ERGODIC DECOMPOSITIONS OF GEOMETRIC MEASURES ON ANOSOV HOMOGENEOUS SPACES.

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Abstract. Let G be a connected semisimple real algebraic group and Γ a Zariski dense Anosov subgroup of G with respect to a minimal parabolic subgroup P. Let N be the maximal horospherical subgroup of G given by the unipotent radical of P. We describe the N-ergodic decompositions of all Burger-Roblin measures as well as the A-ergodic decompositions of all Bowen-Margulis-Sullivan measures on $\Gamma\backslash G$. As a consequence, we obtain the following refinement of the main result of [16]: the space of all *non-trivial* N-invariant ergodic and P°-quasi-invariant Radon measures on $\Gamma\backslash G$, up to constant multiples, is homeomorphic to $\mathbb{R}^{\operatorname{rank} G-1}\times\{1,\cdots,k\}$ where k is the number of P°-minimal subsets in $\Gamma\backslash G$.

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1. Introduction

Let G be a connected semisimple real algebraic group, i.e., the identity component of the group of real points of a semisimple algebraic group defined over $\mathbb{R}$. Let $\Gamma < G$ be a Zariski dense Anosov subgroup of G with respect to a minimal parabolic subgroup P. Fix a Langlands decomposition $P = MAN$ where N is the unipotent radical of P, A is the identity component of a maximal real split torus of G and M is the maximal compact subgroup of P commuting with A. The subgroup N is a maximal horospherical subgroup of G, and in fact, any maximal horospherical subgroup of G arises in this way.

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In our earlier paper [16], we showed that all NM-invariant Burger-Roblin measures on $\Gamma\backslash G$, parameterized by $\mathbb{R}^{\operatorname{rank} G-1}$, are NM-ergodic and that they describe precisely all non-trivial NM-invariant ergodic and P°-quasi-invariant Radon (i.e., locally finite Borel) measures on $\Gamma\backslash G$, where P° is the identity component of P. One cannot replace NM by N in these statements, as the Burger-Roblin measures are not N-ergodic in general. The main aim of this paper is to describe the N-ergodic decompositions of Burger-Roblin measures as well as to classify all non-trivial N-invariant ergodic and P°-quasi-invariant Radon measures on $\Gamma\backslash G$. When G has rank one, the class of Anosov subgroups of G coincides with that of convex cocompact subgroups. If P is connected in addition, which is equivalent to saying $G \not\simeq \mathrm{SL}_2(\mathbb{R})$, then there exists a unique non-trivial N-invariant ergodic measure, as shown by Burger, Roblin and Winter ([4], [20], [27]). This unique measure is called the Burger-Roblin measure. We also mention that when $\Gamma < G$ is a lattice, the classification of ergodic invariant measures for a maximal horospherical subgroup action was obtained by Furstenberg, Veech and Dani ([9], [25], [7]), prior to Ratner's more general measure classification theorem for any connected unipotent subgroup action [19].

We begin by recalling the definition of an Anosov subgroup. Let $\mathcal{F} := G/P$ denote the Furstenberg boundary, and $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F}\times\mathcal{F}$. A Zariski dense discrete subgroup $\Gamma < G$ is called an *Anosov subgroup* (with respect to P) if it is a finitely generated word hyperbolic group which admits a Γ-equivariant continuous embedding ζ of the Gromov boundary $\partial\Gamma$ into $\mathcal{F}$ such that $(\zeta(x),\zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y$ in $\partial\Gamma$ ([14], [10], [13], [26]). The class of Anosov subgroups include the Zariski dense images of representations in the Hitchin component as well as Zariski dense Schottky subgroups.

Denote by $\mathfrak{a}$ the Lie algebra of A and fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ so that $\log N$ is the sum of positive root subspaces. Fix a maximal compact subgroup K of G as in section 2, so that the Cartan decomposition $G = KA^+K$ holds for $A^+ = \exp \mathfrak{a}^+$ (Def. 2.9).

Let $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ denote the limit cone of Γ (Def. 2.8), which is known to be a convex cone with non-empty interior by Benoist [1]. Let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R}\cup\{-\infty\}$ be the growth indicator function of Γ as defined by Quint (Def. 4.1). Consider the following set of linear forms on $\mathfrak{a}$:

$$D_\Gamma^\star := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma, \psi(v) = \psi_\Gamma(v) \text{ for some } v \in \operatorname{int} \mathcal{L}_\Gamma\}.$$

For each $\psi \in D_\Gamma^\star$, we denote by m_ψ^{BR} and m_ψ^{BMS} respectively the Burger-Roblin measure and the Bowen-Margulis-Sullivan measure on $\Gamma\backslash G$ associated to ψ (see (4.6) and (4.8)). The Burger-Roblin measures are all supported on the unique P-minimal subset of $\Gamma\backslash G$:

$$\mathcal{E} := \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

where $\Lambda \subset \mathcal{F}$ denotes the limit set of Γ. In [16], we showed that for Γ Anosov, each m_ψ^{BR} is NM-ergodic and the map

$$\psi \mapsto m_\psi^{\mathrm{BR}}$$

gives a homeomorphism between $D_\Gamma^\star$ and the space of all NM-invariant ergodic and P-quasi invariant Radon measures supported on $\mathcal{E}$, up to constant multiples. We also showed that all m_ψ^{BMS}, $\psi \in D_\Gamma^\star$, are AM-ergodic.

Denote by $\mathfrak{Y}_\Gamma$ the collection of all P°-minimal subsets of $\Gamma\backslash G$. Fixing $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, we set

$$P_\Gamma := \{p \in P : \mathcal{E}_0 p = \mathcal{E}_0\}.$$

By the work of Guivarc'h and Raugi [11], the subgroup P_Γ is independent of the choice of $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, and is a co-abelian subgroup of P containing P°. It follows that for any $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, the map $[p] \mapsto \mathcal{E}_0 p$ defines a bijection between P/P_Γ and $\mathfrak{Y}_\Gamma$. Considering the partition $\mathcal{E} = \bigsqcup_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \mathcal{E}_0$, the following is our main theorem:

Theorem 1.1. *For any Anosov subgroup $\Gamma < G$ and $\psi \in D_\Gamma^\star$,*

(1) *$m_\psi^{\mathrm{BR}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ is an N-ergodic decomposition;*
(2) *$m_\psi^{\mathrm{BMS}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$ is an A-ergodic decomposition.*

In particular, the number of the N-ergodic components of m_ψ^{BR} as well as the A-ergodic components of m_ψ^{BMS} are given by $\#\mathfrak{Y}_\Gamma = [P : P_\Gamma]$, independent of ψ.

See the subsection 7.6 and Theorem 4.4 for the proofs of (1) and (2) respectively.

As $P^\circ \subset P_\Gamma$, P_Γ is of the form $M_\Gamma AN$ where

$$M_\Gamma := \{m \in M : \mathcal{E}_0 m = \mathcal{E}_0\}.$$

Moreover, by [3, Prop. 4.9(a)], the subgroup M_Γ can be explicitly described as follows:

$$M_\Gamma = \text{closure of } \{m \in M : g^{-1}hamng \in \Gamma \text{ for some } h \in N^+, a \in A, n \in N\}$$

for any $g \in G$ such that $g\Gamma g^{-1} \cap \operatorname{int} A^+ M \neq \emptyset$, where N^+ denotes the opposite horospherical subgroup to N. The subgroup M_Γ is not equal to M in general: there exists a Zariski dense Schottky subgroup Γ with $M_\Gamma \neq M$ [2], and for an Anosov subgroup Γ which arises as the image of a Hitchin representation into $\mathrm{PSL}_n(\mathbb{R})$, it is known that $M_\Gamma = \{e\}$ [14].

Since each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$ is a second countable topological space, almost all orbits are dense with respect to an ergodic measure with full support in $\mathcal{E}_0$. Hence Theorem 1.1 implies:

Corollary 1.2. *Let $\mathcal{E}_0$ be a P°-minimal subset of $\Gamma\backslash G$. Then*

(1) *for $m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ almost all $x \in \mathcal{E}_0$, xN is dense in $\mathcal{E}_0$;*
(2) *for $m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$ almost all $x \in \mathcal{E}_0$, xA is dense in $\operatorname{supp} m_\psi^{\mathrm{BMS}} \cap \mathcal{E}_0$.*

Indeed, Corollary 1.2(2) holds for A^+-orbits as well (see Corollary 4.11). In view of our earlier work [16], Theorem 1.1 implies:

Theorem 1.3. *The space of all N-invariant ergodic and P°-quasi-invariant Radon measures on $\mathcal{E}$, up to constant multiples, is given by $\{m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0} : \psi \in D_\Gamma^\star, \mathcal{E}_0 \in \mathfrak{Y}_\Gamma\}$ and hence homeomorphic to $\mathbb{R}^{rank\, G-1} \times \{1, \cdots, \# M/M_\Gamma\}$.*

We mention a recent measure classification result [15] which is based on the above theorem.

On the proofs. For each $\psi \in D_\Gamma^\star$, there exists a unique (Γ, ψ)-Patterson-Sullivan measure, say, ν_ψ, on the limit set $\Lambda \subset G/P$. Denote by $\tilde{\nu}_\psi$ the M-invariant lift of ν_ψ to G/P°. We first show that the Γ-ergodic components of $\tilde{\nu}_\psi$ and the A-ergodic components of m_ψ^{BMS} are respectively given by their restrictions to Γ-minimal subsets of G/P° and to P°-minimal subsets of $\Gamma \backslash G$; hence Theorem 1.1(2). We define the closed subgroup, say E_{ν_ψ} of AM, consisting of all ν_ψ-essential values (Definition 6.1), and show that elements of the generalized length spectrum of Γ, whose ψ-images are sufficiently large, are contained in E_{ν_ψ} (Proposition 7.8). By Proposition 7.4, this implies that AM° is contained in E_{ν_ψ}, from which we deduce Theorem 1.1(1), using the NM-ergodicity of m_ψ^{BR}.

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2. Preliminaries

Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. We use the notation o for the coset $[K]$ in the associated Riemannian symmetric space G/K. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, and set $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, we also set $A^+ := \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M and we set N to be the contracting horospherical subgroup: for $a \in \operatorname{int} A^+$, $N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}$. Note that $\log N$ is the sum of all positive root subspaces for our choice of A^+. Similarly, we also consider the expanding horospherical subgroup N^+: for $a \in \operatorname{int} A^+$, $N^+ := \{g \in G : a^n g a^{-n} \to e \text{ as } n \to +\infty\}$. We set $P = MAN$ which is a minimal parabolic subgroup of G. The quotient $\mathcal{F} = G/P$ is known as the Furstenberg boundary of G and is isomorphic to K/M. We let $\Lambda \subset \mathcal{F}$ denote the limit set of Γ as defined in [1] (see also [16, Lem. 2.13] for an equivalent definition), which is known to be the unique Γ-minimal subset of $\mathcal{F}$.

We fix an element w_0 of the normalizer of $\mathfrak{a}$ such that $\operatorname{Ad}_{w_0}\mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $\mathrm{i}:\mathfrak{a}\to\mathfrak{a}$ is defined as $\mathrm{i}(u) = -\operatorname{Ad}_{w_0} u$.

Definition 2.1 (Visual maps)**.** For each $g\in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$

For all $g\in G$ and $m\in M$, observe that $g^\pm = (gm)^\pm = g(e^\pm)$. Let $\mathcal{F}^{(2)}$ denote the unique open G-orbit in $\mathcal{F}\times\mathcal{F}$:

$$\mathcal{F}^{(2)} = G(e^+,e^-) = \{(g^+,g^-)\in\mathcal{F}\times\mathcal{F} : g\in G\}.$$

We say that $\xi,\eta\in\mathcal{F}$ are in general position if $(\xi,\eta)\in\mathcal{F}^{(2)}$.

2.1. **A-valued cocycles.**

Definition 2.2. The A-valued Iwasawa cocycle $\sigma^A : G\times\mathcal{F}\to A$ is defined as follows: for $(g,\xi)\in G\times\mathcal{F}$, let $\sigma^A(g,\xi)\in A$ be the unique element satisfying

$$gk \in K\sigma^A(g,\xi)N \tag{2.1}$$

where $k\in K$ is such that $\xi = k^+$.

Definition 2.3. The A-valued Busemann function $\beta^A : \mathcal{F}\times G\times G\to A$ is defined as follows: for $\xi\in\mathcal{F}$ and $g_1,g_2\in G$, set

$$\beta^A_\xi(g_1,g_2) := \sigma^A(g_1^{-1},\xi)\,\sigma^A(g_2^{-1},\xi)^{-1}.$$

2.2. **AM-valued cocycles.** The product map $N^+\times P\to G$ is a diffeomorphism onto its image which is Zariski open and dense in G. Hence for each $\xi\in N^+e^+$, we can define $h_\xi\in N^+$ to be the unique element such that

$$\xi = h_\xi e^+. \tag{2.2}$$

Similarly, the product map $K\times A\times N\to G$ is a diffeomorphism, giving the Iwasawa decomposition $G = KAN$. We can therefore define $k_\xi\in K$ to be the unique element such that

$$h_\xi \in k_\xi AN. \tag{2.3}$$

Definition 2.4 (Bruhat cocycle and Iwasawa cocycle)**.** Let $g\in G$ and $\xi\in\mathcal{F}$ be such that $\xi, g\xi\in N^+e^+$.

(1) We define the Bruhat cocycle $b(g,\xi)\in AM$ to be the unique element satisfying

$$gh_\xi \in N^+ b(g,\xi)N.$$

Note that the condition $\xi\in N^+e^+$ allows us to get $h_\xi\in N^+$ and the condition $g\xi\in N^+e^+$ implies $gh_\xi\in N^+AMN$.

(2) We define the Iwasawa cocycle $\sigma^{AM}(g,\xi)\in AM$ to be the unique element satisfying

$$gk_\xi \in k_{g\xi}\sigma^{AM}(g,\xi)N.$$

Note that $gh_\xi \in h_{g\xi}b(g,\xi)N$.

We remark that although $\log \sigma^A(g,\xi)$ was defined as the Iwasawa cocycle in [16], we find it more convenient to use the above notation in this paper. In order to define the AM-valued Iwasawa cocycle, it is necessary to choose a Borel section of the projection $K \simeq G/AN \to K/M \simeq G/P$. In the above definition, we have used a section $\mathsf{s} : G/P \to G/AN$ given by $\mathsf{s}(hP) = hAN$ for all $h \in N^+$, so that it is continuous on $N^+e^+ \subset \mathcal{F}$. It follows that for each fixed $g \in G$, the maps $\xi \mapsto b(g,\xi)$ and $\xi \mapsto \sigma^{AM}(g,\xi)$ are continuous on the set $\{\xi \in N^+e^+ : g\xi \in N^+e^+\}$.

Definition 2.5 (AM-valued Busemann map)**.** For $(\xi, g_1, g_2) \in \mathcal{F} \times G \times G$ such that $\xi, g_1^{-1}\xi, g_2^{-1}\xi \in N^+e^+$, we define

$$\beta_\xi^{AM}(g_1,g_2) := \sigma^{AM}(g_1^{-1},\xi)\sigma^{AM}(g_2^{-1},\xi)^{-1}.$$

Remark 2.6. For fixed $g_1, g_2 \in G$, the map $\xi \mapsto \beta_\xi^{AM}(g_1,g_2)$ is continuous on the set $\{\xi \in N^+e^+ : g_1^{-1}\xi, g_2^{-1}\xi \in N^+e^+\}$.

