THE ASYMPTOTIC DISTRIBUTION OF CIRCLES IN THE ORBITS OF KLEINIAN GROUPS

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ABSTRACT. Let $\mathcal{P}$ be a locally finite circle packing in the plane $\mathbb{C}$ invariant under a non-elementary Kleinian group Γ and with finitely many Γ-orbits. When Γ is geometrically finite, we construct an explicit Borel measure on $\mathbb{C}$ which describes the asymptotic distribution of small circles in $\mathcal{P}$, assuming that either the critical exponent of Γ is strictly bigger than 1 or $\mathcal{P}$ does not contain an infinite bouquet of tangent circles glued at a parabolic fixed point of Γ. Our construction also works for $\mathcal{P}$ invariant under a geometrically *infinite* group Γ, provided Γ admits a finite Bowen-Margulis-Sullivan measure and the Γ-skinning size of $\mathcal{P}$ is finite. Some concrete circle packings to which our result applies include Apollonian circle packings, Sierpinski curves, Schottky dances, etc.

1. Introduction

A circle packing in the plane $\mathbb{C}$ is simply a union of circles (here a line is regarded as a circle of infinite radius). As we allow circles to intersect with each other, our definition of a circle packing is more general than the conventional definition of a circle packing.

For a given circle packing $\mathcal{P}$ in the plane, we are interested in counting and distribution of small circles in $\mathcal{P}$. A natural size of a circle is measured by its radius. We will use the curvature of a circle, that is, the reciprocal of its radius, instead.

We suppose that $\mathcal{P}$ is locally finite in the sense that for any $T > 1$, there are only finitely many circles in $\mathcal{P}$ of curvature at most T in any fixed bounded subset of $\mathbb{C}$. Geometrically, $\mathcal{P}$ is locally finite if there is no infinite sequence of circles in $\mathcal{P}$ converging to a fixed circle. For instance, if the circles of $\mathcal{P}$ have disjoint interiors as in Fig. 1, $\mathcal{P}$ is locally finite.

For a bounded subset E of $\mathbb{C}$ and $T > 1$, we set

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \ \mathrm{Curv}(C) < T\}$$

where $\mathrm{Curv}(C)$ denotes the curvature of a circle C. The local finiteness assumption on $\mathcal{P}$ implies that $N_T(\mathcal{P}, E) < \infty$. Our question is then if there exists a Borel measure $\omega_{\mathcal{P}}$ on $\mathbb{C}$ such that for all nice Borel subsets

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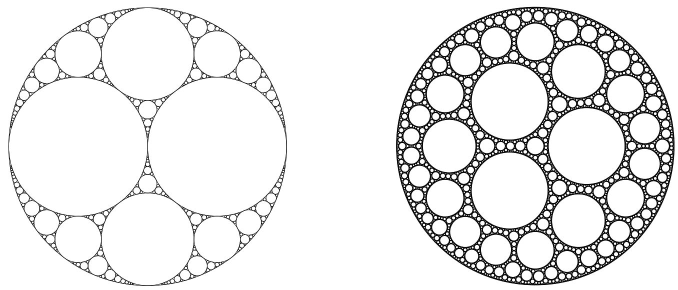

FIGURE 1. Apollonian circle packing and Sierpinski curve (by C. McMullen)

$E_1, E_2 \subset \mathbb{C}$,

$$\frac{N_T(\mathcal{P}, E_1)}{N_T(\mathcal{P}, E_2)} \sim_{T\to\infty} \frac{\omega_{\mathcal{P}}(E_1)}{\omega_{\mathcal{P}}(E_2)},$$

assuming $N_T(\mathcal{P}, E_2) > 0$ and $\omega_{\mathcal{P}}(E_2) > 0$.

Our main theorem applies to a very general packing $\mathcal{P}$, provided $\mathcal{P}$ is invariant under a non-elementary (i.e., non virtually-abelian) Kleinian group satisfying certain finiteness conditions.

Recall that a Kleinian group is a discrete subgroup of $G := \mathrm{PSL}_2(\mathbb{C})$ and G acts on the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$ and $z \in \hat{\mathbb{C}}$. A Möbius transformation maps a circle to a circle and by the Poincare extension, G can be identified with the group of all orientation preserving isometries of $\mathbb{H}^3$. Considering the upper-half space model $\mathbb{H}^3 = \{(z, r) : z \in \mathbb{C}, r > 0\}$, the geometric boundary $\partial_\infty(\mathbb{H}^3)$ is naturally identified with $\hat{\mathbb{C}}$.

For a Kleinian group Γ, we denote by $\Lambda(\Gamma) \subset \hat{\mathbb{C}}$ its limit set, that is, the set of accumulation points of an orbit of Γ in $\hat{\mathbb{C}}$, and by $0 \le \delta_\Gamma \le 2$ its critical exponent. For Γ non-elementary, it is known that $\delta_\Gamma > 0$. Let $\{\nu_x : x \in \mathbb{H}^3\}$ be a Γ-invariant conformal density of dimension δ_Γ on $\Lambda(\Gamma)$, which exists by the work of Patterson [23] and Sullivan [30].

In order to present our main theorem on the asymptotic of $N_T(\mathcal{P}, E)$ we introduce two invariants associated to Γ and $\mathcal{P}$. The first one is a Borel measure on $\mathbb{C}$ depending only on Γ.

Definition 1.1. Define a Borel measure ω_Γ on $\mathbb{C}$: for $\psi \in C_c(\mathbb{C})$

$$\omega_\Gamma(\psi) = \int_{z\in\mathbb{C}} \psi(z) e^{\delta_\Gamma \beta_z(x,(z,1))}\, d\nu_x(z)$$

where $x \in \mathbb{H}^3$ and $\beta_z(x_1, x_2)$ is the signed distance between the horospheres based at $z \in \mathbb{C}$ and passing through $x_1, x_2 \in \mathbb{H}^3$.

By the conformal property of $\{\nu_x\}$, ω_Γ is well-defined independent of the choice of $x \in \mathbb{H}^n$.

We have a simple formula: for $j = (0,1) \in \mathbb{H}^3$,

$$d\omega_\Gamma = (|z|^2 + 1)^{\delta_\Gamma} d\nu_j.$$

For a vector u in the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^3)$, denote by $u^+ \in \hat{\mathbb{C}}$ (resp. $u^- \in \hat{\mathbb{C}}$) the forward (resp. backward) end point of the geodesic determined by u. On the contracting horosphere $H_\infty^-(j) \subset \mathrm{T}^1(\mathbb{H}^3)$ consisting of upward unit normal vectors on the horizontal plane $\{(z,1) : z \in \mathbb{C}\}$, the normal vector based at $(z,1)$ is mapped to z via the map $u \mapsto u^-$. Under this correspondence, the measure ω_Γ on $\mathbb{C}$ is equal to the density of the Burger-Roblin measure $\tilde{m}^{\mathrm{BR}}$ (see Def. 2.3) on $H_\infty^-(j)$.

The second invariant is a number in $[0,\infty]$ measuring a certain size of $\mathcal{P}$.

Definition 1.2 (The Γ-skinning size of $\mathcal{P}$)**.** For a circle packing $\mathcal{P}$ invariant under Γ, define $0 \le \mathrm{sk}_\Gamma(\mathcal{P}) \le \infty$ as follows:

$$\mathrm{sk}_\Gamma(\mathcal{P}) := \sum_{i\in I} \int_{s\in \mathrm{Stab}_\Gamma(C_i^\dagger)\backslash C_i^\dagger} e^{\delta_\Gamma \beta_{s^+}(x,\pi(s))} d\nu_x(s^+)$$

where $x \in \mathbb{H}^3$, $\pi : \mathrm{T}^1(\mathbb{H}^3) \to \mathbb{H}^3$ is the canonical projection, $\{C_i : i \in I\}$ is a set of representatives of Γ-orbits in $\mathcal{P}$, $C_i^\dagger \subset \mathrm{T}^1(\mathbb{H}^3)$ is the set of unit normal vectors to the convex hull $\hat{C}_i$ of C_i and $\mathrm{Stab}_\Gamma(C_i^\dagger)$ denotes the set-wise stabilizer of $C_i^\dagger$ in Γ. Again by the conformal property of $\{\nu_x\}$, the definition of $\mathrm{sk}_\Gamma(\mathcal{P})$ is independent of the choice of x and the choice of representatives $\{C_i\}$.

We remark that the value of $\mathrm{sk}_\Gamma(\mathcal{P})$ can be zero or infinite in general and we do not assume any condition on $\mathrm{Stab}_\Gamma(C_i^\dagger)$'s (they may be trivial).

We denote by m_Γ^{BMS} the Bowen-Margulis-Sullivan measure on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$ associated to the density $\{\nu_x\}$ (Def. 2.2). When Γ is geometrically finite, i.e., Γ admits a finite sided fundamental domain in $\mathbb{H}^3$, Sullivan showed that $|m_\Gamma^{\mathrm{BMS}}| < \infty$ [31] and that δ_Γ is equal to the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ [30]. A point in $\Lambda(\Gamma)$ is called a parabolic fixed point of Γ if it is fixed by a parabolic element of Γ.

Definition 1.3. By an *infinite bouquet of tangent circles glued at a point* $\xi \in \mathbb{C}$, we mean a union of two collections, each consisting of infinitely many pairwise internally tangent circles with the common tangent point ξ and their radii tending to 0, such that the circles in each collection are externally tangent to the circles in the other at ξ (see Fig. 2).

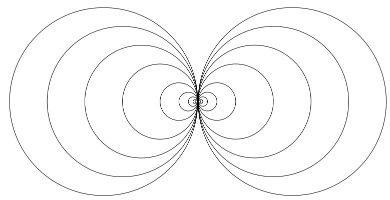

FIGURE 2. Infinite bouquet of tangent circles

Theorem 1.4. *Let $\mathcal{P}$ be a locally finite circle packing in $\mathbb{C}$ invariant under a non-elementary geometrically finite group Γ and with finitely many Γ-orbits. If $\delta_\Gamma \le 1$, we further assume that $\mathcal{P}$ does not contain an infinite bouquet of tangent circles glued at a parabolic fixed point of Γ. Then $\mathrm{sk}_\Gamma(\mathcal{P}) < \infty$ and for any bounded Borel subset E of $\mathbb{C}$ with $\omega_\Gamma(\partial(E)) = 0$,*

$$\lim_{T\to\infty} \frac{N_T(\mathcal{P}, E)}{T^{\delta_\Gamma}} = \frac{\mathrm{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \omega_\Gamma(E).$$

If $\mathcal{P}$ has infinitely many circles, then $\mathrm{sk}_\Gamma(\mathcal{P}) > 0$.

Remark 1.5.
(1) Given a finite collection $\{C_1, \cdots, C_m\}$ of circles in the plane $\mathbb{C}$ and a non-elementary geometrically finite group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$, Theorem 1.4 applies to $\mathcal{P} := \cup_{i=1}^m \Gamma(C_i)$, provided $\mathcal{P}$ contains neither infinitely many circles converging to a fixed circle nor any infinite bouquet of tangent circles.
(2) In the case when $\delta_\Gamma \le 1$ and $\mathcal{P}$ contains an infinite bouquet of tangent circles glued at a parabolic fixed point of Γ, we have $\mathrm{sk}_\Gamma(\mathcal{P}) = \infty$ [19]. In that case if the interior of E intersects $\Lambda(\Gamma)$ non-trivially, the growth order of $N_T(\mathcal{P}, E)$ is $T \log T$ if $\delta_\Gamma = 1$, and it is T if $\delta_\Gamma < 1$ [21].
(3) We note that the asymptotic of $N_T(\mathcal{P}, E)$ depends only on Γ, except for the Γ-skinning size of $\mathcal{P}$. This is rather surprising in view of the fact that there are circle packings with completely different configurations but invariant under the same group Γ.
(4) Theorem 1.4 implies that the asymptotic distribution of small circles in $\mathcal{P}$ is completely determined by the measure ω_Γ: for any bounded Borel sets E_1, E_2 with $\omega_\Gamma(E_2) > 0$ and $\omega_\Gamma(\partial(E_i)) = 0$, $i = 1, 2$, as $T \to \infty$,

$$\frac{N_T(\mathcal{P}, E_1)}{N_T(\mathcal{P}, E_2)} \sim \frac{\omega_\Gamma(E_1)}{\omega_\Gamma(E_2)}.$$

(5) Suppose that all circles in $\mathcal{P}$ can be oriented so that they have disjoint interiors whose union is equal to the domain of discontinuity $\Omega(\Gamma) := \hat{\mathbb{C}} - \Lambda(\Gamma)$. If either $\mathcal{P}$ is bounded or ∞ is a parabolic fixed point for Γ, then δ_Γ is equal to the circle packing exponent $e_{\mathcal{P}}$ defined

as:

$$e_{\mathcal{P}} = \inf\{s : \sum_{C\in\mathcal{P}} \mathrm{Curv}^{-s} < \infty\} = \sup\{s : \sum_{C\in\mathcal{P}} \mathrm{Curv}(C)^{-s} = \infty\}.$$

This was proved by Parker [22] extending the earlier works of Boyd [3] and Sullivan [31] on bounded Apollonian circle packings.

