ERGODIC DECOMPOSITIONS OF BURGER-ROBLIN MEASURES ON ANOSOV HOMOGENEOUS SPACES.

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Abstract. Let $G$ be a connected semisimple real algebraic group and $\Gamma$ a Zariski dense Anosov subgroup of $G$. Let $N$ be a maximal horospherical subgroup of $G$ and $P$ be the normalizer of $N$ with a fixed Langlands decomposition $P = MAN$. We show that each $N$-invariant Burger-Roblin measure on $\Gamma \backslash G$ decomposes into at most $[M : M^c]$-number of $N$-ergodic components, and deduce the following refinement of the main result of [15]: the space of all non-trivial $N$-invariant ergodic and $M^cA$-quasi-invariant Radon measures on $\Gamma \backslash G$, modulo the translations by $M$, is homeomorphic to $\mathbb{R}^{\text{rank } G - 1}$.

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1. INTRODUCTION

Let $G$ be a connected real semisimple algebraic group, i.e., the identity component of the group of real points of a semisimple algebraic group defined over $\mathbb{R}$. Let $N < G$ be a maximal horospherical subgroup of $G$, which is unique up to conjugation. In this paper, we are interested in the study of $N$-invariant ergodic measures on an Anosov homogeneous space $\Gamma \backslash G$. By an Anosov homogeneous space, we mean a homogeneous space $\Gamma \backslash G$ where $\Gamma$ is an Anosov subgroup of $G$. For instance, what are examples of nontrivial $N$-invariant ergodic measures on $\Gamma \backslash G$? Let $P$ denote the normalizer of $N$, which is a minimal parabolic subgroup of $G$. Fix a Langlands decomposition $P = MAN$ where $A$ is a maximal real split torus and $M$ is a maximal compact subgroup of $P$ commuting with $A$. In [15], we proved that the $NM$-invariant Burger-Roblin measures on $\Gamma \backslash G$, parametrized by $\mathbb{R}^{\text{rank } G - 1}$,

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are all $NM$-ergodic. In this paper, we show that these Burger-Roblin measures decompose into at most $[M : M^0]$-number of $\nu$-ergodic components, where $M^0$ denotes the identity component of $M$. In particular, when $M$ is connected, these measures are $\nu$-ergodic. Our work extends the earlier result of Burger, Roblin and Winter for geometrically finite groups in rank one cases ([4], [19], [27]). We also refer to ([1], [2], [14], [23], [12], [13], [16], etc.) for related results on geometrically infinite groups.

We begin by recalling the definition of an Anosov subgroup. Denoting by $F := G/P$ the Furstenberg boundary, let $F^{(2)}$ be the unique open $G$-orbit in $F \times F$. A Zariski dense discrete subgroup $\Gamma < G$ is called an Anosov subgroup if it is a finitely generated word hyperbolic group which admits a $G$-equivariant embedding $\zeta$ of the Gromov boundary $\partial G$ into $F$ such that $(\zeta(x), \zeta(y)) \in F^{(2)}$ for all $x \neq y$ in $\partial G$ (see [11], [7], [10], [26]). When $G$ has rank one, $\Gamma$ is Anosov if and only if $\Gamma$ is a convex-cocompact subgroup of $G$.

Denote by $a$ the Lie algebra of $A$ and fix a positive Weyl chamber $a^+ \subset a$ so that $\log N$ is the sum of positive root subspaces. Fix a maximal compact subgroup $K$ of $G$ so that the Cartan decomposition $G = K(\exp a^+)K$ holds. Let $\Lambda$ denote the limit set of $\Gamma$, which is the unique $\Gamma$-minimal subset of $F$ with $\dim_{\Gamma} \Lambda = 1$. For a linear form $\psi \in a^*$, a Borel probability measure $\nu$ on $\Lambda$ is called a $(\Gamma, \psi)$-PS measure for all $\gamma \in \Gamma$ and $\xi \in F$,

\[
\frac{d\gamma_* \nu}{d\nu}(\xi) = \exp(\log \beta_A(e, \gamma))
\]

where $\beta_A : F \times G \times G \to A$ denotes the $A$-valued Busemann function (Def. 2.3). We call $\nu$ a $\Gamma$-PS measure if it is a $(\Gamma, \psi)$-measure for some $\psi \in a^*$.

Let $L_\Gamma \subset a^*$ denote the limit cone of $\Gamma$, and $\psi_T : a \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$ (see Def. 3.2). Set

\[
D^*_T := \{ \psi \in a^* : \psi \geq \psi_T, \psi(v) = \psi_T(v) \text{ for some } v \in \text{int } L_\Gamma \}.
\]

For each linear form $\psi \in D^*_T$, Quint constructed a $(\Gamma, \psi)$-Patterson-Sullivan measure, say, $\nu_\psi$. For an Anosov group $\Gamma$, it was shown in [15] that the map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D^*_T$ and the space of all $\Gamma$-PS measures. We denote by $\nu_\psi$ the $M$-invariant lift of $\nu_\psi$ on $F \simeq K/M$ to $K$ by abuse of notation. The Burger-Roblin measure $\tilde{m}^{BR}_\psi$ on $\Gamma \backslash G$ is induced from the following $\Gamma$-invariant measure $\tilde{m}^{BR}_\psi$ on $G$: for $g = k(\exp b)n \in KAN$,

\[
d\tilde{m}^{BR}_\psi(g) = e^{\psi(b)}dn \, db \, d\nu_\psi(k)
\]

where $dn$ and $db$ are Lebesgue measures on $N$ and $a$ respectively.

**Theorem 1.1.** For each $\psi \in D^*_T$, there exists an $N$-invariant ergodic and $AM^0$ quasi-invariant measure $m_\psi$ on $\Gamma \backslash G$ such that

\[
m_\psi^{BR} = \sum_{s \in M^0 \backslash M} m_\psi \cdot s
\]

where $(m_\psi \cdot s)(B) = m_\psi(Bs)$ for any Borel subset $B \subset \Gamma \backslash G$. In particular, if $M$ is connected, $m_\psi^{BR}$ is $N$-ergodic.
We call an $N$-invariant measure on $\Gamma \backslash G$ non-trivial if it is supported on the unique $P$-minimal subset of $\Gamma \backslash G$. In view of our earlier work [15], Theorem 1.1 implies:

**Corollary 1.2.** For any Anosov subgroup $\Gamma < G$, the space of all non-trivial $N$-invariant ergodic and $AM^\circ$-quasi-invariant Radon measures on $\Gamma \backslash G$, up to constant multiples and translations by elements of $M$, is homeomorphic to $\mathbb{R}^{\text{rank} G - 1}$.

We also describe the $A$-ergodic decomposition of the Bowen-Margulis-Sullivan measures $m_{\psi}^{\text{BMS}}$: there exists an $A$-invariant ergodic and $AM^\circ$ semi-invariant measure $\mu_{\psi}$ such that $m_{\psi}^{\text{BMS}} = \sum_{s \in M/M^\circ} \mu_{\psi}.s$ (see Theorem 3.3).

We define the closed subgroup, say $E_{\nu_{\psi}}$ of $AM$ consisting of $\nu_{\psi}$-essential values (Def. 5.1), and the key point of our proof of Theorem 1.1 is to verify that the connected component $M^\circ$ is contained in $E_{\nu_{\psi}}$ (Proposition 6.2); we rely on the $NM$-ergodicity of $m_{\psi}^{\text{BR}}$ [15]. We begin by introducing the generalized length spectrum $\hat{\lambda}(\Gamma) < AM$ of a Zariski dense discrete subgroup $\Gamma < G$ (Def. 2.4) and deduce from the work of Guivarch and Raugi [8] that the closed subgroup generated by $\hat{\lambda}(\Gamma)$ contains $AM^\circ$ (Corollary 2.9).

We then show that $E_{\nu_{\psi}}$ contains the generalized length spectrum of some Zariski dense subgroup $\Gamma_{\psi}$ of $\Gamma$ (Proposition 6.5). This involves establishing the $A$-ergodic decomposition of $m_{\psi}^{\text{BMS}}$ as in Theorem 3.3 and equicontinuity of a certain family of Busemann functions as described in Proposition 4.4.

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## 2. Generalized length spectrum and transitivity groups

Let $G$ be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. We fix, once and for all, a Cartan involution $\theta$ of the Lie algebra $\mathfrak{g}$ of $G$ and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of $\theta$, respectively. We denote by $K$ the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of $A$ in $K$ is denoted by $M$ and we set $N$ to be the contracting horospherical subgroup: for $a \in \text{int } A^+$, $N = \{ g \in G : a^{-n} ga^n \rightarrow e \text{ as } n \rightarrow +\infty \}$. Note that $\log N$ is the sum of all positive root subspaces for our choice of $A^+$. Similarly, we also consider the expanding horospherical subgroup $N^+$: for $a \in \text{int } A^+$, $N^+ = \{ g \in G : a^n ga^{-n} \rightarrow e \text{ as } n \rightarrow +\infty \}$. We set $P = MAN$ which is a minimal parabolic subgroup of $G$. The quotient $\mathcal{F} = G/P$ is known as the Furstenberg boundary of $G$ and is isomorphic to $K/M$. We let $\Lambda$ the unique $\Gamma$-minimal subset of $\mathcal{F}$, called the limit set of $\Gamma$. 

We fix a left $G$-invariant and right $K$-invariant Riemannian metric $d$ on $G$. For a subgroup $H < G$, we denote by $H_\varepsilon = \{ h \in H : d(e, h) < \varepsilon \}$. We sometimes use the notation $H_{O(\varepsilon)}$ to mean $H_{C\varepsilon}$ for some absolute constant $C > 0$.

We fix an element $w_0$ of the normalizer of $a$ such that $\text{Ad}_{w_0} a^+ = -a^+$. The opposition involution $i : a \rightarrow a$ is defined as $i(u) = -\text{Ad}_{w_0} u$.

**Definition 2.1** (Visual map). For each $g \in G$, we define
$$ g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P. $$

For all $g \in G$ and $m \in M$, observe that $g^\pm = (gm)^\pm = g(e^\pm)$. Let $\mathcal{F}(2)$ denote the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$:
$$ \mathcal{F}(2) = G(e^+, e^-) = \{ (g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G \}. $$

**AM-valued cocycles.** The product map $N^+ \times P \rightarrow G$ is a diffeomorphism onto its image which is Zariski open and dense in $G$. Hence for each $\xi \in N^+ e^+$, we can define $h_\xi \in N^+$ to be the unique element such that $\xi = h_\xi e^+$. Similarly, the product map $K \times A \times N \rightarrow G$ is a diffeomorphism, giving the Iwasawa decomposition $G = KAN$. We can therefore define $k_\xi \in K$ to be the unique element such that $h_\xi k_\xi AN$.

**Definition 2.2** (Bruhat cocycle and Iwasawa cocycle). Let $g \in G$ and $\xi \in \mathcal{F}$ be such that $\xi, g\xi \in N^+ e^+$.

1. We define the Bruhat cocycle $b(g, \xi) \in AM$ to be the unique element satisfying
$$ gh_\xi \in N^+ b(g, \xi)N. $$

   Note that $\xi \in N^+ e^+$ allows us to get $h_\xi \in N^+$ and $g\xi \in N^+ e^+$ implies $gh_\xi \in N^+ AMN$.

2. We define the Iwasawa cocycle $\sigma^AM(g, \xi) \in AM$ to be the unique element satisfying
$$ gk_\xi \in k_\xi \sigma^AM(g, \xi)N. $$

Note that $gh_\xi \in h_\xi b(g, \xi)N$ and $gk_\xi = k_\xi \sigma^AM(g, \xi)N$.