We have the following whenever both sides are defined: for any $g_1, g_2, g_3 \in G$ and $\xi \in \mathcal{F}$,

(1) (cocycle identity) $\beta_\xi^{AM}(g_1,g_3) = \beta_\xi^{AM}(g_1,g_2)\,\beta_\xi^{AM}(g_2,g_3)$;
(2) (equivariance) $\beta_{g_3\xi}^{AM}(g_3g_1, g_3g_2) = \beta_\xi^{AM}(g_1,g_2)$.

We define β^M to be the projection of β^{AM} to M; we then have $\beta_\xi^{AM}(g_1,g_2) = \beta_\xi^A(g_1,g_2)\beta_\xi^M(g_1,g_2)$. It is simple to check the following:

Example 2.7. If $g = hamn \in N^+AMN$, then $\beta_{g^+}^M(e,g) = m$.

2.3. **Jordan projection and Cartan projection.** Recall that for any loxodromic element $g \in G$, there exists $\varphi \in G$ such that

$$g = \varphi a m \varphi^{-1}$$

for some element $am \in \operatorname{int} A^+M$. Moreover such φ belongs to a unique coset in G/AM. We set

$$y_g := \varphi^+ \in \mathcal{F}$$

which is called the attracting fixed point of g. The element $a \in \operatorname{int} A^+$ is uniquely determined and called the Jordan projection of g. We denote it by $\lambda(g)$. For a general element $g \in G$, g can be written as a commuting product $g_hg_ug_e$ where g_h, g_u and g_e are hyperbolic, unipotent and elliptic respectively. The hyperbolic element g_h belongs to AM up to conjugation, and the Jordan projection $\lambda(g)$ of g is defined as the unique element of $\mathfrak{a}^+$ such that $g_h \in \varphi \exp\lambda(g) m \varphi^{-1}$ for some $\varphi \in G$ and $m \in M$.

Definition 2.8. The limit cone $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ is defined as the smallest closed cone containing all $\lambda(\gamma) \in \mathfrak{a}^+$, $\gamma \in \Gamma$.

This is known to be a convex cone with non-empty interior [1].

Definition 2.9 (Cartan projection)**.** For each $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, such that

$$g \in K \exp(\mu(g))K.$$

3. Generalized length spectrum

In this section, we fix a discrete Zariski dense subgroup Γ of G.

3.1. P°-minimal subsets of $\Gamma\backslash G$.

Since Λ is the unique Γ-minimal subset of $\mathcal{F}$, it follows that the set

$$\mathcal{E} := \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\} \tag{3.1}$$

is the unique P-minimal subset of $\Gamma\backslash G$. We refer to [11, Thm. 2 and Thm. 1.9] for results in this subsection. Set $\mathcal{F}^\circ = G/P^\circ$. For any $g \in G$ with $g^+ \in \Lambda$, the closure of $\Gamma g[P^\circ]$ is a Γ-minimal subset of $\mathcal{F}^\circ$. Moreover the following closed subgroup of M is well-defined:

$$M_\Gamma := \{m \in M : \Lambda_0 m = \Lambda_0\} \tag{3.2}$$

for any Γ-minimal subset Λ_0 of $\mathcal{F}^\circ$. The subgroup M° is a co-abelian subgroup of M and M_Γ/M° is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^p$ for some $0 \le p \le \dim A$.

For any Γ-minimal subset Λ_0 of $\mathcal{F}_0$, the map $s \mapsto \Lambda_0 s$ gives a bijection between $M_\Gamma\backslash M$ and the collection $\mathcal{Y}_\Gamma$ of all Γ-minimal subsets of $\mathcal{F}^\circ$. If we set $\tilde\Lambda := \{gP^\circ \in \mathcal{F}^\circ : gP \in \Lambda\}$, then

$$\tilde\Lambda = \bigsqcup_{\Lambda_0 \in \mathcal{Y}_\Gamma} \Lambda_0.$$

These results can be translated into statements about P°-minimal subsets of $\Gamma\backslash G$ by duality. Each $\Lambda_0 \in \mathcal{Y}_\Gamma$ is of the form $E(\Lambda_0)/P^\circ$ for some left Γ-invariant and right P°-invariant closed subset $E(\Lambda_0)$ of G. The map $\Lambda_0 \mapsto \Gamma\backslash E(\Lambda_0)$ gives a bijection between $\mathcal{Y}_\Gamma$ and the collection of all P°-minimal subsets of $\Gamma\backslash G$, say $\mathfrak{Y}_\Gamma$. Moreover, if we set

$$P_\Gamma := M_\Gamma A N, \tag{3.3}$$

then $P_\Gamma = \{p \in P : \mathcal{E}_0 p = \mathcal{E}_0\}$ for all $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$. We also have

$$\mathcal{E} = \bigsqcup_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \mathcal{E}_0.$$

We remark that each P°-minimal subset of $\Gamma\backslash G$ is in fact AN-minimal; this follows from [11, Thm. 2].

3.2. Generalized length spectrum.

We define

$$\Gamma^\star := \{\gamma \in \Gamma : \text{there exists } \varphi \in N^+N \text{ with } \gamma \in \varphi(\operatorname{int} A^+M)\varphi^{-1}\}. \tag{3.4}$$

Note that if $\gamma \in \Gamma$ is loxodromic and $y_\gamma \in N^+e^+$, then $\gamma \in \Gamma^\star$. As Γ is Zariski dense, the set of loxodromic elements of Γ is Zariski dense in G [1]. It follows that $\Gamma^\star$ is Zariski dense in G as well.

Definition 3.1. For $\gamma \in \Gamma^\star$, we define its *generalized Jordan projection* $\hat\lambda(\gamma)$ to be the unique element of $\operatorname{int} A^+M$ such that

$$\gamma = \varphi\hat\lambda(\gamma)\varphi^{-1} \quad \text{for some } \varphi \in N^+N.$$

Definition 3.2. We call the following set the *generalized length spectrum* of Γ:

$$\hat{\lambda}(\Gamma) := \{\hat{\lambda}(\gamma) \in AM : \gamma \in \Gamma^\star\}.$$

We denote by

$$\mathsf{s}(\Gamma)$$

the closed subgroup of AM generated by $\hat{\lambda}(\Gamma)$.

We refer to Remark 3.8 for the independence of $\mathsf{s}(\Gamma)$ on some choices.

Lemma 3.3. *For all $\gamma \in \Gamma^\star$, we have*

$$\hat{\lambda}(\gamma) = b(\gamma, y_\gamma) = \beta^{AM}_{y_\gamma}(e, \gamma).$$

Proof. Since $\gamma \in \Gamma^\star$, we have $\gamma = \varphi\hat{\lambda}(\gamma)\varphi^{-1}$ for some $\varphi = hn$, where $h \in N^+$ and $n \in N$. Set $\xi := y_\gamma = \varphi^+$. In particular, $h_\xi = h$ and $h \in k_\xi AN$. The defining relations for $b(\gamma, \xi)$ and $\beta^{AM}_\xi(e, \gamma)$ are

$$\gamma h \in hb(\gamma, \xi)N \text{ and } \gamma k_\xi \in k_\xi \beta^{AM}_\xi(e, \gamma)N.$$

Now observe that

$$\gamma h = \varphi\hat{\lambda}(\gamma)\varphi^{-1}h = hn\hat{\lambda}(\gamma)n^{-1} \in h\hat{\lambda}(\gamma)N \text{ and}$$
$$\gamma k_\xi = \varphi\hat{\lambda}(\gamma)\varphi^{-1}k_\xi = k_\xi(k_\xi^{-1}h)n\hat{\lambda}(\gamma)n^{-1}(h^{-1}k_\xi) \in k_\xi\hat{\lambda}(\gamma)N.$$

Therefore $\hat{\lambda}(\gamma) = b(\gamma, \xi) = \beta^{AM}_\xi(e, \gamma)$. □

For each $\xi \in \Lambda \cap N^+e^+$, we define $b_\xi(\Gamma)$ to be the closed subgroup of AM generated by all $b(\gamma, \xi)$ where $\gamma \in \Gamma$ and $\gamma\xi \in N^+e^+$.

Lemma 3.4. *The subgroup $b_\xi(\Gamma) < AM$ is independent of $\xi \in \Lambda \cap N^+e^+$.*

Proof. Let $\xi_1, \xi_2 \in \Lambda \cap N^+e^+$. To show that $b_{\xi_1}(\Gamma) = b_{\xi_2}(\Gamma)$, it suffices to check that $b(\gamma, \xi_2) \in b_{\xi_1}(\Gamma)$ for any $\gamma \in \Gamma$ such that $\gamma\xi_2 \in N^+e^+$. Since Λ is Γ-minimal, there exists a sequence $\gamma_n \in \Gamma$ such that $\lim_{n\to\infty} \gamma_n\xi_1 = \xi_2$. Since N^+e^+ is open and $\xi_2, \gamma\xi_2 \in N^+e^+$, , we have $\gamma_n\xi_1, \gamma\gamma_n\xi_1 \in N^+e^+$ for all large n and $b(\gamma\gamma_n, \xi_1) = b(\gamma, \gamma_n\xi_1)b(\gamma_n, \xi_1)$. Hence

$$b(\gamma, \xi_2) = \lim_{n\to\infty} b(\gamma, \gamma_n\xi_1) = \lim_{n\to\infty} b(\gamma\gamma_n, \xi_1)b(\gamma_n, \xi_1)^{-1} \in b_{\xi_1}(\Gamma),$$

from which the lemma follows. □

By Lemma 3.4, we may define

$$b(\Gamma) := b_\xi(\Gamma) \quad \text{for any } \xi \in \Lambda \cap N^+e^+.$$

In the rest of this section, we assume that

$$\Gamma \cap \operatorname{int} A^+M \neq \emptyset.$$

Lemma 3.5. *We have $b(\Gamma) = \mathsf{s}(\Gamma)$.*

Proof. We first claim that $b(\Gamma) \subset \mathsf{s}(\Gamma)$. By Lemma 3.4, it suffices to show that $b(\gamma, e^+) \in \mathsf{s}(\Gamma)$ for any $\gamma \in \Gamma$ with $\gamma e^+ \in N^+e^+$. Set $s_0 := a_0 m_0 \in \Gamma \cap \operatorname{int} A^+M$. Since γe^+ and e^- are in general position, for all sufficiently large n, $s_0^n\gamma$ is a loxodromic element and $x_n := y_{s_0^n\gamma}$ converges to e^+ as $n \to \infty$. Since $y_{s_0^n\gamma} \in N^+e^+$, we have $s_0^n\gamma \in \Gamma^\star$ for all large n. Now the claim follows from

$$\begin{aligned} b(\gamma, e^+) &= \lim_{n\to\infty} b(\gamma, x_n) = \lim_{n\to\infty} b(s_0^n, \gamma x_n)^{-1} b(s_0^n\gamma, x_n) \\ &= \lim_{n\to\infty} \hat{\lambda}(s_0^n)^{-1}\hat{\lambda}(s_0^n\gamma) \in \mathsf{s}(\Gamma) \end{aligned}$$

We next claim $\mathsf{s}(\Gamma) \subset b(\Gamma)$. Let $\gamma \in \Gamma^\star$ be arbitrary. Note that $y_\gamma \in N^+e^+$. By Lemma 3.3, $\hat{\lambda}(\gamma) = b(\gamma, y_\gamma) \in b_{y_\gamma}(\Gamma)$. Since $b(\Gamma) = b_{y_\gamma}(\Gamma)$ by Lemma 3.4, we have $\hat{\lambda}(\gamma) \in b(\Gamma)$, proving the claim. □

Proposition 3.6. *We have*

1. $b(\Gamma) = b(g^{-1}\Gamma g)$ *for all* $g \in G$ *with* $g^\pm \in \Lambda$*;*
2. $b(\Gamma)$ *is a co-abelian subgroup of* AM *containing* AM°*;*
3. $b(\Gamma) = AM_\Gamma$*.*

Proof. Claims (1) and (2) are proved in [11, Thm. 1.9]. Claim (3) follows since $A \subset b(\Gamma)$ by (2) and the closure of $\{m \in M : \Gamma \cap N^+AmN \neq \emptyset\}$ is equal to M_Γ [3, Prop. 4.9(a)]. □

Hence we deduce the following from Lemma 3.5 and Proposition 3.6.

Corollary 3.7. *We have*

$$\mathsf{s}(\Gamma) = AM_\Gamma.$$

Remark 3.8. We mention that as long as $g \in G$ satisfies $g^\pm \in \Lambda$, we can use $\varphi \in g^{-1}N^+N^-$ and $\xi \in \Lambda \cap g^{-1}N^+e^+$ in defining $\Gamma^\star$, $\hat{\lambda}(\gamma)$ and $b_\xi(\Gamma)$, and get the same $\mathsf{s}(\Gamma) = AM_\Gamma$ by [11, Prop. 1.8 and Thm. 1.9].

4. A-ergodic decompositions of BMS-measures

As before, let Γ be a discrete Zariski dense subgroup of G.

Definition 4.1 (Growth indicator function)**.** The growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as follows: for any vector $u \in \mathfrak{a}^+$,

$$\psi_\Gamma(u) := \|u\| \cdot \inf_{\substack{\text{open cones } \mathcal{C} \subset \mathfrak{a}^+ \\ u \in \mathcal{C}}} \tau_{\mathcal{C}}$$

where $\tau_{\mathcal{C}}$ is the abscissa of convergence of the series $\sum_{\gamma\in\Gamma, \mu(\gamma)\in\mathcal{C}} e^{-t\|\mu(\gamma)\|}$.

We consider ψ_Γ as a function on $\mathfrak{a}$ by setting $\psi_\Gamma = -\infty$ outside of $\mathfrak{a}^+$.

For a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on Λ is called a (Γ, ψ)-Patterson-Sullivan measure if for all $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\log \beta_\xi^A(e,\gamma))}. \tag{4.1}$$

Set

$$D_\Gamma^\star := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma, \psi(u) = \psi_\Gamma(u) \text{ for some } u \in \operatorname{int} \mathcal{L}_\Gamma\}.$$

For each linear form $\psi \in D_\Gamma^\star$, Quint constructed a (Γ, ψ)-Patterson-Sullivan measure, say, ν_ψ [18, Thm. 4.10]. For an Anosov group Γ, it was shown in [16, Thm. 1.3] that the map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D_\Gamma^\star$ and the space of all Γ Patterson-Sullivan measures.

4.1. **Antipodality of Γ.** When Γ is Anosov, we have the following so-called antipodal property from its definition:

$$\{(\xi, \eta) \in \Lambda \times \Lambda : \xi \neq \eta\} \subset \mathcal{F}^{(2)}. \tag{4.2}$$

Lemma 4.2. *Let Γ be Anosov. If $g \in G$ satisfies $g^- \in \Lambda$, then $g^{-1}\Lambda \subset N^+e^+ \cup \{e^-\}$.*

Proof. Suppose that $\xi \in \Lambda$ and $g^{-1}\xi \neq e^-$. Then $\xi \neq g^-$ in Λ. Hence by (4.2), $(\xi, g^-) \in \mathcal{F}^{(2)}$, or equivalently, $(g^{-1}\xi, e^-) \in \mathcal{F}^{(2)}$. Since $\{\eta \in \mathcal{F} : (\eta, e^-) \in \mathcal{F}^{(2)}\} = N^+e^+$, $g^{-1}\xi \in N^+e^+$, proving the claim. □

Corollary 4.3. *Let $\psi \in D_\Gamma^\star$. For any $g \in G$ with $g^\pm \in \Lambda$,*

$$\nu_\psi(\Lambda \cap gN^+e^+) = 1.$$

Proof. By Lemma 4.2, $\Lambda - \{g^-\} = \Lambda \cap gN^+e^+$. Hence the claim follows from the fact that ν_ψ is atom-free [16, Lem. 7.8]. □

In the rest of this section, we assume that $\Gamma < G$ is an Anosov subgroup. We will assume that

$$\Gamma \cap \operatorname{int} A^+M \neq \emptyset;$$

this can be achieved by replacing Γ by one of its conjugates, and hence we do not lose any generality of our discussion by making such an assumption.