In the proof of Theorem 1.4, the geometric finiteness assumption on Γ is used only to ensure the finiteness of the Bowen-Margulis-Sullivan measure m_Γ^{BMS}. We have the following more general theorem:

Theorem 1.6. *Let $\mathcal{P}$ be a locally finite circle packing invariant under a non-elementary Kleinian group Γ and with finitely many Γ-orbits. Suppose that*

$$|m_\Gamma^{\mathrm{BMS}}| < \infty \quad \text{and} \quad \mathrm{sk}_\Gamma(\mathcal{P}) < \infty.$$

Then for any bounded Borel subset E of $\mathbb{C}$ with $\omega_\Gamma(\partial(E)) = 0$,

$$\lim_{T\to\infty} \frac{N_T(\mathcal{P}, E)}{T^{\delta_\Gamma}} = \frac{\mathrm{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \omega_\Gamma(E).$$

If $\mathcal{P}$ is infinite, then $\mathrm{sk}_\Gamma(\mathcal{P}) > 0$.

Since there is a large class of geometrically infinite groups with $|m_\Gamma^{\mathrm{BMS}}| < \infty$ [24], Theorem 1.6 is not subsumed by Theorem 1.4.

We remark that the condition on the finiteness of m_Γ^{BMS} implies that the density $\{\nu_x\}$ is determined uniquely up to homothety (see [26, Coro. 1.8]).

Remark 1.7.
1. The assumption of $|m_\Gamma^{\mathrm{BMS}}| < \infty$ implies that ν_x (and hence ω_Γ) is atom-free [26, Sec. 1.5], and hence the above theorem works for any bounded Borel subset E intersecting $\Lambda(\Gamma)$ only at finitely many points.
2. It is not hard to show that Γ is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$ considered as a real algebraic group if and only if $\Lambda(\Gamma)$ is not contained in a circle in $\hat{\mathbb{C}}$. In such a case, any proper real subvariety of $\hat{\mathbb{C}}$ has zero ν_x-measure. This is shown in [7, Cor.1.4] for Γ geometrically finite but its proof works equally well if ν_x is Γ-ergodic, which is the case when $|m_\Gamma^{\mathrm{BMS}}| < \infty$. Hence Theorem 1.6 holds for any Borel subset E whose boundary is contained in a countable union of real algebraic curves.

We now describe some concrete applications of Theorem 1.4.

1.1. **Apollonian gasket.** Three mutually tangent circles in the plane determine a curvilinear triangle, say, $\mathcal{T}$. By a theorem of Apollonius of Perga (c. 200 BC), one can inscribe a unique circle into the triangle $\mathcal{T}$, tangent to all of the three circles. This produces three more curvilinear triangles inside $\mathcal{T}$ and we inscribe a unique circle into each triangle. By continuing to add circles in this way, we obtain an infinite circle packing of $\mathcal{T}$, called the Apollonian gasket for $\mathcal{T}$, say, $\mathcal{A}$ (see Fig. 3).

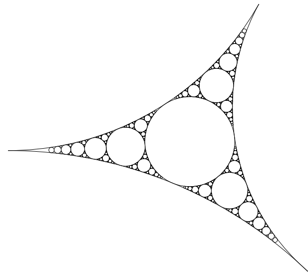

FIGURE 3. Apollonian gasket

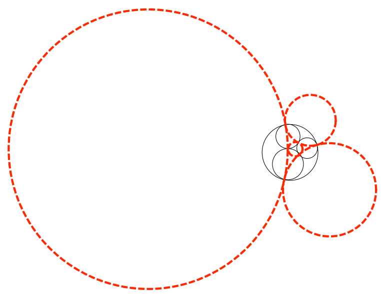

FIGURE 4. Dual circles

By adding *all* the circles tangent to three of the given ones, not only those within $\mathcal{T}$, one obtains an Apollonian circle packing $\mathcal{P} := \mathcal{P}(\mathcal{T})$, which may be bounded or unbounded (cf. [10] [9], [27], [28], [12]).

Fixing four mutually tangent circles in $\mathcal{P}$, consider the four dual circles determined by the six intersection points (see Fig. 4 where the dotted circles are dual circles to the solid ones), and denote by $\Gamma_{\mathcal{P}}$ the intersection of $\mathrm{PSL}_2(\mathbb{C})$ and the group generated by the inversions with respect to those dual circles. Then $\Gamma_{\mathcal{P}}$ is a geometrically finite Zariski dense subgroup of the real algebraic group $\mathrm{PSL}_2(\mathbb{C})$ preserving $\mathcal{P}$, and its limit set in $\hat{\mathbb{C}}$ coincides with the residual set of $\mathcal{P}$ (cf. [12]).

We denote by α the Hausdorff dimension of the residual set of $\mathcal{P}$, which is known to be 1.3056(8) according to McMullen [16].

Corollary 1.8. *Let $\mathcal{T}$ be a curvilinear triangle determined by three mutually tangent circles and $\mathcal{A}$ the Apollonian gasket for $\mathcal{T}$. Then for any Borel subset $E \subset \mathcal{T}$ whose boundary is contained in a countable union of real algebraic curves,*

$$\lim_{T\to\infty} \frac{N_T(E)}{T^\alpha} = \frac{\mathrm{sk}_{\Gamma_{\mathcal{P}}}(\mathcal{P})}{\alpha \cdot |m^{\mathrm{BMS}}_{\Gamma_{\mathcal{P}}}|} \cdot \omega_{\Gamma_{\mathcal{P}}}(E)$$

where $N_T(E) := \#\{C \in \mathcal{A} : C \cap E \neq \emptyset, \;\; \mathrm{Curv}(C) < T\}$ and $\mathcal{P} = \mathcal{P}(\mathcal{T})$.

Either when $\mathcal{P}$ is bounded and E is the disk enclosed by the largest circle of $\mathcal{P}$, or when $\mathcal{P}$ lies between two parallel lines and E is the whole period,

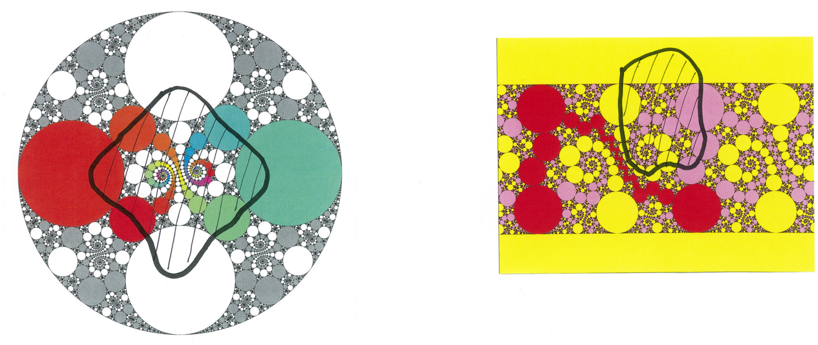

FIGURE 5. Regions whose boundary intersects $\Lambda(\Gamma)$ at finitely many points (background pictures are reproduced with permission from Indra's Pearls, by D.Mumford, C. Series and D. Wright, copyright Cambridge University Press 2002)

it was proved in [12] that $N_T(\mathcal{P}, E) \sim c \cdot T^\alpha$ for some $c > 0$. This implies that [1] $N_T(\mathcal{T}) \asymp T^\alpha$. The approach in [12] was based on the Descartes circle theorem in parameterizing quadruples of circles of curvature at most T as vectors of maximum norm at most T in the cone defined by the Descartes quadratic equation. We remark that the fact that α is strictly bigger than 1 was crucial in the proof of [12] as based on the L^2-spectral theory of $\Gamma_{\mathcal{P}}\backslash\mathbb{H}^3$.

1.2. **Counting circles in the limit set** $\Lambda(\Gamma)$**.** If X is a finite volume hyperbolic 3-manifold with totally geodesic boundary, then its fundamental group $\Gamma := \pi_1(X)$ is geometrically finite and X is homeomorphic to $\Gamma\backslash\mathbb{H}^3 \cup \Omega(\Gamma)$ where $\Omega(\Gamma) := \hat{\mathbb{C}} - \Lambda(\Gamma)$ is the domain of discontinuity [11]. The set $\Omega(\Gamma)$ is a union of countably many disjoint open disks in this case and has finitely many Γ-orbits by the Ahlfors finiteness theorem [1]. Hence Theorem 1.4 applies to counting these open disks in $\Omega(\Gamma)$ with respect to the curvature.

[1] $\asymp$ means that the ratio of the two sides is between two uniform constants

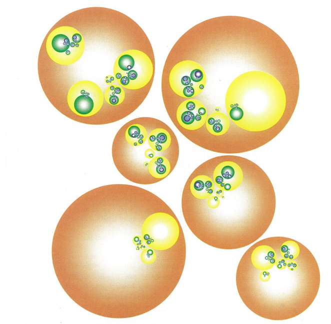

FIGURE 6. Schottky dance (reproduced with permission from Indra's Pearls, by D.Mumford, C. Series and D. Wright, copyright Cambridge University Press 2002)

For example, for the group Γ generated by reflections in the sides of a unique regular tetrahedron whose convex core is bounded by four $\frac{\pi}{4}$ triangles and four right hexagons, $\Omega(\Gamma)$ is illustrated in the second picture in Fig. 1 (see [15, P.9] for details). This circle packing is called a *Sierpinski curve*, being homeomorphic to the well-known Sierpinski carpet [5].

Two pictures in Fig. 5 can be found in the beautiful book *Indra's pearls* by Mumford, Series and Wright (see P. 269 and P. 297 of [17]) where one can find many more circle packings to which our theorem applies. The book presents explicit geometrically finite Schottky groups Γ whose limit sets are illustrated in Fig. 5. The boundaries of the shaded regions meet $\Lambda(\Gamma)$ only at finitely many points. Hence our theorem applies to counting circles in these shaded regions.

1.3. **Schottky dance.** Another class of examples is obtained by considering the images of Schottky disks under Schottky groups. Take $k \geq 1$ pairs of mutually disjoint closed disks $\{(D_i, D_i') : 1 \leq i \leq k\}$ in $\mathbb{C}$ and for each $1 \leq i \leq k$, choose a Möbius transformation γ_i which maps the interior of D_i to the exterior of D_i' and the interior of D_i' to the exterior of D_i. The group, say, Γ, generated by $\{\gamma_i : 1 \leq i \leq k\}$ is called a Schottky group of genus k (cf. [13, Sec. 2.7]). The Γ-orbit of the disks D_i and D_i''s nests down onto the limit set $\Lambda(\Gamma)$ which is totally disconnected. If we denote by $\mathcal{P}$ the union $\cup_{1\le i\le k}\Gamma(C_i)\cup\Gamma(C_i')$ where C_i and C_i' are the boundaries of D_i and D_i' respectively, then $\mathcal{P}$ is locally finite, as the nesting disks become smaller and smaller. The common exterior of hemispheres above the initial disks D_i and D_i' is a fundamental domain for Γ in the upper half-space $\mathbb{H}^3$, and hence Γ is geometrically finite. Since $\mathcal{P}$ contains no infinite bouquet of tangent circles, Theorem 1.4 applies to $\mathcal{P}$; for instance, we can count circles in the picture in Fig. 6 ([17, Fig. 4.11]).