We note that for any $\xi \in \mathcal{F}$, there exists a unique element $\sigma(g, \xi) \in A$ such that $gk_\xi \in K\sigma(g, \xi)N$ where $\xi = [k_\xi] \in K/M = \mathcal{F}$. The logarithm of $\sigma(g, \xi)$ was defined as the Iwasawa cocycle in [15]. In order to define the $AM$-valued Iwasawa cocycle, it is necessary to choose a section of the projection $K \simeq G/AN \rightarrow K/M \simeq G/P$. In the above definition, we have used a section $s : G/P \rightarrow G/AN$ such that $s(hP) = hAN$ for all $h \in N^+$, so that it is continuous on $N^+ e^+ \subset \mathcal{F}$.

It follows that for each fixed $g \in G$, the maps $\xi \mapsto b(g, \xi)$ and $\xi \mapsto \sigma^AM(g, \xi)$ are continuous on the set $\{ \xi \in N^+ e^+ : g\xi \in N^+ e^+ \}$.

**Definition 2.3** ($AM$-valued Busemann map). For $(\xi, g_1, g_2) \in \mathcal{F} \times G \times G$ such that $\xi, g_1^{-1} \xi, g_2^{-1} \xi \in N^+ e^+$, we define
$$ \beta^AM(\xi, g_1, g_2) := \sigma^AM(g_1^{-1} , \xi)\sigma^AM(g_2^{-1} , \xi)^{-1}. $$
We define $\beta^A$ and $\beta^M$ as the projections of $\beta^{AM}$ to $A$ and $M$ respectively. For simplicity, we sometimes drop the superscript $AM$ when its meaning is clear from the context.

For fixed $g_1, g_2 \in G$, the map $\xi \mapsto \beta_\xi(g_1, g_2)$ is continuous on $\{\xi \in N^+e^+ : g_1^{-1} \xi, g_2^{-1} \xi \in N^+e^+\}$. We have the following whenever both sides are defined: for any $g_1, g_2, g_3 \in G$ and $\xi \in F$,

(1) (cocycle identity) $\beta_\xi(g_1, g_3) = \beta_\xi(g_1, g_2) \beta_\xi(g_2, g_3)$;

(2) (equivariance) $\beta_{g\xi}(g_3g_1, g_3g_2) = \beta_\xi(g_1, g_2)$.

**Generalized length spectrum.** Recall that for any loxodromic element $g \in G$, there exists $\varphi \in G$ such that $g = \varphi am \varphi^{-1}$

for some element $am \in \text{int } A^+ M$. Moreover such $\varphi$ belongs to a unique coset in $G/AM$. We set

$y_g^+ := \varphi^+ \in F$

which is called the attracting fixed point of $g$. The element $a \in \text{int } A^+$ is uniquely determined and called the Jordan projection of $g$. We denote it by $\lambda(g)$. On the other hand, only the conjugacy class of $m \in M$ is determined.

In the rest of this section, we fix a discrete subgroup $\Gamma$ of $G$. We define

$\Gamma^* := \{\gamma \in \Gamma : \text{there exists } \varphi \in N^+N \text{ with } \gamma \in \varphi(\text{int } A^+)M\varphi^{-1}\}$.

Note that if $\gamma \in \Gamma$ and $y_\gamma^+ \in N^+e^+$, then $\gamma \in \Gamma^*$. When $\Gamma$ is Zariski dense, the set of loxodromic elements is known to be Zariski dense in $G$ [3]. It follows that $\Gamma^*$ is Zariski dense in $G$ as well. For $\gamma \in \Gamma^*$, we define its generalized Jordan component $\hat{\lambda}(\gamma) \in \text{int } A^+ M$ to be the unique element such that

$\gamma = \varphi \hat{\lambda}(\gamma) \varphi^{-1}$ for some $\varphi \in N^+N$.

**Definition 2.4.** We call the following set the *generalized length spectrum* of $\Gamma$:

$\hat{\lambda}(\Gamma) := \{\hat{\lambda}(\gamma) \in AM : \gamma \in \Gamma^*\}$.

We denote by $s(\Gamma)$ the closed subgroup of $AM$ generated by $\hat{\lambda}(\Gamma)$.

**Lemma 2.5.** For all $\gamma \in \Gamma^*$, we have

$\hat{\lambda}(\gamma) = b(\gamma, y_\gamma^+) = \beta_{y_\gamma^+}(e, \gamma)$.

**Proof.** Since $\gamma \in \Gamma^*$, we have $\gamma = \varphi \hat{\lambda}(\gamma) \varphi^{-1}$ for some $\varphi = hn$, where $h \in N^+$ and $n \in N$. Set $\xi := y_\gamma^+ = \varphi^+$. In particular, $h_\xi = h$ and $h \in k_\xi AN$. The defining relations for $b(\gamma, \xi)$ and $\beta^{AM}_\xi(e, \gamma)$ are

$\gamma h \in hb(\gamma, \xi)N$ and $\gamma k_\xi \in k_\xi \beta_{\xi}(e, \gamma)N$.

Now observe that

$\gamma h = \varphi \hat{\lambda}(\gamma) \varphi^{-1} h = hn \hat{\lambda}(\gamma)n^{-1} \in h\hat{\lambda}(\gamma)N$ and

$\gamma k_\xi = \varphi \hat{\lambda}(\gamma) \varphi^{-1} k_\xi = k_\xi(k_\xi^{-1} h)n \hat{\lambda}(\gamma)n^{-1}(h^{-1} k_\xi) \in k_\xi \hat{\lambda}(\gamma)N$. 

Therefore \( \hat{\lambda}(\gamma) = b(\gamma, \xi) = \beta_{\xi}(e, \gamma). \)

For each \( \xi \in \Lambda \cap N^+e^+ \), we define \( b_\xi(\Gamma) \) to be the closed subgroup of \( AM \) generated by all \( b(\gamma, \xi) \) where \( \gamma \in \Gamma \) and \( \gamma \xi \in N^+e^+ \).

Lemma 2.6. The subgroup \( b_\xi(\Gamma) < AM \) is independent of \( \xi \in \Lambda \cap N^+e^+ \).

\textbf{Proof.} Let \( \xi_1, \xi_2 \in \Lambda \cap N^+e^+ \). To show that \( b_{\xi_1}(\Gamma) = b_{\xi_2}(\Gamma) \), it suffices to check that \( b(\gamma, \xi_2) \in b_{\xi_1}(\Gamma) \) for any \( \gamma \in \Gamma \) such that \( \gamma \xi_2 \in N^+e^+ \). Since \( \Lambda \) is \( \Gamma \)-minimal, there exists a sequence \( \gamma_n \in \Gamma \) such that \( \lim_{n \to \infty} \gamma_n \xi_1 = \xi_2 \). Since \( N^+e^+ \) is open and \( \xi_2, \gamma \xi_2 \in N^+e^+ \), we have \( \gamma_n \xi_1, \gamma \gamma_n \xi_1 \in N^+e^+ \) for all large \( n \) and \( b(\gamma \gamma_n, \xi_1) = b(\gamma, \gamma_n \xi_1) b(\gamma_n, \xi_1) \). Hence

\[
b(\gamma, \xi_2) = \lim_{n \to \infty} b(\gamma, \gamma_n \xi_1) = \lim_{n \to \infty} b(\gamma \gamma_n, \xi_1) b(\gamma_n, \xi_1)^{-1} \in b_{\xi_1}(\Gamma),
\]

from which the lemma follows. \( \square \)

By Lemma 2.6, we may define

\[
b(\Gamma) := b_\xi(\Gamma) \quad \text{for any} \quad \xi \in \Lambda \cap N^+e^+.
\]

In the rest of this section, we assume that \( \Gamma \) is a Zariski dense subgroup of \( G \) containing a loxodromic element contained in \( \text{int} A^+M \).

Lemma 2.7. We have \( b(\Gamma) = s(\Gamma) \).

\textbf{Proof.} We first claim that \( b(\Gamma) \subseteq s(\Gamma) \). By Lemma 2.6, it suffices to show that \( b(\gamma, e^+) \in s(\Gamma) \) for any \( \gamma \in \Gamma \) with \( \gamma e^+ \in N^+e^+ \). Set \( s_0 := a_0 m_0 \in \Gamma \cap \text{int} A^+M - M \). Then for all sufficiently large \( n \), \( s_0^n \gamma \) is a loxodromic element and \( x_n := y_{s_0^n \gamma}^+ \) converges to \( e^+ \) as \( n \to \infty \). Since \( y_{s_0^n \gamma}^+ \in N^+e^+ \), we have \( s_0^n \gamma \in \Gamma^* \) for all large \( n \). Now the claim follows from

\[
b(\gamma, e^+) = \lim_{n \to \infty} b(\gamma, x_n) = \lim_{n \to \infty} b(s_0^n, \gamma x_n)^{-1} b(s_0^n \gamma, x_n) = \lim_{n \to \infty} \hat{\lambda}(s_0^n)^{-1} \hat{\lambda}(s_0^n \gamma) \in s(\Gamma)
\]

We next claim \( s(\Gamma) \subseteq b(\Gamma) \). Let \( \gamma \in \Gamma^* \) be arbitrary. Note that \( y_\gamma^+ \in N^+e^+ \). By Lemma 2.5, \( \hat{\lambda}(\gamma) = b(\gamma, y_\gamma^+) \in b_{y_\gamma^+}(\Gamma) \). Since \( b(\Gamma) = b_{y_\gamma^+}(\Gamma) \) by Lemma 2.6, we have \( \hat{\lambda}(\gamma) \in b(\Gamma) \), proving the claim. \( \square \)

The following is proved in [8, Thm 1.9]:

**Proposition 2.8.** We have \( b(\Gamma) = b(g^{-1} \Gamma g) \) for all \( g \in G \) with \( g^\pm \in \Lambda \), and \( b(\Gamma) \) is a normal subgroup of \( AM \) containing \( AM^\circ \).

Hence we deduce the following from Lemma 2.7 and Proposition 2.8.

**Corollary 2.9.** The subgroup \( s(\Gamma) \) is a normal subgroup of \( AM \) containing \( AM^\circ \).
Definition 2.10 (Transitivity group). For \( g \in G \) with \( g^\pm \in \Lambda \), define the subset \( \mathcal{H}_T^\pm(g) < AM \) as follows: \( a_m \in \mathcal{H}_T^\pm(g) \) if and only if there exist \( \gamma \in \Gamma \) and a sequence \( h_i \in N \cup N^+, i = 1, \ldots, k \) such that
\[(gh_1h_2 \ldots h_r)^\pm \in \Lambda \text{ for all } 1 \leq r \leq k \quad \text{and} \quad \gamma gh_1h_2 \ldots h_k = gam.
\]
It is not hard to check that \( \mathcal{H}_T^\pm(g) \) is a subgroup (cf. [27, Lem. 3.1]); it is called the strong transitivity subgroup. We set \( \mathcal{H}_T^w(g) \) to be the projection of \( \mathcal{H}_T^\pm(g) \) to \( M \) and call it the weak transitivity subgroup.

The notion of transitivity groups was used in [27] in which the following corollary was proved for rank one case by a different approach.

Corollary 2.11. For any \( g \in G \) with \( g^\pm \in \Lambda \), the closure of \( \mathcal{H}_T^\pm(g) \) contains \( AM^\circ \). In particular, \( M^\circ \subset \mathcal{H}_T^w(g) \).

