GROWTH INDICATORS, CONFORMAL MEASURES, AND CONICAL SETS

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Abstract. Let $G$ be a connected semisimple real algebraic group, and $\Gamma < G$ a Zariski dense $\theta$-transverse subgroup. We introduce the $\theta$-growth indicator $\psi^\theta_{\Gamma} : a_\theta \to \{-\infty\} \cup [0, \infty)$ and prove the following extension of a classical theorem of Sullivan (1979). Suppose that there exists a $(\Gamma, \psi)$-conformal measure $\nu$ on $F_\theta = G/P_\theta$ for $\psi \in a_\theta^*$.

1. If $\psi$ is $(\Gamma, \theta)$-proper, then
   
   $\psi \geq \psi^\theta_{\Gamma}$.

2. Letting $\Lambda_\theta^{con} \subset F_\theta$ denote the $\theta$-conical set of $\Gamma$, the following are equivalent:
   
   (a) $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$ (resp. $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$);
   (b) $\nu(\Lambda_\theta^{con}) = 1$ (resp. $\nu(\Lambda_\theta^{con}) = 0$).

In the former case, $\psi$ is tangent to $\psi^\theta_{\Gamma}$ and $\nu$ is the unique $(\Gamma, \psi)$-conformal measure on $F_\theta$.

When $\Gamma$ is $\theta$-Anosov, we show that either $\Lambda_\theta = F_\theta$ or $\text{Leb}_\theta(\Lambda_\theta) = 0$, and the former case arises only when $\Gamma$ projects to a cocompact lattice in a rank one factor of $G$. We also establish the uniqueness of the $(\Gamma, \psi)$-conformal measure on $F_\theta$ for each $\psi \in a_\theta^*$ tangent to $\psi^\theta_{\Gamma}$ and obtain that for any $\theta$-Anosov subgroup $\Gamma_0 < \Gamma$ with $\Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma)$, there exists no common tangent form and $\psi^\theta_{\Gamma_0} < \psi^\theta_{\Gamma}$ on the interior of the $\theta$-limit cone of $\Gamma$.

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1. Introduction

Patterson-Sullivan theory on conformal measures of a discrete subgroup of a rank one simple real algebraic group $G$ has played a pivotal role in the study of dynamics on rank one homogeneous spaces. One of the basic results due to Sullivan in 1979 is the relation between the support of a conformal measure and its dimension, which we recall for $G = \text{SO}^0(n,1)$ which is the identity component of the special orthogonal group $\text{SO}(n,1)$. The group $\text{SO}^0(n,1)$ is the group of orientation-preserving isometries of the real hyperbolic space $\mathbb{H}^n$. The geometric boundary of $\mathbb{H}^n$ can be identified with the sphere $S^{n-1}$. For a discrete subgroup $\Gamma \subset G$, denote by $\Lambda_{\text{con}} \subset S^{n-1}$ the conical set of $\Gamma$, which consists of the endpoints of all geodesic rays in $\mathbb{H}^n$ which accumulate modulo $\Gamma$. Let $\delta_{\Gamma}$ denote the critical exponent of $\Gamma$, which is the abscissa of convergence of the Poincaré series $s \to \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)}$, $o \in \mathbb{H}^n$. The following remarkable theorem says that the convergence behavior of the Poincaré series of $\Gamma$ at $s$ yields the zero-one law for the size of the conical set for all $\Gamma$-conformal measures of dimension $s$.

Theorem 1.1 (Sullivan, [36, Corollaries 4, 20, Theorem 21], see also [1], [8], [33]). Let $\Gamma \subset \text{SO}^0(n,1)$, $n \geq 2$, be a non-elementary discrete subgroup. Suppose that there exists a $\Gamma$-conformal measure $\nu$ on $S^{n-1}$ of dimension $s \geq 0$.

1. We have $s \geq \delta_{\Gamma}$.

2. The following are equivalent:
   (a) $\sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} = \infty$ (resp. $\sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)} < \infty$).
   (b) $\nu(\Lambda_{\text{con}}) = 1$ (resp. $\nu(\Lambda_{\text{con}}) = 0$).

In the former case, $s = \delta_{\Gamma}$ and $\nu$ is the unique $\Gamma$-conformal measure of dimension $\delta_{\Gamma}$.

The main aim of this paper is to establish an analogous result for a class of discrete subgroups of a general connected semisimple real algebraic group $G$. Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where $A$ is a maximal real split torus of $G$, $M$ is the maximal compact subgroup of $P$ commuting with $A$ and $N$ is the unipotent radical of $P$. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebra of $G$ and $A$. Fix a positive Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection defined by the condition $g \in K \exp \mu(g)K$ for $g \in G$. Let $\Pi$ denote the set of all simple roots for $(\mathfrak{g}, \mathfrak{a}^+)$. As usual, the Weyl group of $\mathfrak{a}$ is the quotient of the normalizer of $\mathfrak{a}$ by the centralizer of $\mathfrak{a}$. Let $i : \mathfrak{a} \to \mathfrak{a}$ denote the opposition involution, that is, $i(u) = -\text{Ad}_{w_0}(u)$ for all $u \in \mathfrak{a}$ where $w_0$ is the longest Weyl element. It induces an involution on $\Pi$ which we denote by the same notation $i$. We fix
a non-empty subset \( \theta \subset \Pi \).