On the structure of the proof. In [12], the counting problem for a bounded Apollonian circle packing was related to the equidistribution of expanding closed horospheres on the hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$. For a general circle packing, there is no analogue of the Descartes circle theorem which made such a relation possible. The main idea in our paper is to relate the counting problem for a general circle packing $\mathcal{P}$ invariant under Γ with the equidistribution of orthogonal translates of a closed totally geodesic surface in $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$. Let C_0 denote the unit circle centered at the origin and H the stabilizer of C_0 in $\mathrm{PSL}_2(\mathbb{C})$. Thus $H\backslash G$ may be considered as the space of totally geodesic planes of $\mathbb{H}^3$. The important starting point is to describe certain subset $B_T(E)$ in $H\backslash G$ so that the number of circles in the packing $\mathcal{P}:=\Gamma(C_0)$ of curvature at most T intersecting E can be interpreted as the number of points in $B_T(E)$ of a discrete Γ-orbit on $H\backslash G$. We then describe the weighted limiting distribution of orthogonal translates of an H-period $(H\cap\Gamma)\backslash H$ (which corresponds to a properly immersed hyperbolic surface which may be of infinite area) along these sets $B_T(E)$ in terms of the Burger-Roblin measure (Theorem 4.3) using the main result in [19] (see Thm. 2.5). To translate the weighted limiting distribution result into the asymptotic for $N_T(\mathcal{P},E)$, we relate the density of the Burger-Roblin measure of the contracting horosphere $H_\infty^-(j)$ with the measure ω_Γ.

A version of Theorem 1.4 in a weaker form, and some of its applications stated above were announced in [18]. We remark that the methods of this paper can be easily generalized to prove a similar result for a sphere packing in the n-dimensional Euclidean space invariant under a non-elementary discrete subgroup of $\mathrm{Isom}(\mathbb{H}^{n+1})$.

Acknowledgment. We would like to thank Curt McMullen for inspiring discussions. The applicability of our other paper [19] in the question addressed in this paper came up in the conversation of the first named author with him during her one month visit to Harvard in October, 2009. She thanks the Harvard mathematics department for the hospitality. We would also like to thank Yves Benoist for helpful discussions.

2. Expansion of a hyperbolic surface by orthogonal geodesic flow

We use the following coordinates for the upper half space model for $\mathbb{H}^3$:

$$\mathbb{H}^3=\{z+rj=(z,r):z\in\mathbb{C},r>0\}$$

where $j = (0, 1)$. The isometric action of $G = \mathrm{PSL}_2(\mathbb{C})$, via the Poincare extension of the linear fractional transformations, is explicitly given as the following (cf. [6]):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z + rj) = \frac{(az+b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{|cz+d|^2 + |c|^2 r^2} + \frac{r}{|cz+d|^2 + |c|^2r^2} j. \tag{2.1}$$

In particular, the stabilizer of j is the following maximal compact subgroup of G:

$$K := \mathrm{PSU}(2) = \{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \}.$$

We set

$$A := \{ a_t := \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \}, \quad M := \{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \}$$

and

$$N := \{ n_z := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \}, \quad N^- = \{ n_z^- := \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} : z \in \mathbb{C} \}.$$

We can identify $\mathbb{H}^3$ with G/K via the map $g(j) \mapsto gK$. Denoting by $X_0 \in \mathrm{T}^1(\mathbb{H}^3)$ the upward unit normal vector based at j, we can also identify the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^3)$ with $G.X_0 = G/M$: here $g.X_0$ is given by $d\lambda(g)(X_0)$ where $\lambda(g) : G/K \to G/K$ is the left translation $\lambda(g)(g'K) = gg'K$ and $d\lambda(g)$ is its derivative at j.

The geodesic flow $\{g^t\}$ on $\mathrm{T}^1(\mathbb{H}^3)$ corresponds to the right translation by a_t on G/M:

$$g^t(gM) = g a_t M.$$

For a circle C in $\mathbb{C}$, denote by $\hat{C}$ its convex hull, which is the northern hemisphere above C.

Set C_0 to be the unit circle in $\mathbb{C}$ centered at the origin. The set-wise stabilizer of $\hat{C}_0$ in G is given by

$$H = \mathrm{PSU}(1,1) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{PSU}(1,1)$$

where

$$\mathrm{PSU}(1,1) = \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \}.$$

Note that H is equal to the stabilizer of C_0 in G and hence $\hat{C}_0$ can be identified with $H/H \cap K$.

We have the following generalized Cartan decomposition (cf. [29]): for $A^+ = \{a_t : t \geq 0\}$,

$$G = HA^+K$$

in the sense that every element of $g \in G$ can be written as $g = hak$, $h \in H, a \in A^+, k \in K$ and $h_1a_1k_1 = h_2a_2k_2$ implies that $a_1 = a_2$, $h_1 = h_2 m$ and $k_1 = m^{-1}k_2$ for some $m \in H \cap K \cap Z_G(A) = M$.

As X_0 is orthogonal to the tangent space $\mathrm{T}_j(\hat{C}_0)$, $H.X_0 = H/M$ corresponds to the set of unit normal vectors to $\hat{C}_0$, which we will denote by $C_0^\dagger$. Note that $C_0^\dagger$ has two connected components, depending on their directions. For $t \in \mathbb{R}$, the set $g^t(C_0^\dagger) = (H/M)a_t = (Ha_tM)/M$ corresponds to a union of two surfaces consisting of the orthogonal translates of $\hat{C}_0$ by distance $|t|$ in each direction, both having the same boundary C_0.

Let $\Gamma < G$ be a non-elementary discrete subgroup. As in the introduction, let $\{\nu_x : x \in \mathbb{H}^3\}$ be a Γ-invariant conformal density on $\hat{\mathbb{C}}$ of dimension δ_Γ, that is, each ν_x is a finite measure on $\hat{\mathbb{C}}$ satisfying that for any $x, y \in \mathbb{H}^3$, $z \in \hat{\mathbb{C}}$ and $\gamma \in \Gamma$,

$$\gamma_* \nu_x = \nu_{\gamma x}; \quad \text{and} \quad \frac{d\nu_y}{d\nu_x}(z) = e^{-\delta_\Gamma \beta_z(y,x)}.$$

Here $\gamma_*\nu_x(R) = \nu_x(\gamma^{-1}(R))$ for a Borel subset $R \subset \hat{\mathbb{C}}$ and the Busemann function $\beta_z(y_1, y_2)$ is given by $\lim_{t\to\infty} d(y_1, \xi_t) - d(y_2, \xi_t)$ for a geodesic ray ξ_t toward z.

For $u \in \mathrm{T}^1(\mathbb{H}^3)$, we define $u^+ \in \hat{\mathbb{C}}$ (resp. $u^- \in \hat{\mathbb{C}}$) to be the forward (resp. backward) end point of the geodesic determined by u and $\pi(u) \in \mathbb{H}^3$ to be the basepoint. Fixing $o \in \mathbb{H}^3$, the map $u \mapsto (u^+, u^-, t := \beta_{u^-}(\pi(u), o))$ is a homeomorphism between $\mathrm{T}^1(\mathbb{H}^3)$ and $(\hat{\mathbb{C}} \times \hat{\mathbb{C}} - \{(\xi, \xi) : \xi \in \hat{\mathbb{C}}\}) \times \mathbb{R}$.

Definition 2.2. The Bowen-Margulis-Sullivan measure m_Γ^{BMS} associated to $\{\nu_x\}$ ([2], [14], [31]) is the measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^3)$ induced by the following Γ-invariant measure on $\mathrm{T}^1(\mathbb{H}^3)$: for $x \in \mathbb{H}^3$,

$$d\tilde{m}^{\mathrm{BMS}}(u) = e^{\delta_\Gamma \beta_{u^+}(x,\pi(u))}\ e^{\delta_\Gamma \beta_{u^-}(x,\pi(u))}\ d\nu_x(u^+) d\nu_x(u^-) dt.$$

By the conformal properties of $\{\nu_x\}$, this definition is independent of the choice of $x \in \mathbb{H}^3$.

We also denote by $\{m_x : x \in \mathbb{H}^3\}$ a G-invariant conformal density of dimension 2, which is unique up to homothety: each m_x a finite measure on $\hat{\mathbb{C}}$ which is invariant under $\mathrm{Stab}_G(x)$ and $dm_x(z) = e^{-2\beta_z(y,x)} dm_y(z)$ for any $x, y \in \mathbb{H}^3$ and $z \in \hat{\mathbb{C}}$.

Definition 2.3. The Burger-Roblin measure m_Γ^{BR} associated to $\{\nu_x\}$ and $\{m_x\}$ ([4], [26]) is the measure on $\mathrm{T}^1(\Gamma\backslash\mathbb{H}^n)$ induced by the following Γ-invariant measure on $\mathrm{T}^1(\mathbb{H}^n)$:

$$d\tilde{m}^{\mathrm{BR}}(u) = e^{2\beta_{u^+}(x,\pi(u))}\ e^{\delta_\Gamma \beta_{u^-}(x,\pi(u))}\ dm_x(u^+) d\nu_x(u^-) dt$$

for $x \in \mathbb{H}^3$. By the conformal properties of $\{\nu_x\}$ and $\{m_x\}$, this definition is independent of the choice of $x \in \mathbb{H}^3$.

For any circle C, let

$$H_C = \{g \in G : gC = C\} = \{g \in G : gC^\dagger = C^\dagger\}.$$

We consider the following two measures on $C^\dagger$: Fix any $x \in \mathbb{H}^3$, and let

$$d\mu^{\mathrm{Leb}}_{C^\dagger}(s) := e^{2\beta_{s^+}(x,\pi(s))} dm_x(s) \text{ and } d\mu^{\mathrm{PS}}_{C^\dagger}(s) := e^{\delta_\Gamma \beta_{s^+}(x,\pi(s))} d\nu_x(s^+). \tag{2.4}$$

These definitions are independent of the choice of x and $\mu^{\mathrm{Leb}}_{C^\dagger}$ (resp. $\mu^{\mathrm{PS}}_{C^\dagger}$) is left-invariant by H_C (resp. $H_C \cap \Gamma$)). Hence we may consider the measures $\mu^{\mathrm{Leb}}_{C^\dagger}$ and $\mu^{\mathrm{PS}}_{C^\dagger}$ on the quotient $(H \cap \Gamma)\backslash C^\dagger$.

We denote by $\mathrm{sk}_\Gamma(C)$ the total mass of $\mu^{\mathrm{PS}}_{C^\dagger}$; that is,

$$\mathrm{sk}_\Gamma(C) := \int_{s \in \Gamma \cap H \backslash C_0^\dagger} e^{\delta_\Gamma \beta_{s^+}(x,\pi(s))} d\nu_x(s^+).$$

In general, $\mathrm{sk}_\Gamma(C)$ may be zero or infinite.

Theorem 2.5 ([19, Theorem 1.9]). *Suppose that the natural projection map $\Gamma \cap H_C \backslash \hat{C} \to \Gamma \backslash \mathbb{H}^3$ is proper. Assume that $|m^{\mathrm{BMS}}_\Gamma| < \infty$ and $\mathrm{sk}_\Gamma(C) < \infty$. Then for any $\psi \in C_c(\Gamma \backslash G/M)$, as $t \to \infty$,*

$$e^{(2-\delta_\Gamma)t} \int_{s \in (\Gamma \cap H_C)\backslash C^\dagger} \psi(s a_t) d\mu^{\mathrm{Leb}}_{C^\dagger}(s) \sim \frac{\mathrm{sk}_\Gamma(C)}{|m^{\mathrm{BMS}}_\Gamma|} m^{\mathrm{BR}}_\Gamma(\psi).$$

Moreover $\mathrm{sk}_\Gamma(C) > 0$ *if* $[\Gamma : H_C \cap \Gamma] = \infty$.

Note that if $|m^{\mathrm{BMS}}_\Gamma| < \infty$, then Γ is of divergence type; that is, the Poincare series of Γ diverges at δ_Γ. When Γ is of divergence type, the Γ-invariant conformal density $\{\nu_x\}$ of dimension δ_Γ is unique up to homothety (see [26, Remark following Corollary 1.8]): explicitly ν_x can be taken as the weak-limit as $s \to \delta_\Gamma^+$ of the family of measures

$$\nu_x(s) := \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(j,\gamma j)}} \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma j)} \delta_{\gamma j}.$$

Recall that $g \in \mathrm{PSL}_2(\mathbb{C})$ is parabolic if and only if g has a unique fixed point in $\hat{\mathbb{C}}$.

Theorem 2.6 ([19, Theorem 5.2]). *Let Γ be geometrically finite. Suppose that the natural projection map $\Gamma \cap H_C \backslash \hat{C} \to \Gamma \backslash \mathbb{H}^3$ is proper. Then $\mathrm{sk}_\Gamma(C) < \infty$ if and only if either $\delta_\Gamma > 1$ or any parabolic fixed point of Γ lying on C is fixed by a parabolic element of $H_C \cap \Gamma$.*

Proof. Note that in the notation of [19, Theorem 5.2], if we put $E = \hat{C}$, which is a complete totally geodesic submanifold of $\mathbb{H}^3$ of codimension 1, then $\partial(\pi(\tilde{E})) = C$, $\tilde{E} = C^\dagger$, $\Gamma_{\tilde{E}} = H_C \cap \Gamma$, and $|\mu^{\mathrm{PS}}_E| = \mathrm{sk}_\Gamma(C)$. Hence the conclusion is immediate. □

3. Reformulation into the orbital counting problem on the space of hyperbolic planes

Let $G = \mathrm{PSL}_2(\mathbb{C})$ and $\Gamma < G$ be a non-elementary discrete subgroup. Let C be a circle in $\hat{\mathbb{C}}$ and H_C denote the set-wise stabilizer of C in G.