Proof. By Proposition 2.8, it suffices to show that if \( g \in G \) satisfies \( g^\pm \in \Lambda \), then \( b(g^{-1} \Gamma g) \) is contained in the closure of \( \mathcal{H}_T^\pm(g) \). By Lemma 2.6, it is again enough to show that, fixing \( \xi \in g^{-1} \Lambda \cap N^+ e^+ \), \( b(g^{-1} \gamma g, \xi) \) is contained in \( \mathcal{H}_T^\pm(g) \) for any \( \gamma \in \Gamma \). If \( \xi = he^+ \) for \( h \in N^+ \) and \( am = b(g^{-1} \gamma g, \xi) \), then \( g^{-1} \gamma h = h_1 amn_1 \) for some \( h_1 \in N^+ \) and \( n_1 \in N \). We can rewrite it as \( gam = \gamma ghn_2h_2 \) where \( n_2 \in N \) and \( h_2 \in N^+ \). Observe that \( (\gamma gh)^{-1} = \gamma g^{-1} \in \Lambda \) and \( (\gamma gh)^{+} = \gamma g \xi \in \Lambda \) as \( \xi \in g^{-1} \Lambda \). Moreover, \( (\gamma ghn_2)^{-} = (\gamma ghn_2h_2)^{-} = g^{-1} \in \Lambda \) and \( (\gamma ghn_2)^{+} = (\gamma gh)^{+} \in \Lambda \). Therefore \( am \in \mathcal{H}_T^\pm(g) \). This proves the claim. \( \square \)

3. A-ergodic decompositions of BMS-measures

Let \( \Gamma < G \) be an Anosov subgroup. The limit cone \( \mathcal{L}_\Gamma \subset \mathfrak{a}^+ \) may be defined as the smallest closed cone containing the Jordan components \( \lambda(\gamma) \), \( \gamma \in \Gamma \). It is a convex cone with non-empty interior [3].

Definition 3.1 (Cartan projection). For each \( g \in G \), there exists a unique element \( \mu(g) \in \mathfrak{a}^+ \), called the Cartan projection of \( g \), such that
\[ g \in K \exp(\mu(g)) K. \]

Definition 3.2 (Growth indicator function). The growth indicator function \( \psi_T : \mathfrak{a}^+ \rightarrow \mathbb{R} \cup \{-\infty\} \) is defined as a homogeneous function, i.e., \( \psi_T(tu) = t\psi_T(u) \), such that for any unit vector \( u \in \mathfrak{a}^+ \),
\[ \psi_T(u) := \inf_{u \in \mathcal{C}_{\text{open cones}} \subset \mathfrak{a}^+} \limsup_{t \to \infty} \frac{1}{t} \log \# \{ \gamma \in \Gamma : \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \leq t \}. \]

We consider \( \psi_T \) as a function on \( \mathfrak{a} \) by setting \( \psi_T = -\infty \) outside of \( \mathfrak{a}^+ \).

Set
\[ D_T^t := \{ \psi \in \mathfrak{a}^* : \psi \geq \psi_T, \psi(u) = \psi_T(u) \text{ for some } u \in \text{int} \mathcal{L}_\Gamma \}. \]

We fix \( \psi \in D_T^t \) in the rest of this section. Denote by \( \nu_\psi \) the unique \((\Gamma, \psi)\) PS-measure on \( \Lambda \) ([18, Thm. 4.10], [22], see also [15, Thm 4.3]). The map
\[ g \mapsto (g^+, g^-, \beta_{g^+}^{AM}(e, g)) \]
gives a Borel isomorphism between $G$ and $\mathcal{F}^{(2)} \times AM$. The following defines a left $\Gamma$-invariant and right $AM$-invariant measure on $G$:

$$d\tilde{m}_{BMS}^\psi(g) = e^{\psi(\log \beta^A_+ (e,g)+1\log \beta^A_- (e,g))} d\nu_\psi(g^+)d\nu_{\psi0i}(g^-) \, da \, dm.$$  

We denote by $m_{BMS}^\psi$ the measure on $\Gamma \backslash G$ induced by $\tilde{m}_{BMS}^\psi$, we call this the Bowen-Margulis-Sullivan measure (associated to $\psi$). Note that its support is equal to

$$\Omega := \{ x \in \Gamma \backslash G : x^\pm \in \Lambda \}.$$  

In [15], we showed that $m_{BMS}^\psi$ is an $AM$-ergodic infinite measure. In this section, we show:

**Theorem 3.3.** Let $\Gamma < G$ be an Anosov subgroup. There exists an $AM^o$-invariant and $A$-ergodic measure $\mu_\psi$ on $\Omega$ such that

$$m_{BMS}^\psi = \sum_{s \in M_0 \setminus M} \mu_{\psi\cdot s}.$$  

In particular, if $M$ is connected, then $m_{BMS}^\psi$ is $A$-ergodic.

Set

$$\tilde{\Omega} := \{ g \in G : \Gamma g \in \Omega \}.$$  

Let $B$ denote the Borel $\sigma$-algebra on $G$. We set

$$\Sigma_\pm := \{ B \cap \tilde{\Omega} : B \in B \text{ with } B = \Gamma B A N^\pm \} \quad \text{and} \quad \Sigma := \Sigma_+ \wedge \Sigma_-.$$  

In other words, $B \in \Sigma$ if there exists $B_\pm \in \Sigma_\pm$ such that $m_{BMS}^\psi(B \Delta B_\pm) = 0$. Let

$$\Sigma_0 := \{ B \cap \tilde{\Omega} : B \in B \text{ with } B = \Gamma B A M^o \}.$$  

The following is a main technical ingredient of the proof of Theorem 3.3:

**Lemma 3.4.** We have $\Sigma \subset \Sigma_0 \mod m_{BMS}^\psi$; that is, for all $B \in \Sigma$, there exists $B_0 \in \Sigma_0$ such that $m_{BMS}^\psi(B - B_0) = 0$.

This lemma follows if we show that any bounded $\Sigma$-measurable function on $\tilde{\Omega}$ is $\Sigma_0$-measurable modulo $m_{BMS}^\psi$.

Let $f$ be any bounded $\Sigma$-measurable function on $\tilde{\Omega}$. We may assume without loss of generality that $f$ is strictly left $\Gamma$-invariant and right $A$-invariant [28, Prop. B.5]. There exist bounded $\Sigma^\pm$-measurable functions $f_\pm$ such that $f = f_\pm$ for $m_{BMS}^\psi$-a.e. We may assume that $f_\pm$ satisfy $f_\pm(gn) = f_\pm(g)$ whenever $g, gn \in \tilde{\Omega}$ with $n \in N^\pm$. Set

$$E := \left\{ g A M : f|_{g A M} \text{ is measurable and} \right.$$  

$$\left. f(gm) = f_+(gm) = f_-(gm) \right\} \subset \tilde{\Omega}/AM. $$

By Fubini’s theorem, $E$ has a full measure on $\tilde{\Omega}/AM \simeq \Lambda^{(2)}$ with respect to the measure $d\nu_\psi \, d\nu_{\psi0i}$. For all small $\varepsilon > 0$, define functions $f^\varepsilon, f^\varepsilon_\pm : \tilde{\Omega} \to \mathbb{R}$
Note that if \( gAM \in E \), then \( f^\varepsilon \) and \( f^\varepsilon_\pm \) are continuous and identical on \( gAM \). Moreover, as \( M \) normalizes subgroups \( A \) and \( N^\pm \), \( f^\varepsilon \) is strictly left \( \Gamma \)-invariant, right \( A \)-invariant and \( f^\varepsilon_\pm(gn) = f^\varepsilon_\pm(g) \) whenever \( g, gn \in \tilde{\Omega} \) with \( n \in N^\pm \). Using the isomorphism between \( \tilde{\Omega}/AM \) and \( \Lambda^2 \) given by \( gAM \mapsto (g^+, g^-) \), we may consider \( E \) as a subset of \( \Lambda^2 \). We then define
\[
E^- := \{ \xi \in \Lambda : (\xi, \eta') \in E \text{ for } \nu_{\psi\eta}' \text{-a.e. } \eta' \in \Lambda \};
\]
\[
E^+ := \{ \eta \in \Lambda : (\xi', \eta) \in E \text{ for } \nu_{\psi\xi} \text{-a.e. } \xi' \in \Lambda \}.
\]
Then \( E^- \) is \( \nu_{\psi}\eta \)-conull and \( E^+ \) is \( \nu_{\psi\xi}' \)-conull by Fubini’s theorem. Set
\[
E^-_\eta := \{ \xi \in \Lambda : (\xi, \eta) \subset E \} \quad \text{and} \quad E^+_{\xi'} := \{ \eta \in \Lambda : (\xi', \eta) \subset E \}.
\]
Note that \( E^-_\eta \) is \( \nu_{\psi}\eta \)-conull for all \( \eta \in E^+ \) and that \( E^+_{\xi'} \) is \( \nu_{\psi\xi}' \)-conull for all \( \xi' \in E^- \).

**Lemma 3.5.** Let \( g \in \tilde{\Omega} \) be such that \( gAM \in E \) and \( g^\pm \in E^\pm \). Then for any \( \varepsilon > 0 \), \( f^\varepsilon(gm_0) = f^\varepsilon(g) \) for all \( m_0 \in \mathcal{H}_\Gamma^w(g) \). Moreover, \( f^\varepsilon|_{gAM} \) is \( M^\varepsilon \)-invariant for any \( \varepsilon > 0 \).

**Proof.** We apply similar arguments as in [27, Lem. 4.1]. For any \( m \in \mathcal{H}_\Gamma^w(g) \), there exists \( \gamma \in G \), a sequence \( h_1, \ldots, h_k \in N \cup N^+ \), and \( a \in A \) such that
\[
gh_1 \cdots h_k \in \tilde{\Omega} \quad \text{for all } 1 \leq i \leq k \quad \text{and} \quad gh_1 \cdots h_k = \gamma gam.
\]
If \( gh_1 \cdots h_i AM \in E \) for all \( 1 \leq i \leq k \) in addition, we call such a sequence permissible. In this case,
\[
f^\varepsilon(gm) = f^\varepsilon(\gamma gam) = f^\varepsilon(gh_1 \cdots h_{r-1}) = f^\varepsilon(g) \quad \text{using the } N^+ \text{-invariance of } f^\varepsilon_+, \text{the invariance of } f \text{ by } \Gamma \text{ and } A \text{ and the fact that all three agree on } E.
\]
In general, we need an approximation of the sequence by permissible ones.

Let \( m \in \mathcal{H}_\Gamma^w(g) \) be arbitrary. Let \( n_i \in N \), \( h_i \in N^+ \), \( 1 \leq i \leq k \), \( a \in A \) and \( \gamma \in \Gamma \) be such that for each \( 1 \leq i \leq k \),
\[
\xi_i := gm_i h_1 \cdots h_i^- \in \Lambda, \eta_i := gm_i h_1 \cdots h_i^+ \in \Lambda, gn_i h_1 \cdots n_i h_k = \gamma gam.
\]
We also set \( \xi_0 := g^- \) and \( \eta_0 := g^+ \). For \( 0 \leq i \leq k \), we now define sequences \( \{ \xi_i^\ell : \ell \in \mathbb{N} \} \) and \( \{ \eta_i^\ell : \ell \in \mathbb{N} \} \). Set \( \xi_i^0 := \xi_0 \) and \( \eta_i^0 := \eta_0 \) for all \( \ell \in \mathbb{N} \). Next, choose a sequence \( \xi_i^\ell \in E^- \cap E_{\eta_i^\ell}^+ \) such that \( \xi_i^\ell \to \xi_i \) as \( \ell \to \infty \). This is possible because \( E^- \cap E_{\eta_i^\ell}^+ \) is \( \nu_{\psi} \)-conull from the hypothesis \( \eta_0 = g^+ \in E^+ \) and hence dense in \( \Lambda \). Let \( n_i^\ell \in N \) be the unique element such that \( \xi_i^\ell = (gn_i^\ell)^- \). Note that for all \( \ell \in \mathbb{N} \),
\[
(1) \quad (gn_i^\ell)^- = \xi_i^\ell \in E^-,
(2) \quad (gn_i^\ell)^+ = \eta_i^\ell \in E^+,
(3) \quad gn_i^\ell AM \in E, \quad \text{and}
\]

\[
f^\varepsilon(gm_0) = f^\varepsilon_\pm(gm_0) = f^\varepsilon_\pm(gm_0) = f^\varepsilon_\pm(g) \quad \text{for all } 1 \leq i \leq k,
\]

\[
f^\varepsilon(gm) = f^\varepsilon(\gamma gam) = f^\varepsilon(gh_1 \cdots h_{r-1}) = f^\varepsilon(g).
\]
(4) $n_{\ell} \to 1$ as $\ell \to \infty$.

