\ell, \quad \tilde{f}_\varepsilon(g) := \frac{1}{\text{vol}(AM)_\varepsilon} \int_{(AM)_\varepsilon} \tilde{f}_\varepsilon(g\ell) d\ell$$

where $(AM)_\varepsilon$ denotes the $\varepsilon$-ball around $e$ in $AM$ and $d\ell$ is the Haar measure on $AM$. Note that if $gAM \in E$, then $F^\varepsilon$ and $\tilde{f}_\varepsilon$ are continuous and identical on $gAM$. Moreover, $F^\varepsilon$ is left $\Gamma$-invariant and $\tilde{f}_\varepsilon$ are $N^\pm$-invariant, as $AM$ normalizes $N^\pm$. Using the isomorphism between $G/AM$ and $\mathcal{F}(2)$ given by $gAM \mapsto (g^+, g^-)$, we may consider $E$ as a subset of $\mathcal{F}(2)$. We then define

$$E^+ := \{ \xi \in \Lambda : (\xi, \eta') \in E^\varepsilon \text{ for } \nu_{\psi_1}\text{-a.e. } \eta' \in \Lambda \};$$

$$E^- := \{ \eta \in \Lambda : (\xi', \eta) \in E^\varepsilon \text{ for } \nu_{\psi}\text{-a.e. } \xi' \in \Lambda \}.$$

Then $E^+$ is $\nu_{\psi}$-conull and $E^-$ is $\nu_{\psi_1}$-conull by Fubini’s theorem. By a similar argument as in [19, Lem. 4.6], we can show that for any $gAM \in E$, with $h^\pm \in E^\pm$, and any $\varepsilon > 0$, $F^\varepsilon|_{gAM}$ is $AM_\Gamma$-invariant, using the fact that the closure of $\mathcal{H}_\varepsilon^\varepsilon(g)$ contains $AM_\Gamma$ (Lemma 4.4). It follows that $F$ is $\Sigma_0$-measurable where

$$\Sigma_0 := \{ B \in \mathcal{B} : B = \Gamma BAM_\Gamma \}.$$

We claim that if $f$ is $M$-invariant, then $F$ is constant on the $m'$-conull set $E^\dagger := \{ g \in G : gAM \in E^\pm \}$. Using Hopf’s ratio ergodic theorem once more, this would in turn imply that $m'$ is $M\{a_1\}$-ergodic. Assume $f$ is $M$-invariant. Since $F = \lim_{\varepsilon \to 0} F^\varepsilon$ $m'$-a.e. by the Lebesgue differentiation theorem, it suffices to show that $F^\varepsilon$ is constant on $E^\dagger$. Since $F^\varepsilon$ is $AM$-invariant on $E^\dagger$ and $F^\varepsilon(gh) = F^\varepsilon(g)$ for all $g \in E^\dagger$ and $h \in N^\pm$ with $gh \in E^\dagger$, it is again enough to show that for any $g_1, g_2 \in E^\dagger$, there exist $h_1, h_2, h_3 \in N^+ \cup N^-$ such that $g_1h_1h_2h_3 \in g_2AM$ and $g_1h_1, g_1h_2h_3 \in E^\dagger$.

We note that if $(\xi, \eta_1), (\xi, \eta_2) \in \mathcal{F}(2)$, then there exist $g \in G, h \in N$ such that $(\xi, \eta_1) = ((g^+, g^-))$ and $(\xi, \eta_2) = ((gh)^+, (gh)^-)$. Similarly, if $(\xi_1, \eta), (\xi_2, \eta) \in \mathcal{F}(2)$, then there exist $g \in G, h \in N^+$ such that $(\xi_1, \eta) = (g^+, g^-)$ and $(\xi_2, \eta) = ((gh)^+, (gh)^-)$. 


Note that $E^+$ is $\Gamma$-invariant. Since the limit set $\Lambda$ is the unique $\Gamma$-minimal subset of $F$, the closure of $E^+$ contains $\Lambda$, and in particular it is Zariski dense. Therefore we can choose $\xi \in E^+$ such that $(\xi, g_1^-), (\xi, g_2^-) \in F^{(2)}$. Let $h_1, h_2, h_3 \in N^+ \cup N^-$ such that
\[
(\xi, g_1^-) = (g_1 h_1^+, g_1 h_1^-)
(\xi, g_2^-) = (g_1 h_1 h_2^+, g_1 h_1 h_2^-)
(g_2^+, g_2^-) = (g_1 h_1 h_2 h_3^+, g_1 h_1 h_2 h_3^-).
\]
Hence the claim is proved. In particular, $m'$ is $AM$-ergodic.

Let $\tilde{Y} \subset G$ be the $\Gamma$-invariant lift of $Y$. In order to show that $m'|_Y$ is $\{a_t\}$-ergodic, it suffices to show that $F$, associated to an arbitrary function $f \in C_c(\Gamma \backslash G)$, is constant on $\tilde{Y}$. It follows from the $AM$-ergodicity of $m'$ that $\Sigma_0$ is $\tilde{m}'$-equivalent to a finite $\sigma$-algebra generated by $\{B.s : s \in M_1 \backslash M\}$ for some $B \in \Sigma_0$. Since $\{\check{Y}.s : s \in M_1 \backslash M\} \subset \Sigma_0$ and the $\check{Y}$.s are mutually disjoint, it follows that $\check{Y} = B.s \mod \tilde{m}'$ for some $s \in M_1 \backslash M$.

Since $F$ is constant on $B.s$, being $\Sigma_0$-measurable, it implies that $F$ is constant on $\check{Y}$, concluding that $m'|_Y$ is $\{a_t\}$-ergodic.