It is clear:

Lemma 3.1. *If $\Gamma(C)$ is infinite, then $[\Gamma : H_C \cap \Gamma] = \infty$.*

Lemma 3.2. *The following are equivalent:*

(1) *A circle packing $\Gamma(C)$ is locally finite;*
(2) *the natural projection map $f : \Gamma \cap H_C \backslash \hat{C} \to \Gamma \backslash \mathbb{H}^3$ is proper;*
(3) *$H_C \backslash H_C \Gamma$ is discrete in $H_C \backslash G$.*

Proof. We observe that the properness of f is equivalent to the condition that only finitely many distinct hemispheres in $\Gamma(\hat{C})$ intersects a given compact subset of $\mathbb{H}^3$. Note that any compact subset of $\mathbb{H}^3$ is contained in a compact subset the form $E \times [r_1, r_2] = \{(z, r) : z \in E, r_1 \leq r \leq r_2\}$ for $E \subset \mathbb{C}$ compact and $0 < r_1 < r_2 < \infty$, and that the radius of a circle in $\mathbb{C}$ is same as the height of its convex hull in $\mathbb{H}^3$. Hence the properness of the map f is again equivalent to the condition that for any $r > 0$ and a compact subset $E \subset \mathbb{C}$, there are only finitely many distinct circles in $\Gamma(C)$ intersecting E and of radii at least r, that is, $\Gamma(C)$ being locally finite, proving the equivalence of (1) and (2).

It is straightforward to verify that the properness of f and that of the projection map $\Gamma \cap H_C \backslash C^\dagger \to \Gamma \backslash \mathrm{T}^1(\mathbb{H}^3)$ are equivalent. Let $X_C \in C^\dagger$ such that $X_C^+ = \infty \in \hat{\mathbb{C}}$. Let $M_C = \{g \in G : gX_C = X_C\}$. Since $\hat{C}$ is the unique totally geodesic submanifold of $\mathbb{H}^3$ orthogonal to X_C, M_C is contained in H_C. We identify G/M_C with $\mathrm{T}^1(\mathbb{H}^3)$ via $gM_C \mapsto gX_C$. Since H/M_C identifies with $C^\dagger$, the canonical map $\Gamma \cap H_C \backslash H_C / M_C \to \Gamma \backslash G / M_C$ is proper. Since M_C is compact, it follows that $\Gamma \cap H_C \backslash H_C \to \Gamma \backslash G$ is proper. Equivalently ΓH_C is closed in G (see [19] for the equivalence). As Γ is countable, this is again equivalent to the discreteness of $H_C \backslash H_C \Gamma$ in $H_C \backslash G$. This proves the equivalence of (2) and (3). $\square$

Remark 3.3. If $\Gamma \cap H_C$ is a lattice in H_C, then ΓH_C is closed in G ([25, §1]), and hence $\Gamma(C)$ is a locally finite circle packing. In this case, by [19, Theorem 1.11], we have $\mathrm{sk}_\Gamma(C) < \infty$.

Proposition 3.4. *Let $\xi \in C$ be a parabolic fixed point of Γ. Suppose that $\Gamma(C)$ does not contain an infinite bouquet of tangent circles glued at ξ. Then ξ is a parabolic fixed point for $H_C \cap \Gamma$.*

Proof. Suppose that there exists a parabolic element $\gamma \in \Gamma - H_C$ fixing $\xi \in \mathbb{C}$. By sending ξ to $\infty \in \hat{\mathbb{C}}$ by an element of G, we may assume that $\xi = \infty$ and γ acts as a translation on $\mathbb{C}$. Since $\gamma C \neq C$ and C is a circle passing through ∞, we have that $\{\gamma^k C : k \in \mathbb{Z}\}$ is an infinite collection of parallel lines. By sending ∞ back to the original ξ, we see that $\{\gamma^k C : k \in \mathbb{Z}\}$ is an infinite bouquet of tangent circles glued at ξ. $\square$

3.1. **Deduction of Theorem 1.4 from Theorem 1.6.** We only need to ensure that $\mathrm{sk}_\Gamma(\mathcal{P}) < \infty$, or equivalently, $\mathrm{sk}_\Gamma(C) < \infty$ for each $C \in \mathcal{P}$. By the assumption in Theorem 1.4, if $\xi \in C$ is any parabolic fixed point of Γ, then by Proposition 3.4, ξ is a parabolic fixed point for $H_C \cap \Gamma$. Therefore by Theorem 2.6, $\mathrm{sk}_\Gamma(C) < \infty$. $\square$

3.2. **Relating counting on a single Γ-orbit to a set $B_T(E) \subset H\backslash G$.** In the rest of this section, let C_0 denote the unit circle in $\mathbb{C}$ centered at the origin and let $H := \mathrm{Stab}(\hat{C}_0)$. We follow notations from Section 2. We assume that $\Gamma(C_0)$ is a locally finite circle packing of $\mathbb{C}$.

Let E be a bounded subset in $\mathbb{C}$ and set

$$N_T(\Gamma(C_0), E) := \#\{C \in \Gamma(C_0) : C \cap E \neq \emptyset, \quad \mathrm{Curv}(C) < T\}.$$

For $s > 0$, we set

$$A_s^+ := \{a_t : 0 \le t \le s\}; \quad A_s^- := \{a_{-t} : 0 \le t \le s\}.$$

For a subset $E \subset \mathbb{C}$, we set $N_E := \{n_z : z \in E\}$.

Definition 3.5 (Definition of $B_T(E)$)**.** For $E \subset \mathbb{C}$ and $T > 1$, we define the subset $B_T(E)$ of $H\backslash G$ to be the image of the set

$$KA^+_{\log T} N_{-E} = \{k a_t n_{-z} \in G : k \in K, 0 \le t < \log T, z \in E\}$$

under the canonical projection $G \to H\backslash G$.

For a bounded circle C in $\mathbb{C}$, C° denotes the open disk enclosed by C. We will not need this definition for a line since there can be only finitely many lines intersecting a fixed bounded subset in a locally finite circle packing.

Definition 3.6. For a given circle packing $\mathcal{P}$, a bounded subset $E \subset \mathbb{C}$ is said to be $\mathcal{P}$-*admissible* if, for any bounded circle $C \in \mathcal{P}$, $C^\circ \cap E \neq \emptyset$ implies $C^\circ \subset E$, possibly except for finitely many circles.

The following translation of $N_T(\Gamma(C_0), E)$ as the number of points in $[e]\Gamma \cap B_T(E)$, where $[e] = H \in H\backslash G$, is crucial in our approach:

Proposition 3.7. *If E is $\Gamma(C_0)$-admissible, there exists $m_0 \in \mathbb{N}$ such that for all $T \gg 1$,*

$$\#[e]\Gamma \cap B_T(E) - m_0 \le N_T(\Gamma(C_0), E) \le \#[e]\Gamma \cap B_T(E) + m_0.$$

Proof. Observe that

$$\begin{aligned} \#[e]\Gamma \cap B_T(E) &= \#\{[\gamma] \in \Gamma \cap H\backslash\Gamma : H\gamma \cap KA^+_{\log T} N_{-E} \neq \emptyset\} \\ &= \#\{[\gamma] \in \Gamma/\Gamma \cap H : \gamma HK \cap N_E A^-_{\log T} K \neq \emptyset\} \\ &= \#\{\gamma(\hat{C}_0) : \gamma HK \cap N_E A^-_{\log T} K \neq \emptyset\} \end{aligned}$$

where the second equality is obtained by taking the inverse. Since

$$N_E A^-_{\log T} j = \{(z, r) \in \mathbb{H}^3 : T^{-1} < r \le 1, z \in E\}$$

and K is the stabilizer of j in G, it follows that

$$\#[e]\Gamma \cap B_T(E) = \#\{\gamma(\hat{C}_0) : \gamma(\hat{C}_0) \text{ contains } (z, r) \text{ with } z \in E,\ \ T^{-1} < r \le 1\}.$$

By the admissibility assumption on E, we observe that $\gamma(\hat{C}_0)$ contains (z, r) with $z \in E$ and $T^{-1} < r \le 1$ if and only if the center of $\gamma(C_0)$ lies in E and the radius of $\gamma(C_0)$ is greater than T^{-1}, possibly except for finitely many number (say, m_0) of circles. $\square$

4. Uniform distribution along the family $B_T(E)$ and the Burger-Roblin measure

We keep the notations $C_0, H, K, M, A^+, X_0, G, \{m_x : x \in \mathbb{H}^3\}$, etc., from section 2. Denoting by dm the probability invariant measure on M,

$$dh = d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s)dm \tag{4.1}$$

is a Haar measure on $H \cong C_0^\dagger \times M$ as $\mu^{\mathrm{Leb}}_{C_0^\dagger}$ is H-invariant, and the following defines a Haar measure on G: for $g = ha_r k \in HA^+K$,

$$dg = 4\sinh r \cdot \cosh r \; dhdrdm_j(k) \tag{4.2}$$

where $dm_j(k) := dm_j(k.X_0^+)$.

We denote by $d\lambda$ the unique measure on $H\backslash G$ which is compatible with the choice of dg and dh: for $\psi \in C_c(G)$,

$$\int_G \psi \; dg = \int_{[g]\in H\backslash G}\int_{h\in H} \psi(h[g]) \; dhd\lambda[g].$$

For a bounded set $E \subset \mathbb{C}$, recall that the set $B_T(E)$ in $H\backslash G$ is the image of the set

$$KA^+_{\log T}N_{-E} = \{ka_t n_{-z} \in G : k \in K, 0 \le t < \log T, z \in E\}$$

under the canonical projection $G \to H\backslash G$.

The goal of this section is to deduce the following theorem 4.3 from Theorem 2.5:

Theorem 4.3. *Let Γ be a non-elementary discrete subgroup of G. Suppose that $|m^{\mathrm{BMS}}_\Gamma| < \infty$ and $\mathrm{sk}_\Gamma(C_0) < \infty$. Suppose that the natural projection map $\Gamma \cap H\backslash \hat{C}_0 \to \Gamma\backslash\mathbb{H}^3$ is proper. Then for any bounded Borel subset $E \subset \mathbb{C}$ and for any $\psi \in C_c(\Gamma\backslash G)$, we have*

$$\lim_{T\to\infty} \frac{1}{T^{\delta_\Gamma}} \int_{g\in B_T(E)}\int_{h\in\Gamma\cap H\backslash H} \psi(hg)dhd\lambda(g) = \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|}\cdot \int_{n\in N_{-E}} m^{\mathrm{BR}}_\Gamma(\psi_n) \; dn$$

where $\psi_n \in C_c(\Gamma\backslash G)^M$ is given by $\psi_n(g) = \int_{m\in M}\psi(gmn)dm$ and dn is the Lebesgue measure on N.

In order to prove this result using Theorem 2.5, it is crucial to understand the shape of the set $B_T(E)$ in the HA^+K decomposition of G. This is one of the important technical steps in the proof.

On the shape of $B_T(E)$: Fix a left-invariant metric on G. For $\epsilon > 0$, let U_ϵ be the ϵ-ball around e in G. For a subset W of G, we set $W_\epsilon = W \cap U_\epsilon$.

Proposition 4.4. (1) *If $a_t \in HKa_sK$ for $s > 0$, then $|t| \le s$.*

(2) *Given any $\epsilon > 0$, there exists $T_0 = T_0(\epsilon)$ such that*

$$\{k \in K : a_t k \in HKA^+ \text{ for some } t > T_0\} \subset K_\epsilon M.$$

Proof. Suppose $a_t = hk_1a_sk_2$ for $h \in H, k_1, k_2 \in K$. We note that, as Aj is orthogonal to $\hat{C}_0$ and $j \in \hat{C}_0$,

$$\begin{aligned}|t| &= d(\hat{C}_0, a_tj) = d(\hat{C}_0, hk_1a_sj)\\ &= d(\hat{C}_0, k_1a_sj) \le d(j, k_1a_sj) = d(j, a_sj) = s,\end{aligned}$$

proving the first claim. For the second claim, suppose $a_tk \in HKa_s$ for some $s \ge 0$. Then $ka_{-s} \in a_{-t}HK$. Applying both sides to $j \in \mathbb{H}^3$, $k(e^{-s}j) \in a_{-t}\hat{C}_0$. Now $a_{-t}\hat{C}_0 = e^{-t}\hat{C}_0$ is the northern hemisphere of Euclidean radius e^{-t} about 0 in $\mathbb{H}^3$.