By Corollary 4.3, this assumption implies that

$$\nu_\psi(\Lambda \cap N^+e^+) = 1 \quad \text{for any } \psi \in D_\Gamma^\star.$$

4.2. **Hopf parametrization of G.** The map $\mathrm{i}(gM) = (g^+, g^-, \beta^A_{g^+}(e, g))$ gives a G-equivariant homeomorphism between G/M and $\mathcal{F}^{(2)} \times A$, where the G-action on the latter is given by

$$g.(\xi, \eta, a) = (g\xi, g\eta, \beta^A_{g\xi}(e, g)a) \quad \text{for } g \in G \text{ and } ((\xi, \eta), a) \in \mathcal{F}^{(2)} \times A.$$

For the principal M-bundle $G \to G/M$, we fix a Borel section $\mathsf{s} : G/M \to G$ so that $\mathsf{s}(hanM) = han$ for all $han \in N^+AN$. Now for any $g \in G$, there exists a unique $m_g \in M$ such that $g = \mathsf{s}(gM)m_g$. Then the map $\mathrm{j}(g) = (\mathrm{i}(gM), m_g)$ gives a G-equivariant Borel isomorphism of G with $\mathcal{F}^{(2)} \times AM$ where the G action on the latter is given by

$$g.(\xi, \eta, am) = (g\xi, g\eta, \beta^{AM}_{g\xi}(e, g)am) \tag{4.3}$$

whenever $\xi, g\xi \in N^+e^+$. We call this map the Hopf parametrization of G (relative to the choice of s). We mention that this map was also considered in [6].

The restriction of j to N^+P is given by

$$\mathrm{j}(g) = (g^+, g^-, \beta^{AM}_{g^+}(e,g)) \quad \text{for } g \in N^+P \tag{4.4}$$

which gives a homeomorphism

$$N^+P \simeq \{(\xi,\eta,am) \in \mathcal{F}^{(2)} \times AM : \xi \in N^+e^+\}.$$

Fix $\psi \in D^\star_\Gamma$ in the rest of this section. For $(\xi_1,\xi_2) \in \mathcal{F}^{(2)}$, define the ψ-Gromov product:

$$[\xi_1,\xi_2]_\psi := \psi(\log \beta^A_{g^+}(e,g) + \mathrm{i} \log \beta^A_{g^-}(e,g)) \tag{4.5}$$

where $g \in G$ is such that $g^+ = \xi_1$ and $g^- = \xi_2$.

In terms of the Hopf parametrization of G, the following defines a left Γ-invariant and right AM-invariant measure on G:

$$\begin{aligned} d\tilde{m}^{\mathrm{BMS}}_\psi(g) &= e^{\psi(\log \beta^A_{g^+}(e,g) + \mathrm{i} \log \beta^A_{g^-}(e,g))}\, d\nu_\psi(g^+) d\nu_{\psi\circ \mathrm{i}}(g^-)\, da\, dm \\ &= e^{[\xi_1,\xi_2]_\psi}\, d\nu_\psi(g^+) d\nu_{\psi\circ \mathrm{i}}(g^-)\, da\, dm. \end{aligned} \tag{4.6}$$

We denote by m^{BMS}_ψ the measure on $\Gamma\backslash G$ induced by $\tilde{m}^{\mathrm{BMS}}_\psi$ and call it the Bowen-Margulis-Sullivan measure (associated to ψ). Note that its support is equal to

$$\Omega := \{x \in \Gamma\backslash G : x^\pm \in \Lambda\}. \tag{4.7}$$

In ([21], [16]), it was noted that m^{BMS}_ψ is an AM-ergodic measure and that it is infinite whenever $\operatorname{rank} G \geq 2$.

Similarly, the Burger-Roblin measure m^{BR}_ψ on $\Gamma\backslash G$ is induced from the following left Γ-invariant and right NM-invariant measure on G:

$$d\tilde{m}^{\mathrm{BR}}_\psi(g) = e^{\psi(\log \beta^A_{g^+}(e,g)) + 2\rho(\log \beta^A_{g^-}(e,g))}\, d\nu_\psi(g^+) dm_o(g^-)\, da\, dm, \tag{4.8}$$

where ρ denotes the half sum of all positive roots with respect to $\mathfrak{a}^+$ and m_o denotes the K-invariant probability measure on G/P. Note that the support m^{BR}_ψ is equal to $\mathcal{E}$, which was defined in (3.1).

By Corollary 4.3,

$$\tilde{m}^{\mathrm{BMS}}_\psi(G - N^+P) = 0 = \tilde{m}^{\mathrm{BR}}_\psi(G - N^+P).$$

4.3. **Ergodic decomposition of m^{BMS}_ψ.** Recall from subsection 3.1:

$$\tilde{\Lambda} = \bigsqcup_{\Lambda_0 \in \mathcal{Y}_\Gamma} \Lambda_0 \quad \text{and} \quad \mathcal{E} = \bigsqcup_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \mathcal{E}_0.$$

We denote by $\tilde{\nu}_\psi$ the M/M°-invariant lift of ν_ψ to $\tilde{\Lambda} \subset \mathcal{F}^\circ$, i.e., for $f \in C(\mathcal{F}^\circ)$,

$$\tilde{\nu}_\psi(f) := \nu_\psi\Big(\sum_{m \in M/M^\circ} m.f\Big) = \nu_\psi\Big(\int_{m\in M} m.f\, dm\Big)$$

where $m.f(x) = f(xm)$.

Theorem 4.4. *Let $\Gamma < G$ be an Anosov subgroup.*

(1) *The restriction* $\tilde{\nu}_\psi$ *to each* Γ*-minimal subset of* $\mathcal{F}^\circ$ *is* Γ*-ergodic. In particular,* $\tilde{\nu}_\psi = \sum_{\Lambda_0 \in \mathcal{Y}_\Gamma} \tilde{\nu}_\psi|_{\Lambda_0}$ *is a* Γ*-ergodic decomposition.*
(2) *The restriction of* m_ψ^{BMS} *to each* P°*-minimal subset of* $\Gamma\backslash G$ *is* A*-ergodic.*

In particular,

$$m_\psi^{\mathrm{BMS}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$$

is an A*-ergodic decomposition.*

The rest of this section is devoted to the proof of this theorem. Set

$$\tilde{\Omega} := \{g \in G : \Gamma g \in \Omega\} = \{g \in G : g^\pm \in \Lambda\}.$$

Let $\mathcal{B}$ denote the Borel σ-algebra on G. We set

$$\Sigma_\pm := \{B \cap \tilde{\Omega} : B \in \mathcal{B} \text{ with } B = \Gamma B A N^\pm\}.$$

We also define Σ to be the collection of all $B \in \mathcal{B}$ such that $m_\psi^{\mathrm{BMS}}(B \triangle B_+) = m_\psi^{\mathrm{BMS}}(B \triangle B_-) = 0$ for some $B_\pm \in \Sigma_\pm$. Recall the subgroup $M_\Gamma < M$ given in (3.2), and define

$$\Sigma_0 := \{B \cap \tilde{\Omega} : B \in \mathcal{B} \text{ with } B = \Gamma B A M_\Gamma\}.$$

The following is a main technical ingredient of the proof of Theorem 4.4:

Lemma 4.5. *We have* $\Sigma \subset \Sigma_0$ *mod* m_ψ^{BMS}*; that is, for all* $B \in \Sigma$*, there exists* $B_0 \in \Sigma_0$ *such that* $m_\psi^{\mathrm{BMS}}(B \triangle B_0) = 0$*.*

This lemma follows if we show that any bounded Σ-measurable function on $\tilde{\Omega}$ is Σ_0-measurable modulo m_ψ^{BMS}.

Let f be any bounded Σ-measurable function on $\tilde{\Omega}$. We may assume without loss of generality that f is strictly left Γ-invariant and right A-invariant [28, Prop. B.5]. There exist bounded $\Sigma^\pm$-measurable functions $f_\pm$ such that $f = f_\pm$ for m_ψ^{BMS}-a.e. Note that $f_\pm$ satisfy $f_\pm(gn) = f_\pm(g)$ whenever $g, gn \in \tilde{\Omega}$ with $n \in N^\pm$. Set

$$E := \left\{ gAM : \begin{array}{c} f|_{gAM} \text{ is measurable and} \\ f(gm) = f_+(gm) = f_-(gm) \\ \text{for Haar a.e. } m \in M \end{array} \right\} \subset \tilde{\Omega}/AM.$$

By Fubini's theorem, E has a full measure on $\tilde{\Omega}/AM \simeq \Lambda^{(2)}$ with respect to the measure $d\nu_\psi \, d\nu_{\psi\circ \mathrm{i}}$. For all small $\varepsilon > 0$, define functions $f^\varepsilon, f^\varepsilon_\pm : \tilde{\Omega} \to \mathbb{R}$ by

$$f^\varepsilon(g) := \tfrac{1}{\mathrm{Vol}(M_\varepsilon)} \int_{M_\varepsilon} f(gm)\, dm \text{ and } f^\varepsilon_\pm(g) := \tfrac{1}{\mathrm{Vol}(M_\varepsilon)} \int_{M_\varepsilon} f_\pm(gm)\, dm$$

where M_ε denotes the ε-ball around e in M. Note that if $gAM \in E$, then f^ε and $f^\varepsilon_\pm$ are continuous and identical on gAM. Moreover, as M normalizes subgroups A and $N^\pm$, f^ε is strictly left Γ-invariant, right A-invariant and $f^\varepsilon_\pm(gn) = f^\varepsilon_\pm(g)$ whenever $g, gn \in \tilde{\Omega}$ with $n \in N^\pm$. Using the isomorphism between $\tilde{\Omega}/AM$ and $\Lambda^{(2)}$ given by $gAM \mapsto (g^+, g^-)$, we may consider E as a subset of $\Lambda^{(2)}$. We then define

$$E^+ := \{\xi \in \Lambda : (\xi, \eta') \in E \quad \text{for } \nu_{\psi\circ\mathrm{i}}\text{-a.e. } \eta' \in \Lambda\};$$
$$E^- := \{\eta \in \Lambda : (\xi', \eta) \in E \quad \text{for } \nu_{\psi}\text{-a.e. } \xi' \in \Lambda\}.$$

Then E^- is $\nu_{\psi\circ\mathrm{i}}$ -conull and E^+ is ν_ψ-conull by Fubini's theorem. Set

$$E^+_\eta := \{\xi \in \Lambda : (\xi, \eta) \subset E\} \quad \text{ and } \quad E^-_\xi := \{\eta \in \Lambda : (\xi, \eta) \subset E\}.$$

Note that E^-_ξ is $\nu_{\psi\circ\mathrm{i}}$-conull for all $\xi \in E^+$ and that E^+_η is ν_ψ-conull for all $\eta \in E^-$.

Lemma 4.6. *Let $g \in \tilde{\Omega}$ be such that $gAM \in E$ and $g^\pm \in E^\pm$. Then for any $\varepsilon > 0$, $f^\varepsilon(gm_0) = f^\varepsilon(g)$ for all $m_0 \in M_\Gamma$.*

Proof. We will use the following observation in the proof. For $am \in AM$, suppose that there exist $\gamma \in \Gamma$, and a sequence $h_1, \cdots, h_k \in N \cup N^+$ such that $\gamma gam = gh_1 \cdots h_k$ and $gh_1 \cdots h_i \in E$ for all $1 \le i \le k$. Then

$$f^\varepsilon(gam) = f^\varepsilon(\gamma gam) = f^\varepsilon(gh_1 \cdots h_r) = f^\varepsilon(gh_1 \cdots h_{r-1}) = \cdots = f^\varepsilon(g),$$

by the $N^\pm$-invariance of $f^\varepsilon_\pm$, the invariance of f by Γ and A and the fact that all three agree on E.

By Proposition 3.6, it suffices to prove that

$$f^\varepsilon(gb(g^{-1}\gamma g, \xi)) = f^\varepsilon(g)$$

for any $\gamma \in \Gamma$ and $\xi \in g^{-1}\Lambda \cap N^+e^+$. Setting $b(g^{-1}\gamma g, \xi) = (am)^{-1}$, we may write $\gamma gam = gh_1n_1h_2$ where $h_1, h_2 \in N^+$ and $n_1 \in N$. Note that $E^\pm$ are Γ-invariant, as the measures ν_ψ and $\nu_{\psi\circ\mathrm{i}}$ are Γ-quasi-invariant. Since $g^\pm \in E^\pm$, we get $\gamma g^\pm \in E^\pm$. Set

$$\begin{aligned} \xi_0 &= g^+, & \eta_0 &= g^-, \\ \xi_1 &= gh_1^+, & \eta_1 &= gh_1n_1^-(= \gamma g^-), \\ \xi_2 &= gh_1n_1h_2^+(= \gamma g^+). \end{aligned}$$

Choose a sequence $\xi_{1,\ell} \in E^+ \cap E^+_{\eta_0} \cap E^+_{\eta_1}$ which converges to ξ_1 as $\ell \to \infty$. This is possible because $E^+ \cap E^+_{\eta_0} \cap E^+_{\eta_1}$ is dense in Λ, as it is ν_ψ-conull from the hypothesis that $\xi_0 = g^- \in E^-$ and $\xi_1 = \gamma g^- \in E^-$. Let $h_{1,\ell} \in N^+$ be the unique element such that $(gh_{1,\ell})^+ = \xi_{1,\ell}$, $n_{1,\ell} \in N$ the unique element such that $(gh_{1,\ell}\, n_{1,\ell})^- = \gamma g^-$, and finally $h_{2,\ell} \in N^+$ the unique element such that $(gh_{1,\ell}\, n_{1,\ell}\, h_{2,\ell})^+ = \gamma g^+$. Since $(gh_{1,\ell}\, n_{1,\ell}\, h_{2,\ell})^\pm = \gamma g^\pm$, we have $gh_{1,\ell}\, n_{1,\ell}\, h_{2,\ell} = \gamma g a_\ell m_\ell$ for some $a_\ell \in A$ and $m_\ell \in M$. Note that $a_\ell m_\ell \to am$ as $\ell \to \infty$ and that $a_\ell m_\ell \in b(g^{-1}\Gamma g)$. The sequences $h_{1,\ell}, n_{1,\ell}, h_{2,\ell} \in N \cup N^+$ satisfy

- $gh_{1,\ell}AM \in E$, as $(gh_{1,\ell})^- = \eta_0$ and $(gh_{1,\ell})^+ = \xi_{1,\ell} \in E^+_{\eta_0}$;
- $gh_{1,\ell}\, n_{1,\ell}AM \in E$, as $(gh_{1,\ell}\, n_{1,\ell})^- = \eta_1$ and $(gh_{1,\ell}\, n_{1,\ell})^+ = \xi_{1,\ell} \in E^+_{\eta_1}$;
- $gh_{1,\ell}\, n_{1,\ell}\, h_{2,\ell}AM = \gamma gAM \in E$, as $gAM \in E$ and E is Γ-invariant.

