RELATIVELY ANOSOV GROUPS: FINITENESS, MEASURE OF MAXIMAL ENTROPY, AND REPARAMETERIZATION

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Abstract. For a geometrically finite Kleinian group Γ, the Bowen-Margulis-Sullivan measure is finite and is the unique measure of maximal entropy for the geodesic flow, as shown by Sullivan and Otal-Peigné respectively. Moreover, it is strongly mixing by Babillot. We obtain a higher rank analogue of this theorem. Given a relatively Anosov subgroup Γ of a semisimple real algebraic group, there is a family of flow spaces parameterized by linear forms tangent to the growth indicator. We construct a reparameterization of each flow space by the geodesic flow on the Groves-Manning space of Γ which has an exponentially expanding property along unstable foliations. Using this reparameterization, we prove that the Bowen-Margulis-Sullivan measure of each flow space is finite and is the unique measure of maximal entropy. Moreover, it is strongly mixing.

Contents

1. Introduction

For a geometrically finite Kleinian group Γ of $\mathrm{SO}^\circ(n,1) = \mathrm{Isom}^+(\mathbb{H}^n)$, $n \geq 2$, it is a classical result of Sullivan ([29], see also [13]) that the associated Bowen-Margulis-Sullivan measure m^{BMS} on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ is finite. Moreover, Otal-Peigné [24] showed that the BMS measure is the unique measure of maximal entropy. It is also strongly mixing

Oh is partially supported by the NSF grant No. DMS-1900101.

by a theorem of Babillot [1]. In this paper, we obtain a higher rank analogue of these theorems. Let G be a connected semisimple real algebraic group. Anosov subgroups and relatively Anosov subgroups of G are considered as higher rank generalizations of convex cocompact and geometrically finite rank one groups respectively. There is an even bigger class of discrete subgroups called transverse subgroups, which are considered to be generalizations of rank one discrete subgroups. For a transverse subgroup Γ, we have a family of Bowen-Margulis-Sullivan measures $\mathsf{m}_\psi^{\mathrm{BMS}}$ parameterized by a certain collection of linear forms ψ. For each BMS measure $\mathsf{m}_\psi^{\mathrm{BMS}}$, there exists a canonical one-dimensional flow base space $(\Omega_\psi, m_\psi, \phi_t)$ such that $\mathsf{m}_\psi^{\mathrm{BMS}}$ is a measure on the fibered dynamical system over $(\Omega_\psi, m_\psi, \phi_t)$ with fiber $\ker\psi$ and $\mathsf{m}_\psi^{\mathrm{BMS}}$ is equal to the product measure $m_\psi \otimes \mathrm{Leb}_{\ker\psi}$. We also call m_ψ the Bowen-Margulis-Sullivan measure on Ω_ψ.

In this paper, we prove that for relatively Anosov subgroups, the base BMS measure system m_ψ is of finite measure and is the unique measure of maximal entropy for the flow $\{\phi_t\}$. We also show that for any transverse subgroup with finite m_ψ, the dynamical system $(\Omega_\psi, m_\psi, \phi_t)$ is strongly mixing. Therefore $(\Omega_\psi, m_\psi, \phi_t)$ is strongly mixing for relatively Anosov subgroups.

To state these results precisely, we fix a Cartan decomposition $G = KA^+K$ where K is a maximal compact subgroup of G and $A^+ = \exp\mathfrak{a}^+$ is a positive Weyl chamber of a maximal split torus A of G. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K\exp\mu(g)K$ for $g \in G$. Let Π be the set of all simple roots for $(\mathrm{Lie}\, G, \mathfrak{a}^+)$. Associated to a non-empty subset $\theta \subset \Pi$, there is the notion of relatively Anosov and transverse subgroup. Let $\mathcal{F}_\theta = G/P_\theta$ where P_θ is the standard parabolic subgroup associated with θ. Let $\Gamma < G$ be a discrete subgroup and let Λ_θ denote the limit set of Γ in $\mathcal{F}_\theta$ as defined in (2.1), which we assume to have at least 3 points, that is, Γ is non-elementary. We assume that Γ is a θ-transverse (or simply, transverse) subgroup in the rest of the introduction, that is, Γ satisfies

- *regularity*: $\liminf_{\gamma\in\Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$;
- *antipodality*: any $\xi \neq \eta \in \Lambda_{\theta\cup \mathrm{i}(\theta)}$ are in general position (see (2.3)).

Here $\mathrm{i} = -\mathrm{Ad}_{w_0} : \Pi \to \Pi$ denotes the opposition involution where w_0 is the longest Weyl element.

Fibered dynamical systems. Let $\mathfrak{a}_\theta = \bigcap_{\alpha\in\Pi-\theta} \ker\alpha$ and $A_\theta = \exp\mathfrak{a}_\theta$. The centralizer of A_θ is a Levi subgroup of P_θ which is a direct product $A_\theta S_\theta$ where S_θ is a compact central extension of a semisimple algebraic subgroup. The right translation action of A_θ on the quotient space G/S_θ is equivariantly conjugate to the $\mathfrak{a}_\theta$-translation action on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ where $\mathcal{F}_\theta^{(2)}$ consists of all pairs $(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)}$ in general position. The left Γ-action on G/S_θ is not properly discontinuously in general. On the other hand, if we set $\Lambda_\theta^{(2)} = (\Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)}) \cap \mathcal{F}_\theta^{(2)}$, then it is shown in [19, Theorem 9.1] that

Γ acts properly discontinuously on the following space:

$$\tilde{\Omega}_\Gamma := \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta \simeq \{gS_\theta \in G/S_\theta : gP_\theta \in \Lambda_\theta, gw_0P_{\mathrm{i}(\theta)} \in \Lambda_{\mathrm{i}(\theta)}\}.$$

Hence

$$\Omega_\Gamma := \Gamma\backslash\tilde{\Omega}_\Gamma.$$

is a second countable locally compact Hausdorff space on which $\mathfrak{a}_\theta$ acts by translations. Moreover, for each (Γ, θ)-proper[1] linear form $\psi \in \mathfrak{a}_\theta^*$, the space Ω_Γ fibers over a one-dimensional flow space $\Omega_\psi := \Gamma\backslash(\Lambda_\theta^{(2)} \times \mathbb{R})$.

More precisely, via the projection $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$, the Γ-action on $\tilde{\Omega}_\Gamma$ descends to a proper discontinuous action on $\Lambda_\theta^{(2)} \times \mathbb{R}$ [19, Theorem 9.2]. Therefore $\Omega_\psi := \Gamma\backslash(\Lambda_\theta^{(2)} \times \mathbb{R})$ is a second countable locally compact Hausdorff space over which Ω_Γ is a trivial $\ker\psi$-bundle:

$$\begin{array}{c}(\Omega_\Gamma, \mathfrak{a}_\theta) \simeq \Omega_\psi \times \ker\psi \\ \Big\downarrow \\ (\Omega_\psi, \mathbb{R})\end{array}$$

The translation flow $\phi_t(\xi, \eta, s) = (\xi, \eta, s + t)$ on $\Lambda_\theta^{(2)} \times \mathbb{R}$ descends to a translation flow on Ω_ψ which we also denote by $\{\phi_t\}$ by abuse of notation.

For a pair of a (Γ, ψ)-Patterson-Sullivan measure ν on Λ_θ and a $(\Gamma, \psi \circ \mathrm{i})$-Patterson-Sullivan measure ν_{i} on $\Lambda_{\mathrm{i}(\theta)}$, we denote by $\mathsf{m}_\psi^{\mathrm{BMS}} = \mathsf{m}_{\nu,\nu_{\mathrm{i}}}^{\mathrm{BMS}}$ the associated A_θ-invariant Bowen-Margulis-Sullivan measure on Ω_Γ, locally equivalent to the product $\nu \otimes \nu_{\mathrm{i}} \otimes \mathrm{Leb}_{\mathfrak{a}_\theta}$. Similarly, we denote by m_ψ the associated $\{\phi_t\}$-invariant Bowen-Margulis-Sullivan measure on Ω_ψ, locally equivalent to the product $\nu \otimes \nu_{\mathrm{i}} \otimes \mathrm{Leb}_{\mathbb{R}}$. Then $\mathsf{m}_\psi^{\mathrm{BMS}} = m_\psi \otimes \mathrm{Leb}_{\ker\psi}$. Since ν and ν_{i} may not be unique, $\mathsf{m}_\psi^{\mathrm{BMS}}$ and m_ψ are not necessarily determined by ψ. Nevertheless, it is convenient to refer to them as BMS measures associated to ψ.

Relatively Anosov groups. A transverse subgroup $\Gamma < G$ is called relatively Anosov (more precisely relatively θ-Anosov) if Γ is a relatively hyperbolic group and there exists a Γ-equivariant homeomorphism between the Bowditch boundary of Γ and the limit set Λ_θ. When Γ is hyperbolic, the Bowditch boundary is the Gromov boundary of Γ, and the relatively Anosov subgroup Γ is simply an Anosov subgroup. When G is of rank one, relatively Anosov subgroups are precisely geometrically finite Kleinian groups. Recall that for a geometrically finite Kleinian group Γ, there is a unique Patterson-Sullivan measure of dimension equal to the critical exponent δ_Γ. Denote by ψ_Γ^θ the growth indicator of Γ, which is a higher rank version of the critical exponent of Γ. For a relatively Anosov subgroup Γ, associated to a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ *tangent* to ψ_Γ^θ, there exists a unique (Γ, ψ)-Patterson-Sullivan measure on Λ_θ. Hence there exists a unique BMS

[1] ψ is called (Γ, θ)-proper if $\psi \circ \mu : \Gamma \to [-\varepsilon, \infty)$ is a proper map for some $\varepsilon > 0$.

measure m_ψ associated with ψ (see [21], [28] for Anosov groups and [10] for relatively Anosov groups).

For Anosov subgroups, Ω_ψ is known to be homeomorphic to the Gromov geodesic flow space and thus compact ([12], [7], [28]). Indeed, for a transverse subgroup, Γ is Anosov if and only if Ω_ψ is compact [19]. In particular, Ω_ψ is not compact for relatively Anosov but not Anosov subgroups. Analogous to the finitness of the Bowen-Margulis-Sullivan measure for a geometrically finite Kleinian group, we have the following:

Theorem 1.1 (Finiteness and mixing). *Let Γ be a relatively Anosov subgroup of G. For any (Γ,θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ tangent to the growth indicator of Γ,*

$$|m_\psi| < \infty.$$

Moreover $(\Omega_\psi, m_\psi, \phi_t)$ is strongly mixing.

Indeed, we prove the strong mixing in a more general setting of transverse subgroups, which may be considered as a higher rank version of Babillot's mixing theorem (see Theorem 4.1).

Since m_ψ is shown to be a finite measure in Theorem 1.1, the metric entropy $h_{m_\psi}(\{\phi_t\})$ of the normalized probability $m_\psi/|m_\psi|$ with respect to the flow $\{\phi_t\}$ is well-defined. For a (Γ,θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$, the ψ-critical exponent

$$\delta_\psi = \limsup_{T\to\infty} \frac{\log \#\{\gamma \in \Gamma : \psi(\mu(\gamma)) < T\}}{T} \in (0,\infty)$$

is well-defined and moreover, $\delta_\psi = 1$ if and only if ψ is tangent to ψ_Γ^θ ([10, Theorem 10.1], [19, Theorem 4.5]).

Theorem 1.2 (Unique measure of maximal entropy). *Let Γ be a relatively Anosov subgroup of G. For any (Γ,θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ tangent to the growth indicator of Γ,*

m_ψ is the unique measure of maximal entropy for $(\Omega_\psi, \{\phi_t\})$

and the entropy $h_{m_\psi}(\{\phi_t\})$ is equal to $\delta_\psi = 1$.

For Anosov groups, this is due to Sambarino ([27], [28]) which he obtains as a consequence of the thermodynamic formalism. As we do not use the thermodynamic formalism, our proof gives an alternative proof for Anosov subgroups.

Remark 1.3. The identity $\delta_\psi = 1$ is due to the normalization that ψ is tangent to ψ_Γ^θ. Indeed, in the rank one setting, ϕ_t corresponds to the geodesic flow g_{t/δ_Γ} for time t/δ_Γ and m_{δ_Γ} is the unique measure of maximal entropy for the geodesic flow with $h_{m_{\delta_\Gamma}}(\{g_t\}) = \delta_\Gamma$. Hence $h_{m_{\delta_\Gamma}}(\{\phi_t\}) = h_{m_{\delta_\Gamma}}(\{g_t\})/\delta_\Gamma = 1$.

The main technical ingredient of Theorem 1.1 and Theorem 1.2 is the following coarse reparameterization theorem, which is also of independent interest. We denote by d_{GM} the distance in the Groves-Manning cusp space of Γ. Let $\mathcal{G}$ be the space of all parameterized bi-infinite geodesics in the Groves-Manning cusp space [15] associated to the relatively hyperbolic group Γ. Let $\varphi_s : \mathcal{G} \to \mathcal{G}$ be the time-s geodesic flow for $s \in \mathbb{R}$, i.e., $(\varphi_s \sigma)(\cdot) = \sigma(\cdot + s)$:

Theorem 1.4 (Reparameterization)**.** *There exists a continuous surjective proper map*

$$\Psi : \Gamma \backslash \mathcal{G} \to \Omega_\psi$$

and a continuous cocycle $\mathsf{t} : \mathcal{G} \times \mathbb{R} \to \mathbb{R}$ *such that for all* $\sigma \in \mathcal{G}$ *and* $s \in \mathbb{R}$*,*

(1) $\Psi([\varphi_s \sigma]) = \phi_{\mathsf{t}(\sigma,s)} \Psi([\sigma])$*;*
(2) $\mathsf{t}(\sigma, s) = -\mathsf{t}(\varphi_s \sigma, -s)$*;*
(3) *for some absolute constant* $B > 0$*,*

$$a|s| - B \leq \mathsf{t}(\sigma, |s|) \leq a'|s| + B$$

where

$$0 < a := \liminf_{\gamma \in \Gamma} \frac{\psi(\mu(\gamma))}{d_{GM}(e, \gamma)} \quad \textit{and} \quad a' := 3 \limsup_{\gamma \in \Gamma} \frac{\psi(\mu(\gamma))}{d_{GM}(e, \gamma)} < \infty.$$

Moreover, ϕ_t *has an exponentially expanding property on unstable foliations of* $\tilde{\Omega}_\psi := \Lambda_\theta^{(2)} \times \mathbb{R}$ *as described in Theorem 7.1.*

The map Ψ gives a thick-thin decomposition of Ω_ψ which is crucial in our proof of the finiteness, together with the work of Canary-Zhang-Zimmer [10] on the critical exponents of peripheral subgroups of Γ. The exponentially expanding property of ϕ_t is crucial in constructing a measurable partition of $\tilde{\Omega}_\psi$ subordinated to unstable foliations (Proposition 9.2), as required in the proof of Theorem 1.2 on the measure of maximal entropy.

Acknowledgements. We were informed that the ongoing work of Blayac, Canary, Zhu and Zimmer [4] contains a different proof of Theorem 1.1.

2. Preliminaries

We review some basic facts about Lie groups, following [19, Section. 2] which we refer for more details. Throughout the paper, let G be a connected semisimple real algebraic group. Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G, M is the maximal compact subgroup of P commuting with A and N is the unipotent radical of P. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of G and A. Fix a positive Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g) K$ for $g \in G$. Let $X = G/K$ be the associated Riemannian symmetric space and $o = [K] \in X$. Fix a K-invariant norm $\|\cdot\|$ on $\mathfrak{g}$. This induces the left G-invariant Riemannian metric d on X.

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots, $\Phi^+ \subset \Phi$ the set of all positive roots, and $\Pi \subset \Phi^+$ the set of all simple roots. Fix a Weyl element $w_0 \in K$ of order 2 in the normalizer of A representing the longest Weyl element so that $\operatorname{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$. The map

$$\mathrm{i} = -\operatorname{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$$

is called the opposition involution. It induces an involution $\Phi \to \Phi$ preserving Π, for which we use the same notation i, such that $\mathrm{i}(\alpha) \circ \operatorname{Ad}_{w_0} = -\alpha$ for all $\alpha \in \Phi$.

Henceforth, we fix a non-empty subset $\theta \subset \Pi$. Let P_θ denote a standard parabolic subgroup of G corresponding to θ; that is, P_θ is generated by MA and all root subgroups U_α, where α ranges over all positive roots which are not $\mathbb{Z}$-linear combinations of $\Pi - \theta$. Hence $P_\Pi = P$. Let

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha, \qquad \mathfrak{a}_\theta^+ = \mathfrak{a}_\theta \cap \mathfrak{a}^+,$$

$$A_\theta = \exp \mathfrak{a}_\theta, \quad \text{and} \quad A_\theta^+ = \exp \mathfrak{a}_\theta^+.$$

Let $p_\theta : \mathfrak{a} \to \mathfrak{a}_\theta$ denote the projection invariant under all Weyl elements fixing $\mathfrak{a}_\theta$ pointwise. We write $\mu_\theta := p_\theta \circ \mu : G \to \mathfrak{a}_\theta^+$. The space $\mathfrak{a}_\theta^* = \operatorname{Hom}(\mathfrak{a}_\theta, \mathbb{R})$ can be identified with the subspace of $\mathfrak{a}^*$ which is p_θ-invariant: $\mathfrak{a}_\theta^* = \{\psi \in \mathfrak{a}^* : \psi \circ p_\theta = \psi\}$. We have the Levi-decomposition $P_\theta = L_\theta N_\theta$ where L_θ is the centralizer of A_θ and $N_\theta = R_u(P_\theta)$ is the unipotent radical of P_θ. We set $M_\theta = K \cap P_\theta \subset L_\theta$.

Limit set Λ_θ. We set

$$\mathcal{F}_\theta = G/P_\theta.$$

The subgroup K acts transitively on $\mathcal{F}_\theta$, and hence $\mathcal{F}_\theta \simeq K/M_\theta$.

Definition 2.1. For a sequence $g_i \in G$ and $\xi \in \mathcal{F}_\theta$, we write $\lim_{i\to\infty} g_i = \xi$ and say g_i converges to ξ if

- for each $\alpha \in \theta$, $\alpha(\mu(g_i)) \to \infty$ as $g_i \to \infty$;
- $\lim_{i\to\infty} \kappa_i \xi_\theta = \xi$ in $\mathcal{F}_\theta$ for some $\kappa_i \in K$ such that $g_i \in \kappa_i A^+ K$.

The θ-limit set of a discrete subgroup Γ can be defined as follows:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) := \{\lim \gamma_i \in \mathcal{F}_\theta : \gamma_i \in \Gamma\} \tag{2.1}$$

where $\lim \gamma_i$ is defined as in Definition 2.1. If Γ is Zariski dense, this is the unique Γ-minimal subset of $\mathcal{F}_\theta$ ([2], [26]).

Jordan projections. Any $g \in G$ can be written as the commuting product $g = g_h g_e g_u$ where g_h is hyperbolic, g_e is elliptic and g_u is unipotent. The hyperbolic component g_h is conjugate to a unique element $\exp \lambda(g) \in A^+$ and $\lambda(g)$ is called the Jordan projection of g. We write $\lambda_\theta := p_\theta \circ \lambda$.

Theorem 2.2. [3] *For any Zariski dense subgroup $\Gamma < G$, the subgroup generated by $\{\lambda(\gamma) \in \mathfrak{a}^+ : \gamma \in \Gamma\}$ is dense in $\mathfrak{a}$.*

Busemann map and Gromov product. The $\mathfrak{a}$-valued Busemann map $\beta : \mathcal{F}_\Pi \times G \times G \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi)$$

where $\sigma(g^{-1}, \xi) \in \mathfrak{a}$ is the unique element such that $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for any $k \in K$ with $\xi = kP$. For $(\xi, g, h) \in \mathcal{F}_\theta \times G \times G$, we define

$$\beta^\theta_\xi(g, h) := p_\theta(\beta_{\xi_0}(g, h)) \tag{2.2}$$

for any $\xi_0 \in \mathcal{F}_\Pi$ projecting to ξ. This is well-defined independent of the choice of ξ_0 [26, Lemma 6.1].

Two points $\xi \in \mathcal{F}_\theta$ and $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ are said to be in general position if

$$\xi = gP_\theta \text{ and } \eta = gw_0 P_{\mathrm{i}(\theta)} \text{ for some } g \in G. \tag{2.3}$$

We set

$$\mathcal{F}^{(2)}_\theta = \{(\xi, \eta) \in \mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)} : \xi, \eta \text{ are in general position}\} \tag{2.4}$$

which is the unique open G-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{\mathrm{i}(\theta)}$ under the diagonal G-action.

For $(\xi, \eta) \in \mathcal{F}^{(2)}_\theta$, we define the $\mathfrak{a}_\theta$-valued Gromov product as

$$\langle \xi, \eta \rangle = \beta^\theta_\xi(e, g) + \mathrm{i}(\beta^{\mathrm{i}(\theta)}_\eta(e, g)) \tag{2.5}$$

where $g \in G$ satisfies $(gP_\theta, gw_0 P_{\mathrm{i}(\theta)}) = (\xi, \eta)$. This does not depend on the choice of g [19, Lemma 9.13].

