ERGODIC DECOMPOSITIONS OF GEOMETRIC MEASURES ON ANOSOV HOMOGENEOUS SPACES.

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ABSTRACT. Let G be a connected semisimple real algebraic group and Γ a Zariski dense Anosov subgroup of G. Let N be a maximal horospherical subgroup of G and P the normalizer of N with a fixed Langlands decomposition $P = MAN$. We prove that for any non-trivial NM-invariant ergodic and P-quasi invariant measure μ on $\Gamma\backslash G$, $\mu = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \mu|_{\mathcal{E}_0}$ describes the N-ergodic decomposition, where $\mathfrak{Y}_\Gamma$ denotes the collection of all P°-minimal subsets of $\Gamma\backslash G$. As a consequence, we deduce that the space of all non-trivial N-invariant ergodic and P°-quasi-invariant Radon measures on $\Gamma\backslash G$, up to positive constant multiples, is homeomorphic to $\mathbb{R}^{\operatorname{rank} G - 1} \times \{1, \cdots, \#\mathfrak{Y}_\Gamma\}$, refining the main result of [14].

CONTENTS

1. INTRODUCTION

Let G be a connected semisimple real algebraic group, i.e., the identity component of the group of real points of a semisimple algebraic group defined over $\mathbb{R}$. We let $\Gamma < G$ be a Zariski dense Anosov subgroup of G. Let $N < G$ be a maximal horospherical subgroup of G, which is unique up to conjugation. Let P denote the normalizer of N, which is a minimal parabolic subgroup of G. Fix a Langlands decomposition $P = MAN$ where A is a maximal real split torus of G and M is a maximal compact subgroup of P commuting with A. In our earlier paper [14], we showed that all Burger-Roblin measures on $\Gamma\backslash G$, parametrized by $\mathbb{R}^{\operatorname{rank} G-1}$, are NM-ergodic and

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that they form precisely all non-trivial NM-invariant ergodic and P°-quasi-invariant Radon measures on $\Gamma\backslash G$. One cannot replace NM by N in these statements, as Burger-Roblin measures are not N-ergodic in general. The main aim of this paper is to describe N-ergodic decompositions of Burger-Roblin measures as well as to classify all non-trivial N-invariant ergodic and P°-quasi-invariant Radon measures on $\Gamma\backslash G$. When G has rank one, the class of Anosov subgroups of G coincides with that of convex cocompact subgroups. If P is connected in addition, e.g., when G is the group of all orientation preserving isometries of a rank one symmetric space, then there exists a unique non-trivial N-invariant ergodic measure, which is *the* Burger-Roblin measure, proved by Burger, Roblin and Winter ([4], [18], [26]). We also mention that when $\Gamma < G$ is a lattice, classification of ergodic measures for maximal horospherical subgroup action was first obtained by Furstenberg, Veech and Dani ([8], [24], [6]), prior to Ratner's more general measure classification theorem for any connected unipotent subgroups [17].

We begin by recalling the definition of an Anosov subgroup. Let $\mathcal{F} := G/P$ denote the Furstenberg boundary, and $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F}\times\mathcal{F}$. A Zariski dense discrete subgroup $\Gamma < G$ is called an *Anosov subgroup* if it is a finitely generated word hyperbolic group which admits a Γ-equivariant continuous embedding ζ of the Gromov boundary $\partial\Gamma$ into $\mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y$ in $\partial\Gamma$ ([13], [9], [12], [25]). The class of Anosov subgroups include the Zariski dense images of representations in the Hitchin component as well as Zariski dense Schottky subgroups.

Denote by $\mathfrak{a}$ the Lie algebra of A and fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ so that $\log N$ is the sum of positive root subspaces. Fix a maximal compact subgroup K of G so that the Cartan decomposition $G = KA^+K$ holds for $A^+ = \exp \mathfrak{a}^+$.

Let $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ denote the limit cone of Γ, which is the smallest cone containing the Cartan projection of Γ. Let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R}\cup\{-\infty\}$ denote the growth indicator function of Γ (see Def. 4.1). Set

$$D_\Gamma^\star := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma, \psi(v) = \psi_\Gamma(v) \text{ for some } v \in \operatorname{int}\mathcal{L}_\Gamma\}.$$

For each $\psi \in D_\Gamma^\star$, we denote by m_ψ^{BR} and m_ψ^{BMS} the Burger-Roblin measure and the Bowen-Margulis-Sullivan measure on $\Gamma\backslash G$ associated to ψ ((4.2) and (4.4)). The Burger-Roblin measures are all supported on the unique P-minimal subset of $\Gamma\backslash G$:

$$\mathcal{E} := \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

where $\Lambda \subset \mathcal{F}$ denotes the limit set of Γ. In [14], we showed that for Γ Anosov, each m_ψ^{BR} is NM-ergodic and the map $\psi \mapsto m_\psi^{\mathrm{BR}}$ gives a homeomorphism between $D_\Gamma^\star$ and the space of all NM-invariant ergodic and P-quasi invariant measures supported on $\mathcal{E}$. We also showed that all m_ψ^{BMS} are AM-ergodic.

Denote by $\mathfrak{Y}_\Gamma$ the collection of all P°-minimal subsets of $\Gamma\backslash G$. For any $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, we set

$$P_\Gamma := \{p \in P : \mathcal{E}_0 p = \mathcal{E}_0\}.$$

By the work of Guivarc'h and Raugi [10], P_Γ is independent of the choice of $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, and is a co-abelian subgroup of P containing P°. It follows that for any $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, the map $s \mapsto \mathcal{E}_0 s$ defines a bijection between P/P_Γ and $\mathfrak{Y}_\Gamma$.

Noting that $\{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma\}$ gives a partition of $\mathcal{E}$ into $\#\mathfrak{Y}_\Gamma$-number of closed subsets, the following is our main theorem:

Theorem 1.1. *For any Anosov subgroup $\Gamma < G$ and $\psi \in D_\Gamma^\star$,*

(1) $m_\psi^{\mathrm{BR}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ *is an N-ergodic decomposition;*
(2) $m_\psi^{\mathrm{BMS}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$ *is an A-ergodic decomposition.*

In particular, the numbers of ergodic components of m_ψ^{BR} and of m_ψ^{BMS} are given by $\#\mathfrak{Y}_\Gamma$, independent of ψ.

As $P^\circ \subset P_\Gamma$, P_Γ is of the form $M_\Gamma AN$ where

$$M_\Gamma := \{m \in M : \mathcal{E}_0 m = \mathcal{E}_0\}.$$

For any $g \in G$ such that $g\Gamma g^{-1}$ contains a loxodromic element in $(\operatorname{int} A^+)M$,

$$M_\Gamma = \text{Closure of } \{m \in M : g^{-1}hamng \in \Gamma \text{ for some } h \in N^+, a \in A, n \in N\}$$

where N^+ is the opposite horospherical subgroup to N, i.e., the maximal horospherical subgroup such that the intersection of its normalizer and P is equal to AM [3, Prop. 4.9(a)]. The subgroup M_Γ is not equal to M in general: there exists a Zariski dense Schottky subgroup Γ with $M_\Gamma \neq M$ [2], and for an Anosov subgroup Γ which is the image of a Hitchin representation into $\mathrm{PSL}_n(\mathbb{R})$, it is known that $M_\Gamma = \{e\}$ [13].

Since each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$ is a second countable topological space, almost all orbits are dense with respect to an ergodic measure. Hence Theorem 1.1 implies:

Corollary 1.2. *Let $\mathcal{E}_0$ be a P°-minimal subset of $\Gamma\backslash G$. Then*

(1) *for $m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ almost all $x \in \mathcal{E}_0$, xN is dense in $\mathcal{E}_0$;*
(2) *for $m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$ almost all $x \in \mathcal{E}_0$, xA is dense in $\mathcal{E}_0 \cap \operatorname{supp} m_\psi^{\mathrm{BMS}}$.*

Indeed, Corollary 1.2(2) holds for A^+-orbits as well (see Corollary 4.11).

In view of our earlier work [14], Theorem 1.1 implies:

Theorem 1.3. *The space of all N-invariant ergodic and P°-quasi-invariant Radon measures on $\mathcal{E}$, up to positive constant multiples, is homeomorphic to $\mathbb{R}^{\operatorname{rank} G - 1} \times \{1, \cdots, \#M/M_\Gamma\}$.*

Summary of proofs. For each $\psi \in D_\Gamma^\star$, there exists a unique (Γ, ψ)-Patterson-Sullivan measure, say, ν_ψ, on the limit set $\Lambda \subset \mathcal{F}$. Denote by $\tilde{\nu}_\psi$ the M-invariant lift of ν_ψ to G/P°. Using that the closure of the weak-transitivity subgroup of M contains $M \cap P_\Gamma$ (Corollary 3.8), we show that Γ-ergodic components of $\tilde{\nu}_\psi$ and A-ergodic components of m_ψ^{BMS} are respectively given by their restrictions to Γ-minimal subsets of G/P° and to

P°-minimal subsets of $\Gamma\backslash G$ (Theorem 4.4). We define the closed subgroup, say E_{ν_ψ} of AM consisting of ν_ψ-essential values (Def. 6.1), and show that elements of the generalized length spectrum of Γ, whose images under ψ are sufficiently large, are contained in E_{ν_ψ} (Proposition 7.8). This implies that AM° is contained in E_{ν_ψ}, from which we deduce Theorem 1.1(1), using the NM-ergodicty of m_ψ^{BR}.

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2. Preliminaries

Let G be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, set $A := \exp \mathfrak{a}$ and $A^+ := \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M and we set N to be the contracting horospherical subgroup: for $a \in \operatorname{int} A^+$, $N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}$. Note that $\log N$ is the sum of all positive root subspaces for our choice of A^+. Similarly, we also consider the expanding horospherical subgroup N^+: for $a \in \operatorname{int} A^+$, $N^+ := \{g \in G : a^n g a^{-n} \to e \text{ as } n \to +\infty\}$. We set $P = MAN$ which is a minimal parabolic subgroup of G. The quotient $\mathcal{F} = G/P$ is known as the Furstenberg boundary of G and is isomorphic to K/M. We let Λ denote the unique Γ-minimal subset of $\mathcal{F}$, called the limit set of Γ.

We fix an element w_0 of the normalizer of $\mathfrak{a}$ such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The opposition involution $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ is defined as $\mathrm{i}(u) = -\mathrm{Ad}_{w_0} u$.

Definition 2.1 (Visual maps)**.** For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$

For all $g \in G$ and $m \in M$, observe that $g^\pm = (gm)^\pm = g(e^\pm)$. Let $\mathcal{F}^{(2)}$ denote the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$:

$$\mathcal{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}.$$

2.1. A-valued cocycles.

Definition 2.2. The A-valued Iwasawa cocycle $\sigma : G \times \mathcal{F} \to A$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$, $\sigma^A(g, \xi) \in A$ is the unique element satisfying

$$gk \in K\sigma^A(g, \xi)N \tag{2.1}$$

where $k \in K$ is such that $\xi = k^+$.

Definition 2.3. The A-valued Busemann function $\beta^A : \mathcal{F} \times G \times G \to A$ is defined as follows: for $\xi \in \mathcal{F}$ and $g_1, g_2 \in G$,

$$\beta_\xi^A(g_1, g_2) := \sigma^A(g_1^{-1}, \xi)\sigma^A(g_2^{-1}, \xi)^{-1}.$$

2.2. AM**-valued cocycles.** The product map $N^+ \times P \to G$ is a diffeomorphism onto its image which is Zariski open and dense in G. Hence for each $\xi \in N^+e^+$, we can define $h_\xi \in N^+$ to be the unique element such that $\xi = h_\xi e^+$. Similarly, the product map $K \times A \times N \to G$ is a diffeomorphism, giving the Iwasawa decomposition $G = KAN$. We can therefore define $k_\xi \in K$ to be the unique element such that $h_\xi \in k_\xi AN$.

Definition 2.4 (Bruhat cocycle and Iwasawa cocycle)**.** Let $g \in G$ and $\xi \in \mathcal{F}$ be such that $\xi, g\xi \in N^+e^+$.

(1) We define the Bruhat cocycle $b(g,\xi) \in AM$ to be the unique element satisfying

$$gh_\xi \in N^+ b(g,\xi)N.$$

Note that $\xi \in N^+e^+$ allows us to get $h_\xi \in N^+$ and $g\xi \in N^+e^+$ implies $gh_\xi \in N^+AMN$.

(2) We define the Iwasawa cocycle $\sigma^{AM}(g,\xi) \in AM$ to be the unique element satisfying

$$gk_\xi \in k_{g\xi}\sigma^{AM}(g,\xi)N.$$

Note that $gh_\xi \in h_{g\xi} b(g,\xi)N$.

We note that for any $\xi \in \mathcal{F}$, there exists a unique element $\sigma(g,\xi) \in A$ such that $gk_\xi \in K\sigma(g,\xi)N$ where $\xi = [k_\xi] \in K/M = \mathcal{F}$. The logarithm of $\sigma(g,\xi)$ was defined as the Iwasawa cocycle in [14]. In order to define the AM-valued Iwasawa cocycle, it is necessary to choose a section of the projection $K \simeq G/AN \to K/M \simeq G/P$. In the above definition, we have used a section $\mathsf{s} : G/P \to G/AN$ such that $\mathsf{s}(hP) = hAN$ for all $h \in N^+$, so that it is continuous on $N^+e^+ \subset \mathcal{F}$.

It follows that for each fixed $g \in G$, the maps $\xi \mapsto b(g,\xi)$ and $\xi \mapsto \sigma^{AM}(g,\xi)$ are continuous on the set $\{\xi \in N^+e^+ : g\xi \in N^+e^+\}$.

Definition 2.5 (AM-valued Busemann map)**.** For $(\xi, g_1, g_2) \in \mathcal{F} \times G \times G$ such that $\xi, g_1^{-1}\xi, g_2^{-1}\xi \in N^+e^+$, we define

$$\beta_\xi^{AM}(g_1,g_2) := \sigma^{AM}(g_1^{-1},\xi)\sigma^{AM}(g_2^{-1},\xi)^{-1}.$$

We define β^M to be the projection of β^{AM} to M; we then have $\beta_\xi^{AM}(g_1,g_2) = \beta_\xi^A(g_1,g_2)\beta_\xi^M(g_1,g_2)$. For simplicity, we sometimes drop the superscript AM from β^{AM} when its meaning is clear from the context.

Example 2.6. If $g = hamn \in N^+AMN$, then $\beta_{g^+}^M(e,g) = m$.

For fixed $g_1, g_2 \in G$, the map $\xi \mapsto \beta_\xi(g_1,g_2)$ is continuous on $\{\xi \in N^+e^+ : g_1^{-1}\xi, g_2^{-1}\xi \in N^+e^+\}$. We have the following whenever both sides are defined: for any $g_1, g_2, g_3 \in G$ and $\xi \in \mathcal{F}$,

(1) (cocycle identity) $\beta_\xi(g_1,g_3) = \beta_\xi(g_1,g_2)\,\beta_\xi(g_2,g_3)$;
(2) (equivariance) $\beta_{g_3\xi}(g_3g_1, g_3g_2) = \beta_\xi(g_1,g_2)$.