Let \( a_\theta = \bigcap_{\alpha \in \Pi \setminus \theta} \ker \alpha \) and let \( p_\theta : a \to a_\theta \) be the unique projection, invariant under all Weyl elements fixing \( a_\theta \) pointwise. Let \( P_\theta \) be the standard parabolic subgroup corresponding to \( \theta \) (our convention is that \( P = P_{\Pi} \)) and consider the \( \theta \)-boundary:

\[
F_\theta = G/P_\theta.
\]

We say that \( \xi \in F_\theta \) and \( \eta \in F_{i(\theta)} \) are in general position if the pair \((\xi, \eta)\) belongs to the unique open \( G \)-orbit in \( F_\theta \times F_{i(\theta)} \) under the diagonal action of \( G \).

Let \( \Gamma < G \) be a Zariski dense discrete subgroup. We define the following properties of \( \Gamma \) which are natural to impose in studying analogues of Theorem 1.1 for \( \Gamma \)-conformal measures on the \( \theta \)-boundary \( F_\theta \).

Let \( \Lambda_\theta = \Lambda_\theta(\Gamma) \) denote the \( \theta \)-limit set of \( \Gamma \), which is the unique \( \Gamma \)-minimal subset of \( F_\theta \) (Definition 5.1).

**Definition 1.2.**

(1) We say that \( \Gamma \) is \( \theta \)-discrete if the composition \( \mu_\theta = p_\theta \circ \mu : \Gamma \to a_\theta^+ \) is proper.

(2) A \( \theta \)-discrete subgroup \( \Gamma \) is said to be \( \theta \)-transverse if it is \( \theta \)-regular, i.e., \( \liminf_{\gamma \in \Gamma} \alpha(\mu(\gamma)) = \infty \) for all \( \alpha \in \theta \), and \( \theta \)-antipodal, i.e., if any \( \xi \neq \eta \) in \( \Lambda_{\theta, i(\theta)} \) are in general position.

**Remark 1.3.**

- Note that \( \Gamma \) is \( \theta \)-discrete if and only if the counting measure on \( \mu_\theta(\Gamma) \) weighted with multiplicity is a locally finite Borel measure on \( a_\theta^+ \).

  Any discrete subgroup of \( G \) is \( \Pi \)-discrete. If \( G \) is simple, then \( p_\theta|_{a^+} \) is a proper map and hence a discrete subgroup of \( G \) is \( \theta \)-discrete for any \( \theta \subset \Pi \). However when \( G \) is not simple, a discrete subgroup is not necessarily \( \theta \)-discrete for a general \( \theta \).

- The class of \( \theta \)-transverse subgroups includes all discrete subgroups of rank one Lie groups, \( \theta \)-Anosov subgroups and their relative versions. This class is regarded as a generalization of all rank one discrete subgroups while Anosov subgroups are regarded as higher rank analogues of convex cocompact subgroups. We also note that a subgroup of a \( \theta \)-transverse subgroup is also \( \theta \)-transverse.

We assume that \( \Gamma \) is \( \theta \)-discrete in the rest of the introduction. We define the \( \theta \)-growth indicator \( \psi^\theta_\Gamma : a_\theta \to [-\infty, \infty] \) as follows: fixing any norm \( \| \cdot \| \) on \( a_\theta \), if \( u \in a_\theta \) is non-zero,

\[
\psi^\theta_\Gamma(u) = \| u \| \inf_{u \in C} \tau^\theta_C \tag{1.1}
\]

where \( \tau^\theta_C \) is the abscissa of convergence of the series \( \sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in C} e^{-s\| \mu_\theta(\gamma) \|} \) and \( C \subset a_\theta \) ranges over all open cones containing \( u \). Set \( \psi^\theta_\Gamma(0) = 0 \). This
and concave function. It also follows from (1.2) that
\[ \psi^\theta_T \circ p_\theta \geq \psi_T. \]  
(1.2)

We show that \( \psi^\theta_T < \infty \), and \( \psi^\theta_T \) is a homogeneous, upper semi-continuous
and concave function. It also follows from (1.2) that
\[ \{ \psi^\theta_T \geq 0 \} = \mathcal{L}_\theta \quad \text{and} \quad \psi^\theta_T > 0 \quad \text{on int} \mathcal{L}_\theta \]  
(1.3)
where \( \mathcal{L}_\theta = \mathcal{L}_2(G) \) is the \( \theta \)-limit cone of \( G \) (Theorem 3.3). See Example 3
and Corollary 3.13 for explicit upper bounds of \( \psi^\theta_T \) for \( G = \text{PSL}_d(\mathbb{R}) \).

Denote by \( \mathfrak{a}^{\ast}_{\theta} = \text{Hom}(\mathfrak{a}_\theta, \mathbb{R}) \) the space of all linear forms on \( \mathfrak{a}_\theta \). For \( \psi \in \mathfrak{a}^{\ast}_{\theta} \),
a Borel probability measure \( \nu \) on \( \mathcal{F}_\theta \) is called a \((\Gamma, \psi)\)-conformal measure if
\[ \frac{d\gamma_{\ast}\nu}{d\nu}(\xi) = e^{\psi(\beta^\theta_{\ast}(\xi, \gamma))} \quad \text{for all} \quad \gamma \in \Gamma \quad \text{and} \quad \xi \in \mathcal{F}_\theta \]
where \( \gamma_{\ast}\nu(D) = \nu(\gamma^{-1}D) \) for any Borel subset \( D \subset \mathcal{F}_\theta \) and \( \beta^\theta_{\ast} \) denotes the \( \mathfrak{a}_\theta \)-valued Busemann map defined in (5.3). The linear form \( \psi \) is referred to
as the dimension of \( \nu \).