Patterson-Sullivan measures. For $\psi \in \mathfrak{a}^*_\theta$, a (Γ, ψ)-conformal measure is a Borel probability measure on $\mathcal{F}_\theta$ such that

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta^\theta_\xi(e, \gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_\theta \tag{2.6}$$

where $\gamma_*\nu(D) = \nu(\gamma^{-1}D)$ for any Borel subset $D \subset \mathcal{F}_\theta$ and β^θ_ξ denotes the $\mathfrak{a}_\theta$-valued Busemann map defined in (2.2). A (Γ, ψ)-conformal measure supported on Λ_θ is called a (Γ, ψ)-Patterson Sullivan measure.

Growth indicator. Let $\Gamma < G$ be a θ-discrete subgroup, that is, $\mu_\theta|_\Gamma$ is a proper map. The θ-growth indicator $\psi^\theta_\Gamma : \mathfrak{a}_\theta \to [-\infty, \infty)$ is a higher rank version of the critical exponent, which is defined as follows: If $u \in \mathfrak{a}_\theta$ is non-zero,

$$\psi^\theta_\Gamma(u) = \|u\| \inf_{u \in \mathcal{C}} \tau^\theta_\mathcal{C} \tag{2.7}$$

where $\tau^\theta_\mathcal{C}$ is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s\|\mu_\theta(\gamma)\|}$ and $\mathcal{C} \subset \mathfrak{a}_\theta$ ranges over all open cones containing u. Set $\psi^\theta_\Gamma(0) = 0$. This definition was given in [19], extending Quint's growth indicator [25] to a general θ.

For Γ transverse and ψ (Γ, θ)-proper, it is proved in [19] that if there exists a (Γ, ψ)-conformal measure on $\mathcal{F}_\theta$, then

$$\psi \geq \psi^\theta_\Gamma.$$

We say that $\psi \in \mathfrak{a}_\theta^*$ is tangent to ψ_Γ^θ if the equality is achieved by some non-zero element of $\mathfrak{a}_\theta$. As in the rank one setting, interesting geometry and dynamics occur at the critical case that ψ is a tangent form.

3. Vector bundle structure of the non-wandering set Ω_Γ

We fix a non-empty subset θ of Π. In this section, we assume that $\Gamma < G$ is a non-elementary θ-transverse subgroup, that is, Γ satisfies

- (non-elementary): $\#\Lambda_\theta \geq 3$;
- (regularity): $\liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty$ for all $\alpha \in \theta$; and
- (antipodality): any two distinct $\xi, \eta \in \Lambda_{\theta \cup \mathrm{i}(\theta)}$ are in general position.

We will define a locally compact Hausdorff space Ω_Γ which is the non-wandering set for the action of A_θ. Recall that the centralizer of A_θ is the direct product $A_\theta S_\theta$ where S_θ is a compact central extension of a connected semisimple real algebraic subgroup. Note that S_θ is compact if and only if $\theta = \Pi$.

The homogeneous space G/S_θ can be identified with the space $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ via the map

$$gS_\theta \mapsto (gP_\theta, gw_0 P_{\mathrm{i}(\theta)}, \beta^\theta_{gP_\theta}(e, g)),$$

and the left G-action on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ given by

$$g(\xi, \eta, v) = (g\xi, g\eta, v + \beta^\theta_\xi(g^{-1}, e))$$

makes the above identification G-equivariant. Since S_θ commutes with A_θ, the diagonal subgroup A_θ acts on G/S_θ on the right, and this action is conjugate to the action of $\mathfrak{a}_\theta$ on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ by the translation on the last component. Since S_θ is not compact in general, the action of Γ on $\mathcal{F}_\theta^{(2)} \times \mathfrak{a}_\theta$ is not properly discontinuous. However for Γ transverse, the Γ-action restricted to the subspace $\tilde{\Omega}_\Gamma := \Lambda_\theta^{(2)} \times \mathfrak{a}_\theta$ turns out to be properly discontinuous where $\Lambda_\theta^{(2)} = \mathcal{F}_\theta^{(2)} \cap (\Lambda_\theta \times \Lambda_{\mathrm{i}(\theta)})$ [19, Theorem 9.1]. Hence we obtain the locally compact second countable Hausdorff space

$$\Omega_\Gamma := \Gamma \backslash \tilde{\Omega}_\Gamma,$$

which is the non-wandering set for the right A_θ-action.

For each (Γ, θ)-proper form $\psi \in \mathfrak{a}_\theta^*$, Ω_Γ admits a $\ker \psi$-bundle structure over a non-wandering set Ω_ψ for a one-dimensional flow. More precisely,

Theorem 3.1. [19, Theorem 9.2] *The Γ-action on the space $\tilde{\Omega}_\psi := \Lambda_\theta^{(2)} \times \mathbb{R}$ given by*

$$\gamma(\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \psi(\beta^\theta_\xi(\gamma^{-1}, e)))$$

is properly discontinuous. Thus the space

$$\Omega_\psi := \Gamma \backslash \tilde{\Omega}_\psi = \Gamma \backslash (\Lambda_\theta^{(2)} \times \mathbb{R}) \tag{3.1}$$

is a locally compact second countable Hausdorff space equipped with the translation flow $\{\phi_t\}$ on the $\mathbb{R}$-component.

Explicitly, the translation flow $\{\phi_t\}$ is defined as follows: for $t \in \mathbb{R}$ and $(\xi, \eta, s) \in \tilde{\Omega}_\psi$,

$$\phi_t(\xi, \eta, s) = (\xi, \eta, s + t).$$

This flow $\{\phi_t\}$ on $\tilde{\Omega}_\psi$ commutes with the Γ-action, and hence induces the one-dimensional flow on Ω_ψ which we also denote by ϕ_t by abusing notations.

Consider the projection $\Omega_\Gamma \to \Omega_\psi$ induced by the Γ-equivariant projection $\tilde{\Omega}_\Gamma \to \tilde{\Omega}_\psi$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$. This is a principal $\ker \psi$-bundle, which is trivial since $\ker \psi$ is a vector space. It follows that there exists a $\ker \psi$-equivariant homeomorphism between Ω_Γ and $\Omega_\psi \times \ker \psi$.

$$\begin{array}{l} \Omega_\Gamma \simeq \Omega_\psi \times \ker \psi \\ \Big\downarrow \\ \Omega_\psi \end{array}$$

Let ν and ν_{i} be a pair of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-Patterson-Sullivan measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. The Bowen-Margulis-Sullivan measure $\mathsf{m}_\psi^{\mathrm{BMS}}$ on Ω_Γ associated with the pair (ν, ν_{i}) is the A_θ-invariant measure induced by the Γ-invariant measure $d\tilde{\mathsf{m}}_\psi^{\mathrm{BMS}}(\xi, \eta, v) := e^{\psi(\langle \xi, \eta \rangle)} d\nu(\xi) d\nu_{\mathrm{i}}(\eta) d\,\mathrm{Leb}_{\mathfrak{a}_\theta}$ on $\tilde{\Omega}_\Gamma$, where $\langle \cdot, \cdot \rangle$ denotes the Gromov product (2.5) and $d\,\mathrm{Leb}_{\mathfrak{a}_\theta}$ denotes the Lebesgue measure on $\mathfrak{a}_\theta$.

We also have a $\{\phi_t\}$-invariant Radon measure m_ψ on Ω_ψ induced by the Γ-invariant measure

$$d\tilde{m}_\psi(\xi, \eta, s) := e^{\psi(\langle \xi, \eta \rangle)} d\nu(\xi) d\nu_{\mathrm{i}}(\eta) ds \tag{3.2}$$

on $\tilde{\Omega}_\psi$ where ds denotes the Lebesgue measure on $\mathbb{R}$. The measure m_ψ is also referred to Bowen-Margulis-Sullivan measure on Ω_ψ associated with the pair (ν, ν_{i}). By the $\ker \psi$-equivariant homeomorphism $\Omega_\Gamma \simeq \Omega_\psi \times \ker \psi$, $\mathsf{m}_\psi^{\mathrm{BMS}}$ disintegrates over the measure m_ψ with conditional measure being the Lebesgue measure $\mathrm{Leb}_{\ker \psi}$ so that

$$\mathsf{m}_\psi^{\mathrm{BMS}} = m_\psi \otimes \mathrm{Leb}_{\ker \psi}.$$

4. Strong mixing for transverse groups with finite BMS measure

Let $\Gamma < G$ be a non-elementary θ-transverse subgroup. Fix a (Γ, θ)-proper form $\psi \in \mathfrak{a}_\theta^*$ and a pair (ν, ν_{i}) of (Γ, ψ) and $(\Gamma, \psi \circ \mathrm{i})$-Patterson-Sullivan measures on Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ respectively. Let Ω_ψ be as in Theorem 3.1 and $m_\psi = m_\psi(\nu, \nu_{\mathrm{i}})$ denote a BMS measure on Ω_ψ associated to a pair (ν, ν_{i}).

This section is devoted to the proof of the following:

Theorem 4.1. *If $|m_\psi| < \infty$, then $(\Omega_\psi, m_\psi, \phi_t)$ is strongly mixing. That is, for any $f_1, f_2 \in L^2(\Omega_\psi, m_\psi)$,*

$$\lim_{|t| \to \infty} \int f_1(\phi_t(x)) f_2(x)\ dm_\psi(x) = \frac{1}{|m_\psi|} \int f_1\ dm_\psi \int f_2\ dm_\psi.$$

We begin by observing the ergodicity of m_ψ:

Theorem 4.2. *If $|m_\psi| < \infty$, then $(\Omega_\psi, m_\psi, \phi_t)$ is ergodic.*

Proof. By the Poincaré recurrence theorem, the dynamical system $(\Omega_\psi, m_\psi, \phi_t)$ is conservative. Hence it follows from the higher rank Hopf-Tsuji-Sullivan dichotomy [19, Theorem 10.2] that $(\Omega_\psi, m_\psi, \phi_t)$ is ergodic. $\square$

Although the flow space Ω_ψ was not considered, Theorem 4.2 can also be deduced from [11] once Ω_ψ is shown to make sense. See also [22] and [28] for Anosov cases.

θ-transitivity subgroups. For $g \in G$, we set $g^+ := gP_\theta \in \mathcal{F}_\theta$ and $g^- := gw_0P_{\mathrm{i}(\theta)} \in \mathcal{F}_{\mathrm{i}(\theta)}$. Set $N_\theta^+ = w_0 N_{\mathrm{i}(\theta)} w_0^{-1}$. We use the following notion of θ-transitivity subgroup:

Definition 4.3. For $g \in G$ with $(g^+, g^-) \in \Lambda_\theta^{(2)}$, we define the subset $\mathcal{H}_\Gamma^\theta(g)$ of A_θ as follows: for $a \in A_\theta$, $a \in \mathcal{H}_\Gamma^\theta(g)$ if and only if there exist $\gamma \in \Gamma$, $s \in S_\theta$ and a sequence $n_1, \cdots, n_k \in N_\theta \cup N_\theta^+$, such that

(1) $((gn_1 \cdots n_r)^+, (gn_1 \cdots n_r)^-) \in \Lambda_\theta^{(2)}$ for all $1 \le r \le k$; and
(2) $gn_1 \cdots n_k = \gamma g a s$.

It is not hard to see that $\mathcal{H}_\Gamma^\theta(g)$ is a subgroup (cf. [31, Lemma 3.1]).

Proposition 4.4. *For any $g \in G$ such that $(g^+, g^-) \in \Lambda_\theta^{(2)}$, the subgroup $\psi(\log \mathcal{H}_\Gamma^\theta(g))$ is dense in $\mathbb{R}$.*

Proof. It was shown in [18, Proposition 7.3] that if Γ is a Zariski dense θ-transverse subgroup and if $g \in G$ is such that $(g^+, g^-) \in \Lambda_\theta^{(2)}$, then the subgroup $\mathcal{H}_\Gamma^\theta(g)$ is dense in A_θ, by proving that for a Schottky subgroup $\Gamma_0 < \Gamma$, the set of Jordan projections $\lambda_\theta(\Gamma_0)$ is contained in $\log \mathcal{H}_\Gamma^\theta(g)$. The Zariski dense hypothesis was used to guarantee that Γ_0 can be taken to be Zariski dense, and hence $\lambda_\theta(\Gamma_0)$ generates a dense subgroup in $\mathfrak{a}_\theta$ ([3], Theorem 2.2).

In general, let H be the Zariski closure of Γ and consider the Levi decomposition of H: $H = LU$ where L is a reductive algebraic subgroup and U the unipotent radical of H. Moreover, we have a Cartan decomposition $G = KA^+K$ so that $L = (K \cap L)(A^+ \cap L)(K \cap L)$ by [23]. If $\pi : H \to L$ denotes the projection, then $\pi(\Gamma)$ is Zariski dense in L and hence its Jordan projection generates a dense subgroup of $\mathfrak{a} \cap \operatorname{Lie} L$. This allows the same proof of [18, Proposition 7.3] to work within L, and hence the claim follows. $\square$

Contractions by flow on Ω_ψ. For $g \in G$, we write

$$[g] := (g^+, g^-, \psi(\beta_{g^+}^\theta(e, g))) \in \mathcal{F}_\theta^{(2)} \times \mathbb{R}.$$

We mainly consider the case when $[g] \in \tilde{\Omega}_\psi = \Lambda_\theta^{(2)} \times \mathbb{R}$, that is, when $(g^+, g^-) \in \Lambda_\theta^{(2)}$. For $[g] \in \tilde{\Omega}_\psi$, we denote by $\Gamma[g] \in \Omega_\psi$ the element of Ω_ψ obtained as the projection of $[g]$ by $\tilde{\Omega}_\psi \to \Omega_\psi$.

We set for $g \in G$ such that $[g] \in \tilde{\Omega}_\psi$,

$$\begin{aligned} \tilde{W}^+([g]) &:= \{[gn] \in \tilde{\Omega}_\psi : n \in N_\theta^+\}; \\ \tilde{W}^-([g]) &:= \{[gn] \in \tilde{\Omega}_\psi : n \in N_\theta\}. \end{aligned} \tag{4.1}$$

The elements of $\tilde{W}^\pm([g])$ can be described as follows:

Lemma 4.5. [18, Lemma 7.4] *Let $g \in G$, $n \in N_\theta^+$, and $n' \in N_\theta$. Then*

$$\begin{aligned} [gn] &= \left((gn)^+, g^-, \psi\left(\beta^\theta_{g^+}(e,g) + \langle (gn)^+, g^-\rangle - \langle (g^+, g^-)\rangle\right)\right); \\ [gn'] &= \left(g^+, (gn')^-, \psi\left(\beta^\theta_{g^+}(e,g)\right)\right). \end{aligned}$$

These are leaves of foliations $\tilde{W}^\pm := \{\tilde{W}^+([g]) : [g] \in \tilde{\Omega}_\psi\}$. For $z \in \Omega_\psi$, we set

$$W^+(z) := \Gamma\backslash\tilde{W}^+([g]), \quad \text{and} \quad W^-(z) := \Gamma\backslash\tilde{W}^-([g]) \tag{4.2}$$

where $g \in G$ is such that $\Gamma[g] = z$. The following proposition says that we may consider $W^+ := \{W^+(z) : z \in \Omega_\psi\}$ and $W^- := \{W^-(z) : z \in \Omega_\psi\}$ as unstable and stable foliations for the flow ϕ_t in Ω_ψ: note that since Ω_ψ is a locally compact second countable Hausdorff space by Theorem 3.1, so is its one-point compactification Ω_ψ^*. Hence Ω_ψ^* is metrizable. Therefore, we can choose a metric d on Ω_ψ which is a restriction of a metric on Ω_ψ^*. That we can use this kind of metric d to prove the following proposition was first observed in [4].

Proposition 4.6. [18, Proposition 7.6] *Let $z \in \Omega_\psi$. We have*

(1) *if $x, y \in W^+(z)$, then*

$$\mathsf{d}(\phi_{-t}(x), \phi_{-t}(y)) \to 0 \quad \text{as } t \to +\infty.$$

(2) *if $x, y \in W^-(z)$, then*

$$\mathsf{d}(\phi_t(x), \phi_t(y)) \to 0 \quad \text{as } t \to +\infty.$$

Moreover, the convergence is uniform on compact subsets.

Proof of Theorem 4.1. We are now ready to prove the strong mixing. We recall the following lemma proved by Babillot:

Lemma 4.7. [1, Lemma 1] *Let $(\mathcal{X}, m, \{T_t\}_{t\in\mathbb{R}})$ be a probability measure-preserving system. Let $f \in L^2(\mathcal{X}, m)$ be such that $\int f dm = 0$. Suppose that $f \circ T_{t_i} \not\to 0$ weakly*[2] *for some $t_i \to \infty$. Then there exists a non-constant function F such that by passing to a subsequence,*

$$f \circ T_{t_i} \to F \quad \text{and} \quad f \circ T_{-t_i} \to F \quad \text{weakly} \quad \text{as } i \to \infty.$$

[2] $f_n \to 0$ weakly if and only if $\int f_n g \, dm \to 0$ for all $g \in L^2(\mathcal{X}, m)$

The following is an easy observation in measure theory:

Lemma 4.8. *Let $(\mathcal{X}, m)$ be a probability measure space. If $f_i \to F$ weakly in $L^2(\mathcal{X}, m)$, then there exists a subsequence f_{i_j} such that the Cesaro average converges:*

$$\frac{1}{\ell^2}\sum_{j=1}^{\ell^2} f_{i_j} \to F \quad m\text{-a.e.}$$

Now going back to our setting, let $f_1, f_2 \in L^2(\Omega_\psi, m_\psi)$. We may assume that m_ψ is a probability measure. It suffices to show that for any $f \in L^2(\Omega_\psi, m_\psi)$ with $\int f dm_\psi = 0$, we have $f \circ \phi_t \to 0$ weakly as $|t| \to \infty$. Since $C_c(\Omega_\psi)$ is dense in $L^2(\Omega_\psi, m_\psi)$, we may assume without loss of generality that f is a continuous function with compact support on Ω_ψ. Suppose that $f \circ \phi_t \not\to 0$ weakly as $t \to \infty$. By Lemma 4.7 and Lemma 4.8, there exists a non-constant function $F : \Omega_\psi \to \mathbb{R}$ and a subsequence $t_i \to \infty$ such that

$$\frac{1}{\ell^2}\sum_{i=1}^{\ell^2} f \circ \phi_{t_i} \to F \quad \text{and} \quad \frac{1}{\ell^2}\sum_{i=1}^{\ell^2} f \circ \phi_{-t_i} \to F \quad m_\psi\text{-a.e. as } \ell \to \infty. \tag{4.3}$$

We claim that F is invariant under the flow ϕ_t; this yields a contradiction to the ergodicity of $(\Omega_\psi, m_\psi, \phi_t)$ obtained in Theorem 4.2.

Let $W_0 = \{x \in \Omega_\psi : (4.3) \text{ holds}\}$, which is m_ψ-conull. Since f is uniformly continuous, it follows from Proposition 4.6 that if $g \in G$ and $n \in N_\theta \cup N_\theta^+$ are such that $[g], [gn] \in \tilde{\Omega}_\psi$ and $\Gamma[g], \Gamma[gn] \in W_0$, then

$$F(\Gamma[g]) = F(\Gamma[gn]).$$

Denote by $\tilde{W}_0$ and $\tilde{F}$ the Γ-invariant lifts of W_0 and F to $\tilde{\Omega}_\psi$ respectively. We set

$$W_1 := \{(\xi, \eta) : (\xi, \eta, t) \in \tilde{W}_0 \text{ for Leb-a.e. } t\}.$$

We also set

$$W = \{(\xi, \eta) \in W_1 : (\xi, \eta'), (\xi', \eta) \in W_1 \text{ for } \nu\text{-a.e. } \xi' \text{ and } \nu_{\mathrm{i}}\text{-a.e. } \eta'\}.$$

Recall that we also denote by $\{\phi_t\}$ the translation flow on $\tilde{\Omega}_\psi$. For any $\varepsilon > 0$ and $x \in \tilde{\Omega}_\psi$, let

$$F_\varepsilon(x) := \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon} \tilde{F}(\phi_s(x))\, ds.$$

Then F_ε is continuous on each $\{\phi_t\}$-orbit and as $\varepsilon \to 0$, we have the convergence $F_\varepsilon \to \tilde{F}$ m_ψ-a.e. Hence it suffices to show that F_ε is invariant under the flow ϕ_t.

By the definition of W and the observation on W_0 made above, we have that if $g \in G$ and $n \in N_\theta \cup N_\theta^+$ are such that $[g], [gn] \in W \times \mathbb{R} \subset \tilde{\Omega}_\psi$, then $F_\varepsilon([g]) = F_\varepsilon([gn])$. Fix $g \in G$ such that $[g] \in W \times \mathbb{R}$ and let $t_0 \in \psi(\log \mathcal{H}_\Gamma^\theta(g))$ and $a \in \mathcal{H}_\Gamma^\theta(g)$ such that $\psi(\log a) = t_0$. We then have $\phi_{t_0}([g]) = [ga]$. By the definition of the θ-transitivity subgroup, there exist $\gamma \in \Gamma$, $s \in S_\theta$, and a sequence $n_1, \cdots, n_k \in N_\theta \cup N_\theta^+$, such that

(1) $((gn_1 \cdots n_r)^+, (gn_1 \cdots n_r)^-) \in \Lambda_\theta^{(2)}$ for all $1 \le r \le k$;
(2) $gn_1 \cdots n_k = \gamma gas$.