2.3. **Jordan projection and Cartan projection.** Recall that for any loxodromic element $g \in G$, there exists $\varphi \in G$ such that

$$g = \varphi a m \varphi^{-1}$$

for some element $am \in \operatorname{int} A^+ M$. Moreover such φ belongs to a unique coset in G/AM. We set

$$y_g^+ := \varphi^+ \in \mathcal{F}$$

which is called the attracting fixed point of g. The element $a \in \operatorname{int} A^+$ is uniquely determined and called the Jordan projection of g. We denote it by $\lambda(g)$.

The limit cone $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ is defined as the smallest closed cone containing all $\lambda(\gamma)$, $\gamma \in \Gamma$, which is known to be a convex cone with non-empty interior [1].

Definition 2.7 (Cartan projection)**.** For each $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, such that

$$g \in K \exp(\mu(g)) K.$$

The limit cone $\mathcal{L}_\Gamma$ coincides with the smallest closed cone containing all $\mu(\gamma)$, $\gamma \in \Gamma$.

3. Generalized length spectrum and transitivity groups

In this section, we fix a discrete Zariski dense subgroup Γ of G.

3.1. **P°-minimal subsets of $\Gamma\backslash G$.** Note that the limit set Λ of Γ is the unique Γ-minimal subset of $\mathcal{F}$. It follows that the set

$$\mathcal{E} := \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}$$

is the unique P-minimal subset of $\Gamma\backslash G$.

We refer to [10, Thm. 2 and Thm 1.9] for results in this subsection. Set $\mathcal{F}^\circ = G/P^\circ$. For any $g \in G$ with $g^+ \in \Lambda$, the closure of $\Gamma g P^\circ$ is a Γ-minimal subset of $\mathcal{F}^\circ$. Moreover the following closed subgroup of M is well-defined:

$$M_\Gamma := \{m \in M : \Lambda_0 m = \Lambda_0\} \tag{3.1}$$

for a Γ-minimal subset Λ_0 of $\mathcal{F}^\circ$. The subgroup M_Γ is a co-abelian subgroup of M containing M° and M_Γ/M° is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^p$ for $0 \le p \le \dim A$.

For any Γ-minimal subset Λ_0 of $\mathcal{F}_0$, the map $s \mapsto \Lambda_0 s$ gives a bijection between $M_\Gamma\backslash M$ and the collection $\mathcal{Y}_\Gamma$ of all Γ-minimal subsets of $\mathcal{F}^\circ$. If we set $\tilde\Lambda := \{gP^\circ : gP \in \Lambda\}$, then $\tilde\Lambda = \bigcup_{\Lambda_0 \in \mathcal{Y}_\Gamma} \Lambda_0$.

These results can be translated into statements about P°-minimal subsets of $\Gamma\backslash G$ by the duality. Each $\Lambda_0 \in \mathcal{Y}_\Gamma$ is of the form $E(\Lambda_0)/P^\circ$ for some left Γ-invariant and right P°-invariant closed subset $E(\Lambda_0)$ of G. The map $\Lambda_0 \mapsto \Gamma\backslash E(\Lambda_0)$ gives a bijection between $\mathcal{Y}_\Gamma$ and the collection of all P°-minimal subsets of $\Gamma\backslash G$, say $\mathfrak{Y}_\Gamma$. Moreover, if we set

$$P_\Gamma := M_\Gamma A N, \tag{3.2}$$

then $P_\Gamma = \{p \in P : \mathcal{E}_0 p = \mathcal{E}_0\}$ for all $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$. We also have

$$\mathcal{E} = \bigcup_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \mathcal{E}_0.$$

We remark that a P°-minimal subset is in fact an AN-minimal subset; this follows from [10, Thm.2].

3.2. **Generalized length spectrum.** We define

$$\Gamma^\star := \{\gamma \in \Gamma : \text{there exists } \varphi \in N^+N \text{ with } \gamma \in \varphi(\operatorname{int} A^+)M\varphi^{-1}\}.$$

Note that if $\gamma \in \Gamma$ and $y_\gamma^+ \in N^+e^+$, then $\gamma \in \Gamma^\star$. As Γ is Zariski dense, the set of loxodromic elements is Zariski dense in G [1]. It follows that $\Gamma^\star$ is Zariski dense in G as well. For $\gamma \in \Gamma^\star$, we define its *generalized Jordan component* $\hat{\lambda}(\gamma)$ to be the unique element of $\operatorname{int} A^+M$ such that

$$\gamma = \varphi\hat{\lambda}(\gamma)\varphi^{-1} \quad \text{for some } \varphi \in N^+N.$$

Definition 3.1. We call the following set the *generalized length spectrum* of Γ:

$$\hat{\lambda}(\Gamma) := \{\hat{\lambda}(\gamma) \in AM : \gamma \in \Gamma^\star\}.$$

We denote by $\mathsf{s}(\Gamma)$ the closed subgroup of AM generated by $\hat{\lambda}(\Gamma)$.

Lemma 3.2. *For all $\gamma \in \Gamma^\star$, we have*

$$\hat{\lambda}(\gamma) = b(\gamma, y_\gamma^+) = \beta^{AM}_{y_\gamma^+}(e,\gamma).$$

Proof. Since $\gamma \in \Gamma^\star$, we have $\gamma = \varphi\hat{\lambda}(\gamma)\varphi^{-1}$ for some $\varphi = hn$, where $h \in N^+$ and $n \in N$. Set $\xi := y_\gamma^+ = \varphi^+$. In particular, $h_\xi = h$ and $h \in k_\xi AN$. The defining relations for $b(\gamma,\xi)$ and $\beta^{AM}_\xi(e,\gamma)$ are

$$\gamma h \in hb(\gamma,\xi)N \text{ and } \gamma k_\xi \in k_\xi \beta_\xi(e,\gamma)N.$$

Now observe that

$$\begin{aligned}\gamma h &= \varphi\hat{\lambda}(\gamma)\varphi^{-1}h = hn\hat{\lambda}(\gamma)n^{-1} \in h\hat{\lambda}(\gamma)N \text{ and}\\ \gamma k_\xi &= \varphi\hat{\lambda}(\gamma)\varphi^{-1}k_\xi = k_\xi(k_\xi^{-1}h)n\hat{\lambda}(\gamma)n^{-1}(h^{-1}k_\xi) \in k_\xi\hat{\lambda}(\gamma)N.\end{aligned}$$

Therefore $\hat{\lambda}(\gamma) = b(\gamma,\xi) = \beta^{AM}_\xi(e,\gamma)$. ☐

For each $\xi \in \Lambda \cap N^+e^+$, we define $b_\xi(\Gamma)$ to be the closed subgroup of AM generated by all $b(\gamma,\xi)$ where $\gamma \in \Gamma$ and $\gamma\xi \in N^+e^+$.

Lemma 3.3. *The subgroup $b_\xi(\Gamma) < AM$ is independent of $\xi \in \Lambda \cap N^+e^+$.*

Proof. Let $\xi_1, \xi_2 \in \Lambda \cap N^+e^+$. To show that $b_{\xi_1}(\Gamma) = b_{\xi_2}(\Gamma)$, it suffices to check that $b(\gamma,\xi_2) \in b_{\xi_1}(\Gamma)$ for any $\gamma \in \Gamma$ such that $\gamma\xi_2 \in N^+e^+$. Since Λ is Γ-minimal, there exists a sequence $\gamma_n \in \Gamma$ such that $\lim_{n\to\infty}\gamma_n\xi_1 = \xi_2$. Since N^+e^+ is open and $\xi_2, \gamma\xi_2 \in N^+e^+$, , we have $\gamma_n\xi_1, \gamma\gamma_n\xi_1 \in N^+e^+$ for all large n and $b(\gamma\gamma_n,\xi_1) = b(\gamma,\gamma_n\xi_1)b(\gamma_n,\xi_1)$. Hence

$$b(\gamma,\xi_2) = \lim_{n\to\infty} b(\gamma,\gamma_n\xi_1) = \lim_{n\to\infty} b(\gamma\gamma_n,\xi_1)b(\gamma_n,\xi_1)^{-1} \in b_{\xi_1}(\Gamma),$$

from which the lemma follows. ☐

By Lemma 3.3, we may define

$$b(\Gamma) := b_\xi(\Gamma) \quad \text{for any } \xi \in \Lambda \cap N^+e^+.$$

In the rest of this section, we assume that Γ contains a loxodromic element in $\operatorname{int} A^+M$.

Lemma 3.4. *We have $b(\Gamma) = \mathsf{s}(\Gamma)$.*

Proof. We first claim that $b(\Gamma) \subset \mathsf{s}(\Gamma)$. By Lemma 3.3, it suffices to show that $b(\gamma, e^+) \in \mathsf{s}(\Gamma)$ for any $\gamma \in \Gamma$ with $\gamma e^+ \in N^+e^+$. Set $s_0 := a_0 m_0 \in \Gamma \cap \operatorname{int} A^+M - M$. Then for all sufficiently large n, $s_0^n\gamma$ is a loxodromic element and $x_n := y^+_{s_0^n\gamma}$ converges to e^+ as $n \to \infty$. Since $y^+_{s_0^n\gamma} \in N^+e^+$, we have $s_0^n\gamma \in \Gamma^\star$ for all large n. Now the claim follows from

$$\begin{aligned} b(\gamma, e^+) &= \lim_{n\to\infty} b(\gamma, x_n) = \lim_{n\to\infty} b(s_0^n, \gamma x_n)^{-1} b(s_0^n\gamma, x_n) \\ &= \lim_{n\to\infty} \hat\lambda(s_0^n)^{-1}\hat\lambda(s_0^n\gamma) \in \mathsf{s}(\Gamma) \end{aligned}$$

We next claim $\mathsf{s}(\Gamma) \subset b(\Gamma)$. Let $\gamma \in \Gamma^\star$ be arbitrary. Note that $y_\gamma^+ \in N^+e^+$. By Lemma 3.2, $\hat\lambda(\gamma) = b(\gamma, y_\gamma^+) \in b_{y_\gamma^+}(\Gamma)$. Since $b(\Gamma) = b_{y_\gamma^+}(\Gamma)$ by Lemma 3.3, we have $\hat\lambda(\gamma) \in b(\Gamma)$, proving the claim. $\square$

Proposition 3.5. *We have*
1. *$b(\Gamma) = b(g^{-1}\Gamma g)$ for all $g \in G$ with $g^\pm \in \Lambda$;*
2. *$b(\Gamma)$ is a co-abelian subgroup of AM containing AM°;*
3. *$b(\Gamma) = AM_\Gamma$.*

Proof. Claims (1) and (2) are proved in [10, Thm 1.9]. Claim (3) follows since $A \subset b(\Gamma)$ by (2) and the closure of $\{m \in M : \Gamma \cap N^+AmN \neq \emptyset\}$ is equal to M_Γ [3, Prop. 4.9(a)]. $\square$

Hence we deduce the following from Lemma 3.4 and Proposition 3.5.

Corollary 3.6. *We have*

$$\mathsf{s}(\Gamma) = AM_\Gamma.$$

3.3. **Transitivity groups.**

Definition 3.7 (Transitivity group)**.** For $g \in G$ with $g^\pm \in \Lambda$, define the subset $\mathcal{H}^s_\Gamma(g) < AM$ as follows: $am \in \mathcal{H}^s_\Gamma(g)$ if and only if there exist $\gamma \in \Gamma$ and a sequence $h_i \in N \cup N^+$, $i = 1, \dots, k$ such that

$$(gh_1h_2\dots h_r)^\pm \in \Lambda \text{ for all } 1 \le r \le k \quad \text{and} \quad \gamma g h_1 h_2 \dots h_k = gam.$$

It is not hard to check that $\mathcal{H}^s_\Gamma(g)$ is a subgroup (cf. [26, Lem. 3.1]); it is called the strong transitivity subgroup. We set $\mathcal{H}^w_\Gamma(g)$ to be the projection of $\mathcal{H}^s_\Gamma(g)$ to M and call it the weak transitivity subgroup.

The notion of transitivity groups was used in [26] in which the following corollary was proved for rank one case by a different approach.

Corollary 3.8. *For any $g \in G$ with $g^{\pm} \in \Lambda$, the closure of $\mathcal{H}^s_\Gamma(g)$ contains AM_Γ. In particular, $M_\Gamma \subset \overline{\mathcal{H}^w_\Gamma(g)}$.*

Proof. By Proposition 3.5, it suffices to show that if $g \in G$ satisfies $g^{\pm} \in \Lambda$, then $b(g^{-1}\Gamma g)$ is contained in the closure of $\mathcal{H}^s_\Gamma(g)$. By Lemma 3.3, it is again enough to show that, fixing $\xi \in g^{-1}\Lambda \cap N^+e^+$, $b(g^{-1}\gamma g, \xi)$ is contained in $\mathcal{H}^s_\Gamma(g)$ for any $\gamma \in \Gamma$. If $\xi = he^+$ for $h \in N^+$ and $am = b(g^{-1}\gamma g, \xi)$, then $g^{-1}\gamma gh = h_1 amn_1$ for some $h_1 \in N^+$ and $n_1 \in N$. We can rewrite it as $gam = \gamma ghn_2h_2$ where $n_2 \in N$ and $h_2 \in N^+$. Observe that $(\gamma gh)^{-1} = \gamma g^- \in \Lambda$ and $(\gamma gh)^+ = \gamma g\xi \in \Lambda$ as $\xi \in g^{-1}\Lambda$. Moreover, $(\gamma ghn_2)^- = (\gamma ghn_2h_2)^- = g^- \in \Lambda$ and $(\gamma ghn_2)^+ = (\gamma gh)^+ \in \Lambda$. Therefore $am \in \mathcal{H}^s_\Gamma(g)$. This proves the claim. □

4. A-ergodic decompositions of BMS-measures

Let $\Gamma < G$ be a Zariski dense discrete subgroup of G.

Definition 4.1 (Growth indicator function)**.** The growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(tu) = t\psi_\Gamma(u)$, such that for any unit vector $u \in \mathfrak{a}^+$,

$$\psi_\Gamma(u) := \inf_{u \in \mathcal{C}, \text{open cones } \mathcal{C} \subset \mathfrak{a}^+} \limsup_{t \to \infty} \tfrac{1}{t} \log \#\{\gamma \in \Gamma \,:\, \mu(\gamma) \in \mathcal{C}\,, \|\mu(\gamma)\| \le t\}.$$

We consider ψ_Γ as a function on $\mathfrak{a}$ by setting $\psi_\Gamma = -\infty$ outside of $\mathfrak{a}^+$.