Next, choose $\eta_{\ell} \in E^+ \cap E_{\xi_{\ell}^i}^+$ such that $\eta_{\ell} \to \eta_1$ as $\ell \to \infty$. Again, this is possible because $E^+ \cap E_{\xi_{\ell}^i}^+$ is $\nu_{\psi_0}$-conull. Note that $\eta_{\ell} = (gn_{\ell}h_{\ell}^i)^+$ for some unique $h_{\ell}^i \in N^+$. We have for all $\ell \in \mathbb{N}$,

1. $(gn_{\ell}h_{\ell}^i)^- = \xi_{\ell}^i \in E^-$,
2. $(gn_{\ell}h_{\ell}^i)^+ = \eta_{\ell} \in E^+$,
3. $gn_{\ell}h_{\ell}^iAM \in E$, and
4. $h_{\ell}^i \to h_1$ as $\ell \to \infty$.

Continuing in this fashion, we can find sequences $\xi_{\ell}^i \in E^- \cap E^+$, $n_{\ell}^i \in N$, $\eta_{\ell}^i \in E^+$, $h_{\ell}^i \in N^+(1 \leq i \leq \ell)$ such that the following holds: for all $\ell \in \mathbb{N}$,

1. $\xi_{\ell}^i = (gn_{\ell}^i h_{\ell}^i \cdots n_{\ell}^i h_{\ell}^i)^-$, $\eta_{\ell}^i = (gn_{\ell}^i h_{\ell}^i \cdots n_{\ell}^i h_{\ell}^i)^+$,
2. $\xi_{\ell+1}^i = (gn_{\ell+1}^i h_{\ell+1}^i \cdots n_{\ell}^i h_{\ell}^i)^-$, $\eta_{\ell}^i = (gn_{\ell}^i h_{\ell}^i \cdots n_{\ell}^i h_{\ell}^i)^+$,
3. $gn_{\ell}^i h_{\ell}^i \cdots n_{\ell}^i h_{\ell}^i AM \in E$, and
4. $n_{\ell} \to n_i$ and $h_{\ell} \to h_i$ as $\ell \to \infty$.

For $i = k$, we could have chosen $\xi_k^i = \gamma g^-$, $\eta_k^i = \gamma g^+$ for all $i$ in the above. This implies that $gn_{\ell}^i h_{\ell}^i \cdots n_{\ell}^i h_{\ell}^i = \gamma g a_\ell M$ for some $a_\ell \in A$ and $m_\ell \in M$. Note that $a_\ell m_\ell \to a_0 m_0$ as $\ell \to \infty$ and hence $m_\ell \in H_{\psi}^\omega (g)$ with permissible sequences $n_{\ell}^i, h_{\ell}^i, \cdots, n_k^i, h_k^i \in N \cap N^+$. Therefore, $f^\omega (gm_\ell) = f^\omega (g)$ by the previous observation. Since $gAM \in E$, $f^\omega$ is continuous on $gAM$ and hence

$$ f^\omega (gm) = \lim_{\ell \to \infty} f^\omega (gm_\ell) = \lim_{\ell \to \infty} f^\omega (g) = f^\omega (g). $$

This finishes the proof of the first claim.

For the second claim, let $am \in AM$ and $m_0 \in H_{\psi}^\omega (gm)$. Since $f^\omega$ is $A$-invariant, it follows from the first part that $f^\omega (gamm_0) = f^\omega (gm_0) = f^\omega (gm) = f^\omega (g)$. Since $f^\omega |_{AM}$ is continuous and $H_{\psi}^\omega (gm)$ contains $M^\circ$ by Corollary 2.11, the second claim follows. \(\square\)

**Proof of Lemma 3.4:** Let $f$ be any bounded $\Sigma$-measurable function on $\tilde{\Omega}$. For any $\varepsilon > 0$, by Lemma 3.5, $f^\omega$ coincides with a $\Sigma_0$-measurable function $m_{\psi}^\omega \text{-a.e.}$ Since $\lim_{\varepsilon \to 0} f^\varepsilon = f$ $m_{\psi}^\omega \text{-a.e.}$, $f$ is a $\Sigma_0$-measurable function $m_{\psi}^\omega \text{-a.e.}$ as well. This proves the lemma. \(\square\)

**Corollary 3.6.** There exists $B \in \Sigma$ such that any two distinct subsets in $\{B.s: s \in M^\circ \setminus M\}$ are measurably disjoint and $\Sigma$ is a finite $\sigma$-algebra generated by $\{B.s: s \in M^\circ \setminus M\} \text{ mod } m_{\psi}^\omega$.

**Proof.** First, note that the $AM$-ergodicity of $m_{\psi}^BS$ implies that the $\sigma$-algebra

$$ \Sigma_1 := \{B \cap \tilde{\Omega} : B \in B \text{ such that } B = \Gamma BAM\} $$

is trivial mod $m_{\psi}^BS$. It follows that for any $B \in \Sigma_0$, and hence for any $B \in \Sigma$ by Lemma 3.4, with $m_{\psi}^BS (B) > 0$, the subset $\cup_{s \in M^\circ \setminus M} B.s$, as $AM$-invariant, is $m_{\psi}^BS$-conull.
Let $\mathcal{P} = \{A_1, \ldots, A_k\}$ be a partition of $\tilde{\Omega}$ with maximal $k$, among all partitions of $\Omega$ satisfying

1. $A_i \in \Sigma$ and $m_\psi^{BMS}(A_i) > 0$,
2. $\tilde{\Omega} = A_1 \cup \cdots \cup A_k \mod m_\psi^{BMS}$ and
3. for any $s \in M^o \setminus M$, we have $A_i, s \in \{A_1, \ldots, A_k\} \mod m_\psi^{BMS}$.

Note that $1 \leq k \leq [M : M^o]$. Setting $B = A_1$, we claim that this proves the corollary. Suppose not. Setting $\sigma(\mathcal{P})$ to be the $\sigma$-algebra generated by $\mathcal{P}$, there exists $B' \in \Sigma - \sigma(\mathcal{P})$ mod $m_\psi^{BMS}$. Then for some $1 \leq j \leq k$, $B' \cap A_j$ is neither null nor conull in $A_j$. Hence by considering $B', s \cap A_i$ and $B'^c, s \cap A_i$ $s \in M^o \setminus M$ and $1 \leq i \leq k$, we get a partition finer than $\mathcal{P}$ satisfying the above three conditions. This contradicts the maximality of $k$. \qed

Set $\Lambda(2) = \Lambda \times \Lambda \cap \mathcal{F}(2)$. The Anosov hypothesis on $\Gamma$ implies that $\Lambda(2) = \{(\xi, \eta) \in \Lambda \times \Lambda : \xi \neq \eta\}$. Consider the action of $\Gamma$ on $\Lambda(2) \times \mathbb{R}$ defined by

$$\gamma. (\xi, \eta, s) = (\gamma \xi, \gamma \eta, s + \psi(\log \beta^A_\xi(\gamma^{-1}, e)), \gamma. \xi, \eta, s + \psi(\log \beta^A_\eta(\gamma^{-1}, e)))$$

which is proper and cocompact ([21, Thm 3.2], [5, Thm A.2])

For $(\xi_1, \xi_2) \in \mathcal{F}(2)$, define

$$[\xi_1, \xi_2]_\psi := \psi(\log \beta^A_\gamma(e, g) + i \log \beta^A_\gamma(e, g))$$

where $g \in G$ is such that $g^+ = \xi_1$ and $g^- = \xi_2$.

Then the measure $d\hat{m}_\psi := e^{[\cdot, \cdot]_\psi} dv_\psi dv_\psi$ descends to a finite $\mathbb{R}$-ergodic measure $m_\psi$ on $\Gamma \setminus \Lambda(2) \times \mathbb{R}$. It follows that the product measure $d\hat{m}_\psi dm$ on $\Lambda(2) \times \mathbb{R} \times M$ is $\Gamma$-invariant for the $\Gamma$-action on $\Lambda(2) \times \mathbb{R} \times M$ given by

$$\gamma. (\xi, \eta, s, m) = (\gamma \xi, \gamma \eta, s + \psi(\log \beta^A_\xi(\gamma^{-1}, e)), \beta^M_\xi(\gamma^{-1}, e)m)$$

where $(\xi, \eta) \in \Lambda(2)$, $s \in \mathbb{R}$ and $m \in M$. We denote by $d\hat{m}_\psi$ the finite measure on $Y := \Gamma \setminus \Lambda(2) \times \mathbb{R} \times M$ induced by $d\hat{m}_\psi dm$.

The BMS-measure $m_\psi^{BMS}$ disintegrates over $\hat{m}_\psi$ via the projection $\Gamma \setminus \Lambda(2) \times A \times M \to \Gamma \setminus \Lambda(2) \times \mathbb{R} \times M$, where each conditional measure is the Lebesque measure on $\exp(\ker \psi)$.

Define the projection $\Psi : \tilde{\Omega} \to \Lambda(2) \times \mathbb{R} \times M$ by

$$\Psi(\xi, \eta, a, m) = (\xi, \eta, \psi(\log a), m).$$

Note that for any $g \in \tilde{\Omega}$, $\Psi(g \gamma m) = \gamma \Psi(g) \tau_\Psi(\log a) \tau_m$ for all $\gamma \in \Gamma$, $a \in A$ and $m \in M$, where $\tau$ stands for the right translation action by elements of $\mathbb{R} \times M$. By abuse of notation, let $\Psi : \tilde{\Omega} \to Y$ denote the map induced by $\Psi$ and $\tau$ denote the action of $\mathbb{R} \times M$ on $Y$ induced by $\tau$.

**Lemma 3.7.** There exists a $\tau_{\mathbb{R}}$-invariant, ergodic and $\tau_{M^o}$-invariant finite measure $m_\psi^o$ on $Y$ such that $\hat{m}_\psi = \sum_{s \in M^o \setminus M} m_\psi^o.s$. 
Proof. Let $f \in C(Y)$ be arbitrary. The Birkhoff average $f_\sharp : Y \to \mathbb{R}$ is defined $\hat{m}_\psi$-a.e. by
\[
 f_\sharp(y) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(y_t) \, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(y_{t-t}) \, dt.
\]
Note that $f_\sharp$ is well defined by the Birkhoff ergodic theorem and is $\tau_\mathbb{R}$-invariant. Hence, $f_\sharp \circ \Psi$ is defined $m_{BMS}^\psi$-a.e. Let $a_t \in A^+$ be such that $\psi(\log a_t) = t$ and $a_t \to \infty$ regularly as $t \to \infty$. Observing that $f \circ \Psi$ is uniformly continuous and that $f(\Psi(x)_{t}) = f(\Psi(xa_t))$, it is standard to show that $f_\sharp \circ \Psi$ coincides with $N^\pm$-invariant functions $m_{BMS}^\psi$-a.e. and hence $\Sigma$-measurable. By Corollary 3.6, $\Sigma$ is generated by $\{B.s : s \in M^{\circ} \setminus M\}$ mod $m_{BMS}^\psi$ for some $B \in \Sigma$. Set $m_\psi^o := \frac{1}{T} \tilde{m}_\psi |_{\Psi(B)}$, where $\ell = \# \{s \in M^{\circ} \setminus M : B.s = B \text{ mod } m_{BMS}^\psi \}$. Since $f_\sharp$ is constant $\hat{m}_\psi$-a.e. on $\Psi(B)$, $m_\psi^o$ is $\tau_\mathbb{R}$-ergodic by the Birkhoff ergodic theorem. Since
\[
 \sum_{s \in M^{\circ} \setminus M} m_\psi^o.s = \sum_{s \in M^{\circ} \setminus M} \frac{1}{T} \tilde{m}_\psi |_{\Psi(B.s)} = \hat{m}_\psi,
\]
the lemma follows. \hfill \qed

Proof of Theorem 3.3. Define $\mu_\psi$ on $\Omega$ by $d\mu_\psi = dm_{\psi}^o d\text{Leb}$ where $m_{\psi}^o$ is given in Lemma 3.7. Since $\mu_\psi$ is $A$-invariant, ergodic and $M^\circ$-invariant, the theorem follows. \hfill \qed

The set of strong Myrberg limit points. Let $Z_\psi$ denote the set of elements with dense $\mathbb{R}^+$ orbit in $\text{supp} \, m_{\psi}^o$ and $\tilde{Z}_\psi$ be its lift in $\Lambda^{(2)} \times \mathbb{R} \times M$. Denote by $\pi : \Lambda^{(2)} \times \mathbb{R} \times M \to \Lambda$ the projection map $\pi(\xi, \eta, t, m) \mapsto \xi$, and set
\[
 \Lambda_{M, \psi} := \pi(\tilde{Z}_\psi)
\]
which we call the set of strong Myrberg limit points.