As in the proof of [18, Proposition 7.8], there exist a sequence $a_j \in A_\theta$ and a sequence of k-tuples $(n_{1,j}, \cdots, n_{k,j}) \in \prod_{i=1}^k N_\theta \cup N_\theta^+$ converging to a and $(n_1, \cdots, n_k)$ respectively as $j \to \infty$, and such that for each $j \ge 1$, we have

$$[gn_{1,j} \cdots n_{r,j}] \in W \times \mathbb{R} \quad \text{for all } 1 \le r \le k \quad \text{and} \quad [gn_{1,j} \cdots n_{k,j}] = [\gamma g a_j].$$

Therefore, we have for each $j \ge 1$ that

$$\begin{aligned} F_\varepsilon([g]) &= F_\varepsilon([gn_{1,j}]) = \cdots = F_\varepsilon([gn_{1,j} \cdots n_{k-1,j}]) = F_\varepsilon([gn_{1,j} \cdots n_{k,j}]) \\ &= F_\varepsilon([\gamma g a_j]) = F_\varepsilon([g a_j]). \end{aligned}$$

Taking the limit $j \to \infty$, it follows from the continuity of F_ε on each $\{\phi_t\}$-orbit that

$$F_\varepsilon([g]) = F_\varepsilon([ga]) = (F_\varepsilon \circ \phi_{t_0})([g]).$$

Since $\psi(\log \mathcal{H}_\Gamma^\theta(g))$ is dense in $\mathbb{R}$ by Proposition 4.4, this implies that

$$F_\varepsilon([g]) = (F_\varepsilon \circ \phi_t)([g]) \quad \text{for all } t \in \mathbb{R}.$$

Since $[g] \in W \times \mathbb{R}$ is arbitrary and $(\nu \otimes \nu_{\mathrm{i}})(W) = 1$, this completes the proof. $\square$

5. Relatively Anosov groups

Relatively Anosov groups are relatively hyperbolic groups as abstract groups, which we now define. Let Γ be a countable group acting on a compact metrizable space $\mathcal{X}$ by homeomorphisms. This action is called a *convergence group action* if for any sequence of distinct elements $\gamma_n \in \Gamma$, there exist a subsequence γ_{n_k} and $a, b \in \mathcal{X}$ such that as $k \to \infty$, $\gamma_{n_k}(x)$ converges to a for all $x \in \mathcal{X} - \{b\}$, uniformly on compact subsets. An element $\gamma \in \Gamma$ of infinite order fixes either exactly two points in $\mathcal{X}$ or exactly one point in $\mathcal{X}$. In the former case, we call γ loxodromic, and parabolic otherwise. An infinite subgroup $P < \Gamma$ is called parabolic if P fixes some point in $\mathcal{X}$ and every infinite order element of P is parabolic.

A point $\xi \in \mathcal{X}$ is called a conical limit point if there exist a sequence of distinct elements $\gamma_n \in \Gamma$ and distinct points $a, b \in \mathcal{X}$ such that as $n \to \infty$, $\gamma_n \xi \to a$ and $\gamma_n^{-1} \eta \to b$ for all $\eta \in \mathcal{X} - \{\xi\}$. A point $\xi \in \mathcal{X}$ is called a parabolic limit point if ξ is fixed by a parabolic subgroup of Γ. We say that a parabolic limit point $\xi \in \mathcal{X}$ is bounded if $\mathrm{Stab}_\Gamma(x) \backslash (\mathcal{X} - \{\xi\})$ is compact. The action of Γ on $\mathcal{X}$ is called a *geometrically finite convergence group action* if every point of $\mathcal{X}$ is either conical or bounded parabolic limit point.

Let Γ be a finitely generated group and $\mathcal{P}$ a finite collection of finitely generated infinite subgroups of Γ. We say that Γ is *hyperbolic relative to* $\mathcal{P}$ (or that $(\Gamma, \mathcal{P})$ is *relatively hyperbolic*), if Γ admits a geometrically finite convergence group action on some compact perfect metrizable space $\mathcal{X}$ and the collection of maximal parabolic subgroups is

$$\mathcal{P}^\Gamma := \{\gamma P \gamma^{-1} : P \in \mathcal{P}, \gamma \in \Gamma\}.$$

Bowditch [6] showed that for Γ hyperbolic relative to $\mathcal{P}$, the space $\mathcal{X}$ satisfying the above hypothesis is unique up to a Γ-equivariant homeomorphism. Hence this space is called Bowditch boundary and denoted by $\partial(\Gamma, \mathcal{P})$.

The Groves-Manning cusp space. Let Γ be a hyperbolic group relative to $\mathcal{P}$. The Groves-Manning cusp space for $(\Gamma, \mathcal{P})$ is a proper geodesic Gromov hyperbolic space constructed by Groves-Manning [15] on which Γ acts properly discontinuously and by isometries. We briefly review the construction of the Groves-Manning cusp space. We first need a notion of combinatorial horoballs: for a graph Y equipped with a simplicial distance d_Y, the combinatorial horoball $\mathcal{H}(Y)$ is the graph with the vertex set $Y^{(0)} \times \mathbb{N}$ and two types of edges: vertical edges between vertices (y, n) and $(y, n+1)$ for $y \in Y$ and $n \in \mathbb{N}$, and horizontal edges between vertices (y_1, n) and (y_2, n) for $y_1, y_2 \in Y$ and $n \in \mathbb{N}$ if $d_Y(y_1, y_2) \leq 2^{n-1}$. We also equip $\mathcal{H}(Y)$ with the simplicial distance.

Now fix a finite generating set S of Γ such that for each $P \in \mathcal{P}$, $S \cap P$ generates P. We denote by $\mathcal{C}(\Gamma, S)$ and $\mathcal{C}(P, S \cap P)$ the Cayley graphs of Γ and P with respect to S and $S \cap P$ respectively. For each $\gamma \in \Gamma$ and $P \in \mathcal{P}$, we glue the horoball $\mathcal{H}(\gamma \mathcal{C}(P, S \cap P))$ to $\mathcal{C}(\Gamma, S)$, by identifying $\gamma \mathcal{C}(P, S \cap P) \subset \mathcal{C}(\Gamma, S)$ with $\gamma \mathcal{C}(P, S \cap P) \times \{1\} \subset \mathcal{H}(\gamma \mathcal{C}(P, S \cap P))$. The resulting graph equipped with the simplicial distance is called the Groves-Manning cusp space for $(\Gamma, \mathcal{P})$ and S, which we denote by $X_{GM}(\Gamma, \mathcal{P}, S)$.

Theorem 5.1. [15, Theorem 3.25] *The space $X_{GM}(\Gamma, \mathcal{P}, S)$ is a proper geodesic Gromov hyperbolic space.*

From the construction, the natural action of Γ on the Cayley graph $\mathcal{C}(\Gamma, S)$ induces the isometric action of Γ on $X_{GM}(\Gamma, \mathcal{P}, S)$ which is properly discontinuous. Hence the induced Γ-action on the Gromov boundary $\partial X_{GM}(\Gamma, \mathcal{P}, S)$ is a convergence group action [5, Lemma 2.11], and moreover is a geometrically finite convergence group action by the construction of $X_{GM}(\Gamma, \mathcal{P}, S)$. Therefore the Gromov boundary of $X_{GM}(\Gamma, \mathcal{P}, S)$ is the Bowditch boundary:

$$\partial X_{GM}(\Gamma, \mathcal{P}, S) = \partial(\Gamma, \mathcal{P}).$$

Relatively Anosov subgroups. Let $\Gamma < G$ be a finitely generated non-elementary θ-transverse subgroup with the limit set Λ_θ and $\mathcal{P}$ a finite collection of finitely generated infinite subgroups of Γ.

Definition 5.2. We say that Γ is *θ-Anosov relative to $\mathcal{P}$* if Γ is hyperbolic relative to $\mathcal{P}$ and there exists a Γ-equivariant homeomorphism $\partial(\Gamma, \mathcal{P}) \to \Lambda_\theta$.

Let Γ be a θ-Anosov relative to $\mathcal{P}$ in the rest of the section. We denote by $X_{GM} := X_{GM}(\Gamma, \mathcal{P}, S)$ the associated Groves-Manning cusp space for some fixed generating set S. We then have the Γ-equivariant homeomorphism

$$f : \partial X_{GM} \to \Lambda_\theta,$$

which has the following property:

Proposition 5.3. [10, Proposition 4.3] *Let $x \in X_{GM}$. For a sequence $\gamma_n \in \Gamma$, if $\gamma_n x \to \xi \in \partial X_{GM}$ as $n \to \infty$, then $\gamma_n \to f(\xi)$ as $n \to \infty$.*

By the antipodality of Γ, the canonical projections $\pi_\theta : \Lambda_{\theta \cup \mathrm{i}(\theta)} \to \Lambda_\theta$ and $\pi_{\mathrm{i}(\theta)} : \Lambda_{\theta \cup \mathrm{i}(\theta)} \to \Lambda_{\mathrm{i}(\theta)}$ are Γ-equivariant homeomorphisms. This implies that being relatively θ-Anosov implies being relatively $\theta \cup \mathrm{i}(\theta)$-Anosov as well as relatively $\mathrm{i}(\theta)$-Anosov. In particular, setting the composition $f_{\mathrm{i}} := \pi_{\mathrm{i}(\theta)} \circ \pi_\theta^{-1} \circ f$, two maps

$$f : \partial X_{GM} \to \Lambda_\theta \quad \text{and} \quad f_{\mathrm{i}} : \partial X_{GM} \to \Lambda_{\mathrm{i}(\theta)}$$

have the property that if $\xi, \eta \in \partial X_{GM}$ are distinct, then $(f(\xi), f_{\mathrm{i}}(\eta)) \in \mathcal{F}_\theta^{(2)}$.

Compatibility of shadows. We first define the shadows in the symmetric space X: for $p \in X$ and $R > 0$, let $B(p, R)$ denote the metric ball $\{x \in X : d(x, p) < R\}$. For $q \in X$, the θ-shadow $O_R^\theta(q, p) \subset \mathcal{F}_\theta$ of $B(p, R)$ viewed from q is defined as

$$O_R^\theta(q, p) = \{gP_\theta \in \mathcal{F}_\theta : g \in G,\ go = q,\ gA^+o \cap B(p, R) \neq \emptyset\}.$$

The following two lemmas will be useful:

Lemma 5.4. [21, Lemma 5.7] *There exists $\kappa > 0$ such that for any $g, h \in G$ and $R > 0$, we have*

$$\sup_{\xi \in O_R^\theta(go, ho)} \|\beta_\xi^\theta(g, h) - \mu_\theta(g^{-1}h)\| \leq \kappa R.$$

Lemma 5.5. [19, Lemma 9.9] *Let $g_n \in G$ and $\xi_n \in \mathcal{F}_\theta$ be sequences both converging to some $\xi \in \mathcal{F}_\theta$. Suppose that there exists a sequence $\eta_n \in \mathcal{F}_{\mathrm{i}(\theta)}$ converging to some $\eta \in \mathcal{F}_{\mathrm{i}(\theta)}$ such that $(\xi, \eta) \in \mathcal{F}_\theta^{(2)}$ and the sequence $g_n^{-1}(\xi_n, \eta_n)$ is precompact in $\mathcal{F}_\theta^{(2)}$. Then there exists $R > 0$ such that*

$$\xi_n \in O_R^\theta(o, g_n o) \quad \textit{for all } n \geq 1.$$

We also consider shadows in Groves-Manning cusp space. Let d_{GM} be the simplicial distance on X_{GM}.

The following theorem is obtained in [10, Theorem 10.1]; although it stated only the lower bound, the upper bound also follows from its proof:

Theorem 5.6. *For any (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$, there exists positive constants c, c' and C such that for all $\gamma \in \Gamma$,*

$$c\, d_{GM}(e, \gamma) - C \leq \psi(\mu_\theta(\gamma)) \leq c'\, d_{GM}(e, \gamma) + C.$$

For $y \in X_{GM}$ and $R > 0$, we denote the R-ball centered at y by

$$B_{GM}(y, R) := \{z \in X_{GM} : d_{GM}(y, z) < R\}.$$

For $x, y \in X_{GM}$ and $R > 0$, we define the shadow of $B_{GM}(y, R)$ viewed from x as follows:

$$O_R^{GM}(x, y) := \left\{ \xi \in \partial X_{GM} : \begin{array}{c} \text{there exists a geodesic ray from } x \text{ to } \xi \\ \text{passing through } B_{GM}(y, R) \end{array} \right\}.$$

We prove the following compatibility of shadows under $f : \partial X_{GM} \to \Lambda_\theta$:

Proposition 5.7. *Let $x \in X_{GM}$ and $o \in X$. For all sufficiently large $R > 1$, there exist $r_1 = r_1(R), r_2 = r_2(R) > 0$ such that for any $\gamma \in \Gamma$, we have*

$$O_{r_1}^\theta(o, \gamma o) \cap \Lambda_\theta \subset f(O_R^{GM}(x, \gamma x)) \subset O_{r_2}^\theta(o, \gamma o) \cap \Lambda_\theta.$$

Moreover, we can take $r_1(R) \to \infty$ as $R \to \infty$.

We begin with some lemmas:

Lemma 5.8. *For any $x \in X_{GM}$, there exists $R_0 > 0$ such that $O_{R_0}^{GM}(x, \gamma x) \neq \emptyset$ for any $\gamma \in \Gamma$.*

Proof. Suppose not. Then there exists an infinite sequence $\gamma_n \in \Gamma$ so that $O_n^{GM}(x, \gamma_n x) = \emptyset$, and hence $O_n^{GM}(\gamma_n^{-1} x, x) = \emptyset$ for all $n \geq 1$. This forces ∂X_{GM} to be a singleton, which contradicts the perfectness of ∂X_{GM}. $\square$

Lemma 5.9. *Let $x \in X_{GM}$ and $R > 0$. Let $\gamma_n \in \Gamma$ and $\xi_n \in \partial X_{GM}$ be sequences such that $\xi_n \in O_R^{GM}(x, \gamma_n x)$ for all $n \geq 1$. If $\gamma_n x \to \xi \in \partial X_{GM}$ as $n \to \infty$, then $\xi_n \to \xi$ as $n \to \infty$.*

Proof. Suppose to the contrary that the sequence ξ_n, after passing to a subsequence, converges to $\xi' \in \partial X_{GM}$ distinct from ξ. Since $\gamma_n x \to \xi$ as $n \to \infty$, this implies that there exist a constant $R' > 0$ and a sequence of geodesic rays $[\gamma_n x, \xi_n]$ from $\gamma_n x$ to ξ_n such that $d_{GM}(x, [\gamma_n x, \xi_n]) < R'$ for all $n \geq 1$. On the other hand, since $\xi_n \in O_R^{GM}(x, \gamma_n x)$, there exists a geodesic ray $[x, \xi_n]$ from x to ξ_n and a point $c_n \in [x, \xi_n]$ such that $d_{GM}(c_n, \gamma_n x) < R$ for all $n \geq 1$. Since the distance between $\gamma_n x$ and c_n is uniformly bounded, the Hausdorff distance between two geodesic rays $[\gamma_n x, \xi_n]$ and $[c_n, \xi_n] \subset [x, \xi_n]$ is uniformly bounded. Since the distance $d_{GM}(x, [\gamma_n x, \xi_n])$ is uniformly bounded, this implies that the distance $d_{GM}(x, [c_n, \xi_n])$ is uniformly bounded as well. Since $[c_n, \xi_n]$ is the geodesic ray contained in the geodesic ray $[x, \xi_n]$, we have that $d_{GM}(x, c_n) = d_{GM}(x, [c_n, \xi_n])$ is uniformly bounded. Therefore, it follows from the uniform boundedness of $d_{GM}(c_n, \gamma_n x)$ that $d_{GM}(x, \gamma_n x)$ is uniformly bounded, which contradicts the hypothesis that $\gamma_n x \to \xi$ as $n \to \infty$. This finishes the proof. $\square$

Proof of Proposition 5.7. Note that the first inclusion and the last claim follow once we show that for any $c > 0$, there exists $C > 0$ such that $O_c^\theta(o, \gamma o) \subset f(O_C^{GM}(x, \gamma x))$ for all $\gamma \in \Gamma$. Suppose not. Then there exist sequences $\gamma_n \in \Gamma$ and $\xi_n \in \partial X_{GM} - O_n^{GM}(x, \gamma_n x)$ such that $f(\xi_n) \in O_c^\theta(o, \gamma_n o)$

for all $n \geq 1$. After passing to a subsequence, we may assume that the sequence $\gamma_n^{-1}x$ converges to some point $\eta \in \partial X_{GM}$ as $n \to \infty$. Since $\gamma_n^{-1}\xi_n \notin O_n^{GM}(\gamma_n^{-1}x, x)$ for all $n \geq 1$, we have that

$$\lim_{n\to\infty} \gamma_n^{-1}\xi_n = \eta. \tag{5.1}$$

On the other hand, by Proposition 5.3, we have $\lim_{n\to\infty} \gamma_n^{-1} = f_{\mathrm{i}}(\eta) \in \Lambda_{\mathrm{i}(\theta)}$. Since $f(\gamma_n^{-1}\xi_n) \in O_c^{\theta}(\gamma_n^{-1}o, o)$ for all $n \geq 1$ and $\lim_{n\to\infty} \gamma_n^{-1} = f_{\mathrm{i}}(\eta)$, it follows from (5.1) and the continuity of higher rank shadows on viewpoints [18, Proposition 3.4] that $f(\eta) = \lim_{n\to\infty} f(\gamma_n^{-1}\xi_n) \in \Lambda_\theta$ is in general position with $f_{\mathrm{i}}(\eta)$. This yields contradiction.

We now prove the second inclusion. Let $R_0 > 0$ be as given by Lemma 5.8 and fix $R > R_0$. Let $x \in X_{GM}$ and $o \in X$. Suppose on the contrary that there exists a sequence $\gamma_n \in \Gamma$ such that

$$f(O_R^{GM}(x, \gamma_n x)) \not\subset O_n^{\theta}(o, \gamma_n o) \quad \text{for all } n \geq 1.$$

This means that there exists a sequence $\xi_n \in O_R^{GM}(x, \gamma_n x)$ such that $f(\xi_n) \notin O_n^{\theta}(o, \gamma_n o)$ for all $n \geq 1$. After passing to a subsequence, we may assume that the sequence $\gamma_n x$ converges to a point $\xi \in \partial X_{GM}$. By Proposition 5.3, we have

$$\gamma_n \to f(\xi) \quad \text{as } n \to \infty. \tag{5.2}$$

In addition, it follows from Lemma 5.9 that $\xi_n \to \xi$ as $n \to \infty$. For each $n \geq 1$, we choose a point $\eta_n \in O_R^{GM}(\gamma_n x, x)$ which is possible by Lemma 5.8. We may assume that the sequence η_n converges to $\eta \in \partial X_{GM}$, after passing to a subsequence. Since $\gamma_n x \to \xi$ as $n \to \infty$ and $\eta_n \in O_R^{GM}(\gamma_n x, x)$ for all $n \geq 1$, we have $\xi \neq \eta$. Therefore, we have the following convergence of the sequence in $\mathcal{F}_\theta^{(2)}$:

$$(f(\xi_n), f_{\mathrm{i}}(\eta_n)) \to (f(\xi), f_{\mathrm{i}}(\eta)) \in \mathcal{F}_\theta^{(2)} \quad \text{as } n \to \infty. \tag{5.3}$$

On the other hand, we also have $\gamma_n^{-1}\xi_n \in O_R^{GM}(\gamma_n^{-1}x, x)$ and $\gamma_n^{-1}\eta_n \in O_R^{GM}(x, \gamma_n^{-1}x)$ for all $n \geq 1$. Together with the Γ-equivariance of f and f_{i}, a similar argument as above implies that

$$\text{the sequence } \gamma_n^{-1}(f(\xi_n), f_{\mathrm{i}}(\eta_n)) \text{ is precompact in } \mathcal{F}_\theta^{(2)}. \tag{5.4}$$

By (5.2), (5.3), and (5.4), we apply Lemma 5.5 and deduce that there exists $R' > 0$ so that $f(\xi_n) \in O_{R'}^{\theta}(o, \gamma_n o)$ for all $n \geq 1$. This contradicts to the choice of the sequence ξ_n that $f(\xi_n) \notin O_n^{\theta}(o, \gamma_n o)$ for all $n \geq 1$. This completes the proof. □

Lemma 5.10. *Let $x \in X_{GM}$ and $R > 0$. Then there exists a compact subset $Q \subset \mathfrak{a}_\theta$ satisfying the following: if $\xi, \eta \in \partial X_{GM}$ are such that $d_{GM}(x, [\xi, \eta]) < R$ for some bi-infinite geodesic $[\xi, \eta]$, then*

$$\langle f(\xi), f_{\mathrm{i}}(\eta) \rangle \in Q$$

where $\langle \cdot, \cdot \rangle$ is the Gromov product defined in (2.5).