For a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on Λ is called a (Γ, ψ)-PS measure if for all $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\log \beta^A_\xi(e,\gamma))}. \tag{4.1}$$

Set

$$D^\star_\Gamma := \{\psi \in \mathfrak{a}^* : \psi \ge \psi_\Gamma, \psi(u) = \psi_\Gamma(u) \text{ for some } u \in \operatorname{int} \mathcal{L}_\Gamma\}.$$

For each linear form $\psi \in D^\star_\Gamma$, Quint constructed a (Γ, ψ)-Patterson-Sullivan measure, say, ν_ψ [16, Thm. 4.10]. For an Anosov group Γ, it was shown in [14, Thm 4.3] that the map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D^\star_\Gamma$ and the space of all Γ-PS measures.

4.1. Antipodality of Γ.

When Γ is Anosov, we have the following so-called anti-podal property:

$$\{(\xi, \eta) \in \Lambda \times \Lambda : \xi \ne \eta\} \subset \mathcal{F}^{(2)}.$$

Lemma 4.2. *Let Γ be Anosov. If $g \in G$ satisfies $g^- \in \Lambda$, then $g^{-1}\Lambda \subset N^+e^+ \cup \{e^-\}$.*

Proof. Suppose that $\xi \in \Lambda$ and $g^{-1}\xi \ne e^-$. Then $\xi \ne g^-$ in Λ. Hence $(\xi, g^-) \in \mathcal{F}^{(2)}$, or equivalently, $(g^{-1}\xi, e^-) \in \mathcal{F}^{(2)}$. Since $\{\eta \in \mathcal{F} : (\eta, e^-) \in \mathcal{F}^{(2)}\} \subset N^+e^+$, $g^{-1}\xi \in N^+e^+$, proving the claim. □

Corollary 4.3. *Let $\psi \in D^\star_\Gamma$. For any $g \in G$ with $g^{\pm} \in \Lambda$,*

$$\nu_\psi(\Lambda \cap gN^+e^+) = 1.$$

Proof. By Lemma 4.2, $\Lambda - \{g^-\} = \Lambda \cap gN^+e^+$. Hence the claim follows from the fact that ν_ψ is atom-free [14, Lem. 7.7]. $\square$

In the rest of this section, we may assume that $\Gamma < G$ is an Anosov subgroup. Without loss of generality, we assume that Γ contains a loxodromic element in $\operatorname{int} A^+M$. This in particular means that $\nu_\psi(\Lambda \cap N^+e^+) = 1$ for any $\psi \in D_\Gamma^\star$.

4.2. **Hopf parametrization.** The map $\mathrm{i}(gM) = (g^+, g^-, \beta^A_{g^+}(e,g))$ gives a G-equivariant homeomorphism between G/M and $\mathcal{F}^{(2)} \times A$, where the G-action on the latter is given by

$$g.(\xi, \eta, a) = (g\xi, g\eta, \beta^A_{g\xi}(e,g)a).$$

For the principal M-bundle $G \to G/M$, we fix a Borel section $\mathsf{s} : G/M \to G$ so that $\mathsf{s}(hanM) = han$ for all $han \in N^+AN$. Now for any $g \in G$, there exists a unique $m_g \in M$ such that $g = \mathsf{s}(gM)m_g$. Then the map $\mathrm{j}(g) = (\mathrm{i}(gM), m_g)$ gives a G-equivariant Borel isomorphism of G with $\mathcal{F}^{(2)} \times AM$ where the G action on the latter is given by $g.(\xi, \eta, am) = (g\xi, g\eta, \beta^{AM}_{g\xi}(e,g)am)$ whenever $\xi, g\xi \in N^+e^+$. The restriction of j to N^+P is a homeomorphism onto its image:

$$\mathrm{j}(g) = (g^+, g^-, \beta^{AM}_{g^+}(e,g)).$$

We call this map the Hopf parametrization of G (relative to the choice of s).

We fix $\psi \in D_\Gamma^\star$ in the rest of this section. In terms of this Hopf parametrization of G, the following defines a left Γ-invariant and right AM-invariant measure on G:

$$d\tilde{m}^{\mathrm{BMS}}_\psi(g) = e^{\psi(\log \beta^A_{g^+}(e,g) + \mathrm{i} \log \beta^A_{g^-}(e,g))}\, d\nu_\psi(g^+) d\nu_{\psi\circ \mathrm{i}}(g^-)\, da\, dm. \tag{4.2}$$

We denote by m^{BMS}_ψ the measure on $\Gamma\backslash G$ induced by $\tilde{m}^{\mathrm{BMS}}_\psi$; we call this the Bowen-Margulis-Sullivan measure (associated to ψ). Note that its support is equal to

$$\Omega := \{x \in \Gamma\backslash G : x^\pm \in \Lambda\}. \tag{4.3}$$

In [14], we showed that m^{BMS}_ψ is an AM-ergodic measure and that it is infinite whenever $\operatorname{rank} G \geq 2$.

Similarly, the Burger-Roblin measure m^{BR}_ψ on $\Gamma\backslash G$ is induced from the following left Γ-invariant and right NM-invariant measure on G:

$$d\tilde{m}^{\mathrm{BR}}_\psi(g) = e^{\psi(\log \beta^A_{g^+}(e,g)) + 2\rho(\log \beta^A_{g^-}(e,g))}\, d\nu_\psi(g^+) dm_o(g^-)\, da\, dm. \tag{4.4}$$

where ρ denotes the half sum of all positive roots with respect to $\mathfrak{a}^+$ and m_o denotes the K-invariant probability measure on G/P. Note that the support m^{BR}_ψ is equal to $\mathcal{E}$.

By Corollary 4.3,

$$\tilde{m}^{\mathrm{BMS}}_\psi(G - N^+P) = 0 = \tilde{m}^{\mathrm{BR}}_\psi(G - N^+P).$$

4.3. **Ergodic decomposition of m_ψ^{BMS}.** Recall the notation $\mathcal{Y}_\Gamma$ and $\mathfrak{Y}_\Gamma$ from subsection 3.1. In particular, $\tilde{\Lambda} = \bigcup_{\Lambda_0 \in \mathcal{Y}_\Gamma} \Lambda_0$ and $\mathcal{E} = \bigcup_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \mathcal{E}_0$. We denote by $\tilde{\nu}_\psi$ the M/M°-invariant lift of ν_ψ to $\tilde{\Lambda} \subset \mathcal{F}^\circ$, i.e., for $f \in C(\mathcal{F}^\circ)$,

$$\tilde{\nu}_\psi(f) := \nu_\psi(\sum_{m \in M/M^\circ} m.f) = \nu_\psi(\int_{m \in M} m.f \, dm)$$

where $m.f(x) = f(xm)$.

Theorem 4.4. *Let $\Gamma < G$ be an Anosov subgroup.*

(1) *The restriction $\tilde{\nu}_\psi$ to a Γ-minimal subset of $\mathcal{F}^\circ$ is Γ-ergodic. In particular, $\tilde{\nu}_\psi = \sum_{\Lambda_0 \in \mathcal{Y}_\Gamma} \tilde{\nu}_\psi|_{\Lambda_0}$ is a Γ-ergodic decomposition.*

(2) *The restriction of m_ψ^{BMS} to a P°-minimal subset of $\Gamma \backslash G$ is A-ergodic.*

In particular,

$$m_\psi^{\mathrm{BMS}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$$

is an A-ergodic decomposition.

The rest of this section is devoted to the proof of this theorem. Set

$$\tilde{\Omega} := \{g \in G : \Gamma g \in \Omega\} = \{g \in G : g^{\pm} \in \Lambda\}.$$

Let $\mathcal{B}$ denote the Borel σ-algebra on G. We set

$$\Sigma_\pm := \{B \cap \tilde{\Omega} : B \in \mathcal{B} \text{ with } B = \Gamma B A N^{\pm}\}.$$

We also define Σ to be the collection of all $B \in \mathcal{B}$ such that $m_\psi^{\mathrm{BMS}}(B \triangle B_\pm) = 0$ for some $B_\pm \in \Sigma_\pm$. Recall the subgroup $M_\Gamma < M$ given in (3.1), and define

$$\Sigma_0 := \{B \cap \tilde{\Omega} : B \in \mathcal{B} \text{ with } B = \Gamma B A M_\Gamma\}.$$

The following is a main technical ingredient of the proof of Theorem 4.4:

Lemma 4.5. *We have $\Sigma \subset \Sigma_0$ mod m_ψ^{BMS}; that is, for all $B \in \Sigma$, there exists $B_0 \in \Sigma_0$ such that $m_\psi^{\mathrm{BMS}}(B \triangle B_0) = 0$.*

This lemma follows if we show that any bounded Σ-measurable function on $\tilde{\Omega}$ is Σ_0-measurable modulo m_ψ^{BMS}.

Let f be any bounded Σ-measurable function on $\tilde{\Omega}$. We may assume without loss of generality that f is strictly left Γ-invariant and right A-invariant [27, Prop. B.5]. There exist bounded $\Sigma^{\pm}$-measurable functions $f_\pm$ such that $f = f_\pm$ for m_ψ^{BMS}-a.e. We may assume that $f_\pm$ satisfy $f_\pm(gn) = f_\pm(g)$ whenever $g, gn \in \tilde{\Omega}$ with $n \in N^{\pm}$. Set

$$E := \left\{ gAM : \begin{array}{c} f|_{gAM} \text{ is measurable and} \\ f(gm) = f_+(gm) = f_-(gm) \\ \text{for Haar a.e. } m \in M \end{array} \right\} \subset \tilde{\Omega}/AM.$$

By Fubini's theorem, E has a full measure on $\tilde{\Omega}/AM \simeq \Lambda^{(2)}$ with respect to the measure $d\nu_\psi\, d\nu_{\psi\circ \mathrm{i}}$. For all small $\varepsilon > 0$, define functions $f^\varepsilon, f^\varepsilon_\pm : \tilde{\Omega} \to \mathbb{R}$ by

$$f^\varepsilon(g) := \tfrac{1}{\mathrm{Vol}(M_\varepsilon)} \int_{M_\varepsilon} f(gm)\, dm \text{ and } f^\varepsilon_\pm(g) := \tfrac{1}{\mathrm{Vol}(M_\varepsilon)} \int_{M_\varepsilon} f_\pm(gm)\, dm$$

where M_ε denotes the ε-ball around e in M. Note that if $gAM \in E$, then f^ε and $f^\varepsilon_\pm$ are continuous and identical on gAM. Moreover, as M normalizes subgroups A and $N^\pm$, f^ε is strictly left Γ-invariant, right A-invariant and $f^\varepsilon_\pm(gn) = f^\varepsilon_\pm(g)$ whenever $g, gn \in \tilde{\Omega}$ with $n \in N^\pm$. Using the isomorphism between $\tilde{\Omega}/AM$ and $\Lambda^{(2)}$ given by $gAM \mapsto (g^+, g^-)$, we may consider E as a subset of $\Lambda^{(2)}$. We then define

$$\begin{aligned} E^- &: = \{\xi \in \Lambda : (\xi, \eta') \in E \quad \text{for } \nu_{\psi\circ \mathrm{i}}\text{-a.e. } \eta' \in \Lambda\}; \\ E^+ &: = \{\eta \in \Lambda : (\xi', \eta) \in E \quad \text{for } \nu_{\psi}\text{-a.e. } \xi' \in \Lambda\}. \end{aligned}$$

Then E^- is ν_ψ-conull and E^+ is $\nu_{\psi\circ \mathrm{i}}$-conull by Fubini's theorem. Set

$$E^-_\eta := \{\xi \in \Lambda : (\xi, \eta) \subset E\} \quad \text{ and } \quad E^+_\xi := \{\eta \in \Lambda : (\xi, \eta) \subset E\}.$$

Note that E^-_η is ν_ψ-conull for all $\eta \in E^+$ and that E^+_ξ is $\nu_{\psi\circ \mathrm{i}}$-conull for all $\xi \in E^-$.

Lemma 4.6. *Let $g \in \tilde{\Omega}$ be such that $gAM \in E$ and $g^\pm \in E^\pm$. Then for any $\varepsilon > 0$, $f^\varepsilon(gm_0) = f^\varepsilon(g)$ for all $m_0 \in \mathcal{H}^w_\Gamma(g)$. Moreover, $f^\varepsilon|_{gAM}$ is M_Γ-invariant.*

Proof. We apply similar arguments as in [26, Lem. 4.1]. For any $m \in \mathcal{H}^w_\Gamma(g)$, there exists $\gamma \in G$, a sequence $h_1, \cdots, h_k \in N \cup N^+$, and $a \in A$ such that

$$gh_1 \dots h_i \in \tilde{\Omega} \text{ for all } 1 \le i \le k \quad \text{and} \quad gh_1 \dots h_k = \gamma gam.$$

If $gh_1 \cdots h_i AM \in E$ for all $1 \le i \le k$ in addition, we call such a sequence permissible. In this case,

$$f^\varepsilon(gm) = f^\varepsilon(\gamma gam) = f^\varepsilon(gh_1 \cdots h_r) = f^\varepsilon(gh_1 \cdots h_{r-1}) = \cdots = f^\varepsilon(g),$$

using the $N^\pm$-invariance of $f^\varepsilon_\pm$, the invariance of f by Γ and A and the fact that all three agree on E. In general, we need an approximation of the sequence by permissible ones.

Let $m \in \mathcal{H}^w_\Gamma(g)$ be arbitrary. Let $n_i \in N$, $h_i \in N^+$, $1 \le i \le k$, $a \in A$ and $\gamma \in \Gamma$ be such that for each $1 \le i \le k$,

$$\xi_i := gn_1h_1 \cdots n_i^- \in \Lambda, \eta_i := gn_1h_1 \cdots n_ih_i^+ \in \Lambda, gn_1h_1 \cdots n_kh_k = \gamma gam.$$

We also set $\xi_0 := g^-$ and $\eta_0 := g^+$. For $0 \le i \le k$, we now define sequences $\{\xi^\ell_i : \ell \in \mathbb{N}\}$ and $\{\eta^\ell_i : \ell \in \mathbb{N}\}$. Set $\xi^\ell_0 := \xi_0$ and $\eta^\ell_0 := \eta_0$ for all $\ell \in \mathbb{N}$. Next, choose a sequence $\xi^\ell_1 \in E^- \cap E^-_{\eta_0}$ such that $\xi^\ell_1 \to \xi_1$ as $\ell \to \infty$. This is possible because $E^- \cap E^-_{\eta_0}$ is ν_ψ-conull from the hypothesis $\eta_0 = g^+ \in E^+$ and hence dense in Λ. Let $n^\ell_1 \in N$ be the unique element such that $\xi^\ell_1 = (gn^\ell_1)^-$. Note that for all $\ell \in \mathbb{N}$,

(1) $(gn_1^\ell)^- = \xi_1^\ell \in E^-$,
(2) $(gn_1^\ell)^+ = \eta_0^\ell \in E^+$,
(3) $gn_1^\ell AM \in E$, and
(4) $n_1^\ell \to n_1$ as $\ell \to \infty$.