Now to show the converse, assume that $(m'|_Y, \{a_t\})$ is ergodic. Since the quotient map $\Gamma \backslash G \rightarrow \Gamma \backslash G/M$ is a proper map, it suffices to show that $(\Gamma \backslash G/M, m, \{a_t\})$ is conservative when it is ergodic. Assume that $(\Gamma \backslash G/M, m, \{a_t\})$ is ergodic. Then it is either conservative or completely dissipative by the Hopf decomposition theorem [15]. Suppose it is completely dissipative. Then it is isomorphic to a translation on $\mathbb{R}$ with respect to the Lebesgue measure. This implies that $\nu_\psi \otimes \nu_{\psi_0i}$ is supported on a single $\Gamma$-orbit, say, $\Gamma (\xi_0, \eta_0)$ in $F^{(2)}$. It follows that $\nu_\psi$ (resp. $\nu_{\psi_0i}$) is an atomic measure supported on $\Gamma \xi_0$ (resp. $\Gamma \eta_0$). Hence $\nu_\psi \otimes \nu_{\psi_0i}$ is supported on $\Gamma \xi_0 \times \Gamma \eta_0$. To obtain a contradiction, it suffices to prove that $(\Gamma \xi_0 \times \Gamma \eta_0) \cap F^{(2)} \neq \Gamma (\xi_0, \eta_0)$. Suppose not. Then $\Gamma \eta_0 \subset \Gamma \xi_0 \eta_0$ where $\Gamma \xi_0$ denotes the stabilizer of $\xi_0$ in $\Gamma$. This is not possible, since the set of accumulation points of $\Gamma \eta_0$ contains $\Lambda$ which is Zariski dense in $F$, while the set of accumulation points of $\Gamma \xi_0 \eta_0$ is contained in a proper subvariety of $F$. To see the last claim, write $(\xi_0, \eta_0) = (g^+, g^-)$ for some $g \in G$. Then $\Gamma \xi_0 < g P g^{-1}$ and $\eta_0 = g w_0 P$. If $\gamma_i = g p_i g^{-1} \rightarrow \infty$ in $\Gamma \xi_0$, then the limit point of $\gamma_i \eta_0$ is of the form $\lim_{i \rightarrow \infty} g p_i w_0 [P]$. Since $p_i \rightarrow \infty$ in $P$, the limit is not contained in $g P w_0 [P]$. Hence the set of all accumulation points of $\Gamma \xi_0 \eta_0$ is contained in $F - g P w_0 [P]$, which is a proper subvariety of $F$. 

5. Directional Poincare series

Let $\Gamma < G$ be a Zariski dense discrete subgroup. We define the limit cone $L_\Gamma \subset a^+$ as the asymptotic cone of $\mu(\Gamma)$. Then $L_\Gamma$ coincides with the smallest cone containing the Jordan projection of $\Gamma$ and is a convex cone with non-empty interior $[3]$.

Quint [22] introduced the following:
**Definition 5.1.** The growth indicator function \( \psi_T : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\} \) is defined as a homogeneous function, i.e., \( \psi_T(tu) = t\psi_T(u) \) for all \( t > 0 \), such that for any unit vector \( \psi \) defined as a homogeneous function, i.e., \( \psi > \psi \) and \( C \) is the abscissa of convergence of the series \( \sum_{\gamma \in \Gamma, \mu(\gamma) \in C} e^{-\tau |\mu(\gamma)|} \).

We consider \( \psi_T \) as a function on \( \mathfrak{a} \) by setting \( \psi_T = -\infty \) outside \( \mathfrak{a}^+ \).

Quint showed that \( \psi_T \) is upper semi-continuous, \( \psi_T > 0 \) on \( \mathcal{L}_T \), and \( \psi_T = -\infty \) outside \( \mathcal{L}_T \) [22, Thm. IV.2.2].

**Lemma 5.2.** Let \( \psi \in \mathfrak{a}^* \) and \( u \in \text{int} \mathfrak{a}^+ \) be such that \( \psi(u) > \psi_T(u) \). Then for any \( R > 0 \),

\[
\sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} < \infty.
\]

**Proof.** Since \( \psi(u) > \psi_T(u) \), the upper-semi continuity of \( \psi_T \) implies that there exists a small open convex cone \( \mathcal{C} \) containing \( u \) such that \( \overline{\mathcal{C}} \subset \text{int} \mathfrak{a}^+ \) and \( \psi > \psi_T \) on \( \mathcal{C} \). Since \( \psi > \psi_T \) on some open convex cone \( \mathcal{C}' \) containing \( \overline{\mathcal{C}} \), we can choose a continuous homogeneous function \( \theta : \mathfrak{a} \to \mathbb{R} \) such that \( \psi \geq \theta > \psi_T \) on \( \mathcal{C} \) and \( \theta > \psi_T \) on \( \mathfrak{a}^+ \). Since \( \psi_T = -\infty \) outside \( \mathfrak{a}^+ \), we have \( \theta > \psi_T \) on \( \mathfrak{a} - \{0\} \). Applying [22, Lem. III.1.3] to the measure \( \sum_{\gamma \in \Gamma} \delta_{\mu(\gamma)} \) on \( \mathfrak{a}^+ \), we get

\[
\sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} \leq \sum_{\gamma \in \Gamma} e^{-\theta(\mu(\gamma))} < \infty,
\]

Since \( \# \{ \gamma \in \Gamma_{u,R} : \mu(\gamma) \not\in \mathcal{C} \} < \infty \) for any \( R > 0 \), the lemma follows. □

In the rest of this section, we assume that \( \Gamma < G \) is regular. Let \( \psi \in \mathfrak{a}^* \) and fix a pair of \( (\Gamma, \psi) \) and \( (\Gamma, \psi \circ i) \)-conformal measures \( (\nu_\psi, \nu_{\psi \circ i}) \) on \( \mathcal{F} \) respectively. We let \( m \) denote the BMS measure on \( \Gamma\backslash G/M \) associated to \( (\nu_\psi, \nu_{\psi \circ i}) \).