Corollary 3.8. For $\Gamma$ Anosov, we have $\nu_\psi(\Lambda_{M, \psi}) = 1$.

Proof. Since $m_{\psi}^o$ is $\mathbb{R}$-ergodic and finite, $\tilde{Z}_\psi$ has full measure by the Birkhoff ergodic theorem. Since $\pi_* m_{\psi}^o \ll \nu_\psi$, $\pi(\tilde{Z}_\psi)$ has full $\nu_\psi$ measure in $\Lambda$, as desired. \hfill \qed

4. EQUI-CONTINUOUS FAMILY OF BUSEMANN FUNCTIONS

Let $\Pi$ denote the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$. 

Definition 4.1. We write $a_n \to \infty$ regularly in $A^+$ if $\alpha(\log a_n) \to \infty$ as $n \to \infty$ for all $\alpha \in \Pi$.

An important property of Anosov groups which will be used in this section is the following:

Lemma 4.2. Let $\Gamma$ be Anosov. For any $g, h \in G$ and a sequence $\gamma_n \to \infty$ in $\Gamma$, $\mu(g\gamma_nh) \to \infty$ regularly in $A^+$. 

This lemma is a consequence of the fact that the limit cone of $\Gamma$ is contained in the interior of $a^+$, except for $0$ (cf. [15, Thm. 4.3] for references).

When $\mu(g) \in \inter a^+$ and $g = k_1 \mu(g) k_2$, $k_1, k_2$ are determined uniquely up to mod $M$, more precisely, if $g = k'_1 \mu(g) k'_2$, then for some $m \in M$, $k_1 = k'_1 m$ and $k_2 = m^{-1} k'_2$. We write

$$\kappa_1(g) := [k_1] \in K/M \quad \text{and} \quad \kappa_2(g) := [k_2] \in M \setminus K.$$

**Definition 4.3.**  
(1) A sequence $g_n \in G$ is said to converge to $\xi \in F$, if $g_n \to \infty$ regularly in $G$ and $\lim_{n \to \infty} \kappa_1(g_n) = \xi$.

(2) A sequence $p_n = g_n(o) \in X$ is said to converge to $\xi \in F$ if $g_n$ does.

In this section, we prove the following proposition, which will be used in the proof of Theorem 1.1:

**Proposition 4.4 (Equi-continuity).** Let $\Gamma < G$ be an Anosov subgroup. Let $g \in G$ be such that $g^+ \in \Lambda$ and let $\gamma_n \in \Gamma$ be a sequence such that for some $\xi \in \Lambda - \{g^\pm\}$, $\gamma_n^{-1} \xi \to g^+$ and $\gamma_n^{-1} g(o) \to g^-$ as $n \to \infty$. Then the sequence of maps $\eta \mapsto \beta^{AM}_\eta(\gamma_n^{-1}, g)$ is equi-continuous at $g^+$, i.e., for any $\varepsilon > 0$, there exists a neighborhood $U_\varepsilon \subset F$ of $g^+$ such that for all $n \geq 1$,  

$$\beta^{AM}_\eta(\gamma_n^{-1}, g) \subset \beta^{AM}_{g^+}(\gamma_n^{-1}, g)(AM)_\varepsilon \quad \text{for all } \eta \in U_\varepsilon.$$ 

We first prove the following two lemmas using the structure theory of semisimple Lie groups.

**Lemma 4.5.** There exists $C > 0$ such that for all sufficiently small $\varepsilon > 0$,  

$$a G_\varepsilon \subset K_{C \varepsilon} a A_{C \varepsilon} N \quad \text{for all } a \in A^+.$$  

*Proof.* For all sufficiently small $\varepsilon > 0$, we have  

$$G_\varepsilon \subset M_{O(\varepsilon)} N_{O(\varepsilon)} N_{O(\varepsilon)}^+ \subset K_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)}.$$ 

Since $a N_{\varepsilon} a^{-1} \subset N_{\varepsilon}^+$ for any $a \in A^+$, it follows that  

$$a G_\varepsilon \subset M_{O(\varepsilon)} N_{O(\varepsilon)}^+ A_{O(\varepsilon)} N_{O(\varepsilon)} = M_{O(\varepsilon)} (a N_{O(\varepsilon)}^+ a^{-1}) A_{O(\varepsilon)} N_{O(\varepsilon)} \subset M_{O(\varepsilon)} (K_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)}) a A_{O(\varepsilon)} N_{O(\varepsilon)},$$ 

which was to be proved. \hfill $\square$

**Lemma 4.6.** Let $g_n = k_n a_n \ell_n^{-1} \in K A^+ K$ where $a_n \to \infty$ regularly in $A^+$ and $k_n \to k_0$, $\ell_n \to \ell_0$ in $K$ as $n \to \infty$. Assume that both $k_0^+$ and $\ell_0^+$ belong to $N^+ e^+$. Then for all sufficiently large $n$ and small $\varepsilon > 0$, there exist $m_0 \in M$ and neighborhoods $U_\varepsilon$ and $V_\varepsilon$ of $\ell_0^+$ and $k_0^+$, respectively, such that for all $\eta \in U_\varepsilon \cap g_n^{-1} V_\varepsilon$,  

$$\sigma_{AM}(g_n, \eta) \subset a_n m_0 (AM)_\varepsilon.$$ 

*Proof.* Set $\xi = k_0^+$ and $\zeta = \ell_0^+$. By the continuity of the visual maps, there exist neighborhoods $U_\varepsilon$ of $\zeta$ and $V_\varepsilon$ of $\xi$ such that $k_\eta \in k_\zeta K_\varepsilon$ for all $\eta \in U_\varepsilon$.
and \( k_\eta \in k_\xi K_\varepsilon \) for all \( \eta \in V_\varepsilon \). We may assume that \( k_0^{-1}k_\eta, \ell_n^{-1}_0 \in K_\varepsilon \) for all \( n \). Let \( \eta \in U_\varepsilon \cap g_n^{-1}V_\varepsilon \) be arbitrary. By definition,

\[
g_nk_\eta \in k_\eta \sigma(g_n, \eta)N, \quad \text{i.e.,} \quad k_0^{-1}g_nk_\eta \in k_0^{-1}k_\eta \sigma(g_n, \eta)N
\]

Observe that

\[
k_0^{-1}g_nk_\eta \in k_0^{-1}g_nk_\xi K_\varepsilon = (k_0^{-1}k_\eta)a_n(\ell_n^{-1}_0)\ell_n^{-1}k_\xi K_\varepsilon \]

\[
\subset K_\varepsilon a_nK_\varepsilon \ell_0^{-1}k_\xi K_\varepsilon \subset K_\varepsilon a_nK_{O(\varepsilon)} \ell_0^{-1}k_\xi.
\]

On the other hand, since \( g_n\eta \in V_\varepsilon \), the right-hand side belongs to

\[
k_0^{-1}k_\eta \sigma(g_n, \eta)N \subset k_0^{-1}k_\xi K_\varepsilon \sigma(g_n, \eta)N \subset K_{O(\varepsilon)}k_0^{-1}k_\xi \sigma(g_n, \eta)N.
\]

Combining these with the fact \( \ell_0^{-1}k_\xi \in M \),

\[
a_nK_{O(\varepsilon)} \cap K_{O(\varepsilon)}k_0^{-1}k_\xi \sigma(g_n, \eta)(\ell_0^{-1}k_\xi)^{-1}N \neq \emptyset.
\]

Since \( k_0^{-1}k_\xi \in M \) as well, by Lemma 4.5, it follows that

\[
\sigma^A(g_n, \eta) \in a_nA_{O(\varepsilon)}, \quad \text{and} \quad \sigma^M(g_n, \eta) \in (k_0^{-1}k_\xi)^{-1}M_{O(\varepsilon)}\ell_0^{-1}k_\xi \subset (k_0^{-1}k_\xi)^{-1}\ell_0^{-1}k_\xi M_{O(\varepsilon)}.
\]

It remains to set \( m_0 := (k_0^{-1}k_\xi)^{-1}\ell_0^{-1}k_\xi \).

We make the following simple but important observation:

**Lemma 4.7.** Let \( \Gamma \) be Anosov. If \( g \in G \) satisfies \( g^- \in \Lambda \), then \( g^{-1}\Lambda \subset N^+e^+ \cup \{e^-\} \). In other words, for any \( \xi \in \Lambda \), \( \Lambda \subset gN^+e^+ \cup \{\xi\} \) for any \( g \in G \) with \( g^- = \xi \).

**Proof.** Suppose that \( \xi \in \Lambda \) and \( g^{-1}\xi \neq e^- \). Then \( \xi \neq g^- \) in \( \Lambda \). Hence \( (\xi, g^-) \in F(2) \), or equivalently, \( (g^{-1}\xi, e^-) \in F(2) \). Since \( \{\eta \in F : (\eta, e^-) \in F(3)\} \subset N^+e^+ \), \( g^-\xi \in N^+e^+ \), proving the claim. \( \square \)

**Proof of Proposition 4.4:** Set \( g_n := g^{-1}g_n \). Then \( g_n^{-1}(g^{-1}\xi) \to e^+ \) and \( g_n^{-1}(o) \to e^- \) as \( n \to \infty \). By passing to a subsequence, we may write \( g_n = k_n a_n \ell_n^{-1} \in KA^+K \). We may assume that \( k_n \to k_0 \) and \( a_n \to a_0 \) in \( K \). It follows from the hypothesis that \( \Gamma \) is Anosov that \( a_n \to \infty \) regularly in \( A^+ \). Combined with the hypothesis \( g_n^{-1}(o) \to e^- \) as \( n \to \infty \), we have \( \ell_0^{-1} = e^- \), or equivalently, \( \ell_0 \in M \).

We claim that \( k_0^+ = g^{-1}\xi \). Since \( a_n \to \infty \) regularly, for any \( \eta \in N^+e^+ \), \( g_n\eta \to k_0^+ \) as \( n \to \infty \) and the convergence is uniform on a compact subset of \( N^+e^+ \). Since \( g_n^{-1}(g^{-1}\xi) \to e^+ \) as \( n \to \infty \), \( g_n^{-1}(g^{-1}\xi) \) is in a compact subset of \( N^+e^+ \) for all large \( n \). From the previous observation, \( g_n(g_n^{-1}(g^{-1}\xi)) \to k_0^+ \) as \( n \to \infty \), which proves the claim.