Next, choose $\eta_1^\ell \in E^+ \cap E^+_{\xi_1^\ell}$ such that $\eta_1^\ell \to \eta_1$ as $\ell \to \infty$. Again, this is possible because $E^+ \cap E^+_{\xi_1^\ell}$ is $\nu_{\psi\circ\mathrm{i}}$-conull. Note that $\eta_1^\ell = (gn_1^\ell h_1^\ell)^+$ for some unique $h_1^\ell \in N^+$. We have for all $\ell \in \mathbb{N}$,

(1) $(gn_1^\ell h_1^\ell)^- = \xi_1^\ell \in E^-$,
(2) $(gn_1^\ell h_1^\ell)^+ = \eta_1^\ell \in E^+$,
(3) $gn_1^\ell h_1^\ell AM \in E$, and
(4) $h_1^\ell \to h_1$ as $\ell \to \infty$.

Continuing in this fashion, we can find sequences $\xi_i^\ell \in E^-, n_i^\ell \in N, \eta_i^\ell \in E^+, h_i^\ell \in N^+ (1 \le i \le k)$ such that the following holds: for all $\ell \in \mathbb{N}$,

(1) $\xi_i^\ell = (gn_1^\ell h_1^\ell \cdots n_i^\ell h_i^\ell)^-$, $\eta_i^\ell = (gn_1^\ell h_1^\ell \cdots n_i^\ell h_i^\ell)^+$,
(2) $\xi_{i+1}^\ell = (gn_1^\ell h_1^\ell \cdots n_i^\ell h_i^\ell n_{i+1}^\ell)^-$, $\eta_i^\ell = (gn_1^\ell h_1^\ell \cdots n_i^\ell h_i^\ell n_{i+1}^\ell)^+$,
(3) $gn_1^\ell h_1^\ell \cdots n_i^\ell AM$, $gn_1^\ell h_1^\ell \cdots n_i^\ell h_i^\ell AM \in E$, and
(4) $n_i^\ell \to n_i$ and $h_i^\ell \to h_i$ as $\ell \to \infty$.

For $i = k$, we could have chosen $\xi_k^\ell = \gamma g^-, \eta_k^\ell = \gamma g^+$ for all ℓ in the above. This implies that $gn_1^\ell h_1^\ell \cdots n_k^\ell h_k^\ell = \gamma g a_\ell m_\ell$ for some $a_\ell \in A$ and $m_\ell \in M$. Note that $a_\ell m_\ell \to am$ as $\ell \to \infty$ and hence $m_\ell \in \mathcal{H}^w_\Gamma(g)$ with permissible sequences $n_1^\ell, h_1^\ell, \cdots, n_k^\ell, h_k^\ell \in N \cup N^+$. Therefore, $f^\varepsilon(gm_\ell) = f^\varepsilon(g)$ by the previous observation. Since $gAM \in E$, f^ε is continuous on gAM and hence

$$f^\varepsilon(gm) = \lim_{\ell\to\infty} f^\varepsilon(gm_\ell) = \lim_{\ell\to\infty} f^\varepsilon(g) = f^\varepsilon(g).$$

This finishes the proof of the first claim.

For the second clam, let $am \in AM$ and $m_0 \in \mathcal{H}^w_\Gamma(gm)$. Since f^ε is A-invariant, it follows from the first part that $f^\varepsilon(gamm_0) = f^\varepsilon(gmm_0) = f^\varepsilon(gm) = f^\varepsilon(gam)$. Since $f^\varepsilon|_{gAM}$ is continuous and $\overline{\mathcal{H}^w_\Gamma(gm)}$ contains M_Γ by Corollary 3.8, the second claim follows. $\square$

Proof of Lemma 4.5: Let f be any bounded Σ-measurable function on $\tilde{\Omega}$. For any $\varepsilon > 0$, by Lemma 4.6, f^ε coincides with a Σ_0-measurable function m_ψ^{BMS}-a.e. Since $\lim_{\varepsilon\to 0} f^\varepsilon = f$ m_ψ^{BMS}-a.e., f is a Σ_0-measurable function m_ψ^{BMS}-a.e. as well. This proves the lemma. $\square$

Corollary 4.7. *There exists $B \in \Sigma$ such that any two distinct subsets in $\{B.s : s \in M_\Gamma\backslash M\}$ are measurably disjoint and Σ is a finite σ-algebra generated by $\{B.s : s \in M_\Gamma\backslash M\}$ mod m_ψ^{BMS}.*

Proof. First, note that the AM-ergodicity of m_ψ^{BMS} implies that the σ-algebra

$$\Sigma_1 := \{B \cap \tilde{\Omega} : B \in \mathcal{B} \text{ such that } B = \Gamma BAM\}$$

is trivial mod $m_\psi^{\rm BMS}$. It follows that for any $B\in\Sigma_0$, and hence for any $B\in\Sigma$ by Lemma 4.5, with $m_\psi^{\rm BMS}(B)>0$, the union $\cup_{s\in M_\Gamma\backslash M}B.s$ is $m_\psi^{\rm BMS}$-conull.

Let $\mathcal{P}=\{A_1,\cdots,A_k\}$ be a partition of $\tilde\Omega$ with maximal k, among all partitions of Ω satisfying

1. $A_i\in\Sigma$ and $m_\psi^{\rm BMS}(A_i)>0$,
2. $\tilde\Omega=A_1\cup\cdots\cup A_k$ mod $m_\psi^{\rm BMS}$ and
3. for any $s\in M_\Gamma\backslash M$, we have $A_i.s\in\{A_1,\cdots,A_k\}$ mod $m_\psi^{\rm BMS}$.

Note that $1\le k\le[M:M_\Gamma]$. Setting $B=A_1$, we claim that this proves the corollary. Suppose not. Setting $\sigma(\mathcal{P})$ to be the σ-algebra generated by $\mathcal{P}$, there exists $B'\in\Sigma-\sigma(\mathcal{P})$ mod $m_\psi^{\rm BMS}$. Then for some $1\le j\le k$, $B'\cap A_j$ is neither null nor conull in A_j. Hence by considering $B'.s\cap A_i$ and $B'^c.s\cap A_i$ $s\in M_\Gamma\backslash M$ and $1\le i\le k$, we get a partition finer than $\mathcal{P}$ satisfying the above three conditions. This contradicts the maximality of k. $\square$

4.4. **$\mathbb{R}$-ergodic decomposition of $\hat m_\psi$ on $\Lambda^{(2)}\times\mathbb{R}\times M$.** For $(\xi_1,\xi_2)\in\mathcal{F}^{(2)}$, define

$$[\xi_1,\xi_2]_\psi:=\psi(\log\beta^A_{g^+}(e,g)+\mathrm{i}\log\beta^A_{g^-}(e,g))$$

where $g\in G$ is such that $g^+=\xi_1$ and $g^-=\xi_2$.

Set $\Lambda^{(2)}=\Lambda\times\Lambda\cap\mathcal{F}^{(2)}$. The action of Γ on $\Lambda^{(2)}\times\mathbb{R}$ defined by

$$\gamma.(\xi,\eta,s)=(\gamma\xi,\gamma\eta,t+\psi(\log\beta^A_{\gamma\xi}(e,\gamma)))$$

is proper and cocompact, and the measure $d\tilde m_\psi:=e^{[\cdot,\cdot]_\psi}d\nu_\psi\,d\nu_{\psi\circ\mathrm{i}}\,dt$ on $\Lambda^{(2)}\times\mathbb{R}$ descends to a finite $\mathbb{R}$-ergodic measure m_ψ on $\Gamma\backslash\Lambda^{(2)}\times\mathbb{R}$ ([20, Thm 3.2], [5, Thm A.2]). We denote by $d\hat m_\psi$ the finite measure on

$$Z:=\Gamma\backslash\Lambda^{(2)}\times\mathbb{R}\times M$$

induced by the Γ-invariant product measure $d\tilde m_\psi\,dm$ on $\Lambda^{(2)}\times\mathbb{R}\times M$; here Γ acts on $\Lambda^{(2)}\times\mathbb{R}\times M$ by

$$\gamma.(\xi,\eta,t,m)=(\gamma\xi,\gamma\eta,t+\psi(\log\beta^A_{\gamma\xi}(e,\gamma)),\beta^M_{\gamma\xi}(e,\gamma)m)$$

where $(\xi,\eta)\in\Lambda^{(2)}$, $t\in\mathbb{R}$ and $m\in M$.

Define the Borel map $\Psi:\tilde\Omega\to\Lambda^{(2)}\times\mathbb{R}\times M$ by

$$\Psi(g)=(g^+,g^-,\psi(\beta^A_{g^+}(e,g)),\beta^M_{g^+}(e,g)).$$

Note that for all $\gamma\in\Gamma$, $a\in A$ and $m\in M$, $\Psi(\gamma gam)=\gamma\Psi(g)\tau_{\psi(\log a)}\tau_m$ for $\tilde m_\psi^{\rm BMS}$-almost all $g\in\tilde\Omega$, where τ stands for the right translation action by elements of $\mathbb{R}\times M$. By abuse of notation, let $\Psi:\Omega\to Z$ denote the map induced by Ψ and τ denote the action of $\mathbb{R}\times M$ on Z induced by τ.

Recalling that $\Omega=\bigcup_{\mathcal{E}_0\in\mathfrak{Y}_\Gamma}(\Omega\cap\mathcal{E}_0)$, we set

$$Z_{\mathcal{E}_0}:=\Psi(\Omega\cap\mathcal{E}_0)\quad\text{ for each }\mathcal{E}_0\in\mathfrak{Y}_{\Gamma_0}.$$

Hence $\{Z_{\mathcal{E}_0}:\mathcal{E}_0\in\mathfrak{Y}_\Gamma\}$ give a measurable partition for $(Z,\hat m_\psi)$.

Proposition 4.8. *For each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, the restriction $\hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ is $\mathbb{R}$-ergodic, and $\hat{m}_\psi = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} \hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ is an $\mathbb{R}$-ergodic decomposition. In particular, $\tilde{\nu}_\psi|_{\Lambda_0}$ is Γ-ergodic and $\tilde{\nu}_\psi = \sum_{\Lambda_0 \in \mathcal{Y}_\Gamma} \tilde{\nu}_\psi|_{\Lambda_0}$ is a Γ-ergodic decomposition.*

Proof. By Corollary 4.7, Σ is generated by $\{B.s : s \in M_\Gamma \backslash M\}$ mod m_ψ^{BMS} for some $B \in \Sigma$. We first claim that $\hat{m}_\psi|_{\Psi(B.s)}$ is $\mathbb{R}$-ergodic for each $s \in M_\Gamma \backslash M$.

Let $f \in C(Z)$ be arbitrary. The Birkhoff average $f_\sharp : Z \to \mathbb{R}$ is defined $\hat{m}_\psi$-a.e. by

$$f_\sharp(y) := \lim_{T\to\infty} \frac{1}{T} \int_0^T f(y\tau_t)\, dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(y\tau_{-t})\, dt.$$

Note that $f_\sharp$ is well defined by the Birkhoff ergodic theorem and is $\mathbb{R}$-invariant. Hence, $f_\sharp \circ \Psi$ is defined m_ψ^{BMS}-a.e. The desired ergodicity follows from the Birkhoff ergodic theorem if we show that $f_\sharp \circ \Psi$ is constant m_ψ^{BMS}-a.e. on each $B.s$. Let $u \in \operatorname{int} \mathcal{L}_\Gamma$ be the unique vector such that $\psi(u) = \psi_\Gamma(u) = 1$ and let $a_t = \exp tu$. Observing that $f \circ \Psi$ is uniformly continuous on each $xAN \cap \Omega$ whenever Ψ is continuous at x and that $f(\Psi(x)\tau_t) = f(\Psi(xa_t))$ for all $t \in \mathbb{R}$, it is a standard Hopf argument to show that $f_\sharp \circ \Psi$ coincides with $N^\pm$-invariant functions m_ψ^{BMS}-a.e. Hence $f_\sharp \circ \Psi$ is Σ-measurable, implying that $f_\sharp \circ \Psi$ is constant m_ψ^{BMS}-a.e. on each $B.s$. Therefore this proves the claim.

For each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, $\hat{m}_\psi(\Psi(B.s) \cap Z_{\mathcal{E}_0}) > 0$ for some $s \in M_\Gamma \backslash M$. It follows from the $\mathbb{R}$-ergodicity of $\hat{m}_\psi|_{\Psi(B.s)}$ that $\hat{m}_\psi|_{\Psi(B.s)} = \hat{m}_\psi|_{Z_{\mathcal{E}_0}}$. Therefore the proposition is proved. $\square$

The measure m_ψ^{BMS} disintegrates over $\hat{m}_\psi$ via the projection $\Gamma\backslash\Lambda^{(2)} \times A \times M \to \Gamma\backslash\Lambda^{(2)} \times \mathbb{R} \times M$, where each conditional measure is the Lebesque measure on $\exp(\ker\psi)$.

Proof of Theorem 4.4. Since $dm_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0} = d\hat{m}_\psi|_{Z_{\mathcal{E}_0}}\, d\,\mathrm{Leb}_{\ker\psi}$, the $\mathbb{R}$-ergodicity of $\hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ proved in Proposition 4.8 implies the A-ergodicity of $m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$. $\square$

4.5. **The set of strong Myrberg limit points.** In [14], we defined Myrberg limit points of Γ.

Definition 4.9. We now define the set of *strong* Myrberg limit points as follows:

$$\begin{aligned}(4.5)\quad \Lambda_\psi^\spadesuit = \{\xi \in \Lambda \cap N^+e^+ : \text{for each } \mathcal{E}_0 \in \mathfrak{Y}_\Gamma, \text{ there exist}\\ \eta \in \Lambda, t \in \mathbb{R}, m \in M \text{ s.t. } Z_{\mathcal{E}_0} = \overline{\Gamma(\xi,\eta,t,m)\mathbb{R}_+}\}\end{aligned}$$

Since $\hat{m}_\psi|_{Z_{\mathcal{E}_0}}$ is $\mathbb{R}$-ergodic and finite for each $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, the Birkhoff ergodic theorem for the $\mathbb{R}$-action implies:

Corollary 4.10. *We have $\nu_\psi(\Lambda_\psi^\spadesuit) = 1$.*

The same proof as the proof of [14, Prop. 8.2] shows that if $g \in \mathcal{E}_0$ and $g^+ \in \Lambda_\psi^\spadesuit$,

$$\limsup \Gamma\backslash\Gamma g A^+ = \Omega \cap \mathcal{E}_0.$$

Hence Corollary 4.10 implies (cf. [14, Coro 8.11]):

Corollary 4.11. *For $m_\psi^{\mathrm{BMS}}|_{\mathcal{E}_0}$-almost all $x \in \mathcal{E}_0 \cap \Omega$, each xA^+ and xw_0A^+ is dense in $\mathcal{E}_0 \cap \Omega$.*

Let Π denote the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$.