Therefore, $f^\varepsilon(ga_\ell m_\ell) = f^\varepsilon(g)$ by the observation made in the beginning of the proof. Since $gAM \in E$, f^ε is continuous on gAM and hence

$$f^\varepsilon(gam) = \lim_{\ell\to\infty} f^\varepsilon(ga_\ell m_\ell) = f^\varepsilon(g).$$

This finishes the proof. □

Proof of Lemma 4.5: Let f be any bounded Σ-measurable function on $\tilde{\Omega}$. For any $\varepsilon > 0$, by Lemma 4.6, f^ε coincides with a Σ_0-measurable function m_ψ^{BMS}-a.e. Since $\lim_{\varepsilon\to 0} f^\varepsilon = f$ m_ψ^{BMS}-a.e., f is a Σ_0-measurable function m_ψ^{BMS}-a.e. as well. This proves the lemma. □

Corollary 4.7. *There exists $B \in \Sigma$ such that any two distinct subsets in $\{B.s : s \in M_\Gamma\backslash M\}$ are measurably disjoint and Σ is the finite σ-algebra generated by $\{B.s : s \in M_\Gamma\backslash M\}$ mod m_ψ^{BMS}.*

Proof. First, note that the AM-ergodicity of m_ψ^{BMS} implies that the σ-algebra

$$\Sigma_1 := \{B \cap \tilde{\Omega} : B \in \mathcal{B} \text{ such that } B = \Gamma BAM\}$$

is trivial mod m_ψ^{BMS}. It follows that for any $B \in \Sigma_0$, and hence for any $B \in \Sigma$ by Lemma 4.5, with $m_\psi^{\mathrm{BMS}}(B) > 0$, the union $\cup_{s\in M_\Gamma\backslash M} B.s$ is m_ψ^{BMS}-conull.

Let $\mathcal{P} = \{A_1, \cdots, A_k\}$ be a partition of $\tilde{\Omega}$ with maximal k, among all partitions of Ω satisfying

(1) $A_i \in \Sigma$ and $m_\psi^{\mathrm{BMS}}(A_i) > 0$,
(2) $\tilde{\Omega} = A_1 \cup \cdots \cup A_k$ mod m_ψ^{BMS} and
(3) for any $s \in M_\Gamma\backslash M$, we have $A_i.s \in \{A_1, \cdots, A_k\}$ mod m_ψ^{BMS}.

It remains to set $B = A_1$ to prove the claim. □

4.4. **$\mathbb{R}$-ergodic decomposition of $\hat{m}_\psi$ on $\Lambda^{(2)} \times \mathbb{R} \times M$.** Set $\Lambda^{(2)} = (\Lambda \times \Lambda) \cap \mathcal{F}^{(2)}$. The action of Γ on $\Lambda^{(2)} \times \mathbb{R}$ defined by

$$\gamma.(\xi, \eta, t) = (\gamma\xi, \gamma\eta, t + \psi(\log \beta^A_{\gamma\xi}(e, \gamma)))$$

is proper and cocompact, and the measure $d\tilde{m}_\psi := e^{[\cdot,\cdot]_\psi} d\nu_\psi\, d\nu_{\psi\circ \mathrm{i}}\, dt$ on $\Lambda^{(2)} \times \mathbb{R}$ descends to a finite $\mathbb{R}$-ergodic measure m_ψ on $\Gamma\backslash\Lambda^{(2)} \times \mathbb{R}$ ([22, Thm. 3.2], [5, Thm. A.2]). We denote by $d\hat{m}_\psi$ the finite measure on

$$Z := \Gamma\backslash\Lambda^{(2)} \times \mathbb{R} \times M$$

induced by the Γ-invariant product measure $d\tilde{m}_\psi\, dm$ on $\Lambda^{(2)} \times \mathbb{R} \times M$; here Γ acts on $\Lambda^{(2)} \times \mathbb{R} \times M$ by

$$\gamma.(\xi, \eta, t, m) = (\gamma\xi, \gamma\eta, t + \psi(\log \beta^A_{\gamma\xi}(e, \gamma)), \beta^M_{\gamma\xi}(e, \gamma)m)$$

where $(\xi, \eta) \in \Lambda^{(2)}$, $t \in \mathbb{R}$ and $m \in M$.

Define the Borel map $\Psi : \tilde{\Omega} \to \Lambda^{(2)} \times \mathbb{R} \times M$ by

$$\Psi(g) = (g^+, g^-, \psi(\beta^A_{g^+}(e, g)), \beta^M_{g^+}(e, g)).$$

Note that for all $\gamma \in \Gamma$, $a \in A$ and $m \in M$, $\Psi(\gamma gam) = \gamma\Psi(g)\tau_{\psi(\log a)}\tau_m$ for $\tilde{m}_\psi^{\mathrm{BMS}}$-almost all $g \in \tilde{\Omega}$, where τ stands for the right translation action by elements of $\mathbb{R} \times M$. By abuse of notation, let $\Psi : \Omega \to Z$ denote the map induced by Ψ and τ denote the action of $\mathbb{R} \times M$ on Z induced by τ.

Recalling that $\Omega = \bigsqcup_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma}(\Omega \cap \mathcal{E}_0)$, we set

$$Z_{\mathcal{E}_0} := \Psi(\Omega \cap \mathcal{E}_0) \quad \text{ for each } \mathcal{E}_0 \in \mathfrak{Y}_{\Gamma_0}.$$

Hence the collection $\{Z_{\mathcal{E}_0} : \mathcal{E}_0 \in \mathfrak{Y}_\Gamma\}$ gives a measurable partition for $(Z, \hat{m}_\psi)$.

Proposition 4.8. *For each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, the restriction $\hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ is $\mathbb{R}$-ergodic, and $\hat{m}_\psi = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ is an $\mathbb{R}$-ergodic decomposition. In particular, $\tilde{\nu}_\psi|_{\Lambda_0}$ is Γ-ergodic and $\tilde{\nu}_\psi = \sum_{\Lambda_0 \in \mathcal{Y}_\Gamma} \tilde{\nu}_\psi|_{\Lambda_0}$ is a Γ-ergodic decomposition.*

Proof. By Corollary 4.7, Σ is generated by $\{B.s : s \in M_\Gamma \backslash M\}$ mod m_ψ^{BMS} for some $B \in \Sigma$. We first claim that $\hat{m}_\psi|_{\Psi(B.s)}$ is $\mathbb{R}$-ergodic for each $s \in M_\Gamma \backslash M$.

Let $f \in C(Z)$ be arbitrary. The Birkhoff average $f_\sharp : Z \to \mathbb{R}$ is defined $\hat{m}_\psi$-a.e. by

$$f_\sharp(y) := \lim_{T \to \infty} \frac{1}{T} \int_0^T f(y\tau_t)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(y\tau_{-t})\, dt.$$

Note that $f_\sharp$ is well defined by the Birkhoff ergodic theorem and is $\mathbb{R}$-invariant. Hence, $f_\sharp \circ \Psi$ is defined m_ψ^{BMS}-a.e. The desired ergodicity follows from the Birkhoff ergodic theorem if we show that $f_\sharp \circ \Psi$ is constant m_ψ^{BMS}-a.e. on each $B.s$. Let $u \in \operatorname{int} \mathcal{L}_\Gamma$ be the unique vector such that $\psi(u) = \psi_\Gamma(u) = 1$ and let $a_t = \exp tu$. Observing that $f \circ \Psi$ is uniformly continuous on each $xAN \cap \Omega$ whenever Ψ is continuous at x and that $f(\Psi(x)\tau_t) = f(\Psi(xa_t))$ for all $t \in \mathbb{R}$, it is a standard Hopf argument to show that $f_\sharp \circ \Psi$ coincides with $N^\pm$-invariant functions m_ψ^{BMS}-a.e. Hence $f_\sharp \circ \Psi$ is Σ-measurable, implying that $f_\sharp \circ \Psi$ is constant m_ψ^{BMS}-a.e. on each $B.s$. Therefore this proves the claim.

For each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, $\hat{m}_\psi(\Psi(B.s) \cap Z_{\mathcal{E}_0}) > 0$ for some $s \in M_\Gamma \backslash M$. It follows from the $\mathbb{R}$-ergodicity of $\hat{m}_\psi|_{\Psi(B.s)}$ that $\hat{m}_\psi|_{\Psi(B.s)} = \hat{m}_\psi|_{Z_{\mathcal{E}_0}}$. Therefore the proposition is proved. $\square$

The measure m_ψ^{BMS} disintegrates over $\hat{m}_\psi$ via the projection $\Gamma\backslash\Lambda^{(2)} \times A \times M \to \Gamma\backslash\Lambda^{(2)} \times \mathbb{R} \times M$, where each conditional measure is the Lebesque measure on $\exp(\ker \psi)$.

Proof of Theorem 4.4. Since $dm_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0} = d\hat{m}_\psi|_{Z_{\mathcal{E}_0}}\, d\,\mathrm{Leb}_{\ker\psi}$, the $\mathbb{R}$-ergodicity of $\hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ proved in Proposition 4.8 implies the A-ergodicity of $m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$. $\square$

4.5. **The set of strong Myrberg limit points.** In [16], we defined Myrberg limit points of Γ.

Definition 4.9. We now define the set of *strong* Myrberg limit points as follows:

$$\Lambda_\psi^{\spadesuit} = \{\xi \in \Lambda \cap N^+e^+ : \text{for each } \mathcal{E}_0 \in \mathfrak{Y}_\Gamma, \text{ there exist } \eta \in \Lambda \text{ and } m \in M \text{ such that } Z_{\mathcal{E}_0} = \overline{\Gamma(\xi,\eta,0,m)\mathbb{R}_+}\}. \tag{4.9}$$

Since $\hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ is $\mathbb{R}$-ergodic and finite for each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, the Birkhoff ergodic theorem for the $\mathbb{R}$-action implies:

Corollary 4.10. *We have $\nu_\psi(\Lambda_\psi^{\spadesuit}) = 1$.*

The same proof as the proof of [16, Prop. 8.2] shows that if $g \in \mathcal{E}_0$ and $g^+ \in \Lambda_\psi^{\spadesuit}$,

$$\limsup \Gamma\backslash\Gamma g A^+ = \Omega \cap \mathcal{E}_0.$$

Hence Corollary 4.10 implies (cf. [16, Coro 8.12]):

Corollary 4.11. *For $m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$-almost all $x \in \mathcal{E}_0 \cap \Omega$, each xA^+ and xw_0A^+ is dense in $\mathcal{E}_0 \cap \Omega$.*

Let Π denote the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$.

Definition 4.12. For a sequence $a_n \in A^+$, we write $a_n \to \infty$ regularly in A^+ or $\log a_n \to \infty$ regularly in $\mathfrak{a}^+$, if $\alpha(\log a_n) \to \infty$ as $n \to \infty$ for all $\alpha \in \Pi$.

The following is an important property of Anosov groups:

Lemma 4.13. *Let Γ be Anosov. For any $g, h \in G$ and a sequence $\gamma_n \to \infty$ in Γ, $\mu(g\gamma_n h) \to \infty$ regularly in A^+.*

This lemma is a consequence of the fact that the limit cone of Γ is contained in $\operatorname{int}\mathfrak{a}^+ \cup \{0\}$ (cf. [16, Thm. 4.3] for references).

In the Cartan decomposition $g = k_1(\exp\mu(g))k_2 \in KA^+K$, if $\mu(g) \in \operatorname{int}\mathfrak{a}^+$, then $k_1, k_2 \in K$ are determined uniquely up to mod M, more precisely, if $g = k_1'(\exp\mu(g))k_2'$, then there exists $m \in M$ such that $k_1 = k_1'm$ and $k_2 = m^{-1}k_2'$. We write

$$\kappa_1(g) := [k_1] \in K/M \quad \text{ and } \quad \kappa_2(g) := [k_2] \in M\backslash K.$$

Definition 4.14. Let $o = [K] \in G/K$ and let $g_n \in G$ be a sequence. A sequence $g_n(o) \in G/K$ is said to converge to $\xi \in \mathcal{F}$ if $\mu(g_n) \to \infty$ regularly in $\mathfrak{a}^+$ and $\lim\limits_{n\to\infty} \kappa_1(g_n) = \xi$; we write $\lim_{n\to\infty} g_n(o) = \xi$.

Recall the map j from (4.4):

Lemma 4.15. *Let $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$ and $\tilde{\mathcal{E}}_0 \subset G$ be its Γ-invariant lift. There exists $s_0 \in M/M_\Gamma$ such that*

$$\mathrm{j}(\tilde\Omega \cap \tilde{\mathcal{E}}_0 \cap N^+P) = \{(\xi, \eta, ams_0) \in \Lambda^{(2)} \times AM : \xi \in N^+e^+, am \in AM_\Gamma\}.$$

Proof. Recall that $\Gamma \cap \operatorname{int} A^+M \neq \emptyset$ and hence $e^\pm \in \Lambda$. In particular, $\mathrm{j}(\tilde{\Omega} \cap \tilde{\mathcal{E}}_0 \cap N^+P)$ contains an element of the form $(e^+, e^-, s_0) \in \Lambda^{(2)} \times AM$ for some $s_0 \in M$. Note that for all $\gamma \in \Gamma \cap N^+P$, we have

$$\gamma.(e^+, e^-, s_0) = (\gamma^+, \gamma^-, \beta^{AM}_{e^+}(\gamma^{-1}, e)s_0).$$

Since $\Gamma \cap \operatorname{int} A^+M \neq \emptyset$, M_Γ is equal to the closure of $\{m \in M : \Gamma \cap N^+mAN \neq \emptyset\}$ by [3, Prop. 4.9(a)]. Recall also that for $\gamma \in \Gamma \cap N^+mAN$, $\beta^M_{e^+}(\gamma^{-1}, e) = m$. Therefore, using the fact that $\tilde{\mathcal{E}}_0$ is right $M_\Gamma AN$-invariant, we deduce that the set $\mathrm{j}(\tilde{\Omega} \cap \tilde{\mathcal{E}}_0 \cap N^+P)$ contains

$$\{(\gamma^+, \eta, ams_0) \in \Lambda^{(2)} \times AM : \gamma \in \Gamma \cap N^+P, am \in AM_\Gamma\}.$$

This proves the claim, since $\{\gamma^+ \in \mathcal{F} : \gamma \in \Gamma \cap N^+P\}$ is dense in Λ. $\square$

Lemma 4.16. *Let $p \in G/K$ and $\eta \neq \xi_0 \in \Lambda$. For any $\xi \in \Lambda^{\spadesuit}_\psi - \{\eta\}$, there exists an infinite sequence $\gamma_i \in \Gamma$ such that*

$$\lim_{i\to\infty} \gamma_i^{-1}p = \eta, \quad \lim_{i\to\infty} \gamma_i^{-1}\xi = \xi_0, \quad \textit{and} \quad \lim_{i\to\infty} \beta^M_\xi(\gamma_i, e) = e. \tag{4.10}$$

Moreover, there exists a neighborhood U of ξ_0 such that, as $i \to \infty$, the sequence $\gamma_i\xi'$ converges to ξ uniformly for all $\xi' \in U$.

Proof. Let ξ and η be as in the statement. Fix any $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$. By the definition of $\Lambda^{\spadesuit}_\psi$, there exist $\check{\xi} \in \Lambda$ and $m \in M$ such that $\Gamma(\xi, \check{\xi}, 0, m)\mathbb{R}^+$ is dense in $Z_{\mathcal{E}_0}$. Note that $(\xi_0, \eta, 0, m) \in Z_{\mathcal{E}_0}$ by Lemma 4.15. Therefore there exist sequences $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that

$$\begin{aligned}&\lim_{i\to\infty} \gamma_i^{-1}.(\xi, \check{\xi}, 0 + t_i, m)\\ &= \lim_{i\to\infty} (\gamma_i^{-1}\xi, \gamma_i^{-1}\check{\xi}, \psi(\log \beta^A_\xi(\gamma_i, e)) + t_i, \beta^M_\xi(\gamma_i, e)m) = (\xi_0, \eta, 0, m).\end{aligned}$$

The last two conditions in (4.10) immediately follow from this and the first condition follows from [16, Lem. 8.9].