Now let \( \varepsilon > 0 \) be arbitrary. Since \( g^- \in \Lambda \), by Lemma 4.7, \( g^{-1}\Lambda \setminus \{e^-\} \subset N^+e^+ \). Hence both \( e^+ \) and \( g^{-1}\xi \) belong to \( N^+e^+ \). Applying Lemma 4.6 to
the sequence \(g_n\), we obtain \(m_0 \in M\), neighborhoods \(U'_\varepsilon\) and \(V'_\varepsilon\) of \(e^+\) and \(g^{-1}\xi\), respectively, such that

\[
\sigma(g_n, \eta) \in a_n m_0(AM)_{\varepsilon/2} \quad \text{for all } \eta \in U'_\varepsilon \cap g_n^{-1}V'_\varepsilon.
\]

By uniform convergence on a compact subset of \(N^+e^+\), we may assume that \(g_n U'_\varepsilon \subset V'_\varepsilon\) for all large \(n\), by shrinking \(U'_\varepsilon\) if necessary. Set \(U_\varepsilon := g U'_\varepsilon\). Then for all \(\eta \in U_\varepsilon\), \(g^{-1}\eta \in U'_\varepsilon \cap g_n^{-1}V'_\varepsilon\) and therefore

\[
\beta_\eta(\gamma_n^{-1}g, g) = \beta_{g^{-1}\eta}(g^{-1}\gamma_n^{-1}g, e) = \sigma(g^{-1}\gamma_n g, g^{-1}\eta) \in a_n m_0(AM)_{\varepsilon/2}.
\]

Since \(g^+ \in U_\varepsilon\) as well, the lemma is proved.

5. Essential Values and Ergodicity

Consider the isomorphism \(G/N \to \mathcal{F} \times AM\) given by

\[
gN \mapsto (g^+, \beta_{g^+}(e, g)).
\]

With respect to (5.1), we have for all \(g \in G\), \(\xi \in \mathcal{F}\) and \(am \in AM\),

\[
g(\xi, am) = (g\xi, \beta_{g^+}(g^{-1}, e)am).
\]

Let \(\nu\) be a \((\Gamma, \psi)\)-PS measure on \(\Lambda\) for \(\psi \in D_\Gamma^\ast\). We define a \(\Gamma\)-invariant locally finite measure on \(G/N\) by

\[
d\hat{\nu}([g]) = d\nu(g^+) e^{\psi(\log a)} da dm
\]

where \(da\) and \(dm\) are Haar measures on \(A\) and \(M\) respectively.

**Definition 5.1.** An element \(am \in AM\) is called a \(\nu\)-essential value, if for any Borel set \(B \subset \mathcal{F}\) with \(\nu(B) > 0\) and any \(\varepsilon > 0\), there exists \(\gamma \in \Gamma\) such that \(\nu(B \cap \gamma^{-1}B \cap \{\xi \in \mathcal{F} : \beta_\varepsilon(\gamma^{-1}, e) \in am(AM)_{\varepsilon}\}) > 0\).

Let \(E = E_\psi\) be the set of all essential values in \(AM\). One can check that \(E\) is a closed subgroup of \(AM\). The proof of the following proposition follows the same line as in [15, Prop. 9.2 and Lem. 9.3].

**Lemma 5.2.** Let \(h : G/N \to [0, 1]\) be a \(\Gamma\)-invariant Borel function. If \(E\) contains the whole conjugacy class of \(am \in AM\), then \(h(xam) = h(x)\) for \(\hat{\nu}\)-a.e. \(x\).

**Proof.** Let \(c > 1\) be a positive number such that for all sufficiently small \(\varepsilon > 0\) and all \(am \in AM\), we have

\[(AM)_{c-1}am \subset am(AM)_{\varepsilon} \subset (AM)_{ce}am.
\]

Now suppose the lemma is false. Then there exists \(a_0 m_0 \in E\) whose conjugacy class is contained in \(E\), while \(\nu\{x : h(xa_0m_0) \neq h(x)\} > 0\). Without loss of generality, after possibly replacing \(h\) by \(-h\), we may assume that \(\nu\{x : h(x) < h(xa_0m_0)\} > 0\). Hence there exist \(r, \varepsilon > 0\) such that

\[Q_{a_0m_0} := \{x \in G/N : h(x) < r - \varepsilon < r + \varepsilon < h(xa_0m_0)\}
\]

has a positive measure. Consider \(\rho_n \in C_c(AM)\) supported on \((AM)_{1/n}\), with \(\int_{AM} \rho_n = 1\). Identifying \(G/N \simeq \mathcal{F} \times AM\), we have \(\lim_{n \to \infty} h * \rho_n = h\).
\[ \nu \text{-a.e. Hence, by replacing } h \text{ with } h \ast p_n \text{ for some } n, \text{ we can further assume that there exists } \varepsilon' > 0 \text{ such that}
\]
\[ |h(\xi, \cdot a) - h(\xi, \cdot)| \leq \varepsilon \quad \text{for all } \xi \in F \text{ and all } am \in (AM)_{2\varepsilon'}.
\]
Since \( \nu(Q_{a_0m_0}) > 0 \), we can find \( O := a'm'(AM)_{\varepsilon'/2c} \subset AM \) such that
\[ \nu((F \times O) \cap Q_{a_0m_0}) > 0
\]
Setting \( F_{a_0m_0} := \{ \xi \in F : \{ \xi \} \times a'm'(AM)_{\varepsilon'} \cap Q_{a_0m_0} \neq \emptyset \} \), we claim that
\[ (5.4)
\]
if \( (\xi, am) \in F_{a_0m_0} \times a'm'(AM)_{\varepsilon'} \), then \( h(\xi, am(a_0m_0)) > r > h(\xi, am) \).

Note that there exists \( a_0m_0m' \in (AM)_{2\varepsilon'} \) such that \( (\xi, am_0m_0m') \in Q_{a_0m_0} \)
and hence,
\[ |h(\xi, am)| \leq |h(\xi, am) - h(\xi, am_0m_0m')| + |h(\xi, am_0m_0m')| < \varepsilon + (r - \varepsilon) \leq r.
\]
Similarly,
\[ |h(\xi, am(a_0m_0))| \geq |h(\xi, am_0m_0m')(a_0m_0))| - |h(\xi, am(a_0m_0)) - h(\xi, am_0m_0m')(a_0m_0))| > (r + \varepsilon) - \varepsilon > r,
\]
where the second last inequality is valid because \( a_0m_0m'a_0m_0 \in a_0m_0m(AM)_{2\varepsilon'}. \)
This verifies the claim \((5.4)\).

We will use the fact that \( (a_0m_0m_0\varepsilon'/(2c))^{-1} \in E \). Together with \( \nu(F_{a_0m_0}) > 0 \), there exists \( \gamma \in \Gamma \) such that
\[ A := F_{a_0m_0} \cap \gamma F_{a_0m_0} \cap \{ \xi \in F : \beta_\xi(e, \gamma) \in (a_0m_0m_0m')^{-1}(AM)_{\varepsilon'/2c} \}
\]
has a positive \( \nu \)-measure. For \( \xi \in A \), set
\[ O_\xi := O \cap O \cdot a_0m_0m_0m'^{-1} \beta_\xi(e, \gamma).
\]
Since \( a_0m_0m_0m'^{-1} \beta_\xi(e, \gamma) \in (AM)_{\varepsilon'/2c} \) for all \( \xi \in A \) and \( O \) is a ball of radius \( \varepsilon'/2c \), the Haar measure of \( O_\xi \) is uniformly bounded from below. Hence \( A^* := \bigcup_{\xi \in A} (\xi) \times O_\xi \) has a positive \( \nu \)-measure inside \( F_{a_0m_0} \times O \). Now we prove that \( h \circ \gamma^{-1} > h \) on \( A^* \), which contradicts the assumption that \( h \) is \( \Gamma \)-invariant \( \nu \)-a.e. Observe that if \( (\xi, a_1m_1) \in A^* \), then it follows from \((5.4)\) that \( r > h(\xi, a_1m_1) \). On the other hand, we have
\[ \beta_\xi(e, \gamma)^{-1}a_1m_1 \in (AM)_{\varepsilon'/2c}(a_0m_0m_0m'^{-1})a_1m_1 \]
\[ \subset (AM)_{\varepsilon'/2c}(a_0m_0m_0m'^{-1})a'm'(AM)_{\varepsilon'/2c} \]
\[ = (AM)_{\varepsilon'/2c}a'm'a_0m_0(AM)_{\varepsilon'/2c} \]
\[ \subset a'm'(AM)_{\varepsilon'}a_0m_0.
\]
Since \( \gamma^{-1}(\xi, a_1m_1) = (\gamma^{-1}\xi, \beta_\xi(e, \gamma)^{-1}a_1m_1) \), we deduce from \((5.4)\) that \( h(\gamma^{-1}(\xi, a_1m_1)) > r \), as desired.
6. N-ergodic decompositions of BR-measures

Let $\Gamma < G$ be an Anosov subgroup. We prove Theorem 1.1 in this section.

The following version of ergodic decomposition of any Radon measure can be deduced from [9, Thm 5.2]. We will use this proposition for BR-measures $m_{\psi}^{\text{BR}}$, $\psi \in D_{\mathbb{H}}^*$, with $H = N$.

**Proposition 6.1** (Ergodic decomposition). Let $H$ be a locally compact second countable group and $M$ be a compact subgroup normalizing $H$. Suppose that $HM$ acts on a standard Borel space $(X, \mathcal{B})$, preserving a Radon measure $\mu$ on $X$.

1. There exists a Borel map $x \mapsto \mu_x$ from $X$ to the space of $H$-invariant ergodic Radon measures on $X$ and an $M$-invariant probability measure $\mu^*$ on $X$ equivalent to $\mu$ with the following properties:
   
   a. $\mu_x = \mu_{xh}$ for every $x \in X$ and $h \in H$.
   
   b. For all nonnegative Borel function $f : X \to \mathbb{R}$, we have
      $$
      \int f \, d\mu_x = \mathbb{E}_{\mu^*} \left( f \frac{d\mu}{d\mu^*} \big| \mathcal{S}_H \right)(x) \text{ for } \mu^*-\text{a.e. } x \in X,
      $$
   
   where $\mathcal{S}_H := \{ B \in \mathcal{B} : B.h = B \text{ for all } h \in H \}$. In particular, we have
   $$
   \mu = \int_{x \in X} \mu_x \, d\mu^*(x).
   $$

   If $\mu$ is finite, we can take $\mu^* = \mu$.

2. Let $\mathcal{T} \subset \mathcal{S}_H$ be the smallest $\sigma$-algebra such that the map $x \mapsto \mu_x$ is $\mathcal{T}$-measurable. Then $\mathcal{T}$ is countably generated, $\mathcal{T} = \mathcal{S}_H \mod \mu$ and
   $$
   \mu_x([y]_\mathcal{T}) = 0 \text{ for all } y \notin [x]_\mathcal{T}, \mu_x([x]_\mathcal{T}) = 0.
   $$

3. For each $m \in M$, we have $\mu_{xm} = \mu_x \cdot m$ for $\mu^*$-a.e. $x \in X$.

**Proof.** Fix an $M$-invariant positive function $\varphi \in L^1(\mu)$ with $\int \varphi \, d\mu = 1$. Then $d\mu^* := \varphi \, d\mu$ defines an $H$-quasi-invariant and $M$-invariant probability measure on $X$. By applying [9, Thm 5.2] to $\mu^*$ with the cocycle $\rho : H \times X \to \mathbb{R}$ given by $\rho(h, y) = \log \frac{\varphi(y h^{-1})}{\varphi(y)}$, we get a Borel map $x \mapsto \mu^*_x$ from $X$ to the space of $H$-ergodic probability measures such that for all nonnegative Borel function $f : X \to \mathbb{R}$, we have

$$
\int f \, d\mu^*_x = \mathbb{E}_{\mu^*}(f \mid \mathcal{S}_H)(x) \text{ for } \mu^*$-a.e. $x \in X$,
$$

and $\frac{d(h \cdot \mu^*_x)}{d\mu^*_x}(y) = \frac{\varphi(y h^{-1})}{\varphi(y)}$. In particular, we have $\mu^* = \int \mu_x^* \, d\mu^*(x)$. Now define a Radon measure $\mu_x$ on $X$ by $d\mu_x := \frac{1}{\varphi} \, d\mu^*_x$. A direct computation shows that $\mu_x$ is $H$-invariant, ergodic for all $x \in X$ and (1) holds. (2) follows from the corresponding statement on $\mu^*_x$ from [9, Thm 5.2].