Definition 4.12. We write $a_n \to \infty$ regularly in A^+ if $\alpha(\log a_n) \to \infty$ as $n \to \infty$ for all $\alpha \in \Pi$.

The following is an important property of Anosov groups:

Lemma 4.13. *Let Γ be Anosov. For any $g, h \in G$ and a sequence $\gamma_n \to \infty$ in Γ, $\mu(g\gamma_n h) \to \infty$ regularly in A^+.*

This lemma is a consequence of the fact that the limit cone of Γ is contained in the interior of $\mathfrak{a}^+$, except for 0 (cf. [14, Thm. 4.3] for references).

When $\mu(g) \in \operatorname{int} \mathfrak{a}^+$ and $g = k_1\mu(g)k_2$, k_1, k_2 are determined uniquely up to mod M, more precisely, if $g = k_1'\mu(g)k_2'$, then for some $m \in M$, $k_1 = k_1'm$ and $k_2 = m^{-1}k_2'$. We write

$$\kappa_1(g) := [k_1] \in K/M \quad \text{ and } \quad \kappa_2(g) := [k_2] \in M\backslash K.$$

Definition 4.14. (1) A sequence $g_n \in G$ is said to converge to $\xi \in \mathcal{F}$, if $g_n \to \infty$ regularly in G and $\lim_{n\to\infty} \kappa_1(g_n) = \xi$.

(2) A sequence $p_n = g_n(o) \in X$ is said to converge to $\xi \in \mathcal{F}$ if g_n does.

Lemma 4.15. *Let $p \in G/K$ and $\eta \neq \xi_0 \in \Lambda$. For any $\xi \in \Lambda_\psi^\spadesuit - \{\eta\}$, there exists an infinite sequence $\gamma_i \in \Gamma$ such that*

$$\lim_{i\to\infty} \gamma_i^{-1}p = \eta, \quad \lim_{i\to\infty} \gamma_i^{-1}\xi = \xi_0, \quad \textit{and} \quad \lim_{i\to\infty} \beta_\xi^M(\gamma_i, e) = e. \tag{4.6}$$

Moreover, there exists a neighborhood U of ξ_0 such that as $i \to \infty$, $\gamma_i\xi'$ converges to ξ uniformly for all $\xi' \in U$.

Proof. Fix $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$. Since $\xi \in \Lambda_\psi^\spadesuit$, there exists $\Gamma(\xi, \check{\xi}, 0, m) \in Z_{\mathcal{E}_0}$ for some $\check{\xi} \neq \xi \in \Lambda$ and $m \in M$. By the definition of $\Lambda_\psi^\spadesuit$, there exist sequences $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that

$$\lim_{i\to\infty} (\gamma_i^{-1}\xi, \gamma_i^{-1}\check{\xi}, \psi(\log \beta_\xi^A(\gamma_i, e)) + t_i, \beta_\xi^M(\gamma_i, e)m) = (\xi_0, \eta, 0, m).$$

The last two conditions in (4.6) are immediate and the first condition can be proved by the same proof of [14, Thm 8.9].

By passing to a subsequence, we may write $\gamma_i = k_ia_i\ell_i^{-1}$ where $k_i \to k_0, \ell_i \to \ell_0$ in K and $a_i \in A^+$. As Γ is Anosov, $a_i \to \infty$ regularly in A^+. We then have $\ell_0^- = \eta$. Note that $\gamma_i\xi' \to k_0^+$ for all $\xi' \in \mathcal{F}$ with $(\xi', \eta) \in \mathcal{F}^{(2)}$ and this convergence is uniform on a compact subset of $\{\xi' : (\xi', \eta) \in \mathcal{F}^{(2)}\}$. Since $(\xi_0, \eta) \in \mathcal{F}^{(2)}$, there exists a neighborhood U of ξ_0 such that $\gamma_i\xi' \to k_0^+$

uniformly for all $\xi' \in U$. Since $\gamma_i^{-1}\xi \to \xi_0$ and hence $\gamma_i^{-1}\xi \in U$ for all large i, we have $\gamma_i(\gamma_i^{-1}\xi) \to k_0^+$. Hence $\xi = k_0^+$. The claim follows. $\square$

5. Equi-continuous family of Busemann functions

We fix a left G-invariant and right K-invariant Riemannian metric d on G. For a subgroup $H < G$, we set $H_\varepsilon = \{h \in H : d(e,h) < \varepsilon\}$.

In this section, we prove the following proposition.

Proposition 5.1 (Equi-continuity). *Let $\Gamma < G$ be an Anosov subgroup. Let $g \in G$ be such that $g^\pm \in \Lambda$ and let $\gamma_n \in \Gamma$ be a sequence such that for some $\xi \in \Lambda - \{g^-\}$, $\gamma_n^{-1}\xi \to g^+$ and $\gamma_n^{-1}g(o) \to g^-$ as $n \to \infty$. Then the sequence of maps $\eta \mapsto \beta_\eta^{AM}(\gamma_n^{-1}g, g)$ is equi-continuous at g^+, i.e., for any $\varepsilon > 0$, there exists a neighborhood U_ε of g^+ in $\mathcal{F}$ such that for all $n \geq 1$,*

$$\beta_\eta^{AM}(\gamma_n^{-1}g, g) \subset \beta_{g^+}^{AM}(\gamma_n^{-1}g, g)(AM)_\varepsilon \quad \text{for all } \eta \in U_\varepsilon.$$

We first prove the following two lemmas using the structure theory of semisimple Lie groups.

Lemma 5.2. *There exists $c > 0$ such that for all sufficiently small $\varepsilon > 0$,*

$$aG_\varepsilon \subset K_{c\varepsilon} a A_{c\varepsilon} N \quad \text{for all } a \in A^+.$$

Proof. In the following proof, we use the notation $H_{O(\varepsilon)}$ to mean $H_{c\varepsilon}$ for some absolute constant $c > 0$. For all sufficiently small $\varepsilon > 0$, we have

$$G_\varepsilon \subset M_{O(\varepsilon)} N^+_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)} \text{ and } N_\varepsilon^+ \subset K_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)}.$$

Since $aN_\varepsilon^+ a^{-1} \subset N_\varepsilon^+$ for any $a \in A^+$, it follows that

$$\begin{aligned} aG_\varepsilon &\subset aM_{O(\varepsilon)} N^+_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)} = M_{O(\varepsilon)} (aN^+_{O(\varepsilon)} a^{-1}) a A_{O(\varepsilon)} N_{O(\varepsilon)} \\ &\subset M_{O(\varepsilon)} (K_{O(\varepsilon)} A_{O(\varepsilon)} N_{O(\varepsilon)}) a A_{O(\varepsilon)} N_{O(\varepsilon)} \subset K_{O(\varepsilon)} a A_{O(\varepsilon)} N, \end{aligned}$$

which was to be proved. $\square$

Lemma 5.3. *Let $g_n = k_n a_n \ell_n^{-1} \in KA^+K$ where $a_n \to \infty$ regularly in A^+ and $k_n \to k_0$, $\ell_n \to \ell_0$ in K as $n \to \infty$. Assume that both k_0^+ and ℓ_0^+ belong to N^+e^+. Then for all sufficiently large $n > 1$ and small $\varepsilon > 0$, there exist $m_0 \in M$ and neighborhoods U_ε and V_ε of ℓ_0^+ and k_0^+, respectively, such that for all $\eta \in U_\varepsilon \cap g_n^{-1} V_\varepsilon$,*

$$\beta_\eta^{AM}(g_n^{-1}, e) \subset a_n m_0 (AM)_\varepsilon.$$

Proof. Set $\xi = k_0^+$ and $\zeta = \ell_0^+$. By the continuity of the visual maps, there exist neighborhoods U_ε of ζ and V_ε of ξ such that $k_\eta \in k_\zeta K_\varepsilon$ for all $\eta \in U_\varepsilon$ and $k_\eta \in k_\xi K_\varepsilon$ for all $\eta \in V_\varepsilon$. We may assume that $k_0^{-1}k_n$, $\ell_n^{-1}\ell_0 \in K_\varepsilon$ for all n. Let $\eta \in U_\varepsilon \cap g_n^{-1} V_\varepsilon$ be arbitrary. By definition,

$$g_n k_\eta \in k_{g_n\eta} \sigma(g_n, \eta) N, \text{ i.e., } k_0^{-1} g_n k_\eta \in k_0^{-1} k_{g_n\eta} \sigma(g_n, \eta) N.$$

Observe that

$$\begin{aligned} k_0^{-1} g_n k_\eta \in k_0^{-1} g_n k_\zeta K_\varepsilon &= (k_0^{-1} k_n) a_n (\ell_n^{-1} \ell_0) \ell_0^{-1} k_\zeta K_\varepsilon \\ &\subset K_\varepsilon a_n K_\varepsilon \ell_0^{-1} k_\zeta K_\varepsilon \subset K_\varepsilon a_n K_{O(\varepsilon)} \ell_0^{-1} k_\zeta . \end{aligned}$$

On the other hand, since $g_n\eta \in V_\varepsilon$, the right-hand side belongs to

$$k_0^{-1} k_{g_n\eta} \sigma(g_n, \eta) N \subset k_0^{-1} k_\xi K_\varepsilon \sigma(g_n, \eta) N \subset K_{O(\varepsilon)} k_0^{-1} k_\xi \sigma(g_n, \eta) N.$$

Combining these with the fact $\ell_0^{-1} k_\zeta \in M$,

$$a_n K_{O(\varepsilon)} \cap K_{O(\varepsilon)} k_0^{-1} k_\xi \sigma(g_n, \eta) (\ell_0^{-1} k_\zeta)^{-1} N \neq \emptyset.$$

Since $k_0^{-1} k_\xi \in M$ as well, by Lemma 5.2, it follows that

$$\begin{aligned} &\sigma^A(g_n, \eta) \in a_n A_{O(\varepsilon)}, \text{ and} \\ &\sigma^M(g_n, \eta) \in (k_0^{-1} k_\xi)^{-1} M_{O(\varepsilon)} \ell_0^{-1} k_\zeta \subset (k_0^{-1} k_\xi)^{-1} \ell_0^{-1} k_\zeta M_{O(\varepsilon)}. \end{aligned}$$

Since $\beta_\eta^{AM}(g_n^{-1}, e) = \sigma^{AM}(g_n, \eta)$, it remains to set $m_0 := (k_0^{-1} k_\xi)^{-1} \ell_0^{-1} k_\zeta$. ☐

Proof of Proposition 5.1: Set $g_n := g^{-1}\gamma_n g$. Then $g_n^{-1}(g^{-1}\xi) \to e^+$ and $g_n^{-1}(o) \to e^-$ as $n \to \infty$. By passing to a subsequence, we may write $g_n = k_n a_n \ell_n^{-1} \in KA^+K$. We may assume that $k_n \to k_0$ and $\ell_n \to \ell_0$ in K. It follows from the hypothesis that Γ is Anosov that $a_n \to \infty$ regularly in A^+. Combined with the hypothesis $g_n^{-1}(o) \to e^-$ as $n \to \infty$, we have $\ell_0^- = e^-$, or equivalently, $\ell_0 \in M$.

We claim that $k_0^+ = g^{-1}\xi$. Since $a_n \to \infty$ regularly, for any $\eta \in N^+e^+$, $g_n\eta \to k_0^+$ as $n \to \infty$ and the convergence is uniform on a compact subset of N^+e^+. Since $g_n^{-1}(g^{-1}\xi) \to e^+$ as $n \to \infty$, $g_n^{-1}(g^{-1}\xi)$ is in a compact subset of N^+e^+ for all large n. From the previous observation, $g_n(g_n^{-1}(g^{-1}\xi)) \to k_0^+$ as $n \to \infty$, which proves the claim.

Now let $\varepsilon > 0$ be arbitrary. Since $g^- \in \Lambda$, by Lemma 4.2, $g^{-1}\Lambda - \{e^-\} \subset N^+e^+$. Hence both e^+ and $g^{-1}\xi$ belong to N^+e^+. Applying Lemma 5.3 to the sequence g_n, we obtain $m_0 \in M$, neighborhoods U'_ε and V'_ε of e^+ and $g^{-1}\xi$, respectively, such that

$$\beta_\eta(g_n^{-1}, e) \in a_n m_0 (AM)_{\varepsilon/2} \quad \text{for all } \eta \in U'_\varepsilon \cap g_n^{-1} V'_\varepsilon.$$

By uniform convergence on a compact subset of N^+e^+, we may assume that $g_n U'_\varepsilon \subset V'_\varepsilon$ for all large n, by shrinking U'_ε if necessary. Set $U_\varepsilon := gU'_\varepsilon$. Then for all $\eta \in U_\varepsilon$, $g^{-1}\eta \in U'_\varepsilon = U'_\varepsilon \cap g_n^{-1} V'_\varepsilon$ and therefore

$$\beta_\eta(\gamma_n^{-1} g, g) = \beta_{g^{-1}\eta}(g_n^{-1}, e) \in a_n m_0 (AM)_{\varepsilon/2}.$$

Since $g^+ \in U_\varepsilon$ as well, the lemma is proved.

6. Essential values and ergodicity

As before, we let $\Gamma < G$ be an Anosov subgroup containing a loxodromic element in $\operatorname{int} A^+M$. Fix $\psi \in D_\Gamma^\star$. Let $\nu = \nu_\psi$ be a (Γ, ψ)-Patterson Sullivan measure on Λ. By Corollary 4.3,

$$\nu(N^+e^+ \cap \Lambda) = 1. \tag{6.1}$$

Fix a Borel isomorphism $G/N \to \mathcal{F} \times AM$ such that for $g \in N^+AM$,

$$gN \mapsto (g^+, \beta_{g^+}^{AM}(e, g)). \tag{6.2}$$

This isomorphism is G-equivariant for a Borel G-action on $\mathcal{F} \times AM$ given by

$$g(\xi, am) = (g\xi, \beta_\xi^{AM}(g^{-1}, e)am)$$

for $am \in AM$, $g \in G$, and $\xi \in N^+e^+$ with $g\xi \in N^+e^+$.

The following then defines a Γ-invariant locally finite measure on G/N by

$$d\hat{\nu}([g]) = d\nu(g^+)e^{\psi(\log a)}\, da\, dm \tag{6.3}$$

where da and dm are Haar measures on A and M respectively.

Motivated by the work of Schmidt [22] (also [18]), we define:

Definition 6.1. An element $am \in AM$ is called a *ν-essential value*, if for any Borel set $B \subset \mathcal{F}$ with $\nu(B) > 0$ and any $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\nu\left(B \cap \gamma^{-1}B \cap \{\xi \in \mathcal{F} : \beta_\xi^{AM}(\gamma^{-1}, e) \in am(AM)_\varepsilon\}\right) > 0. \tag{6.4}$$

In view of (6.1), it suffices to consider Borel subsets $B \subset N^+e^+$ in this definition, and hence $\beta_\xi^{AM}(\gamma^{-1}, e)$ is well-defined for all $\xi \in B \cap \gamma^{-1}B$.

