TEMPEREDNESS OF $L^2(\Gamma \backslash G)$ AND POSITIVE EIGENFUNCTIONS IN HIGHER RANK.

SAM EDWARDS AND HEE OH

Abstract. Let $G = SO^\circ(n, 1) \times SO^\circ(n, 1)$ and $X = \mathbb{H}^n \times \mathbb{H}^n$ for $n \geq 2$. For a pair $(\pi_1, \pi_2)$ of non-elementary convex cocompact representations of a finitely generated group $\Sigma$ into $SO^\circ(n, 1)$, let $\Gamma = (\pi_1 \times \pi_2)(\Sigma)$. Denoting the bottom of the $L^2$-spectrum of the negative Laplacian on $\Gamma \backslash X$ by $\lambda_0$, we show:

1. $L^2(\Gamma \backslash G)$ is tempered and $\lambda_0 = \frac{1}{2}(n-1)^2$;
2. There exists no positive Laplace eigenfunction in $L^2(\Gamma \backslash X)$.

In fact, analogues of (1)-(2) hold for any Anosov subgroup $\Gamma$ in the product of at least two simple algebraic groups of rank one as well as for Hitchin subgroups $\Gamma < \text{PSL}_d(\mathbb{R})$, $d \geq 3$. Moreover, if $G$ is a semisimple real algebraic group of rank at least 2 and has the trivial opposition involution, then (2) holds for any Anosov subgroup $\Gamma$ of $G$.

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1. Introduction

Motivation and background. Let $(\mathbb{H}^n, d)$, $n \geq 2$, denote the $n$-dimensional hyperbolic space of constant curvature $-1$, and let $G = \text{Isom}^+ (\mathbb{H}^n) \simeq \text{SO}^\circ(n, 1)$ denote the group of all orientation preserving isometries of $\mathbb{H}^n$. Let $\Gamma < G$ be a torsion-free$^1$ discrete subgroup. The critical exponent

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$^1$all discrete subgroups in this paper will be assumed to be torsion-free
$0 \leq \delta = \delta_\Gamma \leq n - 1$ is defined as the abscissa of convergence of the Poincaré series $\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}$ for $o \in \mathbb{H}^n$. We denote by $\Delta$ the hyperbolic Laplacian and by $\lambda_0 = \lambda_0(\Gamma \backslash \mathbb{H}^n)$ the bottom of the $L^2$-spectrum of the negative Laplace operator $-\Delta$, which is given as
\begin{equation}
\lambda_0 := \inf \left\{ \frac{\int_{\Gamma \backslash \mathbb{H}^n} \| \text{grad} f \|^2 \, d\text{vol}}{\int_{\Gamma \backslash \mathbb{H}^n} |f|^2 \, d\text{vol}} : f \in C_\infty^c(\Gamma \backslash \mathbb{H}^n) \right\}
\end{equation}
(1.1)
(see [45, Theorem 2.2]). In a series of papers, Elstrodt ([12], [13], [14]) and Patterson ([34], [35], [36]) developed the relationship between $\delta$ and $\lambda_0$, proving the following theorem for $n = 2$. The general case is due to Sullivan [45, Theorem 2.21].

**Theorem 1.1** (Generalized Elstrodt-Patterson I). For any discrete subgroup $\Gamma < SO^0(n, 1)$, the following are equivalent:

1. $\delta \leq \frac{1}{4}(n - 1)$;
2. $\lambda_0 = \frac{1}{4}(n - 1)^2$.

The right translation action of $G$ on the quotient space $\Gamma \backslash G$ equipped with a $G$-invariant measure gives rise to a unitary representation of $G$ on the Hilbert space $L^2(\Gamma \backslash G)$, called a quasi-regular representation of $G$. If we set $K \simeq SO(n)$ to be a maximal compact subgroup of $G$ and identify $\mathbb{H}^n$ with $G/K$, the space of $K$-invariant functions of $L^2(\Gamma \backslash G)$ can be identified with $L^2(\Gamma \backslash H^n)$. The bottom of the $L^2$-spectrum $\lambda_0$ then provides information on which complementary series representation of $G$ can occur in $L^2(\Gamma \backslash G)$. Indeed, it follows from the classification of the unitary dual of $SO^0(n, 1)$ that $\lambda_0 = (n - 1)^2/4$ is equivalent to saying that the quasi-regular representation $L^2(\Gamma \backslash G)$ does not contain any complementary series representation (cf. [45], [11]), which is again equivalent to the *temperedness* of $L^2(\Gamma \backslash G)$. As first introduced by Harish-Chandra [19], a unitary representation $(\pi, \mathcal{H}_{\pi})$ of a semisimple real algebraic group $G$ is tempered (Definition 2.6) if all of its matrix coefficients belong to $L^{2+\varepsilon}(G)$ for any $\varepsilon > 0$, or, equivalently, if $\pi$ is weakly contained\(^2\) in the regular representation $L^2(G)$ ([9], see Proposition 2.7).

Therefore Theorem 1.1 can be rephrased as follows:

**Theorem 1.2** (Generalized Elstrodt-Patterson II). For any discrete subgroup $\Gamma < G$, the following are equivalent:

1. $\delta \leq \frac{1}{4}(n - 1)$;
2. $L^2(\Gamma \backslash G)$ is tempered.

The size of the critical exponent $\delta$ is also related to the existence of square-integrable positive Laplace eigenfunction on $\Gamma \backslash \mathbb{H}^n$. A discrete subgroup $\Gamma < G$ is called convex cocompact if there exists a convex subspace of $\mathbb{H}^n$\(^2\) is weakly contained in a unitary representation $\sigma$ of $G$ if any diagonal matrix coefficients of $\pi$ can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of $\sigma$.\(^2\)
on which $\Gamma$ acts co-compactly. For convex cocompact subgroups of $G$ (more generally for geometrically finite subgroups), Patterson and Sullivan showed the following using their theory of conformal measures on the boundary $\partial \mathbb{H}^n$ ([37], [46], [45, Theorem 2.21]):

**Theorem 1.3** (Sullivan). For a convex cocompact subgroup $\Gamma < \text{SO}^0(n, 1)$, the following are equivalent:

1. $\delta \leq \frac{1}{2}(n - 1)$;
2. There exists no positive Laplace eigenfunction in $L^2(\Gamma \backslash \mathbb{H}^n)$.

Since $\lambda_0$ divides the positive spectrum and the $L^2$-spectrum on $\Gamma \backslash \mathbb{H}^n$ by Sullivan’s theorem [45, Theorem 2.1] (see Theorem 4.1), (2) is equivalent to saying that any $\lambda_0$-harmonic function (i.e., $-\Delta f = \lambda_0 f$) on $\Gamma \backslash \mathbb{H}^n$ is not square-integrable.

**Main results.** The main aim of this article is to discuss analogues of Theorems 1.1, 1.2, and 1.3 for a certain class of discrete subgroups of a connected semisimple real algebraic group of higher rank, i.e., rank at least 2.

We begin by describing a special case of our main theorem when $G = \text{SO}^0(n_1, 1) \times \text{SO}^0(n_2, 1)$ with $n_1, n_2 \geq 2$. Let $X$ be the Riemannian product $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ and $\Delta$ the Laplace-Beltrami operator on $X$. For a torsion-free discrete subgroup $\Gamma < G$, a smooth function $f$ on $\Gamma \backslash X$ is called $\lambda$-harmonic if $-\Delta f = \lambda f$. The number $\lambda_0 = \lambda_0(\Gamma \backslash X)$ is given in the same way as (1.1) replacing $\Gamma \backslash \mathbb{H}^n$ by $\Gamma \backslash X$.

**Theorem 1.4.** Let

$$\Gamma = (\pi_1 \times \pi_2)(\Sigma) = \{(\pi_1(\sigma), \pi_2(\sigma)) \in G : \sigma \in \Sigma\}$$

where $\pi_i : \Sigma \to \text{SO}^0(n_i, 1)$ is a non-elementary convex cocompact representation of a finitely generated group $\Sigma$ for $i = 1, 2$. Then

1. $L^2(\Gamma \backslash G)$ is tempered and $\lambda_0 = \frac{1}{3}((n_1 - 1)^2 + (n_2 - 1)^2)$;
2. There exists no positive Laplace eigenfunction in $L^2(\Gamma \backslash X)$, or equivalently, any $\lambda_0$-harmonic function is not square-integrable.

Even when $\Sigma$ is a surface group and $\pi_1, \pi_2$ are elements of the Teichmüller space $T(\Sigma)$, this theorem is new.

**Remark 1.5.** Theorem 1.4 does not hold for a general subgroup $\Gamma < G$ of infinite co-volume. For example, if $\Gamma < \text{SO}^0(n_1, 1) \times \text{SO}^0(n_2, 1)$ is the product of two convex cocompact subgroups, each of which having critical exponent greater than $\frac{1}{2}(n_i - 1)$, then $L^2(\Gamma \backslash G)$ is not tempered and $L^2(\Gamma \backslash X)$ possesses a positive Laplace eigenfunction.

We now discuss a general setting. Let $G$ be a connected semisimple real algebraic group and $X$ the associated Riemannian symmetric space.

In the rest of the introduction, we assume that $\Gamma < G$ is a torsion-free Zariski dense discrete subgroup. We let $\psi_{\Gamma} : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ denote the *growth indicator function* of $\Gamma$ as defined in (2.4), where $\mathfrak{a}$ is the Lie algebra
of a maximal real split torus of $G$. The function $\psi_T$ can be regarded as a higher rank generalization of the critical exponent of $\Gamma$. Let $\rho$ denote the half sum of all positive roots for $(g, a)$, counted with multiplicity. Analogous to the fact that the critical exponent $\delta$ is always bounded above by $n - 1$ for a discrete subgroup $\Gamma < SO^o(n, 1)$, we have the upper bound $\psi_T \leq 2\rho$ for any discrete subgroup $\Gamma$ of $G$ [41].

The following Theorem 1.6 generalizes Theorems 1.1, 1.2, and 1.3 to Anosov subgroups of a general semisimple real algebraic group. Anosov subgroups of $G$ (with respect to a minimal parabolic subgroup of $G$) may be regarded as higher rank generalizations of convex cocompact subgroups. They were first introduced by Labourie [28] for surface groups and then generalized by Guichard and Wienhard [18]. For $G = SO^o(n_1, 1) \times SO^o(n_2, 1)$, they are precisely given by the class of subgroups considered in Theorem 1.4. We refer to Definition 2.4 for a general case. The norm $\|\rho\|$ is defined via the identification $a^*$ and $a$ using the Killing form on $g$. Denote by $\sigma(\Gamma \setminus X)$ the $L^2$-spectrum of $-\Delta$ on $\Gamma \setminus X$.

**Theorem 1.6.** Let $G$ be a connected semisimple real algebraic group and $\Gamma$ a Zariski dense Anosov subgroup of $G$. The following (1)-(3) are equivalent, and imply (4):

1. $\psi_T \leq \rho$;
2. $L^2(\Gamma \setminus G)$ is tempered and $\lambda_0(\Gamma \setminus X) = \|\rho\|^2$;
3. $L^2(G)$ and $L^2(\Gamma \setminus G)$ are weakly contained in each other and $\sigma(\Gamma \setminus X) = \sigma(X) = [\|\rho\|^2, \infty)$;
4. There exists no positive Laplace eigenfunction in $L^2(\Gamma \setminus X)$.

The implication (1) $\Rightarrow$ (2) is based on the asymptotic behavior of the Haar matrix coefficients for compactly supported continuous functions for Anosov subgroups obtained in [10], using [7] as well as Harish-Chandra’s Plancherel formula (see Theorems 6.4 and 9.4). The implication (2) $\Rightarrow$ (1) is true for a general discrete subgroup (see the proof of Theorem 9.4). The equivalence (2) $\iff$ (3) uses the observation that $L^2(G)$ is weakly contained in $L^2(\Gamma \setminus G)$ whenever the injectivity radius of $\Gamma \setminus G$ is infinite, and $\Gamma \setminus G$ has infinite injectivity radius for any Anosov subgroup $\Gamma < G$, except for cocompact lattices of a rank one Lie group (see Section 8). In order to prove the implication (2) $\Rightarrow$ (4), we first prove that any positive Laplace eigenfunction in $L^2(\Gamma \setminus X)$ is indeed a joint eigenfunction for the whole ring of $G$-invariant differential operators, which then can be studied via $\Gamma$-conformal measures on the Furstenberg boundary of $G$ (see Sections 3 and 6). We extend Sullivan-Thurston’s smearing argument on the associated higher rank version of the Bowen-Margulis-Sullivan measure to deduce the non-existence of square-integrable positive Laplace eigenfunctions under the assumption that $\Gamma$ is Anosov and (2) (see Section 7 and Corollary 7.2).

Although the condition $\psi_T \leq \rho$ may appear quite strong, it was verified in a recent work of Kim-Minsky-Oh [24] for Anosov subgroups in the following setting, and hence we deduce from Theorem 1.6:
Theorem 1.7. Let $\Gamma$ be a Zariski dense Anosov subgroup of the product of at least two simple real algebraic groups of rank one, or a Zariski dense Anosov subgroup of a Hitchin subgroup of $\text{PSL}_d(\mathbb{R})$ for $d \geq 3$. Then (1)-(4) of Theorem 1.6 hold.

It is conjectured in [24] that any Anosov subgroup of a higher rank semisimple real algebraic group satisfies the condition $\psi_\Gamma \leq \rho$. This conjecture suggests that Anosov subgroups in higher rank groups are more like generalizations of convex cocompact subgroups of small critical exponent.

Groups with trivial opposition involution. The opposition involution $i : a \to a$ is defined by

$$i(u) = -\text{Ad}_{w_0}(u),$$

where $w_0$ is a Weyl element such that $\text{Ad}_{w_0}a^+ = -a^+$ for the positive Weyl chamber $a^+$. The opposition involution is non-trivial if and only if $G$ has a simple factor of type $A_n$ ($n \geq 2$), $D_{2n+1}$ ($n \geq 2$) and $E_6$ [49, 1.5.1]. See Corollary 7.3 for a more general version of the following:

Theorem 1.8. Let $G$ be a connected semisimple real algebraic group with rank $G \geq 2$ and trivial opposition involution. For any Zariski dense Anosov subgroup $\Gamma < G$, there exists no positive Laplace eigenfunction in $L^2(\Gamma \backslash X)$.

Groups of the second kind and positive joint eigenfunctions. Sullivan proved the existence of a positive $\lambda$-harmonic function for any $\lambda \leq \lambda_0$ for any discrete subgroup $\Gamma$ which is not cocompact in $G$. We prove a higher-rank strengthening of this result; more precisely, for any linear form $\psi \geq \psi_\Gamma$ we construct a positive joint eigenfunction with character corresponding to $\psi$, for any discrete subgroup of the second kind (see Definition 5.1) whose limit cone is contained in the interior of $a^+$ (see Theorem 5.2).

Organization: In section 2, we review the basic notions and notations which will be used throughout the paper.

In section 3, we show that any positive joint eigenfunction on $\Gamma \backslash X$ (i.e., an eigenfunction for the whole ring of $G$-invariant differential operators) arises from a $(\Gamma, \psi)$-conformal density (Proposition 3.7).

In section 4, we compute the Laplace eigenvalue of a positive joint eigenfunction associated to a $(\Gamma, \psi)$-conformal measure (Proposition 4.2).

In section 5, we introduce the notion of subgroups of the second kind. We then construct positive joint eigenfunctions for any $\psi \geq \psi_\Gamma$ for any subgroup of the second kind with its limit cone contained in $\text{int } a^+ \cup \{0\}$ (Theorem 5.2).

In section 6, we compute the $L^2$-spectrum of $X$ (Theorem 6.3) and show that $\lambda_0 = \|\rho\|^2$ if $L^2(\Gamma \backslash G)$ is tempered (Theorem 6.4). We show that a positive Laplace eigenfunction in $L^2(\Gamma \backslash X)$ is necessarily a joint eigenfunction (Corollary 6.6) and a spherical vector of a unique irreducible subrepresentation of $L^2(\Gamma \backslash G)$ (Theorem 6.8).
In section 7, we use Sullivan-Thurston’s smearing argument to obtain the non-existence theorem of $L^2$-positive Laplace eigenfunctions.

In section 8, we prove the weak containment $L^2(G) \propto L^2(\Gamma \backslash G)$ for all Anosov subgroups $\Gamma$ in higher rank groups.

