GEODESIC PLANES IN GEOMETRICALLY FINITE ACYLINDRICAL 3-MANIFOLDS

YVES BENOIST AND HEE OH

Abstract. Let M be a geometrically finite acylindrical hyperbolic 3-manifold and let M^* denote the interior of the convex core of M. We show that any geodesic plane in M^* is either closed or dense, and that there are only countably many closed geodesic planes in M^*. These results were obtained in [27] and [28] when M is convex cocompact.

As a corollary, we obtain that when M covers an arithmetic hyperbolic 3-manifold M_0, the topological behavior of a geodesic plane in M^* is governed by that of the corresponding plane in M_0. We construct a counterexample of this phenomenon when M_0 is non-arithmetic.

Dedicated to Anatole Katok who was so enthusiastic
in sharing with us his encyclopedic knowledge
and his deep insights in dynamical systems.

Contents

Oh was supported in part by NSF Grant.

1. Introduction

1.1. **Geometrically finite acylindrical manifolds.** Let $M = \Gamma\backslash\mathbb{H}^3$ be a complete, oriented hyperbolic 3-manifold presented as a quotient of hyperbolic space by a discrete subgroup

$$\Gamma \subset G = \mathrm{Isom}^+(\mathbb{H}^3).$$

We denote by $\Lambda = \Lambda(\Gamma)$ the limit set of Γ in the sphere at infinity $\mathbb{S}^2$ and by Ω the domain of discontinuity; $\Omega = \mathbb{S}^2 - \Lambda$. The convex core of M is the smallest closed convex subset containing all periodic geodesics in M, or equivalently

$$\mathrm{core}(M) := \Gamma\backslash\mathrm{hull}(\Lambda) \subset M$$

is the quotient of the convex hull of the limit set Λ by Γ. We denote by M^* the interior of the convex core of M. Note that M^* is non-empty if and only if Γ is Zariski dense in G.

A geodesic plane P in M is the image $f(\mathbb{H}^2) \subset M$ of a totally geodesic immersion $f : \mathbb{H}^2 \to M$ of the hyperbolic plane into M. By a geodesic plane P^* in M^*, we mean the non-trivial intersection

$$P^* = P \cap M^* \neq \emptyset$$

of a geodesic plane in M with the interior of the convex core. Note that a plane P^* in M^* is always connected and that P^* is closed in M^* if and only if P^* is properly immersed in M^* [28, §2].

We say M is geometrically finite if the unit neighborhood of $\mathrm{core}(M)$ has finite volume. When $\mathrm{core}(M)$ is compact, M is called convex cocompact.

The closures of geodesic planes in M^* have been classified for convex cocompact acylindrical hyperbolic 3-manifolds by McMullen, Mohammadi and the second named author in [27] and [28] (see also [18] for a discussion on the higher dimensional case as well as [29] for survey). The main aim of this paper is to extend those classification results to geometrically finite acylindrical manifolds. In concrete terms, the new feature of this paper compared to [27] and [28] is that we allow the existence of cusps in M^*. We mention that [16] also discusses the influence of cusps on those classification results.

For a geometrically finite manifold M, the condition that M is acylindrical is a topological one, it means that its compact core N (called the Scott core) is acylindrical, i.e., N has incompressible boundary, and every essential cylinder in N is boundary parallel [35].

In the case when the boundary of the convex core of M is totally geodesic, we call M *rigid acylindrical.* The class of rigid acylindrical hyperbolic 3-manifolds M includes those for which M^* is obtained by "cutting" a finite volume complete hyperbolic 3-manifold M_0 along a properly embedded compact geodesic surface $S \subset M_0$ (see §12.3 for explicit examples). We remark that a geometrically finite acylindrical manifold is quasiconformal conjugate to a unique geometrically finite rigid acylindrical manifold ([34], [26]).

1.2. **Closures of geodesic planes.** Our main theorem is the following:

Theorem 1.1. *Let M be a geometrically finite acylindrical hyperbolic 3-manifold. Then any geodesic plane P^* is either closed or dense in M^*. Moreover, there are only countably many closed geodesic planes P^* in M^*.*

In the rigid case, each geodesic boundary component has a fundamental group which is a cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$, up to conjugation. This rigid structure forces any closed plane P^* in M^* to be a part of a closed plane P in M:

Theorem 1.2. *If M is rigid in addition, then any geodesic plane P in M intersecting M^* non-trivially is either closed or dense in M.*

We do not know whether Theorem 1.2 can be extended to a non-rigid M or not. This is unknown even when M is convex compact, as remarked in [28].

Added in proof: A recent work of Y. Zhang [38] shows that Theorem 1.2 does not extend to a non-rigid case: there exists a convex cocompact acylindrical 3-manifold M which contains a plane P intersecting M^* such that P^* is closed in M^*, but P is not closed in M.

When M is a cover of an arithmetic hyperbolic 3-manifold M_0, the topological behavior of a geodesic plane in M^* is governed by that of the corresponding plane in M_0:

Theorem 1.3. *Let $M = \Gamma\backslash\mathbb{H}^3$ be a geometrically finite acylindrical manifold which covers an arithmetic manifold $M_0 = \Gamma_0\backslash\mathbb{H}^3$ of finite volume. Let $p : M \to M_0$ be the covering map. Let $P \subset M$ be a geodesic plane with $P^* = P \cap M^* \neq \emptyset$. Then*

1. *P^* is closed in M^* if and only if $p(P)$ is closed in M_0;*
2. *P^* is dense in M^* if and only if $p(P)$ is dense in M_0.*

When M is rigid, we can replace M^* by M in the above two statements.

Theorem 1.3 is not true in general without the arithmeticity assumption on M_0:

Theorem 1.4. *There exists a non-arithmetic closed hyperbolic 3-manifold M_0, covered by a geometrically finite rigid acylindrical manifold M such that there exists a properly immersed geodesic plane P in M with $P \cap M^* \neq \emptyset$ whose image $p(P)$ is dense in M_0, where $p : M \to M_0$ is the covering map.*

The following theorem for a general geometrically finite manifold of infinite volume, describes those geodesic planes for which $P = P^*$:

Theorem 1.5. *Let M be a geometrically finite hyperbolic 3-manifold of infinite volume. Then there are only finitely many geodesic planes in M that are contained in core(M). All these planes are closed with finite area.*

Theorem 1.5 is generally false for a finite volume manifold. For instance, an arithmetic hyperbolic 3-manifold with one closed geodesic plane contains infinitely many closed geodesic planes (see [19, Theorem 9.5.6]).

1.3. **Orbit closures in the space of circles.** We now formulate a stronger version of Theorems 1.1 and 1.2. Let $\mathcal{C}$ denote the space of all oriented circles in $\mathbb{S}^2$. We identify $G := \mathrm{Isom}^+(\mathbb{H}^3)$ with $\mathrm{PSL}_2(\mathbb{C})$, considered as a simple real Lie group, and $H := \mathrm{Isom}^+(\mathbb{H}^2)$ with $\mathrm{PSL}_2(\mathbb{R})$, so that we have a natural isomorphism

$$\mathcal{C} \simeq G/H.$$

The following subsets of $\mathcal{C}$ play important roles in our discussion and we will keep the notation throughout the paper:

$$\mathcal{C}_\Lambda = \{C \in \mathcal{C} : C \cap \Lambda \neq \emptyset\};$$
$$\mathcal{C}^* = \{C \in \mathcal{C} : C \text{ separates } \Lambda\}.$$

The condition C *separates* Λ means that both connected components of $\mathbb{S}^2 - C$ meet Λ. Note that $\mathcal{C}_\Lambda$ is closed in $\mathcal{C}$ and that $\mathcal{C}^*$ is open in $\mathcal{C}$. If the limit set Λ is connected, then $\mathcal{C}^*$ is a dense subset *of* $\mathcal{C}_\Lambda$ [27, Corollary 4.5] but $\mathcal{C}^*$ is not contained in $\mathcal{C}_\Lambda$ in general.

The classification of closures of geodesic planes follows from the classification of Γ-orbit closures of circles (cf. [27]). In fact, the following theorem strengthens both Theorems 1.1 and 1.2 in two aspects: it describes possible orbit closures of single Γ-orbits as well as of any Γ-invariant subsets of $\mathcal{C}^*$.

Theorem 1.6. *Let $M = \Gamma\backslash\mathbb{H}^3$ be a geometrically finite acylindrical manifold. Then*

1. *Any Γ-invariant subset of $\mathcal{C}^*$ is either dense or a union of finitely many closed Γ-orbits in $\mathcal{C}^*$;*
2. *There are at most countably many closed Γ-orbits in $\mathcal{C}^*$; and*
3. *For M rigid, any Γ-invariant subset of $\mathcal{C}^*$ is either dense or a union of finitely many closed Γ-orbits in $\mathcal{C}_\Lambda$.*

We also present a reformulation of Theorem 1.5 in terms of circles:

Theorem 1.7. *Let $M = \Gamma\backslash\mathbb{H}^3$ be a geometrically finite manifold of infinite volume. Then there are only finitely many Γ-orbits of circles contained in Λ. Moreover each of these orbits is closed in $\mathcal{C}_\Lambda$.*

1.4. **Strategy and organization.** Our approach follows the same lines as the approaches of [27] and [28], and we tried to keep the same notation from those papers as much as possible.

Below is the list of subgroups of $G = \mathrm{PSL}_2(\mathbb{C})$ which will be used throughout the paper:

$$\begin{aligned}
H &= \mathrm{PSL}_2(\mathbb{R}) \\
A &= \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\} \\
K &= \mathrm{PSU}(2) \\
N &= \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\} \\
U &= \{u_t : t \in \mathbb{R}\} \quad \text{and} \\
V &= \{u_t : t \in i\mathbb{R}\}.
\end{aligned}$$

We can identify $M = \Gamma\backslash\mathbb{H}^3$ with the double coset space $\Gamma\backslash G/K$ and its oriented frame bundle $\mathrm{F}M$ with the homogeneous space $\Gamma\backslash G$. By the duality between Γ-orbits in $\mathcal{C}$ and H-orbits in $\mathrm{F}M$, Theorem 1.6 follows from the classification of H-invariant closed subsets in F^* where F^* is the H-invariant subset of $\mathrm{F}M$ such that $\Gamma\backslash\mathcal{C}^* = F^*/H$ (cf. [27], [28]). Working in the frame bundle $\mathrm{F}M$ enables us to use the geodesic flow, the horocyclic flow as well as the "imaginary horocyclic flow" given by the right-translation actions of A, U, and V in the space $\Gamma\backslash G$ respectively.

The results in §§2-6 hold for a general Zariski dense discrete subgroup Γ.

In §2, we review some basic definitions and notations to be used throughout the paper. The notation $\mathrm{RF}\, M$ denotes the *renormalized frame bundle of* M, which is the closure of the union of all periodic A-orbits in $\mathrm{F}M$ (see also (2.2)). We consider the H-invariant subset F_Λ of $\mathrm{F}M$ which corresponds to $\mathcal{C}_\Lambda$:

$$\Gamma\backslash\mathcal{C}_\Lambda = F_\Lambda/H.$$

We set $F_\Lambda^* = F_\Lambda \cap F^*$. When Λ is connected, $F_\Lambda^* = F^*$, but not in general.

In §3, we study the closure of an orbit xH in F_Λ which accumulates on an orbit yH with non-elementary stabilizer. We prove that xH is dense in F_Λ if $y \in F^*$ (Proposition 3.5), and also present a condition for the density of xH in F_Λ when $y \notin F^*$ (Proposition 3.8).

In §4, we prove that if the closure $\overline{xH}$ contains a periodic U-orbit in F_Λ^*, then xH is either locally closed or dense in F_Λ (Proposition 4.2).

In §5, we recall a closed A-invariant subset $\mathrm{RF}_k\, M \subset \mathrm{RF}\, M$, $k > 1$, with *k-thick* recurrence properties for the horocycle flow, which was introduced in [28]. This set may be empty in general. We show that the thick recurrence property remains preserved even after we remove a neighborhood of finitely many cusps from $\mathrm{RF}_k\, M$. That is, setting

$$W_{k,R} := \mathrm{RF}_k\, M - \mathscr{H}_R \tag{1.1}$$

where $\mathscr{H}_R$ is a neighborhood of finitely many cusps in $\mathrm{F}M$ of depth $R \gg 1$, we show that for any $x \in W_{k,R}$, the set $T_x := \{t \in \mathbb{R} : xu_t \in W_{k,R}\}$ is

$4k$-thick at ∞, in the sense that for all sufficiently large $r > 1$ (depending on x),

$$T_x \cap ([-4kr, -4r] \cup [r, 4kr]) \neq \emptyset$$

(Proposition 5.4).

In §6, we present a technical proposition (Proposition 6.2) which ensures the density of xH in F_Λ when xH intersects a compact subset $W \subset F_\Lambda^*$ with the property that the return times $\{t : xu_t \in W\}$ is k-thick at ∞ for every $x \in W$. The proof of this proposition uses the notion of U-minimal sets with respect to W, together with the polynomial divergence property of unipotent flows of two nearby points, which goes back to Margulis' proof of the Oppenheim conjecture [21]. In the context when the return times of the horocyclic flow is only k-thick (at ∞), this argument was used in [27] and [28] in order to construct a piece of V-orbit inside the closure of xH, which can then be pushed to the density of xH in F_Λ using Corollary 4.1.

In §§7-10, we make an additional assumption that Γ is geometrically finite, which implies that $\mathrm{RF}\, M$ is a union of a compact set and finitely many cusps. Therefore the set $W_{k,R}$ in (1.1) is compact when $\mathscr{H}_R$ is taken to be a neighborhood of all cusps in $\mathrm{F}M$.

In §7, we apply the result in §6 to an orbit $xH \subset F_\Lambda^*$ intersecting $W_{k,R}$ when $\overline{xH}$ does not contain a periodic U-orbit. Combined with the results in §4, we conclude that any H-orbit intersecting $\mathrm{RF}_k M \cap F^*$ is either locally closed or dense in F_Λ (Theorem 7.1).

In §8, we prove that any locally closed orbit xH intersecting $\mathrm{RF}_k M \cap F^*$ has a non-elementary stabilizer and intersects $\mathrm{RF}_k M \cap F^*$ as a closed subset (Theorem 8.1).

In §9, we give an interpretation of the results obtained so far in terms of Γ-orbits of circles for a general geometrically finite Zariski dense subgroup. (Theorem 9.2).

In §10, we prove that any orbit xH included in $\mathrm{RF}\, M$ is closed of finite volume, and that only finitely many such orbits exist, proving Theorem 1.5 (Theorem 10.1).

In §11, we specialize to the case where $M = \Gamma\backslash\mathbb{H}^3$ is a geometrically finite acylindrical manifold. We show that in this case, *every* H-orbit in F^* intersects a compact subset $W_{k,R}$ for k sufficiently large (Corollary 11.7). We prove Theorem 1.6 in terms of H-orbits on $\mathrm{F}M$ (Theorem 11.8).

In §12, we prove Theorem 1.3 and give a counterexample when M_0 is not arithmetic (Proposition 12.1), proving Theorem 1.4. We also give various examples of geometrically finite rigid acylindrical 3-manifolds.

1.5. **Comparison of the proof of Theorem 1.1 with the convex co-compact case.** For readers who are familiar with the work [28], we finish the introduction with a brief account on some of essential differences in the proofs between the present work and [28] for the case when M is a geometrically finite acylindrical manifold.

For M convex cocompact acylindrical, that $\operatorname{RF} MH = \operatorname{RF}_k MH$ for all sufficiently large k was known in [28]. An important ingredient of our paper will be to extend this result to the geometrically finite acylindrical setting (§11). The proof of this property requires new arguments. When M is a convex cocompact acylindrical manifold, the proof in [28] relies on the fact that Λ is a Sierpinski curve of positive modulus, that is, there exists $\varepsilon > 0$ such that the modulus of the annulus between any two components B_1, B_2 of $\Omega := \mathbb{S}^2 - \Lambda$ satisfy

$$\operatorname{mod}(\mathbb{S}^2 - (\overline{B}_1 \cup \overline{B}_2)) \geq \varepsilon$$

[28, Theorem 3.1].

When M has cusps, the closures of some components of Ω meet each other, and hence Λ is not even a Sierpinski curve. Nevertheless, under the assumption that M is a geometrically finite acylindrical manifold, Λ is still a *quotient of a Sierpinski curve of positive modulus*, in the sense that we can present Ω as the disjoint union $\bigcup_\ell T_\ell$ where T_ℓ's are maximal *trees of components of* Ω so that

$$\inf_{\ell \neq k} \operatorname{mod}(\mathbb{S}^2 - (\overline{T}_\ell \cup \overline{T}_k)) > 0$$

(Theorem 11.5). See [37, Figures 7-8] for some visual images of these limit sets. This analysis enables us to show that every separating circle C intersects Λ as a Cantor set of positive modulus (Theorem 11.6), which immediately implies $\operatorname{RF} MH = \operatorname{RF}_k MH$ for all sufficiently large k.