By passing to a subsequence, we may write $\gamma_i = k_i a_i \ell_i^{-1}$ where $k_i \to k_0, \ell_i \to \ell_0$ in K and $a_i \in A^+$. As Γ is Anosov, $a_i \to \infty$ regularly in A^+. We then have $\ell_0^- = \eta$. Note that $\gamma_i\xi' \to k_0^+$ for all $\xi' \in \mathcal{F}$ with $(\xi', \eta) \in \mathcal{F}^{(2)}$ and this convergence is uniform on a compact subset of $\{\xi' : (\xi', \eta) \in \mathcal{F}^{(2)}\}$. Since $(\xi_0, \eta) \in \mathcal{F}^{(2)}$, there exists a neighborhood U of ξ_0 such that $\gamma_i\xi' \to k_0^+$ uniformly for all $\xi' \in U$. Since $\gamma_i^{-1}\xi \to \xi_0$ and hence $\gamma_i^{-1}\xi \in U$ for all large i, we have $\gamma_i(\gamma_i^{-1}\xi) \to k_0^+$. Hence $\xi = k_0^+$. The claim follows. $\square$

5. Equi-continuous family of Busemann functions

We fix a left G-invariant and right K-invariant Riemannian metric d on G. For a subgroup $H < G$ and $\varepsilon > 0$, we set $H_\varepsilon = \{h \in H : d(e, h) < \varepsilon\}$. We will use the notation $H_{O(\varepsilon)}$ to mean $H_{c\varepsilon}$ for some absolute constant $c > 0$. Recall the notation $o = [K] \in G/K$.

In this section, we prove the following proposition.

Proposition 5.1 (Equi-continuity)**.** *Let $\Gamma < G$ be an Anosov subgroup. Fix $g \in N^+P$ be such that $g^\pm \in \Lambda$. Let $\gamma_n \in \Gamma$ be a sequence such that for some $\xi \in \Lambda - \{g^-\}$, $\gamma_n^{-1}\xi \to g^+$ and $\gamma_n^{-1}g(o) \to g^-$ as $n \to \infty$. Then, up to passing to a subsequence of γ_n, the sequence of maps $\eta \mapsto \beta_\eta^{AM}(\gamma_n^{-1}g, g)$ is equi-continuous at g^+, i.e., for any $\varepsilon > 0$, there exists a neighborhood U_ε of g^+ in $\mathcal{F}$ such that for all $n \geq 1$ and for all $\eta \in U_\varepsilon$,*

$$\beta_\eta^{AM}(\gamma_n^{-1}g, g) \subset \beta_{g^+}^{AM}(\gamma_n^{-1}g, g)(AM)_\varepsilon.$$

We first prove the following two lemmas using the structure theory of semisimple Lie groups.

Lemma 5.2. *There exists $c > 0$ such that for all sufficiently small $\varepsilon > 0$,*

$$aG_\varepsilon \subset K_{c\varepsilon}aA_{c\varepsilon}N \quad \text{ for all } a \in A^+.$$

Proof. For all sufficiently small $\varepsilon > 0$, we have

$$G_\varepsilon \subset M_{O(\varepsilon)}N^+_{O(\varepsilon)}A_{O(\varepsilon)}N_{O(\varepsilon)} \text{ and } N^+_\varepsilon \subset K_{O(\varepsilon)}A_{O(\varepsilon)}N_{O(\varepsilon)}.$$

Since $aN_\varepsilon^+a^{-1} \subset N_\varepsilon^+$ for any $a \in A^+$, it follows that

$$\begin{aligned} aG_\varepsilon &\subset aM_{O(\varepsilon)}N^+_{O(\varepsilon)}A_{O(\varepsilon)}N_{O(\varepsilon)} = M_{O(\varepsilon)}(aN^+_{O(\varepsilon)}a^{-1})aA_{O(\varepsilon)}N_{O(\varepsilon)} \\ &\subset M_{O(\varepsilon)}(K_{O(\varepsilon)}A_{O(\varepsilon)}N_{O(\varepsilon)})aA_{O(\varepsilon)}N_{O(\varepsilon)} \subset K_{O(\varepsilon)}aA_{O(\varepsilon)}N, \end{aligned}$$

which was to be proved. ☐

Lemma 5.3. *Let $g_n = k_na_n\ell_n^{-1} \in KA^+K$ where $a_n \to \infty$ regularly in A^+ and $k_n \to k_0$, $\ell_n \to \ell_0$ in K as $n \to \infty$. Assume that both $\xi := k_0^+$ and $\zeta := \ell_0^+$ belong to N^+e^+, and set $m_0 = m_0[k_0, \ell_0]$ to be*

$$m_0 := k_\xi^{-1}k_0\ell_0^{-1}k_\zeta \in M$$

where $k_\xi, k_\zeta \in K$ are defined as in (2.3). Then for all small $\varepsilon > 0$, there exist neighborhoods V'_ε and U'_ε of ξ and ζ, respectively, such that

$$\{\beta_\eta^{AM}(g_n^{-1}, e) : \eta \in U'_\varepsilon \cap g_n^{-1}V'_\varepsilon\} \subset a_nm_0(AM)_\varepsilon$$

for all sufficiently large $n > 1$.

Proof. By the continuity of the visual maps, there exist neighborhoods V'_ε of ξ and U'_ε of ζ such that $k_\eta \in k_\zeta K_\varepsilon$ for all $\eta \in U'_\varepsilon$ and $k_\eta \in k_\xi K_\varepsilon$ for all $\eta \in V'_\varepsilon$. We may assume without loss of generality that $k_0^{-1}k_n$, $\ell_n^{-1}\ell_0 \in K_\varepsilon$ for all $n \geq 1$. Let $\eta \in U'_\varepsilon \cap g_n^{-1}V'_\varepsilon$ be arbitrary. By definition,

$$g_nk_\eta \in k_{g_n\eta}\sigma^{AM}(g_n, \eta)N, \text{ i.e., } k_0^{-1}g_nk_\eta \in k_0^{-1}k_{g_n\eta}\sigma^{AM}(g_n, \eta)N.$$

Observe that

$$\begin{aligned} k_0^{-1}g_nk_\eta \in k_0^{-1}g_nk_\zeta K_\varepsilon &= (k_0^{-1}k_n)a_n(\ell_n^{-1}\ell_0)\ell_0^{-1}k_\zeta K_\varepsilon \\ &\subset K_\varepsilon a_nK_\varepsilon \ell_0^{-1}k_\zeta K_\varepsilon \subset K_\varepsilon a_nK_{O(\varepsilon)}\ell_0^{-1}k_\zeta. \end{aligned}$$

On the other hand, since $g_n\eta \in V'_\varepsilon$,

$$\begin{aligned} k_0^{-1} g_n k_\eta &\in k_0^{-1} k_{g_n\eta} \sigma^{AM}(g_n,\eta) N \\ &\subset k_0^{-1} k_\xi K_\varepsilon \sigma^{AM}(g_n,\eta) N \subset K_{O(\varepsilon)} k_0^{-1} k_\xi \sigma^{AM}(g_n,\eta) N. \end{aligned}$$

Combining these with the fact that $\ell_0^{-1} k_\zeta \in M$, we get

$$a_n K_{O(\varepsilon)} \cap K_{O(\varepsilon)} k_0^{-1} k_\xi \sigma^{AM}(g_n,\eta)(\ell_0^{-1}k_\zeta)^{-1} N \neq \emptyset.$$

Since $k_0^{-1} k_\xi \in M$ as well, it follows from Lemma 5.2 that

$$\begin{aligned} &\sigma^A(g_n,\eta) \in a_n A_{O(\varepsilon)}, \text{ and} \\ &\sigma^M(g_n,\eta) \in (k_0^{-1}k_\xi)^{-1} M_{O(\varepsilon)} \ell_0^{-1} k_\zeta \subset (k_0^{-1}k_\xi)^{-1} \ell_0^{-1} k_\zeta M_{O(\varepsilon)}. \end{aligned}$$

Since $\beta_\eta^{AM}(g_n^{-1}, e) = \sigma^{AM}(g_n,\eta)$, and $m_0 := (k_0^{-1}k_\xi)^{-1}\ell_0^{-1}k_\zeta$, this implies the claim. $\square$

Proof of Proposition 5.1: Set $g_n := g^{-1}\gamma_n g$. Then $g_n^{-1}(g^{-1}\xi) \to e^+$ and $g_n^{-1}(o) \to e^-$ as $n \to \infty$. By passing to a subsequence, we may write $g_n = k_n a_n \ell_n^{-1} \in KA^+K$ where the sequences k_n and ℓ_n converge to some k_0 and ℓ_0 in K respectively. Since Γ is Anosov, it follows that $a_n \to \infty$ regularly in A^+. Combined with the hypothesis $g_n^{-1}(o) \to e^-$ as $n \to \infty$, we have $\ell_0^- = e^-$, or equivalently, $\ell_0 \in M$. Hence $\ell_0^+ = e^+$.

We claim that $k_0^+ = g^{-1}\xi$. Since $a_n \to \infty$ regularly in A^+, for any $\eta \in N^+e^+$, $g_n\eta \to k_0^+$ as $n \to \infty$ and the convergence is uniform on a compact subset of N^+e^+. Since $g_n^{-1}(g^{-1}\xi) \to e^+$ as $n \to \infty$, $g_n^{-1}(g^{-1}\xi)$ is contained in a compact subset of N^+e^+ for all large n, it follows that $g_n(g_n^{-1}(g^{-1}\xi)) \to k_0^+$ as $n \to \infty$, which proves the claim.

Now let $\varepsilon > 0$ be arbitrary. Since $g^- \in \Lambda$, by Lemma 4.2, $g^{-1}\Lambda - \{e^-\} \subset N^+e^+$. Hence both e^+ and $g^{-1}\xi$ belong to N^+e^+. Applying Lemma 5.3 to the sequence g_n, we obtain $m_0 = m_0[k_0,\ell_0] \in M$, and some bounded neighborhoods $U'_\varepsilon, V'_\varepsilon \subset N^+e^+$ of e^+ and $g^{-1}\xi$ respectively, such that

$$\beta_{\eta'}^{AM}(g_n^{-1}, e) \in a_n m_0 (AM)_{\varepsilon/2} \quad \text{for all } \eta' \in U'_\varepsilon \cap g_n^{-1} V'_\varepsilon.$$

Since $k_0^+ = g^{-1}\xi \in V'_\varepsilon$ and $U'_\varepsilon \subset N^+e^+$, and hence $U'_\varepsilon \times \{\ell_0^-\} \subset \mathcal{F}^{(2)}$, we have $g_n U'_\varepsilon \subset V'_\varepsilon$, and hence $U'_\varepsilon = U'_\varepsilon \cap g_n^{-1} V'_\varepsilon$ for all large $n \gg 1$. Set $U_\varepsilon := gU'_\varepsilon \cap N^+e^+$. Note that $g^+ \in U_\varepsilon$.

Let $\eta \in U_\varepsilon$. Then $g^{-1}\eta \in U'_\varepsilon = U'_\varepsilon \cap g_n^{-1}V'_\varepsilon$ and hence

$$\beta_{g^{-1}\eta}^{AM}(g_n^{-1}, e) \in a_n m_0 (AM)_{\varepsilon/2}. \tag{5.1}$$

Since $g^{-1}\gamma_n\eta = g_n(g^{-1}\eta) \in k_n a_n \ell_n^{-1} U'_\varepsilon$, we have $g^{-1}\gamma_n\eta \to k_0^+ \in N^+e^+$, and hence $g^{-1}\gamma_n\eta \in N^+e^+$ for all large $n \gg 1$. Therefore for all sufficiently large $n > 1$, $\beta_\eta^{AM}(\gamma_n^{-1}g, g)$ is well-defined and

$$\beta_\eta^{AM}(\gamma_n^{-1}g, g) = \beta_{g^{-1}\eta}^{AM}(g^{-1}\gamma_n^{-1}g, e) = \beta_{g^{-1}\eta}^{AM}(g_n^{-1}, e).$$

Hence the lemma follows from the inclusion (5.1).

6. Essential values and ergodicity

As before, we let $\Gamma < G$ be an Anosov subgroup such that $\Gamma \cap \text{int}\, A^+M \neq \{e\}$. Fixing $\psi \in D_\Gamma^\star$, let $\nu = \nu_\psi$ be the unique (Γ, ψ)-Patterson Sullivan measure on Λ. By Corollary 4.3,

$$\nu(N^+e^+ \cap \Lambda) = 1. \tag{6.1}$$

Fix a Borel isomorphism $G/N \to \mathcal{F} \times AM$ given by

$$gN \mapsto (g^+, \beta_{g^+}^{AM}(e, g)) \quad \text{for } g \in N^+AM. \tag{6.2}$$

This isomorphism is G-equivariant for a Borel G-action on $\mathcal{F} \times AM$ given by

$$g(\xi, am) = (g\xi, \beta_\xi^{AM}(g^{-1}, e)am)$$

for $am \in AM$, $g \in G$, and $\xi \in N^+e^+$ with $g\xi \in N^+e^+$.

The following then defines a Γ-invariant locally finite measure on G/N by

$$d\hat{\nu}([g]) = d\nu(g^+)e^{\psi(\log a)}\, da\, dm \tag{6.3}$$

where da and dm are Haar measures on A and M respectively.

Motivated by the work of Schmidt [24] (also [20]), we define:

Definition 6.1. An element $am \in AM$ is called a *ν-essential value*, if for any Borel set $B \subset \mathcal{F}$ with $\nu(B) > 0$ and any $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\nu\{\xi \in B \cap \gamma^{-1}B : \beta_\xi^{AM}(\gamma^{-1}, e) \in am(AM)_\varepsilon\} > 0. \tag{6.4}$$

In view of (6.1), it suffices to consider Borel subsets $B \subset N^+e^+$ in this definition, and hence $\beta_\xi^{AM}(\gamma^{-1}, e)$ is well-defined for all $\xi \in B \cap \gamma^{-1}B$.

Let E_ν denote the set of all ν-essential values in AM. By the following lemma, $am \in \mathsf{E}_\nu$ if and only if $(am)^{-1} \in \mathsf{E}_\nu$; hence the condition $\beta_\xi^{AM}(\gamma^{-1}, e) \in am(AM)_\varepsilon$ in (6.4) can be replaced by $\beta_\xi^{AM}(e, \gamma^{-1}) \in am(AM)_\varepsilon$ in the above definition.