In section 9, we prove the equivalence of the temperedness of $L^2(\Gamma \backslash G)$ and $\psi_\Gamma \leq \rho$ (Theorem 9.4). We also explain how to deduce Theorem 1.6.

Acknowledgements: We would like to thank Marc Burger for bringing the reference [45] to our attention. We would also like to thank Dick Canary, Francois Labourie, Curt McMullen and Dennis Sullivan for useful conversations.

2. Preliminaries and notations

Let $G$ be a connected semisimple real algebraic group, i.e., the identity component of the group of real points of a semisimple algebraic group defined over $\mathbb{R}$. Let $P$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition $P = MAN$ where $A$ is a maximal real split torus of $G$, $M$ is the compact subgroup, which is the centralizer of $A$, and $N$ is the unipotent radical of $P$. We denote by $g, a, n$ respectively the Lie algebras of $G, A, N$.

We fix a positive Weyl chamber $a^+ \subset a$ so that $n$ consists of positive root subspaces. Let $\Sigma^+$ denote the set of all positive roots for $(g, a^+)$. We also write $\Pi \subset \Sigma^+$ for the set of all simple roots. We denote by $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha$ the half sum of the positive roots for $(g, a^+)$, counted with multiplicity. We denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the inner product and norm on $g$ respectively, induced from the Killing form: $B(x, y) = \text{Tr}(\text{ad}x \text{ad}(y))$ for $x, y \in g$.

We fix a maximal compact subgroup $K$ of $G$ so that the Cartan decomposition $G = K(\exp a^+)K$ holds, that is, for any $g \in G$, there exists a unique element $\mu(g) \in a^+$ such that $g \in K \exp \mu(g)K$. We call the map $\mu : G \to a^+$ the Cartan projection map.

Let $w_0 \in K$ be an element of the normalizer of $A$ so that $\text{Ad}_{w_0} a^+ = -a^+$. The opposition involution $i : a \to a$ is defined by

$$i(u) = -\text{Ad}_{w_0}(u) \quad \text{for all } u \in a.$$  

(2.1)

The Riemannian symmetric space $(X, d)$ can be identified with the quotient space $G/K$ with the metric $d$ induced from $\|\cdot\|$. We denote by $d\text{vol}$ the Riemannian volume form on $X$. We also use $dx$ to denote this volume form as well as the Haar measure on $G$, or on $\Gamma \backslash G$. We set $o = [K] \in X$. We then have $\|\mu(g)\| = d(go, o)$ for $g \in G$. We do not distinguish a function on $X$ and a right $K$-invariant function on $G$. Let $F := G/P$ denote the Furstenberg boundary of $G$.

For each $g \in G$, we define the following visual maps:

$$g^+ := gP \in F \quad \text{and} \quad g^- := gw_0P \in F.$$  

(2.2)
The unique open $G$-orbit $\mathcal{F}^{(2)}$ in $\mathcal{F} \times \mathcal{F}$ under the diagonal $G$-action is given by:
$$\mathcal{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}.$$ Two points $\xi, \eta$ in $\mathcal{F}$ are said to be in general position if $(\xi, \eta) \in \mathcal{F}^{(2)}$.

**Conformal measures.** Let $G = KAN$ be the Iwasawa decomposition, $\kappa : G \to K$ the $K$-factor projection of this decomposition, and $H : G \to \mathfrak{a}$ be the Iwasawa cocycle defined by the relation:
$$g \in \kappa(g) \exp(H(g))N.$$ Note that $K$ acts transitively on $\mathcal{F}$ and $K \cap P = M$, and hence we may identify $\mathcal{F}$ with $K/M$. The Iwasawa decomposition can be used to describe both the action of $G$ on $\mathcal{F} = K/M$ and the $\mathfrak{a}$-valued Busemann map as follows: for all $g \in G$ and $[k] \in \mathcal{F}$ with $k \in K$,
$$g \cdot [k] = [\kappa(gk)],$$ and the $\mathfrak{a}$-valued Busemann map is defined by
$$\beta_{[k]}(g(o), h(o)) := H(g^{-1}k) - H(h^{-1}k) \in \mathfrak{a} \quad \text{for all } g, h \in G.$$

**Definition 2.1.** Let $\psi \in \mathfrak{a}^*$, and let $\Gamma < G$ be a closed subgroup.

1. A finite Borel measure $\nu$ on $\mathcal{F} = K/M$ is said to be a $(\Gamma, \psi)$-conformal measure (for the basepoint $o$) if for all $\gamma \in \Gamma$ and $\xi = [k] \in K/M$,
$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{-\psi(\beta_{[\gamma o]} \gamma o, o)} = e^{-\psi(H(\gamma^{-1}k))},$$ or equivalently
$$d\nu([k]) = e^{\psi(H(\gamma k))} d\nu(\gamma \cdot [k])$$ where $\gamma_* \nu(Q) = \nu(\gamma^{-1}Q)$ for any Borel subset $Q \subset \mathcal{F}$.

2. A collection $\{\nu_x : x \in X\}$ of finite Borel measures on $\mathcal{F}$ is called a $(\Gamma, \psi)$-conformal density if for all $x, y \in X$, $\xi \in \mathcal{F}$ and $\gamma \in \Gamma$,
$$\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\psi(\beta_{y}(x, y))} \quad \text{and} \quad d\gamma_* \nu_x = d\nu_y(\gamma). \quad (2.3)$$

A $(\Gamma, \psi)$-conformal measure $\nu$ defines a $(\Gamma, \psi)$-conformal density $\{\nu_x : x \in X\}$ by the formula:
$$d\nu_x(\xi) = e^{-\psi(\beta_{x}(x, o))} d\nu(\xi),$$ and conversely any $(\Gamma, \psi)$-conformal density $\{\nu_x\}$ is uniquely determined by its member $\nu_o$ by (2.3).

**Growth indicator function.** Let $\Gamma < G$ be a Zariski dense discrete subgroup. Following Quint [41], let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$: for any non-zero $v \in \mathfrak{a}$,
$$\psi_\Gamma(v) := \|v\| \inf_{\tau \in \mathcal{C}} \tau_c, \quad (2.4)$$
where the infimum is over all open cones \( C \) containing \( v \) and \( \tau_C \) denotes the abscissa of convergence of the series \( \sum_{\gamma \in \Gamma, \mu(\gamma) \in C} e^{-s\|\mu(\gamma)\|} \). For \( v = 0 \), we let \( \psi_T(0) = 0 \). We note that \( \psi_T \) does not change if we replace the norm \( \| \cdot \| \) by any other norm which is Weyl-group invariant. For any discrete group \( \Gamma \), we have the upper bound \( \psi_T \leq 2\rho \) [41]. On the other hand, when \( \Gamma \) is of infinite co-volume in a simple Lie group of rank at least 2, Quint deduced from [33] that \( \psi_T \leq 2(\rho - \eta_G) \), where \( 2\eta_G \) is the sum of the maximal strongly orthogonal subset of the root system of \( G \) [42].

**Limit cone and limit set.** The limit cone \( L = L_\Gamma \) of \( \Gamma \) is defined as the asymptotic cone of \( \mu(\Gamma) \), i.e.,
\[
L = \{ \lim t_i \mu(\gamma_i) : t_i \to 0, \gamma_i \in \Gamma \}.
\]
For \( \Gamma \) Zariski dense, \( L \) is a convex cone with non-empty interior [2]. Quint [41] showed that \( \psi_\Gamma \) is a concave and upper-semicontinuous function such that \( \psi_\Gamma \geq 0 \) on \( L \), \( \psi_\Gamma > 0 \) on \( \text{int} L \) and, and \( \psi_\Gamma = -\infty \) outside \( L \).

**Definition 2.2.** A sequence \( p_i \in X \) is said to converge to \( \xi \in \mathcal{F} \) if there exists \( g_i \to \infty \) regularly in \( G \) with \( p_i = g_i(o) \) and \( \lim_{i \to \infty} [\kappa_1(g_i)] = \xi \).

We denote by \( \Lambda \subset \mathcal{F} \) the limit set of \( \Gamma \), which is defined as
\[
\Lambda = \{ \lim \gamma_i(o) \in \mathcal{F} : \gamma_i \in \Gamma \}.
\] (2.5)
For \( \Gamma < G \) Zariski dense, this is the unique \( \Gamma \)-minimal subset of \( \mathcal{F} \) ([2], [31]).

**Tangent linear forms.** We set
\[
D_\Gamma = \{ \psi \in a^* : \psi \geq \psi_T \}.
\] (2.6)
A linear form \( \psi \in a^* \) is said to be tangent to \( \psi_T \) at \( u \in a \) if \( \psi \in D_\Gamma \) and \( \psi(u) = \psi_T(u) \). We denote by \( D_\Gamma^* \) the set of all linear forms tangent to \( \psi_T \) at \( L \cap \text{int} a^+ \), i.e.,
\[
D_\Gamma^* := \{ \psi \in D_\Gamma : \psi(u) = \psi_T(u) \text{ for some } u \in L \cap \text{int} a^+ \}.
\] (2.7)
For \( \Gamma < \text{SO}^\circ(n, 1) \) and \( \delta \) its critical exponent, we have \( D_\Gamma^* = \{ \delta \} \) and \( D_\Gamma = \{ s \geq \delta \} \).

Extending the construction of Patterson [37] and Sullivan [44], Quint [40] showed the following:

**Theorem 2.3.** For any \( \psi \in D_\Gamma^* \), there exists a \( (\Gamma, \psi) \)-conformal measure supported on \( \Lambda \).
Anosov subgroups.

**Definition 2.4.** ([18], [22], [17], [3]) A closed subgroup $\Gamma < G$ is called an Anosov subgroup (with respect to $P$) if $\Gamma$ can be realized as the image $\pi(\Sigma)$ of an Anosov representation $\pi : \Sigma \to G$ of a finitely generated group $\Sigma$.

For $\sigma \in \Sigma$, let $|\sigma|$ denote the word length of $\sigma$ for some fixed symmetric generating set of $\Sigma$. A representation $\pi : \Sigma \to G$ is Anosov with respect to $P$ if there exist constants $c_1, c_2 > 0$ such that for all $\sigma \in \Sigma$ and $\alpha \in \Pi$,

$$\alpha(\mu(\pi(\sigma))) \geq c_1|\sigma| - c_2. \quad (2.8)$$

Note that the discreteness of an Anosov subgroup $\Gamma$ is a direct consequence of the property (2.8). Moreover, if $\Gamma = \pi(\Sigma)$ is Anosov, then $\Sigma$ is a Gromov hyperbolic group ([22], [3]).

As mentioned in the introduction, Anosov subgroups of $G$ were first introduced by Labourie for surface groups [28], and then extended by Guichard and Wienhard [18] to general word hyperbolic groups. When $G$ has rank one, the class of Anosov subgroups coincides with that of convex cocompact subgroups, and when $G$ is a product of two rank one simple algebraic groups, any Anosov subgroup arises in a similar fashion to (1.2). Examples of Anosov subgroups include Schottky groups as well as Hitchin subgroups.

**Hitchin subgroups.** Let $\iota_d$ denote the irreducible representation $\text{PSL}_2(\mathbb{R}) \to \text{PSL}_d(\mathbb{R})$, which is unique up to conjugations. A Hitchin subgroup is the image of a representation $\pi : \Sigma \to \text{PSL}_d(\mathbb{R})$ of a uniform lattice $\Sigma < \text{PSL}_2(\mathbb{R})$, which belongs to the same connected component as $\iota_d|\Sigma$ in the character variety $\text{Hom}(\Sigma, \text{PSL}_d(\mathbb{R}))/\sim$ where the equivalence is given by conjugations.

One of the important features of an Anosov subgroup is the following:

**Theorem 2.5.** [39] For any Anosov subgroup $\Gamma < G$, we have

$$\mathcal{L} \subset \text{int } a^+ \cup \{0\}.$$

**Tempered representations.** By definition, a unitary representation of $G$ is a Hilbert space $\mathcal{H}_\pi$ equipped with a strongly continuous homomorphism $\pi$ from $G$ to the group of unitary operators on $\mathcal{H}_\pi$. Given two unitary representations $\pi$ and $\sigma$ of $G$, $\pi$ is said to be weakly contained in $\sigma$ if any diagonal matrix coefficients of $\pi$ can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of $\sigma$. We use the notation $\pi \preceq \sigma$ for the weak containment.

The Harish-Chandra function $\Xi_G : G \to (0, \infty)$ is a bi-$K$-invariant function defined via the formula

$$\Xi_G(g) = \int_K e^{-\rho(H(gk))}dk \quad \text{for all } g \in G$$

where $dk$ denotes the probability Haar measure on $K$. The following estimate is well-known, cf. e.g. [25]: for any $\varepsilon > 0$, there exist $C, C_\varepsilon > 0$ such that for any $g \in G$,

$$Ce^{-\rho(\mu(g))} \leq \Xi_G(g) \leq C_\varepsilon e^{-(1-\varepsilon)\rho(\mu(g))}. \quad (2.9)$$
**Definition 2.6.** A unitary representation \((\pi, \mathcal{H}_\pi)\) of \(G\) is called **tempered** if for any \(K\)-finite unit vectors \(v, w \in \mathcal{H}_\pi\) and any \(g \in G\),

\[
|\langle \pi(g)v, w \rangle| \leq \left( \dim \langle Kv \rangle \dim \langle Kw \rangle \right)^{1/2} \Xi_G(g),
\]

where \(\langle Kv \rangle\) denotes the linear subspace of \(\mathcal{H}_\pi\) spanned by \(Kv\).

**Proposition 2.7.** [9] The following are equivalent for a unitary representation \((\pi, \mathcal{H}_\pi)\) of \(G\):

1. \(\pi\) is tempered;
2. \(\pi \propto L^2(G)\);
3. for any vectors \(v, w \in \mathcal{H}_\pi\), the matrix coefficient \(g \mapsto \langle \pi(g)v, w \rangle\) lies in \(L^{2+\varepsilon}(G)\) for any \(\varepsilon > 0\);
4. for any \(\varepsilon > 0\), \(\pi\) is strongly \(L^{2+\varepsilon}\), i.e., there exists a dense subset of \(\mathcal{H}_\pi\) whose matrix coefficients all belong to \(L^{2+\varepsilon}(G)\).

In the whole paper, the notation \(f(v) \asymp g(v)\) means that the ratio \(f(v)/g(v)\) is bounded uniformly between two positive constants, and \(f \ll g\) means that \(|f| \leq c|g|\) for some \(c > 0\).

3. **Positive Joint Eigenfunctions and Conformal Densities**

Let \(G\) be a connected semisimple real algebraic group and \(\Gamma \subset G\) be a Zariski dense discrete subgroup. The main goal of this section is to obtain Proposition 3.7, which explains the relationship between positive joint eigenfunctions on \(\Gamma \backslash X\) and \(\Gamma\)-conformal measures on the Furstenberg boundary of \(G\).

**Joint invariant eigenfunctions on \(X\).** Let \(\mathcal{D} = \mathcal{D}(X)\) denote the ring of all \(G\)-invariant differential operators on \(X\). We call a real valued function on \(X\) a joint eigenfunction if it is an eigenfunction for all operators in \(\mathcal{D}\). For each joint eigenfunction \(f\), there exists an associated character \(\chi_f : \mathcal{D} \to \mathbb{R}\) such that

\[
\mathcal{D} f = \chi_f(D) f
\]

for all elements \(D \in \mathcal{D}\). The ring \(\mathcal{D}\) is generated by \(\text{rank}(G)\) elements, and the set of all characters of \(\mathcal{D}\) is in bijection with the space \(a^* = \text{Hom}_{\mathbb{R}}(a, \mathbb{R})\) modulo the action of the Weyl group, as we now explain. Denote by \(Z(\mathfrak{g}_\mathbb{C})\) the center of the universal enveloping algebra \(\mathcal{U}(\mathfrak{g}_\mathbb{C})\) of \(\mathfrak{g}_\mathbb{C}\). Recall the well-known fact that the joint eigenfunctions on \(X\) can be identified with the right \(K\)-invariant real-valued \(Z(\mathfrak{g}_\mathbb{C})\)-eigenfunctions on \(G\) (cf. [20]).