Let E_ν denote the set of all ν-essential values in AM. By the following lemma, $am \in \mathsf{E}_\nu$ if and only if $(am)^{-1} \in \mathsf{E}_\nu$; hence the condition $\beta_\xi^{AM}(\gamma^{-1}, e) \in am(AM)_\varepsilon$ in (6.4) can be replaced $\beta_\xi^{AM}(e, \gamma^{-1}) \in am(AM)_\varepsilon$ in the above definition.

Lemma 6.2. *E_ν is a closed subgroup of AM.*

Proof. Since the metric d restricted to M is bi-M-invariant, we have that all $\varepsilon > 0$, $M_\varepsilon^{-1} = M_\varepsilon$, $m^{-1}M_\varepsilon m = M_\varepsilon$ for all $m \in M$ and $M_{\varepsilon/2}M_{\varepsilon/2} \subset M_\varepsilon$. Let $b_1, b_2 \in \mathsf{E}_\nu$. It suffices to show that $b_1b_2^{-1} \in \mathsf{E}_\nu$. Let $B \subset \mathcal{F}$ be a Borel subset with $\nu(B) > 0$ and let $\varepsilon > 0$. As $b_1 \in \mathsf{E}_\nu$, there exists $\gamma_1 \in \Gamma$ such that

$$B_1 := B \cap \gamma_1^{-1}B \cap \{\xi : \beta_\xi(\gamma_1^{-1}, e) \in b_1(AM)_{\varepsilon/2}\}$$

has a positive ν-measure. As $b_2 \in \mathsf{E}_\nu$, there exists $\gamma_2 \in \Gamma$ such that

$$B_2 := B_1 \cap \gamma_2^{-1}B_1 \cap \{\xi : \beta_\xi(\gamma_2^{-1}, e) \in b_2(AM)_{\varepsilon/2}\}$$

has a positive ν-measure. Note that $\gamma_2 B_2 \subset B \cap \gamma_2\gamma_1^{-1}B$ and that for all $\xi \in \gamma_2 B_2$, we have

$$\begin{aligned}\beta_\xi(\gamma_2\gamma_1^{-1}, e) = \beta_{\gamma_2^{-1}\xi}(\gamma_1^{-1}, \gamma_2^{-1}) &= \beta_{\gamma_2^{-1}\xi}(\gamma_1^{-1}, e)\beta_{\gamma_2^{-1}\xi}(e, \gamma_2^{-1}) \\ &\in b_1(AM)_{\varepsilon/2}(AM)_{\varepsilon/2}^{-1} b_2^{-1} \subset b_1 b_2^{-1}(AM)_\varepsilon.\end{aligned}$$

Hence

$$\gamma_2 B_2 \subset B \cap \gamma_2\gamma_1^{-1}B \cap \{\xi : \beta_\xi(\gamma_2^{-1}\gamma_1, e) \in b_1 b_2^{-1}(AM)_\varepsilon\}.$$

Since $\nu(\gamma_2^{-1}B_2) > 0$, it follows that $b_1 b_2^{-1} \in \mathsf{E}_\nu$. This proves that E is a subgroup of AM. Now suppose that $b_i \in \mathsf{M}$ converges to some $b \in AM$. Let $\varepsilon > 0$ and $B \subset \mathcal{F}$ be a Borel subset with $\nu(B) > 0$. Fix i large enough so that $b_i(AM)_{\varepsilon/2} \subset b(AM)_\varepsilon$, and let $\gamma_i \in \Gamma$ be such that $\nu(B \cap \gamma_i^{-1}B \cap \{\xi : \beta_\xi(\gamma_i^{-1}, e) \in b_i(AM)_{\varepsilon/2}\}) > 0$. Then $\nu(B \cap \gamma_i^{-1}B \cap \{\xi : \beta_\xi(\gamma_i^{-1}, e) \in b(AM)_\varepsilon\}) > 0$. This proves $b \in \mathsf{E}_\nu$. Hence E_ν is closed. □

Lemma 6.3. *Let $b_0 \in \mathsf{E}_\nu$ be such that $\{bb_0b^{-1} : b \in AM\} \subset \mathsf{E}_\nu$. Then for any Γ-invariant Borel function $h : G/N \to [0,1]$, we have*

$$h(xb_0) = h(x) \quad \text{for } \hat\nu\text{-a.e. } x.$$

Proof. In view of the homeomorphsim $N^+AMN/N \to N^+e^+ \times AM$ given by $gN \mapsto (g^+, \beta_{g^+}(e, g))$ and (6.1), it suffices to show that for any Γ-invariant Borel function $h : N^+e^+ \times AM \to [0,1]$, $h(\xi, b) = h(\xi, bb_0)$ for ν a.e ξ and for all $b \in AM$.

Suppose not. Then there exists $b_1 \in AM$ such that $\nu\{\xi \in \mathcal{F} : h(\xi, b_1) < h(\xi, b_1b_0)\} > 0$ or $\nu\{\xi \in \mathcal{F} : h(\xi, b_1) < h(\xi, b_1b_0)\} > 0$. We consider the first case; the second case can be treated similarly. Then there exist $r, \varepsilon > 0$ such that

$$Q_{b_0} := \{\xi \in N^+e^+ : h(\xi, b_1) < r - \varepsilon < r + \varepsilon < h(\xi, b_1b_0)\}$$

has a positive ν-measure. By considering the convolution of h with the approximation of identity functions on AM, we may assume without loss of generality that the family $h(\xi, \cdot)$, $\xi \in N^+e^+$, is uniformly equi-continuous on AM. Hence there exists $\varepsilon' > 0$ such that for all $\xi \in Q_{b_0}$ and $b \in (AM)_{\varepsilon'}$,

$$h(\xi, b_1b) < r < h(\xi, b_1b_0b). \tag{6.5}$$

Since $b_1b_0b_1^{-1} \in \mathsf{E}_\nu$ by the hypothesis and $\nu(Q_{b_0}) > 0$, there exists $\gamma \in \Gamma$ such that

$$\mathcal{Q} := Q_{b_0} \cap \gamma^{-1}Q_{b_0} \cap \{\xi \in \mathcal{F} : \beta_\xi(\gamma^{-1}, e) \in b_1b_0b_1^{-1}(AM)_{\varepsilon'/2}\}$$

has a positive ν-measure. We now claim that

$$h(\xi, b_1b) < r < h(\gamma(\xi, b_1b))$$

for all $\xi \in \mathcal{Q}$ and for all $b \in (AM)_{\varepsilon'/2}$. This yields a contradiction to the Γ-invariance of h. Since $\mathcal{Q} \subset Q_{b_0}$, we have $h(\xi, b_1b) < r$ for all $b \in (AM)_{\varepsilon'}$ by (6.5). On the other hand, for all $b \in (AM)_{\varepsilon'/2}$ and $\xi \in \mathcal{Q}$, we have

$$\beta_\xi(\gamma^{-1}, e)b_1b \in b_1b_0b_1^{-1}(AM)_{\varepsilon'/2}b_1b \subset b_1b_0(AM)_{\varepsilon'},$$

since $m^{-1}M_{\varepsilon'/2}mM_{\varepsilon'/2}\subset M_{\varepsilon'}$ for all $m\in M$. Since $\gamma\xi\in Q_{b_0}$ and $\gamma(\xi,b_1b)=(\gamma\xi,\beta_\xi(\gamma^{-1},e)b_1b)$, it follows from (6.5) that $h(\gamma(\xi,b_1b))>r$. This proves the claim.

□

7. N-ergodic decompositions of BR-measures

Let $\Gamma<G$ be an Anosov subgroup. We prove Theorem 1.1 in this section.

7.1. **Ergodic decomposition of an infinite measure.** The following version of ergodic decomposition of any Radon measure can be deduced from [11, Thm 5.2].

Proposition 7.1 (Ergodic decomposition)**.** *Let N be a locally compact second countable group and M be a compact subgroup normalizing N. Suppose that NM acts on a standard Borel space $(X,\mathcal{B})$, preserving a Radon measure μ on X.*

(1) *There exists a Borel map $x\mapsto\mu_x$ from X to the space of N-invariant ergodic Radon measures on X and an M-invariant probability measure μ^* on X equivalent to μ with the following properties:*
   (a) $\mu_x=\mu_{xn}$ for every $x\in X$ and $n\in N$.
   (b) For all nonnegative Borel function $f:X\to\mathbb{R}$, we have
$$\int f\,d\mu_x=\mathbb{E}_{\mu^*}\left(f\frac{d\mu}{d\mu^*}|\mathcal{S}_N\right)(x)\qquad \text{for } \mu\text{-a.e. } x\in X,$$
   where $\mathcal{S}_N:=\{B\in\mathcal{B}:B.n=B \text{ for all } n\in N\}$. In particular, we have
$$\mu=\int_{x\in X}\mu_x\,d\mu^*(x).$$
   If μ is finite, we can take $\mu^*=\mu$.
(2) *Let $\mathcal{T}\subset\mathcal{S}_N$ be the smallest σ-algebra such that the map $x\mapsto\mu_x$ is $\mathcal{T}$-measurable. Then $\mathcal{T}$ is countably generated, $\mathcal{T}=\mathcal{S}_N$ mod μ and $\mu_x([y]_\mathcal{T})=0$ for all $y\notin[x]_\mathcal{T}$, $\mu_x([x]^c_\mathcal{T})=0$; here $[y]_\mathcal{T}=\cap_{y\in C\in\mathcal{T}}C$ denotes the atom of y in $\mathcal{T}$.*
(3) *For each $m\in M$, we have $\mu_{xm}=\mu_x.m$ for μ-a.e. $x\in X$.*

Proof. Fix an M-invariant positive function $\varphi\in L^1(\mu)$ with $\int\varphi\,d\mu=1$. Then $d\mu^*:=\varphi\,d\mu$ defines an N-quasi-invariant and M-invariant probability measure on X. By applying [11, Thm 5.2] to μ^* with the cocycle $\rho:N\times X\to\mathbb{R}$ given by $\rho(n,y)=\log\frac{\varphi(yn^{-1})}{\varphi(y)}$, we get a Borel map $x\mapsto\mu^*_x$ from X to the space of N-ergodic probability measures such that for all nonnegative Borel function $f:X\to\mathbb{R}$, we have
$$\int f\,d\mu^*_x=\mathbb{E}_{\mu^*}(f|\mathcal{S}_N)(x)\quad\text{for } \mu^*\text{-a.e. } x\in X,$$
and $\frac{d(n.\mu^*_x)}{d\mu^*_x}(y)=\frac{\varphi(yn^{-1})}{\varphi(y)}$. In particular, we have $\mu^*=\int\mu^*_x\,d\mu^*(x)$. Now define a Radon measure μ_x on X by $d\mu_x:=\frac{1}{\varphi}\,d\mu^*_x$. A direct computation

shows that μ_x is N-invariant, ergodic for all $x \in X$ and (1) holds. (2) follows from the corresponding statement on μ_x^* from [11, Thm 5.2].

In order to prove (3), we compute that for a non-negative Borel function $f : X \to \mathbb{R}$,

$$\mu_{xm}^*(f) = \mathbb{E}_{\mu^*}(f|\mathcal{S}_N)(xm) = \mathbb{E}_{\mu^*}(m.f|\mathcal{S}_N)(x) = \mu_x^*(m.f);$$

the second equality follows since $\mathcal{S}_N.m = \mathcal{S}_N$ and μ^* is M-invariant. It follows that $\mu_{xm}^* = \mu_x^*.m$ for μ-a.e. $x \in X$; this implies (3). $\square$

7.2. **P°-semi-invariant measures.** In terms of the coordinates $G = G/P^\circ \times AM^\circ N$, we have

$$d\tilde{m}_\psi^{\mathrm{BR}} = d\tilde{\nu}_\psi e^{\psi(\log a)} dadmdn. \tag{7.1}$$

If μ is a P°-semi-invariant measure on $\Gamma\backslash G$, then μ is NM°-invariant and there exists a linear form $\chi_\mu \in \mathfrak{a}^*$ such that for all $a \in A$,

$$a_*\mu = e^{-\chi_\mu(\log a)}\mu.$$

We set $\psi_\mu := \chi_\mu + 2\rho \in \mathfrak{a}^*$.

Proposition 7.2. *Let μ be any N-invariant ergodic and P°-semi invariant Radon measure supported on $\mathcal{E}$. Let $\tilde{\mu}$ denote its lift to $G \simeq G/P^\circ \times AM^\circ N$. Then $\psi_\mu \in D_\Gamma^\star$ and $d\tilde{\mu}$ is proportional to $d\tilde{\nu}_{\psi_\mu}|_{\Lambda_0} e^{\psi_\mu(\log a)} dadmdn$ for some Γ-minimal subset $\Lambda_0 \in \mathcal{Y}_\Gamma$, or equivalently, μ is proportional to $m_{\psi_\mu}^{\mathrm{BR}}|_{\mathcal{E}_0}$ for some $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$.*

Proof. Since $\tilde{\mu}$ is a right P°-semi-invariant measure on $G \simeq G/P^\circ \times AM^\circ N$, up to a positive constant multiple, we have

$$d\tilde{\mu} = e^{\tilde{\chi}(\log a)} d\tilde{\nu}\, da\, dm\, dn$$

for some Radon measure $\tilde{\nu}$ on G/P° and $\tilde{\chi} \in \mathfrak{a}^*$ [14, Proposition 10.25]. Since $a_*\tilde{\mu} = e^{-\chi_\mu(\log a)}\tilde{\mu}$, it follows $\tilde{\chi} = \psi_\mu$. Denote by $\pi : G/P^\circ \to G/P$ the projection map. Since $\tilde{\mu}$ is right N-ergodic, $\tilde{\nu}$ is a Γ-ergodic measure on G/P°. And since $\tilde{\mu}$ is Γ-invariant, $\pi_*\tilde{\nu}$ is a (Γ, ψ_μ)-conformal measure on G/P (cf. [14, Prop. 10.25]). In particular, $\psi_\mu \in D_\Gamma^\star$ by [14, Thm 4.3]. Let $\tilde{\nu}_{\psi_\mu}$ be the M-invariant lift of $\nu_{\psi_\mu} := \pi_*\tilde{\nu}$ to G/P°. Since $\tilde{\nu} \ll \tilde{\nu}_{\psi_\mu}$ and $\tilde{\nu}$ is Γ-ergodic, $\tilde{\nu}$ is proportional to $\tilde{\nu}_{\psi_\mu}|_{\Lambda_0}$ for some Γ-minimal subset $\Lambda_0 \in \mathcal{Y}_\Gamma$ by Proposition 4.8. This completes the proof. $\square$

7.3. **Essential values and Ergodicity.** Fix $\psi \in D_\Gamma^\star$, and let ν_ψ be the unique (Γ, ψ)-Patterson Sullivan measure on Λ. Let E_{ν_ψ} be the set of essential values as defined in Definition 6.1.