We now discuss new features of dynamical aspects of this paper. Consider an orbit xH in F^* and set $X := \overline{xH}$. When M is convex cocompact, the main strategy of [28] is to analyze U-minimal subsets of X with respect to $\operatorname{RF}_k M$. That is, unless xH is locally closed, it was shown, using the thickness of the return times and the polynomial divergence property of the horocyclic flow, that any such U-minimal subset is invariant under a connected semigroup L transversal to U. Then pushing further, one could find an N-orbit inside X, which implies $X = F_\Lambda$. In the present setting, if $X \cap F^*$ happens to contain a periodic U-orbit, which is a generic case a posteriori, then any U-minimal subset of X relative to $\operatorname{RF}_k M$ is a periodic U-orbit and hence is not invariant by any other subgroup but U. Hence the aforementioned strategy does not work. However it turns out that this case is simpler to handle, and that's what §4 is about. See Proposition 4.2.

Another important ingredient of [28] is that a closed H-orbit in F^* has a non-elementary stabilizer; this was used to conclude $X = F_\Lambda$ whenever X contains a closed H-orbit in F^* properly, as well as to establish the countability of such orbits, and its proof relied on the absence of parabolic elements. In the present setting, we show that any locally closed H-orbit in F^* has a non-elementary stabilizer, regardless of the existence of parabolic elements of Γ. Geometrically, this means that every locally closed geodesic plane P that meets M^* has a non-elementary fundamental group. This is Theorem 8.1.

When M is not convex cocompact, $\mathrm{RF}\, M$ is not compact and neither is $\mathrm{RF}_k M$, which presents an issue in the polynomial divergence argument. In Proposition 5.4, we show that the remaining compact set after removing horoballs from $\mathrm{RF}_k M$ has the desired thickness property for U-orbits and hence can be used as a replacement of $\mathrm{RF}_k M$. We remark that the *global* thickness of the return times of U-orbits to $\mathrm{RF}_k M$ was crucial in establishing this step.

The proof of Theorem 1.1 combines these new ingredients with the techniques developed in [28].

We would also like to point out that we made efforts in trying to write this paper in a greater general setting. For instance, we establish (Theorem 9.2):

Theorem 1.8. *Let Γ be a geometrically finite Zariski dense subgroup of G. Then for any $C \in \mathcal{C}^*$ such that $C \cap \Lambda$ contains a uniformly perfect Cantor set, the orbit $\Gamma(C)$ is either discrete or dense in $\mathcal{C}_\Lambda$.*

With an exception of §11, we have taken a more homogeneous dynamics viewpoint overall in this paper, hoping that this perspective will also be useful in some context.

Acknowledgement: The present paper heavily depends on the previous works [27] and [28] by McMullen, Mohammadi, and the second named author. We are grateful to Curt McMullen for clarifying the notion of the acylindricality of a geometrically finite manifold. We would like to thank Curt McMullen and Yair Minsky for their help in writing §11, especially the the proof of Theorem 11.5. We would also like to thank the referee for a careful reading and many useful comments.

2. Preliminaries

In this section, we set up some notation which will be used throughout the paper and review some definitions.

Recall that we keep the notation introduced in §1.4 throughout the paper. In particular, G is the simple connected real Lie group $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$. The action of G on $\mathbb{H}^3 = G/K$ extends continuously to a conformal action of G on the Riemann sphere $\mathbb{S}^2 = \hat{\mathbb{C}} \cup \{\infty\}$ and the union $\mathbb{H}^3 \cup \mathbb{S}^2$ is compact. We denote by $\mathrm{T}^1\, \mathbb{H}^3$ and $\mathrm{F}\, \mathbb{H}^3$ the unit tangent bundle and the oriented frame bundle of $\mathbb{H}^3$ respectively. Fix $o \in \mathbb{H}^3$ whose stabilizer is K and fix a vector $v_o \in \mathrm{T}^1(\mathbb{H}^3)$ based at o whose stabilizer is the centralizer of A in K. Choose a unit vector $w_o \in \mathrm{T}_o(\mathbb{H}^3)$ orthogonal to v_o.

The orbit map $g \mapsto (gv_o, gw_o)$ induces the identification $G = \mathrm{F}\, \mathbb{H}^3$. For $g \in G$, we define $g^+, g^- \in \mathbb{S}^2$ the forward and backward end points of the geodesic determined by $g(v_o)$ respectively. Denote by π the projection $G = \mathrm{F}\, \mathbb{H}^3 \to \mathbb{H}^3$ given by $\pi(g) = g(o)$. The right translation action of A on

$G = \mathrm{F}\,\mathbb{H}^3$ defines the frame flow, and we have

$$g^{\pm} = \lim_{t\to\pm\infty} \pi(ga_t) \tag{2.1}$$

where the limit is taken in the compactification $\mathbb{H}^3 \cup \mathbb{S}^2$.

For $g \in G$, the set $\pi(gH)$ is a geodesic plane in $\mathbb{H}^3$ to which $g(v_o)$ and $g(w_o)$ are tangent and

$$(gH)^+ := \{(gh)^+ : h \in H\} \subset \mathbb{S}^2$$

is an oriented circle which bounds the plane $\pi(gH)$.

The correspondences

$$gH \to (gH)^+ \;\text{ and }\; gH \to \pi(gH)$$

give rise to bijections of G/H with the space $\mathcal{C}$ of all oriented circles in $\mathbb{S}^2$, as well as with the space of all oriented geodesic planes of $\mathbb{H}^3$.

A horosphere (resp. horoball) in $\mathbb{H}^3$ is a Euclidean sphere (resp. open Euclidean ball) tangent to $\mathbb{S}^2$ and a horocycle in $\mathbb{H}^3$ is a Euclidean circle tangent to $\mathbb{S}^2$. A horosphere is the image of an N-orbit under π while a horocycle is the image of a U-orbit under π.

2.1. **Renormalized frame bundle.** Let Γ be a non-elementary discrete subgroup of G and $M = \Gamma\backslash\mathbb{H}^3$. We denote by $\Lambda = \Lambda(\Gamma) \subset \mathbb{S}^2$ the limit set of Γ. We can identify the oriented frame bundle $\mathrm{F}M$ with $\Gamma\backslash G$. With abuse of notation, we also denote by π for the canonical projection $\mathrm{F}M \to M$. Note that, for $x = [g] \in \mathrm{F}M$, the condition $g^{\pm} \in \Lambda$ does not depend on the choice of a representative. The renormalized frame bundle $\mathrm{RF}\,M \subset \mathrm{F}M$ is defined as

$$\mathrm{RF}\,M = \{[g] \in \Gamma\backslash G : g^{\pm} \in \Lambda\}, \tag{2.2}$$

in other words, it is the closed subset of $\mathrm{F}M$ consisting of frames (e_1, e_2, e_3) such that e_1 is tangent to a complete geodesic contained in the convex core of M.

We define the following H-invariant subsets of $\mathrm{F}M$:

$$\begin{aligned} F_\Lambda &= \{[g] \in \Gamma\backslash G : (gH)^+ \cap \Lambda \neq \emptyset\} = (\mathrm{RF}\,M)NH, \\ F^* &= \{[g] \in \Gamma\backslash G : (gH)^+ \text{separates } \Lambda\}, \\ F^*_\Lambda &= F_\Lambda \cap F^*. \end{aligned}$$

Note that $F^* = \{x \in \mathrm{F}M : \pi(xH) \cap M^* \neq \emptyset\}$ is an open subset of $\mathrm{F}M$ and that $F^* \subset F_\Lambda$ when Λ is connected. In particular, when Λ is connected, F_Λ has non-empty interior equal to F^*.

We recall that a circle $C \in \mathcal{C}$ is *separating* or *separates* Λ if both connected components of $\mathbb{S}^2 - C$ intersect Λ. By analogy, a frame $x \in F^*$ will be called a *separating frame*.

In the identification $\mathcal{C} = G/H$, the sets F_Λ, F^* and F^*_Λ satisfy

$$\Gamma\backslash\mathcal{C}_\Lambda = F_\Lambda/H, \;\; \Gamma\backslash\mathcal{C}^* = F^*/H \quad\text{and}\quad \Gamma\backslash\mathcal{C}^*_\Lambda = F^*_\Lambda/H,$$

where

$$\mathcal{C}_\Lambda^* := \mathcal{C}_\Lambda \cap \mathcal{C}^*.$$

Since Γ acts properly on the domain of discontinuity Ω, ΓC is closed in $\mathcal{C}$ for any circle C with $C \cap \Lambda = \emptyset$. For this reason, we only consider Γ-orbits in $\mathcal{C}_\Lambda$ or, equivalently, H-orbits in F_Λ.

2.2. **Geometrically finite groups.** We give a characterization of geometrically finite groups in terms of their limit sets (cf. [7], [20], [25]). A limit point $\xi \in \Lambda$ is called *radial* if any geodesic ray toward ξ has an accumulation point in M and *parabolic* if ξ is fixed by a parabolic element of Γ. In the group $G = \mathrm{PSL}_2(\mathbb{C})$, parabolic elements are precisely unipotent elements of G; any parabolic element in G is conjugate to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. When ξ is parabolic, its stabilizer $\mathrm{Stab}_\Gamma \xi$ in Γ is virtually abelian, and its rank, which is called the rank of ξ, is either 1 or 2. A parabolic fixed point ξ is called *bounded parabolic* if $\mathrm{Stab}_\Gamma \xi$ acts co-compactly on $\Lambda - \{\xi\}$.

We denote by Λ_r the set of radial limit points, by Λ_p the set of parabolic fixed points and by Λ_{bp} the subset of bounded parabolic points. Note that Λ_r and Λ_p are always disjoint. The fixed points of hyperbolic elements of Γ are contained in Λ_r and form a dense subset of Λ. More strongly, the set of pairs of attracting and repelling fixed points of hyperbolic elements is a dense subset of $\Lambda \times \Lambda$ [10].

A Zariski dense discrete subgroup Γ of G is called geometrically finite if it satisfies one of the following equivalent conditions (see [20, Sections 3.6 and 3.11] or [25, Theorem 3.7]):

(1) the convex core of M has finite volume;
(2) $\Lambda = \Lambda_r \cup \Lambda_{bp}$ (Beardon-Maskit condition);
(3) $\Lambda = \Lambda_r \cup \Lambda_p$ (Bishop condition [5]).

If $\Lambda = \Lambda_r$, or equivalently, if the convex core of M is compact, then Γ is called convex cocompact.

We say that a horoball $\mathcal{h}$ is based at $\xi \in \mathbb{S}^2$ if it is tangent to $\mathbb{S}^2$ at ξ. If $\xi \in \Lambda_p$, then for any fixed horoball $\mathcal{h} \subset \mathbb{H}^3$ based at $\xi \in \mathbb{S}^2$, its Γ-orbit $\Gamma\mathcal{h}$ is closed in the space of horoballs in $\mathbb{H}^3$. Given a horoball $\mathcal{h}$ and $R > 0$, we will write $\mathcal{h}_R$ for the horoball contained in $\mathcal{h}$ whose distance to the boundary of $\mathcal{h}$ is R.

Suppose that Γ is geometrically finite. Then Γ has finitely many orbits in Λ_p called cusps, and hence there exist finitely many horoballs $\mathcal{h}^1, \cdots, \mathcal{h}^m$ in $\mathbb{H}^3$ corresponding to these cusps such that the horoballs $\gamma\mathcal{h}^i$, for $\gamma \in \Gamma$ and $1 \le i \le m$, form a disjoint collection of open horoballs, i.e. $\gamma\mathcal{h}^i$ intersects $\mathcal{h}^j$ if and only if $i = j$ and $\gamma\mathcal{h}^i = \mathcal{h}^j$ (cf. [25, Theorem 3.7] and its proof).

We fix

$$\mathscr{H} := \{[g] \in \mathrm{F}M : \pi(g) \in \textstyle\bigcup_{\gamma,i} \gamma\mathcal{h}^i\}. \tag{2.3}$$

For $R \geq 0$, we set

$$\mathcal{H}_R = \{[g] \in \mathrm{F}M : \pi(g) \in \bigcup_{\gamma,i} \gamma \mathcal{h}_R^i\}. \tag{2.4}$$

Then each $\mathcal{H}_R$ provides a union of disjoint neighborhood of the cusps in $\mathrm{RF}\, M$ and hence the subset

$$\mathrm{RF}\, M - \mathcal{H}_R$$

is compact for any $R \geq 0$. See [20, Theorem 3.6.1].

3. Limit circle with non-elementary stabilizer

As noted before, the study of geodesic planes in $M = \Gamma \backslash \mathbb{H}^3$ can be approached in two ways: either via the study of H-orbits in $\mathrm{F}M$ or via the study of Γ-orbits in $\mathcal{C}$. In this section we analyze a Γ-orbit ΓC in $\mathcal{C}^*$ which accumulates on a circle D with a non-elementary stabilizer in Γ. We show that ΓC is dense in $\mathcal{C}_\Lambda$ in the following two cases:

(1) when D separates Λ (Proposition 3.5);
(2) when there exists a sequence of distinct circles $C_n \in \Gamma C$ such that $C_n \cap \Lambda$ does not collapse to a countable set (Proposition 3.8).

3.1. **Sweeping the limit set.** The following proposition is a useful tool, which says that, in order to prove the density of ΓC in $\mathcal{C}_\Lambda$, we only have to find a "sweeping family" of circles in the closure of ΓC.

Proposition 3.1. [27, Corollary 4.2] *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup. If $\mathcal{D} \subset \mathcal{C}$ is a collection of circles such that $\bigcup_{C \in \mathcal{D}} C$ contains a non-empty open subset of Λ, then there exists $C \in \mathcal{D}$ such that $\overline{\Gamma C} = \mathcal{C}_\Lambda$.*

In subsection 3.2, we will construct such a sweeping family $\mathcal{D}$ using a result of Dal'bo in [9] (see Proposition 3.2).

In the subsequent sections, we will use other sweeping families $\mathcal{D}$ as in Corollary 4.1 that are constructed via a more delicate polynomial divergence argument.

3.2. **Influence of Fuchsian groups.** Let $B \subset \mathbb{S}^2$ be a round open disk with a hyperbolic metric ρ_B. We set

$$G^B = \mathrm{Isom}^+(B, \rho_B) \simeq \mathrm{PSL}_2(\mathbb{R}).$$

A discrete subgroup of G^B is a Fuchsian group, and its limit set lies in the boundary ∂B. For a non-empty subset $E \subset \partial B$, we denote by $\mathrm{hull}(E, B) \subset B$ the convex hull of E in B. We emphasise that in our definition the convex hull $\mathrm{hull}(E, B)$ is included in the sphere at infinity $\mathbb{S}^2$.

We denote by $\mathcal{H}(B, E) \subset \mathcal{C}$ the closure of the set of horocycles in B resting on E. The only circle in $\mathcal{H}(B, E)$ which is not a horocycle is ∂B itself.

We first recall the following result of Dal'bo [9, Proposition B]:

Proposition 3.2. *Let $\Gamma \subset G^B$ be a non-elementary discrete subgroup with limit set Λ. If $C \in \mathcal{H}(B,\Lambda)$ is a horocycle in B such that $C \cap \Lambda$ is a radial limit point, then ΓC is dense in $\mathcal{H}(B,\Lambda)$.*

The following proposition is a generalization of [27, Corollary 3.2] from Γ convex cocompact to a general finitely generated group. Note that a finitely generated Fuchsian group is geometrically finite.

Lemma 3.3. *Let $\Gamma \subset G^B$ be a non-elementary finitely generated discrete subgroup. Suppose $C_n \to \partial B$ in $\mathcal{C}$ and $C_n \cap \operatorname{hull}(\Lambda, B) \neq \emptyset$ for all n. Then the closure of $\bigcup \Gamma C_n$ contains $\mathcal{H}(B,\Lambda)$.*

Proof. In view of Proposition 3.2, it suffices to show that the closure of $\bigcup \Gamma C_n$ contains a circle $C \subset \overline{B}$ resting at a radial limit point of Γ. Passing to a subsequence, we will consider two separate cases.

Case 1: $C_n \cap \partial B = \{a_n, b_n\}$ with $a_n \neq b_n$. Note that the angle between C_n and ∂B converges to 0. Note also that since C_n meets the convex hull of Λ in B, the geodesic joining a_n and b_n in B also meets $\operatorname{hull}(\Lambda, B)$.

Since Γ is a finitely generated Fuchsian group, it is geometrically finite. It follows that there exists a compact set $W \subset \Gamma\backslash B$ such that every geodesic on the surface $\Gamma\backslash B$ that intersects its convex core also intersects W. Applying this to each geodesic joining a_n and b_n, we may assume that the sequences a_n and b_n converge to two distinct points $a_0 \neq b_0$, after replacing C_n by $\delta_n C_n$ for a suitable $\delta_n \in \Gamma$ if necessary. The open arcs $(a_n, b_n) := C_n \cap B$ converge to an open arc (a_0, b_0) of ∂B. We consider two separate subcases.

Case 1.A: $(a_0, b_0) \cap \Lambda \neq \emptyset$. Choose a hyperbolic element $\delta \in \Gamma$ whose repelling and attracting fixed points ξ_r and ξ_a lie inside the open arc (a_0, b_0). Denote by L the axis of translation of δ and choose a compact arc $L_0 \subset L$ which is a fundamental set for the action of δ on L. For n large, $C_n \cap L \neq \emptyset$. Pick $c_n \in C_n \cap L$. Then there exist $k_n \in \mathbb{Z}$ such that $\delta^{k_n} c_n \in L_0$. Note that $k_n \to \pm\infty$ since c_n converges to a point in ∂B. Hence $\delta^{k_n} C_n$ converges to a circle contained in $\overline{B}$ resting at ξ_r or at ξ_a.