Letting \(T\) be a maximal torus in \(M\) with Lie algebra \(t\), set \(\mathfrak{h} = (a \oplus t)\). Then \(\mathfrak{h}_\mathbb{C} := (a \oplus t)_\mathbb{C}\) is a Cartan subalgebra of \(\mathfrak{g}_\mathbb{C}\). We let

\[
\iota : Z(\mathfrak{g}_\mathbb{C}) \to S^W(\mathfrak{h}_\mathbb{C})
\]

denote the Harish-Chandra isomorphism from \(Z(\mathfrak{g}_\mathbb{C})\) to the Weyl group-invariant elements of the symmetric algebra \(S(\mathfrak{h}_\mathbb{C})\) of \(\mathfrak{h}\) [25, Theorem 8.18].
For any $\psi \in \mathfrak{a}^*$, we can extend it to $\mathfrak{h}$ by letting $\psi(J) = 0$ for all $J \in \mathfrak{m}$, and then to $\mathcal{S}(\mathfrak{h}_C)$ polynomially. This lets us define a character $\chi_\psi$ on $\mathcal{Z}(\mathfrak{g}_C)$ by
\[
\chi_\psi(Z) := \psi(\iota(Z))
\]
for all $Z \in \mathcal{Z}(\mathfrak{g}_C)$. Conversely, if $f$ is a right $K$-invariant $\mathcal{Z}(\mathfrak{g}_C)$-eigenfunction, then, since $\mathfrak{t}$ acts trivially on $f$, the associated character $\chi_f$ must arise as $\psi \circ \iota$ for some $\psi \in \mathfrak{a}^*$.

**Example 3.1.**
- Consider the hyperbolic space $\mathbb{H}^n = \{(x_1, \cdots, x_n, y) \in \mathbb{R}^{n+1} : y > 0\}$ with the metric $\sqrt{\sum_{i=1}^{n} dx_i^2 + dy^2}$. The Laplacian $\Delta$ on $\mathbb{H}^n$ is $\Delta = -y^2 (\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2})$ and the ring of $\text{SO}^0(n,1)$-invariant differential operators is generated by $\Delta$, i.e., a polynomial in $\Delta$. If $\psi \in \mathfrak{a}^*$ is given by $\psi(v) = \delta v$ for some $\delta \in \mathbb{R}$ under the isomorphism $\mathfrak{a} = \mathbb{R}$, then $\chi_\psi(-\Delta) = 2(n - 1 - \delta)$.
- Let $G = \text{SO}^0(n_1,1) \times \text{SO}^0(n_2,1)$ and $X$ be the Riemannian product $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ for $n_1, n_2 \geq 2$. Then $\mathcal{D}(X)$ is generated by the hyperbolic Laplacians $\Delta_1, \Delta_2$ on each factor $\mathbb{H}^{n_1}$ and $\mathbb{H}^{n_2}$. If we identify $\mathfrak{a}$ with $\mathbb{R}^2$ and if a linear form $\psi \in \mathfrak{a}^*$ is given by $\psi(v) = \langle v, (\delta_1, \delta_2) \rangle$ for some vector $(\delta_1, \delta_2) \in \mathbb{R}^2$, then $\chi_\psi(-\Delta_i) = \delta_i(n_i - 1 - \delta_i)$ for $i = 1, 2$.

**Joint eigenfunctions on $\Gamma \backslash X$.** We now consider joint eigenfunctions on $\Gamma \backslash X$ or, equivalently, $\Gamma$-invariant joint eigenfunctions on $X$.

**Definition 3.2.** Let $\psi \in \mathfrak{a}^*$. Associated to a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, we define the following function $E_\nu$ on $G$: for $g \in G$,
\[
E_\nu(g) := |\nu_{g(o)}| = \int_{\mathcal{F}} e^{-\psi(H(g^{-1}k))} \, d\nu([k]).
\]
Since $|\nu_{\gamma(x)}| = |\nu_x|$ for all $\gamma \in \Gamma$ and $x \in X$, the left $\Gamma$-invariance and right $K$-invariance of $E_\nu$ are clear. Hence we may consider $E_\nu$ as a $K$-invariant function on $\Gamma \backslash G$, or, equivalently, as a function on $\Gamma \backslash X$.

**Proposition 3.3.** For each $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, $E_\nu$ is a positive joint eigenfunction on $\Gamma \backslash X$ with character $\chi_\psi - \rho$. Conversely, any positive joint eigenfunction on $\Gamma \backslash X$ arises in this way for some $\psi \geq \rho$ and a $(\Gamma, \psi)$-conformal measure $\nu$ with $(\psi, \nu)$ uniquely determined.

In order to prove this proposition, we consider the following right $K$-invariant function on $G$ for each $\psi \in \mathfrak{a}^*$ and $h \in G$:
\[
\varphi_{\psi,h}(g) = e^{-\psi(H(g^{-1}h))}
\]
so that
\[
E_\nu(g) = \int_{\mathcal{F}} \varphi_{\psi,h}(g) \, d\nu([k]).
\]

We may also consider $\varphi_{\psi,h}$ as a function on $X$. Hence the first part of Proposition 3.3 is a consequence of the following:
**Lemma 3.4.** ([25, Propositions 8.22 and 9.9]) For any \( \psi \in \mathfrak{a}^* \) and \( h \in G \), the function \( \varphi_{\psi,h} \) is a joint eigenfunction on \( X \) with character \( \chi_{\psi,-\rho} \).

**Proof.** While we refer to [25] for the full proof, we outline some of the key points below, as we will use some part of this proof later. Since the elements of \( Z(\mathfrak{g}_C) \) commute with translation, we simply need to prove that

\[
[Z\varphi_{\psi,e}](e) = \chi_{\psi,-\rho}(Z)\varphi_{\psi,e}(e) \quad \text{for any } Z \in Z(\mathfrak{g}_C);
\]

the same identity will then hold for the function \( g \mapsto \varphi_{\psi,e}(h^{-1}g) \), and thus also for \( \varphi_{\psi,h} \) for any \( h \in G \). Following [25, Chapter VII], we define the (non-unitary) principal series representation \( U^\psi \)

\[
[U^\psi(g)f](k) := e^{-\psi(H(g^{-1}k))} f(\kappa(g^{-1}k))
\]

for all \( g \in G \), \( k \in K \), and \( f \in C(K) \). This extends to a representation \( dU^\psi \) of \( \mathcal{U}(\mathfrak{g}_C) \) on the right \( M \)-invariant functions in \( C^\infty(K) \) by way of the formula

\[
[dU^\psi(X)f](k) = \frac{d}{dt} \bigg|_{t=0} [U^\psi(\exp(tX))f](k) \quad \text{for any } X \in \mathfrak{g}.
\]

Observe that \( [Z\varphi_{\psi,e}](e) = [dU^\psi(Z)1](e) \), so in order to prove the proposition, it suffices to show that \( dU^\psi(Z) = \chi_{\psi,-\rho}(Z) \) for all \( Z \in Z(\mathfrak{g}_C) \).

The next key observation is that

\[
Z(\mathfrak{g}_C) \subset \mathcal{U}(\mathfrak{h}_C) \oplus \mathcal{U}(\mathfrak{g}_C).
\]

We thus write

\[
Z = Y + \sum_i X_i U_i,
\]

where \( Y \in \mathcal{U}(\mathfrak{h}_C) \), \( X_i \in \mathfrak{n} \), and \( U_i \in \mathcal{U}(\mathfrak{g}_C) \). Note that in this decomposition, \( Y \) is uniquely defined. Now, for arbitrary \( X \in \mathfrak{n} \) and \( f \),

\[
[dU^\psi(X)f](e) = \frac{d}{dt} \bigg|_{t=0} [U^\psi(\exp(tX))f](e) = \frac{d}{dt} \bigg|_{t=0} [U^\psi(\exp(tX))f](e)
\]

\[
= \frac{d}{dt} \bigg|_{t=0} e^{-\psi(H(\exp(-tX))f)}(\kappa(\exp(-tX))) = \frac{d}{dt} \bigg|_{t=0} f(e) = 0,
\]

so applying this to the \( X_i \) and functions \( dU^\psi(U_i)f \) gives

\[
[dU^\psi(X_i U_i)f](e) = [dU^\psi(X_i)(dU^\psi(U_i)f)](e) = 0,
\]

hence \( [dU^\psi(Z)f](e) = [dU^\psi(Y)f](e) \). For \( L \in \mathfrak{m} \), we have \( f(\exp(-L)) = f(e) \), so \( [dU^\psi(J)f](e) = 0 \) for all \( J \in \mathfrak{t} \). Thus, it is only the \( \mathfrak{a} \) component of \( Y \) that contributes to \( [dU^\psi(Y)f](e) \). Finally, note that for \( X \in \mathfrak{a} \), we have

\[
[dU^\psi(X)f](e) = \frac{d}{dt} \bigg|_{t=0} e^{-\psi(H(\exp(-tX))f)}(\kappa(\exp(-tX)))
\]

\[
= \frac{d}{dt} \bigg|_{t=0} e^{t\psi(X)}f(e) = \psi(X)f(e).
\]
Since the Harish-Chandra isomorphism consists of projection onto $\mathcal{H}(h_C)$ and then composition with the “$\delta$-shift” $H \mapsto H + \delta(H)1 = H + \rho(H)1$, this shows that $dU\psi(Z) = \chi_{\psi-\rho}(Z)$. □

Letting $h = kan \in KAN$, we see that for any $g \in G$,
\[
\varphi_{\psi,h}(g) = e^{-\psi(H(g^{-1}h))} = e^{-\psi(H(g^{-1}kan))} = e^{-\psi(H(g^{-1}k))} \cdot e^{-\psi(\log(a))},
\]
i.e., the function $\varphi_{\psi,h}$ is a scalar multiple of $\varphi_{\psi,k(h)}$. In fact, the functions $\varphi_{\psi,k}, k \in K$ form a complete set of minimal positive joint eigenfunctions with character $\chi_{\psi-\rho}$ with $\psi \geq \rho$, in the sense that if $f$ is a positive joint eigenfunction on $X$ with character $\chi_{\psi-\rho}$ such that $f \leq \varphi_{\psi,k}$ for some $k \in K$, then
\[
f = c \cdot \varphi_{\psi,k}
\]
for some $c > 0$ (cf. [16, 23], see also [28, Theorem 1]).

As a consequence, we have the following (cf. [28, Theorem 3]):

**Theorem 3.5.** For any positive joint eigenfunction $f$ on $X$, there exist $\psi \in a^*$ with $\psi \geq \rho$ and a Borel measure $\nu$ on $F = K/M$ such that for all $g \in G$,
\[
f(g) = \int_{F} \varphi_{\psi,k}(g) \, d\nu([k]).
\]
Moreover, the pair $(\psi, \nu)$ is uniquely determined by $f$.

**Proof of the second part of Proposition 3.3:** Let $f$ be a $\Gamma$-invariant joint eigenfunction on $X$. By Theorem 3.5, there exist unique $\psi \in a^*$ and a Borel measure $\nu$ on $F$ so that for all $g \in G$,
\[
f(g) = \int_{F} \varphi_{\psi,k}(g) \, d\nu([k]).
\]
Since $f$ is $\Gamma$-invariant, for any $\gamma \in \Gamma$,
\[
f(g) = f(\gamma g) = \int_{F} \varphi_{\psi,k}(\gamma g) \, d\nu([k]) = \int_{F} \varphi_{\psi,\kappa(\gamma^{-1}k)}(g) e^{-\psi(H(\gamma^{-1}k))} \, d\nu([k]) = \int_{F} \varphi_{\psi,\tilde{k}}(g) e^{\psi(H(\gamma \tilde{k}))} \, d\nu(\gamma \cdot [\tilde{k}]).
\]
By the uniqueness of $\nu$ in the integral representation of $f$,
\[
d\nu([k]) = e^{\psi(H(\gamma \tilde{k}))} \, d\nu(\gamma \cdot [k]),
\]
i.e. $\nu$ is a $(\Gamma, \psi)$-conformal measure on $F$, finishing the proof.

We denote by $\psi_\Gamma : a \to \mathbb{R} \cup \{-\infty\}$ the growth indicator function of $\Gamma$ as defined in (2.4).

**Theorem 3.6.** [40, Theorem 8.1]. Let $\Gamma < G$ be Zariski dense. If there exists a $(\Gamma, \psi)$-conformal measure on $F$ for some $\psi \in a^*$, then $\psi \geq \psi_\Gamma$. 

Therefore Proposition 3.3 and Theorem 3.6 yield the following:

**Proposition 3.7.** Let $\Gamma < G$ be a Zariski dense discrete subgroup. If $\nu$ is a $(\Gamma, \psi)$-conformal measure on $\mathcal{F}$ for some $\psi \in a^*$, then $E_\nu$ is a positive joint eigenfunction on $\Gamma \backslash X$ with character $\chi_{\psi - \rho}$. Conversely, any positive joint eigenfunction on $\Gamma \backslash X$ is of the form $E_\nu$ for some $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$ with $\psi \geq \max(\rho, \psi_T)$, with $(\psi, \nu)$ uniquely determined.

### 4. Eigenvalues of positive eigenfunctions

Let $\Gamma$ be a torsion-free discrete subgroup of a connected semisimple real algebraic group $G$. Let $\Delta$ denote the Laplace-Beltrami operator on $X$ or on $\Gamma \backslash X$. Since $\Delta$ is an elliptic differential operator, an eigenfunction is always smooth. We call a smooth function $\lambda$-harmonic if

$$-\Delta f = \lambda f.$$  

Let $C \in Z(g_{\mathbb{C}})$ denote the Casimir operator on $C^\infty(G)$ (or on $C^\infty(\Gamma \backslash G)$) whose restriction to $K$-invariant functions coincides with $\Delta$. Then $K$-invariant $C$-eigenfunctions on $\Gamma \backslash G$ correspond to Laplace eigenfunctions on $\Gamma \backslash X$. In particular, a joint eigenfunction of $\Gamma \backslash X$ is a Laplace eigenfunction.

Define the real number $\lambda_0 = \lambda_0(\Gamma \backslash X) \in [0, \infty)$ as follows:

$$\lambda_0 := \inf \left\{ \frac{\int_{\Gamma \backslash X} \|\nabla f\|^2 \, d\text{vol}}{\int_{\Gamma \backslash X} |f|^2 \, d\text{vol}} : f \in C^\infty_c(\Gamma \backslash X), f \neq 0 \right\}$$  

(4.1)

where $d\text{vol}$ denotes the Riemannian volume form on $\Gamma \backslash X$.

**Positive Laplace eigenfunctions.**

**Theorem 4.1.** [45, Theorem 2.1, 2.2] Suppose that $\Gamma \backslash X$ is not compact.

1. For any $\lambda \leq \lambda_0$, there exists a positive $\lambda$-harmonic function on $\Gamma \backslash X$;
2. For any $\lambda > \lambda_0$, there is no positive $\lambda$-harmonic function on $\Gamma \backslash X$.

We identify $a^*$ with $a$ via the inner product on $a$ induced by the Killing form on $g$. This endows an inner product on $a^*$. More precisely, for each $\psi \in a^*$, there exist a unique $v_\psi \in a$ such that $\psi = \langle v_\psi, \cdot \rangle$. Then $\langle \psi_1, \psi_2 \rangle = \langle v_{\psi_1}, v_{\psi_2} \rangle$. Equivalently, fixing an orthonormal basis $\{H_i\}$ of $a$, we have $\langle \psi_1, \psi_2 \rangle = \sum_i \psi_1(H_i) \psi_2(H_i)$.

For $\psi \in a^*$, we set

$$\lambda_\psi := (\|\rho\|^2 - \|\psi - \rho\|^2).$$  

(4.2)

**Proposition 4.2.**

1. A positive joint eigenfunction on $X$ with character $\chi_{\psi - \rho}$, $\psi \in a^*$, is $\lambda_\psi$-harmonic.
2. A positive Laplace eigenfunction on $X$ is $\lambda_\psi$-harmonic for some $\psi \in a^*$ with $\psi \geq \rho$.