We define the \( \theta \)-conical set of \( \Gamma \) as
\[ \Lambda^{\text{con}}_{\theta} = \left\{ gP_\theta \in \mathcal{F}_\theta : \limsup \Gamma gA^+ \neq \emptyset \right\}, \]
(1.4)
that is, \( \xi \in \Lambda^{\text{con}}_{\theta} \) if and only if for some \( g \in G \) such that \( \xi = gP_\theta \), there
exist infinite sequences \( \gamma_i \in \Gamma \) and \( a_i \in A^+ \) such that the sequence \( \gamma_iga_i \) is
bounded. We mention that if \( \Gamma \) is \( \theta \)-regular, then \( \Lambda^{\text{con}}_{\theta} \subset \Lambda_{\theta} \) (Proposition
5.5).

**Definition 1.4.** We say \( \psi \in \mathfrak{a}^{\ast}_{\theta} \) is \((\Gamma, \theta)\)-proper if \( \psi \circ \mu_\theta : \Gamma \to \mathbb{R} \) is
a proper map for some \( \varepsilon > 0 \).

For example, a linear form \( \psi \) which is positive on \( \mathcal{L}_\theta - \{0\} \) is \((\Gamma, \theta)\)-proper.
For a \((\Gamma, \theta)\)-proper form \( \psi \), we have \( \psi(\mu_\theta(\gamma)) > 0 \) except for finitely many
\( \gamma \in \Gamma \) and hence the critical exponent \( 0 < \delta_\psi = \delta_\psi(\Gamma) \leq \infty \) of the \( \psi \)-Poincaré
series \( \mathcal{P}_\psi(s) = \sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} \) is well-defined and we have
\[ \delta_\psi = \limsup_{t \to \infty} \frac{1}{t} \# \{ \gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < t \} \]
(see Lemma 4.2).

A linear form \( \psi \in \mathfrak{a}^{\ast}_{\theta} \) is said to be \((\Gamma, \theta)\)-critical if \( \psi \) is tangent to the \( \theta \)-
growth indicator \( \psi^\theta_T \), i.e., \( \psi \geq \psi^\theta_T \) and \( \psi(u) = \psi^\theta_T(u) \) for some \( u \in \mathfrak{a}^+_\theta - \{0\} \).

**Main theorems.** The following main theorem of our paper is an extension of Theorem 1.1 to higher ranks:

**Theorem 1.5.** Let \( \Gamma < G \) be a \( \theta \)-transverse subgroup. Suppose that there exists a \((\Gamma, \psi)\)-conformal measure \( \nu \) on \( \mathcal{F}_\theta \) for \( \psi \in \mathfrak{a}^{\ast}_{\theta} \).

1. If \( \psi \) is \((\Gamma, \theta)\)-proper, then
\[ \psi \geq \psi^\theta_T. \]  
(1.5)
(2) The following are equivalent:
(a) \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty \) (resp. \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty \));
(b) \( \nu(\Lambda_{\theta}^{\text{con}}) = 1 \) (resp. \( \nu(\Lambda_{\theta}^{\text{con}}) = 0 \)).
In the former case, any \((\Gamma, \theta)\)-proper \(\psi\) is necessarily \((\Gamma, \theta)\)-critical and \(\nu\) is the unique \((\Gamma, \psi)\)-conformal measure on \(\mathcal{F}_{\theta}\).

Remark 1.6. Canary-Zhang-Zimmer [7] initiated the systematic study of conformal measures for the class of \(\theta\)-transverse subgroups; indeed, the terminology of a transverse subgroup was first introduced in their paper. By incorporating their work into ours, we provide in Theorem 1.5 more equivalent conditions in Theorem 1.5 involving ergodicity and conservativity of the \(\Gamma\)-action on \(\Lambda_{\theta} \times \Lambda_{i(\theta)}\) (see also Remark 1.13).

Remark 1.7. See Corollary 8.4 for the comparison of \(\psi \circ p_{\theta}\) and \(\psi_{\Gamma}\) in the context of Theorem 1.5, which yields an improvement of Quint’s bound ([30, Theorem 8.1], see (8.7)).

Disjoint dimension phenomenon. Let

\[
D_{\Gamma}^{\theta} = \left\{ \psi \in a_{\theta}^* : (\Gamma, \theta)\text{-proper}, \delta_{\psi}(\Gamma) = 1 \text{ and } \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty \right\}.
\]

Inspired by the entropy drop phenomenon proved by Canary-Zhang-Zimmer [7, Theorem 4.1] for \(\theta = i(\theta)\), we deduce from Theorem 1.5 the following disjointness of dimensions, which turns out to be equivalent to the entropy drop phenomenon (Corollary 8.15):

Corollary 1.8 (Disjoint dimensions). Let \(\Gamma < G\) be a \(\theta\)-transverse subgroup. For any Zariski dense subgroup \(\Gamma_0 < \Gamma\) with \(\Lambda_{\theta}(\Gamma_0) \neq \Lambda_{\theta}(\Gamma)\), we have

\[
D_{\Gamma}^{\theta} \cap D_{\Gamma_0}^{\theta} = \emptyset.
\]

In the rank one case, this corollary says that if \(\Lambda(\Gamma_0) \neq \Lambda(\Gamma)\) and \(\Gamma_0 < \Gamma\) are of divergence type, that is, their Poincaré series diverge at the critical exponents, then \(\delta_{\Gamma_0} < \delta_{\Gamma}\). We refer to [7] for a more detailed background on this phenomenon.