Case 1.B: $(a_0, b_0) \cap \Lambda = \emptyset$. In this case, one of these two points, say a_0, is in the limit set and (a_0, b_0) is included in a maximal arc (a_0, b_0') of $\partial B - \Lambda$. Let L be the geodesic of B that connects a_0 and b_0'. Its projection to $\Gamma\backslash B$ is included in the boundary of the surface $S := \operatorname{core}(\Gamma\backslash B)$. Since Γ is geometrically finite, this projection is compact, and hence L is the axis of translation of a hyperbolic element $\delta \in \Gamma$. For n large, $C_n \cap L$ is non-empty, and we proceed as in Case 1.A.

Case 2: $C_n \cap \partial B = \emptyset$ or $\{a_n\}$. Choose any hyperbolic element $\delta \in \Gamma$ and let L be the axis of translation of δ. Then $C_n \cap L \neq \emptyset$ for all n, by passing to a subsequence. Hence we conclude as in Case 1.A. $\square$

Corollary 3.4. *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup. Let $B \subset \mathbb{S}^2$ be a round open disk that meets Λ and whose stabilizer Γ^B is non-elementary and finitely generated. If $C_n \to \partial B$ in $\mathcal{C}$ and*

$$C_n \cap \operatorname{hull}(B, \Lambda(\Gamma^B)) \neq \emptyset, \tag{3.1}$$

then the closure of $\bigcup \Gamma C_n$ *contains* $\mathcal{C}_\Lambda$*.*

Note that Condition (3.1) is always satisfied when Γ^B is a lattice in G^B, as soon as C_n meets B.

Proof. This follows from Proposition 3.1 and Lemma 3.3. $\square$

The following is a useful consequence of the previous discussion.

Proposition 3.5. *Let* $\Gamma \subset G$ *be a Zariski dense discrete subgroup. Let* $C \in \mathcal{C}^*_\Lambda$*. Suppose that there exists a sequence of distinct circles* $C_n \in \Gamma C$ *converging to a circle* $D \in \mathcal{C}^*_\Lambda$ *whose stabilizer* Γ^D *is non-elementary. Then* ΓC *is dense in* $\mathcal{C}_\Lambda$*.*

Proof. Observe that every circle C_n sufficiently near D intersects the set $\mathrm{hull}(B, \Lambda(\Gamma^B))$ for at least one of the two disks B bounded by D. Moreover such a disk B always meets Λ as D separates Λ and $\Gamma^B = \Gamma^D$ as D is an oriented circle. Hence the claim follows from Corollary 3.4. $\square$

Corollary 3.6. *Let* $\Gamma \subset G$ *be a Zariski dense discrete subgroup. If* $C \in \mathcal{C}^*_\Lambda$ *has a non-elementary stabilizer* Γ^C*, then* ΓC *is either discrete or dense in* $\mathcal{C}_\Lambda$*.*

Proof. If the orbit ΓC is not locally closed, there exists a sequence of distinct circles $C_n \in \Gamma C$ converging to C, and hence the claim follows from Proposition 3.5. $\square$

We recall that a subset is *locally closed* when it is open in its closure and that a subset is *discrete* when it is *countable and locally closed*. Hence the conclusion of Corollary 3.6 can also be stated as the dichotomy that ΓC is either locally closed or dense in $\mathcal{C}_\Lambda$.

3.3. **Planes near the boundary of the convex core.** We now explain an analog of Proposition 3.5 when the limit circle D does not separate Λ. The following lemma says that its stabilizer Γ^D is non-elementary as soon as $D \cap \Lambda$ is uncountable.

Lemma 3.7. *Let* $M = \Gamma\backslash\mathbb{H}^3$ *be a geometrically finite manifold with limit set* Λ *and* $D \in \mathcal{C}_\Lambda$ *be a circle that does not separate* Λ*.*

Then

(1) Γ^D *is a finitely generated Fuchsian group;*
(2) *There is a finite set* Λ_0 *such that* $D \cap \Lambda = \Lambda(\Gamma^D) \cup \Gamma^D \Lambda_0$*.*

Proof. This lemma is stated for Γ convex cocompact in [28, Theorem 5.1]. But its proof works for Γ geometrically finite as well with no change. $\square$

Proposition 3.8. *Let* $M = \Gamma\backslash\mathbb{H}^3$ *be a geometrically finite manifold. Consider a sequence of circles* $C_n \to D$ *with* $C_n \in \mathcal{C}^*_\Lambda$ *and* $D \notin \mathcal{C}^*_\Lambda$*. Suppose that* $\liminf_{n\to\infty}(C_n \cap \Lambda)$ *is uncountable. Then* $\bigcup \Gamma C_n$ *is dense in* $\mathcal{C}_\Lambda$*.*

Recall that $\liminf_{n\to\infty}(C_n \cap \Lambda) := \{a_0 \in \mathbb{S}^2 \mid \exists\, a_n \in C_n \cap \Lambda \text{ with } \lim_{n\to\infty} a_n = a_0\}$.

Proof. This proposition is stated for M convex cocompact with incompressible boundary [28, Theorem 6.1]. With Lemmas 3.3 and 3.7 in place, the proof extends verbatim to the case claimed. $\square$

4. H-orbit closures containing a periodic U-orbit

In the rest of the paper, we will use the point of view of H-orbits in the frame bundle $\mathrm{F}M$ in our study of geodesic planes in M. This viewpoint enables us to utilize the dynamics of the actions of the subgroups A, U, and V of G on the space $\mathrm{F}M = \Gamma\backslash G$.

In this section we focus on an orbit xH in F_Λ whose closure contains a periodic U-orbit yU contained in F^* and prove that such xH is either locally closed or dense in F_Λ (Proposition 4.2).

4.1. **One-parameter family of circles.** In proving the density of an orbit xH in F_Λ, our main strategy is to find a point $y \in F^*_\Lambda$ and a one-parameter semigroup $V^+ \subset V$ such that yV^+ is included in $\overline{xH}$ and to apply the following corollary of Proposition 3.1.

Corollary 4.1. *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup and V^+ a one-parameter subsemigroup of V. For any $y \in F^*$, the closure $\overline{yV^+H}$ contains F_Λ.*

Proof. We choose a representative $g \in G$ of $y = [g]$ and consider the corresponding circle $C = (gH)^+$. The union of circles $(gvH)^+$ for $v \in V^+$ contains a disk B bounded by C. Since C separates Λ, B contains a non-empty open subset of Λ. Therefore, by Proposition 3.1, the set $\overline{yV^+H}$ contains F_Λ. $\square$

When the closure of xH contains a periodic U-orbit Y, we will apply Corollary 4.1 to a point $y \in Y$.

4.2. **Periodic U-orbits.**

Proposition 4.2. *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup and $x \in F^*_\Lambda$. Suppose that $\overline{xH} \cap F^*$ contains a periodic U-orbit yU. Then either*

(1) *xH is locally closed and $yH = xH$; or*
(2) *xH is dense in F_Λ.*

Before beginning the proof, let us recall how one can detect that an orbit yS of a subgroup $S < G$ is not locally closed:

$$yS \text{ is not locally closed} \iff \exists\, w_n \to e \text{ in } G - S \text{ with } yw_n \in yS. \tag{4.1}$$

The rest of this section is devoted to a proof of Proposition 4.2. We fix x, y and $Y := yU$ as in Proposition 4.2 and set $X = \overline{xH}$.

Setting

$$S_y := \{g \in G : yg \in X\},$$

we split the proof into the following three cases:

♠1. $S_y \cap O \not\subset VH$ for any neighborhood O of e;

♠2. $S_y \cap O \subset VH$ for some neighborhood O of e and $S_y \cap O \not\subset H$ for any neighborhood O of e;

♠3. $S_y \cap O \subset H$ for some neighborhood O of e.

Case ♠1. In this case, we will need the following algebraic lemma:

Lemma 4.3. *Let S be a subset of $G - VH$ such that $e \in \overline{S}$. Then the closure of USH contains a one-parameter semigroup V^+ of V.*

Proof. Let $\mathfrak{sl}(2, \mathbb{R})$ denote the set of real 2 by 2 traceless matrices, and consider the Lie algebra $\mathcal{V} = \begin{pmatrix} 0 & i\mathbb{R} \\ 0 & 0 \end{pmatrix}$ of V. As $e \in \bar{S}$, there exists s_n in S converging to e.

Since $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{R}) \oplus i\mathfrak{sl}(2, \mathbb{R})$, for all large n, s_n is of the form $(\exp M_n)h_n$ for $M_n \in i\,\mathfrak{sl}(2, \mathbb{R})$ and $h_n \in H$. Moreover $M_n \in i\,\mathfrak{sl}(2, \mathbb{R}) - \mathcal{V}$ as $S \cap VH = \emptyset$. Since M_n tends to 0 as $n \to \infty$, the closure of $\bigcup_n \{uM_nu^{-1} : u \in U\}$ contains a half-line in $\mathcal{V}$. Since USH contains $\bigcup_n \{u(\exp M_n)u^{-1} : u \in U\}$, our claim follows. □

Lemma 4.4. *In case ♠1, xH is dense in F_Λ.*

Proof. The assumption implies that S_y contains a subset S such that $S \cap VH = \emptyset$ and $e \in \overline{S}$. Therefore, by Lemma 4.3, the closure of US_yH contains a one-parameter semigroup V^+. Hence for any $v \in V^+$, one can write

$$v = \lim_{n \to \infty} u_n g_n h_n$$

for some $u_n \in U$, $g_n \in S_y$ and $h_n \in H$. By passing to a subsequence, we may assume that $yu_n^{-1} \to y_0 \in Y$. So

$$yg_nh_n = (yu_n^{-1})(u_ng_nh_n) \to y_0v.$$

Hence y_0v belongs to X, and $yv \in y_0Uv = y_0vU$ belongs to X too. This proves the inclusion $yV^+ \subset X$. Therefore, by Corollary 4.1, since $y \in F^*$, the orbit xH is dense in F_Λ. □

Case ♠2. In this case, we will use the following algebraic fact:

Lemma 4.5. *If $w \in VH$ satisfies $wv \in VH$ for some $v \in V - \{e\}$, then $w \in AN$.*

Proof. Without loss of generality, we can assume that $w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H$ and write $v = \begin{pmatrix} 1 & is \\ 0 & 1 \end{pmatrix} \in V$ with s real and non-zero. If the product wv belongs to VH, the lower row of the matrix wv must be real and this implies $c = 0$ as required. □

Lemma 4.6. *Case ♠2 does not happen.*

Proof. Suppose it happens.

First step: We claim that xH contains a periodic U-orbit.

Indeed if $y \notin xH$, Condition ♠2 says that there exists a non-trivial $v \in V$ such that $y' := yv$ belongs to xH. Since v commutes with U, the orbit $y'U \subset xH$ is periodic.

Therefore, by renaming y' to y, xH contains a periodic orbit yU.

Second step: We now claim that yAU is locally closed.

Assume that this is not the case. By (4.1), for any open neighborhood O of e in G, we can find $p \in AU$ and $w \in O - AU$ such that $yp = yw$. By Condition ♠2, we know that $w \in VH$ and we also know that there exists a sequence $v_n h_n \to e$ in $VH - H$ such that $yv_n \in X$. Since h_n is the real part of the 2 by 2 matrix $v_n h_n$, the sequence $v_n \in V - \{e\}$ also converges to e. Therefore, we get

$$yw(p^{-1}v_n p) = yv_n p \in X.$$

If O is small enough, Condition ♠2 implies that for n large,

$$w(p^{-1}v_n p) \in VH.$$

Therefore, Lemma 4.5 implies that

$$w \in AN.$$

Hence the element $s := wp^{-1}$ belongs to the stabilizer $S := \mathrm{Stab}_{AN}(y)$.

Let $\varphi : AN \to AN/AU = N/U$ be the natural projection. Since the discrete group S intersects U cocompactly, it is included in N and its image $\varphi(S)$ is discrete. Since w can be taken arbitrarily close to e, its image $\varphi(s) = \varphi(w)$ can be made arbitrarily close to e as well, yielding a contradiction.

Third step: We finally claim that xH is locally closed.

Since yAU is locally closed and since the group $K_H := \mathrm{SO}(2)$ is compact, the set $yAUK_H = yH$ is also locally closed. Since $y \in xH$, the H-orbit $xH = yH$ is locally closed. By (4.1), this contradicts Condition ♠2. □

Case ♠3. By assumption, $yO \cap X \subset yH$ for some neighborhood O of e in G. Hence yH is open in X, in other words, $\overline{xH} - yH$ is closed. Hence if y didn't belong to xH, then $\overline{xH} - yH$, being a closed subset containing xH, should be equal to $\overline{xH}$. Therefore $yH = xH$. Since xH is open in X, xH is locally closed. This finishes the proof of Proposition 4.2.

5. Return times of U-orbits

In Section 4, we have described the possible closures $\overline{xH}$ that contain a periodic U-orbit. In this section we recall, for each $k > 1$, a closed A-invariant subset $\mathrm{RF}_k\, M \subset \mathrm{RF}\, M$ consisting of points which have a "thick" set of return times to $\mathrm{RF}_k\, M$ itself under the U-action; this set was introduced in [28].

The main result of this section (Proposition 5.4) is that for the set $W_{k,R} := \mathrm{RF}_k\, M - \mathscr{H}_R$ where $\mathscr{H}_R$ is a sufficiently deep cusp neighborhood, every point $x \in W_{k,R}$ has a thick set of return times to $W_{k,R}$ under the U-action.

This information will be useful for the following reasons that will be explained in the next sections:
∗ When M is geometrically finite, these sets $W_{k,R}$ are compact.
∗ When M is a geometrically finite acylindrical manifold, every H-orbit xH in F^* intersects $W_{k,R}$ for all sufficiently large $k > 1$ (Corollary 11.7).
∗ For any compact set W with a thick set of return times to W, one can develop a polynomial divergence argument for U-orbits yU when $y \in W$, (Proposition 6.2).
∗ This argument can be applied to a U-minimal subset $Y = \overline{yU} \subset \overline{xH}$ with respect to $W_{k,R}$ (Proposition 7.3).

In this section we do not yet assume M to be geometrically finite even though this is the main case we have in mind. We will make this assumption only starting from Section 7.

5.1. **Points with k-thick return times.**

Definition 5.1. *Let $k \geq 1$. Let T be a subset of $\mathbb{R}$ and S be a subset of a circle $C \subset \mathbb{S}^2$.*

(1) *T is k-thick at ∞ if there exists $s_T \geq 0$ such that, for $s > s_T$,*

$$T \cap ([-ks, -s] \cup [s, ks]) \neq \emptyset \tag{5.1}$$

(2) *T is k-thick if the condition (5.1) is satisfied for all $s > 0$.*
(3) *T is globally k-thick if $T \neq \emptyset$ and $T - t$ is k-thick for every $t \in T$.*
(4) *S is k-uniformly perfect if for any homography $\varphi : C \to \mathbb{R} \cup \{\infty\}$ such that $\varphi^{-1}(\infty) \in S$ the set $\varphi(S) \cap \mathbb{R}$ is globally k-thick.*
(5) *S is uniformly perfect if it is k-uniformly perfect for some $k \geq 1$.*

Let $\Gamma \subset G$ be a Zariski dense discrete subgroup and $M = \Gamma\backslash\mathbb{H}^3$. For $x \in \mathrm{F}M$, let

$$T_x = \{t \in \mathbb{R} : xu_t \in \mathrm{RF}\, M\}$$

be the set of return times of x to the renormalized frame bundle $\mathrm{RF}\, M$ under the horocycle flow. Note that $T_{xu_t} = T_x - t$ for all $t \in \mathbb{R}$.

Define

$$\begin{aligned} \mathrm{RF}_k\, M \quad := \quad &\{x \in \mathrm{RF}\, M : T_x \text{ contains a globally} \\ &\qquad\qquad k\text{-thick subset containing } 0\,\}. \end{aligned} \tag{5.2}$$

It is easy to check as in [28, Proposition 4.2] that $\mathrm{RF}_k\, M$ is a closed A-invariant subset such that for any $x \in \mathrm{RF}_k\, M$, the set

$$T_{x,k} := \{t \in \mathbb{R} : xu_t \in \mathrm{RF}_k\, M\}$$

is globally k-thick.

5.2. **Disjoint Horoballs.** When $\mathcal{h}$ is a horoball in $\mathbb{H}^3$ and R is positive, we denote by $\mathcal{h}_R$ the horoball contained in $\mathcal{h}$ whose distance to the boundary of $\mathcal{h}$ is R. Hence the bigger R is, the deeper the horoball $\mathcal{h}_R$ is.

For an interval $J \subset \mathbb{R}$, we denote by ℓ_J the length of J. The following basic lemma says that "the time spent by a horocycle in a deep horoball $\mathcal{h}_R$ is a small fraction of the time spent in the fixed horoball $\mathcal{h}$".