**Proof.** Let $\psi \in a^*$. Recall the functions $\varphi_{\psi,h}$ in (3.3). By Theorem 3.5, (1) follows if we show that for any $h \in G$,

$$-C \varphi_{\psi,h} = \lambda_\psi \varphi_{\psi,h}.$$  

(4.3)
Let \( \{H_i\} \) be an orthonormal basis of \( \mathfrak{a} \). To each \( \alpha \in \Sigma \) corresponds \( H_\alpha \in \mathfrak{a} \) with \( \alpha(x) = B(x, H_\alpha) = \langle x, H_\alpha \rangle \) for all \( x \in \mathfrak{a} \). For each \( \alpha \in \Sigma \), choose a unit root vector \( E_\alpha \in \mathfrak{n} \) so that \( [x, E_\alpha] = \alpha(x)E_\alpha \) for all \( x \in \mathfrak{a} \). We may write

\[
C = \sum_i H_i^2 + \sum_{\alpha \in \Sigma^+} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) + J,
\]

where \( J \in \mathcal{U}(m_C) \) (cf. [26, Proposition 5.28]). Now using \( E_{-\alpha} E_\alpha = E_\alpha E_{-\alpha} - H_\alpha \) gives

\[
C = \sum_i H_i^2 - \sum_{\alpha \in \Sigma^+} H_\alpha + \sum_{\alpha \in \Sigma^+} 2E_\alpha E_{-\alpha} + J.
\]

As in the proof of Lemma 3.4, \( [J \varphi, \psi, h](e) = 0 \), and \( [E_\alpha E_{-\alpha} \varphi, \psi, h](e) = 0 \).

Applying \(-C\) to \( \varphi, \psi, h \) gives

\[
-C \varphi, \psi, h = \left( \sum_i \psi(H_i)^2 - \sum_{\alpha \in \Sigma^+} \psi(H_\alpha) \right) \varphi, \psi, h
\]

\[
= \left( \|\psi\|^2 - 2\langle \rho, \psi \rangle \right) \varphi, \psi, h
\]

\[
= \left( \|\rho\|^2 - \|\psi - \rho\| \right) \varphi, \psi, h,
\]

proving (4.3). Let \( f \) be a positive \( \lambda \)-harmonic function on \( X \), which we consider as a \( K \)-invariant function on \( G \). By [28, Theorem 2], for any \( g \in G \),

\[
f(g) = \int_{\{\psi \geq \rho : \lambda_\psi = \lambda\} \times K/M} \varphi, \psi, k(g) \, d\mu([k], \psi)
\]

for some Borel measure \( \mu \) on \( \{\psi \geq \rho : \lambda_\psi = \lambda\} \times K/M \). By (4.3), this implies (2).

**Corollary 4.3.** For any Zariski dense discrete subgroup \( \Gamma \subset G \),

\[
\sup\{\lambda_\psi : \psi \in D^*_\Gamma\} \leq \lambda_0.
\]

**Proof.** If \( \Gamma \) is cocompact in \( G \), then \( \psi_\Gamma = 2\rho \) and hence \( D^*_\Gamma = \{2\rho\} \). Since \( \lambda_0 = 0 = \lambda_{2\rho} \), the claim follows. In general, it follows from Theorem 2.3 and Proposition 3.7 that for any \( \psi \in D^*_\Gamma \), there exists a positive joint eigenfunction on \( \Gamma \setminus X \) with character \( \chi_{\psi - \rho} \). Hence the claim follows from Theorem 4.1 and Proposition 4.2. \( \square \)

5. **Groups of the second kind and positive joint eigenfunctions**

When \( G \) has rank one in which case the Furstenberg boundary is same as the geometric boundary of \( X \), a discrete subgroup \( \Gamma \subset G \) is said to be of the second kind if \( \Lambda \neq \mathcal{F} \). We extend this definition to higher rank groups as follows:

**Definition 5.1.** A discrete subgroup \( \Gamma \subset G \) is of *the second kind* if there exists \( \xi \in \mathcal{F} \) which is in general position with all points of \( \Lambda \), i.e., \( (\xi, \Lambda) \subset \mathcal{F}(2) \).
Sullivan’s theorem 4.1 provides a positive λ-harmonic function for any \( \lambda \leq \lambda_0 \). The following theorem can be viewed as a higher rank strengthening of this result. The second-kind hypothesis may be interpreted as an analogue of the hypothesis of Theorem 4.1 that \( \Gamma \backslash X \) is non-compact.

**Theorem 5.2.** Let \( \Gamma < G \) be of the second kind with \( \mathcal{L} \subset \text{int} \, a^+ \cup \{0\} \). For any \( \psi \in D_\Gamma \), there exists a positive joint eigenfunction on \( \Gamma \backslash X \) with character \( \chi_{\psi - \rho} \).

By Proposition 3.7, we get the following immediate corollary:

**Corollary 5.3.** Let \( \Gamma < G \) be as above. Then for any \( \psi \geq \max(\psi_\Gamma, \rho) \), there exists a \((\Gamma, \psi)\)-conformal measure on \( \mathcal{F} \).

**Remark 5.4.**

1. Let \( \Gamma_0 < G \) be an Anosov subgroup. Then for any Anosov subgroup \( \Gamma < \Gamma_0 \) with some point \( \xi \in \Lambda_{\Gamma_0} - \Lambda_\Gamma \), \( (\Lambda_\Gamma, \xi) \in \mathcal{F}^{(2)} \), since any two distinct points of \( \Lambda_{\Gamma_0} \) are in general position by the Anosov assumption on \( \Gamma_0 \). Hence \( \Gamma \) is of the second kind.

2. If \( \Lambda \subset gNw_0P \) for some \( g \in G \), then \( (\Lambda, g^+) \subset \mathcal{F}^{(2)} \). One can construct many Schottky groups with \( \Lambda \subset Nw_0P \), which would then be of the second kind.

3. Let \( G = \prod_{i=1}^k G_i \) be a product of simple algebraic groups \( G_i \) of rank one. Then \( \mathcal{F} = \prod_i \mathcal{F}_i \) where \( \mathcal{F}_i = G_i / P_i \), and \( (\xi_i)_i, (\eta_i)_i \in \mathcal{F} \) are in general position if and only if \( \xi_i \neq \eta_i \) for all \( i \). Therefore if there exists \( \xi_i \notin \pi_i(\Lambda) \) where \( \pi_i : \mathcal{F} \rightarrow \mathcal{F}_i \) is the canonical projection, then for \( \xi = (\xi_i)_i \), \( (\Lambda, \xi) \in \mathcal{F}^{(2)} \). Therefore any closed subgroup \( \Gamma < G \) with \( \pi_i(\Lambda) \neq \mathcal{F}_i \) for all \( i \) is of the second kind.

4. The well-known properties of the limit set of a Hitchin subgroup of \( \text{PSL}_d(\mathbb{R}) \) imply that Hitchin groups are not of the second kind for any even \( d \) or \( d = 3 \); we thank Canary andLabourie for communicating this with us.

We will use shadow lemma to prove Theorem 5.2. For \( q \in X \) and \( r > 0 \), we set \( B(q, r) = \{ x \in X : d(x, q) < r \} \). For \( p = g(o) \in X \), the shadow of the ball \( B(q, r) \) viewed from \( p \) is defined as

\[
O_r(p, q) := \{(gk)^+ \in \mathcal{F} : k \in K, \, gk \text{ int } A^+ o \cap B(q, r) \neq \emptyset \}.
\]

Similarly, for \( \xi \in \mathcal{F} \), the shadow the ball \( B(q, r) \) viewed from \( \xi \) is defined by

\[
O_r(\xi, q) := \{h^+ \in \mathcal{F} : h \in G \text{ satisfies } h^- = \xi, \, ho \in B(q, r) \}.
\]

We will use the following:

**Lemma 5.5.** [31, Lemma 5.6, 5.7]

1. If a sequence \( q_i \in X \) converges to \( \eta \in \mathcal{F} \), then for any \( q \in X \), \( r > 0 \) and \( \varepsilon > 0 \),

\[
O_{r-\varepsilon}(q_i, q) \subset O_r(\eta, q) \subset O_{r+\varepsilon}(q_i, q)
\]

for all sufficiently large \( i \).
(2) There exists $\kappa > 0$ such that for any $g \in G$ and $r > 0$,
\[
\sup_{\xi \in \Omega_r(g(o), o)} \|\beta_{\xi}(g(o), o) - \mu(g^{-1})\| \leq \kappa r.
\]

**Lemma 5.6.** If $\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$, then the union $\Gamma(o) \cup \Lambda$ is compact in the topology given in Definition 2.2.

**Proof.** The hypothesis implies that any sequence $\gamma_i \to \infty$ in $\Gamma$ tends to $\infty$ regularly, and hence has a limit in $\mathcal{F}$. Moreover the limit belongs to $\Lambda$ by its definition. \hfill $\square$

**Lemma 5.7.** Suppose that $\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$. If $\xi \in \mathcal{F}$ satisfies that $(\xi, \Lambda) \subset \mathcal{F}^{(2)}$, then there exists $R > 0$ such that
\[
\xi \in \bigcap_{\gamma \in \Gamma} O_R(\gamma(o), o).
\]

**Proof.** We first claim that $\xi \in \bigcap_{\eta \in \Lambda} O_R(\eta, o)$ for some $R > 0$. Note that $\lim_{R \to \infty} O_R(\eta, o) = \{z \in \mathcal{F} : (z, \eta) \in \mathcal{F}^{(2)}\}$. Hence for each $\eta \in \Lambda$, we have $R_\eta = \inf\{R + 1 : \xi \in O_R(\eta, o)\} < \infty$.

It suffices to show that $R := \sup_{\eta \in \Lambda} R_\eta < \infty$. Suppose not; then $R_{\eta_i} \to \infty$ for some sequence $\eta_i \in \Lambda$. By passing to a subsequence, we have $\eta_i$ converges to some $\eta$. This follows that $O_{R_{\eta_i} + 1}(\eta_i, o) \subset O_{R_{\eta_i} + 2}(\eta_i, o)$ for all sufficiently large $i$. Therefore $R_{\eta_i} \leq R_{\eta} + 3$, yielding a contradiction.

We now claim that $\xi \in \bigcap_{\gamma \in \Gamma} O_{R'}(\gamma(o), o)$ for some $R' > 0$. Suppose not; then there exist sequences $\gamma_i \to \infty$ in $\Gamma$ and $R_i \to \infty$ such that $\xi \notin O_{R_i}(\gamma_i o, o)$. By Lemma 5.6, by passing to a subsequence, we may assume that $\gamma_i(o)$ converges to some $\eta \in \Lambda$. By the first claim, we have $\xi \in O_R(\eta, o)$. By Lemma 5.5, we have $\xi \in O_{R}(\eta, o) \subset O_{R+1}(\gamma_i(o), o)$ for all sufficiently large $i$. This is a contradiction since for $i$ large enough so that $R_i > R + 1$, we have $\xi \notin O_{R+1}(\gamma_i(o), o)$. This proves the claim. \hfill $\square$

As an immediate corollary of Lemmas 5.5 and 5.7, we obtain:

**Corollary 5.8.** If $\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$ and $\xi \in \mathcal{F}$ satisfies that $(\xi, \Lambda) \subset \mathcal{F}^{(2)}$,
\[
\sup_{\gamma \in \Gamma} \|\beta_{\xi}(\gamma^{-1}o, o) - \mu(\gamma)\| < \infty.
\]

**Proof of Theorem 5.2:** If $\psi \in D^*_\Gamma$, this follows Theorem 2.3. Hence we assume $\psi \in D_\Gamma - D^*_\Gamma$; this implies that
\[
\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty
\]
by [41, Lem. III. 1.3]. As $\Gamma$ is of the second kind, there exists $\xi \in \mathcal{F}$ such that $(\xi, \eta) \in \mathcal{F}^{(2)}$ for all $\eta \in \Lambda$. By Corollary 5.8, $\|\beta_{\xi}(\gamma^{-1}o, o) - \mu(\gamma)\|$ is bounded uniformly for all $\gamma \in \Gamma$. Therefore (5.1) implies that
\[
\sum_{\gamma \in \Gamma} e^{-\psi(\beta_{\xi}(\gamma^{-1}o, o))} < \infty.
\]
For any fixed $x \in X$, we have $\beta_\xi(\gamma^{-1}x, o) = \beta_\xi(\gamma^{-1}, o, o) + \beta_\xi(x, o)$ and $\|\beta_\xi(x, o)\| \leq d(x, o)$. Hence $e^{-\psi(\beta_\xi(\gamma^{-1}o, o))} \simeq e^{-\psi(\mu(\gamma))}$ with implied constant uniform for all $\gamma \in \Gamma$.

Therefore, by (5.1) the following function $F_\psi = F_{\psi, \xi}$ on $X$ is well-defined: for $x \in X$,

$$F_\psi(x) := \sum_{\gamma \in \Gamma} e^{-\psi(\beta_\xi(\gamma^{-1}x, o))}. \tag{5.3}$$

If we write $\xi = [k_0] \in K/M = F$, then for any $g \in G$,

$$\beta_\xi(\gamma^{-1}g_0, o) = \beta_M(k_0^{-1}\gamma^{-1}g_0, o) = H(g^{-1}\gamma k_0)$$

and hence $e^{-\psi(\beta_\xi(\gamma^{-1}g_0, o))} = \varphi_{\psi, \gamma k_0}(g)$. Therefore $F_\psi = \sum_{\gamma \in \Gamma} \varphi_{\psi, \gamma k_0}$. It now follows from Lemma 2.2 that $F_\psi$ is a positive $\Gamma$-invariant joint eigenfunction on $X$ with eigenvalue $\chi_{\psi - \rho}$. This finishes the proof.

**Remark 5.9.** For $\psi \in D_T - D_T^c$, we have constructed positive joint eigenfunction $F_{\psi, \xi}$ on $\Gamma \backslash X$ of eigenvalue $\chi_{\psi - \rho}$ for any $\xi \in F$ with $(\Lambda, \xi) \in F^{(2)}$.

Hence we get the strengthened version of Corollary 4.3:

**Corollary 5.10.** For $\Gamma$ as in Theorem 5.2, we have

$$\sup\{\lambda_{\psi} : \psi \in D_\Gamma\} \leq \lambda_0. \tag{5.4}$$

**Example 5.11.** Let $\Gamma < \text{SO}^o(n, 1)$ be a discrete subgroup with $\Lambda \neq \partial \mathbb{H}^n$; then $\Gamma$ satisfies the hypothesis of Proposition 5.2. Since $\rho = \frac{(n-1)}{2}$ and $D_\Gamma = \{s \geq \delta\}$, we have

$$\sup\{\|\rho\|^2 - \|\psi - \rho\|^2 : \psi \in D_\Gamma\} = \begin{cases} \delta(n - 1 - \delta) & \text{if } \delta \geq \frac{n-1}{2} \\ (n-1)^2/4 & \text{if } \delta \leq \frac{n-1}{2}. \end{cases} \tag{5.5}$$

It then follows from Proposition 3.7 and Theorem 4.1 that we have equality in (5.4) in this case, as was proved by Sullivan [45, Theorem 2.17].

### 6. The $L^2$-spectrum and uniqueness

Let $\Gamma$ be a torsion-free discrete subgroup of a connected semisimple real algebraic group $G$. The space $L^2(\Gamma \backslash X)$ consists of square-integrable functions together with the inner product $\langle f_1, f_2 \rangle = \int_{\Gamma \backslash X} f_1 f_2 \, d\text{vol}$.

Let $W^1(\Gamma \backslash X) \subset L^2(\Gamma \backslash X)$ denote the closure of $C^\infty_c(\Gamma \backslash X)$ with respect to the norm $\|\cdot\|_{W^1}$ induced by the inner product

$$\langle f_1, f_2 \rangle_{W^1} := \int_{\Gamma \backslash X} f_1 f_2 \, d\text{vol} + \int_{\Gamma \backslash X} \langle \text{grad} f_1, \text{grad} f_2 \rangle \, d\text{vol}$$

for any $f_1, f_2 \in W^1(\Gamma \backslash X)$.

As $\Gamma \backslash X$ is complete, there exists a unique self-adjoint operator on the space $W^1(\Gamma \backslash X)$ extending the Laplacian $\Delta$ on $C^\infty_c(\Gamma \backslash X)$, which we also denote by $\Delta$. The $L^2$-spectrum of $-\Delta$, which we denote by

$$\sigma(\Gamma \backslash X),$$
is the set of all $\lambda \in \mathbb{C}$ such that $\Delta + \lambda$ does not have a bounded inverse $(\Delta + \lambda)^{-1} : L^2(\Gamma \setminus X) \to W^1(\Gamma \setminus X)$. The self-adjointness of $\Delta$ and the fact that $\langle -\Delta f, f \rangle = \int_X \|\text{grad } f\|^2 d\text{vol}$ for all $f \in C^\infty_c(\Gamma \setminus X)$ imply $\sigma(\Gamma \setminus X) \subset [0, \infty)$.