Proof. Suppose not. Then there exists a sequence of bi-infinite geodesics $[\xi_n, \eta_n]$ for some $\xi_n, \eta_n \in \partial X_{GM}$ such that we have $\sup_n d_{GM}(x, [\xi_n, \eta_n]) < R$ and the Gromov products $\langle f(\xi_n), f_{\mathrm{i}}(\eta_n)\rangle$ escape every compact subset of $\mathfrak{a}_\theta$ as $n \to \infty$. After passing to a subsequence, we may assume that $\xi_n \to \xi$ and $\eta_n \to \eta$ in ∂X_{GM}. The hypothesis $\sup_n d_{GM}(x, [\xi_n, \eta_n]) < R$ implies $\xi \neq \eta$. Therefore $(f(\xi), f_{\mathrm{i}}(\eta)) \in \Lambda_\theta^{(2)}$ and hence $\langle f(\xi), f_{\mathrm{i}}(\eta)\rangle \in \mathfrak{a}_\theta$ is well-defined. On the other hand, by the continuity of the Gromov product, we have $\langle f(\xi_n), f_{\mathrm{i}}(\eta_n)\rangle \to \langle f(\xi), f_{\mathrm{i}}(\eta)\rangle \in \mathfrak{a}_\theta$ as $n \to \infty$. This yields a contradiction. $\square$

6. Reparameterization for relatively Anosov groups

Let $\Gamma < G$ be a θ-Anosov subgroup relative to $\mathcal{P}$ and $X_{GM} = X_{GM}(X, \mathcal{P}, S)$ the associated Groves-Manning cusp space for a fixed generating set S. Fix a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$ and consider the space $\Omega_\psi = \Gamma\backslash(\Lambda_\theta^{(2)} \times \mathbb{R})$ defined in section 3. We will relate Ω_ψ with the Groves-Manning cusp space X_{GM} in this section. More precisely, let

$$\mathcal{G} := \{\sigma : \mathbb{R} \to X_{GM} : \text{ bi-infinite geodesic}\}.$$

The space $\mathcal{G}$ admits the geodesic flow $\varphi_s : \mathcal{G} \to \mathcal{G}$ defined by $(\varphi_s\sigma)(\cdot) = \sigma(\cdot + s)$ for $s \in \mathbb{R}$, and the inversion $I : \mathcal{G} \to \mathcal{G}$ defined by $(I\sigma)(s) = \sigma(-s)$ for $s \in \mathbb{R}$. The canonical isometric action of Γ on $\mathcal{G}$ commutes with the geodesic flow and I, and is properly discontinuous. Hence we can also consider the locally compact Hausdorff space $\Gamma\backslash\mathcal{G}$. This section is devoted to the proof of the following reparameterization theorem:

Set

$$a = \liminf_{\gamma\in\Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e, \gamma)} \quad \text{and} \quad a' = 3\limsup_{\gamma\in\Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e, \gamma)}. \tag{6.1}$$

By Theorem 5.6, we have $0 < a \leq a' < \infty$.

Theorem 6.1 (Reparameterization)**.** *There exists a continuous surjective proper map*

$$\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi.$$

Moreover, we have a continuous cocycle $\mathtt{t} : \Gamma\backslash\mathcal{G} \times \mathbb{R} \to \mathbb{R}$ *such that for all* $\sigma \in \mathcal{G}$ *and* $s \in \mathbb{R}$,

(1) $\Psi([\varphi_s\sigma]) = \phi_{\mathtt{t}(\sigma,s)}\Psi([\sigma])$;
(2) $\mathtt{t}(\sigma, s) = -\mathtt{t}(\varphi_s\sigma, -s)$;
(3) *for some absolute constant* $B > 0$,

$$a|s| - B \leq \mathtt{t}(\sigma, |s|) \leq a'|s| + B$$

The reparameterization gives us a thick-thin decomposition of Ω_ψ where the thin part is the union of Ψ-images of bi-infinite geodesics based at the horoballs in $\Gamma\backslash X_{GM}$ corresponding to elements of $\mathcal{P}$.

Thick-thin decomposition of $\mathcal{G}$. For $P \in \mathcal{P}$, let $\xi_P \in \partial X_{GM}$ be the bounded parabolic limit point fixed by P. We consider the open horoball $H_P \subset X_{GM}$ based at ξ_P invariant under P, defined as follows: let $H'_P \subset X_{GM}$ be the subgraph induced by the vertices $\{(g, n) : g \in P, n \geq 2\}$ and $\hat{H}_P \subset X_{GM}$ be the subgraph induced by the vertices $\{(g, 2) : g \in P\}$. We then set

$$H_P := H'_P - \hat{H}_P.$$

For $\gamma \in \Gamma$, we also set

$$H_{\gamma P \gamma^{-1}} := \gamma H_P$$

which is the open horoball based at $\xi_{\gamma P\gamma^{-1}} := \gamma\xi_P$ and invariant under $\gamma P\gamma^{-1} \in \mathcal{P}^\Gamma$. The boundary $\partial H_{\gamma P\gamma^{-1}}$ consists of the vertices $\gamma\{(g, 2) : g \in P\}$. We then have the Γ-invariant family $\{H_P : P \in \mathcal{P}^\Gamma\}$ of open horoballs with disjoint closures.

We define the following subsets of $\mathcal{G}$: for $P \in \mathcal{P}^\Gamma$, let

$$\mathcal{G}_P := \{\sigma \in \mathcal{G} : \sigma(0) \in H_P\};$$
$$\partial\mathcal{G}_P := \{\sigma \in \mathcal{G} : \sigma(0) \in \partial H_P\}.$$

We have the thick-thin decomposition of $\mathcal{G}$:

$$\mathcal{G}_{thin} := \bigcup_{P\in\mathcal{P}^\Gamma} \mathcal{G}_P \quad \text{and} \quad \mathcal{G}_{thick} := \mathcal{G} - \mathcal{G}_{thin}.$$

Since the Groves-Manning cusp space X_{GM} is constructed by attaching combinatorial horoballs to the Cayley graph of Γ, the Γ-action on $X_{GM} - \bigcup_{P\in\mathcal{P}^\Gamma} H_P$ is cocompact. Hence the Γ-action on $\mathcal{G}_{thick}$ which consists of bi-infinite geodesics based at $X_{GM} - \bigcup_{P\in\mathcal{P}^\Gamma} H_P$ is also cocompact.

We also introduce the following subsets of $\partial\mathcal{G}_P$ for each $P \in \mathcal{P}^\Gamma$:

$$\partial^+\mathcal{G}_P := \{\sigma \in \partial\mathcal{G}_P : \sigma(t) \in H_P \text{ for all sufficiently small } t > 0\};$$
$$\partial^-\mathcal{G}_P := \{\sigma \in \partial\mathcal{G}_P : \sigma(-t) \in H_P \text{ for all sufficiently small } t > 0\}.$$

Note that $\partial^+\mathcal{G}_P \cap \partial^-\mathcal{G}_P = \emptyset$. For $\sigma \in \partial^+\mathcal{G}_P$, we set

$$T^+_\sigma := \min\{t \in (0, \infty] : \sigma(t) \notin H_P\},$$

and for $\sigma \in \partial^-\mathcal{G}_P$, we set

$$T^-_\sigma := \max\{t \in [-\infty, 0) : \sigma(t) \notin H_P\},$$

which are the escaping times for the horoball H_P. We then have

$$\mathcal{G}_P = \left(\bigcup_{\sigma\in\partial^+\mathcal{G}_P} \bigcup_{t\in(0,T^+_\sigma)} \varphi_t\sigma \right) \cup \left(\bigcup_{\sigma\in\partial^-\mathcal{G}_P} \bigcup_{t\in(T^-_\sigma,0)} \varphi_t\sigma \right).$$

Construction of the reparameterization. To construct the reparameterization, we consider the trivial bundle

$$\mathcal{G} \times \mathbb{R}_+ \to \mathcal{G}.$$

Given $\sigma \in \mathcal{G}$, we denote by $\sigma^+ = \sigma(\infty) \in \partial X_{GM}$ and $\sigma^- = \sigma(-\infty) \in \partial X_{GM}$ the forward and backward endpoint of the bi-infinite geodesic σ. Noting that we have Γ-equivariant homeomorphisms $f : \partial X_{GM} \to \Lambda_\theta$ and $f_{\mathrm{i}} : \partial X_{GM} \to \Lambda_{\mathrm{i}(\theta)}$, we identify ∂X_{GM}, Λ_θ, and $\Lambda_{\mathrm{i}(\theta)}$ in this section via the homeomorphisms. We define the Γ-action on $\mathcal{G} \times \mathbb{R}_+$ as follows: for $\gamma \in \Gamma$ and $(\sigma, v) \in \mathcal{G} \times \mathbb{R}_+$,

$$\gamma(\sigma, v) = \left(\gamma\sigma, v e^{\psi\left(\beta^\theta_{\sigma^+}(\gamma^{-1}, e)\right)}\right).$$

This action makes the following projection Γ-equivariant:

$$\begin{aligned} \Psi_0 : \mathcal{G} \times \mathbb{R}_+ &\longrightarrow \tilde{\Omega}_\psi \\ (\sigma, v) &\longmapsto (\sigma^+, \sigma^-, \log v). \end{aligned}$$

We construct the reparameterization $\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi$ in Theorem 6.1 by constructing a nice Γ-equivariant section $u : \mathcal{G} \to \mathcal{G} \times \mathbb{R}_+$ of the trivial bundle so that we obtain a Γ-equivariant map $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$ as follows, with the desired properties:

$$\begin{tikzcd}
\mathcal{G} \times \mathbb{R}_+ \arrow[d] \arrow[dr, "\Psi_0"] & \\
\mathcal{G} \arrow[u, dashed, bend left, "u"] \arrow[r, dashed, "\tilde{\Psi}"'] & \tilde{\Omega}_\psi
\end{tikzcd}$$

Norms on fibers. To construct a section of the trivial bundle $\mathcal{G} \times \mathbb{R}_+ \to \mathcal{G}$, we define a continuous family of Γ-equivariant norms on fibers. More precisely, we define a Γ-invariant continuous function

$$\| \cdot \| : \mathcal{G} \times \mathbb{R}_+ \to \mathbb{R}_+$$

such that for each $\sigma \in \mathcal{G}$, $\|(\sigma, \cdot)\|$ is the restriction of a norm on $\mathbb{R}$ to $\mathbb{R}_+$. We simply write

$$\| \cdot \|_\sigma := \|(\sigma, \cdot)\| \quad \text{for each } \sigma \in \mathcal{G}.$$

Once we define the norm, we will define a section $u : \mathcal{G} \to \mathcal{G} \times \mathbb{R}_+$ by $u(\sigma) = (\sigma, v_\sigma)$ where $v_\sigma \in \mathbb{R}_+$ is the unique unit vector with respect to the norm $\| \cdot \|_\sigma$, i.e., $\|v_\sigma\|_\sigma = 1$. The Γ-equivariance and the continuity of the norms imply that the section u is also Γ-equivariant and continuous. To make the reparameterization $\tilde{\Psi} = \Psi_0 \circ u$ satisfy the conditions in Theorem 6.1, our norms should have a property that the contraction rate along the geodesic flow is bounded from both *above* and *below* by uniform exponential functions.

Our construction of the family of norms is motivated by [32] which considered flat bundles for relatively Anosov subgroups of $\mathrm{SL}(n, \mathbb{R})$ with respect to a maximal parabolic subgroup. Our proof of the contraction property is motivated by ([9], [32]) where the upper bound of the contraction rate of norms on flat bundles for relatively Anosov subgroups of $\mathrm{SL}(n, \mathbb{R})$ with respect to a maximal parabolic subgroup was proved. We also remark that the contraction property was earlier studied in ([30], [12]) for Anosov subgroups.

We now define a family of norms as follows (compare to a similar construction in [32]): first we fix a continuous family of Γ-equivariant norms $\|\cdot\|_\sigma$ for $\sigma \in \mathcal{G}_{thick}$ such that $\|\cdot\|_\sigma = \|\cdot\|_{I\sigma}$ for all $\sigma \in \mathcal{G}_{thick}$. Let $\sigma \in \mathcal{G}_{thin}$. Then $\sigma \in \mathcal{G}_P$ for some $P \in \mathcal{P}^\Gamma$. There are two cases: let $c > 0$ be the constant given by Theorem 5.6.

Case 1. If $\sigma = \varphi_t \sigma_0$ for some $\sigma_0 \in \partial^+ \mathcal{G}_P$ and $t \in (0, T^+_{\sigma_0})$, we write $T := T^+_{\sigma_0}$ and

- if $t \in \left(0, \frac{1}{3}T\right]$, we set
$$\|\cdot\|_\sigma := e^{-ct} \|\cdot\|_{\sigma_0}.$$
- if $t \in \left[\frac{2}{3}T, T\right)$, we set
$$\|\cdot\|_\sigma := e^{c(T-t)} \|\cdot\|_{\varphi_T \sigma_0}.$$
- if $t \in \left(\frac{1}{3}T, \frac{2}{3}T\right)$, we set
$$\|\cdot\|_\sigma := \|\cdot\|_{\varphi_{T/3}\sigma_0}^{2-\frac{3}{T}t} \|\cdot\|_{\varphi_{2T/3}\sigma_0}^{\frac{3}{T}t-1}.$$

Case 2. If $\sigma = \varphi_s \tilde{\sigma}_0$ for some $\tilde{\sigma}_0 \in \partial^- \mathcal{G}_P$ and $s \in (T^-_{\tilde{\sigma}_0}, 0)$, we write $T := T^-_{\tilde{\sigma}_0}$ and

- if $s \in \left[\frac{1}{3}T, 0\right)$, we set
$$\|\cdot\|_\sigma := e^{-cs} \|\cdot\|_{\tilde{\sigma}_0}.$$
- if $s \in \left(T, \frac{2}{3}T\right]$, we set
$$\|\cdot\|_\sigma := e^{c(T-s)} \|\cdot\|_{\varphi_T \tilde{\sigma}_0}.$$
- if $s \in \left(\frac{2}{3}T, \frac{1}{3}T\right)$, we set
$$\|\cdot\|_\sigma := \|\cdot\|_{\varphi_{2T/3}\tilde{\sigma}_0}^{\frac{3}{T}s-1} \|\cdot\|_{\varphi_{T/3}\tilde{\sigma}_0}^{2-\frac{3}{T}s}.$$

Note that both cases can happen at the same time, and in that case two definitions coincide. The resulting family of norms is continuous and Γ-equivariant.

Contraction rate along geodesic flow. For $\sigma \in \mathcal{G}$, there exists a unique $v_\sigma \in \mathbb{R}_+$ such that $\|v_\sigma\|_\sigma = 1$. For $t \in \mathbb{R}$, we define

$$\kappa_t(\sigma) := \|v_\sigma\|_{\varphi_t \sigma}; \tag{6.2}$$

this measures the contraction rates of norms under the geodesic flow. It is easy to see that for $\sigma \in \mathcal{G}$ and $t, s \in \mathbb{R}$, we have

$$v_{\varphi_t \sigma} = \frac{v_\sigma}{\|v_\sigma\|_{\varphi_t \sigma}} \quad \text{and} \quad \kappa_{t+s}(\sigma) = \kappa_s(\varphi_t \sigma) \kappa_t(\sigma). \tag{6.3}$$

Moreover, $\kappa_t(\cdot)$ is Γ-invariant.

Lemma 6.2. *For $\sigma \in \mathcal{G}$, $t \in \mathbb{R}$, and $\gamma \in \Gamma$, we have*

$$\kappa_t(\gamma\sigma) = \kappa_t(\sigma).$$

Proof. By the Γ-equivariance of the norm, we have

$$1 = \|v_\sigma\|_\sigma = \left\| v_\sigma e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))} \right\|_{\gamma\sigma}.$$

This implies

$$v_{\gamma\sigma} = v_\sigma e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))}.$$

Since $\varphi_t \gamma\sigma = \gamma\varphi_t\sigma$, we have

$$\begin{aligned}
\kappa_t(\gamma\sigma) &= \|v_{\gamma\sigma}\|_{\varphi_t\gamma\sigma} = \|v_\sigma\|_{\gamma\varphi_t\sigma} e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))} \\
&= \left\| v_\sigma e^{\psi(\beta^\theta_{\gamma\sigma^+}(\gamma, e))} \right\|_{\varphi_t\sigma} e^{\psi(\beta^\theta_{\sigma^+}(\gamma^{-1}, e))} \\
&= \|v_\sigma\|_{\varphi_t\sigma} = \kappa_t(\sigma)
\end{aligned}$$

as desired. $\square$

The following is the desired estimate on the contraction rate:

Theorem 6.3. *There exists $b > 1$ such that for all $\sigma \in \mathcal{G}$ and $t \geq 0$, we have*

$$\frac{1}{b} e^{-a't} \leq \kappa_t(\sigma) \leq b e^{-at}$$

where $a = \liminf_{\gamma\in\Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e,\gamma)}$ and $a' = 3 \limsup_{\gamma\in\Gamma} \frac{\psi(\mu_\theta(\gamma))}{d_{GM}(e,\gamma)}$.

We begin by observing that the recurrence to a compact subset implies the exponential contraction:

Lemma 6.4. *For any compact subset $Q \subset X_{GM}$, there exists $C_Q > 1$ such that if $\sigma \in \mathcal{G}$, $t \geq 0$, and $\gamma \in \Gamma$ satisfy $\sigma(0), \gamma^{-1}\sigma(t) \in Q$, then*

$$\frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \leq \kappa_t(\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}.$$

Proof. Suppose not. Then there exist sequences $\sigma_n \in \mathcal{G}$, $t_n \geq 0$, and $\gamma_n \in \Gamma$ such that $\sigma_n(0), \gamma_n^{-1}\sigma_n(t_n) \in Q$ for all $n \geq 1$ while the sequence

$$\log\left(\kappa_{t_n}(\sigma_n) e^{\psi(\mu_\theta(\gamma_n))}\right) = \psi(\mu_\theta(\gamma_n)) + \log \kappa_{t_n}(\sigma_n) \quad \text{is unbounded.} \tag{6.4}$$

In particular, γ_n is an infinite sequence and $t_n \to \infty$ as $n \to \infty$.

By the hypothesis that $\sigma_n(0), \gamma_n^{-1}\sigma_n(t_n) \in Q$, there exist $q \in Q$ and $R > 0$ depending on Q so that we have $\sigma_n^+ \in O_R^{GM}(q, \gamma_n q)$ for all $n \geq 1$. It follows from Proposition 5.7 that for some $r > 0$, we have $\sigma_n^+ \in O_r^\theta(o, \gamma_n o)$ for all $n \geq 1$. By Lemma 5.4, we deduce from (6.4) that the sequence

$$\psi\left(\beta^\theta_{\sigma_n^+}(e, \gamma_n)\right) + \log \kappa_{t_n}(\sigma_n) \quad \text{is unbounded.} \tag{6.5}$$

On the other hand, by the Γ-equivariance of the norms $\|\cdot\|$, we have

$$\begin{aligned}\kappa_{t_n}(\sigma_n) = \|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} &= \left\| v_{\sigma_n} e^{\psi\left(\beta^\theta_{\sigma_n^+}(\gamma_n, e)\right)} \right\|_{\gamma_n^{-1}\varphi_{t_n}\sigma_n} \\ &= e^{\psi\left(\beta^\theta_{\sigma_n^+}(\gamma_n, e)\right)} \|v_{\sigma_n}\|_{\gamma_n^{-1}\varphi_{t_n}\sigma_n}\end{aligned}$$

and therefore

$$\psi\left(\beta^\theta_{\sigma_n^+}(e, \gamma_n)\right) + \log \kappa_{t_n}(\sigma_n) = \log \|v_{\sigma_n}\|_{\gamma_n^{-1}\varphi_{t_n}\sigma_n}. \tag{6.6}$$

Since both $\sigma_n(0)$ and $\gamma_n^{-1}\sigma_n(t_n) = (\gamma_n^{-1}\varphi_{t_n}\sigma_n)(0)$ belong to the compact subset Q for all $n \geq 1$, there exists a compact subset of $\mathcal{G}$ containing σ_n and $\gamma_n^{-1}\varphi_{t_n}\sigma_n$ for all $n \geq 1$. Therefore, the sequence (6.6) is uniformly bounded, which contradicts (6.5). Hence the claim follows. ☐

We obtain the following estimate of the contraction rate between the entrance and exit of a horoball.

Corollary 6.5. *There exists a constant $c_0 \geq 1$ such that if $\sigma \in \partial^+\mathcal{G}_P$ for some $P \in \mathcal{P}^\Gamma$ with $T^+_\sigma < \infty$, then*

$$\frac{1}{c_0} e^{-c' T^+_\sigma} \leq \kappa_{T^+_\sigma}(\sigma) \leq c_0 e^{-c T^+_\sigma}$$

where c and c' are given by Theorem 5.6.

Proof. Let $P \in \mathcal{P}^\Gamma$ and $\sigma \in \partial^+\mathcal{G}_P$ with $T^+_\sigma < \infty$. By Lemma 6.2, we may assume that $P \in \mathcal{P}$ and $\sigma(0) = (e, 2)$ in the combinatorial horoball attached to a Cayley graph of P. We then have $\sigma(T^+_\sigma) = (\gamma, 2)$ for some $\gamma \in P$. Setting $Q = \overline{B_{GM}(e, 1)}$ which is a compact subset of X_{GM}, we have $\sigma(0), \gamma^{-1}\sigma(T^+_\sigma) \in Q$. Hence by Lemma 6.4, we have

$$\frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \leq \kappa_{T^+_\sigma}(\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}$$

where C_Q is the constant therein. On the other hand, it follows from Theorem 5.6 that

$$\begin{aligned}\psi(\mu_\theta(\gamma)) &\geq c d_{GM}(e, \gamma) - C \\ &\geq c(d_{GM}((e, 2), (\gamma, 2)) - 2) - C \\ &= cT^+_\sigma - (2c + C)\end{aligned}$$

with the constants c, C in loc. cit. Therefore, we have

$$\kappa_{T^+_\sigma}(\sigma) \leq C_Q e^{2c+C} e^{-cT^+_\sigma}.$$

Similarly, we have

$$\begin{aligned}\psi(\mu_\theta(\gamma)) &\leq c' d_{GM}(e, \gamma) + C \\ &\leq c'(d_{GM}((e, 2), (\gamma, 2)) + 2) + C \\ &= c' T^+_\sigma + (2c' + C)\end{aligned}$$

where c' is given in Theorem 5.6. Therefore, we have

$$\kappa_{T_\sigma^+}(\sigma) \geq \frac{1}{C_Q} e^{-(2c'+C)} e^{-c'T_\sigma^+}.$$

This finishes the proof. $\square$

We now show estimate the contraction rate in the thin part.