Lemma 5.2. *For any $\alpha > 0$, there exists $R = R(\alpha) > 0$ such that for any horoball $\mathcal{h}$ in $\mathbb{H}^3$ and any $g \in G$, we have*

$$J_g \pm \alpha \cdot \ell_{J_g} \subset I_g \tag{5.3}$$

where $I_g = \{t \in \mathbb{R} : \pi(gu_t) \in \mathcal{h}\}$ and $J_g = \{t \in \mathbb{R} : \pi(gu_t) \in \mathcal{h}_R\}$.

Proof. We use the upper half-space model of $\mathbb{H}^3$. Since

$$(g\mathcal{h})_R = g\mathcal{h}_R \quad \text{for any } g \in G,$$

we may assume without loss of generality that the horoball $\mathcal{h}$ is defined by $x^2 + y^2 + (z - h)^2 \leq h^2$ and that $\pi(gU)$ is the horizontal line

$$L = \{\pi(gu_t) = (t, y_0, 1) : t \in \mathbb{R}\} \subset \mathbb{H}^3$$

of height one. As I_g and J_g are symmetric intervals, it suffices to show that $\ell_{I_g} \geq (\alpha + 1)\ell_{J_g}$ assuming that $J_g \neq \emptyset$. The boundaries of $\mathcal{h}$ and $\mathcal{h}_R$ are defined by $x^2 + y^2 = 2zh - z^2$ and $x^2 + y^2 = 2zhe^{-R} - z^2$ respectively. Hence the intersections $L \cap \mathcal{h}$ and $L \cap \mathcal{h}_R$ are respectively given by

$$t_1^2 = 2h - 1 - y_0^2 \quad \text{and} \quad t_2^2 = 2he^{-R} - 1 - y_0^2.$$

Note that $\ell_{I_g} = 2|t_1|$ and $\ell_{J_g} = 2|t_2|$. We compute

$$t_2^2 \leq e^{-R} t_1^2.$$

Therefore if we take R so that $e^R > (\alpha + 1)^2$, then we gets $\ell_{I_g} \geq (\alpha + 1)\ell_{J_g}$. ☐

We will now use notation $\mathcal{H}$ and $\mathcal{H}_R$ as in (2.3) and (2.4), even though we have not yet assumed Γ to be geometrically finite: we just fix a collection of open horoballs $\mathcal{h}^i$, $i \in \mathcal{I}$, in $\mathbb{H}^3$ such that $\gamma\mathcal{h}^i$ for $\gamma \in \Gamma$ and $i \in \mathcal{I}$ form a disjoint family of horoballs. We set

$$\mathcal{H} := \{[g] \in F : \pi(g) \in \textstyle\bigcup_{\gamma,i} \gamma\mathcal{h}^i\},$$

$$\mathcal{H}_R := \{[g] \in F : \pi(g) \in \textstyle\bigcup_{\gamma,i} \gamma\mathcal{h}^i_R\}.$$

Corollary 5.3. *Given any $\alpha \geq 0$, there exists $R > 0$ satisfying the following: for all $x \in \mathrm{F}M$, we write the set of return times of xU in $\mathcal{H}$ and $\mathcal{H}_R$ as disjoint unions of open intervals*

$$\{t \in \mathbb{R} : xu_t \in \mathcal{H}\} = \bigcup I_n \quad \text{and} \quad \{t \in \mathbb{R} : xu_t \in \mathcal{H}_R\} = \bigcup J_n \tag{5.4}$$

so that $J_n \subset I_n$ for all n (J_n may be empty). Then

$$J_n \pm \alpha\ell_{J_n} \subset I_n. \tag{5.5}$$

5.3. **Thickness of the set of return times.** The following proposition roughly says that for a point x for which the horocyclic flow returns often to $\mathrm{RF}_k\, M$, its U-flow also returns often to the set

$$W_{k,R} := \mathrm{RF}_k\, M - \mathscr{H}_R.$$

Proposition 5.4. *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup. Then, there exists $R \geq 0$ such that for any $x \in W_{k,R}$, the set*

$$\{t \in \mathbb{R} : xu_t \in W_{k,R}\}$$

is $4k$-thick at ∞.

Proof. Writing $x = [g] \in \mathrm{RF}_k\, M$, we may assume $g^+ = \infty$ and $g^- = 0$ without loss of generality. This means that gU is a horizontal line. It follows from the definition of $\mathrm{RF}_k\, M$ that the set

$$T_{x,k} := \{t \in \mathbb{R} : xu_t \in \mathrm{RF}_k\, M\}.$$

is globally k-thick. Let $R = R(2k)$ be as given by Lemma 5.2. We again use the decomposition (5.4) of the sets of return times as a union of disjoint open intervals: $\{t \in \mathbb{R} : xu_t \in \mathscr{H}\} = \bigcup I_n$ and $\{t \in \mathbb{R} : xu_t \in \mathscr{H}_R\} = \bigcup J_n$ with $J_n \subset I_n$. Since xU is not contained in $\mathscr{H}_R$, the intervals J_n have finite length. Write $\ell_n = \ell_{J_n}$. It follows from Corollary 5.3 that for each n,

$$J_n \pm 2k\ell_n \subset I_n. \tag{5.6}$$

If $x \notin \mathscr{H}$, set $s_x = 0$. If $x \in \mathscr{H}$, let n_x be the integer such that $0 \in I_{n_x}$ and set $s_x = \sup\{|t| \mid t \in I_{n_x}\}$.

We claim that for all $s > s_x$,

$$(T_{x,k} - \bigcup J_n) \cap ([-4ks, -s] \cup [s, 4ks]) \neq \emptyset. \tag{5.7}$$

This means that $\{u \in U : xu \in \mathrm{RF}_k\, M - \mathscr{H}_R\}$ is $4k$-thick at ∞.

Since $T_{x,k}$ is k-thick, there exists

$$t \in T_{x,k} \cap ([-2ks, -2s] \cup [2s, 2ks]).$$

If t belongs to none of these intervals J_n, then claim (5.7) holds. Hence suppose that $t \in J_{n_0}$ for some n_0. By the choice of s_x, we have

$$0 \notin I_{n_0}.$$

By (5.6) and by the fact that I_{n_0} does not contain 0, we have

$$2k\ell_{n_0} \leq |t|.$$

By the global k-thickness of $T_{x,k}$, there exists

$$t' \in T_{x,k} \cap (t \pm [\ell_{n_0}, k\ell_{n_0}]).$$

By (5.6), we have $t' \in I_{n_0}$. Clearly, t' is not in J_{n_0}. And t' belongs to none of these intervals J_n. In order to prove (5.7), it remains to prove $s \leq |t'| \leq 4ks$. For this, note that $|t| - k\ell_{n_0} \leq |t'| \leq |t| + k\ell_{n_0}$. Hence

$$|t'| \leq 2|t| \leq 4ks \quad \text{and} \quad |t'| \geq |t|/2 \geq s,$$

proving the claim. $\square$

Note that in the above proof, we set $s_x = 0$ if $x \notin \mathscr{H}$. Therefore we have the following corollary of the proof:

Corollary 5.5. *For any $x \in \mathrm{RF}_k\, M - \mathscr{H}$, the set*

$$\{t \in \mathbb{R} : xu_t \in \mathrm{RF}_k\, M - \mathscr{H}_R\}$$

is $4k$-thick.

6. Homogeneous dynamics

In this section we explain the polynomial divergence argument for Zariski dense subgroups Γ. The main assumption requires that the set of return times of the horocyclic flow in a suitable compact set is k-thick at ∞.

We begin by the following lemma which is analogous to Lemmas 4.3 and 4.5 and that we will apply to a set T of return times in the proof of Proposition 6.2.

Lemma 6.1. *Let $T \subset U$ be a subset which is k-thick at ∞.*

(1) *If $g_n \to e$ in $G - VH$, then $\limsup_{n\to\infty} Tg_nH$ contains a sequence $v_n \to e$ in $V - \{e\}$.*
(2) *If $g_n \to e$ in $G - AN$, then $\limsup_{n\to\infty} Tg_nU$ contains a sequence $\ell_n \to e$ in $AV - \{e\}$.*

Proof. Lemma 6.1 is a slight modification of [27, Thm 8.1 and 8.2] where it was stated for a sequence T_n of k-thick subsets instead of a single set T. If all T_n are equal to a fixed T, the k-thickness at ∞ is sufficient for the proof. Indeed the same proof works, provided s_{T_n} as defined in (5.1) can be taken uniformly bounded for all T_n. $\square$

For a compact subset W of $\mathrm{F}M = \Gamma\backslash G$, a U-invariant closed subset $Y \subset \Gamma\backslash G$ is said to be *U-minimal with respect to W* if $Y \cap W \neq \emptyset$ and $\overline{yU} = Y$ for any $y \in Y \cap W$. Such a minimal set Y always exists.

Proposition 6.2. *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup with limit set Λ. Let $W \subset F_\Lambda^*$ be a compact subset, and $X \subset \mathrm{F}M$ a closed H-invariant subset intersecting W. Let $Y \subset X$ be a U-minimal subset with respect to W. Assume that*

(1) *there exists $k \geq 1$ such that for any $y \in Y \cap W$, $T_y = \{t \in \mathbb{R} : yu_t \in W\}$ is k-thick at ∞;*
(2) *there exists $y \in Y \cap W$ such that $X - yH$ is not closed;*
(3) *there exists $y \in Y \cap W$ such that yU is not periodic;*
(4) *there exist $y \in Y \cap W$ and $t_n \to +\infty$ such that ya_{-t_n} belongs to W.*

Then

$$X = F_\Lambda.$$

For clarification, we comment that the points y in (2), (3) and (4) need not be the same element.

The proof of this proposition is based on the polynomial divergence argument, which allows us to find an orbit yV^+ sitting inside X. Combined with Corollary 4.1, this implies that X is dense.

Proof. **First step**: We claim that there exist $v_n \to e$ in $V - \{e\}$ such that $Yv_n \subset X$.

We follow the proof of [27, Lem. 9.7]. Let $y \in Y \cap W$ be as in (2). Since yH is not open in X, there exists a sequence $g_n \to e$ in $G - H$ such that $yg_n \in X$.

If $g_n \in VH$, the claim follows easily.

If $g_n \notin VH$, then, by Lemma 6.1, there exist $t_n \in T_y$, $h_n \in H$ such that $u_{-t_n} g_n h_n \to v$ for some arbitrarily small $v \in V - \{e\}$. Since $yu_{t_n} \in Y \cap W$ and $Y \cap W$ is compact, the sequence yu_{t_n} converges to some $y_0 \in Y \cap W$, by passing to a subsequence. It follows that the element $y_0 v = \lim_{n\to\infty} yg_n h_n$ is contained in X. Then the orbit closure $\overline{y_0 vU} = \overline{y_0 U}v$ is included in X and hence $Yv \subset X$ since Y is U-minimal with respect to W. Since v can be taken arbitrarily close to e, this proves our first claim.

Second step: We claim that there exists a one-parameter semigroup $L \subset AV$ such that $YL \subset Y$. Let $y \in Y \cap W$ be as in (3). We follow the proof of [27, Theorem 9.4, Lemma 9.5]. By (1), there exists a sequence $t_n \in T_y$ tending to ∞ such that $yu_{t_n} \to y_0 \in Y \cap W$. Write $yu_{t_n} = y_0 g_n$ where $g_n \to e$. Since yU is not periodic, $g_n \in G - U$.

If $g_n \in AN$, then write $g_n = q_n u_n$ where $q_n \in AV$ and $u_n \in U$. Since $q_n \notin U$, $q_n \neq e$. As $y_0 q_n \in Y$ and Y is U-invariant, $y_0 q_n \in Y$. Then the closed subgroup generated these q_n's contains a one-parameter semigroup L as desired. If $g_n \notin AN$, then, by Lemma 6.1, there exist $s_n \in T_{y_0}$, $u'_n \in U$ such that $u_{-s_n} g_n u'_n \to \ell$ for some non-trivial $\ell \in AV$. Since $y_0 u_{s_n} \in Y \cap W$ and $Y \cap W$ is compact, passing to a subsequence, the sequence $y_0 u_{s_n}$ converges to some $y_1 \in Y \cap W$. It follows that the point

$$y_1 \ell = \lim_{n\to\infty} y_0 g_n u'_n = \lim_{n\to\infty} yu_{t_n} u'_n$$

belongs to Y. Since Y is U-minimal with respect to W, one has the inclusion $Y\ell = \overline{y_1 U}\ell = \overline{y_1 \ell U} \subset Y$.

As ℓ can be taken arbitrarily close to e, this yields the desired one-parameter semigroup L and proves our second claim.

Third step: We claim that there exists an interval V_I of V containing e such that $YV_I \subset X$.

By an interval V_I of V we mean an infinite connected subset of V.

We follow the proof of [28, Theorem 7.1]. We use the second step. A one-parameter semigroup $L \subset AV$ is either a semigroup $V^+ \subset V$, a semigroup $A^+ \subset A$ or $v^{-1}A^+v$ for some non-trivial $v \in V$.

If $L = V^+$, our third claim is clear.

If $L = v^{-1}A^+v$, we have $Yv^{-1}A^+vA \subset X$. Now the set $v^{-1}A^+vA$ contains an interval V_I of V containing e. This proves our third claim.

If $L = A^+$, this semigroup is not transversal to H and we need to also use the first step. Indeed we have

$$Y(\bigcup A^+v_nA) \subset X.$$

Now the closure of $\bigcup A^+v_nA$ contains an interval V_I of V containing e. This proves our third claim.

Fourth step: We claim that there exist $w \in W$ and a one-parameter semigroup $V^+ \subset V$ such that $wV^+ \subset X$.

Here V^+ is a one-parameter semigroup of V intersecting V_I as a non-trivial interval. Let y, t_n be as in (4). By compactness of W, the sequence ya_{-t_n} converges to a point $w \in X \cap W$, by passing to a subsequence. For every $v \in V^+$, there exists a sequence $v_n \in V_I$ such that $a_{t_n}v_na_{-t_n} = v$ for n large. It follows that

$$wv = \lim_{n\to\infty} yv_na_{-t_n} \in X$$

and hence $wV^+ \subset X$. This proves the fourth claim.

Now, since $W \subset F^*_\Lambda$, Corollary 4.1 implies that $X = F_\Lambda$. $\square$

7. H-orbits are locally closed or dense

In this section, we explain how the polynomial divergence argument can be applied to the orbit closure of xH for $x \in \mathrm{RF}_kM \cap F^*$, when Γ is geometrically finite. The main advantage of the assumption that Γ is geometrically finite is that $\mathrm{RF}_k\, M - \mathcal{H}_R$ is compact.

The key result of this section is the following theorem:

Theorem 7.1. *Let $\Gamma \subset G$ be a geometrically finite Zariski dense subgroup. If $x \in \mathrm{RF}_k\, M \cap F^*$ for some $k \geq 1$, then xH is either locally closed or dense in F_Λ.*

In Theorem 7.1 the assumptions on Γ are very general, but we point out that the conclusion concerns only those H-orbits intersecting the set $\mathrm{RF}_kM \cap F^*$.

Fix $R \geq 1$ as given by Proposition 5.4 and set

$$W^*_{k,R} := (\mathrm{RF}_kM - \mathcal{H}_R) \cap F^*.$$

Proposition 7.2. *Let $\Gamma \subset G$ be a geometrically finite Zariski dense subgroup. Let $x \in \mathrm{RF}_k\, M \cap F^*$. If $\overline{xH} \cap W^*_{k,R}$ is not compact, then xH is dense in F_Λ.*

Proof. This follows from Proposition 3.8; see the proof of [28, Coro. 6.2]. $\square$

Proposition 7.3. *Let $\Gamma \subset G$ be a geometrically finite Zariski dense subgroup. Let $x \in \mathrm{RF}_k\, M \cap F^*$. If $\overline{xH} \cap F^*$ contains no periodic U-orbit and $(\overline{xH} - xH) \cap W^*_{k,R}$ is non-empty, then xH is dense in F_Λ.*

Proof. Using Proposition 7.2, we may assume that the set $\overline{xH} \cap W^*_{k,R}$ is compact. By assumption, the set $(\overline{xH} - xH) \cap W^*_{k,R}$ is non-empty.

We introduce the compact set

$$W = \begin{cases} (\overline{xH} - xH) \cap W^*_{k,R} & \text{if } xH \text{ is locally closed} \\ \overline{xH} \cap W^*_{k,R} & \text{if } xH \text{ is not locally closed.} \end{cases} \tag{7.1}$$

Let $Y \subset \overline{xH}$ be a U-minimal subset with respect to W. We want to apply Proposition 6.2. We check that its four assumptions are satisfied:

1. By Proposition 5.4, for any $y \in Y \cap W$, the set $T_y := \{t \in \mathbb{R} : yu_t \in Y \cap W\}$ is $4k$-thick at ∞.

2. We can find $y \in Y \cap W$ such that $\overline{xH} - yH$ is not closed. To see this, if $Y \cap W \not\subset xH$, we choose $y \in Y \cap W - xH$. If $\overline{xH} - yH$ were closed, it would be a proper closed subset of $\overline{xH}$ containing xH, yielding a contradiction.

If $Y \cap W \subset xH$, by (7.1), the orbit xH is not locally closed and we choose any $y \in Y \cap W$.