We will be using Weyl’s criterion (cf. [48, Lemma 2.17]) to determine $\sigma(\Gamma \setminus X)$:

**Theorem 6.1.** For $\lambda \in \mathbb{R}$, we have $\lambda \in \sigma(\Gamma \setminus X)$ if and only if there exists a sequence of unit vectors $F_n \in W^1(\Gamma \setminus X)$ such that $\|(\Delta + \lambda)F_n\| \to 0$.

The number $\lambda_0 = \lambda_0(\Gamma \setminus X)$ defined in (4.1) is the bottom of the $L^2$-spectrum $\sigma(\Gamma \setminus X)$:

**Theorem 6.2.** [45, Theorem 2.1, 2.2] We have $\lambda_0 \in \sigma(\Gamma \setminus X) \subset [\lambda_0, \infty)$.

Using Harish-Chandra’s Plancherel formula, we can identify $\lambda_0(X)$ and $\sigma(X)$ for the symmetric space $X = G/K$:

**Proposition 6.3.** We have $\lambda_0(X) = \|\rho\|^2$. Moreover, $\sigma(X) = [\|\rho\|^2, \infty)$.

**Proof.** It is shown in [23] that there are no positive Laplace eigenfunctions on $X$ with eigenvalue strictly bigger than $\|\rho\|^2$; hence the inequality $\lambda_0(X) \leq \|\rho\|^2$ follows from Theorem 4.1 for $\Gamma = \{e\}$. On the other hand, as seen in the proof of (1), $\varphi_{\rho,h}$ is a positive $\|\rho\|^2$-harmonic function (for any $h \in G$), hence $\lambda_0(X) = \|\rho\|^2$ by Theorem 4.1. We now deduce the second claim $\sigma(X) = [\|\rho\|^2, \infty)$ from Harish-Chandra’s Plancherel theorem (cf. e.g. [43]).

For $\psi \in a^*$, define $\Phi_\psi \in C^\infty(K \setminus G/K)$ by

$$\Phi_\psi(g) = \int_K \varphi_{\rho + i\psi,k}(g) dk,$$

where $\varphi_{\rho + i\psi,k}(g) = e^{-(\rho + i\psi)(H(g^{-1}k))}$.

Then by the same computation as (4.3), we have

$$-C\Phi_\psi = -\Delta \Phi_\psi = (\|\rho\|^2 + \|\psi\|^2)\Phi_\psi.$$

Given any $f \in C^\infty_c(a^*)$, we can define a function $F \in L^2(X)$ by the formula

$$F(g) = \int_{a^*} f(\psi)\Phi_\psi(g) \frac{d\psi}{|c(\psi)|^2};$$

here $d\psi$ denotes the Lebesgue measure on $a^*$, $c(\psi)$ denotes the Harish-Chandra c-function. The Plancherel formula says

$$\|F\|_{L^2(X)}^2 = \int_{a^*} |f(\psi)|^2 \frac{d\psi}{|c(\psi)|^2}.$$
(see [43]). Let $\lambda \in [||\rho||^2, \infty)$ be any number. Choose $\psi_0 \in \mathfrak{a}^*$ so that $\lambda = ||\rho||^2 + ||\psi_0||^2$. We then choose a sequence of non-negative functions $\{f_n\} \subset C_c^\infty(\mathfrak{a}^*)$ with supp $f_n \subset B_{1/n}(\psi_0)$ and $||F_n||_{L^2(X)} = 1$.

Then

$$
(\Delta + \lambda)F_n = \int_{\mathfrak{a}^*} f_n(\psi)(\Delta + \lambda)\Phi_\psi(g) \frac{d\psi}{|c(\psi)|^2} = \int_{\mathfrak{a}^*} f_n(\psi)(\lambda - ||\rho||^2 - ||\psi||^2)\Phi_\psi(g) \frac{d\psi}{|c(\psi)|^2}.
$$

This gives

$$
|| (\Delta + \lambda)F_n ||_{L^2(X)}^2 = \max_{\psi \in B_{1/n}(\psi_0)} ||\psi_0||^2 - ||\psi||^2)^2.
$$

Consequently,

$$
\lim_{n \to \infty} || (\Delta + \lambda)F_n ||_{L^2(X)} = 0.
$$

By Weyl’s criterion (Theorem 6.1), this implies that $\lambda \in \sigma(X)$. This proves the claim.

**Theorem 6.4.** If $L^2(\Gamma \backslash G)$ is tempered, then

$$
\lambda_0(\Gamma \backslash X) = ||\rho||^2.
$$

**Proof.** Note that $\lambda_0 = \lambda_0(\Gamma \backslash X) - \lambda_0(X) = ||\rho||^2$ by Proposition 6.3. Assume that $\lambda_0 < ||\rho||^2$. By Theorem 6.1, we can then find a $K$-invariant unit vector $f \in L^2(\Gamma \backslash G)_K$ such that

$$
|| (\Delta - \lambda_0) f || < \frac{||\rho||^2 - \lambda_0}{2}.
$$

This gives

$$
|| \mathcal{C} f || = || \Delta f || \leq || (\Delta - \lambda_0) f || + \lambda_0 < \frac{||\rho||^2 + \lambda_0}{2} < ||\rho||^2.
$$

On the other hand, consider the direct integral representation of $L^2(\Gamma \backslash G) = \int_{\mathcal{Z}} (\pi_\zeta, \mathcal{H}_\zeta) d\mu(\zeta)$ into irreducible unitary representations of $G$ which are tempered, by the hypothesis on the temperedness of $L^2(\Gamma \backslash G)$. Hence

$$
|| \mathcal{C} f ||^2 = \int_{\mathcal{Z}} ||d\pi_\zeta(C)f_\zeta||^2 d\mu(\zeta) \geq \left( \min_{\sigma \text{ spherical tempered}} |d\pi(\mathcal{C})|^2 \right),
$$

where $d\pi$ denotes the derived representation of $\mathcal{U}(\mathfrak{g}_C)$ induced by $\pi$. By Schur’s lemma, there exists a character $\chi_\pi$ of $\mathcal{Z}(\mathfrak{g}_C)$ such that $d\pi(Z) = \chi_\pi(Z)$ for all $Z \in \mathcal{Z}(\mathfrak{g}_C)$. Moreover, for any spherical $\pi$, there exists $\psi_\pi \in \mathfrak{a}_C^*$ such that $\chi_\pi = \chi_{\psi_\pi}$ (cf. (3.1)). Now, by Harish-Chandra’s Plancherel formula (cf. e.g. [21]), for any tempered spherical representation, we have

$$
\psi_\pi = \rho + i \text{Im}(\psi_\pi),
$$

where $\rho = \text{Re}(\pi)$, $\psi_\pi \in \mathfrak{a}_C^*$, and $\text{Im}(\psi_\pi)$ is the imaginary part of $\psi_\pi$. Therefore,

$$
|| \mathcal{C} f ||^2 \geq \left( \min_{\sigma \text{ spherical tempered}} |d\pi(\mathcal{C})|^2 \right).$$

By Proposition 6.3 and the previous inequality, we have

$$
||\mathcal{C} f ||^2 < ||\rho||^2,
$$

and hence

$$
|| \mathcal{C} f || < ||\rho||. $$

By Weyl’s criterion (Theorem 6.1), this implies that $\lambda_0 \in \sigma(X)$. This proves the claim.
Thus for any spherical tempered representation \((\pi, C)\), we have \(d\pi(C) \in \sigma(X)\) and hence, by Proposition 6.3,
\[
\min_{\pi \text{ spherical tempered}} |d\pi(C)| \geq \|\rho\|^2,
\]
giving a contradiction. \(\square\)

**Theorem 6.5.** [45, Theorem 2.8 and Corollary 2.9]

1. Any positive Laplace eigenfunction in \(L^2(\Gamma \setminus X)\) is \(\lambda_0\)-harmonic.
2. If there exists a \(\lambda_0\)-harmonic function in \(L^2(\Gamma \setminus X)\), then the space of \(\lambda_0\)-harmonic functions in \(\Gamma \setminus X\) is one-dimensional and generated by a positive function.

**Proof.** Sullivan’s proof in [45] uses the heat operator and superharmonic functions. We provide a more direct proof here.

Note that if \(f \in L^2(\Gamma \setminus X) \cap C^\infty(\Gamma \setminus X)\) is a \(\lambda\)-harmonic function, then \(f \in W^1(\Gamma \setminus X)\), since
\[
\int_{\Gamma \setminus X} \|\nabla f\|^2 \, d\text{vol} = -\int_{\Gamma \setminus X} f \Delta f \, d\text{vol} = \lambda \int_{\Gamma \setminus X} f^2 \, d\text{vol}.
\]

The key fact for us is that \(\lambda_0\) may also be expressed as an infimum over functions in \(W^1(\Gamma \setminus X)\); for \(f \neq 0\) in \(W^1(\Gamma \setminus X)\), define \(R(f)\) by
\[
R(f) = \frac{\|f\|_{W^1}^2}{\|f\|^2} - 1 \geq 0
\]
where \(\|\cdot\|\) denotes the \(L^2(\Gamma \setminus X)\) norm. For any \(f \neq 0 \in W^1(\Gamma \setminus X)\), and all \(\varphi\) with \(\|f - \varphi\|_{W^1}\) small enough, we have
\[
\frac{\|\varphi\|_{W^1} - \|f - \varphi\|_{W^1}}{\|\varphi\| + \|f - \varphi\|_{W^1}} - 1 \leq R(f) \leq \frac{\|\varphi\|_{W^1} + \|f - \varphi\|_{W^1}}{\|\varphi\| - \|f - \varphi\|_{W^1}} - 1,
\]
i.e. \(f \mapsto R(f)\) is continuous at each \(f \neq 0 \in W^1(\Gamma \setminus X)\). The density of \(C^\infty_c(\Gamma \setminus X)\) in \(W^1(\Gamma \setminus X)\) then gives
\[
\lambda_0 = \inf_{f \in C^\infty_c(\Gamma \setminus X) \setminus \{0\}} R(f) = \inf_{f \in W^1(\Gamma \setminus X) \setminus \{0\}} R(f).
\]

Now suppose that \(\phi \in L^2(\Gamma \setminus X)\) is a positive \(\lambda\)-harmonic function; so \(\phi \in W^1(\Gamma \setminus X)\). We claim that \(\lambda = \lambda_0\). By Green’s identity we have
\[
\lambda_0 \leq R(\phi) = \frac{\int_{\Gamma \setminus X} \|\nabla \phi\|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\phi|^2 \, d\text{vol}} = \frac{\int_{\Gamma \setminus X} \phi (-\Delta \phi) \, d\text{vol}}{\int_{\Gamma \setminus X} |\phi|^2 \, d\text{vol}} = \lambda
\]
(cf. Proposition 4.2). On the other hand, for any \(\varphi \in C^\infty_c(\Gamma \setminus X)\),
\[
\frac{\int_{\Gamma \setminus X} \|\nabla \varphi\|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\varphi|^2 \, d\text{vol}} = \frac{\int_{\Gamma \setminus X} \|\nabla \left(\varphi \cdot \frac{\psi}{\overline{\psi}}\right)\|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\varphi|^2 \, d\text{vol}}.
\]
By Barta’s identity [1],
\[\int_{\Gamma \setminus X} \|\text{grad } (\phi \cdot \frac{x}{\phi})\|^2 \, d\text{vol} = \int_{\Gamma \setminus X} \phi^2 \|\text{grad } \frac{x}{\phi}\|^2 \, d\text{vol} - \int_{\Gamma \setminus X} \left(\frac{x}{\phi}\right)^2 \phi \Delta \phi \, d\text{vol},\]
so
\[\int_{\Gamma \setminus X} \|\text{grad } \varphi\|^2 \, d\text{vol} \geq \int_{\Gamma \setminus X} \left(\frac{x}{\phi}\right)^2 \phi (\Delta \phi) \, d\text{vol} = \lambda \int \varphi^2 \, d\text{vol},\]
i.e.
\[\lambda \leq \frac{\int_{\Gamma \setminus X} \|\text{grad } \varphi\|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} \varphi^2 \, d\text{vol}}\]
for any \(\varphi \in C_c^\infty(\Gamma \setminus X)\), showing that \(\lambda_0 \geq \lambda\). Hence \(\lambda = \lambda_0\).

In order to prove (2), we first claim that \(f \in W^1(\Gamma \setminus X)\) satisfies \(-\Delta f = \lambda_0 f\) if and only if \(R(f) = \lambda_0\).

Suppose that \(R(f) = \lambda_0\). We will then show that for any \(\varphi \in C_c^\infty(\Gamma \setminus X)\), we have
\[\langle f, -\Delta \varphi \rangle = \lambda_0 \langle f, \varphi \rangle;\]  
(6.1)
this implies \(f\) is \(\lambda_0\)-harmonic. Let \(\varphi \in C_c^\infty(\Gamma \setminus X)\). Since \(R(f) = \lambda_0\), \(f\) minimizes \(R\), so for any \(\varphi \in C_c^\infty(\Gamma \setminus X)\), the function \(F : \mathbb{R} \to \mathbb{R}_{\geq 0}\) defined by \(F(x) = R(f + x\varphi)\) has a local minimum at \(x = 0\), hence \(F'(0) = 0\). Now computing \(F'(0)\) gives
\[F'(0) = \frac{2\langle f, \varphi \rangle_{W^1} \|f\|^2 - 2\langle f, \varphi \parallel f\parallel_{W^1}^2}{\|f\|^4} = 0.\]
From \(R(f) = \lambda_0\), we obtain \(\|f\|^2_{W^1} = (\lambda_0 + 1)\|f\|^2\), which, when entered into the identity above, gives
\[\langle f, \varphi \rangle_{W^1} = (\lambda_0 + 1)\langle f, \varphi \rangle.\]  
(6.2)
Letting \(\{f_i\}_{i \in \mathbb{N}} \subset C_c^\infty(\Gamma \setminus X)\) be a sequence converging to \(f\) in \(W^1(\Gamma \setminus X)\), Green’s identity again gives
\[\langle f, \varphi \rangle_{W^1} = \lim_{i \to \infty} \langle f_i, \varphi \rangle_{W^1} = \lim_{i \to \infty} \int_{\Gamma \setminus X} f_i \varphi + \langle \text{grad } f_i, \text{grad } \varphi \rangle \, d\text{vol}\]
\[= \lim_{i \to \infty} \int_{\Gamma \setminus X} f_i \varphi + f_i (\Delta \varphi) \, d\text{vol} = \langle f, \varphi \rangle + \langle f, -\Delta \varphi \rangle.\]  
(6.3)
Combined with (6.2), this gives \(\langle f, -\Delta \varphi \rangle = \lambda_0 \langle f, \varphi \rangle\) as in (6.1).

Conversely, if \(f \in W^1(\Gamma \setminus X)\) satisfies \(-\Delta f = \lambda_0 f\), then for any \(\varphi \in C_c^\infty(\Gamma \setminus X)\), we have (as in (6.3))
\[\langle f, \varphi \rangle_{W^1} = \langle f, \varphi \rangle + \langle f, -\Delta \varphi \rangle = (\lambda_0 + 1)\langle f, \varphi \rangle,\]
hence
\[\|f\|^2_{W^1} = \sup_{\varphi \in C_c^\infty(\Gamma \setminus X)} \langle f, \varphi \rangle_{W^1} = \sup_{\varphi \in C_c^\infty(\Gamma \setminus X)} (\lambda_0 + 1)\langle f, \varphi \rangle = (\lambda_0 + 1)\|f\|^2,\]
giving \(R(f) = \lambda_0\). This proves the claim.