\(\theta\)-Anosov subgroups. A finitely generated subgroup \(\Gamma < G\) is a \(\theta\)-Anosov subgroup if there exists \(C > 0\) such that for all \(\gamma \in \Gamma\),

\[
\min_{\alpha \in \theta} \alpha(\mu(\gamma)) \geq C^{-1}|\gamma| - C
\]

where \(|\gamma|\) denotes the word length of \(\gamma\) with respect to a fixed finite generating set of \(\Gamma\) ([22], [13], [16], [17], [18]). All \(\theta\)-Anosov subgroups are \(\theta\)-transverse and \(\Lambda_{\theta} = \Lambda_{\theta}^{\text{con}}\) ([14], [17]). We deduce the following from Theorem 1.5.

Theorem 1.9. Let \(\Gamma < G\) be a \(\theta\)-Anosov subgroup. Suppose that there exists a \((\Gamma, \psi)\)-conformal measure \(\nu\) on \(\mathcal{F}_{\theta}\) for \(\psi \in a_{\theta}^*\). We have:

(1) The linear form \(\psi\) is \((\Gamma, \theta)\)-proper and \(\psi \geq \psi_{\Gamma}\).
The following are equivalent to each other:

(a) \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty \) (resp. \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty \));
(b) \( \nu(\Lambda_\theta) = 1 \) (resp. \( \nu(\Lambda_\theta) = 0 \));
(c) \( \psi \) is \((\Gamma, \theta)\)-critical (resp. \( \psi \) is not \((\Gamma, \theta)\)-critical).

For each \((\Gamma, \theta)\)-critical \( \psi \in a^*_\theta \), there exists a unique \((\Gamma, \psi)\)-conformal measure on \( F_\theta \), which is necessarily supported on \( \Lambda_\theta \).

The equivalence \((a) \Leftrightarrow (b)\) in (2) answers a question asked by Sambarino [35, Remark 5.10].

**Analogue of Ahlfors measure conjecture for \( \theta \)-Anosov groups.** We denote by \( \text{Leb}_\theta \) Lebesgue measure on \( F_\theta \), which is the unique \( K \)-invariant probability measure on \( F_\theta \). The following corollary is motivated by the Ahlfors measure conjecture [2].

**Corollary 1.10.** If \( \Gamma < G \) is \( \theta \)-Anosov, then

either \( \Lambda_\theta = F_\theta \) or \( \text{Leb}_\theta(\Lambda_\theta) = 0 \).

Moreover, in the former case, \( \theta \) is the simple root of a rank one factor, say \( G_0 \), of \( G \) and \( \Gamma \) projects to a cocompact lattice of \( G_0 \).

**Critical forms and conformal measures.** We set

\[ T^\theta_\Gamma := \{ \psi \in a^*_\theta : \psi \text{ is } (\Gamma, \theta)\text{-critical} \}. \]

Note that \( D^\theta_\Gamma \subset T^\theta_\Gamma \) (Corollary 4.6). For \( \theta \)-Anosov subgroups, we further have \( T^\theta_\Gamma = D^\theta_\Gamma \), which is again same as the set of all \( \psi \in a^*_\theta \) for which there exists a \((\Gamma, \psi)\)-conformal measure supported on \( \Lambda_\theta \) (Lemma 9.3). Using Sambarino’s parametrization of the space of all conformal measures on \( \Lambda_\theta \) as \( \{ \delta_\psi = 1 \} \) [35, Theorem A], we deduce:

**Corollary 1.11.** For any \( \theta \)-Anosov subgroup \( \Gamma < G \), we have a one-to-one correspondence among

1. the set \( T^\theta_\Gamma \) of all \((\Gamma, \theta)\)-critical forms on \( a_\theta \);
2. the set of all unit vectors in \( \text{int} \mathcal{L}_\theta \);
3. the set of all \( \Gamma \)-conformal measures supported on \( \Lambda_\theta \);
4. the set of all \( \Gamma \)-conformal measures on \( F_\theta \) of critical dimensions.

More precisely, for any \( \psi \in T^\theta_\Gamma \), there exists a unique unit vector \( u_\psi \in a_\theta^+ \) such that \( \psi(u_\psi) = \psi^\theta_\Gamma(u_\psi) \); moreover \( u_\psi \in \text{int} \mathcal{L}_\theta \). There also exists a unique \((\Gamma, \psi)\)-conformal measure \( \nu_\psi \) on \( F_\theta \), which is necessarily is supported on \( \Lambda_\theta \). Moreover every \( \Gamma \)-conformal measure supported on \( \Lambda_\theta \) arises in this way.