Proposition 7.3. *If $M^\circ \subset \mathsf{E}_{\nu_\psi}$, then for any $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$, $m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ is N-ergodic. In particular, Theorem 1.1(1) holds.*

Proof. Let $m_\psi^{\mathrm{BR}} = \int_X \mathsf{m}_x \, d\mathsf{m}^*(x)$ be an N-ergodic decomposition as given by Proposition 7.1 with $X = \Gamma\backslash G$. Let $f \in C_c(\Gamma\backslash G)$ and consider the map $h(g) := \mathsf{m}_{[g]}(f)$ for all $[g] \in X$. Note that h defines a Γ-invariant

Borel function on G/N. Since M° is a normal subgroup of AM, Lemma 6.3 implies that h is M°-invariant for $\hat{\nu}_\psi$-almost all. By Proposition 7.1(3), it follows that $M^\circ < \mathrm{Stab}_M(\mathsf{m}_x)$ for almost all x; without loss of generality, we may assume that $M^\circ \subset \mathrm{Stab}_M(\mathsf{m}_x)$ for all $x \in X$. Hence the finite group $S := M^\circ \backslash M$ acts on $\{\mathsf{m}_x : x \in X\}$. Set

$$\tilde{\mathsf{m}}_x := \frac{1}{[M : M^\circ]} \sum_{s \in M^\circ \backslash M} \mathsf{m}_x.s.$$

Since m_ψ^{BR} is M-invariant, we have $m_\psi^{\mathrm{BR}} = \int_X \tilde{\mathsf{m}}_x d\mathsf{m}^*(x)$. As $\mathsf{m}_{xm} = \mathsf{m}_x.m$ for all $m \in M$, the map $x \mapsto \tilde{\mathsf{m}}_x$ is NM-invariant. Since m_ψ^{BR} is NM-ergodic, $\tilde{\mathsf{m}}_x$ is constant m-a.e. $x \in X$. Therefore we may fix $x_0 \in X$ so that $m_\psi^{\mathrm{BR}} = \tilde{\mathsf{m}}_{x_0}$. Set $M_* := \mathrm{Stab}_M(\mathsf{m}_{x_0})$. Then

$$m_\psi^{\mathrm{BR}} = \frac{1}{[M : M_*]} \sum_{s \in M_* \backslash M} \mathsf{m}_{x_0} s$$

where $\mathsf{m}_{x_0} s$ are mutually singular to each other. We claim that each $\mathsf{m}_{x_0}.s$ is A-semi-invariant with $\psi_{\mathsf{m}_{x_0}.s} = \psi$ for each $s \in M_* \backslash M$. It suffices to consider the case when $s = [M^*]$. Let

$$A' := \{a \in A : a \text{ preserves the measure class of } \mathsf{m}_{x_0}\}.$$

As A' is a closed subgroup of A, it suffices to show that for any unit vector $u \in \mathfrak{a}$ and any $\varepsilon > 0$, $\exp tu \in A'$ forsome $0 < t < \varepsilon$. Let $a = \exp \frac{\varepsilon u}{n+2}$ for $n = \# M/M^*$. Since m^{BR} is quasi-invariant under a and has n number of ergodic components, it follows that for some $1 \le k \le n+1$, $a^k.\mathsf{m}_{x_0}$ is in the same measure class as m_{x_0}, implying that $a^k \in A'$. Hence $A = A'$. As m_ψ^{BR} is semi-invariant under A, the claim follows. Therefore, by Proposition 7.2, m_{x_0} is proportional to $m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ for some $\mathcal{E}_0 \in \mathfrak{Y}_\Gamma$. Hence $M_* = \mathrm{Stab}_M\, m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0} = M_\Gamma$. Since the measures $\mathsf{m}_{x_0}.s$ are mutually singular to each other, all $\mathcal{E}_0$'s are distinct. Therefore $m_\psi^{\mathrm{BR}} = \sum_{\mathcal{E}_0 \in \mathfrak{Y}_\Gamma} c(\mathcal{E}_0) \cdot m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ for some constant $c(\mathcal{E}_0) > 0$. It remains to observe $c(\mathcal{E}_0) = 1$ as the supports of $m_\psi^{\mathrm{BR}}|_{\mathcal{E}_0}$ are mutually disjoint from each other. $\square$

Proof of Theorem 1.3. Let $\mathcal{O}_\Gamma$ denote the space of all N-invariant ergodic and P°-quasi-invariant Radon measures supported on $\mathcal{E}$, up to constant multiples. We write $\mathfrak{Y}_\Gamma = \{\mathcal{E}_i : 1 \le i \le k\}$ with $k = \#\mathfrak{Y}_\Gamma = \# M/M_\Gamma$. Consider the map $\iota : D_\Gamma^\star \times \{1, \cdots, k\} \to \mathcal{O}_\Gamma$ defined by $\iota(\psi, i) = m_\psi^{\mathrm{BR}}|_{\mathcal{E}_i}$. By Proposition 7.3, ι is well-defined. Since any measure contained in $\mathcal{O}_\Gamma$ must be P° semi-invariant, being N-ergodic, Proposition 7.2 implies that ι is surjective. That ι is indeed a homeomorphism now follows because the map $\psi \mapsto m_\psi^{\mathrm{BR}}$ is a homeomorphism between $D_\Gamma^\star$ and the space of all NM-invariant ergodic and A-quasi-invariant Radon measures supported on $\mathcal{E}$, up to constant multiples, as shown in [14]. This implies Theorem 1.3 as $D_\Gamma^\star$ is homeomorphic to $\mathbb{R}^{\mathrm{rank}\, G - 1}$ (cf. [14]).

7.4. **The largeness of the length spectrum.** Without loss of generality, we may assume that $\Gamma \cap \operatorname{int} A^+M \not\subset M$ for the rest of section. We will need the following:

Proposition 7.4. *For any $C > 1$, the closed subgroup of AM generated by $\{\hat\lambda(\gamma_0) : \gamma_0 \in \Gamma^\star, \psi(\lambda(\gamma_0)) > C\}$ contains AM°.*

By Corollary 3.6 and Lemma 7.5 and Lemmas 7.3, this proposition follows from the following lemma.

Lemma 7.5. *For any $C > 1$, there exists a Zariski dense subgroup $\Gamma_\psi < \Gamma$, depending on C, such that $\Gamma_\psi \cap A^+M \not\subset M$ and*

$$\psi(\lambda(\gamma)) > C \quad \text{for all } \gamma \in \Gamma_\psi - \{e\}.$$

In particular, $\hat\lambda(\Gamma_\psi^\star) \subset \{\hat\lambda(\gamma_0) : \gamma_0 \in \Gamma^\star, \psi(\lambda(\gamma_0)) > C\}$.

Proof. Recall that Π is the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$. By [1, Lem. 4.3(b)], there exist $\varepsilon > 0$ and $\{s_1, \cdots, s_k\} \subset \Gamma$ such that $s_1 \in \operatorname{int} A^+M - M$, and for each $m \geq 1$, $s_1^m, \cdots, s_k^m$ are (Π, ε)-Schottky generators and the subgroup $\Gamma_m = \langle s_1^m, \cdots, s_k^m \rangle$ is a Zariski-dense (Π, ε)-Schottky subgroup of Γ (see [1, Def. 4.1] for terminologies).

Fix $m > 1$ and let $z \in \lambda(\Gamma_m) - \{0\}$. Then $z = \lambda(w)$ for some $w = g_1^{n_1} \cdots g_\ell^{n_\ell}$ with $g_i \in \{s_1^{\pm m}, \cdots, s_k^{\pm m}\}$, $n_i \in \mathbb{N}$, $g_i \neq g_{i+1}^{-1} (i = 1, \cdots, \ell)$ where we interpret $g_{\ell+1} := g_1$; this is because every element of a (Π, ε)-Schottky group is conjugate to a word of such form. By [1, Lem. 4.1], there exists $R = R(\varepsilon) > 0$ (independent of $w \in \Gamma_1$) such that

$$\|\lambda(w) - \sum_{i=1}^{\ell} n_i \lambda(g_i)\| \leq \ell R.$$

Since $\psi(\lambda(s_j^{\pm 1})) > 0$ and $\lambda(s_j^{\pm m}) = m\lambda(s_j^{\pm 1})$, we can choose m_0 such that $\psi(\lambda(s_j^{\pm m_0})) > \|\psi\| R + C$ for all $j = 1, \cdots, k$. Set $\Gamma_\psi := \Gamma_{m_0}$. Then for any $z = \lambda(w) \in \lambda(\Gamma_\psi) - \{0\}$ as above,

$$\psi(z) \geq \sum_{i=1}^{\ell} n_i \psi(\lambda(g_i)) - \|\psi\| \ell R \geq \sum_{i=1}^{\ell} n_i \Big(\psi(\lambda(g_i)) - \|\psi\| R \Big) > C.$$

The lemma follows. $\square$

7.5. **Proof of Main proposition.** Recall the Gromov product: for any $\xi \neq \eta$ in Λ,

$$\mathcal{G}(\xi, \eta) := \log \beta^A_{h^+}(e, h) + \mathrm{i} \log \beta^A_{h^-}(e, h)$$

for $h \in G$ satisfying that $h^+ = \xi$ and $h^- = \eta$. Set $o = [K] \in G/K$. For any fixed $p = g(o) \in G/K$, the following

$$d_{\psi,p}(\xi, \eta) := e^{-\psi(\mathcal{G}(g^{-1}\xi, g^{-1}\eta))} \quad \text{for any } \xi \neq \eta \text{ in } \Lambda$$

defines a virtual visual metric on Λ, satisfying a version of triangle inequality [14, Lemma 6.11]. For $\xi \in \Lambda$ and $r > 0$, set

$$\mathbb{B}_p(\xi, r) := \{\eta \in \Lambda : d_{\psi,p}(\xi, \eta) < r\}.$$

We recall the following two lemmas:

Lemma 7.6. [14, Lemma 6.12] *There exists $N_0(\psi, p) \geq 1$ satisfying the following: for any finite collection $\mathbb{B}_p(\xi_1, r_1), \cdots, \mathbb{B}_p(\xi_n, r_n)$ with $\xi_i \in \Lambda$ and $r_i > 0$, there exists a disjoint subcollection $\mathbb{B}_p(\xi_{i_1}, r_{i_1}), \cdots, \mathbb{B}_p(\xi_{i_\ell}, r_{i_\ell})$ such that*

$$\mathbb{B}_p(\xi_1, r_1)\cup\cdots\cup\mathbb{B}_p(\xi_n, r_n) \subset \mathbb{B}_p(\xi_{i_1}, 3N_0(\psi, p)r_{i_1})\cup\cdots\cup\mathbb{B}_p(\xi_{i_\ell}, 3N_0(\psi, p)r_{i_\ell}).$$

Moreover, $N_0(\psi, p)$ can be taken uniformly for all p in a fixed compact subset of X.

Lemma 7.7. [14, Lemma 10.6]. *There exists a compact subset $\mathcal{C} \subset G$ such that for any $\xi \in \Lambda$, there exists $g \in \mathcal{C}$ such that $g^+ = \xi$ and $g^- \in \Lambda$.*

We set

$$N_0 := \max_{p \in \mathcal{C}(o)} N_0(\psi, p) < \infty$$

with $N_0(\psi, p)$ and $\mathcal{C}$ given by Lemmas 7.6 and 7.7 respectively.

By Proposition 7.4 and Proposition 7.3, Theorem 1.1 now follows from:

Proposition 7.8 (Main Proposition)**.** *For all $\gamma_0 \in \Gamma^\star$ satisfying $\psi(\lambda(\gamma_0)) > \log 3N_0 + 1$, we have $\hat{\lambda}(\gamma_0) \in \mathsf{E}_{\nu_\psi}$.*

The rest of the section is devoted to the proof of Proposition 7.8.

Definition of $\mathcal{B}_R(\gamma_0, \varepsilon)$. We now fix $\varepsilon > 0$ as well as an element $\gamma_0 \in \Gamma^\star$ such that

$$\psi(\lambda(\gamma_0)) > \log 3N_0 + 1.$$

Note that $y^{\pm}_{\gamma\gamma_0\gamma^{-1}} = \gamma y^{\pm}_{\gamma_0}$. We can choose $g \in \mathcal{C}$ such that $g^+ = y^+_{\gamma_0}$ and $g^- \in \Lambda$. Set

$$p := g(o), \quad \eta := g^-, \text{ and } \xi_0 := g^+.$$

For any $\xi \in \Lambda - \{\eta, e^-\}$, we claim that there is $R_\varepsilon = R_\varepsilon(\xi) > 0$ such that

$$\beta^{AM}_{\xi'}(g, e) \in \beta^{AM}_{\xi}(g, e)(AM)_\varepsilon$$

for all $\xi' \in \mathbb{B}_p(\xi, e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon} R_\varepsilon)$. Indeed, since $e^- \notin \{\xi, g^{-1}\xi\}$, we have $\xi, g^{-1}\xi \in N^+e^+$ by Lemma 4.2. The claim follows as the map $\xi' \mapsto \beta^{AM}_{\xi'}(g, e)$ is continuous at ξ.

By [14, Lem. 6.11], the family $\{\mathbb{B}_p(\xi, r) : \xi \in \Lambda, r > 0\}$ forms a basis of topology in Λ. For $\gamma \in \Gamma$, let $r_g(\gamma)$ be the supremum of $r \geq 0$ such that for all $\xi \in \mathbb{B}_p(\gamma\xi_0, 3N_0 r)$,

$$\beta^{AM}_{\xi}(g, \gamma\gamma_0\gamma^{-1}g) \in \beta^{AM}_{\gamma\xi_0}(g, \gamma\gamma_0\gamma^{-1}g)(AM)_\varepsilon. \tag{7.2}$$

If $\gamma\xi_0 \notin \{e^-, g^-\}$ and hence $\gamma\xi_0, g^{-1}\gamma\xi_0 \in N^+e^+$, then $r_g(\gamma) > 0$.

For each $R > 0$, we define the family of virtual balls as follows:

$$\mathcal{B}_R(\gamma_0, \varepsilon) = \{\mathbb{B}_p(\gamma\xi_0, r) : \gamma \in \Gamma, 0 < r < \min(R, r_g(\gamma))\}.$$

We remark that the difference of the definition of $\mathcal{B}_R$ in this paper and our previous paper [14] lies in the definition of $r_g(\gamma)$; in [14], we used the A-valued Busemann function in (7.2) whereas $r_g(\gamma)$ is defined in terms of the AM-valued Busemann function here.

Theorem 7.9. [14, Thm 5.3] *There exists $C = C(\psi, p) > 0$ such that for all $\gamma \in \Gamma$ and $\xi \in \Lambda$,*

$$-\psi(\underline{a}(p, \gamma p)) - C \le \psi(\log \beta_\xi^A(\gamma p, p)) \le \psi(\underline{a}(\gamma p, p)) + C.$$

where $\underline{a}(p, q) := \mu(g^{-1}h)$ for $p = g(o)$ and $q = h(o)$.

