Euclidean traveller in hyperbolic worlds

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Erdos lecture for students

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We will discuss possible closures of a Euclidean line in various geometric spaces.

Imagine Euclidean traveller visiting many different geometric worlds.

**Question**

Which places does she get to see in each world?
Places that she will visit

1. Circle=1-dim torus
2. Euclidean torus (Kronecker, 1884)
3. Closed hyperbolic surface (Hedlund, 1936)
4. Closed hyp. mfld of higher dim (Ratner, 1991)
Rotations of the circle

\[ S^1 = \{ |z| = 1 \} = \mathbb{R}/\mathbb{Z}, \quad \text{via} \quad e^{2\pi i \alpha} \mapsto \alpha \]

\[ R_\theta : S^1 \to S^1, \quad R_\theta(x) = xe^{2\pi i \theta} \]

\[ R_\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad R_\theta(x) = x + \theta \mod \mathbb{Z}. \]

The orbit of \( x \in \mathbb{R}/\mathbb{Z} \) under \( R_\theta \) is

\[ \{ x + n\theta \mod 1 : n \in \mathbb{Z} \} \]

**Theorem**

*Any orbit of \( R_\theta \) is closed (\( \theta \in \mathbb{Q} \)) or dense (\( \theta \notin \mathbb{Q} \)).*
Line on the torus

\[ \mathbb{T}^2 = S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2 \]

\[ \pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2. \]

Line in \( \mathbb{T}^2 \): \( L_{\omega_1, \omega_2} = \pi(\mathbb{R}(\omega_1, \omega_2)) \).
Kronecker’s theorem, 1884

Theorem

Any line $L_{\omega_1,\omega_2}$ in $\mathbb{T}^2$ is

- either closed ($\omega_1/\omega_2 \in \mathbb{Q}$)
- or dense (otherwise).
Line on the $n$-torus

$$\pi : \mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n.$$ 

Line in $\mathbb{T}^n$: $L_{\omega_1, \ldots, \omega_n} = \pi(\mathbb{R}(\omega_1, \ldots, \omega_n)).$

Theorem (Kronecker, 1884)

**Line** = a $k$-dim. subtorus of $\mathbb{T}^n$

where $k = \text{dim}_\mathbb{Q}(\mathbb{Q}\omega_1 + \cdots + \mathbb{Q}\omega_n)$. 

$T^*_k \subset T^n$
Line on the $n$-torus

Let $\Gamma < \mathbb{R}^n$ be a cocompact discrete subgp, i.e., $\Gamma = \sum_{i=1}^{n} \mathbb{Z}v_i$.

$$\pi : \mathbb{R}^n \rightarrow \mathbb{T}^n = \mathbb{R}^n / \Gamma.$$ 

Theorem (Kronecker, 1884)

*For any line $L \subset \mathbb{R}^n / \Gamma$, there exists a Euclidean subspace $V < \mathbb{R}^n$ s.t.*

$$\overline{L} = V / (V \cap \Gamma).$$
Hyperbolic plane: unique simply connected two dim. mfld of sectional curvature $\leq 1$

Hyperbolic surface $\mathbb{H}^2$

Upper half-plane

$$\mathbb{H}^2 = \{(x, y) : y > 0\}$$

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

$$d(p, q) = \inf \int_0^1 \frac{\|c'(t)\|}{y(t)} \, dt \mid c : [0, 1] \to \mathbb{H}^2, c(0) = p, c(1) = q, c(t) = (x(t), y(t))$$

Geodesics $= \text{hyperbolic lines}$

$\mathbb{H}^2 = \{x \in \mathbb{R}^2 : y > 0\}$
Hyperbolic plane: unique simply connected two dim. mfd of sectional curvature \(-1\)

Disk

\[ \mathbb{H}^2 = \{ x^2 + y^2 < 1 \} \]

\[ ds = \frac{2\sqrt{dx^2 + dy^2}}{1 - (x^2 + y^2)} \]
How does a closed (=complete and compact) hyperbolic surface look like?

- A Euclidean torus is \( \Gamma \backslash \mathbb{R}^2 \)
  where \( \Gamma \) is a cocompact discrete subgp of \( \mathbb{R}^2 \).

- A closed hyperbolic surface is \( \Gamma \backslash \mathbb{H}^2 \)
  where \( \Gamma \) is a cocompact discrete subgp of \( \mathbb{H}^2 \cong \text{Isom}^+(\mathbb{H}^2) \).
Isometry group of $\mathbb{H}^2$

$\text{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}^2 = \{z = x + iy : y > 0\}$ by:

$$
\begin{pmatrix}
    a & b \\
    c & d
\end{pmatrix}
\begin{pmatrix}
    z
\end{pmatrix}
= \frac{az + b}{cz + d}
$$

- $\text{PSL}_2(\mathbb{R}) = \text{Isom}^+(\mathbb{H}^2)$
- $\text{PSL}_2(\mathbb{R})/\text{SO}(2) = \mathbb{H}^2$ via $g \mapsto g(i)$
Closed hyperbolic surfaces

Any closed hyp. surface is of the form

\[ S = \Gamma \backslash \mathbb{H}^2 \]

where \( \Gamma < \text{PSL}_2(\mathbb{R}) \) is a co-cpt disc. subgp.

\[ \Gamma = \langle \tau_A, \tau_B, \tau_C, \tau_D \rangle < \text{PSL}_2(\mathbb{R}) \]
Closed hyperbolic surfaces

Topologically, \( S = \Gamma \backslash \mathbb{H}^2 \cong S_g \) for some \( g \geq 2 \):