Let \(f \in W^1(\Gamma \setminus X) \cap C^\infty(\Gamma \setminus X)\) now be a \(\lambda_0\)-harmonic function. Then \(|f| \in W^1(\Gamma \setminus X)\) and \(R(|f|) = \lambda_0\). As shown above, \(|f|\) is also a \(\lambda_0\)-harmonic
function. Hence either $f$ is a constant multiple of $|f|$ or $f$ must change sign at some point $x_0$, hence $|f(x)| \geq |f(x_0)| = 0$ for all $x \in \Gamma \setminus X$. However, since $\Delta |f| = -\lambda_0 |f| \leq 0$, the strong minimum principle (cf. e.g. [38, Theorem 66, p. 280]) gives that if $|f|$ attains its infimum, then $|f|$ is in fact constant (in this case equal to zero). We therefore conclude that any $\lambda_0$-harmonic function in $L^2(\Gamma \setminus X)$ is a constant multiple of a positive function. This then implies that the space of $\lambda_0$-harmonic functions must be one-dimensional as two positive functions cannot be orthogonal to each other. □

The uniqueness in the above theorem has the following implications for joint eigenfunctions:

**Corollary 6.6.**

1. There exists at most one positive joint eigenfunction in $L^2(\Gamma \setminus X)$ up to a constant multiple.
2. If there exists a positive joint eigenfunction in $L^2(\Gamma \setminus X)$ with character $\chi_{\psi - \rho}$, $\psi \in a^*$, then $\lambda_0 = \lambda_\psi$.
3. There exists a positive Laplace eigenfunction in $L^2(\Gamma \setminus X)$ if and only if there exists a positive joint eigenfunction in $L^2(\Gamma \setminus X)$ of character $\chi_{\psi - \rho}$ with $\lambda_\psi = \lambda_0$.

**Proof.** We only need to verify the third claim. Suppose that $\phi \in L^2(\Gamma \setminus X)$ is a positive Laplace eigenfunction. Via the identification $L^2(\Gamma \setminus X) = L^2(\Gamma \setminus G) K$, we may consider $\phi \in L^2(\Gamma \setminus G) K$ as a positive $C$-eigenfunction for the Casimir operator $C$. By Theorem 6.5, $C \phi = -\lambda_0 \phi$. Let $D \in Z(g_c)$. Then $C \circ D \phi = D \circ C \phi = -\lambda_0 D \phi$. By the uniqueness in Theorem 6.5, it follows that $D \phi$ is a constant multiple of $\phi$; and hence $\phi$ is an eigenfunction for $D$ as well. Therefore $\phi$ is a joint eigenfunction. □

**Spherical unitary representations contained in $L^2(\Gamma \setminus G)$.** We let $C_c(G//K)$ denote the Hecke algebra of $G$, i.e.

$$C_c(G//K) = \{ f \in C_c(G) : f(k_1 g k_2) = f(g) \quad \text{for all } g \in G, k_1, k_2 \in K \}.$$ 

Each element of $C_c(G//K)$ acts on $C(G)$ via right convolution $\ast$.

**Lemma 6.7.** A positive $K$-invariant joint eigenfunction on $G$ is an eigenfunction for the action of the Hecke algebra. More precisely, if

$$\phi(g) = \int_F \varphi_{\psi,k}(g) \, d\nu([k]), \quad g \in G,$$

for some $\psi \in a^*$ and a $(\Gamma, \psi)$-conformal measure $\nu$ on $F = K/M$, then for all $f \in C_c(G//K)$,

$$(\phi \ast f)(g) = \left( \int_G f(h) e^{-\psi(H(h))} \, dh \right) \phi(g).$$
Proof. Given \( f \in C_c(G//K) \), we have
\[
(\phi * f)(g) = \int_G \phi(gh^{-1}) f(h) \, dh = \int_G \int_F \varphi_{\psi,k}(gh^{-1}) f(h) \, d\nu([k]) \, dh
\]
\[
= \int_F \int_G f(h)e^{-\psi(H(\kappa^{-1}k))} \, dh \, d\nu([k]).
\]
Now using \( H(\kappa^{-1}k) = H(h\kappa(g^{-1}k)) + H(g^{-1}k) \) and then the change of variables \( h' = h\kappa(g^{-1}k) \) gives
\[
(\phi * f)(g) = \int_F \left( \int_G f(\kappa(g^{-1}k)) e^{-\psi(H(h))} \, dh \right) e^{-\psi(H(g^{-1}k))} \, d\nu([k])
\]
\[
= \int_F \left( \int_G f(h) e^{-\psi(H(h))} \, dh \right) e^{-\psi(H(g^{-1}k))} \, d\nu([k])
\]
\[
= \left( \int_G f(h) e^{-\psi(H(h))} \, dh \right) \phi(g),
\]
since \( f \in C(G//K) \), and is thus right \( K \)-invariant. In total, we have shown that \( \phi \) is an eigenfunction of the \( f \)-action, with eigenvalue \( \int_G f(h) e^{-\psi(H(h))} \, dh \).
\[\square\]

**Theorem 6.8.** If \( \phi \in L^2(\Gamma \setminus G)_K \) is a positive Laplace eigenfunction of norm one, there exists a unique irreducible spherical unitary subrepresentation \( (\pi, \mathcal{H}_\phi) \) of \( L^2(\Gamma \setminus G) \), and \( \phi \) is the unique \( K \)-invariant unit vector in \( \mathcal{H}_\phi \).

**Proof.** By Corollary 6.6, \( \phi \) is given by (6.4) for some \( \psi \in \mathfrak{a}^* \). Define \( \Phi : G \to \mathbb{C} \) by
\[
\Phi(g) := \langle g, \phi, \phi \rangle
\]
for all \( g \in G \) where the \( g \) action on \( L^2(\Gamma \setminus G) \) is via the translation action of \( G \) on \( \Gamma \setminus G \) from the right. Given \( f \in C_c(G//K) \), we then have, using Lemma 6.7,
\[
(\Phi * f)(g) = \int_G \Phi(gh^{-1}) f(h) \, dh = \int_G \langle (gh^{-1}), \phi, \phi \rangle f(h) \, dh
\]
\[
= \int_G \langle f(h)h^{-1}, \phi, g^{-1}\phi \rangle \, dh = \langle \phi * f, g^{-1}\phi \rangle
\]
\[
= \left( \int_G f(h)e^{-\psi(H(h))} \, dh \right) \Phi(g),
\]
i.e. \( \Phi \) is also a \( C_c(G//K) \)-eigenfunction. Also note that \( \Phi(e) = 1 \), and since \( \phi \) is right \( K \)-invariant, \( \Phi \) is bi-\( K \)-invariant. Moreover, being the matrix coefficient of a unitary representation, \( \Phi \) is also positive definite, i.e., for any \( g_1, \cdots, g_n \in G \) and \( z_1, \cdots, z_n \in \mathbb{C} \),
\[
\sum_{1 \leq i,j \leq n} z_i \bar{z}_j \Phi(g_j^{-1} g_i) \geq 0.
\]
We have thus shown that $\Phi$ is a positive definite spherical function. Letting $\mathcal{H}_\phi$ denote the closure of $\text{span}\{g, \phi : g \in G\}$ in $L^2(\Gamma \backslash G)$, by [29, Chapter IV §5, Corollary of Theorem 9], $\mathcal{H}_\phi$ is an irreducible (spherical) unitary subrepresentation of the quasi-regular representation $L^2(\Gamma \backslash G)$. The uniqueness follows from Corollary 6.6.

We require the following lemma in the proof of Theorem 6.10:

**Lemma 6.9.** Let $\psi \geq \rho$ and $\psi \not\in \mathbb{R}\rho$. Denote by $\psi'$ be the element of the line $\mathbb{R}\psi$ closest to $\rho$. Then $\psi' \geq \rho$.

**Proof.** Let $\phi := \psi - \rho$. Note that $\phi \geq 0$ on $a$ by the hypothesis. Then

$$
\psi' = \frac{\langle \psi, \rho \rangle}{\|\psi\|^2} \psi = \frac{\langle \rho + \phi, \rho \rangle}{\|\rho + \phi\|^2} \psi = \left(1 - \frac{\|\phi\|^2 + \langle \rho, \phi \rangle}{\|\rho + \phi\|^2}\right) \psi,
$$

i.e. $\psi' = t\psi$ with $0 < t < 1$. Now, if $\psi' \geq \rho$, we could repeat the process with $\psi'$ in place of $\psi$ to find another, different, closest vector in $\mathbb{R}\psi$ to $\rho$, which is not possible. □

**Theorem 6.10.** Let $\Gamma < G$ be of the second kind with $\mathcal{L} \subset \text{int } a^+ \cup \{0\}$. If there exists a $\lambda_0$-harmonic function in $L^2(\Gamma \backslash X)$, then

$$
\lambda_0 = \lambda_\psi
$$

for some $\psi \in D^*_\Gamma \cup \{\rho\}$.

**Proof.** Suppose that $\psi \in D^*_\Gamma \setminus \{\rho\} \cup D^*_\Gamma$ and that $\psi \geq \rho$. Assume that there exists a positive joint eigenfunction $\phi \in L^2(\Gamma \backslash X)$ with character $\chi_{\psi - \rho}$. By Corollary 6.6,

$$
\lambda_0 = \lambda_\psi = \|\rho\|^2 - \|\psi - \rho\|^2. \quad (6.5)
$$

Since $\psi_\Gamma$ is concave, there exists $0 < c < 1$ such that $c\psi(u) = \psi_\Gamma(u)$ for some $u \in \mathcal{L}$. So $\psi_0 := c\psi \in D^*_\Gamma$. Since $\psi \not\in D^*_\Gamma$, we have $0 < c < 1$. There exists a unique $s_0 \in \mathbb{R}$ such that

$$
\|s_0\psi_0 - \rho\| = \min\{\|s\psi - \rho\| : s \in \mathbb{R}\}, \quad (6.6)
$$

that is, $s_0\psi_0$ be the element on the line $\mathbb{R}\psi_\Gamma$ that is closest to $\rho$.

We claim that $s_0c \leq 1$; since $0 < c < 1$, this implies that $\max\{1, s_0\} < c^{-1}$. If $\psi \in \mathbb{R}\rho$, then $s_0\psi_0 = \rho$. Since $\psi_0 = c\psi$, we get $s_0c\psi = \rho$. By the hypothesis $\rho \leq \psi$, $s_0c \leq 1$. Now suppose $\psi \not\in \mathbb{R}\rho$. Assume that $s_0c > 1$. Then $s_0\psi_0 = s_0c\psi > \psi$. Hence $s_0c\psi \in D^*_\Gamma$. By Corollary 5.10 and (6.5), we get

$$
\|s_0c\psi - \rho\| \geq \|\psi - \rho\|.
$$

By the choice of $s_0$ in (6.6), it follows that $\|s_0c\psi - \rho\| = \|\psi - \rho\|$. Since $s_0c\psi > \psi \geq \rho$, this yields a contradiction. Therefore the claim $s_0c \leq 1$ follows.

We now choose $t$ so that $\max\{1, s_0\} < t < c^{-1}$. Since $t > 1$ and $\psi_0 \in D^*_\Gamma$, $t\psi_0 \in D^*_\Gamma$. Note also that $s \mapsto \lambda_{s\psi_0}$ is strictly decreasing on the interval $[s_0, \infty)$. Since $s_0 < t < c^{-1}$ and $c^{-1}\psi_0 = \psi$, we get

$$
\lambda_0 = \lambda_\psi < \lambda_{t\psi_0}.
$$
This contradicts Corollary 5.10. This implies the claim by Corollary 6.6. □

If we use the norm on so(n, 1) which endows the constant curvature -1 metric on H^n, then for any non-elementary discrete subgroup Γ < SO^0(n, 1), D^*_Γ = {δ} and hence the above theorem says that if a λ_0-harmonic function belongs to L^2(Γ\H^n), then λ_0 must be given by either δ(n−1−δ) or 1/4(n−1)^2.

7. Smearing argument in higher rank

Let Γ be a discrete subgroup of a connected semisimple real algebraic group G. Recall the notation i for the opposition involution of G from (2.1). The goal of this section is to prove the following, which implies Theorem 1.8 by Theorem 2.5.

**Theorem 7.1.** Let ψ ∈ DΓ be stabilized by i, i.e., ψ ◦ i = ψ. If L ≠ a^+, then no positive joint eigenfunction of character χ_ψ−ρ belongs to L^2(Γ\X).

**Corollary 7.2.** If L ≠ a^+ and λ_0(Γ\X) = ||ρ||^2, then there exists no positive Laplace eigenfunction in L^2(Γ\X).

**Proof.** Suppose that there exists a positive Laplace eigenfunction ψ in L^2(Γ\X). By Corollary 6.6, ψ is a joint eigenfunction in L^2(Γ\X) of character χ_ψ−ρ for some ψ ∈ a^+ satisfying λ_ψ = λ_0. Since λ_0 = ||ρ||^2 = λ_ψ = ||ρ||^2 − ||ρ − ψ||^2, it follows that ψ = ρ. Since ρ is invariant under i, Theorem 7.1 implies the ψ cannot belong to L^2(Γ\X), yielding a contradiction. □

**Corollary 7.3.** Suppose that i is trivial. For any discrete subgroup Γ < G with L ≠ a^+, there exists no positive Laplace eigenfunction in L^2(Γ\X).

Theorem 7.1 will be deduced from Theorem 7.5, the proof of which is based on the smearing argument of Thuston and Sullivan (see [46] and also [47] for historical remarks and the origin of the name ”smearing argument”).

In the rest of this section, we fix ψ ∈ DΓ. For each x ∈ X, define the following analogue d_x = d_φ,x of the visual metric on F: for any (ξ, η) ∈ F(2),

\[ d_x(ξ, η) = e^{−(β_ε(x,go)+iβ_δ(x,go))} \]

where g ∈ G is any element such that g^+ = ξ and g^- = η; this definition is independent of the choice of such g. The following G-equivariance property follows from that of the Busemann function: for any h ∈ G,

\[ d_x(ξ, η) = d_{h^x}(hξ, hη). \]

**Definition 7.4** (Hopf parameterization). The homeomorphism G/M → F(2) × a given by gM ↦ (g^+, g^−, b = β_g (e, g)) is called the Hopf parameterization of G/M.

Fix {ν_x : x ∈ X} and {ν̂_x : x ∈ X} be respectively (Γ, ψ) and (Γ, ψ ◦ i)-conformal densities on F. Using the Hopf parametrization, we define the following locally finite Borel measure ñ_µ,ρ on G/M: for (ξ, η, v) ∈ F(2) × a,

\[ dñ_µ,ρ(ξ, η, v) = \frac{1}{d_x(ξ, η)} dv_x(ξ) dν̂_x(η) dv \]  (7.1)
where $dv$ is the Lebesgue measure on $a$ and $x \in X$ is any element; it follows from the $\Gamma$-conformality of $\{\nu_x\}$ and $\{\bar{\nu}_x\}$ that this definition is independent of $x \in X$. The measure $\tilde{m}_{\nu,\bar{\nu}}$ is left $\Gamma$-invariant and right $A$-invariant. We denote by $m_{\nu,\bar{\nu}}$ the $AM$-invariant Borel measure on $\Gamma \backslash G$ induced by $\tilde{m}_{\nu,\bar{\nu}}$; this measure is called the Bowen-Margulis-Sullivan measure associated to the pair $(\nu, \bar{\nu})$ [10].

**Theorem 7.5.** For any pair $(\nu, \bar{\nu})$ of $(\Gamma, \psi)$ and $(\Gamma, \psi \circ i)$-conformal measures on $F$ respectively, we have

$$m_{\nu,\bar{\nu}}(\Gamma \backslash G) \ll \int_{\Gamma \backslash X} E_\nu(x)E_{\bar{\nu}}(x) \, d\text{vol}.$$  

**Proof.** We extend the smearing argument due to Sullivan and Thurston ([46], [8]). Let $Z = G/K \times F^{(2)}$. For any $(\xi, \eta) \in F^{(2)}$, we write $[\xi, \eta] = gA_o \subset X$ for any $g \in G$ such that $g^+ = \xi$ and $g^- = \eta$; $[\xi, \eta]$ is a maximal flat in $X$ defined independently of the choice of $g \in G$. We also denote by $W_{\xi,\eta} \subset X$ the one neighborhood of $[\xi, \eta]$. Consider the following locally finite Borel measure $\alpha$ on $Z$ defined as follows: for any $f \in C_c(Z)$,

$$\alpha(f) = \int_{(\xi, \eta) \in F^{(2)}} \int_{z \in W_{\xi,\eta}} f(z, \xi, \eta) \, dz \, dm(\xi, \eta)$$

where $dz$ is the $G$-invariant measure on $X$, and $dm(\xi, \eta) = \frac{1}{d\lambda(\xi,\eta)} d\nu_x(\xi)d\bar{\nu}_x(\eta)$ (independent of $x$); in other words,

$$d\alpha(z, \xi, \eta) = d\lambda_{\xi,\eta}(z)dm(\xi, \eta)$$

where $\lambda_{\xi,\eta}$ is the restriction of $\lambda$ to $W_{\xi,\eta}$. Consider natural diagonal action of $\Gamma$ on $Z$. Since $dz$ and $dm$ are both left $\Gamma$-invariant, $\alpha$ is also left $\Gamma$-invariant and hence induces a measure the quotient space $\Gamma \backslash Z$, which we also denote by $\alpha$ by abuse of notation.