**Corollary 1.12** (Disjoint critical dimensions). For any Zariski dense \( \theta \)-Anosov subgroups \( \Gamma_0 < \Gamma \) such that \( \Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma) \), we have

\[ T^\theta_\Gamma \cap T^\theta_{\Gamma_0} = \emptyset \text{ and } \psi^\theta_{\Gamma_0} < \psi^\theta_\Gamma \text{ on } \text{int} \mathcal{L}_\theta(\Gamma). \]

Indeed, the above two conditions are equivalent to each other by the vertical tangency of \( \psi^\theta_{\Gamma_0} \) (Theorem 9.2).
Remark 1.13. Related dichotomy properties for conformal measures were studied in \[10, 4, 24, 11, 35, 7, \] etc. In particular, when \(\Gamma\) is \(\Pi\)-Anosov, Theorem 1.9, Corollaries 1.10 and 1.11 were proved by Lee-Oh \[24, \text{Theorems 1.3, 1.4}\]. The papers \[10, 35, 7, \] study conformal measures supported on the limit set \(\Lambda_{g}\) and the papers \[4, 11, \] study the role of directional conical sets in the ergodic behavior of conformal measures. In particular, for \(\theta\) symmetric, that is, \(\theta = i(\theta)\), and for conformal measures supported on \(\Lambda_{\theta}\), the equivalence of \((a)\) and \((b)\) in Theorem 1.5 was proved by Canary, Zhang and Zimmer \[7, \text{see Theorem 8.7}\]. Our focus on this paper is to address general conformal measures without restriction on their supports following \[24, \text{and to study the relationship between the dimensions of conformal measures and \(\theta\)-growth indicators so as to establish an analogue of Sullivan’s theorem (Theorem 1.1) and the analogue of the Ahlfors measure conjecture. We also emphasize that the \(\theta\)-growth indicator is first introduced and studied in our paper.}

Finally, we mention that there is a plethora of examples of \(\theta\)-transverse subgroups which are not \(\theta\)-Anosov. They include the images of cusped Hitchin representations of geometrically finite Fuchsian groups by \[5, \text{Another important examples are self-joinings of geometrically finite subgroups of rank one Lie groups, that is, } \Gamma = \langle \prod_{i=1}^{k} \rho_{i} \rangle(\Delta) = \{ (\rho_{i}(g))_{i} : g \in \Delta \} \text{ where } \Delta \text{ is a geometrically finite subgroup of a rank one simple real algebraic group } G_{0} \text{ and } \rho_{i} : \Delta \rightarrow G_{i} \text{ is a type-preserving isomorphism onto its image } \rho_{i}(\Delta) \text{ which is a geometrically finite subgroup of a rank one simple real algebraic group } G_{i} \text{ for each } 1 \leq i \leq k. \text{ It follows from } [37, \text{Theorem 3.3}] \text{ and } [9, \text{Theorem A.4}] \text{ (see also } [38, \text{Theorem 0.1}]\text{) that there exists a } \rho_{i}-\text{equivariant homeomorphism between the limit set of } \Delta \text{ and the limit set of } \rho_{i}(\Delta) \text{ for each } 1 \leq i \leq k. \text{ This implies that } \Gamma \text{ is } \Pi\text{-transverse.}

Organization.

- In section 2, we introduce the notion of convergence of elements of \(G\) to those of \(F_{\Delta}\) and present some basic lemmas which will be used in the proof of our main theorems.
- In section 3, we define its \(\theta\)-growth indicator \(\psi_{\Gamma}^{\theta}\) for a \(\theta\)-discrete subgroup \(\Gamma < G\). Properties of the \(\theta\)-growth indicator and its relationship with Quint’s growth indicator \[29, \text{are also discussed.}\]
- In section 4, we introduce \((\Gamma, \theta)\)-proper linear forms and \((\Gamma, \theta)\)-critical linear forms and discuss properties of their critical exponents.
- In section 5, we define the \(\theta\)-limit set and \(\theta\)-conical set of \(\Gamma\). For \(\theta\)-regular subgroups, we show that the \(\theta\)-conical is a subset of the \(\theta\)-limit set and construct conformal measures supported on the \(\theta\)-limit set for each \(\psi \in D_{\Gamma}^{\theta}\).
- In section 6, we prove that for \(\theta\)-transverse subgroups, \(\theta\)-shadows with bounded width have bounded multiplicity, which is one of the key technical ingredients of our main results.
In section 7, we show that if $\Gamma$ is a $\theta$-transverse subgroup, the dimension of a $\Gamma$-conformal measure is at least $\psi_{\theta}\Gamma$ (Theorem 7.1).

In section 8, we prove the zero-one law for the $\nu$-size of the conical set depending on whether or not the associated Poincaré series diverges at its dimension (Theorem 8.1). We also prove Corollary 1.10.

Finally, in section 9 we discuss how our theorems are applied for $\theta$-Anosov groups.

Acknowledgement. We would like to thank Jean-François Quint for telling us about Lemma 3.12.

2. Convergence in $G \cup \mathcal{F}_0$.

In the whole paper, let $G$ be a connected semisimple real algebraic group. Let $P < G$ be a minimal parabolic subgroup with a fixed Langlands decomposition $P = MAN$ where $A$ is a maximal real split torus of $G$, $M$ is the maximal compact subgroup of $P$ commuting with $A$ and $N$ is the unipotent radical of $P$. Let $\mathfrak{g}$ and $\mathfrak{a}$ respectively denote the Lie algebras of $G$ and $A$. Fix a positive Weyl chamber $\mathfrak{a}^+ < \mathfrak{a}$ so that $\log N$ consists of positive root subspaces and set $A^+ = \exp \mathfrak{a}^+$. We fix a maximal compact subgroup $K < G$ such that the Cartan decomposition $G = KA^+K$ holds. We denote by

$$\mu : G \to \mathfrak{a}^+$$

the Cartan projection defined by the condition $g \in K \exp \mu(g)K$ for $g \in G$.