In order to prove (3), we compute that for a non-negative Borel function $f : X \to \mathbb{R}$,

$$
\mu^*_{xm}(f) = \mathbb{E}_{\mu^*}(f \mid \mathcal{S}_H)(xm) = \mathbb{E}_{\mu^*}((m.f) \mid \mathcal{S}_H)(x) = \mu^*_x(m.f);
$$

...
the second equality follows since $S_H.m = S_H$ and $\mu^*$ is $M$-invariant. It follows that $\mu^* \cdot m = \mu^* \cdot m$ for $\mu$-a.e. $x \in X$; this implies (3). \hfill \Box

Fix $\psi \in D^\gamma$, and let $\nu_\psi$ the unique $(\Gamma, \psi)$-measure on $\Lambda$. Let $E_{\nu_\psi}$ be the set of essential values as defined in Definition 5.1.

**Proposition 6.2.** If $M^\circ \subset E_{\nu_\psi}$, then Theorem 1.1 holds.

**Proof.** Let $m^\BR_\psi = \int_X m_x \, dm^*(x)$ be an $N$-ergodic decomposition as given by Proposition 6.1 where $X = \Gamma \backslash G$. Let $f \in C_c(\Gamma \backslash G)$ and consider the map $h(g) := m_{[g]}(f)$ for all $[g] \in X$. Note that $h$ defines a $\Gamma$-invariant Borel function on $G/N$. Since $M^\circ$ is a normal subgroup of $AM$, Lemma 5.2 implies that $h$ is $M^\circ$-invariant for almost all $\nu_\psi$. It follows that $M^\circ < \Stab_M(m_x)$ for almost all $x$; without loss of generality, we may assume that $M^\circ \subset \Stab_M(m_x)$ for all $x \in X$. Hence the finite group $S := M^\circ \backslash M$ acts on $\{m_x : x \in X\}$. Let $\{s_1, \ldots, s_k\} \subset M$ be a set of representatives for $S$ and set $\tilde{m}_x := \frac{1}{k} \sum_{i=1}^{k} m_x, s_i$. Since $m^\BR_\psi$ is $M$-invariant, we have $n^\BR_\psi = \int_X \tilde{m}_x \, dm^*(x)$. As $m_x, s_i \in M$ for all $m \in M$, the map $x \mapsto \tilde{m}_x$ is $NM$-invariant. Since $m^\BR_\psi$ is $NM$-ergodic, $\tilde{m}_x$ is constant $m$-a.e. $x \in X$. Therefore we may fix $x_0 \in X$ so that $m^\BR_\psi = \tilde{m}_{x_0}$. It now suffices to set $m_\psi := \frac{1}{k} m_{x_0}$. \hfill \Box

Recall from [15] that there exists a compact subset $C \subset G$ such that for any $\xi \in \Lambda$, there exists $g \in C$ such that $g^+ = \xi$ and $g^- \in \Lambda$. Let $N_0 = N_0(\psi, C(o))$ be the constant from [15, Lemma 6.12].

Without loss of generality, we may assume that $\Gamma \cap \int A^+ M \not\subset M$.

**Lemma 6.3.** For any $C > 1$, there exists a Zariski dense subgroup $\Gamma_\psi \subset \Gamma$, depending on $C$, such that $\Gamma_\psi \cap A^+ M \not\subset M$ and

$$\psi(\lambda(\gamma)) > C \quad \text{for all } \gamma \in \Gamma_\psi \setminus \{e\}.$$

**Proof.** Recall that $\Pi$ is the set of all simple roots of $\mathfrak{g}$ with respect to $a^+$. By [3, Lem. 4.3(b)], there exist $\varepsilon > 0$ and $\{s_1, \ldots, s_k\} \subset \Gamma$ such that $s_1 \in \int A^+ M - M$, and for each $m \geq 1$, $s_1^m, \ldots, s_k^m$ are $(\Pi, \varepsilon)$-Schottky generators and the subgroup $\Gamma_m = \langle s_1^m, \ldots, s_k^m \rangle$ is a Zariski-dense $(\Pi, \varepsilon)$-Schottky subgroup of $\Gamma$ (see [3, Def. 4.1] for terminologies).

Fix $m > 1$ and let $z \in \lambda(\Gamma_m) \setminus \{0\}$. Then $z = \lambda(w)$ for some $w = g_1^n \cdots g_\ell^n$ with $g_i \in \{s_1^{\pm m}, \ldots, s_k^{\pm m}\}$, $n_i \in \mathbb{N}$, $g_i \neq g_i + 1$ ($i = 1, \ldots, \ell$) where we interpret $g_{\ell+1} := g_1$: this is because every element of a $(\Pi, \varepsilon)$-Schottky group is conjugate to a word of such form. By [3, Lem. 4.1], there exists $R = R(\varepsilon) > 0$ (independent of $w \in \Gamma_1$) such that

$$||\lambda(w) - \sum_{i=1}^\ell n_i \lambda(g_i)|| \leq \ell R.$$

Since $\psi(\lambda(s_j^{\pm 1})) > 0$ and $\lambda(s_j^{\pm m}) = m \lambda(s_j^{\pm 1})$, we can choose $m_0$ such that $\psi(\lambda(s_j^{\pm m_0})) > ||\psi||R + C$ for all $j = 1, \ldots, k$. Set $\Gamma_\psi := \Gamma_{m_0}$. Then for any
For $\xi \in G$ where $y$ such that

$$\psi(z) \geq \sum_{i=1}^{\ell} n_i \psi(\lambda(g_i)) - \|\psi\| R \geq \sum_{i=1}^{\ell} n_i \left( \psi(\lambda(g_i)) - \|\psi\| R \right) > C.$$  

The lemma follows.  

By Corollary 2.9 and Lemma 6.3 and Lemmas 6.2, we obtain:  

**Corollary 6.4.** For any $C > 1$, the closed subgroup of $AM$ generated by $\{\hat{\lambda}(\gamma_0) : \gamma_0 \in \Gamma^*, \psi(\lambda(\gamma_0)) > C\}$ contains $AM^\circ$.  

By Corollary 6.4, Proposition 6.2, Theorem 1.1 now follows from:  

**Proposition 6.5.** For all $\gamma_0 \in \Gamma^*$ satisfying $\psi(\lambda(\gamma_0)) > \log 3N_0 + 1$, we have $\hat{\lambda}(\gamma_0) \in E_{\nu_\psi}$.  

The rest of the section is devoted to the proof of Proposition 6.5.  

**Definition of $B_R(\gamma_0, \epsilon)$.** We now fix $\epsilon > 0$ as well as an element $\gamma_0 \in \Gamma^*$ such that

$$\psi(\lambda(\gamma_0)) > \log 3N_0 + 1.$$  

Note that $y_{\gamma_0 \gamma^{-1}} = \gamma y_{0}$. We can choose $g \in C$ such that $g^+ = y_{0}^+$ and $g^- \in \Lambda$. Set

$$p := g(o), \quad \eta := g^-, \quad \text{and} \quad \xi_0 := g^+.$$  

For $\xi \in \Lambda$ and $r > 0$, set

$$B_p(\xi, r) := \{\eta \in \Lambda : d_{\psi, p}(\xi, \eta) < r\}$$  

where $G(\xi, \eta) := \beta_{\lambda^+}^p(e, h) + i \beta_{\lambda^-}^p(e, h)$ for $h \in G$ satisfying that $h^+ = \xi$ and $h^- = \eta$, and $d_{\psi, p}(\xi, \eta) := e^{-\psi(G(\gamma_0, g, \gamma_0^{-1}))}$ is the virtual visual metric defined in [15].

For any $\xi \in \Lambda - \{\eta, e^\epsilon\}$, we claim that there is $R_\epsilon = R_\epsilon(\xi) > 0$ such that

$$\beta_{\lambda^+}^p(g, e) \in \beta_{\lambda^+}^p(g, e)(AM)_\epsilon$$  

for all $\xi' \in B_p(\xi, e^{\psi(\lambda(\gamma_0)) + \lambda(\gamma_0^{-1})})$. Indeed, the claim follows as the map $\xi' \mapsto \beta_{\lambda^+}^p(g, e)$ is continuous at $\xi$.  

By [15], the family $\{B_p(\xi, r) : \xi \in \Lambda, r > 0\}$ forms a basis of topology in $\Lambda$. For $\gamma \in \Gamma$, let $r_g(\gamma)$ be the supremum of $r \geq 0$ such that for all $\xi \in B_p(\gamma \xi_0, 3N_0r)$,

\begin{equation}
(6.1) \quad \beta_{\lambda^+}^p(g, \gamma \gamma_0 \gamma^{-1} g) \in \beta_{\lambda^+}^p(g, \gamma \gamma_0 \gamma^{-1} g)(AM)_\epsilon.
\end{equation}

If $\gamma \xi_0 \notin \{e^\epsilon, g^\epsilon\}$ and hence $\gamma \xi_0, g^{-1} \gamma_0 \xi_0 \in N^+e^\epsilon$, then $r_g(\gamma) > 0$. For each $R > 0$, we define the family of virtual balls as follows:

$$B_R(\gamma_0, \epsilon) = \{B_p(\gamma \xi_0, r) : \gamma \in \Gamma, 0 < r < \min(R, r_g(\gamma))\}.$$  

We remark that the difference of the definition of $B_R$ in this paper and our previous paper [15] lies in the definition of $r_g(\gamma)$; in [15], we used the
A-valued Busemann function in (6.1) whereas \( r_s(\gamma) \) is defined in terms of the \( AM \)-valued Busemann function here.

Let \( C = C(\psi, p) > 0 \) be as in [15, Thm 5.3] and \( \kappa > 0 \) be the constant from [15, Lem 5.7]. Since \( \xi_0 \) belongs to the shadow \( O_{\epsilon/(8\kappa)}(\eta, p) \), we can choose \( 0 < s = s(\gamma_0) < R \) small enough such that

\[
B_p(\xi_0, e^{-(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))}+\frac{1}{2}\|\psi\|^{2}C\kappa s) \subset O_{\epsilon/(8\kappa)}(\eta, p).
\]

Next, observe that the function \( \xi' \mapsto \beta_{\xi'}(g, \gamma_0 g) \) is continuous at \( \xi_0 \), as \( g^{-1}\xi_0 = e^+ \in N+e^+ \). Hence we may further assume \( s \) is small enough so that

\[
\beta_{\xi'}^{AM}(g, \gamma_0 g) = \beta_{\xi_0}^{AM}(g, \gamma_0 g)(AM)_{\epsilon} \quad \text{for all } \xi' \in B_p(\xi_0, e^{2C\kappa s}).
\]

For each \( \gamma \in \Gamma \), set

\[
D(\gamma \xi_0, r) := B_p(\gamma \xi_0, \frac{1}{3N_0}e^{-\psi(\mu(g^{-1}\gamma g)+\mu(g^{-1}\gamma^{-1}g))r}), \quad \text{and}
\]

\[
3N_0D(\gamma \xi_0, r) := B_p(\gamma \xi_0, e^{-\psi(\mu(g^{-1}\gamma g)+\mu(g^{-1}\gamma^{-1}g))r}).
\]

**Lemma 6.6.** For any \( \xi \in \Lambda_M - \{\eta\} \), there exists an infinite sequence \( \gamma_i \in \Gamma \) such that

\[
\lim_{i \to \infty} \gamma_i^{-1}p = \eta, \quad \lim_{i \to \infty} \gamma_i^{-1}\xi = \xi_0, \quad \text{and} \quad \lim_{i \to \infty} \beta_{\xi_i}^{M}(\gamma_i, e) = e.
\]

Moreover, there exists a neighborhood \( U \) of \( \xi_0 \) such that as \( i \to \infty \), \( \gamma_i \xi' \) converges to \( \xi \) uniformly for all \( \xi' \in U \).