We fix a unit vector \( u \in \text{int} \mathfrak{a}^+ \) such that \( \psi(u) > 0 \), and set

\[
a_t := \exp(tu) \text{ and } \delta := \psi(u).
\]

For an interval \( I \subset \mathbb{R} \), we sometimes write \( a_I = \{a_t : t \in I\} \). We make the following simple observation: for any \( R > 0 \),

\[
\sum_{\gamma \in \Gamma_{(u),R}} e^{-\psi(i(\mu(\gamma))))} = \sum_{\gamma^{-1} \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma^{-1}))} = \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))}.
\] (5.1)

**Lemma 5.3.** If \( \max(\nu_\psi(\Lambda_u), \nu_{\psi \circ i}(\Lambda_{i(u)})) > 0 \), then there exists \( R > 0 \) such that

\[
\sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} = \infty = \sum_{\gamma \in \Gamma_{(u),R}} e^{-\psi(i(\mu(\gamma))))}.
\]
Proof. Without loss of generality, we may assume that \( \nu_\psi(\Lambda_u) > 0 \). Recall that \( \Lambda_u = \cup_{n \in \mathbb{N}} \Lambda_{u,n} \) where

\[
\Lambda_{u,n} = \bigcap_{m=1}^\infty \bigcup_{\|\mu(\gamma)\| \geq m, \gamma \in \Gamma_{u,n}} O_n(o, \gamma o).
\]

Hence \( \nu_\psi(\Lambda_{u,n}) > 0 \) for some \( n \). Now by Lemma 3.4, we have for all \( m \geq 1 \),

\[
0 < \nu_\psi(\Lambda_{u,n}) \leq \sum_{\|\mu(\gamma)\| \geq m, \gamma \in \Gamma_{u,n}} \nu_\psi(O_n(o, \gamma o)) \ll \sum_{\|\mu(\gamma)\| \geq m, \gamma \in \Gamma_{u,n}} e^{-\psi(\mu(\gamma))}.
\]

Since the implicit constant above is independent of \( m \), it follows that the series \( \sum_{\gamma \in \Gamma_{u,n}} e^{-\psi(\mu(\gamma))} \) diverges, which implies the claim by (5.1). \( \square \)

The rest of this section is devoted to the proof of the following:

**Proposition 5.4.** Suppose that \( m \) is \( u \)-balanced as defined in (1.2). If \( \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} = \infty \) for some \( R > 0 \), then

\[
\nu_\psi(\Lambda_u) = 1 = \nu_\psi(1)(\Lambda_{i(u)}).
\]

Proof of this proposition involves investigating the relation between the \( u \)-directional Poincare series and the correlation function of \( m \) for the \( a_t \)-action.

**Multiplicity of shadows.**

**Lemma 5.5.** For any \( R > 0 \) and \( D > 0 \), we have

\[
\sup_{T>0} \sum_{\|\mu(\gamma)\| \leq T+D, \gamma \in \Gamma_{u,R}} 1_{O_R(o, \gamma o)} < \infty.
\]

Proof. Suppose that there exist \( \gamma_1, \cdots, \gamma_m \in \Gamma_{u,R} \) and \( k \in K \) such that \( k^+ \in \cap_{i=1}^m O_R(o, \gamma_i o) \) and \( T \leq \psi(\mu(\gamma_i)) \leq T + D \). By Lemma 3.5, for all \( 1 \leq i \leq m \), there exists \( t_i \geq 0 \) such that \( k a_{t_i} o \in B(\gamma_i o, (2d+1)R) \). Since \( \gamma_i^{-1} k a_{t_i} \in G(1_{(2d+1)}R) \), we have \( \|\mu(\gamma_i) - t_i u\| \leq d(2d+1)R \) by Lemma 2.5. In particular,

\[
t_i \psi(u) \leq \psi(\mu(\gamma_i)) + \|\psi\|d(2d+1)R \leq T + D + \|\psi\|d(2d+1)R,
\]

and similarly

\[
t_i \psi(u) \geq T - \|\psi\|d(2d+1)R.
\]

Hence \( |\psi(u)(t_i - t_1)| < 2\|\psi\|d(2d+1)R + D \). Note that as \( \psi(u) > 0 \), for all \( 1 \leq i \leq m\),

\[
d(\gamma_i o, \gamma_1 o) \leq d(\gamma_i o, k a_{t_i} o) + d(k a_{t_i} o, k a_{t_1} o) + d(k a_{t_1} o, \gamma_1 o) \\
\leq 2d(2d+1)R + |t_i - t_1| \\
\leq S := 2d(2d+1)R + (\psi(u))^{-1}(2\|\psi\|d(2d+1)R + D).
\]
Since there are only finitely many \( \gamma_i o \) in a bounded ball of radius \( S \), it follows that \( m \) is bounded above by a constant depending only on \( S \). This proves the claim. \( \square \)

**Corollary 5.6.** For any large enough \( R > 0 \), we have, for any \( D > 0 \),

\[
\sup_{T > 0} \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} < \infty.
\]

**Proof.** By Lemmas 3.4 and 3.5, there exists \( C = C(\psi) > 0 \) such that for all \( R \) large enough, and any \( T > 0 \),

\[
\sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} \leq C \cdot \nu(\mu(\gamma)) < \infty
\]

by Lemma 5.5. \( \square \)

**Directional Poincare series.** For \( r > 0 \) and \( g \in G \), we set

\[
Q_r := G_r A_r = K A_r K A_r, \quad \text{and} \quad \mathcal{L}_r(o,g(o)) := \{(h^+, h^-) \in \mathcal{F}^{(2)} : h \in G_r \cap gG_r \exp(R - u)\}.
\]

**Lemma 5.7.** For any \( r > 0 \), we have \( Q_r \subset G_{2r} \).

**Proof.** Let \( g \in Q_r \) be arbitrary. By definition, \( g = k_1 a_1 k_2 a_2 \) for some \( k_1, k_2 \in K \) and \( a_1, a_2 \in A_r \). Since

\[
d(go,o) = d(a_1 k_2 a_2 o, o) \leq d(a_1 k_2 a_2 o, a_1 k_2 o) + d(a_1 k_2 o, o)
\]

\[
= d(a_2 o, o) + d(o, a_1^{-1} o) < 2r,
\]

the lemma follows. \( \square \)

The following is the main ingredient of the proof of Proposition 5.4:

**Proposition 5.8.** Suppose that \( \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi(\mu(\gamma))} = \infty \) for some \( R > 0 \). If \( r \) is large enough, we have the following for any \( T > 1 \):

\[
\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \hat{m}(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t}) \, dt \, ds \ll \left( \sum_{\gamma \in \Gamma_{u,4dr}, \psi(\mu(\gamma)) \leq \delta T} e^{-\psi(\mu(\gamma))} \right)^2;
\]

\[
\int_0^T \sum_{\gamma \in \Gamma} \hat{m}(Q_{4rv} \cap \gamma Q_{4va}_{-t}) \, dt \gg \sum_{\gamma \in \Gamma_{u,v}, \psi(\mu(\gamma)) \leq \delta T} e^{-\psi(\mu(\gamma))}
\]

where the implied constants are independent of \( T \).