3. The U-orbit of any point $y \in Y \cap W \subset \overline{xH} \cap F^*$ is never periodic.

4. Since T_y is uncountable while Λ_p is countable, there exists $y_0 := [g_0] \in yU \cap W$ such that g_0^- defined in (2.1) is a radial limit point. Since g_0^- is a radial limit point, there exists a sequence $t_n \to +\infty$ such that $y_0 a_{-t_n} \notin \mathcal{H}_R$. Since both $\mathrm{RF}_k\, M \cap F^*$ and xH are A-invariant, we have $y_0 a_{-t_n} \in W$.

Hence Proposition 6.2 implies that xH is dense in F_Λ. $\square$

Proof of Theorem 7.1. When $\overline{xH} \cap F^*$ contains a periodic U-orbit, the claim follows from Proposition 4.2.

We now assume that $\overline{xH} \cap F^*$ contains no periodic U-orbit. Suppose that xH is not dense in F_Λ. Then, by Proposition 7.3, $\overline{xH} \cap W^*_{k,R} = xH \cap W^*_{k,R}$; in particular, the set

$$W := xH \cap W^*_{k,R} \tag{7.2}$$

is compact. We now repeat almost verbatim the same proof of Proposition 7.3, with this compact set W. Let $Y \subset \overline{xH}$ be a U-minimal subset relative to W. We assume that the orbit xH is not locally closed. The four assumptions of Proposition 6.2 are still valid:

1. For any $y \in Y \cap W$, $\{t : yu_t \in W\}$ is $4k$-thick at ∞.
2. For any $y \in Y \cap W$, the set $\overline{xH} - yH$ is not closed.
3. For any $y \in Y \cap W$, yU is not periodic.
4. Choose any $y = [g] \in Y \cap W$ such that g^- is a radial limit point. Then $ya_{-t_n} \in W$ for some $t_n \to \infty$.

Hence, by Proposition 6.2, the orbit xH is dense in F_Λ, yielding a contradiction. $\square$

8. Locally closed H-orbits have non-elementary stabilizer

In this section we give more information on locally closed H-orbits intersecting the set $\mathrm{RF}_k\, M \cap \mathrm{F}^*$. In particular we show that they have non-elementary stabilizer.

As in the previous section, let $k \geq 1$ and fix $R \geq 0$ as given by Proposition 5.4 and set $W_{k,R}^* := (\mathrm{RF}_k\, M - \mathcal{H}_R) \cap F^*$. The main aim of this section is to prove the following:

Theorem 8.1. *Let $\Gamma \subset G$ be a geometrically finite Zariski dense subgroup. Suppose that $xH \subset \mathrm{F}M$ is a locally closed subset intersecting $\mathrm{RF}_k\, M \cap F^*$. Then*

(1) $\overline{xH} \cap (\mathrm{RF}_k\, M)H \cap F^* \subset xH$, *i.e., xH is closed in $(\mathrm{RF}_k\, M)H \cap F^*$;*
(2) *The stabilizer H_x of x in H is non-elementary.*

In particular, there are only countably many locally closed H-orbits intersecting $\mathrm{RF}_k\, M \cap F^*$.

We remark that the condition that xH intersects $\mathrm{RF}_k\, M \cap F^*$ is necessary in Theorem 8.1. For instance, in the Apollonian gasket $\mathcal{P}$, any circle C whose intersection with $\mathcal{P}$ consists of 3 tangent points of 3 mutually tangent circles inside $\mathcal{P}$ is a separating circle such that ΓC is closed but $\mathrm{Stab}_\Gamma(C)$ is elementary, where $\Gamma = \{g \in G : g(\mathcal{P}) = \mathcal{P}\}$ is the Apollonian group.

We will apply the following lemma with $L = U$.

Lemma 8.2. *Let L be a non-compact one-parameter subgroup of G, and $Y \subset \mathrm{F}M$ be an L-minimal subset with respect to a compact subset $W \subset \mathrm{F}M$. Fix $y \in Y \cap W$ and suppose that $\{\ell \in L : y\ell \in Y \cap W\}$ is unbounded. Then there exists a sequence $\ell_n \to \infty$ in L such that $y\ell_n \to y$.*

Proof. The set $Z := \{z \in Y : \exists \ell_n \to \infty \text{ in } L \text{ such that } y\ell_n \to z\}$ of ω-limit points of the orbit yL is a closed L-invariant subset of Y that intersects W. Therefore, by minimality, it is equal to Y and hence contains y. ☐

We will also need the following lemma:

Lemma 8.3. *For any $R \geq 0$, the set $\mathcal{H}_R$ never contains an A-orbit. In particular, if xH intersects $\mathrm{RF}_k\, M$, it also intersects $W_{k,R}$.*

Proof. The claim follows because no horoball in $\mathbb{H}^3$ contains a complete geodesic and the set $\mathrm{RF}_k\, M$ is A-invariant. ☐

Fix $R \geq 1$ as given by Proposition 5.4. The condition that xH is locally closed implies that the orbit map

$$H_x \backslash H \to xH \tag{8.1}$$

given by $[h] \mapsto xh$ is a proper map when xH is endowed with the induced topology from $\mathrm{F}M$ (cf. [39, Lemma 2.1.15]).

Proof of Theorem 8.1. Let xH be a locally closed set with $x \in \mathrm{RF}_k\, M \cap F^*$.

First case : $\overline{xH} \cap F^*$ contains no periodic U-orbit.

(1) In this case, since xH is locally closed, Propositions 7.2 and 7.3 imply that the intersection $W := \overline{xH} \cap W_{k,R}^*$ is compact and is contained in xH. Since this is true for any R large enough, the intersection $\overline{xH} \cap \mathrm{RF}_k\, M \cap F^*$ is also contained in xH.

(2) As in the proof of Theorem 7.1, we will use the following compact set

$$W = xH \cap W^*_{k,R}.$$

By Lemma 8.3, W is non-empty.

We first construct a sequence of elements δ_n in the stabilizer H_x. Let $Y \subset \overline{xH}$ be a U-minimal subset with respect to W. Fix $y \in Y \cap W$. Since $W \subset xH$, by replacing x by y, we may assume $x = y$ without loss of generality.

By Proposition 5.4, $\{t \in \mathbb{R} : xu_t \in Y \cap W\}$ is unbounded. Hence by Lemma 8.2, one can find a sequence $u_n \to \infty$ in U such that $yu_n \to y$. By the properness of the map in (8.1), there exists a sequence $\delta_n \to \infty$ in H_x such that

$$\delta_n u_n \to e \quad \text{in } H. \tag{8.2}$$

Let $J \subset H$ denote the Zariski closure of H_x. We want to prove $J = H$.

We first claim that $U \subset J$. Since the homogeneous space $J\backslash H$ is real algebraic, any U-orbit in $J\backslash H$ is locally closed [8, 3.18]. Therefore, since the sequence $[e]u_n$ converges to $[e]$ in $J\backslash H$, the stabilizer of $[e]$ in U is non trivial and hence $U \subset J$.

We now claim that $J \not\subset AU$. Indeed, if the elements δ_n were in AU one would write $\delta_n = a_n u_n'$ with $a_n \in A$ and $u_n' \in U$. Since xH contains no periodic U-orbit, the stabilizer H_x does not contain any unipotent element, and hence $a_n \neq e$. Since the sequence $a_n u_n' u_n$ converges to e, the sequence a_n also converges to e. Therefore, δ_n is a sequence of hyperbolic elements of a discrete subgroup of AU whose eigenvalues go to 1. Contradiction.

Since any algebraic subgroup of H containing U but not contained in AU is only H itself, we obtain $J = H$. Therefore H_x is Zariski dense in H.

Second case : $\overline{xH} \cap F^*$ contains a periodic U-orbit yU.

(1) By Proposition 7.2, the set $W := \overline{xH} \cap W_{k,R}$ is compact. We claim that W is contained in xH. If not, the set $(\overline{xH} - xH) \cap W^*_{k,R}$ would contain an element x'. Since xH is locally closed, it is not included in $\overline{x'H}$. By Proposition 4.2, any periodic U-orbit of $\overline{xH}$ is contained in xH. Hence

$$\overline{x'H} \text{ contains no periodic } U\text{-orbit}.$$

Since $x'H$ cannot be dense in F_Λ, Theorem 7.1 implies that $x'H$ is locally closed. By the first case considered above, the stabilizer $H_{x'}$ is non-elementary. Therefore, since $x' \in F^*_\Lambda$, Proposition 3.5 implies that xH is dense in F_Λ. Since xH is locally closed, this is a contradiction. Therefore we obtain:

$$W = xH \cap W^*_{k,R}. \tag{8.3}$$

Since this is true for any R large enough, the intersection $\overline{xH} \cap (\mathrm{RF}_k\, M \cap F^*)$ is contained in xH.

(2) Since xH is locally closed, by Proposition 4.2, yU is contained in xH. This proves that the stabilizer H_x contains a non-trivial unipotent element.

We now construct a non-trivial hyperbolic element $\delta \in H_x$. We write $x = [g]$ and denote by $C = (gH)^+$ the corresponding circle. We again use the compact set W given by (8.3). Since x is in $\mathrm{RF}_k M$, the set

$$\{(gh)^+ \in C : h \in H \text{ with } xh \in W\}$$

is uncountable and hence contains at least two radial limit points. By considering an A-orbit which connects these two limit points, we get $z \in xH$ such that $zA \subset xH \cap (\mathrm{RF}_k M \cap F^*)$. Since the set $\{t \in \mathbb{R} : za_t \in \mathscr{H}_R\}$ is a disjoint union of bounded open intervals, we can find a sequence $a_n \in A$ such that $za_n \in W$ and $a'_n := a_n^{-1} a_{n+1} \to \infty$.

Since $xH \cap W$ is compact, by passing to a subsequence the sequence $z_n := za_n$ converges to a point $z' \in xH \cap W$.

Therefore we have

$$z_n \to z' \quad \text{and} \quad z_n a'_n = z_{n+1} \to z'.$$

By the properness of the map in (8.1), we can write $z_n = z'\varepsilon_n$ with $\varepsilon_n \to e$ in H. Therefore the following product belongs to the stabilizer of z' :

$$\delta_n := \varepsilon_n a'_n \varepsilon_{n+1}^{-1} \in H_{z'}. \tag{8.4}$$

Since $a'_n \to \infty$, the elements δ_n are non-trivial hyperbolic elements of $H_{z'}$ for all n large enough. Since $z' \in xH$, the stabilizer H_x also contains non-trivial hyperbolic elements.

Since a discrete subgroup of H containing simultaneously non-trivial unipotent elements and non-trivial hyperbolic elements is non-elementary, the group H_x is non-elementary.

If $x = [g]$, then $gHg^{-1} \cap \Gamma$ is non-elementary, and hence contains a Zariski dense finitely generated subgroup of gHg^{-1}. Now the last claim on the countability follows because there are only countably many finitely generated subgroups of Γ and H has index two in its normalizer. $\square$

9. A uniformly perfect subset of a circle

In this section we give an interpretation of Theorems 7.1 and 8.1 in terms of Γ-orbits of circles.

According to Section 5.1, the union $\mathrm{RF}_\infty M := \cup_{k \geq 1} \mathrm{RF}_k M$ can be described as

$$\begin{aligned} \mathrm{RF}_\infty M \quad = \quad & \{[g] \in \mathrm{RF}\, M : (gH)^+ \cap \Lambda \supset S \supset \{g^\pm\} \\ & \qquad \text{for a uniformly perfect set } S\ \}. \end{aligned}$$

Hence an H-orbit $[g]H$ intersects $\mathrm{RF}_\infty M$ if and only if the intersection $C \cap \Lambda$ contains a uniformly perfect subset S where C is the circle given by $(gH)^+$.

Putting together Theorems 7.1 and 8.1, we obtain:

Corollary 9.1. *Let $\Gamma \subset G$ be a geometrically finite Zariski dense subgroup. If $x \in \mathrm{RF}_\infty M \cap F^*$, then xH is either locally closed or dense in F_Λ. When xH is locally closed, it is closed in $(\mathrm{RF}_\infty M)H \cap F^*$ and H_x is non-elementary.*

Let $\mathcal{C}_{\mathrm{perf}}$ denote the set of all circles $C \in \mathcal{C}$ such that $C \cap \Lambda$ contains a uniformly perfect subset of C, and let

$$\mathcal{C}^*_{\mathrm{perf}} := \mathcal{C}_{\mathrm{perf}} \cap \mathcal{C}^*.$$

This is a Γ-invariant set and in general, this set is neither open nor closed in $\mathcal{C}$. The following theorem is equivalent to Corollary 9.1.

Theorem 9.2. *Let $\Gamma \subset G$ be a geometrically finite Zariski dense subgroup. For any $C \in \mathcal{C}^*_{\mathrm{perf}}$, the orbit ΓC is either discrete or dense in $\mathcal{C}_\Lambda$. Moreover, a discrete orbit ΓC is closed in $\mathcal{C}^*_{\mathrm{perf}}$ and its stabilizer Γ^C is non-elementary.*

10. H-orbits contained in $\mathrm{RF}\, M$

In this section, we prove Theorem 1.5. We assume that $\Gamma \subset G$ is geometrically finite and of infinite co-volume. We describe the H-orbits xH which are contained in the renormalized frame bundle $\mathrm{RF}\, M$. Equivalently we describe the Γ-orbits ΓC contained in the limit set Λ.

Theorem 10.1. *Let $\Gamma \subset G$ be a geometrically finite subgroup of infinite co-volume and let $x \in \mathrm{F}M$.*

1. *If xH is contained in $\mathrm{RF}\, M$, then xH is closed and has finite volume.*
2. *If $xH \subset \mathrm{F}M$ has finite volume then xH is closed and contained in $\mathrm{RF}\, M$.*
3. *There are only finitely many H-orbits contained in $\mathrm{RF}\, M$.*

This theorem implies that the closed subset

$$F_0 := \{x \in F : xH \subset \mathrm{RF}\, M\}$$

is a union of finitely many closed H-orbits. Equivalently, the closed subset

$$\mathcal{C}_0 := \{C \in \mathcal{C} : C \subset \Lambda\}$$

is a union of finitely many closed Γ-orbits.

We will deduce Theorem 10.1 from the following three lemmas. We write $x = [g]$ and denote by C the corresponding circle $C := (gH)^+$.

Lemma 10.2. *Let $\Gamma \subset G$ be a geometrically finite subgroup of infinite co-volume. Any xH included in $\mathrm{RF}\, M$ is closed.*

Proof. Remember that, by assumption, the circle C is contained in the limit set Λ. Let $F_0^* := F_0 \cap F^*$.

Case 1: $x \notin F_0^*$. In this case the circle C is a boundary circle. This means that C is included in Λ and bounds a disk B in $\Omega = \mathbb{S}^2 - \Lambda$. The surface $\Gamma^C \backslash \mathrm{hull}(C)$ is then a connected component of the boundary of the convex core of M. Hence the orbit xH is closed.

Case 2: $x \in F_0^*$. By Definition 5.2, we have

$$F_0 \subset \mathrm{RF}_k\, M$$

for all $k \geq 1$. Since x belongs to $\mathrm{RF}_k\, M \cap F^*$, Theorems 7.1 and 8.1 imply that either the orbit xH is dense in F_Λ or this orbit is closed in F_0^*.

Since Γ is geometrically finite of infinite covolume, the limit set Λ is not the whole sphere $\mathbb{S}^2$ and therefore $\mathrm{RF}\, M$ is strictly smaller than F_Λ. Therefore the orbit xH cannot be dense in F_Λ.

Hence xH is closed in F_0^*. If xH were not closed in F_0, there would exist an H-orbit $yH \subset \overline{xH} \subset F_0$ which does not lie in F_0^*. By Case 1, the orbit yH corresponds to a boundary circle and hence it is of finite volume. Then by Corollary 3.4, the orbit xH has to be dense in F_Λ. Contradiction. □

Lemma 10.3. *Let $\Gamma \subset G$ be a geometrically finite subgroup. Any closed xH contained in* $\mathrm{RF}\, M$ *has finite volume.*

Proof. Since xH is closed, the inclusion map $xH \to \mathrm{RF}\, M$ is proper, and the the corresponding map $\Gamma^C \backslash C \to \mathrm{core} M$, say, f, is also proper. The properly immersed geodesic plane S, which is the image of f, is geometrically finite by [30, Theorem 4.7]; so its convex core $\mathrm{core} S$ has finite area.