On the other hand $A^-j = (0,1]j$ for $A^- = \{a_{-s} : s \ge 0\}$ and $K_\epsilon\{(0,1]j\}$ consists of geodesic rays in $\mathbb{H}^3$ joining j and points of $K_\epsilon(0) \subset \mathbb{C}$. Now $K_\epsilon(0)$ contains a disk of radius, say $r_\epsilon > 0$, centered at 0 in $\mathbb{C}$, and hence $K_\epsilon\{(0,1]j\}$ contains a Euclidean half ball of radius $r_\epsilon > 0$ centered at 0 in $\mathbb{H}^3$.

Therefore for $t > T_0(\epsilon) := -\log(r_\epsilon)$, $k(e^{-s}j) \in a_{-t}\hat{C}_0$ implies that $k(e^{-s}j) \in K_\epsilon\{(0,1]j\}$, in other words, $ka_{-s}K \subset K_\epsilon A^-K$. By the uniqueness of the left K-component, modulo the right multiplication by M, in the decomposition $G = KA^-K$, it follows that $k \in K_\epsilon M$, proving the second claim. ◻

For $t \in \mathbb{R}$ and $T > 1$, set

$$K_T(t) := \{k \in K : a_tk \in HKA^+_{\log T}\}.$$

As a consequence of Proposition 4.4, we have the following.

Corollary 4.5. (1) *For all $0 \le t < \log T$, $e \in K_T(t)$.*
(2) *For all $t > \log T$, $K_T(t) = \emptyset$.*
(3) *For any $\epsilon > 0$, there exists $T_0(\epsilon) \ge 1$ such that we have*

$$K_T(t) \subset K_\epsilon M \quad \textit{for all } t > T_0(\epsilon).$$

Thus for any $T > 1$,

$$HKA^+_{\log T} = \cup_{0 \le t < \log T} Ha_tK_T(t). \tag{4.6}$$

Since $B_T(E) = H\backslash HKA^+_{\log T}N_{-E}$, (4.6) together with Corollary 4.5 shows that $B_T(E)$ is essentially of the form $H\backslash Ha_{\log T}K_\epsilon MN_{-E}$. The following proposition shows that $B_T(E)$ can be basically controlled by the set $H\backslash Ha_{\log T}N_{-E}$.

Proposition 4.7. *Fix a bounded subset E of $\mathbb{C}$. There exists $\ell = \ell(E) \ge 1$ such that for all sufficiently small $\epsilon > 0$,*

$$a_tkmn_z \in H_{\ell\epsilon}ma_tn_zU_{\ell\epsilon}$$

holds for any $m \in M$, $t > 0$, $z \in E$, and $k \in K_\epsilon$.

Proof. Recalling that N^- denotes the lower triangular subgroup of G, we note that the product map $N^- \times A \times M \times N \to G$ is a diffeomorphism at a neighborhood of e, in particular, bi-Lipschitz. Hence there exists $\ell_1 > 1$ such that for all small $\epsilon > 0$,

$$K_\epsilon \subset N^-_{\ell_1\epsilon} A_{\ell_1\epsilon} M_{\ell_1\epsilon} N_{\ell_1\epsilon}. \tag{4.8}$$

Similarly due to the $H \times A \times N$ product decomposition of G_ϵ, there exists $\ell_2 > 1$ such that

$$U_\epsilon \subset H_{\ell_2\epsilon} A_{\ell_2\epsilon} N_{\ell_2\epsilon} \tag{4.9}$$

for all small $\epsilon > 0$ ([8, Lem 2.4]). We also have $\ell_3 > 1$ such that for all small $\epsilon > 0$,

$$A_{(\ell_1+\ell_2)\epsilon} N_{(\ell_1+\ell_2)\epsilon} M_{\ell_1\epsilon} \subset U_{\ell_3\epsilon}. \tag{4.10}$$

Now let $t > 0, k \in K_\epsilon, m \in M, n \in N$. Then by (4.8), we may write

$$k = n_1^- b_1 m_1 n_1 \in N^-_{\ell_1\epsilon} A_{\ell_1\epsilon} M_{\ell_1\epsilon} N_{\ell_1\epsilon}.$$

Since $a_t n_1^- a_{-t} \in N_\epsilon^-$ for $t > 0$, we have, by (4.9),

$$a_t n_1^- a_{-t} = h_2 b_2 m_2 n_2 \in H_{\ell_2\epsilon} A_{\ell_2\epsilon} M_{\ell_2\epsilon} N_{\ell_2\epsilon}.$$

Therefore

$$\begin{aligned} a_t kmn &= (a_t n_1^- a_{-t})(a_t b_1 m_1 n_1) mn \\ &= (h_2 b_2 m_2 n_2) a_t b_1 m_1 n_1 mn \\ &= h_2 b_2 m_2 (a_t b_1 b_1^{-1} a_{-t}) n_2 a_t b_1 m_1 n_1 mn \\ &= h_2 a_t (b_2 m_2) b_1 (b_1^{-1} a_{-t} n_2 a_t b_1) m_1 n_1 mn \\ &\in h_2 a_t A_{(\ell_1+\ell_2)\epsilon} M_{\ell_2\epsilon} N_{(\ell_1+\ell_2)\epsilon} M_{\ell_1\epsilon} mn \quad \text{by (4.10)} \\ &\subset h_2 a_t U_{\ell_3\epsilon} mn. \end{aligned}$$

As E is bounded, there exists $\ell = \ell(E) > \ell_2$ such that for all small $\epsilon > 0$ and for all $z \in E$,

$$U_{\ell_3\epsilon} m n_z \subset m n_z U_{\ell\epsilon}.$$

Since a_t commutes with m, we obatin for all $z \in E$ that

$$a_t kmn_z \subset H_{\ell\epsilon} m a_t n_z U_{\ell\epsilon}.$$

□

Proof of Theorem 4.3. Let $\ell = \ell(E) \geq 1$ be as in Proposition 4.7. For $\psi \in C_c(\Gamma\backslash G)$ and $\epsilon > 0$, we define $\psi_\epsilon^\pm \in C_c(\Gamma\backslash G)$,

$$\psi_\epsilon^+(g) := \sup_{u \in U_{\ell\epsilon}} \psi(gu) \quad \text{and} \quad \psi_\epsilon^-(g) := \inf_{u \in U_{\ell\epsilon}} \psi(gu).$$

For a given $\eta > 0$, there exists $\epsilon = \epsilon(\eta) > 0$ such that for all $g \in \Gamma\backslash G$,

$$|\psi_\epsilon^+(g) - \psi_\epsilon^-(g)| \leq \eta$$

by the uniform continuity of ψ.

On the other hand, by Theorem 2.5, we have $T_1(\eta) \gg 1$ such that for all $t > T_1(\eta)$,

$$\begin{aligned}(4.11)\qquad &\int_{h\in\Gamma\cap H\backslash H} \psi_\epsilon^+(ha_t n)dh \\ &= \int_{s\in\Gamma\cap H\backslash C_0^\dagger}\int_{m\in M}\psi_\epsilon^+(sa_t mn)dmd\mu^{\mathrm{Leb}}_{C_0^\dagger}(s) \\ &= (1+O(\eta))\frac{\mathrm{sk}_\Gamma(C_0)}{|m^{\mathrm{BMS}}|}m_\Gamma^{\mathrm{BR}}(\psi_{\epsilon,n}^+)e^{(\delta_\Gamma-2)t}\end{aligned}$$

where $\psi^+_{\epsilon,n}(g) = \int_{m\in M}\psi_\epsilon^+(gmn)dm$.

As N_{-E} is relatively compact, the implied constant can be taken uniformly over all $n\in N_{-E}$. Let $T_0(\epsilon) > T_1(\eta)$ be as in Proposition 4.4. For $[e] = H \in H\backslash G$ and $s > 0$, set

$$V_T(s) := \cup_{s\le t<\log T}[e]a_t K_T(t)N_{-E}$$

so that

$$B_T(E) = V_T(s) \cup (B_T(E) - V_T(s)).$$

Setting

$$\psi^H(g) := \int_{h\in\Gamma\cap H\backslash H}\psi(hg)dh,$$

note that ψ^H is left H-invariant as dh is a Haar measure. We will show that

$$\limsup_{T\to\infty}\frac{1}{T^\delta}\int_{[g]\in V_T(T_0(\epsilon))}\psi^H(g)d\lambda(g) = (1+O(\eta))\frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot|m^{\mathrm{BMS}}|}\cdot\int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\psi_n)dn.$$

By Corollary 4.5, we have

$$V_T(T_0(\epsilon)) \subset \cup_{T_0(\epsilon)\le t\le\log T}[e]a_t K_\epsilon M N_{-E}.$$

Let $[g] \in V_T(T_0(\epsilon))$, so $[g] = [e]a_t kmn$ with $T_0(\epsilon) \le t \le \log T$, $k \in K_\epsilon$, $m \in M$ and $n \in N_{-E}$. By Proposition 4.7, there exist $h_0 \in H$ and $u \in U_{\ell\epsilon}$ such that

$$a_t kmn = h_0 m a_t n u$$

so that $[g] = [e]a_t nu$, since $M \subset H$.

We have

$$\psi^H(g) = \int_{h\in\Gamma\cap H\backslash H}\psi(ha_t nu)dh \le \int_{h\in\Gamma\cap H\backslash H}\psi_\epsilon^+(ha_t n)dh.$$

The measure $e^{2t}dtdn$ is a right invariant measure of AN and $[e]AN$ is an open subset in $H\backslash G$. Hence $d\lambda(a_t n)$ (restricted to $[e]AN$) and $e^{2t}dtdn$ are constant multiples of each other. It follows from the formula of dg that $d\lambda(a_t n) = e^{2t}dtdn$. Therefore

$$\int_{[g]\in V_T(T_0(\epsilon))}\psi^H(g)d\lambda(g) \le \int_{n\in N_{-E}}\int_{T_0(\epsilon)<t\le\log T}\int_{h\in\Gamma\cap H\backslash H}\psi_\epsilon^+(ha_t n)dhe^{2t}dtdn.$$

By the choice of $\epsilon = \epsilon(\eta)$, we also have

$$m_\Gamma^{\mathrm{BR}}(\psi_{\epsilon,n}^+) = (1 + O(\eta)) m_\Gamma^{\mathrm{BR}}(\psi_n)$$

where the implied constant depends only on ψ. Hence by (4.11),

$$\int_{n\in N_{-E}} \int_{T_0(\epsilon)<t<\log T} \int_{h\in\Gamma\cap H\backslash H} \psi_{\epsilon,n}^+(ha_t) dh e^{2t} dt dn$$
$$= (1 + O(\eta)) \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \cdot \int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\psi_n) dn \cdot (T^{\delta_\Gamma} - e^{\delta_\Gamma T_0(\epsilon)}).$$

Hence

$$\limsup_T \frac{1}{T^{\delta_\Gamma}} \int_{[g]\in V_T(T_0(\epsilon))} \psi^H(g) d\lambda(g) = (1+O(\eta)) \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \cdot \int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\psi_n) dn.$$

On the other hand, since $\Gamma\backslash\Gamma H$ is a closed subset of $\Gamma\backslash G$, so is $\cup_{0\le t\le s}\Gamma\backslash\Gamma H a_t K N_{-\overline{E}}$ for any fixed $s > 0$; in particular, its intersection with a compact subset of $\Gamma\backslash G$ is compact.