**Lemma 5.9.** If \( Q_r \cap \gamma Q_r a_{-t} \neq \emptyset \) for \( \gamma \in \Gamma \) and \( t, r > 0 \), then

\[
\|\mu(\gamma) - tu\| \leq 4dr.
\]
Proof. If \( Q_t \cap \gamma Q_r a_{-t} \neq \emptyset \), there exist \( q_0, q_0' \in Q_t \) such that \( q_0 = \gamma q_0' a_{-t} \). By Lemma 2.5 and Lemma 5.7,
\[
t u = \mu(a_t) = \mu(q_0^{-1} \gamma q_0') \in \mu(\gamma) + a_{4dr}.
\]

In order to prove Proposition 5.8, we will bound the integrals appearing in the lemma from below and above using shadows, and then apply the shadow lemma (Lemma 3.4). For this purpose, we observe several relations between the sets defined in (5.2) and shadows.

**Lemma 5.10.** If \( g \in Q_t \cap \gamma Q_r a_{-t} \) for \( \gamma \in \Gamma \) and \( t, r > 0 \), then
\[
\begin{align*}
(1) & \quad (g^+, g^-) \in L_r(\alpha, \gamma o); \\
(2) & \quad |\psi(G(g^+, g^-))| < 2||\psi||r \text{ where } c \text{ is from Lemma 2.3}; \\
(3) & \quad [g] A \cap Q_r \cap \gamma Q_r a_{-t} \subset [g] A_{2dr}.
\end{align*}
\]

**Proof.** (1) is immediate from the definition of \( L_r(\alpha, \gamma o) \). Since \( g \in Q_r \), \( g_0 \in B(o, 2r) \) and hence \( ||G(g^+, g^-)|| < 2cr \) by Lemma 2.3 and (2) follows. (3) follows from the stronger inclusion \( g A \cap Q_r \subset g A_{2dr} \) which follows from Lemma 2.5 and Lemma 5.7.

**Lemma 5.11.** For any \( g \in G \) and \( r > 0 \), we have
\[
L_r(\alpha, g(o)) \subset O_4(r(\alpha, g(o))) \times O_4(r(g(o), o)).
\]

**Proof.** Let \( (h^+, h^-) \in L_r(\alpha, g(o)) \); so \( h \in B(o, 2r) \) such that \( ha_0 o \in B(g(o), 2r) \) for some \( t \geq 0 \). Write \( o = ha_0 n_0 o \) for some \( a_0 n_0 \in AN \). Since the Hausdorff distance between \( a_0 n_0 A^+ o \) and \( A^+ o \) is \( d(a_0 n_0 o, o) \) [9, 1.6.6 (4)], we can find \( q' \in ha_0 n_0 A^+ o \) such that \( d(q', ha_0 o) < d(ha_0 n_0, ho) < 2r \). Hence, \( d(g(o), q') < d(g(o), ha_0 o) + d(ha_0 o, q') < 4r \) and it follows that \( h^+ \in O_4(r(\alpha, g(o))) \). A similar argument shows that \( h^- \in O_4(r(g(o), o)) \).

**Lemma 5.12.** For all large enough \( r > 1 \), we have for any \( t > 1 \),
\[
\bar{m}(Q_t \cap \gamma Q_r a_{-t}) \ll e^{-\psi(\mu(\gamma))}
\]
where the implied constant is independent of \( t > 1 \).

**Proof.** If \( r \) is large enough, we get by Lemma 5.11, Lemma 3.4 and Lemma 5.10:
\[
\begin{align*}
\bar{m}(Q_t \cap \gamma Q_r a_{-t}) \\
= \int \chi_{Q_t \cap \gamma Q_r a_{-t}}(\mathbb{F}b) e^{\psi(G(g^+, g^-))} d\nu_\psi(g^+) d\nu_\psi(g^-) db \\
= \int_{L_r(\alpha, \gamma o)} \left( \int_A \chi_{Q_t \cap \gamma Q_r a_{-t}}(\mathbb{F}b) \right) e^{\psi(G(g^+, g^-))} d\nu_\psi(g^+) d\nu_\psi(g^-) \\
\leq c \nu_\psi(O_4(r(\alpha, \gamma o))) \text{ Vol}(A_{2dr}) e^{2||\psi||r} \\
\ll e^{-\psi(\mu(\gamma))}.
\end{align*}
\]
Lemma 5.13. If \(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s} \neq \emptyset\) for \(\gamma, \gamma' \in \Gamma\) and \(r, t, s > 0\), then

\[
\begin{align*}
(1) \ |\mu(\gamma) - tu|, & \ |\mu(\gamma^{-1} \gamma') - su|, \ |\mu(\gamma') - (t + s)u| \leq 4dr; \\
(2) \ \psi(\mu(\gamma)) + \psi(\mu(\gamma^{-1} \gamma')) & \leq \psi(\mu(\gamma')) + 12dr\|\psi\|.
\end{align*}
\]

Proof. Note that from the hypothesis, the intersections

\[Q_r \cap \gamma Q_r a_{-t}, \ Q_r \cap \gamma^{-1} \gamma' Q_r a_{-s}, \ Q_r \cap \gamma' Q_r a_{-t-s}\]

are all nonempty. By Lemma 5.9, we obtain (1).

(2) follows since

\[
|\psi(\mu(\gamma)) + \psi(\mu(\gamma^{-1} \gamma')) - \psi(\mu(\gamma'))| = |\psi(\mu(\gamma) - tu) + \psi(\mu(\gamma^{-1} \gamma') - su) - \psi(\mu(\gamma') - (t + s)u)| \\
\leq 4dr\|\psi\| + 4dr\|\psi\| + 4d\|\psi\| = 12dr\|\psi\|.
\]

Proof of (5.3) in Proposition 5.8. Fix \(r, t > 0\). Let \(r\) be large enough so that \(\sum_{\gamma \in \Gamma_{u,4dr}} e^{-\psi(\mu(\gamma))} = \infty\). In the following proof, the notation \(\sum\) means the sum over all \((\gamma, \gamma') \in \Gamma_{u,4dr} \times \Gamma\) satisfying:

\[
\begin{align*}
\gamma^{-1} \gamma' & \in \Gamma_{u,4dr}; \\
\psi(\mu(\gamma)) & \in (\delta t - 4dr\|\psi\|, \delta t + 4dr\|\psi\|); \text{ and} \\
\psi(\mu(\gamma^{-1} \gamma')) & \in (\delta s - 4dr\|\psi\|, \delta s + 4dr\|\psi\|).
\end{align*}
\]