For each \( g \geq 2 \),

\[
\{ \text{closed hyp. surfaces} \cong S_g \} \cong \mathbb{R}^{6g-6}.
\]
Euclidean lines in $\mathbb{H}^2$

$$\mathbb{H}^2 = \{(x, y) : y > 0\} \quad ds = \frac{\sqrt{dx^2 + dy^2}}{y}$$

Euclidean lines = horocycles
Where does a Euclidean traveller get to visit in the hyperbolic surface?
Euclidean traveller in $S_g$

Theorem (Hedlund, 1936)

Any Euc. line in a closed hyp. surface is dense.
Hyperbolic lines can be very wild

\[ \text{geod} \text{ can be very wild:} \]
$T^1(\mathbb{H}^2) = \text{PSL}_2(\mathbb{R}) \supset gU$

$\mathbb{H}^2 = \text{PSL}_2(\mathbb{R})/\text{SO}(2) \supset \text{Euc. line}$

$\mathbf{U} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$
Homogeneous dynamics

\[ U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \]

\[ T^1(\Gamma \setminus \mathbb{H}^2) = \Gamma \setminus \text{PSL}_2(\mathbb{R}) \supset xU \]

\[ \Gamma \setminus \mathbb{H}^2 = \Gamma \setminus \text{PSL}_2(\mathbb{R}) / \text{SO}(2) \supset \text{Euc. line} \]
Hedlund’s theorem

Theorem (Hedlund, 1936)

For any $x \in \Gamma \backslash PSL_2(\mathbb{R})$,

$$\overline{xU} = \Gamma \backslash PSL_2(\mathbb{R}).$$
Hyperbolic $n$-manifold

\[ \mathbb{H}^n = \{(x_1, \cdots, x_{n-1}, y) : y > 0\}, \quad ds = \sqrt{dx_1^2 + \cdots + dx_{n-1}^2 + dy^2} / y \]

- \[ I \{ g \in SL_n(\mathbb{R}) \mid g(I_{n-1})g^{-1} = (I_{n-1})^3 \} \]

- $\text{Isom}^+(\mathbb{H}^n) = \text{SO}^0(n, 1)$;
- any closed hyp. $n$-mfld is

\[ M = \Gamma \backslash \mathbb{H}^n \]

where $\Gamma < \text{SO}^0(n, 1)$ is a discrete cocpt subgp.
Mostow rigidity thm (1968) implies that $\exists$ only c’bly many closed hyp. $n$-mflds for $n \geq 3$.

**Question**

What are the possible closures of a Euclidean line in $M$?
Orbit closures

$G$ : simple Lie gp (e.g., $\text{SL}_n(\mathbb{R})$, $\text{SO}^\circ(n, 1)$), $\Gamma < G$ a discrete subgp.

Any subgp $U < G$ acts on

$$\Gamma \backslash G \curvearrowright U$$

Question

For a given $x \in \Gamma \backslash G$, what is $\overline{xU}$?

Moore's ergodicity thm (1966): If $\text{Vol}(\Gamma \backslash G) < \infty$ and $U$ non-cpt, for a.e. $x \in \Gamma \backslash G$,

$$\overline{xU} = \Gamma \backslash G.$$
Let $\text{Vol}(\Gamma \backslash G) < \infty$ and $U < G$ be a conn subgp gen. by unipotent elts. For any $x \in \Gamma \backslash G$, 

$$xU = xH$$

for some closed $H < G$.

A matrix $u$ is unipotent if all of its eigenvalues are 1.

E.g., $u = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is not unipotent.
Euc. lines in closed hyp. mflds

\[
F(\Gamma \backslash \mathbb{H}^n) = \Gamma \backslash SO^\circ(n, 1) \supset \overline{xU} \\
\downarrow \quad \downarrow \\
\Gamma \backslash \mathbb{H}^n \supset \text{Euc. line}
\]

\[
U = \left\{ \begin{pmatrix} 1 & (t, 0, \cdots, 0) \\ 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -t^2/2 \\ -t, 0, \cdots, 0 \end{pmatrix}^T : t \in \mathbb{R} \right\}
\]
Theorem (Ratner, Shah 1991)

Let $M = \Gamma \backslash \mathbb{H}^n$ be a closed hyp. mfld.

Euc. line = closed hyp. submfd, up to some translation.

Recall (Kronecker’s thm): In $\mathbb{T}^n = \Gamma \backslash \mathbb{R}^n$,

Euc. line = subtorus.
What about in $\infty$-volume hyperbolic manifolds? Does the topological rigidity of a Euclidean line persist?

- **No** for a general hyp. mfld of $\infty$-volume (e.g., $M \simeq S_g \times \mathbb{R}$); some Euclidean lines have wild closures.

- **Yes** for hyperbolic mflds with "Fuchsian ends"
Hyp. surfaces with Fuchsian ends

For $n = 2$,

**Theorem (Dalbo, 2000)**

*If $S$ is a hyp. surface with Fuchsian ends, any Euclidean line in $S$ is closed or dense.*
For $n \geq 3$, 

\[
\begin{align*}
\{ & \text{c’bly many closed} \\
& \text{hyp. } n\text{-mflds} \} \supset \{ & \text{clsd hyp. } n\text{-mflds} \\
& \text{with codim 1 hyp. submflds} \} \sim \{ & \text{hyp. } n\text{-mflds} \\
& \text{with Fuchsian ends} \}
\end{align*}
\]
Hyperbolic mflds with Fuchsian ends
Theorem (McMullen-Mohammadi-O. 2015, Lee-O. 2020)

Let \( M = \Gamma \backslash \mathbb{H}^n \) be a hyp. mfld with Fuchsian ends. Any Euc. line in \( M \) is closed or

\[ \text{Euc. line} = \text{hyp. submfld} \quad \text{up to translations.} \]