Lemma 6.6. *There exists a constant $c_1 \geq 1$ with the following property: if $\sigma \in \mathcal{G}_{thin}$ is such that $\varphi_s \sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$, then*

$$c_1^{-1} e^{-(3c'-2c)t} \leq \kappa_t(\sigma) \leq c_1 e^{-ct}$$

where $c \leq c'$ are given by Theorem 5.6.

Proof. We fix $\sigma \in \mathcal{G}_{thin}$ such that $\varphi_s \sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$. Then there exists $P \in \mathcal{P}^\Gamma$ so that $\varphi_s \sigma \in \mathcal{G}_P$ for all $0 \leq s \leq t$. There are three cases to consider:

Case 1. Suppose that $\sigma([0, \infty)) \subset \mathcal{G}_P$. Then $\sigma = \varphi_s \sigma_0$ for some $\sigma_0 \in \partial^+ \mathcal{G}_P$ and $s > 0$. In this case, by the definition of the norm, we have

$$\| \cdot \|_{\varphi_t \sigma} = \| \cdot \|_{\varphi_{t+s}\sigma_0} = e^{-c(t+s)} \| \cdot \|_{\sigma_0} = e^{-ct} \| \cdot \|_\sigma.$$

This implies $\kappa_t(\sigma) = e^{-ct}$.

Case 2. Suppose that $\sigma((-\infty, 0]) \subset \mathcal{G}_P$. Then $\sigma = \varphi_s \tilde{\sigma}_0$ for some $\tilde{\sigma}_0 \in \partial^- \mathcal{G}_P$ and $s < 0$. We then have

$$\| \cdot \|_{\varphi_t \sigma} = e^{-c(s+t)} \| \cdot \|_{\tilde{\sigma}_0} = e^{-ct} \| \cdot \|_\sigma,$$

and hence $\kappa_t(\sigma) = e^{-ct}$.

Case 3. Suppose that neither $\sigma([0, \infty)) \subset \mathcal{G}_P$ nor $\sigma((-\infty, 0]) \subset \mathcal{G}_P$ holds. In this case, we have $\sigma = \varphi_s \sigma_0$ for some $s > 0$ and $\sigma_0 \in \partial^+ \mathcal{G}_P$ such that $T_{\sigma_0}^+ < \infty$. We simply write $T := T_{\sigma_0}^+$ and $\sigma_1 = \varphi_T \sigma_0$. We first consider the following three subcases:

- if $s, s+t \in \left(0, \frac{1}{3}T\right]$, then

  $$\| \cdot \|_{\varphi_t \sigma} = \| \cdot \|_{\varphi_{s+t}\sigma_0} = e^{-c(s+t)} \| \cdot \|_{\sigma_0} = e^{-ct} \| \cdot \|_\sigma,$$

  and hence $\kappa_t(\sigma) = e^{-ct}$.

- if $s, s+t \in \left[\frac{2}{3}T, T\right)$, then

  $$\| \cdot \|_{\varphi_t \sigma} = e^{c(T-(t+s))} \| \cdot \|_{\sigma_1} = e^{-ct} \| \cdot \|_\sigma,$$

  and hence $\kappa_t(\sigma) = e^{-ct}$.

- if $s, s+t \in \left[\frac{1}{3}T, \frac{2}{3}T\right]$, then we first observe that

$$\begin{aligned}
\|\cdot\|_\sigma &= \|\cdot\|_{\varphi_{T/3}\sigma_0}^{2-\frac{3}{T}s}\|\cdot\|_{\varphi_{2T/3}\sigma_0}^{\frac{3}{T}s-1} \\
&= \left(e^{-c\frac{T}{3}}\|\cdot\|_{\sigma_0}\right)^{2-\frac{3}{T}s}\left(e^{c\frac{T}{3}}\|\cdot\|_{\sigma_1}\right)^{\frac{3}{T}s-1} \\
&= e^{c(2s-T)}\|\cdot\|_{\sigma_0}^{2-\frac{3}{T}s}\|\cdot\|_{\sigma_1}^{\frac{3}{T}s-1}
\end{aligned}$$

and similarly that

$$\|\cdot\|_{\varphi_t\sigma} = e^{c(2(s+t)-T)}\|\cdot\|_{\sigma_0}^{2-\frac{3}{T}(s+t)}\|\cdot\|_{\sigma_1}^{\frac{3}{T}(s+t)-1}.$$

Combining the above two computations, we obtain

$$\|\cdot\|_{\varphi_t\sigma} = \|\cdot\|_\sigma e^{2ct}\|\cdot\|_{\sigma_0}^{-\frac{3}{T}t}\|\cdot\|_{\sigma_1}^{\frac{3}{T}t}.$$

Evaluating at v_{σ_0}, the above becomes

$$\kappa_{t+s}(\sigma_0) = \kappa_s(\sigma_0)e^{2ct}\kappa_T(\sigma_0)^{\frac{3}{T}t}.$$

Since $\kappa_{t+s}(\sigma_0) = \kappa_t(\sigma)\kappa_s(\sigma_0)$ by (6.3), it follows from Corollary 6.5 and $0 \le t \le \frac{T}{3}$ that

$$\begin{aligned}
\kappa_t(\sigma) &= e^{2ct}\kappa_T(\sigma_0)^{\frac{3}{T}t} \\
&\le e^{2ct}(c_0e^{-cT})^{\frac{3}{T}t} = e^{2ct}c_0^{\frac{3}{T}t}e^{-3ct} \\
&\le \max(1, c_0)e^{-ct}.
\end{aligned}$$

Similarly, we also obtain from Corollary 6.5 and $0 \le t \le \frac{T}{3}$ that

$$\begin{aligned}
\kappa_t(\sigma) &= e^{2ct}\kappa_T(\sigma_0)^{\frac{3}{T}t} \\
&\ge e^{2ct}(c_0^{-1}e^{-c'T})^{\frac{3}{T}t} = e^{2ct}c_0^{\frac{-3}{T}t}e^{-3c't} \\
&\ge \min(1, c_0^{-1})e^{-(3c'-2c)t}.
\end{aligned}$$

We now set $c_1 := \max(1, c_0)$. Note also that $c' \ge c$ and hence $e^{-(3c'-2c)t} \le e^{-ct}$ for all $t \ge 0$. In general, we consider the following three consecutive subintervals

$$[s, s+t] \cap \left(0, \tfrac{1}{3}T\right], \quad [s, s+t] \cap \left[\tfrac{1}{3}T, \tfrac{2}{3}T\right], \quad \text{and} \quad [s, s+t] \cap \left[\tfrac{2}{3}T, T\right],$$

and then apply the each of the above three subcases to each subintervals. Then by (6.3), we get

$$c_1^{-1}e^{-(3c'-2c)t} \le \kappa_t(\sigma) \le c_1e^{-ct}$$

as desired. □

We now combine estimates on the thick and thin parts and prove Theorem 6.3. We give proofs of the lower bound and the upper bound separately:

Proof of the lower bound in Theorem 6.3. Let $\sigma \in \mathcal{G}$ and $t \geq 0$. If $\varphi_s \sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$, then by Lemma 6.6, we have

$$\kappa_t(\sigma) \geq c_1^{-1} e^{-(3c'-2c)t} \tag{6.7}$$

where constants c_1, c', c are given in loc. cit. Now suppose that $\varphi_s \sigma \in \mathcal{G}_{thick}$ for some $s \in [0,t]$ and set

$$\begin{aligned} s_1 &:= \min\{s \in [0,t] : \varphi_s \sigma \in \mathcal{G}_{thick}\}; \\ s_2 &:= \max\{s \in [0,t] : \varphi_s \sigma \in \mathcal{G}_{thick}\} \end{aligned}$$

which are well-defined. It follows from (6.3) and Lemma 6.6 that

$$\begin{aligned} \kappa_t(\sigma) &= \kappa_{t-s_2}(\varphi_{s_2}\sigma)\kappa_{s_2}(\sigma) \\ &= \kappa_{t-s_2}(\varphi_{s_2}\sigma)\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)\kappa_{s_1}(\sigma) \\ &\geq c_1^{-1} e^{-(3c'-2c)(t-s_2)} \kappa_{s_2-s_1}(\varphi_{s_1}\sigma) c_1^{-1} e^{-(3c'-2c)s_1} \\ &= c_1^{-2} e^{-(3c'-2c)t} e^{(3c'-2c)(s_2-s_1)} \kappa_{s_2-s_1}(\varphi_{s_1}\sigma). \end{aligned} \tag{6.8}$$

To estimate $\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)$, we fix a compact fundamental domain $Q \subset X_{GM} - \bigcup_{P \in \mathcal{P}^\Gamma} H_P$ for the Γ-action. We may assume that $e \in Q$. By the definition of s_1 and s_2, there exist $\gamma_1, \gamma_2 \in \Gamma$ such that $(\varphi_{s_1}\sigma)(0) \in \gamma_1 Q$ and $(\varphi_{s_2}\sigma)(0) \in \gamma_2 Q$. In other words, we have $(\gamma_1^{-1}\varphi_{s_1}\sigma)(0) \in Q$ and $(\gamma_1^{-1}\varphi_{s_2}\sigma)(0) \in \gamma_1^{-1}\gamma_2 Q$. Since $(\gamma_1^{-1}\varphi_{s_1}\sigma)(0) = \gamma_1^{-1}\sigma(s_1)$ and $(\gamma_1^{-1}\varphi_{s_2}\sigma)(0) = \gamma_1^{-1}\sigma(s_2)$, this implies that for some constant $q > 0$ depending on Q, we have $|d_{GM}(e, \gamma_1^{-1}\gamma_2) - (s_2 - s_1)| \leq q$. Setting $\gamma := \gamma_1^{-1}\gamma_2$, this is rephrased as

$$|d_{GM}(e, \gamma) - (s_2 - s_1)| \leq q. \tag{6.9}$$

Moreover, noting that $(\varphi_{s_2}\sigma)(0) = (\varphi_{s_1}\sigma)(s_2 - s_1)$, we have

$$(\gamma_1^{-1}\varphi_{s_1}\sigma)(0), \gamma^{-1}(\gamma_1^{-1}\varphi_{s_1}\sigma)(s_2 - s_1) \in Q.$$

Hence, by Lemma 6.2 and Lemma 6.4, we have

$$\begin{aligned} \kappa_{s_2-s_1}(\varphi_{s_1}\sigma) &= \kappa_{s_2-s_1}(\gamma_1^{-1}\varphi_{s_1}\sigma) \\ &\geq \frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \end{aligned}$$

with the constant C_Q given by Lemma 6.4. By Theorem 5.6 and (6.9), we deduce

$$\kappa_{s_2-s_1}(\varphi_{s_1}\sigma) \geq \frac{1}{C_Q} e^{-c' d_{GM}(e,\gamma) - C} \geq \frac{e^{-c'q-C}}{C_Q} e^{-c'(s_2-s_1)}.$$

Together with (6.8), we have

$$\begin{aligned} \kappa_t(\sigma) &\geq c_1^{-2} e^{-(3c'-2c)t} e^{(3c'-2c)(s_2-s_1)} \frac{e^{-c'q-C}}{C_Q} e^{-c'(s_2-s_1)} \\ &= \frac{c_1^{-2} e^{-c'q-C}}{C_Q} e^{-(3c'-2c)t} e^{(2c'-2c)(s_2-s_1)} \geq \frac{c_1^{-2} e^{-c'q-C}}{C_Q} e^{-(3c'-2c)t} \end{aligned} \tag{6.10}$$

where the last inequality is due to $c' \geq c$ and $s_2 \geq s_1$.

Now note that $a' \geq 3c' - 2c$ by Theorem 5.6 and choose $b > 1$ such that $b^{-1} \leq \min\left(c_1^{-1}, \frac{c_1^{-2}e^{-c'q-C}}{C_Q}\right)$. Then it follows from (6.7) and (6.10) that

$$\kappa_t(\sigma) \geq \frac{1}{b} e^{-a't}$$

as desired. $\square$

Proof of the upper bound in Theorem 6.3. Let $\sigma \in \mathcal{G}$ and $t \geq 0$. If $\varphi_s \sigma \in \mathcal{G}_{thin}$ for all $0 \leq s \leq t$, then by Lemma 6.6, we have

$$\kappa_t(\sigma) \leq c_1 e^{-ct} \tag{6.11}$$

where c_1 and c are constants given in Lemma 6.6. We now assume that $\varphi_s \sigma \in \mathcal{G}_{thick}$ for some $s \in [0, t]$. As in the proof of the lower bound, we set

$$\begin{aligned} s_1 &:= \min\{s \in [0,t] : \varphi_s \sigma \in \mathcal{G}_{thick}\}; \\ s_2 &:= \max\{s \in [0,t] : \varphi_s \sigma \in \mathcal{G}_{thick}\} \end{aligned}$$

We then have from (6.3) and Lemma 6.6 that

$$\begin{aligned} \kappa_t(\sigma) &= \kappa_{t-s_2}(\varphi_{s_2}\sigma)\kappa_{s_2-s_1}(\varphi_{s_1}\sigma)\kappa_{s_1}(\sigma) \\ &\leq c_1^2 e^{-ct} e^{c(s_2-s_1)} \kappa_{s_2-s_1}(\varphi_{s_1}\sigma). \end{aligned} \tag{6.12}$$

By the similar argument as in the proof of the lower bound, we have

$$\kappa_{s_2-s_1}(\varphi_{s_1}\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}$$

where $Q \subset X_{GM} - \bigcup_{P \in \mathcal{P}^\Gamma} H_P$ is a compact fundamental domain for the Γ-action, C_Q is the constant given by Lemma 6.4, and $\gamma \in \Gamma$ is such that $|d_{GM}(e, \gamma) - (s_2 - s_1)| \leq q$ for some constant $q \geq 0$ depending only on Q. By Theorem 5.6, this implies

$$\kappa_{s_2-s_1}(\varphi_{s_1}\sigma) \leq C_Q e^{-c d_{GM}(e,\gamma) + C} \leq C_Q e^{cq+C} e^{-c(s_2-s_1)}$$

with the constant C therein. Plugging this into (6.12), we have

$$\kappa_t(\sigma) \leq c_1^2 C_Q e^{cq+C} e^{-ct}. \tag{6.13}$$

We then choose $b \geq \max\left(c_1, c_1^2 C_Q e^{cq+C}\right)$. By (6.11) and (6.13), we finally obtain

$$\kappa_t(\sigma) \leq b e^{-ct}.$$

Since $a = c$ by Theorem 5.6, this completes the proof. $\square$

Proof of Theorem 6.1. As described above, we define the Γ-equivariant continuous section $u : \mathcal{G} \to \mathcal{G} \times \mathbb{R}_+$ by setting $u(\sigma) = (\sigma, v_\sigma)$, and set $\bar{\Psi} = \Psi_0 \circ u$ so that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{G} \times \mathbb{R}_+ \arrow[d] \arrow[dr, "\Psi_0"] & \\ \mathcal{G} \arrow[u, dashed, bend left, "u"] \arrow[r, dashed, "\bar{\Psi}"'] & \tilde{\Omega}_\psi \end{tikzcd}$$

In other words, $\tilde{\Psi}(\sigma) = (\sigma^+, \sigma^-, \log v_\sigma)$. Since $\tilde{\Psi}$ is Γ-equivariant and continuous, it descends to a continuous map

$$\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi.$$

We first prove that $\tilde{\Psi}$ is proper, from which the properness of Ψ follows. Suppose not. Then there exists a sequence $\sigma_n \in \mathcal{G}$ such that σ_n escapes every compact subset of $\mathcal{G}$ as $n \to \infty$ while $\tilde{\Psi}(\sigma_n) = (\sigma_n^+, \sigma_n^-, \log v_{\sigma_n})$ converges in $\tilde{\Omega}_\psi$. Since the sequence (σ_n^+, σ_n^-) converges in $\Lambda_\theta^{(2)}$, two sequences σ_n^+ and σ_n^- converge to two distinct points in ∂X_{GM}. This implies that there exist a sequence $t_n \in \mathbb{R}$ and a compact subset $Q \subset \mathcal{G}$ so that $\varphi_{t_n}\sigma_n \in Q$ for all $n \geq 1$. Moreover, since the sequence $\tilde{\Psi}(\sigma_n) = (\sigma_n^+, \sigma_n^-, \log v_{\sigma_n})$ converges in $\tilde{\Omega}_\psi$, the sequence v_{σ_n} converges in $\mathbb{R}_+$. This implies that, after passing to a subsequence,

(6.14) the sequence $\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n}$ converges to a positive number.

On the other hand, since the sequence σ_n escapes any compact subset of $\mathcal{G}$ as $n \to \infty$, we have either $t_n \to \infty$ or $t_n \to -\infty$ as $n \to \infty$, after passing to a subsequence. Suppose first that $t_n \to \infty$ as $n \to \infty$. It follows from Theorem 6.3 that for all sufficiently large $n \geq 1$,

$$\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} = \kappa_{t_n}(\sigma_n) \leq be^{-at_n} \to 0 \quad \text{as } n \to \infty.$$

This contradicts (6.14). We now assume that $t_n \to -\infty$ as $n \to \infty$. Then for all sufficiently large $n \geq 1$, we have

$$\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} = \frac{1}{\|v_{\varphi_{t_n}\sigma_n}\|_{\sigma_n}} = \frac{1}{\kappa_{-t_n}(\varphi_{t_n}\sigma_n)} \geq b^{-1}e^{-at_n}$$

by (6.3) and Theorem 6.3. Therefore, $\|v_{\sigma_n}\|_{\varphi_{t_n}\sigma_n} \to \infty$ as $n \to \infty$, contradicting (6.14). This proves the properness.

We now prove items (1), (2), and (3). Since the Γ-action on $\mathcal{G}$ and $\tilde{\Omega}_\psi$ commute with flows on $\mathcal{G}$ and $\tilde{\Omega}_\psi$, it suffices to prove the statement for $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$. For $(\sigma, s) \in \mathcal{G} \times \mathbb{R} \to \mathbb{R}$, define

$$\tilde{\mathtt{t}}(\sigma, s) := \log v_{\varphi_s\sigma} - \log v_\sigma.$$

By (6.3), we have

$$v_{\varphi_s\sigma} = \frac{v_\sigma}{\|v_\sigma\|_{\varphi_s\sigma}} = \frac{v_\sigma}{\kappa_s(\sigma)}.$$

Therefore

$$\tilde{\mathtt{t}}(\sigma, s) = -\log \kappa_s(\sigma), \tag{6.15}$$

is Γ-invariant (Lemma 6.2) and hence induces a continuous map $\mathtt{t} : \Gamma\backslash\mathcal{G} \times \mathbb{R} \to \mathbb{R}$. By the definition of $\tilde{\Psi}$, we have

$$\tilde{\Psi}(\varphi_s\sigma) = \phi_{\tilde{\mathtt{t}}(\sigma,s)}\Psi(\sigma),$$

from which (1) follows. The cocycle property of $\mathtt{t}$ follows from (6.3). Moreover, by Theorem 6.3 and (6.15), we have that for all $s \geq 0$,

$$as - \log b \leq \tilde{\mathtt{t}}(\sigma, s) \leq a's + \log b \tag{6.16}$$

where $a, a' > 0$ and $b \geq 1$ are given in Theorem 6.3. This shows (3).