Note that

\[
\sum_{\gamma, \gamma' \in \Gamma} \tilde{m}(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s})
\]

\[
= \sum_{\gamma, \gamma' \in \Gamma} \tilde{m}(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s}) \text{ by Lemma 5.13(1)}
\]

\[
\ll \sum_{\gamma, \gamma' \in \Gamma} e^{-\psi(\mu(\gamma))} \text{ by Lemma 5.12}
\]

\[
\ll \sum_{\gamma, \gamma' \in \Gamma} e^{-\psi(\mu(\gamma))} e^{-\psi(\mu(\gamma^{-1} \gamma'))} \text{ by Lemma 5.13(2)}
\]

\[
\ll \left( \sum_{\gamma \in \Gamma_{u,4dr}, \psi(\mu(\gamma)) \in (\delta t - c_0, \delta t + c_0)} e^{-\psi(\mu(\gamma))} \right) \left( \sum_{\gamma' \in \Gamma_{u,4dr}, \psi(\mu(\gamma')) \in (\delta s - c_0, \delta s + c_0)} e^{-\psi(\mu(\gamma'))} \right)
\]

where \(c_0 = 4dr\|\psi\|\).
Let $I_\gamma$ denote the interval $\delta^{-1}[\psi(\mu(\gamma)) - c_0, \psi(\mu(\gamma)) + c_0]$. Note that $I_\gamma \cap [0, T] \neq \emptyset$ implies that $\psi(\mu(\gamma)) \leq \delta T + c_0$. Hence

$$\int_0^T \left( \sum_{\gamma \in \Gamma_{u,adr}} e^{-\psi(\mu(\gamma))} \right) dt = \sum_{\gamma \in \Gamma_{u,adr}} e^{-\psi(\mu(\gamma))} \int_0^T 1_t(t) dt \ll \sum_{\gamma \in \Gamma_{u,adr}, \psi(\mu(\gamma)) \leq \delta T + c_0} e^{-\psi(\mu(\gamma))}. $$

Putting these two together along with Corollary 5.6, used in order to remove $c_0$, concludes the proof of (5.3).

\[ \square \]

**Lemma 5.14.** For any $S > 0$, there exists $\ell(S) < \infty$ such that for any $\gamma \in \Gamma$ such that $\|\mu(\gamma)\| > \ell(S)$, any point $(\xi, \eta) \in O_S(o, \gamma_0) \times O_S(\gamma_0, o)$ satisfies $\|G(\xi, \eta)\| < \ell(S)$.

**Proof.** Suppose not. Then there exists sequences $\gamma_i \to \infty$ in $\Gamma$ and $(\xi_i, \eta_i) \in O_S(o, \gamma_i) \times O_S(\gamma_i, o)$ such that $\|G(\xi_i, \eta_i)\| \to \infty$ as $i \to \infty$. We may write $\gamma_i = k_i a_i \ell_i$ in $KA^+K$ decomposition, and assume that $k_i \to k_0$ after passing to a subsequence. It follows that $\xi_i \to k_0^+ \ell$ and $\eta_i \to \gamma_0$ for some $\eta_0 \in F$ such that $(k_0^+, \eta_0) \in F^{(2)}$. Hence $\lim_{i \to \infty} \|G(\xi_i, \eta_i)\| = \|G(k_0^+, \eta_0)\| < \infty$, which is a contradiction.

In the following, we fix a large number $S_0$ which satisfies Lemma 3.4, and $\ell_0 = \ell_0(S_0) > 0$ provided by Lemma 5.14 such that for any $(\xi, \eta) \in \bigcup_{\gamma \in \Gamma, \|\mu(\gamma)\| \geq \ell_0} O_S(o, \gamma_0) \times O_S(\gamma_0, o)$, we have $\|G(\xi, \eta)\| < \ell_0$.

**Lemma 5.15.** If $r > 1$ is large enough, the following holds: for any $(\xi, \eta) \in O_{S_0}(o, \gamma_0) \times O_{S_0}(\gamma_0, o)$ for some $\gamma \in \Gamma_{u,r}$ with $\|\mu(\gamma)\| \geq f_0$, there exist $t \in \mathbb{R}$ and $g \in Q_{2r}$ such that

$$ga|_{[t-1,t+1]} \subset \gamma Q_{2r} \quad \text{and} \quad (g^+, g^-) = (\xi, \eta).$$

**Proof.** Let $(\xi, \eta)$ be as in the statement. Then by Lemma 3.5, there exists $t \geq 0$ and $k \in K$ such that $\xi = k^+, ka_0 \in B(\gamma_0, r + (d + 1)S_0)$. Let $g \in G$ be such that $(g^+, g^-) = (\xi, \eta)$. Since $\|\mu(\gamma)\| > \ell_0$, by replacing $g \in G$ by an element of $gA$, we may assume that $d(go, o) < c\ell_0 + c'$ where $c$ and $c'$ are as in Lemma 2.3.

As $g^+ = k^+$ and hence $k^{-1}g \in P$, it follows by [9, 1.6.6 (4)] that $d(ga_0, o, ka_0) \leq d(go, o)$ for all $t \geq 0$.

Hence for all $s \in [t - 1, t + 1]$,

$$d(ga_0, o, ka_0) \leq d(ga_0, o, ga_0) + d(ga_0, ka_0) \leq 1 + d(go, o) < 1 + \epsilon \ell_0 + c'.$$

It follows that $ga|_{[t-1,t+1]} \in \gamma G_{r+(d+1)S_0+c\ell_0+c'+1}$. Now if $r$ is large enough, $ga|_{[t-1,t+1]} \subset \gamma Q_{2r}$.