Define the projection $\pi' : Z \to G/M$ as follows: for $(x, \xi, \eta) \in X \times F^{(2)}$, choose $g \in G$ so that $g^+ = \xi$ and $g^- = \eta$. Then there exists a unique element $a \in A$ such that

$$d(x, gao) = d(x, gA_o) = \inf_{b \in A} d(x, gbo);$$

this follows from [5, Proposition 2.4] since $X$ is a CAT(0) space and $gA(o)$ is a convex complete subspace of $X$. In other words, the point $gao$ is the orthogonal projection of $x$ to the flat $[\xi, \eta] = gA_o$. We then set

$$\pi'(x, \xi, \eta) = gaM \in G/M;$$

this is well-defined independent of the choice of $g \in G$.

Noting that $\pi'$ is $\Gamma$-equivariant, we denote by

$$\pi : \text{supp}(\alpha) \subset \Gamma \backslash Z \to \text{supp} (m_{\nu,\bar{\nu}}) \subset \Gamma \backslash G/M$$

the map induced by $\pi'$.
Fixing \([ga] \in \Gamma\backslash G/M\), the fiber \(\pi^{-1}[ga]\) is of the form \([(gaD_0,g^+,g^-)]\) where

\[D_0 = \{ s \in X : d(s,o) \leq 1, \]

the geodesic connecting \(s\) and \(o\) is orthogonal to \(Ao\) at \(o\).

Noting that each fiber \(\pi^{-1}(v), v \in \text{supp } m_{\nu,\bar{\nu}},\) is isometric to \(D_0\), we have for any Borel subset \(S \subset \text{supp } m_{\nu,\bar{\nu}},\)

\[\alpha(\pi^{-1}(S)) = c \cdot m_{\nu,\bar{\nu}}(S) \quad (7.2)\]

where \(c = \text{Vol}(D_0)\); the volume of \(D_0\) being computed with respect to the volume form induced by the \(G\)-invariant measure on \(X\).

Consider now the map \(p : \text{supp}(\alpha) \to \Gamma\backslash X\) defined by \(p([\langle z, \xi, \eta \rangle]) = [z]\) for any \((z, \xi, \eta) \in \text{supp}(\alpha)\).

Let \(F = \pi^{-1}(\text{supp } m_{\nu,\bar{\nu}}) \subset \text{supp}(\alpha)\). We write

\[\alpha(F) = \int_{\Gamma\backslash X} \alpha_x(p^{-1}(x) \cap F) \, dx,\]

where \(\alpha_x\) is a conditional measure on the fiber \(p^{-1}(x)\).

We claim that there exists a constant \(c > 0\) such that for any \(x \in \Gamma\backslash X,\)

\[\alpha_x(p^{-1}(x)) \leq c E_{\nu}(x) \cdot E_{\bar{\nu}}(x). \quad (7.3)\]

This implies that \(\alpha(F) \ll \int_{\Gamma\backslash X} E_{\nu}(x) E_{\bar{\nu}}(x) \, dx\), which then finishes the proof by (7.2).

Note that \(V_{(h(o))} := \{ (\xi, \eta) \in \mathcal{F}^{(2)} : [\xi, \eta] \cap B(h(o), 1) \neq \emptyset \}\) is a compact subset of \(\mathcal{F}^{(2)}\); if \(g_i \in G\) such that \(d(g_i a_i o, h(o)) \leq 1\) for some \(a_i \in A\), then \((g_i^+, g_i^-) \to (g_0^+, g_0^-) \in \mathcal{F}^{(2)}\) as \(i \to \infty\), from which the compactness of \(V_{(h(o))}\) follows. It follows that

\[\kappa := \inf \{ d_o(\xi, \eta) : (\xi, \eta) \in V_o \} > 0.\]

By the equivariance \(d_{h(o)}(\xi, \eta) = d_o(h^{-1}\xi, h^{-1}\eta)\), we have for any \(h \in G,\)

\[\kappa = \inf \{ d_{h(o)}(\xi, \eta) : (\xi, \eta) \in V_{(h(o))} \}.\]

Note that if \(x = [h(o)] \in \Gamma\backslash X\) for \(h \in G\), then

\[p^{-1}(x) = \{ [(h(o), \xi, \eta)] \in \text{supp}(\alpha) : [\xi, \eta] \cap B(h(o), 1) \neq \emptyset \} \simeq V_{(h(o))}.\]

Therefore for any \(x = [h(o)] \in \Gamma\backslash X,\)

\[\alpha_x(p^{-1}(x)) = \alpha_x(V_{(h(o))})\]

\[= \int_{(\xi,\eta) \in V_{(h(o))}} \frac{1}{d_{h(o)}(\xi,\eta)} \, d\nu_{h(o)}(\xi) d\bar{\nu}_{h(o)}(\eta)\]

\[\leq \frac{1}{\kappa} \int_{(\xi,\eta) \in V_{h(o)}} \, d\nu_{h(o)}(\xi) d\bar{\nu}_{h(o)}(\eta)\]

\[\leq \frac{1}{\kappa} \left| \nu_{h(o)} \right| \cdot |\bar{\nu}_{h(o)}| = \frac{1}{\kappa} E_{\nu}(x) \cdot E_{\bar{\nu}}(x).\]
This proves (7.3), and hence finishes the proof. □

**Proof of Theorem 7.1.** Suppose that $\phi \in L^2(\Gamma \backslash X)$ is a positive joint eigenfunction with character $\chi_{\psi - \rho}$. By Proposition 3.7, $\phi = E_\nu$ for some $(\Gamma, \psi)$-conformal measure $\nu$. Since $\psi \circ i = \psi$, we may form the measure $m_{\nu, \nu}$ and apply Theorem 7.5. Since $E_\nu \in L^2(\Gamma \backslash G)$, it follows that $m_{\nu, \nu}$ is a finite $MA$-invariant Borel measure on $\Gamma \backslash G$. Since $m_{\nu, \nu}$ is finite, it is conservative for any one-parameter subgroup of $A$. In particular, for any non-zero $v \in a^+$, there exist sequences $t_i \to +\infty$ and $\gamma_i \in \Gamma$ such that the sequence $\gamma_i \exp(t_i v)$ is convergent. This implies that $t_i^{-1} \mu(\gamma_i)$ converges to $v$, and hence $v \in \mathcal{L}$. Therefore $L = a^+$. This finishes the proof.

**Remark 7.6.** Suppose that $\Gamma < G$ is Zariski dense and that $\psi > \psi_\Gamma$. Then, by [41, Lem. III. 1.3], we have

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty.$$  

On the other hand, by Theorem 1.4 of [6], the finiteness of $m_{\nu, \nu}$ implies that $\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty$, yielding a contradiction. Therefore the same conclusion of Theorem 7.1 holds in this setting.

### 8. Injectivity radius and $L^2(G) \propto L^2(\Gamma \backslash G)$

As before let $G$ be a connected semisimple real algebraic group. Recall from Proposition 6.3 that $\sigma(X) = [[[\rho]]^2, \infty)$. In this section, we prove the following:

**Theorem 8.1.** Let $\Gamma < G$ be an Anosov subgroup. We suppose that $\Gamma$ is not a cocompact lattice in a rank one group $G$. Then

$$L^2(G) \propto L^2(\Gamma \backslash G) \text{ and } \sigma(X) = [[[\rho]]^2, \infty) \subset \sigma(\Gamma \backslash X).$$

Note that if $\Gamma < G$ is Anosov, $\Gamma \backslash G$ has infinite volume except when $\Gamma$ is a cocompact lattice in a rank one group $G$. The latter case has to be ruled out from the above theorem since the conclusions are not true in that case; $L^2(\Gamma \backslash G)$ contains the constant function and $\sigma(\Gamma \backslash X)$ is countable. In the rank one case, an Anosov subgroup $\Gamma < G$ is simply a convex cocompact subgroup, in which case this theorem is well-known due to the work of Lax and Phillips [30].

We will need the following lemma: when $G$ is of rank one, we may write $A$ is a one-parameter subgroup $A = \{a_t : t \in \mathbb{R}\}$. A loxodromic element $g \in G$ is of the form $g = h a_t m h^{-1}$ for some $t \neq 0$, $m \in M$ and $h \in G$. The translation axis of $g$ is then given by $hA(a)$.

**Lemma 8.2.** Let $G$ be a simple real algebraic group of rank one. For any loxodromic element $g \in G$ with translation axis $L$, we have $d(x, gx) \to \infty$ as $d(x, L) \to \infty$. 

Proof. Write $A = \{a_t : t \in \mathbb{R}\}$. Without loss of generality, we may assume $g = m^{-1}a_{-s_0} \in MA$ with $s_0 \neq 0$ so that $L = A(o)$. Let $x_i \in X$ be a sequence such that $d(x_i, A(o)) \rightarrow \infty$. Write $x_i = n_ia_{-t_i}(o)$ with $n_i \in N$ and $t_i \in \mathbb{R}$.

We may then write

$$d(gx_i, x_i) = d(a_{t_i}h_in_ia_{-t_i}, a^{-1}o).$$

where $h_i = ma_{s_0}n_i^{-1}a_{-s_0}m^{-1} \in N$. As $d(x_i, A(o)) \rightarrow \infty$, we have $a_{t_i}n_ia_{-t_i} \rightarrow \infty$. It suffices to show $a_{t_i}h_in_ia_{-t_i} \rightarrow \infty$.

By the assumption that $G$ has rank one, there is only one simple root, say $\alpha$ and $n$ is the sum of at most two root subspaces $n = n_\alpha + n_{2\alpha}$ where $[n, n] = n_{2\alpha}$. Note that when $N$ is abelian, $n_{2\alpha} = \{0\}$. Hence we have that for any $X, Y \in n$,

$$\log \left( \exp(X) \exp(Y) \right) = X + Y + \frac{1}{2}[X, Y]. \quad (8.1)$$

Write $n_i = Y_i + Z_i$ with $Y_i \in n_\alpha$ and $Z_i \in n_{2\alpha}$. Since $\text{Ad}_m$ preserves $n_\alpha$ and $n_{2\alpha}$, we have

$$\log h_i = -\text{Ad}_{ma_{s_0}} \log n_i = -\alpha(s_0)\text{Ad}_mY_i - e^{2\alpha(s_0)}\text{Ad}_mZ_i.$$ 

Therefore by (8.1), we get

$$\log h_in_i = (1 - e^{\alpha(s_0)}\text{Ad}_m)Y_i + (1 - e^{2\alpha(s_0)}\text{Ad}_m)Z_i - \frac{1}{2}[\alpha(s_0)\text{Ad}_mY_i, Y_i].$$

Hence

$$\text{Ad}_{a_{t_i}} \log h_in_i = (1 - e^{\alpha(s_0)}\text{Ad}_m)e^{\alpha(t_i)}Y_i + (1 - e^{2\alpha(s_0)}\text{Ad}_m)e^{2\alpha(t_i)}Z_i - [e^{\alpha(s_0)}\text{Ad}_me^{\alpha(t_i)}Y_i, e^{\alpha(t_i)}Y_i].$$

Now suppose that $a_{t_i}h_in_ia_{-t_i}$ does not go to infinity as $i \rightarrow \infty$. By passing to a subsequence, we may assume that $\text{Ad}_{a_{t_i}} \log h_in_i$ is uniformly bounded. It follows that both sequences $(1 - e^{\alpha(s_0)}\text{Ad}_m)e^{\alpha(t_i)}Y_i$ and $(1 - e^{2\alpha(s_0)}\text{Ad}_m)e^{2\alpha(t_i)}Z_i - [e^{\alpha(s_0)}\text{Ad}_me^{\alpha(t_i)}Y_i, e^{\alpha(t_i)}Y_i]$ are uniformly bounded. Since $\alpha(s_0) \neq 0$, we have $e^{\alpha(t_i)}Y_i$ is uniformly bounded, which then implies that $e^{2\alpha(t_i)}Z_i$ is uniformly bounded.

This implies that $\text{Ad}_{a_{t_i}} \log n_i = e^{\alpha(t_i)}Y_i + e^{2\alpha(t_i)}Z_i$ is uniformly bounded, contradicting the hypothesis $d(a_{t_i}n_ia_{-t_i}) \rightarrow \infty$. This proves the claim. \qed

Let $\Gamma < G$ be a discrete subgroup. For $x = [g] \in \Gamma \backslash G$, the injectivity radius $\text{inj}(x)$ is defined as the supremum $r > 0$ such that the ball $B_r(g) = \{h \in G : d(h, g) < r\}$ injects to $\Gamma \backslash G$ under the canonical quotient map $G \rightarrow \Gamma \backslash G$. The injectivity radius of $\Gamma \backslash G$ is defined as $\text{inj}(\Gamma \backslash G) = \sup_{x \in \Gamma \backslash G} \text{inj}(x)$.

**Proposition 8.3.** For any Anosov subgroup $\Gamma < G$ which is not a cocompact lattice in a rank one group $G$, we have $\text{inj}(\Gamma \backslash G) = \infty$. 

Proof. If $G$ has rank one, $\Gamma$ is a convex cocompact subgroup which is not a cocompact lattice. In this case, take any $\xi \in \partial X$ which is not a limit point, and any $g_i \in G$ such that $g_i(o) \to \xi$. Then $\text{inj}(g_i(o)) \to \infty$.

Now suppose rank $G \geq 2$. Then $\text{Vol}(\Gamma \backslash G) = \infty$; otherwise, $\Gamma < G$ is a cocompact lattice as Anosov subgroups consists only of loxodromic elements. Since any Anosov subgroup $\Gamma$ is a Gromov hyperbolic group as an abstract group ([22], [3]), it follows that $G$ is a Gromov hyperbolic space and hence must be of rank one, which contradicts the hypothesis. Hence, if every simple factor of $G$ has rank at most 2, the claim follows from a more general result of Fraczyk and Gelander [15] which applies to all discrete subgroups of infinite co-volume.

Therefore it suffices to consider the case where $G = G_1 \times G_2$ where $G_1$ and $G_2$ are respectively semisimple real algebraic subgroups of rank at least one and precisely one. Let $\Sigma$ be a finitely generated group and $\pi : \Sigma \to G$ be an Anosov representation with $\Gamma = \pi(\Sigma)$ as in Definition 2.4. Let $\pi_i : \Sigma \to G_i$ be the composition of $\pi$ and the projection $G \to G_i$ for each $i$. It follows from (2.8) that $\pi_i(\Sigma)$ is a discrete subgroup of $G_i$ for each $i = 1, 2$. Let $X_i$ denote the rank one symmetric space associated to $G_i$ and set $X$ denote the Riemannian product $X = X_1 \times X_2$. Let $R > 0$ be an arbitrary number. We will find a point $x \in X$ with $\text{inj}(x) \geq R$, i.e., $d(x, \gamma x) > R$ for all non-trivial $\gamma \in \Gamma$; this implies the claim. Choose any $x_1 \in X_1$. By the discreteness of $\pi_1(\Sigma)$, the set $\{\sigma \in \Sigma - \{e\} : d_1(\pi_1(\sigma)x_1, x_1) < R\}$ is finite, which we write as $\{\sigma_1, \cdots, \sigma_m\}$. For each $\sigma \in \Sigma \setminus \{e\}$, define a subset $T_2(\sigma) \subset X_2$ by

$$T_2(\sigma) = \{z \in X_2 : d_2(\pi_2(\sigma)z, z) < R\}.$$ 

Note that $\pi_2(\sigma)$ is a loxodromic element of $G_2$ and $T_2(\sigma)$ is contained in a bounded neighborhood of the translation axis of $\pi_2(\sigma)$ by Lemma 8.2. In particular, the symmetric space $X_2$ is not covered by the finite union $\bigcup_{j=1}^m T_2(\sigma_j)$. Hence we may choose $x_2 \in X_2$ outside of $\bigcup_{j=1}^m T_2(\sigma_j)$. We claim that the injectivity radius at $(x_1, x_2)$ is at least $R$; suppose not. Then for some $\sigma \in \Sigma - \{e\}$, $d((\pi_1(\sigma)x_1), x) < R$. In particular, for $i = 1, 2$, $d_i(\pi_i(\sigma)x_i, x_i) < R$. It follows that $\sigma = \sigma_j$ for some $1 \leq j \leq m$ and $x_2 \in T_2(\sigma_j)$, contradicting the choice of $x_2$. This proves the claim.