To see the surjectivity of Ψ, note first that for each $(\xi, \eta, t_0) \in \tilde{\Omega}_\psi$, there exists $\sigma \in \mathcal{G}$ with $\sigma^+ = \xi$ and $\sigma^- = \eta$ as X_{GM} is a proper geodesic Gromov hyperbolic space. For $s_0 \geq 0$, it follows from (6.16) that

$$\tilde{\mathfrak{t}}(\sigma, s_0) \geq a s_0 - \log b \quad \text{and} \quad \tilde{\mathfrak{t}}(\varphi_{-s_0}\sigma, s_0) \geq a s_0 - \log b.$$

Since $\tilde{\mathfrak{t}}(\varphi_{-s_0}\sigma, s_0) = -\tilde{\mathfrak{t}}(\sigma, -s_0)$ due to the cocycle property (6.3), we have

$$\tilde{\mathfrak{t}}(\sigma, s_0) \geq a s_0 - \log b \quad \text{and} \quad \tilde{\mathfrak{t}}(\sigma, -s_0) \leq -a s_0 + \log b.$$

Since $\tilde{\Psi}$ is continuous, this implies that the image of $\tilde{\Psi}$ restricted on $\{\varphi_s \sigma : -s_0 \leq s \leq s_0\}$ contains $\{\phi_t \tilde{\Psi}(\sigma) : -a s_0 + \log b \leq t \leq a s_0 - \log b\}$. Since $\sigma^+ = \xi$ and $\sigma^- = \eta$, $\tilde{\Psi}(\sigma) = (\xi, \eta, t_1)$ for some $t_1 \in \mathbb{R}$. We then take s_0 large enough so that

$$-a s_0 + \log b + t_1 \leq t_0 \leq a s_0 - \log b + t_1.$$

Then $(\xi, \eta, t_0) \in \{\phi_t \tilde{\Psi}(\sigma) : -a s_0 + \log b \leq t \leq a s_0 - \log b\}$, and hence (ξ, η, t_0) belongs to the image of $\tilde{\Psi}$. Therefore, $\tilde{\Psi}$ is surjective, showing that Ψ is surjective. This completes the proof. $\square$

7. Exponential expansion on unstable foliations

Let $\Gamma < G$ be a θ-Anosov subgroup relative to $\mathcal{P}$. Fix a (Γ, θ)-proper linear form $\psi \in \mathfrak{a}_\theta^*$. Recall the space $\tilde{\Omega}_\psi = \Lambda_\theta^{(2)} \times \mathbb{R}$ equipped with the Γ-action given by

$$\gamma(\xi, \eta, s) = (\gamma\xi, \gamma\eta, s + \psi(\beta_\xi^\theta(\gamma^{-1}, e)))$$

for $\gamma \in \Gamma$ and $(\xi, \eta, s) \in \Lambda_\theta^{(2)} \times \mathbb{R}$, and $\Omega_\psi = \Gamma \backslash \tilde{\Omega}_\psi$ as defined in section 3. Recall from (4.1) and (4.2) the unstable and stable foliations $W^\pm$ on Ω_ψ and their lifts $\tilde{W}^\pm$ on $\tilde{\Omega}_\psi$. The goal of this section is to establish the following exponential expansion (resp. contraction) property of the flow $\{\phi_t\}$ on unstable (resp. stable) foliations.

Theorem 7.1. *We have the following:*

1. *There exist a Γ-invariant non-negative symmetric function $d^+ : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ and constants $\alpha, \alpha' > 0$ and $b \geq 1$ such that for $z \in \tilde{\Omega}_\psi$, the restriction of d^+ defines a semi-metric[3] on $\tilde{W}^+(z)$ and for any $w_1, w_2 \in \tilde{W}^+(z)$ and $t \geq 0$,*
$$\frac{1}{b} e^{\alpha t} d^+(w_1, w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq b e^{\alpha' t} d^+(w_1, w_2).$$
2. *Similarly, there exists a Γ-invariant non-negative symmetric function $d^- : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ such that for $z \in \tilde{\Omega}_\psi$, the restriction of d^-*

[3]A semi-metric on $\mathcal{X}$ is a non-negative symmetric function $\mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that vanishes precisely on the diagonal.

defines a semi-metric on $\tilde{W}^-(z)$ *and for any* $w_1, w_2 \in \tilde{W}^-(z)$ *and* $t \geq 0$,

$$\frac{1}{b} e^{-\alpha' t} d^-(w_1, w_2) \leq d^-(\phi_t w_1, \phi_t w_1) \leq b e^{-\alpha t} d^-(w_1, w_2).$$

(3) *For any small enough* $\varepsilon > 0$, *there exists a non-negative symmetric function* $d_\varepsilon^+ : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ *such that for* $z \in \tilde{\Omega}_\psi$, *the restriction of* d_ε^+ *defines a metric on* $\tilde{W}^+(z)$. *Moreover, for any compact subset* $Q \subset \tilde{\Omega}_\psi$, *there exists a constant* $c_Q \geq 1$ *such that for any* $w_1, w_2 \in Q$,

$$\frac{1}{c_Q} d^+(w_1, w_2)^\varepsilon \leq d_\varepsilon^+(w_1, w_2) \leq c_Q d^+(w_1, w_2)^\varepsilon.$$

Remark 7.2. Even though Theorem 7.1 states the exponential expansion and contraction for $t \geq 0$, replacing w_1 and w_2 with $\phi_{-t} w_1$ and $\phi_{-t} w_2$ implies the corresponding estimates for negative-time flow.

The proof of Theorem 7.1 is based on our coarse reparameterization (Theorem 6.1) and the coarse geometry of the Groves-Manning cusp space as a Gromov hyperbolic space.

Groves-Manning cusp space as a Gromov hyperbolic space. Let X_{GM} be the associated Groves-Manning cusp space of $(\Gamma, \mathcal{P})$, which is a proper geodesic Gromov hyperbolic space ([15, Theorem 3.25], Theorem 5.1). We refer to [8, Chapter III.H] for general facts about Gromov hyperbolic spaces.

Recall that $\mathcal{G}$ is the space of all parameterized bi-infinite geodesics in X_{GM}. We define $d^\pm : \mathcal{G} \times \mathcal{G} \to [0, \infty)$ as follows: for $\sigma_1, \sigma_2 \in \mathcal{G}$,

$$\begin{aligned} d^+(\sigma_1, \sigma_2) &:= \limsup_{t \to \infty} e^{d_{GM}(\sigma_1(t), \sigma_2(t)) - 2t}; \\ d^-(\sigma_1, \sigma_2) &:= \limsup_{t \to \infty} e^{d_{GM}(\sigma_1(-t), \sigma_2(-t)) - 2t}. \end{aligned} \tag{7.1}$$

Their well-definedness follows once we explain another formula for $d^\pm$ using Gromov products and Busemann functions on X_{GM}. For $x, p, q \in X_{GM}$, we define the Gromov product of p, q with respect to x by

$$(p|q)_x := \frac{1}{2}(d_{GM}(x, p) + d_{GM}(x, q) - d_{GM}(p, q)) \geq 0.$$

This extends to ∂X_{GM} as follows: for $\xi, \eta \in \partial X_{GM}$, we set

$$(\xi|\eta)_x := \sup \liminf_{i,j \to \infty} (p_i | q_j)_x$$

where the supremum is taken over all sequences $p_i, q_j \in X_{GM}$ such that $p_i \to \xi$ and $q_j \to \eta$ as $i, j \to \infty$. Since X_{GM} is Gromov hyperbolic, there exists a uniform constant $\delta > 0$ such that for any $x \in X_{GM}$, $\xi, \eta \in \partial X_{GM}$, and sequences $p_i, q_j \in X_{GM}$ with $\xi = \lim_{i \to \infty} p_i$ and $\eta = \lim_{j \to \infty} q_j$, we have

$$(\xi|\eta)_x - \frac{\delta}{2} \leq \liminf_{i,j \to \infty} (p_i | q_j)_x \leq (\xi|\eta)_x. \tag{7.2}$$

For $\sigma \in \mathcal{G}$ and $p, q \in X_{GM}$, the following Busemann function is well-defined:

$$\beta_{\sigma^+}(p,q) := \lim_{t\to\infty} d_{GM}(p,\sigma(t)) - d_{GM}(q,\sigma(t)).$$

We have for any $x \in X_{GM}$ that

$$d^+(\sigma_1,\sigma_2) = e^{\beta_{\sigma_1^+}(x,\sigma_1(0))+\beta_{\sigma_2^+}(x,\sigma_2(0))} \limsup_{t\to\infty} e^{-2(\sigma_1(t)|\sigma_2(t))_x}. \tag{7.3}$$

Since $(\sigma_1(t)|\sigma_2(t))_x \geq 0$ for all t, it follows that $d^+(\sigma_1,\sigma_2) < \infty$. Since

$$d^-(\sigma_1,\sigma_2) = d^+(I\sigma_1, I\sigma_2), \tag{7.4}$$

d^- is well-defined as well. The definition of $d^\pm$ is motivated by the Hamenstädt distance in a negatively curved compact manifold [16].

Since Γ acts on X_{GM} by isometries, both d^+ and d^- are Γ-invariant. The geodesic flow on $\mathcal{G}$ exponentially expand and contract d^+ and d^- respectively:

Lemma 7.3. *Let $\sigma_1, \sigma_2 \in \mathcal{G}$ and $s_1, s_2 \in \mathbb{R}$. Then we have*

$$e^{-\delta} e^{s_1+s_2} d^+(\sigma_1,\sigma_2) \leq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta} e^{s_1+s_2} d^+(\sigma_1,\sigma_2);$$
$$e^{-\delta} e^{-(s_1+s_2)} d^-(\sigma_1,\sigma_2) \leq d^-(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta} e^{-(s_1+s_2)} d^-(\sigma_1,\sigma_2).$$

Proof. Fix $x \in X_{GM}$. By (7.3) and (7.2), we have

$$\begin{aligned} d^+(\sigma_1,\sigma_2) &\geq e^{\beta_{\sigma_1^+}(x,\sigma_1(0))+\beta_{\sigma_2^+}(x,\sigma_2(0))} e^{-2(\sigma_1^+|\sigma_2^+)_x}; \\ d^+(\sigma_1,\sigma_2) &\leq e^{\delta} e^{\beta_{\sigma_1^+}(x,\sigma_1(0))+\beta_{\sigma_2^+}(x,\sigma_2(0))} e^{-2(\sigma_1^+|\sigma_2^+)_x}. \end{aligned} \tag{7.5}$$

By the definition of β, we have

$$\begin{aligned} \beta_{\sigma_1^+}(x,(\varphi_{s_1}\sigma_1)(0)) &= \beta_{\sigma_1^+}(x,\sigma_1(0)) + \beta_{\sigma_1^+}(\sigma_1(0),\sigma_1(s_1)) \\ &= \beta_{\sigma_1^+}(x,\sigma_1(0)) + s_1, \end{aligned} \tag{7.6}$$

and similarly

$$\beta_{\sigma_2^+}(x,(\varphi_{s_2}\sigma_2)(0)) = \beta_{\sigma_2^+}(x,\sigma_2(0)) + s_2. \tag{7.7}$$

Since $\varphi_{s_1}\sigma_1^+ = \sigma_1^+$ and $\varphi_{s_2}\sigma_2^+ = \sigma_2^+$, it follows from (7.5), (7.6), and (7.7) that

$$e^{-\delta} e^{s_1+s_2} d^+(\sigma_1,\sigma_2) \leq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta} e^{s_1+s_2} d^+(\sigma_1,\sigma_2).$$

The exponential contraction of d^- follows from the exponential expansion of d^+ shown above and (7.4). ∎

We fix a basepoint $x \in X_{GM}$. It is a standard fact about Gromov hyperbolic spaces that for $\varepsilon > 0$ small enough, there exists $0 < c_\varepsilon < 1$ and a metric d_ε on ∂X_{GM} such that

$$c_\varepsilon e^{-2\varepsilon(\xi|\eta)_x} \leq d_\varepsilon(\xi,\eta) \leq e^{-2\varepsilon(\xi|\eta)_x} \tag{7.8}$$

for all $\xi, \eta \in \partial X_{GM}$, with the convention that $e^{-\infty} = 0$ [8, Proposition 3.21]. We fix one such $\varepsilon > 0$ and a metric d_ε as above.

Lemma 7.4. *For any compact subset $Q \subset \mathcal{G}$, there exists a constant $b_Q \geq 1$ such that for any $\sigma_1, \sigma_2 \in Q$, we have*

$$\frac{1}{b_Q} d^+(\sigma_1, \sigma_2)^\varepsilon \leq d_\varepsilon(\sigma_1^+, \sigma_2^+) \leq b_Q d^+(\sigma_1, \sigma_2)^\varepsilon.$$

Proof. First note that for any $\sigma \in \mathcal{G}$,

$$|\beta_{\sigma^+}(x, \sigma(0))| \leq d_{GM}(x, \sigma(0)).$$

Given a compact subset $Q \subset \mathcal{G}$, we set

$$b' := \sup_{\sigma \in Q} d_{GM}(x, \sigma(0)) < \infty.$$

Then it follows from (7.8) and (7.5) that

$$\begin{aligned} d_\varepsilon(\sigma_1^+, \sigma_2^+) &\leq e^{-\varepsilon\left(\beta_{\sigma_1^+}(x, \sigma_1(0)) + \beta_{\sigma_2^+}(x, \sigma_2(0))\right)} d^+(\sigma_1, \sigma_2)^\varepsilon \\ &\leq e^{2\varepsilon b'} d^+(\sigma_1, \sigma_2)^\varepsilon. \end{aligned}$$

Similarly, we also have

$$d_\varepsilon(\sigma_1^+, \sigma_2^+) \geq c_\varepsilon e^{-\varepsilon(\delta + 2b')} d^+(\sigma_1, \sigma_2)^\varepsilon$$

where $0 < c_\varepsilon < 1$ is given in (7.8). Setting $b_Q := e^{\varepsilon(\delta+2b')}/c_\varepsilon$ completes the proof. $\square$

Reparameterization revisited. Recall the reparameterization $\Psi : \Gamma \backslash \mathcal{G} \to \Omega_\psi$ in Theorem 6.1, which is induced from the Γ-equivariant map $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$. Since $\tilde{\Psi}$ is proper and surjective, for $w_1, w_2 \in \tilde{\Omega}_\psi$, we define

$$\begin{aligned} d^+(w_1, w_2) &:= \sup_{\sigma_1 \in \tilde{\Psi}^{-1}(w_1),\ \sigma_2 \in \tilde{\Psi}^{-1}(w_2)} d^+(\sigma_1, \sigma_2); \\ d^-(w_1, w_2) &:= \sup_{\sigma_1 \in \tilde{\Psi}^{-1}(w_1),\ \sigma_2 \in \tilde{\Psi}^{-1}(w_2)} d^-(\sigma_1, \sigma_2). \end{aligned} \tag{7.9}$$

Since $\tilde{\Psi}$ is Γ-equivariant, if $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, then $\gamma\sigma_1 \in \tilde{\Psi}^{-1}(\gamma w_1)$ and $\gamma\sigma_2 \in \tilde{\Psi}^{-1}(\gamma w_2)$ for all $\gamma \in \Gamma$. Since $d^\pm(\gamma\sigma_1, \gamma\sigma_2) = d^\pm(\sigma_1, \sigma_2)$ as well, we have

$$d^\pm(\gamma w_1, \gamma w_2) = d^\pm(w_1, w_2) \quad \text{for all } \gamma \in \Gamma. \tag{7.10}$$

We also have the following expansion and contraction of d^+ and d^- via the flow $\{\phi_t\}$ respectively:

Lemma 7.5. *There exist $\alpha, \alpha' > 0$ and $b \geq 1$ such that for any $w_1, w_2 \in \tilde{\Omega}_\psi$ and $t \geq 0$, we have*

$$\begin{aligned} \frac{1}{b} e^{\alpha t} d^+(w_1, w_2) &\leq d^+(\phi_t w_1, \phi_t w_2) \leq b e^{\alpha' t} d^+(w_1, w_2); \\ \frac{1}{b} e^{-\alpha' t} d^-(w_1, w_2) &\leq d^-(\phi_t w_1, \phi_t w_2) \leq b e^{-\alpha t} d^-(w_1, w_2). \end{aligned} \tag{7.11}$$

Proof. Let $w_1, w_2 \in \tilde{\Omega}_\psi$ and $t \geq 0$. Let $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$. By Theorem 6.1, there exist $s_1, s_2 \in \mathbb{R}$ such that

$$\varphi_{s_1}\sigma_1 \in \tilde{\Psi}^{-1}(\phi_t w_1) \quad \text{and} \quad \varphi_{s_2}\sigma_2 \in \tilde{\Psi}^{-1}(\phi_t w_2),$$

and moreover, for constants $a, a', B > 0$ in loc. cit., we have:

(1) if $s_1 \geq 0$, then

$$a s_1 - B \leq t \leq a' s_1 + B$$

(resp. if $s_2 \geq 0$, then $a s_2 - B \leq t \leq a' s_2 + B$).

(2) if $s_1 \leq 0$, then

$$a' s_1 - B \leq t \leq a s_1 + B$$

(resp. if $s_2 \leq 0$, then $a' s_2 - B \leq t \leq a s_2 + B$).

By Lemma 7.3, we have

$$e^{-\delta} e^{s_1+s_2} d^+(\sigma_1, \sigma_2) \leq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta} e^{s_1+s_2} d^+(\sigma_1, \sigma_2). \tag{7.12}$$

Suppose first that $s_1, s_2 \geq 0$. Then by (1) above, we deduce from (7.12) that

$$d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta} e^{\frac{2B}{a}} e^{\frac{2t}{a}} d^+(\sigma_1, \sigma_2) \leq e^{\delta} e^{\frac{2B}{a}} e^{\frac{2t}{a}} d^+(w_1, w_2).$$

Since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, $\varphi_{s_1}\sigma_1$ and $\varphi_{s_2}\sigma_2$ are arbitrary elements of $\tilde{\Psi}^{-1}(\phi_t w_1)$ and $\tilde{\Psi}^{-1}(\phi_t w_2)$ respectively. Hence we have

$$d^+(\phi_t w_1, \phi_t w_2) \leq e^{\delta} e^{\frac{2B}{a}} e^{\frac{2t}{a}} d^+(w_1, w_2). \tag{7.13}$$

Similarly, we deduce from (1) and (7.12) that

$$d^+(\phi_t w_1, \phi_t w_2) \geq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \geq e^{-\delta} e^{-\frac{2B}{a'}} e^{\frac{2t}{a'}} d^+(\sigma_1, \sigma_2).$$

Since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, we have

$$d^+(\phi_t w_1, \phi_t w_2) \geq e^{-\delta} e^{-\frac{2B}{a'}} e^{\frac{2t}{a'}} d^+(w_1, w_2). \tag{7.14}$$

Now consider the case when at least one of s_1 and s_2 is negative. Then by (2), we must have $0 \leq t \leq B$, and hence we deduce from (1) and (2) that $s_1, s_2 \in [-B/a, 2B/a]$. It then follows from (7.12) that

$$d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \leq e^{\delta} e^{\frac{4B}{a}} d^+(\sigma_1, \sigma_2) \leq e^{\delta} e^{\frac{4B}{a}} d^+(w_1, w_2)$$

and that

$$d^+(\phi_t w_1, \phi_t w_2) \geq d^+(\varphi_{s_1}\sigma_1, \varphi_{s_2}\sigma_2) \geq e^{-\delta} e^{-\frac{2B}{a}} d^+(\sigma_1, \sigma_2).$$

Again, since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, these imply

$$e^{-\delta} e^{-\frac{2B}{a}} d^+(w_1, w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq e^{\delta} e^{\frac{4B}{a}} d^+(w_1, w_2).$$

Since $0 \leq t \leq B$, we in particular have

$$e^{-\delta} e^{-\frac{2B}{a} - \frac{2B}{a'}} e^{\frac{2t}{a'}} d^+(w_1, w_2) \leq d^+(\phi_t w_1, \phi_t w_2) \leq e^{\delta} e^{\frac{4B}{a}} e^{\frac{2t}{a}} d^+(w_1, w_2). \tag{7.15}$$

Combining (7.13), (7.14), and (7.15), the inequalities for d^+ in (7.11) follows. The inequalities for d^- in (7.11) can be shown by a similar argument. $\square$

For $w_1, w_2 \in \tilde{\Omega}_\psi$, we also define

$$d_\varepsilon^+(w_1, w_2) := d_\varepsilon(\sigma_1^+, \sigma_2^+) \tag{7.16}$$

where $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$. Since every elements of $\tilde{\Psi}^{-1}(w)$ has the common forward endpoint for each $w \in \tilde{\Omega}_\psi$, this is well-defined.

Lemma 7.6. *For any compact subset $Q \subset \tilde{\Omega}_\psi$, there exists a constant $c_Q \geq 1$ such that for any $w_1, w_2 \in Q$, we have*

$$\frac{1}{c_Q} d^+(w_1, w_2)^\varepsilon \leq d_\varepsilon^+(w_1, w_2) \leq c_Q d^+(w_1, w_2)^\varepsilon.$$

Proof. Let $Q \subset \tilde{\Omega}_\psi$ be a compact subset. Since $\tilde{\Psi}$ is proper, it follows from Lemma 7.4 that there exists a uniform constant $c_Q \geq 1$ such that if $w_1, w_2 \in Q$ and $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, then

$$\frac{1}{c_Q} d^+(\sigma_1, \sigma_2)^\varepsilon \leq d_\varepsilon^+(w_1, w_2) \leq c_Q d^+(\sigma_1, \sigma_2)^\varepsilon \leq c_Q d^+(w_1, w_2)^\varepsilon.$$

Since $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$ are arbitrary, the claim follows. $\square$

Proof of Theorem 7.1. Let $d^\pm : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ be functions defined in (7.9). From the definition, $d^\pm$ are non-negative and symmetric. Moreover, they are Γ-invariant by (7.10).

Let $z \in \tilde{\Omega}_\psi$. We show that the restriction on d^+ defines a semi-metric on $\tilde{W}^+(z)$; the corresponding statement for d^- can be shown by the same argument. It suffices to show that for $w_1, w_2 \in \tilde{W}^+(z)$, $d^+(w_1, w_2) = 0$ if and only if $w_1 = w_2$. Suppose first that $w_1 = w_2$. Then for any $\sigma_1, \sigma_2 \in \tilde{\Psi}^{-1}(w_1) = \tilde{\Psi}^{-1}(w_2)$, we have $\sigma_1^+ = \sigma_2^+$. This implies $(\sigma_1|\sigma_2)_x = \infty$. Hence, by (7.5), we have $d^+(\sigma_1, \sigma_2) = 0$. Since $\sigma_1, \sigma_2 \in \tilde{\Psi}^{-1}(w_1) = \tilde{\Psi}^{-1}(w_2)$ are arbitrary, we have $d^+(w_1, w_2) = 0$. Conversely, suppose that $d^+(w_1, w_2) = 0$. Let $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$. We then have $d^+(\sigma_1, \sigma_2) = 0$, and hence $(\sigma_1^+|\sigma_2^+)_x = \infty$ by (7.5), from which we deduce $\sigma_1^+ = \sigma_2^+$. Since $\tilde{\Psi}(\sigma_1) = w_1$ and $\tilde{\Psi}(\sigma_2) = w_2$, it follows from $w_1, w_2 \in \tilde{W}^+(z)$ and Lemma 4.5 that $w_1 = w_2$, showing the claim.