We now claim that $\Lambda(\Gamma^C) = C \cap \Lambda$. As $xH \subset \mathrm{RF}\, M$, we have $C = C \cap \Lambda$. Therefore the claim implies $S = \mathrm{core} S$, finishing the proof. As $\Lambda_r(\Gamma^C) = C \cap \Lambda_r$ by [30, Lemma 4.5], it suffices to show that $C \cap \Lambda_p \subset \Lambda(\Gamma^C)$. Suppose that there exists $\xi \in (C \cap \Lambda_p) - \Lambda(\Gamma^C)$. Using the upper-half space model of $\mathbb{H}^3 = \{(x, y, z) : z > 0\}$, we may assume $\xi = \infty$, and hence C is a vertical plane. As $\xi \notin \Lambda(\Gamma^C)$, there exists $r > 1$ such that the restriction of the canonical projection $p : C \to \Gamma^C \backslash C$ to the half-plane $C \cap \{(x, y, z) : z \geq r\}$ is a proper map. On the other hand, as ∞ is a bounded parabolic fixed point for Γ, the image of the line $C \cap (\mathbb{R}^2 \times \{r\})$ is bounded in $\mathrm{core} M$ under $f \circ p$. This is a contradiction to the properness of f. □

Lemma 10.4. *Let $\Gamma \subset G$ be a discrete group with limit set $\Lambda \neq \mathbb{S}^2$. Then the following set $\mathcal{C}_a$ is discrete in $\mathcal{C}$:*

$$\mathcal{C}_a := \{C \in \mathcal{C} : \Gamma^C \backslash \mathrm{hull}(C) \textit{ is of finite area}\}.$$

Proof. Suppose $C \in \mathcal{C}_a$ is not isolated. Then there exists a sequence of distinct circles $C_n \in \mathcal{C}_a$ converging to C. Since $\Lambda(\Gamma^C) = C$, by Corollary 3.4, the closure of $\bigcup \Gamma C_n$ contains $\mathcal{C}_\Lambda$. Since $C_n \subset \Lambda$, $\mathcal{C}_\Lambda$ is contained in Λ. On the other hand, the closure of the set of all circles in $\mathcal{C}_\Lambda$ is equal to $\mathbb{S}^2$. Therefore $\mathbb{S}^2 = \Lambda$, contradicting hypothesis. □

Proof of Theorem 10.1. (1) follows from Lemmas 10.2 and 10.3.

(2) Since the quotient $\Gamma^C \backslash \mathrm{hull}(C)$ has finite area, the limit set $\Lambda(\Gamma^C)$ is equal to C and the orbit xH is contained in $\mathrm{RF}\, M$. By (1), xH is closed.

(3) By (1) and (2) we have the equality $\mathcal{C}_0 = \mathcal{C}_a$. Therefore by Lemma 10.4, the set $\mathcal{C}_0$ is closed and discrete.

Now since Γ is geometrically finite, by Lemma 8.3, any H-orbit contained in $\mathrm{RF}\, M$ intersects the compact set $W := \mathrm{RF}\, M - \mathscr{H}$, where $\mathscr{H}$ is equal to $\mathscr{H}_R$ with $R = 0$. Hence there exists a compact set $\mathcal{K} \subset \mathcal{C}$ such that, for any circle $C \in \mathcal{C}_0$, there exists γ in Γ such that γC is in $\mathcal{K}$. Since the intersection $\mathcal{C}_0 \cap \mathcal{K}$ is discrete and compact, it is finite. Therefore Γ has only finitely many orbits in $\mathcal{C}_0$, and there are only finitely many H-orbits in F_0. □

11. Planes in geometrically finite acylindrical manifolds

In this section, we assume that $M = \Gamma\backslash\mathbb{H}^3$ is a geometrically finite acylindrical manifold, and we prove Theorems 1.1 and 1.2. Under this assumption, every separating H-orbit xH intersects $\mathrm{RF}_k\, M \cap F^*$ and hence $W^*_{k,R}$ for k large enough (Corollary 11.7) which enables us to apply Theorems 7.1 and 8.1.

11.1. **Geometrically finite acylindrical manifolds.** We first want to define the acylindrical geometrically finite hyperbolic 3-manifolds.

We begin with the definition of an acylindrical compact manifold. Let D^2 denote a closed 2-disk and let $C^2 = S^1 \times [0, 1]$ be a cylinder. A compact 3-manifold N with boundary ∂N is called *acylindrical*

(1) if the boundary of N is incompressible, i.e., any continuous map $f : (D^2, \partial D^2) \to (N, \partial N)$ can be deformed into ∂N or equivalently if the inclusion $\pi_1(R) \to \pi_1(N)$ is injective for any component R of ∂N; and
(2) if any essential cylinder of N is boundary parallel, i.e., any continuous map $f : (C^2, \partial C^2) \to (N, \partial N)$, injective on π_1, can be deformed into ∂N.

Now let $M = \Gamma\backslash\mathbb{H}^3$ be a geometrically finite manifold, and consider the associated Kleinian manifold

$$\overline{M} = \Gamma\backslash(\mathbb{H}^3 \cup \Omega).$$

Recall that a compact connected submanifold N of codimension 0 in M is called a *compact core of* M if the inclusion $\pi_1(N) \to \pi_1(M)$ is an isomorphism and each component of the boundary ∂N is equal to the full boundary of a non-compact component of $M - N$ [20, 3.12]. This is sometimes called the Scott core of M and is unique up to a homeomorphism.

It is convenient to choose an explicit model of the compact core of M. As M is geometrically finite, $\overline{M}$ is compact except for a finite number of bounded rank one cusps and of rank two cusps. The bounded rank one cusps correspond to pairs of punctures on $\partial\overline{M}$ which are arranged so that each pair determines a solid pairing tube in $\overline{M}$, and the rank two cusps determine solid cusp tori in $\overline{M}$ (see [20, §3.6]). We can choose them to be disjoint. A choice $N_0 \subset M$ of the compact core N of M can be obtained by removing from the convex core of M the interiors of these solid pairing tubes and solid cusp tori. This compact core N_0 can also be seen as the thick part of the convex core of M.

The boundary ∂N_0 is then a closed surface whose components have genus at least 1. The pairing cylinders, which are the boundary of the solid pairing tubes, give rise to "marked" cylinder P_k that sit on the union S of components of ∂N_0 of genus at least 2. The cusp tori, which are the boundary of the solid cusp tori, are the components of ∂N_0 of genus 1. Note that

$$\text{the surface } S - \cup P_k \text{ is homeomorphic to } \Gamma\backslash\Omega = \partial\overline{M}. \tag{11.1}$$

We say that a geometrically finite manifold M is *acylindrical* if its compact core N_0 is acylindrical ([35], [20, §4.7])

Remark 11.1. The Apollonian manifold $\mathcal{A}\backslash\mathbb{H}^3$ is not acylindrical, because its compact core is a handle body of genus 2, and hence it is not boundary incompressible.

For the rest of this section, we let

$$\begin{aligned} M &= \Gamma\backslash\mathbb{H}^3 \quad \text{be a geometrically finite} \\ & \quad\ \text{acylindrical manifold of infinite volume.} \end{aligned}$$

Write the domain of discontinuity $\Omega = \mathbb{S}^2 - \Lambda$ as a union of components:

$$\Omega = \cup_i B_i,$$

and let Δ_i denote the stabilizer of B_i in Γ.

Recall that a quasi-fuchsian group is a Kleinian group which leaves a Jordan curve invariant. A finitely generated quasi-fuchsian group whose limit set is a Jordan curve is a quasi-conformal deformation of a lattice of $\mathrm{PSL}_2(\mathbb{R})$ and hence its limit set is a quasi-circle.

Lemma 11.2. *Let M be a geometrically finite acylindrical manifold of infinite volume.*

1. *For each i, Δ_i is a finitely generated quasi-fuchsian group and B_i is an open Jordan disk with $\partial B_i = \Lambda(\Delta_i)$.*
2. *There are countably infinitely many B_i's with finitely many Γ-orbits and $\Lambda = \overline{\bigcup_i \partial B_i}$.*
3. *For each $i \neq j$, $\overline{B_i} \cap \overline{B_j}$ is either empty or a parabolic fixed point of rank one; and any rank one parabolic fixed point of Γ arises in this way.*
4. *No subset of $\{\overline{B}_i\}$ forms a loop of tangent disks, i.e., if $B_{i_1}, \cdots, B_{i_\ell}$, $\ell \geq 3$ is a sequence of distinct disks such that $\overline{B}_{i_j} \cap \overline{B}_{i_{j+1}} \neq \emptyset$ for all j, then $\overline{B_{i_\ell}} \cap \overline{B_{i_1}} = \emptyset$.*

When all B_i are round open disks, M is called rigid acylindrical.

Proof. For (1), we first claim that the limit set Λ is connected or, equivalently, that each B_i is simply connected. Indeed the quotient $\Delta_i\backslash B_i$ is homeomorphic to a component S_i of $S - \cup P_k$. Since N_0 is boundary incompressible, Δ_i is the fundamental group of S_i and B_i is simply connected.

By Ahlfors [3, Lemma 2], the group Δ_i is finitely generated and B_i is a component of $\Omega(\Delta_i)$. Since Δ_i has two invariant components, it is quasi-fuchsian and $\partial B_i = \Lambda(\Delta_i)$ (cf. [23, Theorem 3]).

For (2), since Γ is not quasi-fuchsian by the acylindricality assumption, there are countably infinitely many B_i's. There are only finitely many Γ-orbits of B_i's by Ahlfors' finiteness theorem. Since $\bigcup_i \partial B_i$ is a Γ-invariant subset of Λ, the last claim in (2) is clear.

For (3), note that as Δ_i is a component subgroup of Γ, we have by [24, Theorem 3],

$$\Lambda(\Delta_i \cap \Delta_j) = \Lambda(\Delta_i) \cap \Lambda(\Delta_j). \tag{11.2}$$

On the other hand, as N_0 is acylindrical, a loxodromic element can preserve at most one of the B_i's. As no rank two parabolic fixed point lies in $\bigcup_i \partial B_i$, it follows that for all $i \neq j$, $\Delta_i \cap \Delta_j$ is either trivial or the stabilizer of a parabolic limit point of Γ of rank one. Therefore the first claim in (3) follows from (1) and (11.2). The second claim in (3) follows from a standard fact about geometrically finite groups [20, §3].

For (4), suppose not. Then, using the identifications (11.1), one gets a non-trivial loop on ∂N_0 which bounds a disk in N_0, contradicting the boundary-incompressible condition on N_0. □

11.2. **Lower bounds for moduli of annuli and Cantor sets.** We have seen that the closures of the components of Ω may intersect with each other. The aim of this section is to regroup the components of Ω into maximal trees of disks and to show that the closures of these maximal trees of disks are not only disjoint but are uniformly apart in the language of moduli.

For a closed surface S' of ∂N_0 of genus at least 2, its fundamental group $\pi_1(S')$ injects to Γ as S' is an incompressible surface. We have already seen that S' comes with finitely many marked cylinders P_k's and each connected component of $S' - \cup P_k$ corresponds to a component of $\partial \overline{M}$.

For two connected components of Ω, we write $B_i \sim B_j$ if their closures intersect each other. This spans an equivalence relation on the collection $\{B_i\}$. We write

$$\Omega = \bigcup T_\ell$$

where T_ℓ is the union of all disks in the same equivalence class, i.e., T_ℓ is a maximal tree of disks.

The T_ℓ's fall into finitely many Γ-orbits corresponding to the components S' of genus at least 2 of the surface ∂N_0. Moreover, the stabilizer Γ_ℓ of T_ℓ in Γ is conjugate to $\pi_1(S_\ell) \subset \Gamma$ for the corresponding component S_ℓ. We refer to [1], [14], [36] for various aspects on these manifolds.

The following lemma is a crucial ingredient of Theorem 11.5.

Lemma 11.3. *Let M be a geometrically finite acylindrical manifold of infinite volume. We have:*

1. *For each ℓ, $\partial T_\ell = \Lambda(\Gamma_\ell)$.*
2. *For each ℓ, $\overline{T}_\ell$ is connected, locally contractible and simply connected.*
3. *For each $k \neq \ell$, $\overline{T}_k \cap \overline{T}_\ell = \emptyset$.*
4. *For each ℓ, the subgroup Γ_ℓ has infinite index in Γ.*

Proof. For (1), we set $K := \overline{T_\ell}$, and consider the set

$$R_\ell := \{p_{ij} = \overline{B_i} \cap \overline{B_j} : i \neq j, B_i, B_j \subset T_\ell\}.$$

Each p_{ij} is fixed by a parabolic element, say, $\gamma_{ij} \in \Gamma_\ell$.

By the uniformization theorem, $\pi_1(S_\ell)$ can be realized as a cocompact lattice Σ_ℓ in $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{H}^2$ as isometries. Let $c_{ij} \in \mathbb{H}^2$ be the geodesic stabilized by a hyperbolic element $\iota(\gamma_{ij})$ of Σ_ℓ for the isomorphism $\iota : \Gamma_\ell \simeq \Sigma_\ell$. Let K_0 be the compact set obtained from $\overline{\mathbb{H}^2}$ by collapsing geodesic arcs c_{ij} to single points, for all the points p_{ij} in R_ℓ. We denote by ∂K_0 the image of S^1 in this collapsing process, so that K_0 is an "abstract tree of disks" while ∂K_0 is an "abstract tree of circles".

As Γ_ℓ is a finitely generated subgroup of a geometrically finite group Γ whose covolume is infinite, this subgroup Γ_ℓ itself is geometrically finite by a theorem of Thurston (cf. [25, Theorem 3.11]). So we can apply a theorem of Floyd [14] to Γ_ℓ to obtain a continuous equivariant surjective map $\psi_\ell : S^1 = \partial\mathbb{H}^2 \to \Lambda(\Gamma_\ell)$ conjugating Σ_ℓ to Γ_ℓ, which is 2 to 1 onto rank one parabolic fixed points of Γ_ℓ and injective everywhere else. Moreover, this map ψ_ℓ factors through the map $S^1 \to \partial K_0$ described above, since each p_{ij} is a rank one parabolic fixed point of Γ_ℓ. It follows that $\Lambda(\Gamma_\ell)$ is equal to the closure of $\cup_{B_j \subset T_\ell} \partial B_j$, and hence $\Lambda(\Gamma_\ell) = \partial K$ follows as claimed in (1).

For (2), we denote by $\psi_\ell^* : \partial K_0 \to \partial K$ the continuous surjective map induced by ψ_ℓ. We claim that ψ_ℓ^* is injective by the acylindrical condition on N_0. Suppose not. Then there exists a point $p \in \Lambda(\Gamma_\ell) - \cup_j \partial B_j$ over which ψ_ℓ^* is not injective. By the aforementioned theorem of Floyd, p is a fixed point of a parabolic element $\gamma \in \Gamma_\ell$ and the geodesic in $\mathbb{H}^2$ whose two end points are mapped to p by ψ_ℓ is stabilized by a hyperbolic element of Σ_ℓ corresponding to γ. Hence this gives rise to a non-trivial closed curve α in $S_\ell \subset \partial N_0$ which is homotopic to a boundary component β of a pairing cylinder in ∂N_0. Since $p \notin \cup_j \partial B_j$, α and β are not homotopic within S_ℓ, yielding an essential cylinder in N_0 which is not boundary parallel. This contradicts the acylindrical condition on N_0. Therefore ψ_ℓ^* is injective.

Now the map $\psi_\ell^* : \partial K_0 \mapsto \partial K$ can be extended to a continuous injective map $K_0 \mapsto K$ inducing homeomorphisms between each $\overline{B}_{0,j}$ and $\overline{B}_j$ where $B_{0,j}$ denotes the connected component of $K_0 - \partial K_0$ above B_j. Hence K_0 and K are homeomorphic. This implies claim (2).

For (3), note that for all $\ell \neq k$,

$$\Gamma_\ell \cap \Gamma_k = \{e\}$$

by the acylindrical condition on N_0.

By [25, Theorem 3.14], we have

$$\Lambda(\Gamma_\ell) \cap \Lambda(\Gamma_k) = \Lambda(\Gamma_\ell \cap \Gamma_k) \cup P$$

where P is the set of $\zeta \in \mathbb{S}^2 - \Lambda(\Gamma_\ell \cap \Gamma_k)$ such that $\mathrm{Stab}_{\Gamma_\ell} \zeta$ and $\mathrm{Stab}_{\Gamma_k} \zeta$ generates a parabolic subgroup of rank 2 and $\mathrm{Stab}_{\Gamma_\ell} \zeta \cap \mathrm{Stab}_{\Gamma_k} \zeta = \{e\}$. If $\zeta \in P$, then ζ, being a parabolic fixed point of Γ_ℓ, must arise as the intersection $\overline{B_i} \cap \overline{B}_j$ for some $i \neq j$ by Floyd's theorem as discussed above. But every such point is a rank one parabolic fixed point of Γ by Lemma

11.2(3); so $P = \emptyset$. Hence we deduce that $\Lambda(\Gamma_\ell) \cap \Lambda(\Gamma_k) = \emptyset$, which implies claim (3) by claim (1).

Claim (4) follows from the fact that the group Γ_ℓ is a geometrically finite group which is not acylindrical. Indeed, the compact core of Γ_ℓ is homeomorphic to a product $N_\ell \sim [0,1] \times S_\ell$ where S_ℓ is a compact surface of genus at least 2. The boundary ∂N_ℓ of this compact core is a union of two copies of S_ℓ. One of them contains pairing cylinders, the other does not. $\square$

An annulus $A \subset \mathbb{S}^2$ is an open region whose complement consists of two components.

Corollary 11.4. *Let M be a geometrically finite acylindrical manifold of infinite volume. Then,*

1. *for each ℓ, $\mathbb{S}^2 - \overline{T_\ell}$ is a disk,*
2. *for each $\ell \neq k$, $\mathbb{S}^2 - (\overline{T_\ell} \cup \overline{T_k})$ is an annulus.*

Proof. We may assume without loss of generality that $\overline{T_\ell}$ contains ∞ in $\hat{\mathbb{C}} = \mathbb{S}^2$. Now by (2) of Lemma 11.3, and by Alexander duality (cf. [11, §3, Theorem 3.44]), $U := \mathbb{S}^2 - \overline{T_\ell}$ is a connected open subset of $\mathbb{C}$. By the Riemann mapping theorem [2, §6.1], a connected open subset of $\mathbb{C}$ whose complement in $\mathbb{S}^2$ is connected is a disk. As $\overline{T_\ell}$ is connected, it follows that U is a disk, proving (1).