**Proof.** Since \( \xi \in \Lambda_M, \psi \), there exists \( (\xi, \xi, 0, m_0) \in \tilde{Z}_\psi \) for some \( \xi \neq \xi \in \Lambda \) and \( m_0 \in M \). By definition of \( \tilde{Z}_\psi \), there exist sequences \( \gamma_i \in \Gamma \) and \( t_i \to +\infty \) such that

\[
\lim_{i \to \infty} (\gamma_i^{-1}\xi, \gamma_i^{-1}\xi, \psi(\log \beta_{\xi_i}^A(\gamma_i, e)) + t_i, \beta_{\xi_i}^{M}(\gamma_i, e)m_0) = (\xi_0, \eta, 0, m_0).
\]

The last two conditions in (6.4) are immediate and the first condition can be proved by the same proof of [15, Thm 8.9].

By passing to a subsequence, we may write \( \gamma_i = k_i a_i \xi_i^{-1} \) where \( k_i \to k_0, a_i \to a_0 \) in \( K \) and \( a_i \in A^+ \). As \( \Gamma \) is Anosov, \( a_i \to \infty \) regularly in \( A^+ \).

We then have \( \xi_i^{-1} = \eta \). Note that \( \gamma_i \xi' \to k_0^+ \) for all \( \xi' \in F \) with \( (\xi', \eta) \in F^{(2)} \) and this convergence is uniform on a compact subset of \( \{\xi': (\xi', \eta) \in F^{(2)}\} \).

Since \( (\xi_0, \eta) \in F^{(2)} \), there exists a neighborhood \( U \) of \( \xi_0 \) such that \( \gamma_i \xi' \to k_0^+ \) uniformly for all \( \xi' \in U \). Since \( \gamma_i^{-1}\xi \to \xi_0 \) and hence \( \gamma_i^{-1}\xi \in U \) for all large \( i \), we have \( \gamma_i(\gamma_i^{-1}\xi) \to k_0^+ \). Hence \( \xi = k_0^+ \). The claim follows. \( \square \)

**Lemma 6.7.** Let \( R > 0 \) and \( \xi \in \Lambda - \{\eta\} \). Suppose that \( \gamma_i^{-1}p \to \eta, \gamma_i^{-1}\xi \to \xi_0 \) and \( \beta_{\xi_i}^{M}(\gamma_i, e) \to e \) as \( i \to \infty \) for some \( \gamma_i \in \Gamma \). Then for all sufficiently small \( r > 0 \), there exists \( i_0 = i_0(r) > 0 \) such that for all \( i \geq i_0 \), the following holds:

1. \( \xi \in D(\gamma_i \xi_0, r) \subset B_R(\gamma_0, \epsilon) \); in particular, for any \( R > 0 \), \( \Lambda_M, \psi \subset \bigcup_{D \in B_R(\gamma_0, \epsilon)} D \).

2. \( \beta_{\xi_i}^{AM}(\gamma_i \gamma_0 \gamma_i^{-1}) \in \lambda(\gamma_0)(AM)_{O(\epsilon)} \) for all \( \xi' \in 3N_0D(\gamma_i \xi_0, r) \).
Proof. Note that $\gamma_i^{-1}g_0 \to \eta = g^-$ and $\gamma_i^{-1}\xi \to \xi_0 = g^+$. Let $U_\varepsilon$ be a neighborhood of $\xi_0$ associated to the sequence $\gamma_i$, as in Proposition 4.4. Since $\xi_0 \in U_\varepsilon$, there exists $R_1 > 0$ such that

$$\B_p(\xi_0, e^{2C} R_1), \gamma_0^{-1} U_p(\xi_0, e^{2C} R_1) \subset U_\varepsilon.$$  

Let $0 < r < \min(s(\gamma_0), R_1/2, R_1, R)$. In view of [15, Lem 10.12], we have $3N_0 D(\gamma_i \xi_0, r) \subset \gamma_i \B_p(\xi_0, e^{2C} r)$. In order to show $D(\gamma_i \xi_0, r) \in \B_R(\gamma_0, \varepsilon)$, it suffices to check that for all $\xi' \in \B_p(\xi_0, e^{2C} r)$,

$$\beta^M(\gamma_i^{-1} g, \gamma_0 \gamma_i^{-1} g) \in \beta^M(\gamma_i^{-1} g, \gamma_0 \gamma_i^{-1} g) M_\varepsilon;$$

this implies that $r < r_\varepsilon(\gamma_0)$.

We start by noting that since $r \leq s(\gamma_0)$, $\beta^M(\xi, \gamma_0 g) \in \beta^M_0(\xi, \gamma_0 g) M_\varepsilon$.

Since $\xi', \gamma_0^{-1} \xi' \in U_\varepsilon$, by Proposition 4.4, for all sufficiently large $i$,

$$\beta^M(\xi', \gamma_0 g) = \beta^M(\gamma_i^{-1} g, \gamma_0 g) \beta^M(\gamma_0 g, \gamma_i^{-1} g)
= \beta^M(\xi, \gamma_0 g) \beta^M(\gamma_0 g, \gamma_i^{-1} g)
\in \beta^M_0(\xi, \gamma_0 g) \beta^M(\gamma_i^{-1} g, \gamma_0 g) \beta^M(\gamma_0 g, \gamma_i^{-1} g) M(\varepsilon)
= \beta^M(\gamma_i^{-1} g, \gamma_0 g) M(\varepsilon),$$

which verifies that $D(\gamma_i \xi_0, r)$ belongs to the family $\B_R(\gamma_0, \varepsilon)$. The claim that $\xi \in D(\gamma_i \xi_0, r)$ can be shown in the same way as in the proof of [15, Lem. 10.12]. This proves (1).

(1) implies that for all sufficiently large $i$ and $\xi' \in 3N_0 D(\gamma_i \xi_0, r)$, we have

$$\beta^A(\xi, \gamma_0 g) \in \beta^A_0(\gamma_0 g) (AM)_\varepsilon.$$

Now note that for all $\xi' \in 3N_0 D(\gamma_0 \xi_0, r)$,

$$\beta^A(\xi', \gamma_0 g) = \beta^A(\gamma_0 g) \beta^A(\gamma_0 g, \xi_0) M(\varepsilon)
= \beta^A(\xi, \gamma_0 g) \beta^A(\gamma_0 g, \xi_0) M(\varepsilon),$$

on the other hand,

$$d_p(\gamma_0 g, \xi) \leq e^{-\psi(\log \beta^A(\gamma_0 g, \xi_0)) + 2\varepsilon\psi(\varepsilon)} d_p(\xi, \gamma_0 g)
\leq e^{\psi(\log(\gamma_0 g, \xi_0)) + 2\varepsilon} d_p(\xi, \gamma_0 g),$$

hence

$$\xi', \gamma_0 g \xi' \in \B_p(\xi_0, e^{\psi(\log(\gamma_0 g, \xi_0)) + 2\varepsilon} \varepsilon).$$

Since $\gamma_i \xi_0 \to \xi$ as $i \to \infty$ by Lemma 6.6 and $r < R_\varepsilon/2$, for all sufficiently large $i$ and all $\xi' \in 3N_0 D(\gamma_i \xi_0, r)$, $\gamma_0 g \gamma_i^{-1} \xi'$, and $\gamma_i \xi_0$ all belong to $\B_p(\xi, e^{\psi(\log(\gamma_0 g, \xi_0)) + 2\varepsilon} \varepsilon R_\varepsilon)$ and hence

$$\beta^A(\gamma_0 g, \xi_0) \beta^A(\gamma_0 g, \xi_0) \beta^A(\gamma_0 g, \xi_0) \beta^A(\gamma_0 g, \xi_0) M(\varepsilon),$$

Combining (5.6), (6.6) and (6.7), it follows that for all $\xi' \in 3N_0 D(\gamma_i \xi_0, r)$,

$$\beta^A(\gamma_0 g, \xi_0) \beta^A(\gamma_0 g, \xi_0) \beta^A(\gamma_0 g, \xi_0) \beta^A(\gamma_0 g, \xi_0) M(\varepsilon).$$
Since $\beta^M_{\xi_0}(\gamma^{-1}_i, e) \to e$ as $i \to \infty$ and

$$
\beta^M_{\gamma\xi_0}(e, \gamma\gamma_0\gamma^{-1}_i) = \beta^M_{\gamma\xi_0}(e, \gamma_i)\beta^M_{\gamma\xi_0}(\gamma_i, \gamma\gamma_0)\beta^M_{\gamma\xi_0}(\gamma_i, \gamma\gamma_0)\gamma^{-1}_i
$$

we obtain $\beta^M_{\xi_0}(e, \gamma\gamma_0\gamma^{-1}_i) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon$, as desired. \qed

**Lemma 6.8.** Let $B \subset F$ be a Borel set with $\nu_\psi(B) > 0$. Then for $\nu_\psi$-a.e. $\xi \in B$,

$$
\limsup_{R \to 0} \left\{ \frac{\nu_\psi(B \cap D)}{\nu_\psi(D)} : \begin{array}{l}
\xi \in D = D(\gamma\xi_0, r), r < R, \text{ and } \\
\beta^M_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon \\
\text{for all } \xi' \in 3N_0D(\gamma\xi_0, r).
\end{array} \right\} = 1.
$$

**Proof.** For a Borel function $H : G/P \to \mathbb{R}$, we associate a function $H^* : G/P \to \mathbb{R}$ defined by

$$
H^*(\xi) = \limsup_{R \to 0} \left\{ \frac{1}{\nu_\psi(D)} \int_D H \psi_\xi : \begin{array}{l}
\xi \in D = D(\gamma\xi_0, r), r < R, \text{ and } \\
\beta^M_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon \\
\text{for all } \xi' \in 3N_0D(\gamma\xi_0, r).
\end{array} \right\}.
$$

By Lemma 6.6 and 6.7, $H^*$ is well defined on $\Lambda_{M, \psi} - \{\eta\}$ and hence $\nu_\psi$-a.e. on $G/P$ by Corollary 3.8. The rest of the proof is same as in [15]. \qed

**Proof of Proposition 6.5.** Let $B \subset F$ be a Borel set such that $\nu_\psi(B) > 0$ and let $\varepsilon > 0$ be arbitrary. By Lemma 6.8, for $\nu_\psi$-a.e. $\xi \in B$, there exist $\gamma \in \Gamma^*_{\text{lox}}$ and $D = D(\gamma\xi_0, r) \in B_R(\gamma_0, \varepsilon)$ containing $\xi$ such that

1. $\nu_\psi(D \cap B) > (1 + e^{-\psi(\Lambda(\gamma^{-1})-\|\psi\|\varepsilon)})^{-1}\nu_\psi(B)$, and
2. $\beta^M_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon$ for all $\xi' \in 3N_0D(\gamma\xi_0, r)$.

We claim that

$$
B \cap \gamma\gamma_0\gamma^{-1}B \cap \{\xi : \beta^M_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon\}
$$

has a positive $\nu_\psi$-measure, which will finish the proof.

We have $\gamma\gamma_0\gamma^{-1}D \subset D$ by [15, Proof of Prop. 10.7]. Together with (2) above, it follows that

$$
\beta^M_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon \text{ for all } \xi \in \gamma\gamma_0\gamma^{-1}D.
$$

Consequently, (6.8) contains

$$
(D \cap B) \cap \gamma\gamma_0\gamma^{-1}(D \cap B),
$$

which has a positive $\nu_\psi$-measure by [15, Proof of Prop. 10.7]. This proves the claim. \qed
ERGODIC DECOMPOSITIONS

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