**Lemma 2.1.** [3, Lemma 4.6] For any compact subset $Q \subset G$, there exists $C = C(Q) > 0$ such that for all $g \in G$,

$$\sup_{q_1, q_2 \in Q} \| \mu(q_1 g q_2) - \mu(g) \| \leq C.$$ 

Let $X = G/K$ be the associated Riemannian symmetric space, and set $o = [K] \in X$. Fix a $K$-invariant norm $\| \cdot \|$ on $\mathfrak{g}$ induced from the Killing form on $\mathfrak{g}$ and let $d$ denote the Riemannian metric on $X$ induced by $\| \cdot \|$.

Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{a})$ denote the set of all roots, $\Phi^+ \subset \Phi$ the set of all positive roots, and $\Pi \subset \Phi^+$ the set of all simple roots. We denote by $N_K(A)$ and $C_K(A)$ the normalizer and centralizer of $A$ in $K$ respectively. Consider the Weyl group $W = N_K(A)/C_K(A)$. Fix an element

$$w_0 \in N_K(A)$$

representing the longest Weyl element so that $\text{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$ and $w_0^{-1} = w_0$. Hence the map

$$i = -\text{Ad}_{w_0} : \mathfrak{a} \to \mathfrak{a}$$

defines an involution of $\mathfrak{a}$ preserving $\mathfrak{a}^+$; this is called the opposition involution. It induces a map $\Phi \to \Phi$ preserving $\Pi$, for which we use the same notation $i$, such that $i(\alpha) \circ \text{Ad}_{w_0} = -\alpha$ for all $\alpha \in \Phi$. We have

$$\mu(g^{-1}) = i(\mu(g)) \quad \text{for all } g \in G.$$  

(2.1)
In the whole paper, fix a non-empty subset \( \theta \subset \Pi \). Let \( P_{\theta} \) denote a standard parabolic subgroup of \( G \) corresponding to \( \theta \); that is, \( P_{\theta} \) is generated by \( MA \) and all root subgroups \( U_\alpha \), \( \alpha \in \Phi^{+} \cup [\Pi - \theta] \) where \([\Pi - \theta]\) denotes the set of all roots in \( \Phi \) which are \( \mathbb{Z} \)-linear combinations of \( \Pi - \theta \). Hence \( P_{\Pi} = P \).

The subgroup \( P_{\theta} \) is equal to its own normalizer; for \( g \in G \), \( gP_{\theta}g^{-1} = P_{\theta} \) if and only if \( g \in P_{\theta} \). Let

\[
a_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha, \quad a_\theta^+ = a_\theta \cap a^+, \quad A_\theta = \exp a_\theta, \quad A_\theta^+ = \exp a_\theta^+.
\]

Let

\[
p_{\theta} : a \to a_\theta
\]

denote the projection, invariant under \( w \in W \) fixing \( a_\theta \) pointwise. Let \( L_\theta \) denote the centralizer of \( A_\theta \); it is a Levi subgroup of \( P_\theta \) and \( P_\theta = L_\theta N_\theta \) where \( N_\theta = R_u(P_{\theta}) \) is the unipotent radical of \( P_{\theta} \). We set \( M_{\theta} = K \cap P_{\theta} \subset L_\theta \). We may then write \( L_\theta = A_\theta S_\theta \) where \( S_\theta \) is an almost direct product of a connected semisimple real algebraic subgroup and a compact subgroup. Then \( B_\theta = S_\theta \cap A \) is a maximal \( \mathbb{R} \)-split torus of \( S_\theta \) and \( \Pi - \theta \) is the set of simple roots for \((\text{Lie} S_\theta, \text{Lie} B_\theta)\). Letting

\[
B_\theta^+ = \{ b \in B_\theta : \alpha(\log b) \geq 0 \text{ for all } \alpha \in \Pi - \theta \},
\]

we have the Cartan decomposition of \( S_\theta \):

\[
S_\theta = M_{\theta} B_\theta^+ M_{\theta}.
\]

Any \( u \in a \) can be written as \( u = u_1 + u_2 \) for unique \( u_1 \in a_\theta \) and \( u_2 \in \log B_{\theta} \), and we have \( p_{\theta}(u) = u_1 \). In particular, we have

\[
A = A_{\theta} B_{\theta} \quad \text{and} \quad A^+ \subset A_{\theta}^+ B_{\theta}^+.
\]

**The \( \theta \)-boundary \( \mathcal{F}_{\theta} \) and convergence to \( \mathcal{F}_{\theta} \).** We set

\[
\mathcal{F}_{\theta} = G/P_{\theta} \quad \text{and} \quad \mathcal{F} = G/P.
\]

Let

\[
\pi_{\theta} : \mathcal{F} \to \mathcal{F}_{\theta}
\]

denote the canonical projection map given by \( gP \mapsto gP_{\theta} \), \( g \in G \). We set

\[
\xi_{\theta} = [P_{\theta}] \in \mathcal{F}_{\theta}.
\]

By the Iwasawa decomposition \( G = KP = KAN \), the subgroup \( K \) acts transitively on \( \mathcal{F}_{\theta} \), and hence

\[
\mathcal{F}_{\theta} \simeq K/M_{\theta}.
\]

We consider the following notion of convergence of a sequence in \( G \) to an element of \( \mathcal{F}_{\theta} \).