Claim (2) follows from (1) and Lemma 11.3(3). $\square$

When neither component of $\mathbb{S}^2 - A$ is a point, an annulus A is conformally equivalent to a unique round annulus $\{z \in \mathbb{C} : 1 < |z| < R\}$, and its modulus mod (A) is defined to be $\frac{\log R}{2\pi}$ [17].

Theorem 11.5. *Let M be a geometrically finite acylindrical manifold of infinite volume. Then there exists $\delta > 0$ such that*

$$\inf_{\ell \neq k} \quad \text{mod } (\mathbb{S}^2 - (\overline{T_\ell} \cup \overline{T_k})) \geq \delta. \tag{11.3}$$

Proof. Suppose that the claim does not hold. Since there are only finitely many Γ-orbits of T_ℓ's, and the Γ-action is conformal, without loss of generality, we may assume that there exists T_{k_0} and T_{ℓ_0} such that for some infinite sequence $T_\ell \in \Gamma(T_{\ell_0})$, mod $(\mathbb{S}^2 - (\overline{T_{k_0}} \cup \overline{T_\ell})) \to 0$ as $\ell \to 0$. For ease of notation, we set $k_0 = 1$.

Consider the disk $V_1 := \mathbb{S}^2 - \overline{T_1}$. Since Γ_1 is the stabilizer of T_1 with $\Lambda(\Gamma_1) = \overline{T}_1 - T_1$, Γ_1 acts on V_1 properly discontinuously. Hence by the uniformization theorem together with the fact that Γ_1 is conjugate to $\pi_1(S)$ of a non-toroidal closed surface S of ∂N_0, $V := \Gamma_1 \backslash V_1$ is a closed hyperbolic surface with $\pi_1(V) = \Gamma_1$.

Since T_ℓ's have disjoint closures, each T_ℓ maps injectively into V. We denote its image by T'_ℓ.

We claim that the diameter of T'_ℓ, measured in the hyperbolic metric of V, goes to 0 as $\ell \to \infty$. As $T_\ell \in \Gamma(T_{\ell_0})$ by assumption, we may write $T_\ell = \delta_\ell(T_{\ell_0})$ for $\delta_\ell \in \Gamma$. Since Γ_1 acts cocompactly on V_1, there is a compact fundamental domain, say F in V_1, such that $\gamma_\ell(\partial T_\ell) \cap F \neq \emptyset$ for some $\gamma_\ell \in \Gamma_1$. Suppose that the diameter of T_ℓ in the hyperbolic metric of V_1 does not tend to 0. Then by the compactness of F, up to passing to a subsequence, $\gamma_\ell(\partial T_\ell) = \gamma_\ell \delta_\ell(\partial T_{\ell_0})$ converges to a closed set $L \subset \Lambda$ in the Hausdorff topology where L has positive diameter and intersects F non-trivially. In particular, $\mathrm{hull}(L) \cap \mathbb{H}^3 \neq \emptyset$. Fixing $x_0 \in \mathrm{hull}(L) \cap \mathbb{H}^3$, we have a sequence $s_\ell \in \mathrm{hull}(\partial T_{\ell_0}) \cap \mathbb{H}^3$ such that $\gamma_\ell \delta_\ell(s_\ell) \to x_0$. Moreover, we can find $\gamma'_\ell \in \Gamma_{\ell_0}$ so that $(\gamma'_\ell)^{-1} s_\ell$ belongs to a fixed compact subset, say, K, of $\mathbb{H}^3$. Indeed, if the injectivity radius of x_0 is $\epsilon > 0$, then all s_ℓ should lie in the $\epsilon/2$-thick part of the convex hull of $\Lambda(\Gamma_{\ell_0})$ on which Γ_{ℓ_0} acts cocompactly as it is geometrically finite.

Hence $\gamma_\ell \delta_\ell \gamma'_\ell(K)$ accumulates on a neighborhood of x_0. As the group Γ acts properly discontinuously on $\mathbb{H}^3$, this means that $\{\gamma_\ell \delta_\ell \gamma'_\ell\}$ is a finite set, and hence $\gamma_\ell T_\ell = \gamma_\ell \delta_\ell \gamma'_\ell T_{\ell_0}$ is a finite set. It follows that, up to passing to a subsequence, for all ℓ, $\gamma_\ell T_\ell$ is a constant sequence, containing a point in F. As T_ℓ's are disjoint, $\gamma_\ell \in \Gamma_1$ must be an infinite sequence. On the other hand, if γ_ℓ is an infinite sequence, $T_\ell \cap \gamma_\ell^{-1} F \neq \emptyset$ implies that $\overline{T}_\ell \cap \Lambda(\Gamma_1) \neq \emptyset$, yielding a contradiction. This proves the claim.

Now if the diameter of T'_ℓ is smaller than one quarter of the injectivity radius of V, then we can find a disk D_ℓ in V containing T'_ℓ whose diameter is 1/2 of the injectivity radius of V; this gives a uniform lower bound, say, δ_0 for the modulus of $\mathbb{S}^2 - (\overline{T_1} \cup \overline{T_\ell})$ for all $\ell \geq \ell_0$ for some $\ell_0 > 1$, contradicting our assumption. This proves the theorem. □

If K is a Cantor set in a circle C, we say K has modulus ϵ if

$$\inf_{i \neq j} \ \mathrm{mod}\,(\mathbb{S}^2 - (\overline{I}_i \cup \overline{I}_j)) \geq \epsilon \tag{11.4}$$

where $C - K = \bigcup I_j$ is a disjoint union of intervals with disjoint closures. Note that a Cantor set $K \subset C$ has a positive modulus if and only if K is a uniformly perfect subset as in Definition 5.1. Indeed the relationship between "positive modulus" and "global thickness of return time" is explained in [28, Propositions 3.2 and 4.3].

Theorem 11.6. *Let M be a geometrically finite acylindrical manifold of infinite volume. Then for any $C \in \mathcal{C}^*$, $C \cap \Lambda$ contains a Cantor set of modulus δ' where $\delta' > 0$ depends only on δ as in* (11.3).

Proof. This can be proved by a slight adaptation of the proof of [28, Theorem 3.4], in view of Theorem 11.5. Recalling $\Omega = \bigcup_{\ell=1}^{\infty} T_\ell$, we write $U := C - \Lambda = \cup U_\ell$ where $U_\ell = C \cap T_\ell$. Note that distinct U_ℓ have disjoint closures. We may assume that U is dense without loss of generality. We will say that an open interval $I = (a, b) \subset C$ with $a \neq b$ is a bridge of type ℓ if $a, b \in \partial U_\ell$. We can then construct a sequence of bridges I_j with disjoint closures precisely in the same way as in proof of [28, Theorem 3.4]. We first assume that $|I_1| > |C|/2$ by using a conformal map $g \in G^C$. We let I_2 be a bridge of maximal length among all those which are disjoint from I_1 and of a different type from I_1. We then enlarge I_1 to a maximal bridge of the same type disjoint from I_2. For $k \geq 3$, we proceed inductively to define I_k to be a bridge of maximal length among all bridges disjoint from $I_1, \cdots, I_{k-1}$. We have $|I_1| \geq |I_2| \geq |I_3| \cdots, |I_k| \to 0$ and $\cup I_k$ is dense in C. Now $K := C - \cup I_k$ is the desired Cantor set.

Pick two indices $i < j$. If the intervals I_i and I_j have same type, then mod $(\mathbb{S}^2 - (\overline{I}_i \cup \overline{I}_j)) \geq \delta'$ for some universal constant $\delta' > 0$, because there must be a bridge I_k with $1 < k < i$ such that $I_1 \cup I_k$ separates I_i from I_j. Now if I_i and I_j have different types, say ℓ and k, then as the annulus $\mathbb{S}^2 - (\overline{T}_\ell \cup \overline{T}_k)$ separates ∂I_i from ∂I_j, we have by [17, Ch II, Thm 1.1],

$$\text{mod } (\mathbb{S}^2 - (\overline{I}_i \cup \overline{I}_j)) \geq \quad \text{mod } (\mathbb{S}^2 - (\overline{T}_\ell \cup \overline{T}_k)) \geq \delta$$

proving the claim. □

11.3. **Applications to H-orbits.** We now reformulate this information in terms of H-orbits.

Corollary 11.7. *Let $M = \Gamma \backslash \mathbb{H}^3$ be a geometrically finite acylindrical manifold. Then for all sufficiently large $k > 1$,*

$$F^* \subset (\mathrm{RF}_k M) H.$$

In particular, every H-orbit in F^* intersects $W_{k,R}$ for any $R \geq 0$.

Proof. Every H-orbit xH in F^* corresponds to a separating circle C. By Theorem 11.6, there exists $\delta' > 0$ depending only on M such that $C \cap \Lambda$ contains a Cantor set which has modulus δ'. Therefore, for some $k > 1$ depending only on δ', this intersection $C \cap \Lambda$ is k-uniformly perfect. This tells us that xH meets $\mathrm{RF}_k M$.

By Lemma 8.3 it also meets $W_{k,R}$ for any $R \geq 0$. □

Theorem 11.8. *Let $M = \Gamma \backslash \mathbb{H}^3$ be a geometrically finite acylindrical manifold.*

1. *For $x \in F^*$, xH is either closed in F^* or dense in F_Λ.*
2. *If xH is closed in F^*, the stabilizer $H_x = \{h \in H : xh = x\}$ is non-elementary.*
3. *There are only countably many closed H-orbits in F^*.*
4. *Any H-invariant subset in F^* is either a union of finitely many closed H-orbits in F^*, or is dense in F_Λ.*

Proof. Claim (1) is immediate from Corollary 9.1 and Corollary 11.7.

The statements (2) and (3) follow from Theorem 8.1 combined with Corollary 11.7.

To prove (4), let $E \subset F^*$ be an H-invariant subset, and let X be the closure of E. Suppose X is not F_Λ. Then every H-orbit in E is closed in F^*. We claim that there are only finitely many H-orbits in E. Suppose not. Then by Theorem 11.8.3, the set E consists of infinitely and countably many distinct closed H-orbits in F^*, so $E = \bigcup_{n \geq 1} x_n H$. We claim that $X = F_\Lambda$, which would yield a contradiction. Let $K > 1$ and $R \geq 0$ as in Corollary 11.7 so that we may choose the points x_n in these H-orbits inside $W^*_{k,R}$.

As in Proposition 7.2, the set $X \cap W^*_{k,R}$ is compact, and hence, after extraction, the sequence x_n has a limit point, say, $x_\infty \in X \cap W^*_{k,R}$. We may assume that x_∞ does not lie in E by replacing E by $E - x_\infty H$ if necessary.

When $X \cap F^*$ does not contain periodic U-orbit, since the set $(X - E) \cap W^*_{k,R}$ is non-empty, we can repeat the proof of Proposition 7.3 to get $X = F_\Lambda$.

When $X \cap F^*$ contains a periodic U-orbit, we apply the following Lemma 11.9 and also get $X = F_\Lambda$. These contradictions finish the proof of (4). $\square$

We have just used the following variation of Proposition 4.2.

Lemma 11.9. *Let $\Gamma \subset G$ be a Zariski dense discrete subgroup and $E \subset F^*_\Lambda$ be a H-invariant subset. Assume that $\overline{E} \cap F^*$ contains a periodic U-orbit yU. Then either*

(1) *yH is locally closed and $yH \subset E$; or*
(2) *E is dense in F_Λ.*

Proof. Same proof as Proposition 4.2. $\square$

Theorem 1.2 is a special case of the following theorem:

Theorem 11.10. *Let $M = \Gamma\backslash\mathbb{H}^3$ be a geometrically finite rigid acylindrical manifold. For any $x \in F^*$, the orbit xH is either closed or dense in F_Λ.*

Proof. We suppose xH is not closed in F_Λ. By Theorem 11.8, it suffices to consider the case where xH is closed in F^*_Λ. This means that there exists an H-orbit $yH \subset \overline{xH}$ which is not contained in F^*. We write $y = [g]$ and set $C = (gH)^+$ to be the corresponding circle. Since this circle C is non-separating, it is contained in the closure of a component B_0 of the domain of discontinuity Ω and is tangent to its boundary ∂B_0. Since M is rigid, this boundary ∂B_0 is a circle whose stabilizer Γ^{B_0} acts on B_0 with a finite area fundamental domain. Therefore, by Corollary 3.4, this forces $\overline{\Gamma C} = \mathcal{C}_\Lambda$. In terms of H-orbits, this means that $\overline{xH} = F_\Lambda$. $\square$

12. Arithmetic and Non-arithmetic manifolds

In this section, we will prove Theorem 1.3 and present a counterexample to it when M_0 is not arithmetic.

A good background reference for this section is [19].

12.1. **Proof of Theorem 1.3.** Recall $G = \mathrm{PSL}_2(\mathbb{C})$ and $H = \mathrm{PSL}_2(\mathbb{R})$. A real Lie group $\tilde{G} \subset \mathrm{GL}_N(\mathbb{R})$ is said to be a real algebraic group defined over $\mathbb{Q}$ if there exists a polynomial ideal $I \subset \mathbb{Q}[a_{ij}, \det(a_{ij})^{-1}]$ in N^2+1 variables such that

$$\tilde{G} = \{(a_{ij}) \in \mathrm{GL}_N(\mathbb{R}) : p(a_{ij}, \det(a_{ij})^{-1}) = 0 \text{ for all } p \in I\}.$$

Let $\Gamma_0 \subset G$ be an arithmetic subgroup. This means that there exist a semisimple real algebraic group $\tilde{G} \subset \mathrm{GL}_N(\mathbb{R})$ defined over $\mathbb{Q}$ and a surjective real Lie group homomorphism $\psi : \tilde{G} \to G$ with compact kernel such that Γ_0 is commensurable with $\psi(\tilde{G} \cap \mathrm{GL}_N(\mathbb{Z}))$ (cf. [6]).

Let $\Gamma \subset \Gamma_0$ be a geometrically finite acylindrical group, and denote by $p : \Gamma\backslash G \to \Gamma_0\backslash G$ the canonical projection map. In order to prove Theorem 1.3, we first note that the orbit xH is either closed in F^* or dense in F_Λ by Theorem 11.8.

Suppose that xH is closed in F^*. Write $x = [g]$ and let $C := (gH)^+$ be the corresponding circle. Then, by Theorem 11.8, the stabilizer Γ^C of C in Γ is a non-elementary subgroup and hence is Zariski dense in G^C. Hence the group $\psi^{-1}(G^C)$ is defined over $\mathbb{Q}$ and the group $\psi^{-1}(\Gamma_0^C)$ is commensurable with $\psi^{-1}(G^C) \cap \mathrm{GL}_N(\mathbb{Z})$. By Borel and Harish-Chandra's theorem [6, Corollary 13.2], Γ_0^C is a lattice in G^C. Hence the orbit $p(x)H$ has finite volume and is closed in $\Gamma_0\backslash G$.

When xH is dense in F_Λ, its image $p(x)H$ is dense in $p(F_\Lambda)$. On the other hand, F_Λ has non-empty interior as Λ is connected. Therefore $p(F_\Lambda) \subset \Gamma_0\backslash G$ is an A-invariant subset with non-empty interior. Since $\Gamma_0\backslash G$ contains a dense A-orbit, it follows that the set $p(F_\Lambda)$ is dense in $\Gamma_0\backslash G$. Therefore the image $p(x)H$ is also dense in $\Gamma_0\backslash G$. The other implications are easy to see.

12.2. **Cutting finite volume hyperbolic manifolds.** In the rest of this section, we will construct an example of a *non-arithmetic* manifold M_0 and a rigid acylindrical manifold M which covers M_0 for which Theorem 1.3 fails, as described in Proposition 12.1.

We first explain how to construct a geometrically finite *rigid* acylindrical manifold M starting from a hyperbolic manifold M_0 of finite volume. We also explain the construction of an arithmetic hyperbolic manifold M_0 which admits a properly immersed geodesic surface. Let Γ_0 be a lattice in G such that $\Delta := H \cap \Gamma_0$ is a cocompact lattice in H. This gives rise to a properly immersed compact geodesic surface $S_0 = \Delta\backslash\mathbb{H}^2$ in the orbifold $M_0 = \Gamma_0\backslash\mathbb{H}^3$. According to [19, Theorem 5.3.4], by passing to a subgroup of finite index in Γ_0, we may assume that M_0 is a manifold and that S_0 is properly embedded in M_0.

We cut M_0 along S_0. The completion N_0 of a connected component, say, M', of the complement $M_0 - S_0$ is a hyperbolic 3-manifold whose boundary ∂N_0 is totally geodesic and is the union of one or two copies of S_0.