Lemma 6.2. *E_ν is a closed subgroup of AM.*

Proof. Since the metric d restricted to M is bi-M-invariant, we have that for all $\varepsilon > 0$, $M_\varepsilon^{-1} = M_\varepsilon$, $m^{-1}M_\varepsilon m = M_\varepsilon$ for all $m \in M$ and $M_{\varepsilon/2}M_{\varepsilon/2} \subset M_\varepsilon$. Let $b_1, b_2 \in \mathsf{E}_\nu$. Let $B \subset \mathcal{F}$ be a Borel subset with $\nu(B) > 0$ and let $\varepsilon > 0$. Since $b_i \in \mathsf{E}_\nu$ for $i = 1, 2$, there exists $\gamma_i \in \Gamma$ such that

$$B_1 := \{\xi \in B \cap \gamma_1^{-1}B : \beta_\xi^{AM}(\gamma_1^{-1}, e) \in b_1(AM)_{\varepsilon/2}\};$$
$$B_2 := \{\xi \in B_1 \cap \gamma_2^{-1}B_1 : \beta_\xi^{AM}(\gamma_2^{-1}, e) \in b_2(AM)_{\varepsilon/2}\}$$

has a positive ν-measure. Note that $B_2 \subset B \cap \gamma_2^{-1}\gamma_1^{-1}B$ and that for all $\xi \in B_2$, we have

$$\begin{aligned}\beta_\xi^{AM}(\gamma_2^{-1}\gamma_1^{-1}, e) &= \beta_{\gamma_2\xi}^{AM}(\gamma_1^{-1}, \gamma_2) = \beta_{\gamma_2\xi}^{AM}(\gamma_1^{-1}, e)\beta_\xi^{AM}(\gamma_2^{-1}, e) \\ &\in b_1(AM)_{\varepsilon/2}b_2(AM)_{\varepsilon/2} \subset b_1b_2(AM)_\varepsilon.\end{aligned}$$

Hence $b_1 b_2 \in \mathsf{E}_\nu$. This proves that E_ν is a subgroup of AM. Now suppose that a sequence $b_i \in \mathsf{E}_\nu$ converges to some $b \in AM$. Let $\varepsilon > 0$ and $B \subset \mathcal{F}$ be a Borel subset with $\nu(B) > 0$. Fix i large enough so that $b_i(AM)_{\varepsilon/2} \subset b(AM)_\varepsilon$, and let $\gamma_i \in \Gamma$ be such that $\nu\{\xi \in B \cap \gamma_i^{-1}B : \beta_\xi(\gamma_i^{-1}, e) \in b_i(AM)_{\varepsilon/2}\} > 0$. Then $\nu\{\xi \in B \cap \gamma_i^{-1}B : \beta_\xi(\gamma_i^{-1}, e) \in b(AM)_\varepsilon\} > 0$. This proves that $b \in \mathsf{E}_\nu$. Hence E_ν is closed. □

Lemma 6.3. *Let $b_0 \in \mathsf{E}_\nu$ be such that $\{bb_0b^{-1} : b \in AM\} \subset \mathsf{E}_\nu$. Then for any Γ-invariant Borel function $h : G/N \to [0,1]$, we have*

$$h(xb_0) = h(x) \quad \text{for } \hat{\nu}\text{-a.e. } x.$$

Proof. In view of the homeomorphsim $N^+AMN/N \to N^+e^+ \times AM$ given by $gN \mapsto (g^+, \beta_{g^+}(e,g))$ and (6.1), it suffices to show that for any Γ-invariant Borel function $h : N^+e^+ \times AM \to [0,1]$, $h(\xi, b) = h(\xi, bb_0)$ for ν-a.e. ξ and for all $b \in AM$. Suppose not. Then there exists $b_1 \in AM$ such that $\nu\{\xi \in \mathcal{F} : h(\xi, b_1) < h(\xi, b_1b_0)\} > 0$ or $\nu\{\xi \in \mathcal{F} : h(\xi, b_1) < h(\xi, b_1b_0)\} > 0$. We consider the first case; the second case can be treated similarly. Then there exist $r, \varepsilon > 0$ such that

$$Q_{b_0} := \{\xi \in N^+e^+ : h(\xi, b_1) < r - \varepsilon < r + \varepsilon < h(\xi, b_1b_0)\}$$

has a positive ν-measure. By considering the convolution of h with the approximation of identity functions on AM, we may assume without loss of generality that the family $h(\xi, \cdot)$, $\xi \in N^+e^+$, is uniformly equi-continuous on AM. Hence there exists $\varepsilon' > 0$ such that for all $\xi \in Q_{b_0}$ and $b \in (AM)_{\varepsilon'}$,

$$h(\xi, b_1b) < r < h(\xi, b_1b_0b). \tag{6.5}$$

Since $b_1b_0b_1^{-1} \in \mathsf{E}_\nu$ by the hypothesis and $\nu(Q_{b_0}) > 0$, there exists $\gamma \in \Gamma$ such that

$$\mathcal{Q} := \{\xi \in Q_{b_0} \cap \gamma^{-1}Q_{b_0} : \beta_\xi(\gamma^{-1}, e) \in b_1b_0b_1^{-1}(AM)_{\varepsilon'/2}\}$$

has a positive ν-measure. We now claim that

$$h(\xi, b_1b) < r < h(\gamma(\xi, b_1b))$$

for all $\xi \in \mathcal{Q}$ and for all $b \in (AM)_{\varepsilon'/2}$. This yields a contradiction to the Γ-invariance of h. Since $\mathcal{Q} \subset Q_{b_0}$, we have $h(\xi, b_1b) < r$ for all $b \in (AM)_{\varepsilon'}$ by (6.5). On the other hand, for all $b \in (AM)_{\varepsilon'/2}$ and $\xi \in \mathcal{Q}$, we have

$$\beta_\xi(\gamma^{-1}, e)b_1b \in b_1b_0b_1^{-1}(AM)_{\varepsilon'/2}b_1b \subset b_1b_0(AM)_{\varepsilon'},$$

since $m^{-1}M_{\varepsilon'/2}mM_{\varepsilon'/2} \subset M_{\varepsilon'}$ for all $m \in M$. Since $\gamma\xi \in Q_{b_0}$ and $\gamma(\xi, b_1b) = (\gamma\xi, \beta_\xi(\gamma^{-1}, e)b_1b)$, it follows from (6.5) that $h(\gamma(\xi, b_1b)) > r$. This proves the claim. □

7. N-ergodic decompositions of BR-measures

Let $\Gamma < G$ be an Anosov subgroup. We prove Theorem 1.1(2) in this section.

7.1. **Ergodic decomposition of an infinite measure.** The following version of ergodic decomposition of any Radon measure can be deduced from [12, Thm. 5.2].

Proposition 7.1 (Ergodic decomposition)**.** *Let G be a locally compact second countable group. Let $N < G$ be a closed subgroup and $M < G$ be a compact subgroup normalizing N. Suppose that NM acts continuously on a locally compact, σ-compact, standard Borel space $(X, \mathcal{B})$, preserving a Radon measure μ on X.*

(1) *There exists a Borel map $x \mapsto \mu_x$ from X to the space of N-invariant ergodic Radon measures on X and an M-invariant probability measure μ^* on X equivalent to μ with the following properties:*
   (a) *$\mu_x = \mu_{xn}$ for every $x \in X$ and $n \in N$.*
   (b) *For all nonnegative Borel function $f : X \to \mathbb{R}$, we have*
$$\int f\, d\mu_x = \mathbb{E}_{\mu^*}\left(f \frac{d\mu}{d\mu^*} | \mathcal{S}_N\right)(x) \qquad \text{for } \mu\text{-a.e. } x \in X,$$
where $\mathcal{S}_N := \{B \in \mathcal{B} : B.n = B \text{ for all } n \in N\}$. In particular, we have
$$\mu = \int_{x\in X} \mu_x \, d\mu^*(x).$$
If μ is finite, we can take $\mu^* = \mu$.

(2) *Let $\mathcal{T} \subset \mathcal{S}_N$ be the smallest σ-algebra such that the map $x \mapsto \mu_x$ is $\mathcal{T}$-measurable. Then $\mathcal{T}$ is countably generated, $\mathcal{T} = \mathcal{S}_N$ mod μ, $\mu_x([y]_{\mathcal{T}}) = 0$ for all $y \notin [x]_{\mathcal{T}}$, and $\mu_x([x]_{\mathcal{T}}^c) = 0$ for all $x, y \in X$. Here $[y]_{\mathcal{T}} = \cap_{y\in C\in\mathcal{T}} C$ denotes the atom of y in $\mathcal{T}$.*

(3) *For each $m \in M$, we have $\mu_{xm} = \mu_x.m$ for μ-a.e. $x \in X$.*

Proof. Fix an M-invariant positive function $\varphi \in L^1(\mu)$ with $\int \varphi\, d\mu = 1$. Then $d\mu^* := \varphi\, d\mu$ defines an N-quasi-invariant and M-invariant probability measure on X. By applying [12, Thm. 5.2] to μ^* with the cocycle $\rho : N \times X \to \mathbb{R}$ given by $\rho(n, y) = \log \frac{\varphi(yn^{-1})}{\varphi(y)}$, we get a Borel map $x \mapsto \mu_x^*$ from X to the space of N-ergodic probability measures such that for all nonnegative Borel function $f : X \to \mathbb{R}$, we have
$$\int f\, d\mu_x^* = \mathbb{E}_{\mu^*}(f|\mathcal{S}_N)(x) \quad \text{for } \mu^*\text{-a.e. } x \in X,$$
and $\frac{d(n.\mu_x^*)}{d\mu_x^*}(y) = \frac{\varphi(yn^{-1})}{\varphi(y)}$. In particular, we have $\mu^* = \int \mu_x^*\, d\mu^*(x)$. Now define a Radon measure μ_x on X by $d\mu_x := \frac{1}{\varphi}\, d\mu_x^*$. A direct computation shows that μ_x is N-invariant, ergodic for all $x \in X$ and (1) holds. (2) follows from the corresponding statement on μ_x^* from [12, Thm. 5.2].

In order to prove (3), we compute that for a non-negative Borel function $f : X \to \mathbb{R}$,
$$\mu_{xm}^*(f) = \mathbb{E}_{\mu^*}(f|\mathcal{S}_N)(xm) = \mathbb{E}_{\mu^*}(m.f|\mathcal{S}_N)(x) = \mu_x^*(m.f);$$

the second equality follows since $\mathcal{S}_N.m = \mathcal{S}_N$ and μ^* is M-invariant. It follows that $\mu^*_{xm} = \mu^*_x.m$ for μ-a.e. $x \in X$; this implies (3). $\square$

7.2. **P°-semi-invariant measures.** In terms of the coordinates $G = G/P^\circ \times AM^\circ N$, we have

$$d\tilde{m}^{\mathrm{BR}}_\psi = d\tilde{\nu}_\psi e^{\psi(\log a)} dadmdn. \tag{7.1}$$

Recall that a measure μ on $\Gamma\backslash G$ is P°-semi-invariant if there exists a character $\chi : P \to \mathbb{R}_+$ such that for all $p \in P^\circ$, $p_*\mu = \chi(p)\mu$. Since χ must be trivial on NM°, μ is necessarily NM°-invariant and if we set $\chi_\mu \in \mathfrak{a}^*$ to be $-\log(\chi|_A)$, we get that for all $a \in A$,

$$a_*\mu = e^{-\chi_\mu(\log a)}\mu.$$

We set $\psi_\mu := \chi_\mu + 2\rho \in \mathfrak{a}^*$.

Proposition 7.2. *Let μ be a P°-semi invariant and N-ergodic Radon measure supported on $\mathcal{E}$. Let $\tilde{\mu}$ denote its Γ-invariant lift to $G \simeq G/P^\circ \times AM^\circ N$. Then $\psi_\mu \in D^\star_\Gamma$ and $d\tilde{\mu}$ is proportional to $d\tilde{\nu}_{\psi_\mu}|_{\Lambda_0} e^{\psi_\mu(\log a)} da\, dm\, dn$ for some Γ-minimal subset $\Lambda_0 \in \mathcal{Y}_\Gamma$, or equivalently, μ is proportional to $m^{\mathrm{BR}}_{\psi_\mu}|_{\mathcal{E}_0}$ for some $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$.*

Proof. Since $\tilde{\mu}$ is a right P°-semi-invariant measure on $G \simeq G/P^\circ \times AM^\circ N$, up to a positive constant multiple, we have

$$d\tilde{\mu} = e^{\tilde{\chi}(\log a)} d\tilde{\nu}\, da\, dm\, dn$$

for some Radon measure $\tilde{\nu}$ on G/P° and $\tilde{\chi} \in \mathfrak{a}^*$ [16, Proposition 10.25]. Since $a_*\tilde{\mu} = e^{-\chi_\mu(\log a)}\tilde{\mu}$, it follows $\tilde{\chi} = \psi_\mu$. Denote by $\pi : G/P^\circ \to G/P$ the projection map. Since $\tilde{\mu}$ is right N-ergodic, $\tilde{\nu}$ is a Γ-ergodic measure on G/P°. And since $\tilde{\mu}$ is Γ-invariant, $\pi_*\tilde{\nu}$ is a (Γ, ψ_μ)-conformal measure on G/P (cf. [16, Prop. 10.25]). In particular, $\psi_\mu \in D^\star_\Gamma$ by [16, Thm. 7.7]. Let $\tilde{\nu}_{\psi_\mu}$ be the M-invariant lift of $\nu_{\psi_\mu} := \pi_*\tilde{\nu}$ to G/P°. Since $\tilde{\nu} \ll \tilde{\nu}_{\psi_\mu}$ and $\tilde{\nu}$ is Γ-ergodic, $\tilde{\nu}$ is proportional to $\tilde{\nu}_{\psi_\mu}|_{\Lambda_0}$ for some Γ-minimal subset $\Lambda_0 \in \mathcal{Y}_\Gamma$ by Proposition 4.8. This completes the proof. $\square$

7.3. **Essential values and Ergodicity.** We fix $\psi \in D^\star_\Gamma$ for the rest of the section. Let ν_ψ be the unique (Γ, ψ)-Patterson Sullivan measure on Λ. Let E_{ν_ψ} be the set of essential values as defined in Definition 6.1.