Since

$$\cup_{[g]\in B_T(E)-V_T(s)} \Gamma\backslash\Gamma H g \subset \cup_{0\le t\le s}\Gamma\backslash\Gamma H a_t K N_{-\overline{E}},$$

and ψ has compact support, we have, as $T \to \infty$,

$$\int_{[g]\in B_T(E)-V_T(T_0(\epsilon))} \int_{h\in\Gamma\cap H\backslash H} \psi(hg) dh d\lambda(g) = O(1).$$

Therefore

$$\limsup_T \frac{1}{T^{\delta}} \int_{[g]\in B_T(E)} \psi^H(g) d\lambda(g) \le (1+O(\eta)) \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \cdot \int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\psi_n) dn.$$

As $\eta > 0$ is arbitrary and $\epsilon(\eta) \to 0$ as $\eta \to 0$, we have

$$\limsup_T \frac{1}{T^{\delta}} \int_{[g]\in B_T(E)} \psi^H(g) d\lambda(g) \le \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \cdot \int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\psi_n) dn.$$

Similarly we can show that

$$\liminf_T \frac{1}{T^{\delta}} \int_{[g]\in B_T(E)} \psi^H(g) d\lambda(g) \ge \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}|} \cdot \int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\psi_n) dn.$$

□

5. On the measure ω_Γ

In this section we will describe a measure ω_Γ on $\mathbb{C}$ and show that the term

$$\int_{n\in N_{-E}} m_\Gamma^{\mathrm{BR}}(\Psi_n)\, dn,$$

which appears in the asymptotic expression in Theorem 4.3, converges to $\omega_\Gamma(E)$ as the support of Ψ shrinks to $[e]$ with $\int_{\Gamma\backslash G} \Psi\, dg = 1$.

We keep the notations $G, K, M, A^+, N, N^-, a_t, n_z, n_z^-$, etc., from section 2. Throughout this section, we assume that Γ is a non-elementary discrete subgroup of G. Recall that $\{\nu_x = \nu_{\Gamma,x} : x \in \mathbb{H}^3\}$ denotes a Γ-invariant conformal density for Γ of dimension $\delta_\Gamma > 0$.

Definition 5.1. *Define a Borel measure ω_Γ on $\mathbb{C}$ as follows: for $\psi \in C_c(\mathbb{C})$,*

$$\omega_\Gamma(\psi) = \int_{z\in\mathbb{C}} e^{\delta_\Gamma \beta_z(x,z+j)} \psi(z) d\nu_{\Gamma,x}(z)$$

for $x \in \mathbb{H}^3$ and $z + j := (z,1) \in \mathbb{H}^3$.

In order to see that the definition of ω_Γ is independent of the choice of $x \in \mathbb{H}^3$, we observe that for any $x_1, x_2 \in \mathbb{H}^3$ and $z \in \mathbb{C}$,

$$e^{\delta_\Gamma(\beta_z(x_1,z+j)-\beta_z(x_2,z+j))} \frac{d\nu_{x_1}}{d\nu_{x_2}}(z) = e^{\delta_\Gamma\cdot\beta_z(x_1,x_2)} \frac{d\nu_{x_1}}{d\nu_{x_2}}(z) = 1$$

by the conformality of $\{\nu_x\}$.

Lemma 5.2. *For any $x = p + rj \in \mathbb{H}^3$ and $\psi \in C_c(\mathbb{C})$,*

$$\omega_\Gamma(\psi) := \int_{z\in\mathbb{C}} (r^{-1}|z-p|^2 + r)^{\delta_\Gamma} \psi(z) d\nu_x(z).$$

Proof. It suffices to show that

$$\beta_z(p + rj, z + j) = \log \frac{|z-p|^2 + r^2}{r}.$$

We use the fact that the hyperbolic distance d on the upper half space model of $\mathbb{H}^3$ satisfies

$$\cosh(d(z_1 + r_1 j, z_2 + r_2 j)) = \frac{|z_1 - z_2|^2 + r_1^2 + r_2^2}{2r_1 r_2}$$

for $z_i + r_i j \in \mathbb{H}^3$ (cf. [6]).

Note that

$$\begin{aligned}\beta_z(z, z + j) &= \beta_0(j, -z + p + rj) \\ &= \lim_{t\to\infty} t - d(-z + p + rj, e^{-t}j) \\ &= \lim_{t\to\infty} t - d(p + rj, z + e^{-t}j).\end{aligned}$$

Now

$$\cosh d(p + rj, z + e^{-t}j) = \frac{e^t(|z-p|^2 + r^2) + e^{-t}}{2r}$$

and hence

$$e^{d(p+rj,z+e^{-t}j)} + e^{-d(p+rj,z+e^{-t}j)} = \frac{e^t(|z-p|^2 + r^2) + e^{-t}}{r}.$$

Therefore as $t \to \infty$,

$$d(p + rj, z + e^{-t}j) \sim t + \log \frac{|z-p|^2 + r^2}{r}.$$

Hence

$$\beta_z(p + rj, z + j) = \log \frac{|z-p|^2 + r^2}{r}.$$

□

Definition 5.3. For a function ψ on $\mathbb{C}$ with compact support, define a function $\mathfrak{R}_\psi$ on $MAN^-N \subset G$ by

$$\mathfrak{R}_\psi(ma_t n_x^- n_z) = e^{-\delta_\Gamma t}\psi(-z)$$

for $m \in M, t \in \mathbb{R}, x, z \in \mathbb{C}$. If ψ is the characteristic function of $E \subset \mathbb{C}$, we put $\mathfrak{R}_E = \mathfrak{R}_{\chi_E}$.

Since the product map $M \times A \times N^- \times N \to G$ has a diffeomorphic image, the above function is well-defined.

Proposition 5.4. *For any* $\psi \in C_c(\mathbb{C})$,

$$\omega_\Gamma(\psi) = \int_{k \in K/M} \mathfrak{R}_\psi(k^{-1}) d\nu_j(k(0)).$$

Proof. If $k \in K$ with $k^{-1} = ma_t n_x^- n_z \in MAN^-N$, since MAN fixes 0,

$$k(0) = n_{-z}(0) = -z.$$

We note that $\lim_{s\to\infty} a_{-s}(j) = 0$ and compute

$$\begin{aligned}
0 &= \beta_{-z}(k(j), j) \\
&= \beta_{-z}(n_{-z}n_{-x}^- a_{-t}j, j) \\
&= \beta_0(n_{-x}^- a_{-t}j, n_z(j)) \\
&= \lim_{s\to\infty} d(n_{-x}^- a_{-t}j, a_{-s}j) - d(n_z(j), a_{-s}j) \\
&= \lim_{s\to\infty} d((a_s n_{-x}^- a_{-s})a_{s-t}j, j) - d(n_z(j), a_{-s}j) \\
&= \lim_{s\to\infty} d(a_{s-t}j, j) - d(n_z(j), a_{-s}j) \\
&= \lim_{s\to\infty} s - t - d(n_z(j), a_{-s}j)
\end{aligned}$$

and hence

$$-t = \lim_{s\to\infty} d(n_z(j), a_{-s}j) - s = \beta_0(n_z(j), j) = \beta_{-z}(j, -z + j).$$

Hence for $k^{-1} \in K \cap MAN^-N$,

$$\mathfrak{R}_\psi(k^{-1}) = e^{\delta_\Gamma \beta_{k(0)}(j, n_{k(0)}(j))}\psi(k(0)).$$

Since the complement of NN^-AM/M in K/M is a single point and ν_j is atom-free, we have

$$\begin{aligned}
&\int_{k\in K/M} \mathfrak{R}_E(k^{-1})d\nu_j(k(0))\\
&= \int_{k\in (K\cap NN^-AM)/M} \mathfrak{R}_E(k^{-1})d\nu_j(k(0))\\
&= \int_{z\in\mathbb{C}} e^{\delta_\Gamma\beta_{k(0)}(j,k(0)+j)}\psi(k(0))d\nu_j(k(0))\\
&= \int_{z\in\mathbb{C}} e^{\delta_\Gamma\beta_{-z}(j,-z+j)}\psi(-z)d\nu_j(-z)\\
&= \int_{z\in\mathbb{C}} e^{\delta_\Gamma\beta_{z}(j,z+j)}\psi(z)d\nu_j(z) = \omega_\Gamma(\psi).
\end{aligned}$$

$\square$

Lemma 5.5. *If* $(ma_tn_x^-n_z)(m_1a_{t_1}n_{x_1}^-n_{z_1}) = m_0a_{t_0}n_{x_0}^-n_{z_0}$ *in the* MAN^-N *coordinates, then*

$$t_0 = t + t_1 + 2\log(|1 + e^{-t_1}x_1z'|)$$

for some $z' \in \mathbb{C}$ *with* $|z| = |z'|$.

Proof. Note that if $m_1 = \text{diag}(e^{i\theta_1}, e^{-i\theta_1})$, then

$$a_tn_x^-n_zm_1 = m_1a_tn^-_{e^{i\theta_1}x}n_{e^{i\theta_1}z}.$$

Hence we may assume $m_1 = m = e$ without loss of generality. We use the following simple identity for $z, x \in \mathbb{C}$:

$$n_zn_x^- = \begin{pmatrix} 1+xz & 0\\ 0 & (1+xz)^{-1}\end{pmatrix} n^-_{x(1+xz)}n_{z(1+xz)^{-1}}. \tag{5.6}$$

Hence we have

$$\begin{aligned}
&(a_tn_x^-n_z)(a_{t_1}n_{x_1}^-n_{z_1})\\
&= (a_{t+t_1})(a_{t_1}^{-1}n_x^-a_{t_1})(a_{t_1}^{-1}n_za_{t_1})n_{x_1}^-n_{z_1}\\
&= a_{t+t_1}n^-_{e^{t_1}x}n_{e^{-t_1}z}n_{x_1}^-n_{z_1}\\
&= a_{t+t_1}n^-_{e^{t_1}x}\begin{pmatrix} 1+e^{-t_1}x_1z & 0\\ 0 & (1+e^{-t_1}x_1z)^{-1}\end{pmatrix} n^-_{x_1(1+e^{-t_1}x_1z)}n_{e^{-t_1}z(1+e^{-t_1}zx_1)^{-1}}n_{z_1}\\
&= ma_{t+t_1+2\log(|1+e^{-t_1}x_1z|)}n_{x_2}^-n_{z_2}
\end{aligned}$$

for appropriate $m \in M$ and $x_2, z_2 \in \mathbb{C}$. $\square$

Let $E \subset \mathbb{C}$ be a bounded subset and $U_\epsilon \subset G$ a symmetric ϵ-neighborhood of e in G. For $\epsilon > 0$, set

$$E_\epsilon^+ := U_\epsilon(E) \quad \text{and} \quad E_\epsilon^- := \cap_{u\in U_\epsilon} u(E). \tag{5.7}$$

Lemma 5.8. *There exists $\ell > 0$ such that for all small $\epsilon > 0$ and any $g \in U_{\ell\epsilon}$,*

$$\int_{k\in K/M} \mathfrak{R}_E(k^{-1}g)d\nu_j(k(0)) = (1+O(\epsilon))\cdot\omega_\Gamma(E_\epsilon^{\pm})$$

where the implied constant depends only on E.

Proof. Write $k^{-1} = ma_t n_x^- n_z$ and $g = m_1 a_{t_1} n_{x_1}^- n_{z_1} \in U_\epsilon$. By Lemma 5.5, we have $k^{-1}g = m_0 a_{t_0} n_{x_0} n_{z_0}$ where $t_0 = t + t_1 + 2\log(|1+e^{-t_1}x_1 z|)$. Since $\mathfrak{R}_E(k^{-1}g) = e^{-\delta_\Gamma t_0}\chi_E(g^{-1}k(0))$, we have

$$\begin{aligned}
&\int_{k\in K/M} \mathfrak{R}_E(k^{-1}g)d\nu_j(k(0)) \\
&= \int_{k(0)\in g(E)} e^{-\delta_\Gamma t_0} d\nu_j(k(0)) \\
&= \int_{k(0)\in g(E)} e^{-\delta_\Gamma t} e^{-\delta(t_1 + 2\log(|1+e^{t_1}x_1 z|))} d\nu_j(k(0)) \\
&= (1+O(\epsilon)) \int_{k(0)\in E_\epsilon^{\pm}} e^{-\delta_\Gamma t} e^{-\delta(t_1 + 2\log(|1+e^{t_1}x_1 z|))} d\nu_j(k(0)).
\end{aligned}$$

Since $t_1 = O(\epsilon), x_1 = O(\epsilon)$ and $z = -k(0) \in -g(E) \subset -E_\epsilon^+$,

$$t_1 + 2\log(|1+e^{-t_1}x_1 z|) = O(\epsilon)$$

where the implied constant depends only on E. Hence

$$\begin{aligned}
&\int_{k\in K/M} \mathfrak{R}_E(k^{-1}g)d\nu_j(k(0)) \\
&= (1+O(\epsilon)) \int_{k(0)\in E_\epsilon^{\pm}} e^{-\delta_\Gamma t} d\nu_j(k(0)) \\
&= (1+O(\epsilon)) \int_{k\in K} \mathfrak{R}_{E_\epsilon^{\pm}}(k^{-1}) d\nu_j(k(0)) \\
&= (1+O(\epsilon))\cdot \omega_\Gamma(E_\epsilon^{\pm}).
\end{aligned}$$