Theorem 8.1 follows from Proposition 8.3 and the following proposition, which was suggested by C. McMullen.

**Proposition 8.4.** Let $\Gamma < G$ be a discrete subgroup with $\text{inj}(\Gamma \backslash G) = \infty$. Then

$$L^2(G) \propto L^2(\Gamma \backslash G) \quad \text{and} \quad \sigma(X) \subset \sigma(\Gamma \backslash X).$$

**Proof.** Let $v \in L^2(G)$. We may choose a sequence $f_i \in C_c(G)$ such that $f_i$ vanishes outside $B_{R_i}(e)$ and $\|v - f_i\| \to 0$ as $i \to \infty$. Since the matrix coefficients $\langle gf_i, f_i \rangle$ converges to $\langle gv, v \rangle$ uniformly on compact subsets.
For each $i$, consider the function $F_i \in C_c(\Gamma \backslash G)$ given by

$$F_i(x) = \sum_{\gamma \in \Gamma} g_i^{-1} f_i(\gamma h) \text{ for any } x = [h] \in \Gamma \backslash G.$$ 

Since $B_{R_i}(g_i)$ injects to $\Gamma \backslash G$, we have for any $g \in G$,

$$\langle g, F_i, F_i \rangle_{L^2(\Gamma \backslash G)} = \int_{\Gamma \backslash G} F_i(xg) F_i(x) dx$$

$$= \int_{\Gamma \backslash G} F_i(\Gamma h g) \left( \sum_{\gamma \in \Gamma} f_i(\gamma h) \right) d(\Gamma h) = \int_{h \in G} F_i(\Gamma h g) f_i(h) dh$$

$$= \int_{h \in B_{R_i}(g_i)} \left( \sum_{\gamma \in \Gamma} f_i(\gamma h g g_i^{-1}) \right) f_i(h) dh = \int_{B_{R_i}(g_i)} g_i^{-1} f_i(h g) f_i(h) dh$$

$$= \langle g^{-1} f_i, f_i \rangle_{L^2(G)}.$$ 

Therefore the diagonal matrix coefficient $g \mapsto \langle gv, v \rangle$ can be approximated by the matrix coefficients in $L^2(\Gamma \backslash G)$ uniformly on compact subsets. This implies the first claim.

In order to prove the second claim, let $W^1(\Gamma \backslash X) \subset L^2(\Gamma \backslash X)$ be as defined in the proof of Theorem 6.5.

Let $\lambda \in \sigma(X)$. By Weyl’s criterion (Theorem 6.1), there exists a sequence of $L^2(X)$-unit vectors $\{u_n\}_{n \in \mathbb{N}} \subset W^1(X)$ such that

$$\lim_{n \to \infty} \| (\Delta + \lambda) u_n \|_{L^2(X)} = 0.$$ 

Since $C^\infty_c(X)$ is dense in $W^1(X)$ with respect to $\| \cdot \|_{W^1(X)}$, we may assume that $\{u_n\}_{n \in \mathbb{N}} \subset C^\infty_c(X)$. Denoting the support of $u_n$ by $B_n$, since $\Gamma \backslash X$ has infinite injectivity radius, for each $n \in \mathbb{N}$ we can find $g_n \in G$ so that $g_n B_n$ injects to $\Gamma \backslash G$. We may therefore define $\{v_n\}_{n \in \mathbb{N}} \subset W^1(\Gamma \backslash X)$ by

$$v_n(\Gamma g_n x) = \begin{cases} u_n(x) & \text{if } x \in B_n \\ 0 & \text{otherwise.} \end{cases}$$ 

The $G$-invariance of $\Delta$ then gives

$$\lim_{n \to \infty} \| (\Delta + \lambda) v_n \|_{L^2(\Gamma \backslash X)} = \lim_{n \to \infty} \| (\Delta + \lambda) u_n \|_{L^2(X)} = 0;$$

and so using Weyl’s criterion again yields $\lambda \in \sigma(\Gamma \backslash X)$, hence $\sigma(X) \subset \sigma(\Gamma \backslash X)$, as claimed. \qed

9. TEMPEREDNESS OF $L^2(\Gamma \backslash G)$

Let $G$ be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. The goal of this section is to prove Theorem 9.4 and Corollary 9.6.
Burger-Roblin measures. We set $N^+ = w_0 N w_0^{-1}$ and $N^- = N$. For a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, we denote by $m_\nu^{\text{BR}+}$ and $m_\nu^{\text{BR}−}$ the associated $N^+$ and $N^-$-invariant Burger-Roblin measures on $\Gamma \backslash G$ respectively, as defined in [10]. By [10, Lem. 4.9], it can also be defined as follows: for any $f \in C_c(\Gamma \backslash G)$,
\[ m_\nu^{\text{BR}+}(f) = \int_{[k]m(\exp a)n \in K/M \times MAN^+} f([k]m(\exp a)n)e^{-\psi_i(a)} \, d\nu(k^-) \, dm \, da \, dn \]
and
\[ m_\nu^{\text{BR}−}(f) = \int_{[k]m(\exp a)n \in K/M \times MAN^-} f([k]m(\exp a)n)e^{\psi(a)} \, d\nu(k^+) \, dm \, da \, dn \]
where $dm$, $da$, $dn$ are Haar measures on $M$, $a$, $N^\pm$ respectively.

We denote by $dx$ the $G$-invariant measure on $\Gamma \backslash G$ which is defined using the $(G, 2\rho)$-conformal measure, that is, the $K$-invariant probability measure on $\mathcal{F}$ (see [10, (3.11)]). For functions $f_1, f_2$ on $\Gamma \backslash G$, we write
\[ \langle f_1, f_2 \rangle = \int_{\Gamma \backslash G} f_1(x)f_2(x) \, dx \]
whenever the integral converges. We write $C_c(\Gamma \backslash G)_K$ for the space of $K$-invariant compactly supported continuous functions on $\Gamma \backslash G$.

**Lemma 9.1.** For a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$ and any $f \in C_c(\Gamma \backslash G)_K$, we have
\[ m_\nu^{\text{BR}+}(f) = \langle f, E_\nu \rangle = m_\nu^{\text{BR}−}(f). \]

**Proof.** If $g = (\exp b)nk \in AN^+K$, then $\beta_{e^−}(go,o) = \beta_{e^+}(\exp(−i(b)),o) = i(b)$. Hence
\[ m_\nu^{\text{BR}+}(f) = \int_{KAN^+} \int_K f(k \exp bnk_0)e^{-\psi_i(b)} \, dk_0 \, d\nu(k^-) \, db \, dn \]
\[ = \int_G \int_K f(kg)e^{-\psi(\beta_{e^−}(go,o))} \, d\nu(k^-) \, dg \]
\[ = \int_G f(g) \int_K e^{-\psi(\beta_{e^−}(go,o))} \, d\nu(k^-) \, dg = \langle f, E_\nu \rangle \]
If $g = (\exp b)nk \in ANK$, then $\beta_{e^+}(go,o) = −b$ and using this, the second identity can be proved similarly. □

**Local matrix coefficients for Anosov subgroups.** In the rest of this section, we assume that
\[ \Gamma < G \] is a Zariski dense Anosov subgroup.

**Lemma 9.2.** For any $\psi \in D_\Gamma$, there exists a unique unit vector $u \in a^+$ and $0 < c \leq 1$ such that $c\psi(u) = \psi_\Gamma(u)$ and $u \in \text{int} \mathcal{L}$.

**Proof.** Since $\psi_\Gamma$ is strictly concave [39, Propositions 4.6, 4.11], there exists $0 < c \leq 1$ and unique $u \in \mathcal{L}$ such that $c\psi(u) = \psi_\Gamma(u)$. Moreover there is no linear form tangent to $\psi_\Gamma$ at $\partial\mathcal{L}$ [39], and hence $u \in \text{int} \mathcal{L}$. □
For each \( v \in \text{int} \mathcal{L} \), there exists a unique \( \psi_v \in D^*_\Gamma \) such that \( \psi_v(v) = \psi_T(v) \) and a unique \((\Gamma, \psi_v)\)-conformal probability measure, say, \( \nu_v \) supported on \( \Lambda \) \cite[Corollary 7.8 and Theorem 7.9]{10}.

Hence \cite[Theorem 7.12]{10}, together with Lemma 9.1, implies (let \( r = \text{rank} G \):

**Theorem 9.3.** For any \( v \in \text{int} \mathcal{L} \), there exists \( \kappa_v > 0 \) such that for all \( f_1, f_2 \in C_c(\Gamma \setminus G)_K \) and any \( w \in \ker \psi_v,

\[
\lim_{t \to +\infty} t^{(r-1)/2} e^{t(2\rho - \psi_v)(tv + \sqrt{tw})} \langle \exp(tv + \sqrt{tw})f_1, f_2 \rangle
= \kappa_v e^{-I(w)} \cdot \langle f_1, E_{\nu(v)} \rangle \cdot \langle f_2, E_{\nu} \rangle
\]

where \( I(w) \in \mathbb{R} \) is given as in \cite[7.5]{10}. Moreover, the left-hand side is uniformly bounded over all \((t, w) \in (0, \infty) \times \ker \psi_v \) such that \( tv + \sqrt{tw} \in \mathfrak{a}^+ \).

**Theorem 9.4.**

1. We have \( L^2(\Gamma \setminus G) \) is tempered if and only if \( \psi_T \leq \rho \).

2. If \( L^2(\Gamma \setminus G) \) is tempered, then \( \lambda_0(\Gamma \setminus X) = ||\rho||^2 \) and \( \sigma(\Gamma \setminus X) = [||\rho||^2, \infty) \).

**Proof.** The second claim follows from Theorems 6.4 and 8.1. Suppose that \( \psi_T \leq \rho \). In order to show that \( L^2(\Gamma \setminus G) \) is tempered, by Proposition 2.7, it suffices to show that the matrix coefficients \( g \mapsto \langle g \cdot f_1, f_2 \rangle \) are in \( L^{2+\varepsilon}(G) \) for all \( \varepsilon > 0 \) and for all \( f_1, f_2 \in C_c(\Gamma \setminus G) \), since \( C_c(\Gamma \setminus G) \) is dense in \( L^2(\Gamma \setminus G) \).

Without loss of generality, we may just consider non-negative functions \( f_1, f_2 \in C_c(\Gamma \setminus G) \). Fix any \( \varepsilon > 0 \). Then using the Cartan decomposition \( G = KA^+K \), we have

\[
\int_G \langle g \cdot f_1, f_2 \rangle^{2+\varepsilon} dg = \int_K \int_{A^+} \int_K \langle k_1 \exp(v) k_2 \cdot f_1, f_2 \rangle^{2+\varepsilon} \Xi(v) dk_1 dv dk_2,
\]

where \( \Xi(v) \sim e^{2\rho(v)} \) (cf. \cite{25}). Denoting \( F_i(\Gamma g) = \max_{k \in \mathcal{K}} f_i(\Gamma g k) \in C_c(\Gamma \setminus G)_K \), we then have

\[
\int_G \langle g \cdot f_1, f_2 \rangle^{2+\varepsilon} dg \ll \int_{A^+} \langle \exp(v) \cdot F_1, F_2 \rangle^{2+\varepsilon} e^{2\rho(v)} dv.
\]

Since \( \psi_T \leq \rho \), we have \( \rho \in D^*_\Gamma \). By Lemma 9.2, there exists \( 0 < c \leq 1 \) such that \( c\rho \in D^*_\Gamma \) and a unit vector \( u_0 \in \text{int} \mathcal{L} \) such that

\[
\psi_T(u_0) = c\rho(u_0).
\]

We now parameterize \( \mathfrak{a}^+ \) as follows: for each \( v \in \ker \rho \), define

\[
t_v := \min\{t \in \mathbb{R}_{>0} : tu_0 + \sqrt{tv} \in \mathfrak{a}^+\}.
\]

Substituting \( u = tu_0 + \sqrt{tv} \) for \( t \geq 0 \) and \( v \in b \cap \ker \rho \) gives \( du = s \cdot t^{r-1} \) \( dt \) \( dv \) for some constant \( s > 0 \). Then (letting \( r = \dim(\mathfrak{a}) \))

\[
\int_{A^+} \langle \exp(u) \cdot F_1, F_2 \rangle^{2+\varepsilon} e^{2\rho(u)} du \ll \int_{\ker \rho} \int_{t_v}^\infty \langle \exp(tu_0 + \sqrt{tv}) \cdot F_1, F_2 \rangle^{2+\varepsilon} e^{2\rho(u_0)} t^{(r-1)/2} dt \, dv.
\]
By Theorem 9.3 ([10, Theorem 7.19 (1)]), there exists \( C = C(F_1, F_2) > 0 \) such that
\[
\left(t^{(r-1)/2}e^{(2-c)t\rho(u_0)}\langle \exp(tu_0 + \sqrt{t}v) \cdot F_1, F_2 \rangle \right) \leq C
\]
for all \((v, t) \in \ker \rho \times [t_v, \infty)\).

Combining this with the trivial bound
\[
\langle g \cdot F_1, F_2 \rangle \leq \|F_1\| \|F_2\|,
\]
we have (again, for all \((v, t) \in \ker \rho \times [t_v, \infty)\)),
\[
\langle \exp(tu_0 + \sqrt{t}v) \cdot F_1, F_2 \rangle^{2+\varepsilon} \leq \left( C + \|F_1\| \|F_2\| \right)^{2+\varepsilon} \left( \min\left\{ 1, t^{-(r-1)/2}e^{-(2-c)t\rho(u_0)} \right\} \right)^{2+\varepsilon} \ll \min\{1, e^{-\eta t\rho(u_0)}\} \leq e^{-\eta t\rho(u_0)},
\]
where \( \eta = (2 - c)(2 + \varepsilon) > 2 \). This gives
\[
\int_G \langle g \cdot f_1, f_2 \rangle^{2+\varepsilon} \, dg \ll \int_{t_v}^{\infty} e^{-\eta t\rho(u_0)} e^{2t\rho(u_0)} t^{(r-1)/2} \, dt \, dv \ll \int_{\mathfrak{a}^+} e^{-(\eta-2)\rho(u)} \, du < \infty.
\]
Therefore \( L^2(\Gamma \backslash G) \) is tempered.

The converse holds for a general discrete subgroup. Suppose now that \( L^2(\Gamma \backslash G) \) is tempered. Then by the definition of temperedness and the estimate of \( \Xi_G(g) \) in (2.9), it follows that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that for any \( f_1, f_2 \in L^2(\Gamma \backslash G)_K \) and \( u \in \mathfrak{a}^+ \),
\[
|\langle \exp(u) \cdot f_1, f_2 \rangle| \leq C_\varepsilon \|f_1\| \|f_2\| e^{-(1-\varepsilon)\rho(u)}.
\]
(9.1)

Applying [32, Prop. 7.3], we get \( \psi_\Gamma \leq \rho \).

Now recall the following recent theorem of Kim, Minsky, and Oh [24]:

**Theorem 9.5.** [24] Let \( \Gamma \) be an Anosov subgroup of the product \( G \) of at least two simple real algebraic groups or \( \Gamma < G = \text{PSL}_d(\mathbb{R}) \) be a Zariski dense Anosov subgroup of a Hitchin subgroup. Then
\[
\psi_\Gamma \leq \rho.
\]

Hence by Theorem 9.4, we get:

**Corollary 9.6.** Let \( \Gamma < G \) be as in Theorem 9.5. Then \( L^2(\Gamma \backslash G) \) is tempered.

**Proofs of Theorem 1.6.** The equivalence (1) \( \Leftrightarrow \) (2) is proved in Theorem 9.4. The equivalence (2) \( \Leftrightarrow \) (3) follow from Theorems 8.1 and 9.4. The implication (1) \( + \) (2) \( \Rightarrow \) (4) follows from Corollary 7.2 by Theorem 2.5.
References

[34] S. Patterson. The Laplacian operator on a Riemann surface. Composition Math. 31 (1975), 83-107