**Definition 2.2.** For a sequence \( g_i \in G \) and \( \xi \in \mathcal{F}_{\theta} \), we write \( \lim_{i \to \infty} g_i = \lim g_{i,0} = \xi \) and say \( g_i \) (or \( g_{i,0} \in X \)) converges to \( \xi \) if

- \( \min_{\alpha \in \theta} \alpha(\mu(g_i)) \to \infty \); and
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• lim_{i→∞} κ_θ_i = κ_θ in F_θ for some κ_θ_i ∈ K such that g_i ∈ κ_θ_i A^+ K.

**Points in general position.** Let P_θ^+ be the standard parabolic subgroup of G opposite to P_θ such that P_θ ∩ P_θ^+ = L_θ. We have P_θ^+ = w_0 P_{i(θ)} w_0^{-1} and hence

\[ F_{i(θ)} = G/P_θ^+. \]

In particular, if θ is symmetric in the sense that \( θ = i(θ) \), then \( F_θ = G/P_θ^+ \). Let \( N_θ^+ \) denote the unipotent radical of \( P_θ^+ \). The set \( N_θ^+ P_θ \) is a Zarsiki open and dense subset of \( G \). In particular, \( N_θ^+ ξ_θ \cap h N_θ^+ ξ_θ \neq ∅ \) for any \( h \in G \). The \( G \)-orbit of \(( P_θ, P_θ^+)\) is the unique open \( G \)-orbit in \( G/P_θ \times G/P_θ^+ \) under the diagonal \( G \)-action.

**Definition 2.3.** Two elements \( ξ, η \in G/P_θ \) and \( η \in G/P_θ^+ \) are said to be in general position if \(( ξ, η) \in G.(P_θ, P_θ^+)\), i.e., \( ξ = gP_θ \) and \( η = gP_θ^+ \) for some \( g \in G \).

It follows from the identity \( P_θ^+ = N_θ^+ (P_θ \cap P_θ^+) \) that

\[ (gP_θ, P_θ^+) \in (P_θ, P_θ^+) \] if and only if \( g \in N_θ^+ P_θ \). (2.3)

**Basic lemmas.** We generalize [23 Lemmas 2.9-11] from \( θ = I \) to a general \( θ \) as follows. For subsets \( S_i \subset G \), we often write \( g = g_1 g_2 g_3 \in S_1 S_2 S_3 \) to mean that \( g_i \in S_i \) for each \( i \), in addition to \( g = g_1 g_2 g_3 \).

**Lemma 2.4.** Consider a sequence \( g_i = k_i a_i h_i^{-1} \) where \( k_i \in K, a_i \in A^+, \) and \( h_i \in G \). Suppose that \( k_i \to k_0 \in K, h_i \to h_0 \in G, \) and \( \min_{a_i} a_i(\log a_i) \to \infty, \) as \( i \to \infty \). Then for any \( ξ \in h_0 N_θ^+ ξ_θ \), we have

\[ \lim_{i→∞} g_i ξ = k_0 ξ_θ. \]

**Proof.** Since \( h_i^{-1} ξ \) converges to the element \( h_0^{-1} ξ \in N_θ^+ ξ_θ \) by the hypothesis and \( N_θ^+ ξ_θ \subset F_θ \) is open, we have \( h_i^{-1} ξ \in N_θ^+ ξ_θ \) for all large \( i \). Hence we can write \( h_i^{-1} ξ = n_i ξ_θ \) with \( n_i \in N_θ^+ \) uniformly bounded. Since \( \min_{a_i} a_i(\log a_i) \to \infty \) and \( n_i \in N_θ^+ \) is uniformly bounded, we have \( a_i n_i a_i^{-1} → e \) as \( i → ∞ \). Therefore the sequence \( a_i h_i^{-1} ξ = a_i n_i a_i^{-1} ξ_θ \) converges to \( ξ_θ \). Hence we have

\[ \lim_{i→∞} g_i ξ = \lim_{i→∞} k_i(a_i h_i^{-1} ξ) = k_0 ξ_θ. \]

**Corollary 2.5.** If \( w \in N_K(A) \) is such that \( mw \in N_θ^+ P_θ \) for some \( m \in M_θ \), then \( w \in M_θ \). In particular, if \( wP_θ \) and \( P_θ^+ \) are in general position, then \( w \in M_θ \).

**Proof.** Choose any sequence \( a_i \in A_θ^+ \) such that \( \min_{a_i} a_i(\log a_i) \to ∞ \). Since \( mw ξ_θ \in N_θ^+ ξ_θ \), we deduce from Lemma 2.3 that \( a_i mw ξ_θ \) converges to \( ξ_θ \) as \( i → ∞ \). On the other hand, since \( w \in N_K(A), A \subset P_θ \) and \( m \in M_θ \), we have \( a_i mw ξ_θ = mw(a_i w) ξ_θ = mw ξ_θ \) for all \( i \). Hence \( mw ξ_θ = ξ_θ \). Since \( m \in M_θ \), this implies \( w ξ_θ = ξ_θ \) and hence \( w \in P_θ \cap K = M_θ \). □
It turns out that the convergence of \( g_i \to \xi \) is equivalent to \( g_i p \to \xi \) for any \( p \in X \). More generally, we have