□

For $\epsilon > 0$, let ψ^ϵ be a non-negative continuous function in $C(G)$ with support in U_ϵ with integral one and $\Psi^\epsilon \in C_c(\Gamma\backslash G)$ be the Γ-average of ψ^ϵ:

$$\Psi^\epsilon(\Gamma g) := \sum_{\gamma\in\Gamma} \psi^\epsilon(\gamma g).$$

We define $\Psi_E^\epsilon \in C_c(\Gamma\backslash G)^M$ by

$$\Psi_E^\epsilon(g) := \int_{z\in -E}\int_{m\in M} \Psi^\epsilon(gmn_z)dmdz.$$

Lemma 5.9. *For a bounded Borel subset $E \subset \mathbb{C}$, there exists $c = c(E) > 1$ such that for all small $\epsilon > 0$,*

$$(1 - c\cdot\epsilon)\cdot\omega_\Gamma(E_\epsilon^-) \le m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon) \le (1+c\cdot\epsilon)\cdot\omega_\Gamma(E_\epsilon^+).$$

Proof. Note that N^- is the expanding horospherical subgroup for the right action of a_t, i.e., $N^- = \{g \in G : a_t g a_{-t} \to e \text{ as } t \to \infty\}$. We have for $\psi \in C_c(G)^M$,

$$\tilde{m}_\Gamma^{\mathrm{BR}}(\psi) = \int_{KAN^-} \psi(ka_t n^-) e^{-\delta_\Gamma t} dn dt d\nu_j(k(0))$$

(cf. [19, 6.2]). We note that $d(a_t n_x^- m n_z) = dt dx dm dz$ is the restriction of the Haar measure dg to $AN^-N \subset G/M$.

We deduce

$$\begin{aligned}
m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon) &= \int_{z\in -E} \tilde{m}^{\mathrm{BR}}(\psi_{n_z}^\epsilon) dz \\
&= \int_{z\in -E} \int_{KAN^-} \int_{m\in M} \psi^\epsilon(ka_t n_x^- m n_z) e^{-\delta_\Gamma t} dm dx dt d\nu_j(k(0)) dz \\
&= \int_{k\in K} \int_{AN^-MN} \psi^\epsilon(k(a_t n_x^- m n_z)) \chi_{-E}(z) e^{-\delta_\Gamma t} dx dt dm dz d\nu_j(k(0)) \\
&= \int_{k\in K} \int_{g\in G} \psi^\epsilon(kg) \mathfrak{R}_E(g) dg d\nu_j(k(0)) \\
&= \int_{g\in U_\epsilon} \psi^\epsilon(g) \left(\int_{k\in K} \mathfrak{R}_E(k^{-1}g) d\nu_j(k(0)) \right) dg.
\end{aligned}$$

Hence by Lemma 5.8 and the identity $\int_{U_\epsilon} \psi^\epsilon dg = 1$, we have

$$m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon) = (1 + O(\epsilon)) \omega_\Gamma(E_\epsilon^\pm).$$

□

Corollary 5.10. *If* $\omega_\Gamma(\partial(E)) = 0$, *then*

$$\omega_\Gamma(E) = \lim_{\epsilon \to 0} m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon).$$

Proof. For any $\eta > 0$, there exists $\epsilon = \epsilon(\eta)$ such that $\omega_\Gamma(E_\epsilon^+ - E_\epsilon^-) < \eta$.

Together with Lemma 5.9, it implies that

$$m_\Gamma^{\mathrm{BR}}(\Psi_E^\epsilon) = (1 + O(\epsilon))(1 + O(\eta)) \omega_\Gamma(E)$$

and hence the claim follows. □

6. Conclusion: Counting circles

Let $\Gamma < G := \mathrm{PSL}_2(\mathbb{C})$ be a non-elementary discrete group with $|m_\Gamma^{\mathrm{BMS}}| < \infty$. Suppose that $\mathcal{P} := \Gamma(C)$ is a locally finite circle packing.

Recall that

$$\mathrm{sk}_\Gamma(\mathcal{P}) = \mathrm{sk}_\Gamma(C) := \int_{s \in \mathrm{Stab}_\Gamma(C^\dagger)\backslash C^\dagger} e^{\delta_\Gamma \beta_{s^+}(x,s)} d\nu_{\Gamma,x}(s^+),$$

where $C^\dagger$ is the set of unit normal vectors to $\hat{C}$. It follows from the conformal property of $\{\nu_{\Gamma,x}\}$ that $\mathrm{sk}_\Gamma(C)$ is independent of the choice of $C \in \Gamma(C)$, and hence is an invariant of the packing $\Gamma(C)$.

Theorem 1.6 is an immediate consequence of the following statement.

Theorem 6.1. *Suppose that* $\mathrm{sk}_\Gamma(C) < \infty$*. For any bounded Borel subset* E *of* $\mathbb{C}$ *with* $\omega_\Gamma(\partial(E)) = 0$*, we have*

$$\lim_{T\to\infty} \frac{N_T(\mathcal{P}, E)}{T^{\delta_\Gamma}} = \frac{\mathrm{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\mathrm{BMS}}|} \cdot \omega_\Gamma(E). \tag{6.2}$$

Moreover $\mathrm{sk}_\Gamma(C) > 0$ *if* $\mathcal{P}$ *is infinite.*

The second claim on the positivity of $\mathrm{sk}_\Gamma(C)$ follows from the second claim of Theorem 2.5 and Lemma 3.1.

We will first prove Theorem 6.1 for the case when C is the unit circle C_0 centered at the origin and deduce the general case from that.

The case of $C = C_0$. Fix $\eta > 0$. As $\omega_\Gamma(\partial(E)) = 0$, there exists $\epsilon = \epsilon(\eta) > 0$ such that

$$\omega_\Gamma(E^+_{4\epsilon} - E^-_{4\epsilon}) \le \eta \tag{6.3}$$

where $E^\pm_{4\epsilon}$ is defined as in (5.7): $E^+_{4\epsilon} := U_{4\epsilon}(E)$ and $E^-_{4\epsilon} := \cap_{u\in U_{4\epsilon}} u(E)$.

We can find a $\mathcal{P}$-admissible Borel subset $\tilde{E}^+_\epsilon$ such that $E \subset \tilde{E}^+_\epsilon \subset E^+_\epsilon$ by adding all the open disks inside E^+_ϵ intersecting the boundary of E. Similarly we can find a $\mathcal{P}$-admissible Borel subset $\tilde{E}^-_\epsilon$ such that $E^-_\epsilon \subset \tilde{E}^-_\epsilon \subset E$ by adding all the open disks inside E intersecting the boundary of E^-_ϵ. By the local finiteness of $\mathcal{P}$, there are only finitely many circles intersecting E (resp. $\tilde{E}^-_\epsilon$) which are not contained in $\tilde{E}^+_\epsilon$ (resp. E). Therefore there exists $q_\epsilon \ge 1$ (independent of T) such that

$$N_T(\mathcal{P}, \tilde{E}^-_\epsilon) - q_\epsilon \le N_T(\mathcal{P}, E) \le N_T(\mathcal{P}, \tilde{E}^+_\epsilon) + q_\epsilon. \tag{6.4}$$

Recalling the set $B_T(\tilde{E}^\pm_\epsilon) = H\backslash HKA^+_{\log T} N_{-\tilde{E}^\pm_\epsilon} \subset H\backslash G$, it follows from Proposition 3.7 and (6.4) that for all $T \gg 1$,

$$\#[e]\Gamma \cap B_T(\tilde{E}^-_\epsilon) - m_0 \le N_T(\Gamma(C_0), E) \le \#[e]\Gamma \cap B_T(\tilde{E}^+_\epsilon) + m_0 \tag{6.5}$$

for some fixed $m_0 = m_0(\epsilon) \ge 1$.

Lemma 6.6. *There exists* $\ell > 0$ *such that for all* $T > 1$ *and for all small* $\epsilon > 0$*,*

$$KA^+_{\log T} U_\epsilon \subset KA^+_{\log T+\epsilon} N_{\ell\epsilon}$$

where $N_{\ell\epsilon}$ *is the* $\ell\epsilon$*-neighborhood of* e *in* N*.*

Proof. We may write $U_\epsilon = M_\epsilon N^-_\epsilon A_\epsilon N_\epsilon = K_\epsilon A_\epsilon N_\epsilon$ up to uniform Lipschitz constants. For $u = mn^-an \in M_\epsilon N^-_\epsilon A_\epsilon N_\epsilon$, $a_t u = m(a_t n^- a_{-t}) a_t a n$. Since $a_t n^- a_{-t} \in U_\epsilon$ for $t > 0$, we may write it as $k_1 a_1 n_1 \in K_\epsilon A_\epsilon N_\epsilon$. Hence for $0 < t < \log T$, we have $(a^{-1} a_{-t} n_1 a_t a) \in N_\epsilon$ and

$$a_t u = (mk_1)(a_1 a_t a)(a^{-1} a_{-t} n_1 a_t a) n \in KA^+_{\log T+2\epsilon} N_{2\epsilon}.$$

This proves the claim. $\square$

Lemma 6.7 (Stability of KAN-decomposition)**.** *There exists $\ell_0 > 0$ (depending on E) such that for all $T > 1$ and for all small $\epsilon > 0$,*

$$KA^+_{\log T} N_{-\tilde{E}^+_\epsilon} U_{\ell_0 \epsilon} \subset KA^+_{\log T+\epsilon} N_{-E^+_{2\epsilon}};$$

$$KA^+_{\log T-\epsilon} N_{-E^-_{2\epsilon}} \subset KA^+_{\log T}(\cap_{u \in U_{\ell_0\epsilon}} N_{-\tilde{E}^-_\epsilon} u).$$

Proof. There exists $\ell_0 > 0$ depending on E such that $N_{-\tilde{E}^+_\epsilon} U_{\ell_0\epsilon} \subset U_\epsilon N_{-E^+_{2\epsilon}}$. Hence the first claim follows from Lemma 6.6. The second claim can be proved similarly. □

For $\epsilon > 0$, define functions $F^{\epsilon,\pm}_T$ on $\Gamma\backslash G$:

$$F^{\epsilon,+}_T(g) := \sum_{\gamma \in (H\cap\Gamma)\backslash\Gamma} \chi_{B_{e^\epsilon T}(N_{-E^+_{2\epsilon}})}([e]\gamma g); \quad F^{\epsilon,-}_T(g) := \sum_{\gamma \in (H\cap\Gamma)\backslash\Gamma} \chi_{B_{e^{-\epsilon} T}(N_{-E^-_{2\epsilon}})}([e]\gamma g).$$

Let ℓ_0 be as in Lemma 6.7. Without loss of generality, we may assume that $\ell_0 < \ell$ for ℓ as in Lemma 5.8.

Lemma 6.8. *For all $g \in U_{\ell_0\epsilon}$ and $T \gg 1$,*

$$F^{\epsilon,+}_T(g) - m_0 \le N_T(\Gamma(C_0), E) \le F^{\epsilon,+}_T(g) + m_0. \tag{6.9}$$

Proof. Note that, since $U_{\ell_0\epsilon}$ is symmetric, for any $g \in U_{\ell_0\epsilon}$,

$$\#[e]\Gamma \cap B_T(\tilde{E}^+_\epsilon) \le \#[e]\Gamma \cap B_T(\tilde{E}^+_\epsilon) U_{\ell_0\epsilon} g^{-1} \le \#[e]\Gamma g \cap B_{e^\epsilon T}(N_{-E^+_{2\epsilon}}),$$

by Lemma 6.7, which proves the second inequality by (6.5). The other inequality can be proved similarly. □

For $\epsilon > 0$, let ψ^ϵ be a non-negative continuous function in $C(G)$ with support in $U_{\ell_0\epsilon}$ with integral one and $\Psi^\epsilon \in C_c(\Gamma\backslash G)$ be the Γ-average of ψ^ϵ:

$$\Psi^\epsilon(\Gamma g) := \sum_{\gamma\in\Gamma} \psi^\epsilon(\gamma g).$$

By integrating (6.9) against Ψ^ϵ, we have

$$\langle F^{\epsilon,-}_T, \Psi^\epsilon\rangle - m_0 \le N_T(\Gamma(C_0), E) \le \langle F^{\epsilon,+}_T, \Psi^\epsilon\rangle + m_0.$$

Since

$$\begin{aligned}\langle F^{\epsilon,+}_T, \Psi^\epsilon\rangle &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma\cap H\backslash\Gamma} \chi_{B_{e^\epsilon T}(N_{-E^+_{2\epsilon}})}([e]\gamma g)\Psi^\epsilon(g)\, dg \\ &= \int_{g\in\Gamma\cap H\backslash G} \chi_{B_{e^\epsilon T}(N_{-E^+_{2\epsilon}})}([e]g)\Psi^\epsilon(g)\, dg \\ &= \int_{[g]\in B_{e^\epsilon T}(N_{-E^+_{2\epsilon}})} \int_{h\in\Gamma\cap H\backslash H} \Psi^\epsilon(hg)\, dh d\lambda(g)\end{aligned}$$

we deduce from Theorem 4.3 and Lemma 3.2 that

$$\langle F^{\epsilon,+}_T, \Psi^\epsilon\rangle \sim \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}_\Gamma|} \cdot \int_{n\in N_{-E^+_{2\epsilon}}} m^{\mathrm{BR}}_\Gamma(\Psi^\epsilon_n) dn \cdot T^{\delta_\Gamma} \cdot e^{\epsilon\delta_\Gamma} \tag{6.10}$$

where $\Psi^\epsilon_n(g) = \int_{m\in M} \Psi^\epsilon(gmn)dm$.