Note that the fundamental group Γ of N_0 can be considered as a subgroup of Γ_0. Indeed, let $p_0 : \mathbb{H}^3 \to M_0$ be the natural projection and E_0 be the closure of a connected component of $p_0^{-1}(M')$. Since E_0 is convex, Γ can be identified with the stabilizer of E_0 in Γ_0. The complete manifold $M = \Gamma\backslash\mathbb{H}^3$ is a geometrically finite manifold whose convex core has a boundary which is isometric to ∂N_0. Note that M does not have any rank one cusp. The domain of discontinuity Ω is a dense union of round open disks whose closures are mutually disjoint, that is, Λ is a Sierpinski curve. It follows that M is a rigid geometrically finite acylindrical manifold.

Conversely, any rigid acylindrical geometrically finite manifold M is obtained that way. Indeed, the double M_0 of the convex core N_0 of M along its totally geodesic boundary ∂N_0 is a finite volume hyperbolic 3-manifold.

12.3. **Comparing closures of geodesic planes.** Let

$$q = q(x_1, x_2, x_3, x_4)$$

be a quadratic form with real coefficients and signature $(3, 1)$. Hyperbolic space can be seen as

$$\mathbb{H}^3 \simeq \{[v] : v \in \mathbb{R}^4 \ , \ q(v) < 0\} \tag{12.1}$$

where $[v]$ denotes the real line containing v, and the group G is isomorphic to the identity component $\mathrm{SO}(q)^\circ$ of the special orthogonal group. When $v \in \mathbb{R}^4$ is such that $q(v) > 0$, the restriction $q|_{v^\perp}$ is a quadratic form of signature $(2, 1)$ where $v^\perp$ denotes the orthogonal complement to v with respect to q. Therefore

$$\mathbb{H}_v := v^\perp \cap \mathbb{H}^3$$

is a totally geodesic plane in $\mathbb{H}^3$ and the stabilizer $G_v = \mathrm{SO}(q|_{v^\perp})^\circ$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. Hence the space $\mathcal{C}$ of circles $C \subset \mathbb{S}^2$ or, equivalently, of geodesic planes in $\mathbb{H}^3$, can be seen as

$$\mathcal{C} \simeq \{\mathbb{H}_v : v \in \mathbb{R}^4 \ , \ q(v) > 0\}.$$

We now assume that the coefficients of the quadratic form q belong to a totally real number field $\mathbb{K}$ of degree d such that for each non-trivial embedding σ of $\mathbb{K}$ in $\mathbb{R}$, the quadratic form q^σ has signature $(4, 0)$ or $(0, 4)$, so that the orthogonal group $\mathrm{SO}(q^\sigma)$ is compact.

Now an arithmetic subgroup $\Gamma_0 \subset G$ which contains a cocompact Fuchsian group is commensurable with

$$G \cap \mathrm{SL}_4(\mathbb{O})$$

where $G = \mathrm{SO}(q)^\circ$ and $\mathbb{O}$ is the ring of integers of $\mathbb{K}$ (cf. [19, Section 10.2]). In this case, the corresponding finite volume hyperbolic orbifold $M_0 = \Gamma_0 \backslash \mathbb{H}^3$ is an *arithmetic* 3-orbifold with a properly immersed totally geodesic arithmetic surface. This orbifold is compact if and only if q does not represent 0 over $\mathbb{K}$.

We introduce the set of rational positive lines:

$$\mathcal{C}_{\mathbb{K}} := \{[v] : v \in \mathbb{K}^4 \ , \ q(v) > 0\},$$

which can also be thought as the set of rational planes $\{\mathbb{H}_v : v \in \mathbb{K}^4 \ , \ q(v) > 0\}$.

For $v \in \mathcal{C}_{\mathbb{K}}$, the restriction $q|_{v^\perp}$ is defined over $\mathbb{K}$. The group $G_v(\mathbb{O}) := G_v \cap \mathrm{SL}_4(\mathbb{O})$ is an arithmetic subgroup of G_v. We call $P_v := G_v(\mathbb{O})\backslash\mathbb{H}^2$ an arithmetic geodesic surface of M_0. We recall that every properly embedded geodesic plane Q of M_0 is arithmetic, i.e. we have $Q = P_v$ for some $[v] \in \mathcal{C}_{\mathbb{K}}$.

By passing to a finite cover, we assume P_v is properly embedded in M_0, and let $\Gamma \subset \Gamma_0$ be the fundamental group of a component $M_0 - P_v$ as discussed in the subsection 12.2.

12.4. **Arithmetic examples.** We begin by a very explicit example of arithmetic lattice with $\mathbb{K} = \mathbb{Q}$. We consider the following family of quadratic forms

$$q_a := q_0(x_1, x_2, x_3) + a x_4^2$$

where $q_0(x_1, x_2, x_3) = 7x_1^2 + 7x_2^2 - x_3^2$ and a is a positive integer.

One can check, using the Hasse-Minkowski principle [32, Theorem 8 in Chapter 4], that

(1) q_0 does not represent 0 over $\mathbb{Q}$;
(2) q_a represents 0 over $\mathbb{Q}$ iff a is a square mod 7 and $a \neq -1$ mod 8;

We choose a positive integer a satisfying condition (2). For instance $a = 1$ or $a = 2$ will do. We denote by $\Gamma_{a,0} := \mathrm{SO}(q_a, \mathbb{Z})^\circ$ the corresponding arithmetic lattice and by $M_{a,0} = \Gamma_{a,0} \backslash \mathbb{H}^3$ the corresponding arithmetic hyperbolic orbifold.

We introduce the geodesic plane $\mathbb{H}_{v_0}$ associated to the vector

$$v_0 = (0, 0, 0, 1).$$

Note that, seen in the model (12.1) of $\mathbb{H}^3$ given by the quadratic form q_a, the geodesic plane $\mathbb{H}_{v_0}$ does not depend on a. We fix the geodesic orbifold $S_{a,0} := P_{v_0}$. Passing as above to a finite cover, we may assume that $M_{a,0}$ is a hyperbolic manifold and that $S_{a,0}$ is a properly embedded compact geodesic surface. After cutting along $S_{a,0}$, we get a hyperbolic 3-manifold $N_{a,0}$ whose geodesic boundary $\partial N_{a,0}$ is a union of one or two copies of $S_{a,0}$. We denote by Γ_a the fundamental group of $N_{a,0}$ and by

$$M_a := \Gamma_a \backslash \mathbb{H}^3$$

the corresponding geometrically finite rigid acylindrical hyperbolic manifold whose convex core has compact boundary. As we have seen, the group Γ_a is naturally a subgroup of $\Gamma_{a,0}$ and there is a natural covering map

$$p_a : M_a \to M_{a,0}$$

to which Theorem 1.3 applies.

12.5. **Non-arithmetic examples.** The class of non-arithmetic hyperbolic 3-manifolds that we will now construct are some of those introduced by Gromov and Piatetski-Shapiro in [15].

Choose two positive integers a, a' such that $a/a' \notin \mathbb{Q}^2$, for instance $a = 1$ and $a' = 2$. Since the surfaces $S_{a,0}$ and $S_{a',0}$ are isometric, and the boundaries $\partial N_{a,0}$ and $\partial N_{a',0}$ are union of one or two copies of these surfaces, we can glue one or two copies of the 3-manifolds $N_{a,0}$ and $N_{a',0}$ along their boundaries and get a connected finite volume hyperbolic manifold M_0 with no boundary. We write

$$M_0 = \Gamma_0 \backslash \mathbb{H}^3.$$

By [15], the lattice Γ_0 of G is non-arithmetic. The group Γ_a is also naturally a subgroup of Γ_0 and there is a natural covering map

$$p : M_a \to M_0.$$

Theorem 1.4 follows from the following:

Proposition 12.1. *Let S denote the boundary of the convex core of M_a. Let $P \subset M_a$ be a geodesic plane that intersects S but is neither contained in S nor orthogonal to S. Then the image $p(P)$ is dense in M_0.*

In particular, there exists a closed geodesic plane $P \subset M_a$ that intersects M_a^* and whose image $p(P)$ is dense in M_0.

We first need to compute the angle $\theta_{a,v}$ between two rational planes in $\mathbb{H}^3$; the following lemma follows from a direct computation.

Lemma 12.2. *Let $v = (w, x_4)$ with $w \in \mathbb{R}^3$, $x_4 \in \mathbb{R}$ with $q_a(v) > 0$.*

The intersection of the two geodesic planes $\mathbb{H}_v$ and $\mathbb{H}_{v_0}$ is a geodesic line $\mathbb{D}_w$, independent of a and x_4.

The angle $\theta_{a,v} := \angle(\mathbb{H}_v, \mathbb{H}_{v_0})$ between these geodesic planes is given by

$$\cos^2(\theta_{a,v}) = \frac{\langle v_0, v\rangle_{q_a}^2}{q_a(v_0)q_a(v)} = \frac{a x_4^2}{q_0(w) + a x_4^2}. \tag{12.2}$$

Therefore we have:

Corollary 12.3. *Fix $w \in \mathbb{Q}^3 - \{0\}$ and set*

$$\Theta_{a,w} := \{\cos^2(\theta_{a,(w,x)}) : x \in \mathbb{Q} - \{0\}\}.$$

If a/a' is not a square in $\mathbb{Q}$, then

$$\Theta_{a,w} \cap \Theta_{a',w} = \emptyset.$$

Proof. For $v = (w, x_4)$ and $v' = (w, x_4')$ with x_4 and x_4' in $\mathbb{Q} - \{0\}$, by Formula (12.2), an equality $\cos^2\theta_{a,v} = \cos^2\theta_{a',v'}$ would imply $a x_4^2 = a' {x_4'}^2$, and hence $a/a' \in \mathbb{Q}^2$. This proves the claim. $\square$

Proof of Proposition 12.1. By Theorem 1.2, which is due to Shah and Ratner independently ([33], [31]) for a hyperbolic manifold of finite volume, the geodesic plane $Q := p(P)$ is either closed or dense in M_0. We assume by contradiction that Q is closed in M_0.

Since $Q \cap N_{a,0}$ is closed, as explained in Theorem 1.3, Q must be the image in M_0 of a rational plane $\mathbb{H}_v$ with $v = (w, x_4) \in \mathbb{Q}^4$, in the model (12.1) of $\mathbb{H}^3$ given by the quadratic form q_a. Similarly, since $Q \cap N_{a',0}$ is closed, Q must be the image in M_0 of a rational plane $\mathbb{H}_{v'}$ with $v' = (w', x_4') \in \mathbb{Q}^4$, in the model of $\mathbb{H}^3$ given by the quadratic form $q_{a'}$.

We can choose these lifts such that the two intersection geodesics $\mathbb{D}_w = \mathbb{H}_v \cap \mathbb{H}_{v_0}$ and $\mathbb{D}_{w'} = \mathbb{H}_{v'} \cap \mathbb{H}_{v_0}$ are equal and the corresponding two angles between these intersecting planes are equal. This says that w and w' are equal up to a multiplicative factor, and that $\cos^2\theta_{a,v} = \cos^2\theta_{a',v'}$. This contradicts Corollary 12.3. $\square$

Remark 12.4. After seeing the result proven in the paper of Fisher, Lafont, Miller and Stover [13], we realized that Proposition 12.1, and the main result in [33] together can also be used to show that the non-arithmetic manifolds in section 12.5 can have at most finitely many properly immersed geodesic planes. To see this using the notation of Proposition 12.1, note that by [33], all but finitely many closed planes in M_0 intersect $p(S)$. By Proposition 12.1, they must intersect $p(S)$ orthogonally. On the other hand, for any infinite collection of planes in M_0, their normal bundles form a dense subset in the tangent bundle of M by [33]. Therefore it follows that there are at most finitely many closed planes in M_0. For a more general result in this direction, see more recent preprints ([22], [4]).

References

[1] W. Abikoff, *On boundaries of Teichmüller spaces and on Kleinian groups III.* Acta Math., 134 (1975), 221-237

[2] L. Ahlfors, *Complex Analysis.* McGraw-Hill, 1979

[3] L. Ahlfors, *The structure of a finitely generated Kleinian group.* Acta Math., 122 (1969), 1-17.

[4] U. Bader, D. Fisher, N. Miller, M. Stover. *Arithmeticity, superrigidity, and totally geodesic submanifolds.* Preprint, arXiv:1903.08467, to appear in JEMS.

[5] C. Bishop, *On a theorem of Beardon and Maskit.* Ann. Ac. Sc. Fen. Math., 21 (1996) 383-388.

[6] A. Borel, *Introduction aux groupes arithmétiques.* Hermann (1969).

[7] B. H. Bowditch, *Geometrical finiteness for hyperbolic groups.* J. Funct. Anal., 113 (1993) 245-317.

[8] A. Borel and J. Tits, *Homomorphismes "abstraits" de groupes algébriques simples.* Ann. Math., 97 (1973) 499-571.

[9] F. Dal'bo, *Topologie du feuilletage fortement stable.* Ann. Inst. Fourier. 50 (2000) 981-993.

[10] P. Eberlein, *Geodesic flows on negatively curved manifolds, I.* Ann Math., 95 (1972) 492-510.

[11] A. Hatcher, *Algebraic topology.* Cambr. Univ., 2002.

[12] G. Hedlund, *Fuchsian groups and transitive horocycles.* Duke Math. J., 2 (1936) 530-542.

[13] D. Fisher. J. F. Lafont, N. Miller and M. Stover, *Finiteness of maximal geodesic submanifolds in hyperbolic hybrids.* 2018, arXiv: 1802.04619.

[14] W. J. Floyd, *Group completions and limit sets of Kleinian groups.* Invent. Math., 57 (1980) 205-218.

[15] M. Gromov and I. Piatetski-Shapiro, *Non-arithmetic groups in Lobachevsky spaces.* Publ. Math. IHES, 66 (1988) 93-103.

[16] O. Khalil, *Geodesic planes in geometrically finite manifolds.* Discrete and Cont. Dyn. Syst. Vol 39 (2019) 881-903

[17] O. Lehto and K. J. Virtanen, *Quasiconformal mappings in the plane.* Springer-Verlag, 1973.

[18] M. Lee and H. Oh, *Orbit closures of unipotent flows for hyperbolic manifolds with Fuchsian ends.* 2019, arXiv:1902.06621.

[19] C. Maclachlan and A. Reid, *The arithmetic of hyperbolic 3-manifolds.* GTM, 2003.

[20] A. Marden, *Outer Circles.* Cambr. Univ. 2007.

[21] G. Margulis, *Indefinite quadratic forms and unipotent flows on homogeneous spaces.* Banach Center Publ. 23 (1989) 399-409.

[22] G. Margulis and A. Mohammadi, *Arithmeticity of hyperbolic 3-manifolds containing infinitely many totally geodesic surfaces.* (2019), arXiv:1902.07267.
[23] B. Maskit, *On Boundaries of Teichmuller spaces and on Kleinian groups II.* Annals of Math., 91 (1970) 607-639.
[24] B. Maskit, *Intersections of component subgroups of Kleinian groups.* Ann. Math. St. 79 (1974) 349-367.
[25] K. Matsuzaki and M. Taniguchi, *Hyperbolic manifolds and Kleinian groups.* Oxford Univ., 1998.
[26] C. McMullen, *Iteration on Teichmuller space.* Invent. Math., 99 (1990) 425-454.
[27] C. McMullen, A. Mohammadi and H. Oh, *Geodesic planes in hyperbolic* 3*-manifolds.* Invent. Math., 209 (2017) 425-461.
[28] C. McMullen, A. Mohammadi and H. Oh, *Geodesic planes in the convex core of an acylindrical* 3*-manifold.* (2018) arXiv: 1802.03853, to appear in Duke Math.J.
[29] H. Oh, *Dynamics for discrete subgroups of* $\mathrm{SL}_2(\mathbb{C})$. Preprint (2019), arXiv:1909.02922, to appear in Margulis volume.
[30] H. Oh and N. Shah, *Equidistribution and Counting for orbits of geometrically finite hyperbolic groups.* Journal of the AMS. Vol 26 (2013), 511-562.
[31] M. Ratner, *Raghunathan's topological conjecture and distributions of unipotent flows.* Duke Math. J. 63 (1991), 235-280.
[32] J.P. Serre, *Cours d'arithmétique.* GTM, 1973.
[33] N. Shah, *Closures of totally geodesic immersions in manifolds of constant negative curvature.* In Group Theory from a Geometrical Viewpoint (1991) 718-732.
[34] W. P. Thurston, *Hyperbolic geometry and* 3*-manifolds.* In Low dimensional topology, LMS Lect. Not. 48 (1982), 9-25.
[35] W. P. Thurston, *Hyperbolic structures on* 3*-manifolds I: Deformation of acylindrical manifolds.* Annals. Math. 124 (1986) 203-246.
[36] F. Waldhausen, *On irreducible* 3*-manifolds which are sufficiently large.* Annals. Math. 87 (1968) 56-88.
[37] Y. Zhang, *Construction of acylindrical hyperbolic 3-manifolds with quasifuchsian boundary.* To appear in Experimental Math.
[38] Y. Zhang, *Existence of an exotic plane in an acylindrical 3-manifold.* Preprint, 2020.
[39] R. Zimmer, *Ergodic theory and semisimple groups.* Birkhauser 1984.

CNRS, Universite Paris-Sud, IMO, 91405 Orsay, France
Email address: yves.benoist@u-psud.fr

Mathematics department, Yale university, New Haven, CT 06520 and Korea Institute for Advanced Study, Seoul, Korea
Email address: hee.oh@yale.edu