Similarly, since $go \in G_{c\ell_0+c'}$, we have $g \in Q_{2r}$, which was to be shown. \[ \square \]
Lemma 5.16. If \( r \) is large enough, for any \( g \in G \) such that \((g^+, g^-) \in O_{\mathbb{S}_\mathbb{O}}(\alpha, \gamma_0) \times O_{\mathbb{S}_\mathbb{O}}(\gamma_0, \alpha)\) for some \( \gamma \in \Gamma_{u,r} \) and \( T > 0 \) satisfying

\[ \|\mu(\gamma)\| > t_0 \quad \text{and} \quad 8dr\|\psi\| + \delta < \psi(\mu(\gamma)) < \delta T - 8dr\|\psi\| - \delta, \]

we have

\[ \int_0^T \int_A 1_{Q_{4r} \cap \gamma Q_{4r, a-t}}([gb]) \, db \, dt \geq 2 \text{Vol}(A_{2r}). \quad (5.5) \]

Proof. Note that replacing \( g \) with an element of \( gA \) does not affect the validity of (5.5). Hence by Lemma 5.15, we may assume that \( g \in Q_{2r} \) and \( ga_{[t_0 - 1, t_0 + 1]} \subset \gamma Q_{2r} \) for some \( t_0 \in \mathbb{R} \).

It follows that \( Q_{2r} \cap \gamma Q_{2r, a-t} \neq \emptyset \) for all \( t \in [t_0 - 1, t_0 + 1] \). Note that \(|\psi(\mu(\gamma)) - t_0\delta| \leq 8dr\|\psi\|\) by Lemma 5.9, and hence \([t_0 - 1, t_0 + 1] \subset [0, T]\) by the hypothesis. Since \( g \in Q_{2r} \) and hence \( g \in G_{4r} \) by Lemma 5.7, we have \( gA \cap Q_{4r} \supset gA_{4r} \). Consequently,

\[ \int_A 1_{Q_{4r} \cap \gamma Q_{4r, a-t}}([gb]) \, db \geq \int_{A_{4r}} 1_{\gamma Q_{4r}}([gba_t]) \, db. \quad (5.6) \]

By definition of \( Q_{4r} \), there is a uniform lower bound for (5.6), say \( \text{Vol}(A_{2r}) \), whenever \([gba_t] \cap \gamma Q_{4r} \neq \emptyset\), in particular for all \( t \in [t_0 - 1, t_0 + 1] \) by Lemma 5.15. Hence,

\[ \int_0^T \int_A 1_{Q_{4r} \cap \gamma Q_{4r, a-t}}([gb]) \, db \, dt \geq \int_{t_0 - 1}^{t_0 + 1} \int_A 1_{Q_{4r} \cap \gamma Q_{4r, a-t}}([gb]) \, db \, dt \geq 2 \text{Vol}(A_{2r}). \]

This proves the lemma. \( \square \)

Proof of (5.4) in Proposition 5.8. By definition of \( \tilde{m} \), we have for any \( \gamma \in \Gamma \) and \( r, t > 0 \),

\[
\tilde{m}(Q_{4r} \cap \gamma Q_{4r, a-t}) = \int_{\mathcal{F}} \left( \int_A 1_{Q_{4r} \cap \gamma Q_{4r, a-t}}([gb]) \right) e^{\psi(G(g^+, g^-))} d\nu_{\psi}(g^+) d\nu_{\psi_0}(g^-) \\
\geq \int_{O_{\mathbb{S}_\mathbb{O}}(\alpha, \gamma_0) \times O_{\mathbb{S}_\mathbb{O}}(\gamma_0, \alpha)} \left( \int_A 1_{Q_{4r} \cap \gamma Q_{4r, a-t}}([gb]) \right) e^{\psi(G(g^+, g^-))} d\nu_{\psi}(g^+) d\nu_{\psi_0}(g^-). \]

Now Lemma 5.16 implies that if \( \gamma \in \Gamma_{u,r}, \|\mu(\gamma)\| > t_0 \) and \((8dr\|\psi\| + \delta) < \psi(\mu(\gamma)) < \delta T - (8dr\|\psi\| - \delta),\) then

\[
\int_0^T \tilde{m}(Q_{4r} \cap \gamma Q_{4r, a-t}) \, dt \\
\geq 2 \text{Vol}(A_{2r}) \int_{O_{\mathbb{S}_\mathbb{O}}(\alpha, \gamma_0) \times O_{\mathbb{S}_\mathbb{O}}(\gamma_0, \alpha)} e^{\psi(G(g^+, g^-))} d\nu_{\psi}(g^+) d\nu_{\psi_0}(g^-) \\
\geq 2 \text{Vol}(A_{2r}) e^{-\|\psi\|t_0} \beta(\nu_{\psi_0}) e^{-\delta \|\psi\|} e^{-\delta t_0} e^{-\kappa \|\psi\|} \beta(\nu_{\psi_0}) \epsilon_1^{-1} \epsilon_2^{-\kappa \|\psi\|} \beta(\nu_{\psi_0}),
\]
where the second inequality follows from the lower bound $e^{\psi(G(g^+ g^-))} \geq e^{-\|\psi\|_\ell_0}$ and the last inequality follows from Lemma 3.4. Therefore,

$$
\int_0^T \sum_{\gamma \in \Gamma} \tilde{m}(Q_{4r} \cap \gamma Q_{4r} a \setminus t) \, dt \geq \int_0^T \sum_{\gamma \in \Gamma, \|\mu(\gamma)\| > \ell_0} \tilde{m}(Q_{4r} \cap \gamma Q_{4r} a \setminus t) \, dt
$$

$$
\gg \sum_{\gamma \in \Gamma, \|\mu(\gamma)\| > \ell_0, \psi(\mu(\gamma)) < \delta T - (8d\|\psi\| + \delta)} e^{-\psi(\mu(\gamma))}.
$$

Since $\#\{\gamma \in \Gamma : \|\mu(\gamma)\| \leq \ell_0\}$ is a finite set, this proves the lemma by Corollary 5.6. □

**Proposition 5.8 yields:**

**Corollary 5.17.** Suppose that for any large $r, s \gg 1$, and $T > 1$,

$$
\int_0^T \sum_{\gamma \in \Gamma} \tilde{m}(Q_r \cap \gamma Q_r a \setminus t) \, dt \asymp \int_0^T \sum_{\gamma \in \Gamma} \tilde{m}(Q_s \cap \gamma Q_s a \setminus t) \, dt
$$