For $q \in X$ and $r > 0$, the shadows of the ball $B(q, r)$ viewed from $p \in X$ and $\xi \in \mathcal{F}$ are respectively defined as

$$O_r(p, q) := \{gk^+ \in \mathcal{F} : k \in K,\, gk \operatorname{int} A^+o \cap B(q, r) \neq \emptyset\}$$

where $g \in G$ satisfies $p = g(o)$, and

$$O_r(\xi, q) := \{h^+ \in \mathcal{F} : h^- = \xi, ho \in B(q, r)\}.$$

Lemma 7.10. [14, Lem. 5.7] *There exists $\kappa > 0$ such that for any $p, q \in X$ and $r > 0$, we have*

$$\sup_{\xi \in O_r(p,q)} \|\log \beta_\xi^A(p, q) - \underline{a}(p, q)\| \le \kappa r.$$

We let $C = C(\psi, p) > 0$ and $\kappa > 0$ be the constants given by Theorem 7.9 and Lemma 7.10 respectively. Since ξ_0 belongs to the shadow $O_{\varepsilon/(8\kappa)}(\eta, p)$, we can choose $0 < s = s(\gamma_0) < R$ small enough such that

$$\mathbb{B}_p(\xi_0, e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+\frac{1}{2}\|\psi\|\varepsilon+2C} s) \subset O_{\varepsilon/(8\kappa)}(\eta, p). \tag{7.3}$$

Next, observe that the function $\xi' \mapsto \beta_{\xi'}(g, \gamma_0 g)$ is continuous at ξ_0, as $g^{-1}\xi_0 = e^+ \in N^+e^+$. Hence we may further assume that s is small enough so that

$$\beta_{\xi'}^{AM}(g, \gamma_0 g) \in \beta_{\xi_0}^{AM}(g, \gamma_0 g)(AM)_\varepsilon \quad \text{for all } \xi' \in \mathbb{B}_p(\xi_0, e^{2C} s). \tag{7.4}$$

For each $\gamma \in \Gamma$, set

$$D(\gamma\xi_0, r) := \mathbb{B}_p(\gamma\xi_0, \tfrac{1}{3N_0} e^{-\psi(\mu(g^{-1}\gamma g)+\mu(g^{-1}\gamma^{-1}g))} r) \text{ and}$$
$$3N_0 D(\gamma\xi_0, r) := \mathbb{B}_p(\gamma\xi_0, e^{-\psi(\mu(g^{-1}\gamma g)+\mu(g^{-1}\gamma^{-1}g))} r).$$

Here note that $\underline{a}(\gamma^{-1}p, p) = \mu(g^{-1}\gamma g)$ and $\mathrm{i}\, \underline{a}(\gamma^{-1}p, p) = \mu(g^{-1}\gamma^{-1}g)$.

Lemma 7.11. *Let $R > 0$ and $\xi \in \Lambda - \{\eta\}$. Suppose that $\gamma_i^{-1}p \to \eta$, $\gamma_i^{-1}\xi \to \xi_0$ and $\beta_\xi^M(\gamma_i, e) \to e$ as $i \to \infty$ for some $\gamma_i \in \Gamma$. Then for all sufficiently small $r > 0$, there exists $i_0 = i_0(r) > 0$ such that for all $i \ge i_0$, the following holds:*

(1) $\xi \in D(\gamma_i\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$; *in particular, for any* $R > 0$,

$$\Lambda_\psi^{\spadesuit} \subset \bigcup_{D \in \mathcal{B}_R(\gamma_0,\varepsilon)} D.$$

(2) $\beta_{\xi'}^{AM}(e, \gamma_i\gamma_0\gamma_i^{-1}) \in \hat{\lambda}(\gamma_0)(AM)_{O(\varepsilon)}$ *for all* $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$.

Proof. Note that $\gamma_i^{-1}go \to \eta = g^-$ and $\gamma_i^{-1}\xi \to \xi_0 = g^+$. Let $U_\varepsilon \subset \mathcal{F}$ be a neighborhood of ξ_0 associated to the sequence γ_i, as in Proposition 5.1. Since $\xi_0 \in U_\varepsilon$, there exists $R_1 > 0$ such that

$$\mathbb{B}_p(\xi_0, e^{2C}R_1), \gamma_0^{-1}\mathbb{B}_p(\xi_0, e^{2C}R_1) \subset U_\varepsilon.$$

Let $0 < r < \min(s(\gamma_0), R_\varepsilon/2, R_1, R)$. In view of [14, Lem 10.12], we have $3N_0 D(\gamma_i\xi_0, r) \subset \gamma_i\mathbb{B}_p(\xi_0, e^{2C}r)$. In order to show $D(\gamma_i\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$, it suffices to check that for all $\xi' \in \mathbb{B}_p(\xi_0, e^{2C}r)$,

$$\beta_{\xi'}^M(\gamma_i^{-1}g, \gamma_0\gamma_i^{-1}g) \in \beta_{\xi_0}^M(\gamma_i^{-1}g, \gamma_0\gamma_i^{-1}g)M_\varepsilon;$$

this implies that $r < r_g(\gamma_i)$.

We start by noting that since $r \le s(\gamma_0)$, we have $\beta_{\xi'}^M(g, \gamma_0 g) \in \beta_{\xi_0}^M(g, \gamma_0 g)M_\varepsilon$. Since $\xi', \gamma_0^{-1}\xi' \in U_\varepsilon$, by Proposition 5.1, for all sufficiently large i,

$$\begin{aligned}
\beta_{\xi'}^M(\gamma_i^{-1}g, \gamma_0\gamma_i^{-1}g) &= \beta_{\xi'}^M(\gamma_i^{-1}g, g)\beta_{\xi'}^M(g, \gamma_0 g)\beta_{\xi'}^M(\gamma_0 g, \gamma_0\gamma_i^{-1}g) \\
&= \beta_{\xi'}^M(\gamma_i^{-1}g, g)\beta_{\xi'}^M(g, \gamma_0 g)\beta_{\gamma_0^{-1}\xi'}^M(\gamma_i^{-1}g, g)^{-1} \\
&\in \beta_{\xi_0}^M(\gamma_i^{-1}g, g)\beta_{\xi_0}^M(g, \gamma_0 g)\beta_{\xi_0}^M(\gamma_i^{-1}g, g)^{-1}M_{O(\varepsilon)} \\
&= \beta_{\xi_0}^M(\gamma_i^{-1}g, \gamma_0\gamma_i^{-1}g)M_{O(\varepsilon)},
\end{aligned}$$

which verifies that $D(\gamma_i\xi_0, r)$ belongs to the family $\mathcal{B}_R(\gamma_0, \varepsilon)$. The claim that $\xi \in D(\gamma_i\xi_0, r)$ can be shown in the same way as in the proof of [14, Lem. 10.12]. This proves (1).

(1) implies that for all sufficiently large i and $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$, we have

$$\beta_{\xi'}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1}g) \in \beta_{\gamma_i\xi_0}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1}g)(AM)_\varepsilon. \tag{7.5}$$

Now note that for all $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$,

$$\begin{aligned}
\beta_{\xi'}^{AM}(e, \gamma_i\gamma_0\gamma_i^{-1}) &= \beta_{\xi'}^{AM}(e, g)\beta_{\xi'}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1}g)\beta_{\xi'}^{AM}(\gamma_i\gamma_0\gamma_i^{-1}g, \gamma_i\gamma_0\gamma_i^{-1}) \\
&= \beta_{\xi'}^{AM}(e, g)\beta_{\xi'}^{AM}(g, \gamma_i\gamma_0\gamma_i^{-1}g)\beta_{\gamma_i\gamma_0^{-1}\gamma_i^{-1}\xi'}^{AM}(e, g)^{-1}. \qquad (7.6)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d_p(\gamma_i\gamma_0\gamma_i^{-1}\xi', \gamma_i\xi_0) &= e^{-\psi(\log\beta_{\xi'}^A(\gamma_i\gamma_0^{-1}\gamma_i^{-1}g, g) + \mathrm{i}\log\beta_{\gamma_i\xi_0}^A(\gamma_i\gamma_0^{-1}\gamma_i^{-1}g, g))}d_p(\xi', \gamma_i\xi_0) \\
&\le e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon}d_p(\xi', \gamma_i\xi_0),
\end{aligned}$$

and hence

$$\xi', \gamma_i\gamma_0\gamma_i^{-1}\xi' \in \mathbb{B}_p(\gamma_i\xi_0, e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon}r).$$

Since $\gamma_i\xi_0 \to \xi$ as $i \to \infty$ by Lemma 4.15 and $r < R_\varepsilon/2$, for all sufficiently large i and all $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$, ξ', $\gamma_i\gamma_0\gamma_i^{-1}\xi'$, and $\gamma_i\xi_0$ all belong to $\mathbb{B}_p(\xi, e^{\psi(\lambda(\gamma_0)+\lambda(\gamma_0^{-1}))+2\|\psi\|\varepsilon}R_\varepsilon)$ and hence

$$\beta^{AM}_{\xi'}(e,g), \beta^{AM}_{\gamma_i\gamma_0^{-1}\gamma_i^{-1}\xi'}(e,g), \beta^{AM}_{\gamma_i\xi_0}(e,g) \in \beta^{AM}_{\xi}(e,g)M_\varepsilon. \tag{7.7}$$

Combining (7.5), (7.6) and (7.7), it follows that for all $\xi' \in 3N_0 D(\gamma_i\xi_0, r)$,

$$\beta^{AM}_{\xi'}(e, \gamma_i\gamma_0\gamma_i^{-1}) \in \beta^{AM}_{\gamma_i\xi_0}(e, \gamma_i\gamma_0\gamma_i^{-1})(AM)_{O(\varepsilon)}.$$

Since $\beta^{M}_{\xi_0}(\gamma_i^{-1}, e) \to e$ as $i \to \infty$ and

$$\begin{aligned}\beta^{AM}_{\gamma_i\xi_0}(e, \gamma_i\gamma_0\gamma_i^{-1}) &= \beta^{AM}_{\gamma_i\xi_0}(e, \gamma_i)\beta^{AM}_{\gamma_i\xi_0}(\gamma_i, \gamma_i\gamma_0)\beta^{AM}_{\gamma_i\xi_0}(\gamma_i\gamma_0, \gamma_i\gamma_0\gamma_i^{-1}) \\ &= \beta^{M}_{\xi_0}(\gamma_i^{-1}, e)\hat\lambda(\gamma_0)\beta^{M}_{\xi_0}(\gamma_i^{-1}, e)^{-1},\end{aligned}$$

we obtain $\beta^{AM}_{\xi'}(e, \gamma_i\gamma_0\gamma_i^{-1}) \in \hat\lambda(\gamma_0)(AM)_{O(\varepsilon)}$, as desired. □

Lemma 7.12. *Let $B \subset \mathcal{F}$ be a Borel set with $\nu_\psi(B) > 0$. Then for ν_ψ-a.e. $\xi \in B$,*

$$\limsup_{R\to 0}\left\{ \frac{\nu_\psi(B\cap D)}{\nu_\psi(D)} : \begin{array}{c} \xi \in D = D(\gamma\xi_0, r), r < R, \text{ and} \\ \beta^{AM}_{\xi'}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon \\ \text{for all } \xi' \in 3N_0 D(\gamma\xi_0, r). \end{array} \right\} = 1.$$

Proof. To each Borel function $h : G/P \to \mathbb{R}$, we associate a function $h^* : G/P \to \mathbb{R}$ defined by

$$h^*(\xi) = \limsup_{R\to 0}\left\{ \frac{1}{\nu_\psi(D)}\int_D h\, d\nu_\psi : \begin{array}{c} \xi \in D = D(\gamma\xi_0, r), r < R, \text{ and} \\ \beta^{AM}_{\xi'}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon \\ \text{for all } \xi' \in 3N_0 D(\gamma\xi_0, r). \end{array} \right\}.$$

By Lemma 4.15 and 7.11, h^* is well defined on $\Lambda^{\spadesuit}_\psi - \{\eta\}$ and hence ν_ψ-a.e. on G/P by Corollary 4.10. We may then apply the same argument as in [14, Proof of Prop. 10.17] to deduce $h^* = h$ ν_ψ-a.e. Hence the lemma follows by taking $H = \mathbf{1}_B$. □

Proof of Proposition 7.8. Let $B \subset \mathcal{F}$ be a Borel set such that $\nu_\psi(B) > 0$ and let $\varepsilon > 0$ be arbitrary. By Lemma 7.12, for ν_ψ-a.e. $\xi \in B$, there exist $\gamma \in \Gamma^\star$ and $D = D(\gamma\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$ containing ξ such that

(1) $\nu_\psi(D\cap B) > (1 + e^{-\psi(\lambda(\gamma_0^{-1}))-\|\psi\|\varepsilon})^{-1}\nu_\psi(B)$, and
(2) $\beta^{AM}_{\xi'}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon$ for all $\xi' \in 3N_0 D(\gamma\xi_0, r)$.

We claim that

$$B \cap \gamma\gamma_0\gamma^{-1}B \cap \{\xi : \beta^{AM}_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon\} \tag{7.8}$$

has a positive ν_ψ-measure, which will finish the proof.

We have $\gamma\gamma_0\gamma^{-1}D \subset D$ by [14, Proof of Prop. 10.7]. Together with (2) above, it follows that

$$\beta^{AM}_{\xi}(e, \gamma\gamma_0\gamma^{-1}) \in \hat\lambda(\gamma_0)(AM)_\varepsilon \quad \text{for all } \xi \in \gamma\gamma_0\gamma^{-1}D.$$

Consequently, (7.8) contains

$$(D \cap B) \cap \gamma\gamma_0\gamma^{-1}(D \cap B), \tag{7.9}$$

which has a positive ν_ψ-measure by [14, Proof of Prop. 10.7]. This proves the claim. $\square$

Remark 7.13. We remark that the approach of this paper shows the following result when G has rank one.

Theorem 7.14. *Let G have rank one with M connected (e.g., $G = \mathrm{Isom}^+(X)$ of a rank one symmetric space X). Let $\Gamma < G$ be a Zariski dense discrete subgroup. Let ν_o be an ergodic Γ-conformal probability measure on the limit set of Γ. Let m^{BMS} and m^{BR} be respectively the* BMS *and* BR *measures on $\Gamma\backslash G$ associated to ν_o. Suppose that m^{BMS} is AM-ergodic. Then m^{BMS} is A-ergodic and m^{BR} is N-ergodic.*

In the rank one case, all the properties that we had to establish for Anosov groups hold automatically from the negative curvature property of the associated symmetric space. As Γ is Zariski dense and M is connected, Theorem 4.4 proves that m^{BMS} is A-ergodic. Then the Hopf ratio ergodic theorem for the one-parameter subgroup A implies that ν_o gives full measure on the set of strong Myrberg limit points of Γ, i.e., Corollary 4.11 holds. Now the arguments in section 7 shows that the set of ν_o-essential values is equal to AM, and hence m^{BR} is N-ergodic.

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