Therefore by applying Lemma 5.9 to (6.10) and using (6.3), we deduce

$$\begin{aligned}\limsup_T \frac{\langle F_T^{\epsilon,+}, \Psi^\epsilon\rangle}{T^{\delta_\Gamma}} &\le (1+\epsilon)\frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot |m_\Gamma^{\mathrm{BMS}}|}\cdot \int_{n\in N_{-E^+_{2\epsilon}}} m_\Gamma^{\mathrm{BR}}(\Psi^\epsilon_n)dn \\ &\le (1+\epsilon)(1+c\epsilon)\frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot |m_\Gamma^{\mathrm{BMS}}|}\cdot \omega_\Gamma(E^+_{4\epsilon}) \\ &\le (1+c_1\eta)(1+c_2\epsilon)\frac{\mathrm{sk}_\Gamma(\Gamma(C_0))}{\delta_\Gamma\cdot |m_\Gamma^{\mathrm{BMS}}|}\cdot \omega_\Gamma(E)\end{aligned}$$

where the constants c, c_1, c_2 depend only on E.

Similarly, we have

$$\liminf_T \frac{\langle F_T^{\epsilon,+}, \Psi^\epsilon\rangle}{T^{\delta_\Gamma}} \ge (1-c_1\eta)(1-c_2\epsilon)\frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot |m_\Gamma^{\mathrm{BMS}}|}\cdot \omega_\Gamma(E).$$

As $\eta>0$ is arbitrary and $\epsilon=\epsilon(\eta)\to 0$ as $\eta\to 0$, we have

$$\lim_{T\to\infty} \frac{N_T(\Gamma(C_0), E)}{T^{\delta_\Gamma}} = \frac{\mathrm{sk}_\Gamma(C_0)}{\delta_\Gamma\cdot |m_\Gamma^{\mathrm{BMS}}|}\cdot \omega_\Gamma(E).$$

This proves Theorem 6.1 for $C = C_0$.

The general case. Let $r>0$ be the radius of C and $p\in\mathbb{C}$ the center of C. Set

$$g_0 = n_p a_{\log r} = \begin{pmatrix} 1 & p\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{r} & 0\\ 0 & \sqrt{r^{-1}}\end{pmatrix}.$$

Then $g_0^{-1}(z) = r^{-1}(z-p)$ for $z\in\mathbb{C}$ and $g_0^{-1}(C) = C_0$.

Setting $\Gamma_0 = g_0^{-1}\Gamma g_0$, we have

$$\begin{aligned} N_T(\Gamma(C), E) &= \#\{C\in \Gamma(g_0(C_0)) : C^\circ\cap E\neq\emptyset, \mathrm{Curv}(C)<T\}\\ &= \#\{g_0^{-1}(C)\in\Gamma_0(C_0) : C^\circ\cap E\neq\emptyset, \mathrm{Curv}(C)<T\}\\ &= \#\{C_*\in\Gamma_0(C_0) : C_*^\circ\cap g_0^{-1}(E)\neq\emptyset, \mathrm{Curv}(C_*)<r^{-1}T\}\\ &= N_{r^{-1}T}(\Gamma_0(C_0), g_0^{-1}(E)).\end{aligned}$$

We claim that

(6.11)
$$\frac{1}{|m^{\mathrm{BMS}}_{\Gamma_0}|}\cdot \mathrm{sk}_{\Gamma_0}(\Gamma_0(C_0))\cdot r^{-\delta_\Gamma}\cdot \omega_{\Gamma_0}(g_0^{-1}(E)) = \frac{1}{|m^{\mathrm{BMS}}_{\Gamma}|}\cdot \mathrm{sk}_\Gamma(\Gamma(C))\cdot\omega_\Gamma(E).$$

Note that the each side of the above is independent of the choices of conformal densities of Γ_0 and Γ respectively.

Fixing a Γ-invariant conformal density $\{\nu_{\Gamma,x}\}$ of dimension δ_Γ, set

$$\nu_{\Gamma_0,x} := {g_0}^*\nu_{\Gamma,g_0(x)}$$

where $g_0^*\nu_{\Gamma,g_0(x)}(R) = \nu_{\Gamma,g_0(x)}(g_0(R))$. It is easy to check that $\nu_{\Gamma_0,x}$ is supported on $\Lambda(\Gamma_0) = g_0\Lambda(\Gamma)$ and satisfies

$$\frac{d\nu_{\Gamma_0,x}}{d\nu_{\Gamma_0,y}}(z) = e^{-\delta_\Gamma \beta_z(x,y)}; \quad \gamma_*\nu_{\Gamma_0,x} = \nu_{\Gamma_0,\gamma(x)}$$

for all $x, y \in \mathbb{H}^3$, $\gamma \in \Gamma_0$ and $z \in \hat{\mathbb{C}}$.

Hence $\{\nu_{\Gamma_0,x} : x \in \mathbb{H}^3\}$ is a Γ_0-invariant conformal density of dimension $\delta_\Gamma = \delta_{\Gamma_0}$ and satisfies that for $f \in C_c(\mathbb{C})$

$$\int_{g_0(z)\in E} f(z)d\nu_{\Gamma_0,x}(z) = \int_{z\in E} f(g_0^{-1}(z))d\nu_{\Gamma,g_0(x)}(z).$$

We consider the Bowen-Margulis-Sullivan measures m_Γ^{BMS} and $m_{\Gamma_0}^{\mathrm{BMS}}$ on $\Gamma\backslash \mathrm{T}^1(\mathbb{H}^3)$ and $\Gamma_0\backslash \mathrm{T}^1(\mathbb{H}^3)$ associated to $\{\nu_{\Gamma,x}\}$ and $\{\nu_{\Gamma_0,x}\}$, respectively.

Lemma 6.12. *For a bounded Borel function ψ on $\Gamma\backslash \mathrm{T}^1(\mathbb{H}^3)$, consider a function ψ_{g_0} on $\Gamma_0\backslash \mathrm{T}^1(\mathbb{H}^3)$ given by $\psi_{g_0}(u) := \psi(g_0(u))$. Then*

$$m_{\Gamma_0}^{\mathrm{BMS}}(\psi_{g_0}) = m_\Gamma^{\mathrm{BMS}}(\psi).$$

In particular, $|m_{\Gamma_0}^{\mathrm{BMS}}| = |m_\Gamma^{\mathrm{BMS}}|$.

Proof. Note that if $v = g(u)$, then

$$\beta_{u^\pm}(x, \pi(u)) = \beta_{v^\pm}(g(x), \pi(v)).$$

Since $\nu_{\Gamma_0,x} = g_0^*\nu_{\Gamma,g_0(x)}$, we have

$$\begin{aligned}
&m_{\Gamma_0}^{\mathrm{BMS}}(\psi_{g_0})\\
&= \int_{u\in\Gamma_0\backslash \mathrm{T}^1(\mathbb{H}^n)} \psi(g_0(u))e^{\delta_\Gamma\beta_{u^+}(x,\pi(u))}\, e^{\delta_\Gamma\beta_{u^-}(x,\pi(u))}\, d\nu_{\Gamma_0,x}(u^+)d\nu_{\Gamma_0,x}(u^-)dt\\
&= \int_{v\in\Gamma\backslash \mathrm{T}^1(\mathbb{H}^n)} \psi(v)e^{\delta_\Gamma\beta_{v^+}(g_0(x),\pi(v))}\, e^{\delta_\Gamma\beta_{v^-}(g_0(x),\pi(v))}\, d\nu_{\Gamma,g_0(x)}(v^+)d\nu_{\Gamma,g_0(x)}(v^-)dt\\
&= m_\Gamma^{\mathrm{BMS}}(\psi).
\end{aligned}$$

□

Similarly, we can verify:

Lemma 6.13. *For any $x \in \mathbb{H}^3$,*

$$\int_{s\in\mathrm{Stab}_{\Gamma_0}(C_0^\dagger)\backslash C_0^\dagger} e^{\delta_\Gamma\beta_{s^+}(x,s)}d\nu_{\Gamma_0,x}(s^+) = \int_{s\in\mathrm{Stab}_\Gamma(C^\dagger)\backslash C^\dagger} e^{\delta_\Gamma\beta_{s^+}(g_0(x),s)}d\nu_{\Gamma,g_0(x)}(s^+);$$

that is, $\mathrm{sk}_\Gamma(\Gamma(C)) = \mathrm{sk}_{\Gamma_0}(\Gamma_0(C_0))$.

Lemma 6.14. *For any bounded Borel subset $E \subset \mathbb{C}$,*

$$\omega_{\Gamma_0}(g_0^{-1}(E)) = r^{\delta_\Gamma}\omega_\Gamma(E).$$

Proof. Since $g_0^{-1}(z) = r^{-1}(z-p)$, r is the linear distortion of the map g_0^{-1} in the Euclidean metric, that is, $r = \lim_{w\to w_0} \frac{|g_0^{-1}(w)-g_0^{-1}(w_0)|}{|w-w_0|}$ for any $w_0 \in \mathbb{C}$. Hence

$$d\nu_{\Gamma,g_0(j)}(w) = r^{\delta_\Gamma} \frac{(|w|^2+1)^{\delta_\Gamma}}{(|g_0^{-1}(w)|^2+1)^{\delta_\Gamma}} d\nu_{\Gamma,j}(w).$$

Since $\nu_{\Gamma_0,x} = g_0^* \nu_{\Gamma,g_0(x)}$, we deduce

$$\begin{aligned}
\omega_{\Gamma_0}(g_0^{-1}(E)) &= \int_{z\in g_0^{-1}(E)} (|z|^2+1)^{\delta_\Gamma} d\nu_{\Gamma_0,j}(z) \\
&= \int_{u\in E} (|g_0^{-1}(u)|^2+1)^{\delta_\Gamma} d\nu_{\Gamma,g_0(j)}(u) \\
&= r^{\delta_\Gamma} \int_{u\in E} (|u|^2+1)^{\delta_\Gamma} d\nu_{\Gamma,j}(u) \\
&= r^{\delta_\Gamma} \omega_\Gamma(E).
\end{aligned}$$

□

This concludes a proof of (6.11). Therefore, since $\mathrm{sk}_{\Gamma_0}(C_0) < \infty$ and $|m^{\mathrm{BMS}}_{\Gamma_0}| < \infty$, the previous case of $C = C_0$ yields that

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T^{\delta_\Gamma}} N_T(\Gamma(C), E) &= \lim_{T\to\infty} \frac{1}{T^{\delta_\Gamma}} N_{r^{-1}T}(\Gamma_0(C_0), g_0^{-1}(E)) \\
&= \frac{1}{\delta_{\Gamma_0} \cdot |m^{\mathrm{BMS}}_{\Gamma_0}|} \cdot \mathrm{sk}_{\Gamma_0}(C_0) \cdot r^{-\delta_\Gamma} \cdot \omega_{\Gamma_0}(g_0^{-1}(E)) \\
&= \frac{1}{\delta_\Gamma \cdot |m^{\mathrm{BMS}}_\Gamma|} \cdot \mathrm{sk}_\Gamma(C) \cdot \omega_\Gamma(E).
\end{aligned}$$

This completes the proof of Theorem 6.1. □

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