(5.7)

with the implied constant independent of $T$. If $\sum_{\gamma \in \Gamma, R} e^{-\psi(\mu(\gamma))} = \infty$ for some $R > 0$, then for all sufficiently large $r$, we have for any $T > 1$:

$$
\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \tilde{m}(Q_r \cap \gamma Q_r a \setminus t \cap \gamma' Q_r a \setminus t - s) \, dt \, ds \ll \left( \int_0^T \sum_{\gamma \in \Gamma} \tilde{m}(Q_r \cap \gamma Q_r a \setminus t) \, dt \right)^2.
$$

(5.8)

**Proof.** By Proposition 5.8, we get

$$
\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \tilde{m}(Q_r \cap \gamma Q_r a \setminus t \cap \gamma' Q_r a \setminus t - s) \, dt \, ds \ll \left( \int_0^T \sum_{\gamma \in \Gamma} \tilde{m}(Q_{16dr} \cap \gamma Q_{16dr} a \setminus t) \, dt \right)^2,
$$

which implies the claim in view of the hypothesis 5.7. □

**Proof of Proposition 5.4.** We will apply the following version of the Borel-Cantelli lemma:

**Lemma 5.18.** [2, Lem. 2] Let $(\Omega, M)$ be a finite Borel measure space and $\{P_t : t \geq 0\} \subset \Omega$ be such that $(t, \omega) \mapsto 1_{P_t}(\omega)$ is measurable. Suppose that

1. $\int_0^\infty M(P_t) \, dt = \infty$, and
2. there is a constant $C > 0$ such that

$$
\int_0^T \int_0^T M(P_t \cap P_s) \, dt \, ds \leq C \left( \int_0^T M(P_t) \, dt \right)^2
$$

for all $T \gg 1$.
Then we have
\[ M \left\{ \omega \in \Omega : \int_0^\infty 1_{P_t} (\omega) \, dt = \infty \right\} > \frac{1}{C}. \]

Suppose that \( \sum_{\gamma \in \Gamma_u, R} e^{-\psi(\mu(\gamma))} = \infty \) for some \( R > 0 \). Let \( r > R \) be large enough to satisfy Proposition 5.8, and consider \( Q_r = G_r A_r \). As \( M \) commutes with \( A \) and \( Q_r = KA_r, Q_r \) is an \( M \)-invariant subset. Let 
\[ [Q_r] = \Gamma \backslash Q_r / M \subset \Gamma \backslash G / M. \]
Set
\[ M := m_{[Q_r]} \text{ and } P_t := \Gamma \backslash (Q_r \cap \Gamma Q_r a_- t) \subset \Gamma \backslash G / M. \]

We claim that
\[ \int_0^T \int_0^T M(P_t \cap P_s) \, ds \, dt \ll \left( \int_0^T M(P_t) \, dt \right)^2. \] (5.9)

Since \( m \) is assumed to be \( u \)-balanced, Corollary 5.17 applies, and hence
\[ \int_0^T \int_0^T M(P_t \cap P_{t+s}) \, ds \, dt \ll \left( \int_0^T M(P_t) \, dt \right)^2. \] (5.10)

Therefore
\begin{align*}
\int_0^T \int_0^T M(P_t \cap P_s) \, ds \, dt &= 2 \int_0^T \int_t^T M(P_t \cap P_s) \, ds \, dt \\
&\leq 2 \int_0^T \int_0^T M(P_t \cap P_{t+s}) \, ds \, dt \ll \left( \int_0^T M(P_t) \, dt \right)^2,
\end{align*}
proving the claim. Applying Lemma 5.18 with \( M \) and \( P_t \), we conclude that
\[ m \left\{ \{ g \} \in [Q_r] : \int_0^\infty 1_{[Q_r]} ([g] a_t) \, dt = \infty \right\} > 0. \]

It follows that \( \nu_\psi (\{ g^+ \in F : \limsup [g] a_t \neq \emptyset \}) > 0 \) and hence \( \nu_\psi (\Lambda_u) > 0 \). On the other hand, by (5.1), we have \( \sum_{\gamma \in \Gamma(u), R} e^{-\psi(\mu(\gamma))} = \infty \). By the same argument as above, this implies that
\[ \nu_\psi_0 (\{ g^+ \in F : \limsup [g] \exp(t i(u)) \neq \emptyset \}) > 0 \]
and hence \( \nu_\psi_0 (\Lambda_0(u)) > 0 \). This finishes the proof by Proposition 3.1.

**Proof of Theorem 1.5.** The equivalence (1) \( \iff \) (2) follows from Proposition 3.1. The equivalence (2) \( \iff \) (3) follows from Proposition 4.2. The equivalence (3) \( \iff \) (4) \( \iff \) (5) follows from Proposition 4.5. The implication (1) \( \Rightarrow \) (6) is proved in Lemma 5.3, and the implication (6) \( \Rightarrow \) (7) follows from Lemma 5.3 and Proposition 5.4.

**Remark 5.19.** The asymptotic inequality (5.9) shows that if \( m \) is \( u \)-balanced and \( \sum_{\gamma \in \Gamma(u), R} e^{-\psi(\mu(\gamma))} = \infty \) for some \( R > 0 \), then the measure preserving flow 
\( (\Gamma \backslash G / M, m, \{ a_t \}) \) is rationally ergodic and the following
\[ A_T = \frac{1}{m([Q_r])^2} \int_{\Gamma \backslash G / M} \int_0^T 1_{[Q_r]}(xa_t) \, dt \, dm(x) \]
is the asymptotic type of the flow in the sense of [1] and [2, 5].

6. Dichotomy for Anosov groups

Let $\Gamma < G$ be an Anosov subgroup defined as in the introduction. We then have $\mathcal{L}_\Gamma - \{0\} \subset \text{int} \, a^+$, which implies that $\Gamma$ is regular [20, Prop. 4.6]. For each $v \in \text{int} \, \mathcal{L}_\Gamma$, there exists a unique $\psi_v \in a^*$ such that $\psi_v \geq \psi_\Gamma$ and $\psi_v(v) = \psi_\Gamma(v)$, and a unique $(\Gamma, \psi_v)$-conformal measure $\nu_v$ supported on $\Lambda$ ([27], [10]). Moreover, $\{u \in \text{int} \, a^* : \psi_v(u) = \psi_\Gamma(u)\} = \mathbb{R}_+ v$ ([24], [28]). The assignments $v \mapsto \psi_v$ and $v \mapsto \nu_v$ give bijections among int $\mathcal{L}_\Gamma$, $D_\Gamma^+$ and the space of all $\Gamma$-conformal measures supported on $\Lambda$ [18, Prop. 4.4 and Thm. 7.7].