Proposition 7.3. *If $M^\circ \subset \mathsf{E}_{\nu_\psi}$, then for any $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, $m^{\mathrm{BR}}_\psi|_{\mathcal{E}_0}$ is N-ergodic.*

Proof. Let $m^{\mathrm{BR}}_\psi = \int_X \mathsf{m}_x\, d\mathsf{m}^*(x)$ be an N-ergodic decomposition as given by Proposition 7.1 with $X = \Gamma\backslash G$. Let $f \in C_c(\Gamma\backslash G)$ and consider the map $h(g) := \mathsf{m}_{[g]}(f)$ for all $[g] \in X$. Note that h defines a Γ-invariant Borel function on G/N. Since M° is a normal subgroup of AM, Lemma 6.3 implies that h is M°-invariant for $\hat{\nu}_\psi$-almost all. By Proposition 7.1(3), it follows that $M^\circ < \mathrm{Stab}_M(\mathsf{m}_x)$ for almost all x; without loss of generality, we may assume that $M^\circ < \mathrm{Stab}_M(\mathsf{m}_x)$ for all $x \in X$. Hence the finite group $S := M^\circ \backslash M$ acts on $\{\mathsf{m}_x : x \in X\}$. Set

$$\tilde{\mathsf{m}}_x := \frac{1}{[M : M^\circ]} \sum_{s \in M^\circ \backslash M} \mathsf{m}_x.s.$$

Since $m_\psi^{\rm BR}$ is M-invariant, we have $m_\psi^{\rm BR} = \int_X \tilde{\mathsf{m}}_x d\mathsf{m}^*(x)$. As $\mathsf{m}_{xm} = \mathsf{m}_x.m$ for all $m \in M$, the map $x \mapsto \tilde{\mathsf{m}}_x$ is NM-invariant. Since $m_\psi^{\rm BR}$ is NM-ergodic, $\tilde{\mathsf{m}}_x$ is constant m-a.e. $x \in X$. Therefore we may fix $x_0 \in X$ so that $m_\psi^{\rm BR} = \tilde{\mathsf{m}}_{x_0}$. Set $M_* := \mathrm{Stab}_M(\mathsf{m}_{x_0})$. Then

$$m_\psi^{\rm BR} = \frac{1}{[M : M_*]} \sum_{s \in M_* \backslash M} \mathsf{m}_{x_0}.s$$

where $\mathsf{m}_{x_0}.s$ are mutually singular to each other. We claim that each $\mathsf{m}_{x_0}.s$ is A-semi-invariant with $\psi_{\mathsf{m}_{x_0}.s} = \psi$ for each $s \in M_* \backslash M$. It suffices to consider the case when $s = [M^*]$. Let

$$A' := \{a \in A : a \text{ preserves the measure class of } \mathsf{m}_{x_0}\}.$$

As A' is a closed subgroup of A, it suffices to show that for any unit vector $u \in \mathfrak{a}$ and any $\varepsilon > 0$, $\exp tu \in A'$ for some $0 < t < \varepsilon$. Let $a = \exp \frac{\varepsilon u}{n+2}$ for $n = \# M/M^*$. Since $m_\psi^{\rm BR}$ is quasi-invariant under a and has n number of ergodic components, it follows that for some $1 \le k \le n+1$, $a^k.\mathsf{m}_{x_0}$ is in the same measure class as m_{x_0}, implying that $a^k \in A'$. Hence $A = A'$. As $m_\psi^{\rm BR}$ is semi-invariant under A, the claim follows. Therefore, by Proposition 7.2, m_{x_0} is proportional to $m_\psi^{\rm BR}|_{\mathcal{E}_0}$ for some $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$. Hence $M_* = \mathrm{Stab}_M\, m_\psi^{\rm BR}|_{\mathcal{E}_0} = M_\Gamma$. Since the measures $\mathsf{m}_{x_0}.s$ are mutually singular to each other, all $\mathcal{E}_0$'s are distinct. Therefore $m_\psi^{\rm BR} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} c(\mathcal{E}_0) \cdot m_\psi^{\rm BR}|_{\mathcal{E}_0}$ for some constant $c(\mathcal{E}_0) > 0$. It remains to observe $c(\mathcal{E}_0) = 1$ as the supports of $m_\psi^{\rm BR}|_{\mathcal{E}_0}$ are mutually disjoint from each other. $\square$

Proof of Theorem 1.3. Let $\mathcal{O}_\Gamma$ denote the space of all N-invariant ergodic and P°-quasi-invariant Radon measures supported on $\mathcal{E}$, up to constant multiples. We write $\mathfrak{Y}_\Gamma = \{\mathcal{E}_i : 1 \le i \le k\}$ with $k = \#\mathfrak{Y}_\Gamma = \# M/M_\Gamma$. Consider the map $\iota : D_\Gamma^\star \times \{1, \cdots, k\} \to \mathcal{O}_\Gamma$ defined by $\iota(\psi, i) = m_\psi^{\rm BR}|_{\mathcal{E}_i}$. By Proposition 7.3, ι is well-defined. Since any measure contained in $\mathcal{O}_\Gamma$ must be P°-semi-invariant, being N-ergodic, Proposition 7.2 implies that ι is surjective. That ι is indeed a homeomorphism now follows because the map $\psi \mapsto m_\psi^{\rm BR}$ is a homeomorphism between $D_\Gamma^\star$ and the space of all NM-invariant ergodic and A-quasi-invariant Radon measures supported on $\mathcal{E}$, up to constant multiples, as shown in [16]. This implies Theorem 1.3, as $D_\Gamma^\star$ is homeomorphic to $\mathbb{R}^{\mathrm{rank}\, G - 1}$ [16].

7.4. **The largeness of the length spectrum.** Without loss of generality, we may assume that $\Gamma \cap \mathrm{int}\, A^+ M \neq \emptyset$ for the rest of section. Recall the notation $\Gamma^\star$ from (3.4) and $\hat\lambda(g)$ from Definition 3.1. We will need the following:

Proposition 7.4. *For any $C>1$, the closed subgroup of AM generated by $\{\hat\lambda(\gamma_0)\in AM : \gamma_0\in\Gamma^\star, \psi(\lambda(\gamma_0))>C\}$ contains AM°.*

By Corollary 3.7 applied to Γ_ψ, this proposition follows from the following lemma.

Lemma 7.5. *For any $C>1$, there exists a Zariski dense subgroup $\Gamma_\psi<\Gamma$, depending on C, such that $\Gamma_\psi\cap \operatorname{int} A^+M\neq\emptyset$ and*

$$\psi(\lambda(\gamma))>C \quad \text{for all } \gamma\in\Gamma_\psi-\{e\}.$$

In particular, $\hat\lambda(\Gamma_\psi^\star)\subset\{\hat\lambda(\gamma_0)\in AM : \gamma_0\in\Gamma^\star, \psi(\lambda(\gamma_0))>C\}$.

Proof. Recall that Π is the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$. By [1, Lem. 4.3(b)], there exist $\varepsilon>0$ and $\{s_1,s_2\}\subset\Gamma$ such that $s_1\in\operatorname{int}A^+M$, and for each $m\geq 1$, s_1^m, s_2^m are (Π,ε)-Schottky generators and the subgroup $\Gamma_m=\langle s_1^m,s_2^m\rangle$ is a Zariski-dense (Π,ε)-Schottky subgroup of Γ (see [1, Def. 4.1] for terminologies).

Fix $m>1$ and let $z\in\lambda(\Gamma_m)-\{0\}$. Then $z=\lambda(w)$ for some $w=g_1^{n_1}\cdots g_\ell^{n_\ell}$ with $g_i\in\{s_1^{\pm m},s_2^{\pm m}\}$, $n_i\in\mathbb{N}$, $g_i\neq g_{i+1}^{-1}(i=1,\cdots,\ell)$ where we interpret $g_{\ell+1}:=g_1$; this is because every element of a (Π,ε)-Schottky group is conjugate to a word of such form. By [1, Lem. 4.1], there exists $R=R(\varepsilon)>0$ (independent of $w\in\Gamma_1$) such that

$$\|\lambda(w)-\sum_{i=1}^{\ell} n_i\lambda(g_i)\|\leq \ell R.$$

Since $\psi(\lambda(s_j^{\pm1}))>0$ and $\lambda(s_j^{\pm m})=m\lambda(s_j^{\pm 1})$, we can choose $m_0\in\mathbb{N}$ such that

$$\psi(\lambda(s_j^{\pm m_0}))>\|\psi\|R+C \quad \text{for each } j=1,2.$$

Set

$$\Gamma_\psi:=\Gamma_{m_0}.$$

Then for any $z=\lambda(w)\in\lambda(\Gamma_\psi)-\{0\}$ as above,

$$\psi(z)\geq\sum_{i=1}^{\ell}n_i\psi(\lambda(g_i))-\|\psi\|\ell R\geq\sum_{i=1}^{\ell}n_i\bigg(\psi(\lambda(g_i))-\|\psi\|R\bigg)>C.$$

The lemma follows. $\square$

7.5. **Proof of Main proposition.** Recall the $\mathfrak{a}$-valued Gromov product on $\Lambda^{(2)}$: for any $\xi\neq\eta$ in Λ,

$$\mathcal{G}(\xi,\eta):=\log\beta^A_{h^+}(e,h)+\mathrm{i}\log\beta^A_{h^-}(e,h)$$

for $h\in G$ satisfying that $h^+=\xi$ and $h^-=\eta$. For any fixed $p=g(o)\in G/K$, the following

$$d_{\psi,p}(\xi,\eta):=e^{-\psi(\mathcal{G}(g^{-1}\xi,g^{-1}\eta))} \quad \text{for any } \xi\neq\eta \text{ in } \Lambda$$

defines a virtual visual metric on Λ, satisfying a weak version of triangle inequality [16, Lem. 6.11]. For $\xi \in \Lambda$ and $r > 0$, set

$$\mathbb{B}_p(\xi, r) := \{\eta \in \Lambda : d_{\psi,p}(\xi, \eta) < r\}.$$

We recall the following two lemmas:

Lemma 7.6. [16, Lem. 6.12] *There exists $N_0(\psi, p) \geq 1$ satisfying the following: for any finite collection $\mathbb{B}_p(\xi_1, r_1), \cdots, \mathbb{B}_p(\xi_n, r_n)$ with $\xi_i \in \Lambda$ and $r_i > 0$, there exists a disjoint subcollection $\mathbb{B}_p(\xi_{i_1}, r_{i_1}), \cdots, \mathbb{B}_p(\xi_{i_\ell}, r_{i_\ell})$ such that*

$$\mathbb{B}_p(\xi_1, r_1) \cup \cdots \cup \mathbb{B}_p(\xi_n, r_n) \subset \mathbb{B}_p(\xi_{i_1}, 3N_0(\psi, p) r_{i_1}) \cup \cdots \cup \mathbb{B}_p(\xi_{i_\ell}, 3N_0(\psi, p) r_{i_\ell}).$$

Moreover, $N_0(\psi, p)$ can be taken uniformly for all p in a fixed compact subset of G/K.

Lemma 7.7. [16, Lem. 10.6]. *There exists a compact subset $\mathcal{C} \subset G$ such that for any $\xi \in \Lambda$, there exists $g \in \mathcal{C}$ such that $g^+ = \xi$ and $g^- \in \Lambda$.*

We set

$$N_0 := \max_{p \in \mathcal{C}(o)} N_0(\psi, p) < \infty$$

with $N_0(\psi, p)$ and $\mathcal{C}$ given by Lemmas 7.6 and 7.7 respectively.

Proposition 7.8 (Main Proposition)**.** *For all $\gamma_0 \in \Gamma^\star$ satisfying $\psi(\lambda(\gamma_0)) > \log 3N_0 + 1$, we have $\hat{\lambda}(\gamma_0) \in \mathsf{E}_{\nu_\psi}$.*

7.6. **Proof of Theorem 1.1(1).** By Propositions 7.4 and 7.8, E_{ν_ψ} contains AM°. Therefore Theorem 1.1(1) follows from Proposition 7.3.

The rest of the section is devoted to the proof of Proposition 7.8.

Definition of $\mathcal{B}_R(\gamma_0, \varepsilon)$. We now fix $\varepsilon > 0$ as well as an element $\gamma_0 \in \Gamma^\star$ such that

$$\psi(\lambda(\gamma_0)) > \log 3N_0 + 1.$$

Note that $y_{\gamma\gamma_0^{\pm 1}\gamma^{-1}} = \gamma y_{\gamma_0^{\pm 1}}$ for all $\gamma \in \Gamma$. We can choose $g \in \mathcal{C}$ such that $g^+ = y_{\gamma_0}$ and $g^- \in \Lambda$. Note that $g^+ \in N^+ e^+$, as $\gamma_0 \in \Gamma^\star$. Set

$$p := g(o), \;\; \eta := g^-, \text{ and } \xi_0 := g^+.$$

For any $\xi \in \Lambda - \{\eta, e^-\}$, we claim that there is $R_\varepsilon = R_\varepsilon(\xi) > 0$ such that

$$\beta^{AM}_{\xi'}(g, e) \in \beta^{AM}_{\xi}(g, e)(AM)_\varepsilon$$

for all $\xi' \in \mathbb{B}_p(\xi, e^{\psi(\lambda(\gamma_0) + \lambda(\gamma_0^{-1})) + 2\|\psi\|\varepsilon} R_\varepsilon)$. Indeed, since $e^- \notin \{\xi, g^{-1}\xi\}$, we have $\xi, g^{-1}\xi \in N^+ e^+$ by Lemma 4.2. The claim follows as the map $\xi' \mapsto \beta^{AM}_{\xi'}(g, e)$ is continuous at ξ.

By [16, Lem. 6.11], the family $\{\mathbb{B}_p(\xi, r) : \xi \in \Lambda, r > 0\}$ forms a basis of topology in Λ. For $\gamma \in \Gamma$, let $r_g(\gamma)$ be the supremum of $r \geq 0$ such that for all $\xi \in \mathbb{B}_p(\gamma\xi_0, 3N_0 r)$, $\beta^{AM}_\xi(g, \gamma\gamma_0\gamma^{-1}g)$ is well-defined and

$$\beta^{AM}_\xi(g, \gamma\gamma_0\gamma^{-1}g) \in \beta^{AM}_{\gamma\xi_0}(g, \gamma\gamma_0\gamma^{-1}g)(AM)_\varepsilon. \tag{7.2}$$

If $\gamma\xi_0 \notin \{e^-, g^-\}$ and hence $\gamma\xi_0, g^{-1}\gamma\xi_0 \in N^+e^+$, then $r_g(\gamma) > 0$.

For each $R > 0$, we define the family of virtual balls as follows:

$$\mathcal{B}_R(\gamma_0, \varepsilon) = \{\mathbb{B}_p(\gamma\xi_0, r) : \gamma \in \Gamma, 0 < r < \min(R, r_g(\gamma))\}.$$

We remark that the difference of the definition of $\mathcal{B}_R$ in this paper and our previous paper [16] lies in the definition of $r_g(\gamma)$; in [16], we used the A-valued Busemann function in (7.2) whereas $r_g(\gamma)$ is defined in terms of the AM-valued Busemann function here.

Theorem 7.9. [16, Thm. 5.3] *There exists $C = C(\psi, p) > 0$ such that for all $\gamma \in \Gamma$ and $\xi \in \Lambda$,*

$$-\psi(\underline{a}(p, \gamma p)) - C \le \psi(\log \beta_\xi^A(\gamma p, p)) \le \psi(\underline{a}(\gamma p, p)) + C.$$

where $\underline{a}(p, q) := \mu(g^{-1}h)$ for $p = g(o)$ and $q = h(o)$.

For $q \in G/K$ and $r > 0$, the shadow of the ball $B(q, r)$ viewed from $p = g(o) \in G/K$ and $\xi \in \mathcal{F}$ are respectively defined as

$$O_r(p, q) := \{gk^+ \in \mathcal{F} : k \in K,\ gk \operatorname{int} A^+o \cap B(q, r) \neq \emptyset\}$$

where $g \in G$ satisfies $p = g(o)$, and

$$O_r(\xi, q) := \{h^+ \in \mathcal{F} : h^- = \xi, ho \in B(q, r)\}.$$

Lemma 7.10. [16, Lem. 5.7] *There exists $\kappa > 0$ such that for any $p, q \in G/K$ and $r > 0$, we have*

$$\sup_{\xi \in O_r(p,q)} \|\log \beta_\xi^A(p, q) - \underline{a}(p, q)\| \le \kappa r.$$

We let $C = C(\psi, p) > 0$ and $\kappa > 0$ be the constants given by Theorem 7.9 and Lemma 7.10 respectively. Since ξ_0 belongs to the shadow $O_{\varepsilon/(8\kappa)}(\eta, p)$, we can choose $0 < s = s(\gamma_0) < R$ small enough such that

$$\mathbb{B}_p(\xi_0, e^{\psi(\lambda(\gamma_0) + \lambda(\gamma_0^{-1})) + \frac{1}{2}\|\psi\|\varepsilon + 2C} s) \subset O_{\varepsilon/(8\kappa)}(\eta, p). \tag{7.3}$$

Next, observe that the map $\xi' \mapsto \beta_{\xi'}(g, \gamma_0 g)$ is continuous at ξ_0, as $g^{-1}\xi_0 = e^+ \in N^+e^+$. Hence we may further assume that s is small enough so that

$$\beta_{\xi'}^{AM}(g, \gamma_0 g) \in \beta_{\xi_0}^{AM}(g, \gamma_0 g)(AM)_\varepsilon \quad \text{for all } \xi' \in \mathbb{B}_p(\xi_0, e^{2C}s). \tag{7.4}$$

For each $\gamma \in \Gamma$, set

$$D(\gamma\xi_0, r) := \mathbb{B}_p(\gamma\xi_0, \tfrac{1}{3N_0} e^{-\psi(\mu(g^{-1}\gamma g) + \mu(g^{-1}\gamma^{-1}g))} r) \text{ and}$$
$$3N_0 D(\gamma\xi_0, r) := \mathbb{B}_p(\gamma\xi_0, e^{-\psi(\mu(g^{-1}\gamma g) + \mu(g^{-1}\gamma^{-1}g))} r).$$

Here note that $\underline{a}(\gamma^{-1}p, p) = \mu(g^{-1}\gamma g)$ and $\mathrm{i}\,\underline{a}(\gamma^{-1}p, p) = \mu(g^{-1}\gamma^{-1}g)$.