The inequalities in (1) and (2) follow from Lemma 7.5, finishing the proofs of (1) and (2).

We now show (3). For small enough $\varepsilon > 0$, we consider the function $d_\varepsilon^+ : \tilde{\Omega}_\psi \times \tilde{\Omega}_\psi \to \mathbb{R}$ defined in (7.16), that is, for $w_1, w_2 \in \tilde{\Omega}_\psi$,

$$d_\varepsilon^+(w_1, w_2) = d_\varepsilon(\sigma_1^+, \sigma_2^+)$$

where $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, and d_ε is the visual metric on ∂X_{GM} given in (7.8). Since d_ε is a metric, d_ε^+ is symmetric and satisfies the triangle inequality. Let $z \in \tilde{\Omega}_\psi$ and $w_1, w_2 \in \tilde{W}^+(z)$. As discussed above, for $\sigma_1 \in \tilde{\Psi}^{-1}(w_1)$ and $\sigma_2 \in \tilde{\Psi}^{-1}(w_2)$, we have $w_1 = w_2 \Leftrightarrow \sigma_1^+ = \sigma_2^+$ since $w_1, w_2 \in \tilde{W}^+(z)$. Hence $d_\varepsilon^+(w_1, w_2) = 0$ if and only if $w_1 = w_2$, and therefore the restriction of d_ε^+ defines a metric on $\tilde{W}^+(z)$. The inequality stated in (3) is proved in Lemma 7.6. This completes the proof.

8. Finiteness of Bowen-Margulis-Sullivan measures

Let $\Gamma < G$ be a θ-Anosov subgroup relative to $\mathcal{P}$ and X_{GM} the associated Groves-Manning cusp space. Let $\psi \in \mathfrak{a}_\theta^*$ be a (Γ, θ)-proper linear form tangent to the θ-growth indicator ψ_Γ^θ. By [10], there exists a unique (Γ, ψ)-Patterson-Sullivan measure ν_ψ on Λ_θ and a unique $(\Gamma, \psi \circ \mathrm{i})$-Patterson-Sullivan measure $\nu_{\psi\circ\mathrm{i}}$ on $\Lambda_{\mathrm{i}(\theta)}$. Let m_ψ be the Bowen-Margulis-Sullivan measure on Ω_ψ associated with the pair (ν, ν_{i}) defined in (3.2).

The relatively Anosov subgroups are regarded as the higher rank generalization of geometrically finite subgroups. Indeed, same as geometrically finite subgroups, relatively Anosov subgroups have finite Bowen-Margulis-Sullivan measures:

Theorem 8.1. *We have*

$$|m_\psi| := m_\psi(\Omega_\psi) < \infty.$$

We prove this finiteness of the Bowen-Margulis-Sullivan measure as a consequence of our reparameterization theorem (Theorem 6.1).

Thick-thin decomposition of Ω_ψ. Let $\Psi : \Gamma\backslash\mathcal{G} \to \Omega_\psi$ be the reparameterization given in Theorem 6.1. Via Ψ, the decomposition $\mathcal{G} = \mathcal{G}_{thick} \cup \mathcal{G}_{thin}$ gives the thick-thin decomposition

$$\Omega_\psi = \Psi(\Gamma\backslash\mathcal{G}_{thick}) \cup \Psi(\Gamma\backslash\mathcal{G}_{thin})$$

into the compact thick part $\Psi(\Gamma\backslash\mathcal{G}_{thick})$ and the thin part $\Psi(\Gamma\backslash\mathcal{G}_{thin})$.

The followings are extra ingredients in the proof:

Lemma 8.2 (Shadow lemma)**.** [19, Lemma 7.2] *For all large enough $R > 0$, there exists $c_0 = c_0(\psi, R) \geq 1$ such that for all $\gamma \in \Gamma$,*

$$c_0^{-1} e^{-\psi(\mu_\theta(\gamma))} \leq \nu_\psi(O_R^\theta(o, \gamma o)) \leq c_0 e^{-\psi(\mu_\theta(\gamma))}.$$

We denote by $0 \leq \delta_\psi(\Gamma) \leq \infty$ the abscissa of convergence of the Poincaré series $s \mapsto \sum_{\gamma\in\Gamma} e^{-s\psi(\mu_\theta(\gamma))}$; this is well-defined by the (Γ, θ)-properness hypothesis on ψ. Since ψ is tangent to ψ_Γ^θ, we furthermore have

$$\delta_\psi(\Gamma) = 1$$

([10, Theorem 10.1], [19, Theorem 4.5]). On the other hand, we have the following:

Theorem 8.3 (Canary-Zhang-Zimmer, [10, Corollary 7.2])**.** *The Patterson-Sullivan measure ν_ψ is atomless and for each $P \in \mathcal{P}$, we have*

$$\delta_\psi(P) < 1.$$

Proof of Theorem 8.1. As before, we identify Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ with ∂X_{GM} through the boundary maps. Recall the norm $\|\cdot\|_\sigma$ on $\mathbb{R}_+$ for each $\sigma \in \mathcal{G}$ and the Γ-equivariant surjective proper map $\tilde{\Psi} : \mathcal{G} \to \tilde{\Omega}_\psi$, $\sigma \mapsto (\sigma^+, \sigma^-, \log v_\sigma)$, defined in the proof of Theorem 6.1 where $v_\sigma \in \mathbb{R}_+$ is the unique vector such that $\|v_\sigma\|_\sigma = 1$. We then have

$$\tilde{\Omega}_\psi = \tilde{\Psi}(\mathcal{G}_{thick}) \cup \tilde{\Psi}(\mathcal{G}_{thin}).$$

We will use this specific decomposition to show the finiteness of m_ψ. Since Γ acts cocompactly on $\tilde{\Psi}(\mathcal{G}_{thick})$, it suffices to show that the measure of thin part $m_\psi(\Gamma\backslash\tilde{\Psi}(\mathcal{G}_{thin}))$ is finite. Moreover, since $\mathcal{G}_{thin} = \Gamma \cdot \bigcup_{P\in\mathcal{P}} \mathcal{G}_P$ and $\mathcal{P}$ is a finite collection, it suffices to show $m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) < \infty$ for each $P \in \mathcal{P}$.

Let us fix $P \in \mathcal{P}$ and denote by $\xi_P \in \partial X_{GM}$ the parabolic limit point fixed by P. Since ξ_P is bounded parabolic, we have a compact fundamental domain for the P-action on $\partial X_{GM} - \{\xi_P\}$, which we denote by D. Recall that we have identified Λ_θ and $\Lambda_{\mathrm{i}(\theta)}$ with ∂X_{GM} via the boundary maps. Since ν_ψ and $\nu_{\psi\circ\mathrm{i}}$ are atomless by Theorem 8.3, we have

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) = \sum_{\gamma\in P} \int_{(\gamma D\times D\times\mathbb{R})\cap\tilde{\Psi}(\mathcal{G}_P)} e^{\psi(\langle\xi,\eta\rangle)} d\nu_\psi(\xi) d\nu_{\psi\circ\mathrm{i}}(\eta) dt. \tag{8.1}$$

We first estimate the integration with respect to dt. We claim that there exists $C > 0$ such that for any $\gamma \in P$ and $\sigma \in \mathcal{G}_P$ such that $\sigma^- \in D$ and $\sigma^+ \in \gamma D$, we have

$$-C \le \log v_\sigma \le C + \psi(\mu_\theta(\gamma)). \tag{8.2}$$

Let us fix $\gamma \in P$ and let $\sigma \in \mathcal{G}_P$ be such that $\sigma^+ \in \gamma D$ and $\sigma^- \in D$. Recalling that H_P denotes the open horoball in X_{GM} associated to P, this implies that the following two constants are well-defined:

$$\begin{aligned} s_0 &:= \min\{s < 0 : \sigma(s) \in \partial H_P\} \\ s_1 &:= \max\{s > 0 : \sigma(s) \in \partial H_P\}. \end{aligned}$$

In other words, s_0 is the first time that σ enters into ∂H_P and s_1 is the last time that σ exits ∂H_P. We then have from (6.3) and Theorem 6.3 that

$$\begin{aligned} v_{\varphi_{s_0}\sigma} &= \|v_{\varphi_{s_0}\sigma}\|_\sigma v_\sigma = \kappa_{-s_0}(\varphi_{s_0}\sigma) v_\sigma \\ &\le be^{as_0} v_\sigma \le bv_\sigma; \\ v_{\varphi_{s_1}\sigma} &= \frac{1}{\|v_\sigma\|_{\varphi_{s_1}\sigma}} v_\sigma = \frac{1}{\kappa_{s_1}(\sigma)} v_\sigma \\ &\ge b^{-1} e^{as_1} v_\sigma \ge b^{-1} v_\sigma. \end{aligned}$$

Therefore, we have

$$-\log b + \log v_{\varphi_{s_0}\sigma} \le \log v_\sigma \le \log b + \log v_{\varphi_{s_1}\sigma}. \tag{8.3}$$

Now fix $x \in \partial H_P$. Then there exists $R > 0$ with the following property: for any $\sigma_0 \in \mathcal{G}_P$ such that $\sigma_0^- \in D$, the entering point of σ_0 into ∂H_P, i.e. $\sigma_0(s) \in \partial H_P$ with minimal s, must be contained in the R-ball $B_{GM}(x, R)$.

Indeed, if not, then there exists a sequence $\sigma_n \in \mathcal{G}_P$ such that $\sigma_n^- \in D$ and the entering point of σ_n into ∂H_P is not contained in $B_{GM}(x, n)$ for all $n \geq 1$. However, since $\sigma_n \in \mathcal{G}_P$ and $\sigma_n^- \in D$ for all $n \geq 1$, two sequences σ_n^+ and σ_n^- converge to two distinct points in ∂X_{GM} as $n \to \infty$, after passing to a subsequence. Hence the images of the bi-infinite geodesics σ_n intersect a single ball centered at x, which contradicts the choice of the sequence σ_n.

Hence we have $(\varphi_{s_0}\sigma)(0) = \sigma(s_0) \in B_{GM}(x, R)$. Since $I(\gamma^{-1}\sigma) \in \mathcal{G}_P$ also satisfies that $I(\gamma^{-1}\sigma)^- = \gamma^{-1}\sigma^+ \in D$ and its entering point into ∂H_P is given by $I(\gamma^{-1}\sigma)(-s_1) = \gamma^{-1}\sigma(s_1)$, we also have $\gamma^{-1}\sigma(s_1) \in B_{GM}(x, R)$. In other words, we have $(\gamma^{-1}\varphi_{s_0}\sigma)(s_1 - s_0) \in B_{GM}(x, R)$. Hence we can apply Lemma 6.4 to $\varphi_{s_0}\sigma$ by setting $Q = \overline{B_{GM}(x, R)}$ and obtain

$$\frac{1}{C_Q} e^{-\psi(\mu_\theta(\gamma))} \leq \kappa_{s_1 - s_0}(\varphi_{s_0}\sigma) \leq C_Q e^{-\psi(\mu_\theta(\gamma))}. \tag{8.4}$$

Since

$$v_{\varphi_{s_1}\sigma} = \frac{1}{\|v_{\varphi_{s_0}\sigma}\|_{\varphi_{s_1}\sigma}} v_{\varphi_{s_0}\sigma} = \frac{1}{\kappa_{s_1 - s_0}(\varphi_{s_0}\sigma)} v_{\varphi_{s_0}\sigma}$$

by (6.3), it follows from (8.4) that

$$\log v_{\varphi_{s_1}\sigma} \leq \log C_Q + \log v_{\varphi_{s_0}\sigma} + \psi(\mu_\theta(\gamma)).$$

Hence we deduce from (8.3) that

$$-\log b + \log v_{\varphi_{s_0}\sigma} \leq \log v_\sigma \leq \log(bC_Q) + \log v_{\varphi_{s_0}\sigma} + \psi(\mu_\theta(\gamma)).$$

Since $(\varphi_{s_0}\sigma)(0) \in B_{GM}(x, R)$ where x is fixed and R is determined by x and P, the constant $\log v_{\varphi_{s_0}\sigma}$ is also uniformly bounded. Therefore, the claim (8.2) follows.

By the claim (8.2), we deduce from (8.1) that

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \\ \leq \sum_{\gamma \in P} (2C + \psi(\mu_\theta(\gamma))) \int_{(\gamma D \times D) \cap \{(\sigma^+, \sigma^-) : \sigma \in \mathcal{G}_P\}} e^{\psi(\langle \xi, \eta \rangle)} d\nu_\psi(\xi) d\nu_{\psi \circ \mathrm{i}}(\eta).$$

As we already observed, for $x \in \partial H_P$ and $R > 0$ above, we have that if $\sigma \in \mathcal{G}_P$ is such that $\sigma^- \in D$ and $\sigma^+ \in \gamma D$, then the image of the bi-infinite geodesic σ must intersect $B_{GM}(x, R)$ and $B_{GM}(\gamma x, R)$. Hence it follows from Lemma 5.10 that

$$\psi(\langle \sigma^+, \sigma^- \rangle) \quad \text{is uniformly bounded.} \tag{8.5}$$

Moreover, we also have that $\sigma^+ \in O_{R'}^{GM}(x, \gamma x)$ for some $R' > 0$ depending on x and R. By Proposition 5.7, we then have for some uniform $r > 0$ that

$$\sigma^+ \in O_r^\theta(o, \gamma o). \tag{8.6}$$

By (8.5) and (8.6), we now have

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma \in P} (2C + \psi(\mu_\theta(\gamma))) \nu_\psi(O_r^\theta(o, \gamma o)).$$

Applying the shadow lemma (Lemma 8.2), we finally obtain

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma\in P}(2C+\psi(\mu_\theta(\gamma)))e^{-\psi(\mu_\theta(\gamma))}.$$

Let $0<\varepsilon<1$. Since ψ is (Γ,θ)-proper, $\liminf_{\gamma\in P}\psi(\mu_\theta(\gamma))=\infty$, and hence $\psi(\mu_\theta(\gamma))\ll e^{\varepsilon\psi(\mu_\theta(\gamma))}$. Hence

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma\in P}(2C+\psi(\mu_\theta(\gamma)))e^{-\psi(\mu_\theta(\gamma))} \ll \sum_{\gamma\in P} e^{-(1-\varepsilon)\psi(\mu_\theta(\gamma))}.$$

By Theorem 8.3, for $\varepsilon>0$ sufficiently small, we have

$$m_\psi(P\backslash\tilde{\Psi}(\mathcal{G}_P)) \ll \sum_{\gamma\in P} e^{-(1-\varepsilon)\psi(\mu_\theta(\gamma))} < \infty.$$

This completes the proof of Theorem 8.1. □

9. Unique measure of maximal entropy

Let Γ be a relatively θ-Anosov subgroup and $\psi\in\mathfrak{a}_\theta^*$ a (Γ,θ)-proper form tangent to ψ_Γ^θ. Let m_ψ be the Bowen-Margulis-Sullivan measure on Ω_ψ. This section is devoted to the proof of the following: by Theorem 8.1, m_ψ is of finite measure.

Theorem 9.1. *Let m be a probability $\{\phi_t\}$-invariant measure on Ω_ψ. Then the metric entropy $h_m(\{\phi_t\})$ is at most $\delta_\psi=1$, and $h_m(\{\phi_t\})=1$ if and only if $m=m_\psi/|m_\psi|$, the normalized probability measure of m_ψ.*

We recall some basic notions about entropy; we refer to ([17], [14]) for details.

Measurable partitions and entropy. Let $(\mathcal{X},\mathcal{M},m)$ be a probability space, where $\mathcal{M}$ is a σ-algebra and m is a probability measure. By a partition ζ of $\mathcal{X}$, we mean a collection of disjoint non-empty measurable subsets of $\mathcal{X}$ whose union is $\mathcal{X}$. For a partition ζ of $\mathcal{X}$ and $x\in\mathcal{X}$, we denote by $\zeta(x)$ the element of ζ containing x, called the atom at x. Let $\mathcal{M}_\zeta\subset\mathcal{M}$ be the sub σ-algebra generated by the atoms of ζ. A partition ζ of $\mathcal{X}$ is called m-measurable if it admits a separation by countably many elements in $\mathcal{M}_\zeta$. More precisely, ζ is m-measurable if there exist a m-conull subset $\mathcal{Y}\subset\mathcal{X}$ and a sequence $\{Y_i\in\mathcal{M}_\zeta : i\in\mathbb{N}\}$ such that for any distinct atoms z,z' of ζ, there exists $i\in\mathbb{N}$ such that either $z\cap\mathcal{Y}\subset Y_i$ and $z'\cap\mathcal{Y}\subset\mathcal{X}-Y_i$, or $z\cap\mathcal{Y}\subset\mathcal{X}-Y_i$ and $z'\cap\mathcal{Y}\subset Y_i$.

For an m-measurable partition ζ and m-a.e. $x\in\mathcal{X}$, we denote by $m_{\zeta(x)}$ the conditional measure on the atom $\zeta(x)$ so that the following holds [14, Theorem 5.9]: for any measurable $Y\subset\mathcal{X}$, we have

- $x\mapsto m_{\zeta(x)}(Y\cap\zeta(x))$ is measurable;
- $m(Y)=\int_{\mathcal{X}} m_{\zeta(x)}(Y\cap\zeta(x))\,dm(x)$.

For two m-measurable partitions ζ, ζ', we say that ζ is finer than ζ' and write $\zeta \succ \zeta'$ if for m-a.e. $x \in \mathcal{X}$, $\zeta(x) \subset \zeta'(x)$. For a sequence of m-measurable partitions ζ_i, we denote by $\bigvee_i \zeta_i$ the smallest m-measurable partition finer than all ζ_i.

Given an m-measurable partition ζ and an m-measurable map $\varphi : \mathcal{X} \to \mathcal{X}$, the pull-back $\varphi^{-1}\zeta$ is an m-measurable partition with atoms $(\varphi^{-1}\zeta)(x) = \varphi^{-1}(\zeta(\varphi(x)))$. We say that ζ is φ-decreasing if $\varphi^{-1}\zeta \succ \zeta$ and φ-generating if $\bigvee_{i\in\mathbb{N}} \varphi^{-i}\zeta$ is m-equivalent to the partition consisting of points.

Let $\varphi : \mathcal{X} \to \mathcal{X}$ be an m-measure-preserving transformation. For a countable partition ζ, the entropy of ζ relative to m is

$$H_m(\zeta) := \int_{\mathcal{X}} -\log m(\zeta(x))\ dm(x)$$

with the convention that $\infty \cdot 0 = 0$. The average entropy of ζ is defined as

$$H_m(\varphi, \zeta) := \lim_{n\to\infty} \frac{1}{n} H_m\left(\bigvee_{i=0}^{n-1} \varphi^{-i}\zeta\right).$$

The metric entropy of φ with respect to m is defined as

$$h_m(\varphi) := \sup H_m(\varphi, \zeta)$$

where the supremum is taken over all countable partitions ζ with $H_m(\zeta) < \infty$. For a flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $\mathcal{X}$, we have $h_m(\phi_t) = |t| h_m(\phi_1)$ for all $t \neq 0$. The metric entropy of the flow $\{\phi_t\}$ with respect to m is defined as

$$h_m(\{\phi_t\}) := h_m(\phi_1).$$

For a φ-decreasing m-measurable partition ζ, we also define

$$h_m(\varphi, \zeta) := \int_{\mathcal{X}} -\log m_{\zeta(x)}((\varphi^{-1}\zeta)(x))\ dm(x).$$

Partition realizing the entropy. Recall the foliations $\tilde{W}^{\pm}$ of $\tilde{\Omega}_\psi$ and $W^{\pm}$ of Ω_ψ from (4.1) and (4.2). Let m be a probability measure on Ω_ψ and $\tilde{m}$ the Γ-invariant lift of m to $\tilde{\Omega}_\psi$. A Γ-invariant partition $\tilde{\zeta}$ of $\tilde{\Omega}_\psi$ is called $\tilde{m}$-measurable if the induced partition ζ on Ω_ψ is m-measurable. We say that an $\tilde{m}$-measurable partition $\tilde{\zeta}$ is subordinated to $\tilde{W}^+$ if for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$, there exist precompact open neighborhoods $\tilde{\mathcal{U}}_1$ and $\tilde{\mathcal{U}}_2$ of $\tilde{x}$ in $\tilde{W}^+(\tilde{x})$ such that

$$\tilde{\mathcal{U}}_1 \subset \tilde{\zeta}(\tilde{x}) \subset \tilde{\mathcal{U}}_2$$

Proposition 9.2. *Let $\tau > 0$. Let m be a probability measure on Ω_ψ which is invariant and ergodic under ϕ_τ and $\tilde{m}$ its lift to $\tilde{\Omega}_\psi$. Then there exists a Γ-invariant $\tilde{m}$-measurable partition $\tilde{\zeta}$ of $\tilde{\Omega}_\psi$ subordinated to $\tilde{W}^+$ such that its projection ζ is an m-measurable ϕ_τ-decreasing and generating partition of Ω_ψ which satisfies*

$$h_m(\phi_\tau) = h_m(\phi_\tau, \zeta) < \infty.$$

The most delicate part of the proof of this proposition lies in the construction of the partition which is subordinated to the unstable foliation $\tilde{W}^+$. The exponential expansion property of the flow $\{\phi_t\}$ on Ω_ψ (Theorem 7.1) was obtained precisely for this purpose. Other parts of Proposition 9.2 can be obtained by similar argument in [24].