For each $v \in \text{int} \, \mathcal{L}_\Gamma$, we denote by $m_v$ the BMS measure on $\Gamma \backslash G/M$ associated to $(\nu_v, \nu_{(v)})$. Chow and Sarkar proved the following theorem for $f_1, f_2 \in C_\text{c}(\Gamma \backslash G/M)$.

**Theorem 6.1.** [8] Let $\Gamma < G$ be an Anosov subgroup and let $v \in \text{int} \, \mathcal{L}_\Gamma$. There exists $\kappa_v > 0$ such that for any $f_1, f_2 \in C_\text{c}(\Gamma \backslash G/M)$,

$$
\lim_{t \to +\infty} t^{\frac{\text{rank}(G)-1}{2}} \int_{\Gamma \backslash G/M} f_1(x) f_2(x \exp(tv)) \, d\nu_v(x) = \kappa_v \cdot m_v(f_1) m_v(f_2).
$$

Since $m_v$ is $A$-invariant, the above is equivalent to:

$$
\lim_{t \to +\infty} t^{\frac{\text{rank}(G)-1}{2}} \int_{\Gamma \backslash G/M} f_1(x) f_2(x \exp(-tv)) \, d\nu_v(x) = \kappa_v \cdot m_v(f_1) m_v(f_2).
$$

(6.1)

In particular, for any $v \in \text{int} \, \mathcal{L}_\Gamma$, the measure $m_v$ is $v$-balanced.

**Corollary 6.2.** For any $v \in \text{int} \, \mathcal{L}_\Gamma$ and any bounded Borel subset $Q \subset G/M$ with $\tilde{m}_v(\text{int} \, Q) > 0$, we have

$$
\int_0^\infty \sum_{\gamma \in \Gamma} \tilde{m}_v(Q \cap \gamma Q \exp(-tv)) \, dt = \infty \text{ if and only if rank}(G) \leq 3.
$$

**Proof.** Choose $\tilde{f}_1, \tilde{f}_2 \in C_\text{c}(G/M)$ so that $0 \leq \tilde{f}_1 \leq 1_Q \leq \tilde{f}_2$ and $\tilde{m}_v(\tilde{f}_1) > 0.$

For each $i = 1, 2$, let $f_i \in C_\text{c}(\Gamma \backslash G/M)$ be defined by $f_i([g]) = \sum_{\gamma \in \Gamma} \tilde{f}_i(\gamma g)$.

By (6.1), we get

$$
\int_{\Gamma \backslash G/M} f_i([g] \exp(tv)) f_i([g]) \, d\nu_v[g] = \int_{G/M} \sum_{\gamma \in \Gamma} \tilde{f}_i(g \exp(tv)) \tilde{f}_i(g) \, d\tilde{m}_v(g) \asymp t^{(-\text{rank}(G)+1)/2}.
$$

The claim follows since $\int_1^\infty t^{(-\text{rank}(G)+1)/2} \, dt = \infty$ if and only if rank$(G) \leq 3$.

By Theorem 1.5, the following theorem implies Theorem 1.7:

**Theorem 6.3.** Let $v \in \text{int} \, \mathcal{L}_\Gamma$ and $u \in \text{int} \, a^+$. The following are equivalent:
(1) \( \text{rank}(G) \leq 3 \) and \( Ru = Rv \);
(2) \( \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi_v(\mu(\gamma))} = \infty \) for some \( R > 0 \).

**Proof.** Suppose that \( \text{rank}(G) \leq 3 \) and \( u = v \). Let \( a_t = \exp(tv) \). Let \( Q_r \subset G/M \) be as in (5.3) of Proposition 5.8. Then for \( \delta = \psi_v(v) > 0 \), we have

\[
\int_0^T \int_0^T \sum_{\gamma, \gamma' \in \Gamma} \tilde{m}_v(Q_r \cap \gamma Q_r a_{-t} \cap \gamma' Q_r a_{-t-s}) \, dt \, ds \leq \left( \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi_v(\mu(\gamma))} \right)^2.
\]

Set \( Q_r^- := \cap_{0 \leq s < r/10} Q_r a_{-s} \). We may assume that \( m_v(\text{int} Q_r^-) > 0 \) by increasing \( r \). Note that

\[
\frac{r}{10} \int_0^T \sum_{\gamma \in \Gamma} \tilde{m}_v(Q_r^- \cap \gamma Q_r^- a_{-t}) \, dt \leq \int_0^T \int_{0 \leq s \leq r/10} \sum_{\gamma \in \Gamma} \tilde{m}_v(Q_r \cap \gamma (Q_r \cap Q_r a_{-s}) a_{-t}) \, ds \, dt.
\]

By (6.2), we get

\[
\int_0^T \sum_{\gamma \in \Gamma} \tilde{m}_v(Q_r^- \cap \gamma Q_r^- a_{-t}) \, dt \leq \left( \sum_{\gamma \in \Gamma_{u,R}, \psi_v(\mu(\gamma)) < \delta T} e^{-\psi_v(\mu(\gamma))} \right)^2.
\]

Hence by Corollary 6.2, we get \( \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi_v(\mu(\gamma))} = \infty \).

Now suppose that \( \sum_{\gamma \in \Gamma_{u,R}} e^{-\psi_v(\mu(\gamma))} = \infty \) for some \( R > 0 \). By Lemma 5.2, \( \psi_v(u) = \psi_T(u) \). This implies \( \mathbb{R}v = Ru \), as \( \mathbb{R}v \) is the unique line where \( \psi_v \) and \( \psi_T \) are equal to each other. This also implies \( u \in \text{int} \mathcal{L}_T \). By Proposition 5.8, it follows that \( \int_0^\infty \sum_{\gamma \in \Gamma} \tilde{m}_v(Q_r \cap \gamma Q_r a_{-t}) \, dt = \infty \). Hence \( \text{rank}(G) \leq 3 \) by Corollary 6.2.

**Remark 6.4.** It follows from Theorem 6.3 that when \( \text{rank} G \leq 3 \) and \( v \in \text{int} \mathcal{L}_T \), the flow \( (\Gamma \backslash G/M, m_v, \exp(tv)) \) is rationally ergodic by Remark 5.19.

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