Lemma 7.11. *Let $R > 0$ and $\xi \in \Lambda - \{\eta\}$. Let $\gamma_i \in \Gamma$ be a sequence such that $\gamma_i^{-1}p \to \eta$, $\gamma_i^{-1}\xi \to \xi_0$, and $\beta_\xi^M(\gamma_i, e) \to e$ as $i \to \infty$. Then, by passing to a subsequence, the following holds for all sufficiently small $r > 0$: there exists $i_0 = i_0(r) > 0$ such that for all $i \ge i_0$, we have*

(1) $\xi \in D(\gamma_i\xi_0, r)$ *and* $D(\gamma_i\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$*; in particular, for any* $R > 0$*,*

$$\Lambda_\psi^\spadesuit \subset \bigcup_{D \in \mathcal{B}_R(\gamma_0,\varepsilon)} D.$$

(2) $\{\beta_{\xi'}^{AM}(e, \gamma_i\gamma_0\gamma_i^{-1}) : \xi' \in 3N_0 D(\gamma_i\xi_0, r)\} \subset \hat\lambda(\gamma_0)(AM)_{O(\varepsilon)}$.

Proof. Let $g \in G$ be such that $p = g(o)$. Note that $\gamma_i^{-1} go \to \eta = g^-$ and $\gamma_i^{-1}\xi \to \xi_0 = g^+$. By passing to a subsequence, we have a neighborhood $U_\varepsilon \subset \mathcal{F}$ of ξ_0 associated to the sequence γ_i given by Proposition 5.1. Since $\xi_0 \in U_\varepsilon$, there exists $R_1 > 0$ such that

$$\mathbb{B}_p(\xi_0, e^{2C} R_1), \gamma_0^{-1}\mathbb{B}_p(\xi_0, e^{2C} R_1) \subset U_\varepsilon.$$

Let $0 < r < \min(s(\gamma_0), R_\varepsilon/2, R_1, R)$. In view of [16, Lem. 10.12], we have $3N_0 D(\gamma_i\xi_0, r) \subset \gamma_i \mathbb{B}_p(\xi_0, e^{2C} r)$. In order to show that $D(\gamma_i\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$, it suffices to check that for all $\xi' \in \mathbb{B}_p(\xi_0, e^{2C} r)$,

$$\beta_{\xi'}^M(\gamma_i^{-1} g, \gamma_0\gamma_i^{-1} g) \in \beta_{\xi_0}^M(\gamma_i^{-1} g, \gamma_0\gamma_i^{-1} g) M_\varepsilon;$$

this implies that $r < r_g(\gamma_i)$.

We start by noting that since $r \le s(\gamma_0)$, we have $\beta_{\xi'}^M(g, \gamma_0 g) \in \beta_{\xi_0}^M(g, \gamma_0 g) M_\varepsilon$. Since $\xi', \gamma_0^{-1}\xi' \in U_\varepsilon$, by Proposition 5.1, for all sufficiently large i,

$$\begin{aligned}\beta_{\xi'}^M(\gamma_i^{-1} g, \gamma_0\gamma_i^{-1} g) &= \beta_{\xi'}^M(\gamma_i^{-1} g, g)\beta_{\xi'}^M(g, \gamma_0 g)\beta_{\xi'}^M(\gamma_0 g, \gamma_0\gamma_i^{-1} g)\\ &= \beta_{\xi'}^M(\gamma_i^{-1} g, g)\beta_{\xi'}^M(g, \gamma_0 g)\beta_{\gamma_0^{-1}\xi'}^M(\gamma_i^{-1} g, g)^{-1}\\ &\in \beta_{\xi_0}^M(\gamma_i^{-1} g, g)\beta_{\xi_0}^M(g, \gamma_0 g)\beta_{\xi_0}^M(\gamma_i^{-1} g, g)^{-1} M_{O(\varepsilon)}\\ &= \beta_{\xi_0}^M(\gamma_i^{-1} g, \gamma_0\gamma_i^{-1} g) M_{O(\varepsilon)},\end{aligned}$$

which verifies that $D(\gamma_i\xi_0, r)$ belongs to the family $\mathcal{B}_R(\gamma_0, \varepsilon)$. The claim that $\xi \in D(\gamma_i\xi_0, r)$ can be shown in the same way as in the proof of [16, Lem. 10.12]. This proves (1).

(1) implies that for all sufficiently large i and $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$, we have

$$\beta_{\xi'}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1} g) \in \beta_{\gamma_i\xi_0}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1} g)(AM)_\varepsilon. \tag{7.5}$$

Now note that for all $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$,

$$\begin{aligned}\beta_{\xi'}^{AM}(e, \gamma_i\gamma_0\gamma_i^{-1}) &= \beta_{\xi'}^{AM}(e, g)\beta_{\xi'}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1} g)\beta_{\xi'}^{AM}(\gamma_i\gamma_0\gamma_i^{-1} g, \gamma_i\gamma_0\gamma_i^{-1})\\ &= \beta_{\xi'}^{AM}(e, g)\beta_{\xi'}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1} g)\beta_{\gamma_i\gamma_0^{-1}\gamma_i^{-1}\xi'}^{AM}(e, g)^{-1}.\end{aligned} \tag{7.6}$$

On the other hand,

$$\begin{aligned}d_p(\gamma_i\gamma_0\gamma_i^{-1}\xi', \gamma_i\xi_0) &= e^{-\psi(\log\beta_{\xi'}^A(\gamma_i\gamma_0^{-1}\gamma_i^{-1} g, g) + \mathrm{i}\log\beta_{\gamma_i\xi_0}^A(\gamma_i\gamma_0^{-1}\gamma_i^{-1} g, g))} d_p(\xi', \gamma_i\xi_0)\\ &\le e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon} d_p(\xi', \gamma_i\xi_0),\end{aligned}$$

and hence

$$\xi', \gamma_i\gamma_0\gamma_i^{-1}\xi' \in \mathbb{B}_p(\gamma_i\xi_0, e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon} r).$$

Since

$$\gamma_i\xi_0 \to \xi \quad \text{as } i \to \infty \tag{7.7}$$

by Lemma 4.16 and $r < R_\varepsilon/2$, for all sufficiently large i and all $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$, the elements ξ', $\gamma_i\gamma_0\gamma_i^{-1}\xi'$, and $\gamma_i\xi_0$ all belong to the subset $\mathbb{B}_p(\xi, e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon}R_\varepsilon)$. Hence

$$\beta^{AM}_{\xi'}(e,g), \beta^{AM}_{\gamma_i\gamma_0^{-1}\gamma_i^{-1}\xi'}(e,g), \beta^{AM}_{\gamma_i\xi_0}(e,g) \in \beta^{AM}_{\xi}(e,g)M_\varepsilon. \tag{7.8}$$

Combining (7.5), (7.6) and (7.8), it follows that for all $\xi' \in 3N_0D(\gamma_i\xi_0, r)$,

$$\beta^{AM}_{\xi'}(e, \gamma_i\gamma_0\gamma_i^{-1}) \in \beta^{AM}_{\gamma_i\xi_0}(e, \gamma_i\gamma_0\gamma_i^{-1})(AM)_{O(\varepsilon)}.$$

Note that by Proposition 5.1 and (7.7), we get

$$\begin{aligned}
\beta^{AM}_{\xi_0}(\gamma_i^{-1}, e) &= \beta^{AM}_{\xi_0}(\gamma_i^{-1}, \gamma_i^{-1}g)\beta^{AM}_{\xi_0}(\gamma_i^{-1}g, g)\beta^{AM}_{\xi_0}(g, e)\\
&= \beta^{AM}_{\gamma_i\xi_0}(e,g)\beta^{AM}_{\xi_0}(\gamma_i^{-1}g, g)\beta^{AM}_{\xi_0}(g,e)\\
&\in \beta^{AM}_{\xi}(e,g)\beta^{AM}_{\gamma_i^{-1}\xi}(\gamma_i^{-1}g, g)\beta^{AM}_{\xi_0}(g,e)(AM)_{O(\varepsilon)}\\
&= \beta^{AM}_{\gamma_i^{-1}\xi}(\gamma_i^{-1}, \gamma_i^{-1}g)\beta^{AM}_{\gamma_i^{-1}\xi}(\gamma_i^{-1}g, g)\beta^{AM}_{\gamma_i^{-1}\xi}(g,e)(AM)_{O(\varepsilon)}\\
&= \beta^{AM}_{\gamma_i^{-1}\xi}(\gamma_i^{-1}, e)(AM)_{O(\varepsilon)} \qquad (7.9)
\end{aligned}$$

Since $\beta^{M}_{\gamma_i^{-1}\xi}(\gamma_i^{-1}, e) = \beta^M_\xi(e, \gamma_i) \to e$ as $i \to \infty$ by the hypothesis, (7.9) implies that

$$\beta^M_{\xi_0}(\gamma_i^{-1}, e) \in M_{O(\varepsilon)} \text{ for all large enough } i. \tag{7.10}$$

Since

$$\begin{aligned}
\beta^{AM}_{\gamma_i\xi_0}(e, \gamma_i\gamma_0\gamma_i^{-1}) &= \beta^{AM}_{\gamma_i\xi_0}(e, \gamma_i)\beta^{AM}_{\gamma_i\xi_0}(\gamma_i, \gamma_i\gamma_0)\beta^{AM}_{\gamma_i\xi_0}(\gamma_i\gamma_0, \gamma_i\gamma_0\gamma_i^{-1})\\
&= \beta^M_{\xi_0}(\gamma_i^{-1}, e)\hat\lambda(\gamma_0)\beta^M_{\xi_0}(\gamma_i^{-1}, e)^{-1},
\end{aligned}$$

we deduce from (7.10) that

$$\beta^{AM}_{\xi'}(e, \gamma_i\gamma_0\gamma_i^{-1}) \in \hat\lambda(\gamma_0)(AM)_{O(\varepsilon)}$$

as desired. $\square$

Lemma 7.12. *Let $B \subset \mathcal{F}$ be a Borel set with $\nu_\psi(B) > 0$. Then for ν_ψ-a.e. $\xi \in B$,*

$$\limsup_{R\to 0}\left\{ \frac{\nu_\psi(B \cap D(\gamma\xi_0, r))}{\nu_\psi(D(\gamma\xi_0, r))} : \begin{array}{c} \xi \in D(\gamma\xi_0, r), r < R, \textit{ and} \\ \beta^{AM}_{\xi'}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon \\ \textit{for all } \xi' \in 3N_0D(\gamma\xi_0, r)\end{array}\right\} = 1.$$

Proof. To each Borel function $h : G/P \to \mathbb{R}$, we associate a function $h^* : G/P \to \mathbb{R}$ defined by

$$h^*(\xi) = \limsup_{R\to 0}\left\{ \frac{1}{\nu_\psi(D)}\int_D h\, d\nu_\psi : \begin{array}{c} \xi \in D = D(\gamma\xi_0, r), r < R, \text{ and} \\ \beta^{AM}_{\xi'}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon \\ \text{for all } \xi' \in 3N_0D(\gamma\xi_0, r)\end{array}\right\}.$$

By Lemma 4.16 and 7.11, h^* is well defined on $\Lambda_\psi^{\spadesuit} - \{\eta\}$ and hence ν_ψ-a.e. on G/P by Corollary 4.10. We may then apply the same argument as in [16, Proof of Prop. 10.17] to deduce $h^* = h$ ν_ψ-a.e. Hence the lemma follows by taking $h = \mathbf{1}_B$. □

Proof of Proposition 7.8. Let $B \subset \mathcal{F}$ be a Borel set such that $\nu_\psi(B) > 0$ and let $\varepsilon > 0$ be arbitrary. By Lemma 7.12, for ν_ψ-a.e. $\xi \in B$, there exist $\gamma \in \Gamma^\star$ and $D = D(\gamma\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$ containing ξ such that

(1) $\nu_\psi(D \cap B) > (1 + e^{-\psi(\lambda(\gamma_0^{-1})) - \|\psi\|\varepsilon})^{-1}\nu_\psi(B)$, and
(2) $\beta_{\xi'}^{AM}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon$ for all $\xi' \in 3N_0 D(\gamma\xi_0, r)$.

We claim that

$$\{\xi \in B \cap \gamma\gamma_0\gamma^{-1}B : \beta_\xi^{AM}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon\} \tag{7.11}$$

has a positive ν_ψ-measure, which will finish the proof.

We have $\gamma\gamma_0\gamma^{-1}D \subset D$ by [16, Proof of Prop. 10.7]. Together with (2) above, it follows that

$$\beta_\xi^{AM}(e, \gamma\gamma_0\gamma^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_\varepsilon \quad \text{for all } \xi \in \gamma\gamma_0\gamma^{-1}D.$$

Consequently, (7.11) contains

$$(D \cap B) \cap \gamma\gamma_0\gamma^{-1}(D \cap B), \tag{7.12}$$

which has a positive ν_ψ-measure by [16, Proof of Prop. 10.7]. This proves the claim. □

Remark 7.13. We remark that the approach of this paper shows the following result when G has rank one.

Theorem 7.14. *Let G have rank one, and $\Gamma < G$ be a Zariski dense discrete subgroup. Let ν_o be an ergodic Γ-conformal probability measure on the limit set of Γ. Let m^{BMS} and m^{BR} be respectively the* BMS *and* BR *measures on $\Gamma\backslash G$ associated to ν_o. Suppose that m^{BMS} is AM-ergodic. If $G \not\simeq \mathrm{SL}_2(\mathbb{R})$, then m^{BMS} is A-ergodic and m^{BR} is N-ergodic. If $G \simeq \mathrm{SL}_2(\mathbb{R})$, then m^{BMS} (resp. m^{BR}) is the sum of at most two A-ergodic (resp. N-ergodic) components.*

In the rank one case, all the properties that we had to establish for Anosov groups hold automatically from the negative curvature property of the associated symmetric space. As Γ is Zariski dense, Theorem 4.4 proves that m^{BMS} is the sum of at most $[M : M^\circ]$ number of A-ergodic components. Then the Hopf ratio ergodic theorem for the one-parameter subgroup A implies that ν_o gives full measure on the set of strong Myrberg limit points of Γ, i.e., Corollary 4.11 holds. Now the arguments in section 7 shows that the set of ν_o-essential values is equal to AM, and hence m^{BR} is the sum of at most $[M : M^\circ]$ number of N-ergodic components. When $G \not\simeq \mathrm{SL}_2(\mathbb{R})$, M is connected [27, Lem. 2.4] and for $G \simeq \mathrm{SL}_2(\mathbb{R})$, $M = \{\pm e\}$. Hence Theorem 7.14 follows.

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