Proof of Proposition 9.2. Let $d^\pm$ and d_ε^+ be functions on $\tilde{\Omega}_\psi \times \tilde{\Omega}_\psi$ given in Theorem 7.1 for some fixed $\varepsilon > 0$. Fix $u \in \tilde{\Omega}_\psi$. For $r > 0$, we set

$$\tilde{C}(u,r) = \left\{ v \in \tilde{\Omega}_\psi : \begin{array}{c} \exists s \in (-r,r), w \in \tilde{W}^-(\phi_s u) \text{ with } d^-(\phi_s u, w) < r \\ \text{s.t. } v \in \tilde{W}^+(w) \text{ and } d_\varepsilon^+(w,v) < r \end{array} \right\}.$$

Fix $\rho > 0$ small enough so that the projection $\tilde{\Omega}_\psi \to \Omega_\psi$ is injective on $\tilde{C}(u, 4\rho)$. For $0 < r < 4\rho$, we denote by $C(u,r)$ the image of $\tilde{C}(u,r)$ under the projection $\tilde{\Omega}_\psi \to \Omega_\psi$.

We define a function $\ell : \Omega_\psi \to \mathbb{R}$ as follows: for each $x \in C(u,\rho)$, let $\tilde{x} \in \tilde{C}(u,\rho)$ be the unique lift of x. It follows from the description of $\tilde{W}^\pm$ in Lemma 4.5 that there exist unique $s \in (-\rho,\rho)$ and $\tilde{y} \in \tilde{W}^-(\phi_s u)$ such that $\tilde{x} \in \tilde{W}^+(\tilde{y})$, $d^-(\phi_s u, \tilde{y}) < \rho$ and $d_\varepsilon^+(\tilde{y}, \tilde{x}) < \rho$. We set

$$\ell(x) := \max(s, d^-(\phi_s u, \tilde{y}), d_\varepsilon^+(\tilde{y}, \tilde{x})).$$

For $x \in \Omega_\psi - C(u,\rho)$, we then set $\ell(x) := \rho$.

For each $0 < r < \rho$, let $\tilde{\zeta}_r'$ be the partition of $\tilde{\Omega}_\psi$ with atoms $\gamma \tilde{C}(u,r) \cap \tilde{W}^+(\tilde{x})$ for $\tilde{x} \in \tilde{\Omega}_\psi$, $\gamma \in \Gamma$ and $\tilde{\Omega}_\psi - \Gamma \tilde{C}(u,r)$. We then define

$$\tilde{\zeta}_r := \bigvee_{i=0}^\infty \phi_\tau^i \tilde{\zeta}_r'.$$

Let ζ_r' and ζ_r be the partitions obtained by projecting $\tilde{\zeta}_r'$ and $\tilde{\zeta}_r$ to Ω_ψ respectively. Then $\zeta_r = \bigvee_{i=0}^\infty \phi_\tau^i \zeta_r'$ since the Γ-action commutes with the flow $\{\phi_t\}$. It is clear that ζ_r is ϕ_τ-decreasing. In view of the construction of $\tilde{\zeta}$ which uses atoms $\gamma \tilde{C}(u,r) \cap W^+(\tilde{x})$, we can verity that ζ_r is m-measurable by a same argument as in [24, Proposition 1]. Denote by $\tilde{m}$ is the lift of m to $\tilde{\Omega}_\psi$. Let d be the metric on Ω_ψ considered in Proposition 4.6. By the ergodicity of m, we have that for m-a.e. $x \in \Omega_\psi$, $\phi_\tau^k x \in C(u,r)$ for infinitely many $k \in \mathbb{N}$, and hence $\zeta_r'(\phi_\tau^k x)$ is contained in a uniformly bounded set $C(u,r) \cap W^+(\phi_\tau^k x)$ with respect to d. Since $(\phi_\tau^{-k} \zeta_r)(x) \subset \phi_\tau^{-k}(\zeta_r'(\phi_\tau^k x))$, it follows from Proposition 4.6 that ζ_r is ϕ_τ-generating. Similarly, for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$, we have $\phi_\tau^{-k} \tilde{x} \in \gamma \tilde{C}(u,r)$ for some $k \in \mathbb{N}$ and $\gamma \in \Gamma$. Hence we have $\tilde{\zeta}_r(\tilde{x}) \subset \phi_\tau^k(\tilde{\zeta}_r'(\phi_\tau^{-k}\tilde{x})) \subset \phi_\tau^k \gamma \tilde{C}(u,r) \cap \tilde{W}^+(\tilde{x})$, and therefore $\tilde{\zeta}_r(\tilde{x})$ is a precompact subset of $\tilde{W}^+(\tilde{x})$.

We now show the most delicate part of the proof that we can take $r > 0$ so that $\tilde{\zeta}_r(\tilde{x})$ contains an open neighborhood of $\tilde{x}$ in $W^+(\tilde{x})$ for $\tilde{m}$-a.e. $\tilde{x} \in \tilde{\Omega}_\psi$. We use Theorem 7.1 in a crucial way.

Consider the push-forward $\ell_* m$ of the measure m by ℓ, which is a probability measure on $[0, \rho] \subset \mathbb{R}$. For any $\varepsilon_0 \in (0,1)$, we have that

$$\mathrm{Leb}\left(\left\{ r \in (0,\rho) : \sum_{k=0}^{\infty} (\ell_* m)([r-\varepsilon_0^k, r+\varepsilon_0^k]) < \infty \right\}\right) = \rho$$

by [20, Proposition 3.2]. Since m is ϕ_τ-invariant, this is same to say that

$$\mathrm{Leb}\left(\left\{ r \in (0,\rho) : \sum_{k=0}^{\infty} m(\{x : |\ell(\phi_\tau^{-k} x) - r| < \varepsilon_0^k\}) < \infty \right\}\right) = \rho.$$

We fix a constant $e^{-\varepsilon\alpha\tau} < \varepsilon_0 < 1$ where $\alpha > 0$ is a constant given in Theorem 7.1. We can therefore choose $0 < r < \rho/2$ so that $m(\partial C(u,r)) = 0$ and that

$$\sum_{k=0}^{\infty} m(\{x : |\ell(\phi_\tau^{-k} x) - r| < \varepsilon_0^k\}) < \infty.$$

Let Ω'_ψ be the set of all $x \in \Omega_\psi - \bigcup_{k=0}^{\infty} \phi^k \partial C(u,r)$ satisfying that for some $N_0 = N_0(x) > 0$, we have

$$\ell(\phi_\tau^{-k} x) < r - \varepsilon_0^k \quad \text{or} \quad \ell(\phi_\tau^{-k} x) > r + \varepsilon_0^k \tag{9.1}$$

for all $k \geq N_0$. Since $m(\partial C(u,r)) = 0$, it follows from the classical Borel-Cantelli lemma that $m(\Omega'_\psi) = 1$. Let $x \in \Omega'_\psi$ be an arbitrary point and corresponding $N_0 = N_0(x)$. We fix a lift $\tilde{x} \in \tilde{\Omega}_\psi$ of x.

For $\tilde{y} \in \tilde{\Omega}_\psi$, we write y for its projection to Ω_ψ. Fix a compact subset $Q \subset \tilde{\Omega}_\psi$ containing

$$\bigcup_{v_0 \in \tilde{C}(u,\rho)} \{v \in \tilde{W}^+(v_0) : d^+(v, v_0) \leq b\}$$

where $b \geq 1$ is the constant given in Theorem 7.1.

We set

$$r_1 := \min\left(\frac{1}{2}, \frac{1}{b(2c)^{1/\varepsilon}}\right) > 0$$

where $c = c_Q \geq 1$ is as given in Theorem 7.1(3). Let

$$\tilde{\mathcal{U}} = \{\tilde{y} \in \tilde{W}^+(\tilde{x}) : d^+(\tilde{x}, \tilde{y}) < r_1\};$$

this is a precompact neighborhood of $\tilde{x}$ in $\tilde{W}^+(\tilde{x})$. Let $\mathcal{U}$ be the image of $\tilde{\mathcal{U}}$ in Ω_ψ. We claim that for each $k \geq N_0$, either

$$\phi_\tau^{-k}(\tilde{\mathcal{U}}) \subset \gamma^{-1}\tilde{C}(u,r) \text{ for some } \gamma \in \Gamma \quad \text{or} \quad \phi_\tau^{-k}(\tilde{\mathcal{U}}) \cap \Gamma\tilde{C}(u,r) = \emptyset. \tag{9.2}$$

Fix $k \geq N_0$. Recall that x satisfies either $\ell(\phi_\tau^{-k} x) < r - \varepsilon_0^k$ or $\ell(\phi_\tau^{-k} x) > r + \varepsilon_0^k$. Consider the first case. This implies that there exists $\gamma \in \Gamma$ such that $\gamma\phi_\tau^{-k}\tilde{x} \in \tilde{C}(u, r - \varepsilon_0^k)$. We then have

$$d^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) = d^+(\phi_\tau^{-k}\tilde{x}, \phi_\tau^{-k}\tilde{y}) \leq b e^{-\alpha\tau k} d^+(\tilde{x}, \tilde{y}).$$

by (7.10) and Theorem 7.1(1). In particular, we have $\gamma\phi_\tau^{-k}\tilde{y} \in Q$ and hence

$$d_\varepsilon^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) \le cd^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y})^\varepsilon \le cb^\varepsilon e^{-\varepsilon\alpha\tau k} d^+(\tilde{x},\tilde{y})^\varepsilon \tag{9.3}$$

by Theorem 7.1(3). Let $\tilde{y} \in \tilde{\mathcal{U}}$, and hence $d^+(\tilde{x},\tilde{y}) < r_1$. Since $e^{-\varepsilon\alpha\tau} < \varepsilon_0$, we then have

$$d_\varepsilon^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) < \varepsilon_0^k$$

by (9.3), and therefore $\gamma\phi_\tau^{-k}\tilde{y} \in \tilde{C}(u,r)$. Hence

$$\phi_\tau^{-k}(\tilde{\mathcal{U}}) \subset \gamma^{-1}\tilde{C}(u,r),$$

proving (9.2) in this case.

Now consider the case when $\ell(\phi_\tau^{-k}x) > r + \varepsilon_0^k$. In this case, we claim that $\phi_\tau^{-k}(\tilde{\mathcal{U}}) \cap \Gamma\tilde{C}(u,r) = \emptyset$. Suppose not. Then there exists $\gamma \in \Gamma$ and some $\tilde{y} \in \tilde{W}^+(\tilde{x})$ such that $d^+(\tilde{x},\tilde{y}) < r_1$ and $\gamma\phi_\tau^{-k}\tilde{y} \in \tilde{C}(u,r)$. By the same argument as above, $\gamma\phi_\tau^{-k}\tilde{x} \in Q$ and hence

$$d_\varepsilon^+(\gamma\phi_\tau^{-k}\tilde{x}, \gamma\phi_\tau^{-k}\tilde{y}) \le cb^\varepsilon e^{-\varepsilon\alpha\tau k} d^+(\tilde{x},\tilde{y})^\varepsilon.$$

Since $d^+(\tilde{x},\tilde{y}) < r_1$, we have $\gamma\phi_\tau^{-k}\tilde{x} \in \tilde{C}(u, r+\varepsilon_0^k)$. This is a contradiction since $\ell(\phi_\tau^{-1}x) > r + \varepsilon_0^k$, proving the claim.

The claim (9.2) implies that $\phi_\tau^{-k}(\tilde{\mathcal{U}})$ lies in a single atom of $\tilde{\zeta}_r'$ for each $k \ge N_0$.

Since $\phi_\tau^{-k}\tilde{x} \notin \partial\gamma^{-1}\tilde{C}(u,r)$ for all $k \in \mathbb{N}$ and $\gamma \in \Gamma$, we can find a small neighborhood $\tilde{\mathcal{U}}' \subset \tilde{\mathcal{U}}$ of $\tilde{x}$ in $\tilde{W}^+(\tilde{x})$ such that $\phi_\tau^{-k}(\tilde{\mathcal{U}}')$ is entirely contained in some $\gamma^{-1}\tilde{C}(u,r)$, $\gamma \in \Gamma$ or disjoint from $\Gamma\overline{C(u,r)}$ for each $0 \le k \le N_0$. Therefore $\phi_\tau^{-k}(\tilde{\mathcal{U}}')$ is contained in a single atom of $\tilde{\zeta}_r'$ for all $k \in \mathbb{N}$. This proves that the atom of $\tilde{\zeta}_r$ containing $\tilde{x}$ also contains $\tilde{\mathcal{U}}'$. Since $x \in \Omega_\psi'$ is arbitrary, $\tilde{\zeta}_r$ is subordinated to $\tilde{W}^+$.

The rest of the argument is a similar entropy computation as in the deduction of [24, Proposition 4] from [24, Proposition 1]. □

Proof of Theorem 9.1. The deduction of Theorem 9.1 from Proposition 9.2 can be done similarly to [24].

First, note that $\delta_\psi = 1$ since ψ is tangent to ψ_Γ^θ ([10, Theorem 10.1], [19, Theorem 4.5]). For $g \in G$ such that $[g] \in \tilde{\Omega}_\psi$, we consider the measure $\mu_{\tilde{W}^+([g])}$ on $\tilde{W}^+([g])$ given by

$$d\mu_{\tilde{W}^+([g])}([gn]) = e^{\psi(\beta^\theta_{(gn)^+}(e,gn))} d\nu((gn)^+)$$

for $n \in N_\theta^+$. It follows from the definition that for all $a \in A_\theta$, we have

$$\frac{da_*\mu_{\tilde{W}^+([g])}}{d\mu_{\tilde{W}^+([ga])}}(x) = e^{-\psi(\log a)}. \tag{9.4}$$

We write m^{pr} for the normalized probability measure $m_\psi/|m_\psi|$. Denote by $\tilde{m}^{pr}$ its lift to $\tilde{\Omega}_\psi$. The following can be obtained by directly checking the condition for conditional measures:

Lemma 9.3. *Let $\tilde\zeta$ be an $\tilde m^{pr}$-measurable partition of $\tilde\Omega_\psi$ subordinated to $\tilde W^+$. Then the family of conditional measures of $\tilde m^{pr}$ with respect to $\tilde\zeta$ is given by*

$$d\tilde m^{pr}_{\tilde\zeta(\tilde x)}(w) := \frac{\mathbb{1}_{\tilde\zeta(\tilde x)}(w)}{\mu_{\tilde W^+(\tilde x)}(\tilde\zeta(\tilde x))} d\mu_{\tilde W^+(\tilde x)}(w) \quad \text{for } \tilde x \in \tilde\Omega_\psi.$$

By Theorem 8.1, m_ψ is finite, and hence it follows from Theorem 4.2 that m^{pr} is $\{\phi_t\}$-ergodic. It is a general fact that m^{pr} is ergodic for the transformation ϕ_t for uncountably many t [24, Lemma 7]. Fix $\tau > 0$ so that m^{pr} is ϕ_τ-ergodic. Now let m be a probability $\{\phi_t\}$-invariant measure on Ω_ψ. Considering the ergodic decomposition of m, we may assume that m is ϕ_τ-ergodic without loss of generality [14, (3.5a)].

We now consider the partition $\tilde\zeta$ given by Proposition 9.2 for the measure m, its lift $\tilde m$, and the transformation ϕ_τ. Since $\tilde\zeta$ is subordinated to $\tilde W^+$, the measure

$$d\tilde m^{pr}_{\tilde\zeta(\tilde x)}(w) := \frac{\mathbb{1}_{\tilde\zeta(\tilde x)}(w)}{\mu_{\tilde W^+(\tilde x)}(\tilde\zeta(\tilde x))} d\mu_{\tilde W^+(\tilde x)}(w)$$

and the function

$$\tilde G(\tilde x) := -\log \mu_{\tilde W^+(\tilde x)}(\tilde\zeta(\tilde x))$$

are well-defined for $\tilde m$-a.e. $\tilde x \in \tilde\Omega_\psi$. Note that since $\tilde\zeta$ is a partition for the measure $\tilde m$, it may not be $\tilde m^{pr}$-measurable and hence Lemma 9.3 does not apply to $\tilde\zeta$. It follows from (9.4) that for $\tilde m$-a.e. $\tilde x \in \tilde\Omega_\psi$, we have

$$-\log \tilde m^{pr}_{\tilde\zeta(\tilde x)}((\phi_\tau^{-1}\tilde\zeta)(\tilde x)) = \tau + (\tilde G \circ \phi_\tau)(\tilde x) - \tilde G(\tilde x). \tag{9.5}$$

This implies

$$\tilde G \circ \phi_\tau - \tilde G \geq -\tau$$

$\tilde m$-a.e. Since $\tilde G$ is Γ-invariant, it induces the function $G : \Omega_\psi \to \mathbb{R}$. By [24, Lemme 8], we have $\int G \circ \phi_\tau - G \; dm = 0$ and therefore

$$\int -\log m^{pr}_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x)) \; dm(x) = \tau. \tag{9.6}$$

where $m^{pr}_{\zeta(x)}$ is the measure on $\zeta(x)$ induced by $\tilde m^{pr}_{\tilde\zeta(\tilde x)}$.

We can now show $h_{m^{pr}}(\{\phi_t\}) = 1$. Indeed, if we consider the special case that $m = m^{pr}$, then the partition ζ becomes an m^{pr}-measurable partition given by Proposition 9.2. Hence by Lemma 9.3, the measure $m^{pr}_{\zeta(x)}$ forms the family of conditional measure for m^{pr}. Therefore the above identity (9.6) yields

$$h_{m^{pr}}(\phi_\tau) = h_{m^{pr}}(\phi_\tau, \zeta) = \int -\log m^{pr}_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x)) \; dm(x) = \tau.$$

Hence

$$h_{m^{pr}}(\{\phi_t\}) = h_{m^{pr}}(\phi_\tau)/\tau = 1.$$

It remains to show that for a general m, $h_m(\{\phi_t\}) \le 1$ and that $h_m(\{\phi_t\}) = 1$ implies $m = m^{pr}$. We define the following function: for m-a.e. $x \in \Omega_\psi$,

$$F(x) := \frac{m^{pr}_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x))}{m_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x))} \quad \text{if} \quad m_{\zeta(x)}((\phi_\tau^{-1}\zeta)(x)) > 0,$$

and $F(x) := 0$ otherwise. By [24, Fait 9], both functions F and $\log F$ are m-integrable and $\int F \, dm \le 1$. Since

$$\int \log F \, dm = -\tau + h_m(\phi_\tau, \zeta) = -\tau + h_m(\phi_\tau) = -\tau + \tau h_m(\{\phi_t\})$$

by (9.6) and the choice of ζ, we apply Jensen's inequality and obtain

$$-\tau + \tau h_m(\{\phi_t\}) \le \log\left(\int F \, dm\right) \le 0.$$

This proves

$$h_m(\{\phi_t\}) \le 1.$$

Now suppose that $h_m(\{\phi_t\}) = 1$. This implies that the equality holds in Jensen's inequality, that is, $\log\left(\int F \, dm\right) = 0$, which means that $F = 1$ m-a.e. It follows that the two conditional measures $m^{pr}_{\zeta(x)}$ and $m_{\zeta(x)}$ coincide on the σ-algebra generated by $(\phi_\tau^{-1}\zeta)(x)$ for m-a.e. x. Since this holds after replacing ϕ_τ with ϕ_τ^k for any $k \in \mathbb{N}$ and the partition ζ is ϕ_τ-generating, we have

$$m^{pr}_{\zeta(x)} = m_{\zeta(x)} \quad \text{for } m\text{-a.e. } x \in \Omega_\psi.$$

Then the equality between measures $m = m^{pr}$ follows from the Hopf argument. Indeed, let $f : \Omega_\psi \to \mathbb{R}$ be a compactly supported continuous function. By the Birkhoff ergodic theorem, the set

$$\mathcal{Z} := \left\{ x \in \Omega_\psi : \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\phi_s x) ds = m^{pr}(f) \right\}$$

has a full m^{pr}-measure. Then $\mathcal{Z}$ is invariant under the flow $\{\phi_t\}$ and moreover, since f is uniformly continuous, $x \in \mathcal{Z}$ implies $W^-(x) \subset \mathcal{Z}$ by Proposition 4.6. By the quasi-product structure of the BMS measure m^{pr}, this implies that for all $x \in \Omega_\psi$, $\mathcal{Z} \cap W^+(x)$ has full $\mu_{W^+(x)}$-measure. Hence $\mathcal{Z} \cap \zeta(x)$ has full $m^{pr}_{\zeta(x)}$-measure for m-a.e. $x \in \Omega_\psi$ by the definition of $m^{pr}_{\zeta(x)}$. Hence $\mathcal{Z} \cap \zeta(x)$ has full $m_{\zeta(x)}$-measure for m-a.e. $x \in \Omega_\psi$. Since $m_{\zeta(x)}$ is a conditional measure for m, this implies $m(\mathcal{Z}) = 1$, and therefore $m(f) = m^{pr}(f)$ by applying the Birkhoff ergodic theorem again to m. This finishes the proof. $\square$

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