**Lemma 2.6.** If a sequence \( g_i \in G \) converges to \( \xi \in F_\theta \) and \( p_i \in X \) is a bounded sequence, then

\[
\lim_{i \to \infty} g_i p_i = \xi.
\]

**Proof.** Let \( g'_i \in G \) be such that \( g'_i o = p_i \); then \( g'_i \) is bounded. Since \( \lim g_i = \xi \), we may write \( g_i = k_i a_i \ell_i^{-1} \) with \( k_i, \ell_i \in K \) and \( a_i \in A^+ \) where \( \min_{a \in \theta} \alpha(\log a_i) \to \infty \), and \( k_i \xi_\theta \to \xi \) as \( i \to \infty \). Write \( g_i g'_i = k'_i a'_i (\ell'_i)^{-1} \in KA^+ K \). Since \( g'_i \) is bounded, \( \lim_{i \to \infty} \min_{a \in \theta} \alpha(\log a'_i) = \infty \), by Lemma [2.1]. Let \( q \in K \) be a limit of the sequence \( q_i := k_i^{-1} k'_i \). By passing to a subsequence, we may assume that \( q_i \to q \). Since \( d(o, p_i) = d(g_i o, g'_i p_i) = d(o, a_i^{-1} q_i a'_i o) \), the sequence \( h_i^{-1} := a_i^{-1} q_i a'_i \) is bounded. Passing to a subsequence, we may assume that \( h_i \) converges to some \( h_0 \in G \). Choose any \( \eta \in N^+_\theta \xi_\theta \cap h_0 N^+_\theta \xi_\theta \). By Lemma [2.4] we have

\[
\lim_{i \to \infty} a_i h_i^{-1} \eta = \xi_\theta \quad \text{and} \quad \lim_{i \to \infty} q_i a'_i \eta = q \xi_\theta.
\]

Since \( a_i h_i^{-1} = q_i a'_i \), it follows that \( q \xi_\theta = \xi_\theta \); so \( q \in K \cap P_\theta \). Hence \( \xi = \lim k_i \xi_\theta = \lim k'_i \xi_\theta \). It follows that \( \lim g_i p_i = \xi \). 

**Lemma 2.7.** If a sequence \( g_i \in G \) converges to \( g \) and a sequence \( a_i \in A^+ \) satisfies \( \min_{a \in \theta} \alpha(\log a_i) \to \infty \) as \( i \to \infty \), then for any \( p \in X \), we have

\[
\lim_{i \to \infty} g_i a_i p = g \xi_\theta.
\]

**Proof.** By Lemma [2.6] it suffices to consider the case when \( p = o \). Write \( g_i a_i = k_i b_i \ell_i^{-1} \) with \( k_i, \ell_i \in K \) and \( b_i \in A^+ \). Since the sequence \( g_i \) is bounded,\( \lim_{i \to \infty} \min_{a \in \theta} \alpha(\log b_i) = \infty \). Let \( k_0 \) be a limit of the sequence \( k_i \); without loss of generality, we may assume that \( k_i \) converges to \( k_0 \) as \( i \to \infty \). Then \( \lim_{i \to \infty} g_i a_i o = k_0 \xi_\theta \). We may also assume that \( \ell_i \) converges to some \( \ell_0 \in K \). Choose \( \xi \in \ell_0 N^+_\theta \xi_\theta \cap N^+_\theta \xi_\theta \). Then by Lemma [2.4] as \( i \to \infty \), \( g_i a_i \xi \to k_0 \xi_\theta \) and \( a_i \xi \to \xi_\theta \). Since \( g_i \) converges to \( g \), this implies that \( k_0 \xi_\theta = g \xi_\theta \). This finishes the proof.

### 3. \( \theta \)-Growth Indicators

Let \( \Gamma < G \) be a Zariski dense discrete subgroup. We set

\[
\mu_\theta := p_\theta \circ \mu : G \to \mathfrak{a}_\theta^+.
\]

**Definition 3.1.** We say that \( \Gamma \) is \( \theta \)-discrete if the restriction \( \mu_\theta|_{\Gamma} : \Gamma \to \mathfrak{a}_\theta^+ \) is proper.

The \( \theta \)-discreteness of \( \Gamma \) implies that \( \mu_\theta(\Gamma) \) is a closed discrete subset of \( \mathfrak{a}_\theta^+ \). Indeed, \( \Gamma \) is \( \theta \)-discrete if and only if the counting measure on \( \mu_\theta(\Gamma) \) weighted with multiplicity is a Radon measure on \( \mathfrak{a}_\theta^+ \).
Definition 3.2 ($\theta$-growth indicator). For a $\theta$-discrete subgroup $\Gamma < G$, we define the $\theta$-growth indicator $\psi_\theta^{\Gamma} : a_\theta \rightarrow [-\infty, \infty]$ as follows: if $u \in a_\theta$ is non-zero,

$$\psi_\theta^{\Gamma}(u) = \|u\| \inf_{u \in C} \tau_C^\theta$$

(3.2)

where $C \subset a_\theta$ ranges over all open cones containing $u$, and $\psi_\theta^{\Gamma}(0) = 0$. Here $-\infty \leq \tau_C^\theta \leq \infty$ denotes the abscissa of convergence of the series $P_C^\theta(s) = \sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in C} e^{-s\|\mu_\theta(\gamma)\|}$, that is,

$$\tau_C^\theta = \sup\{s \in \mathbb{R} : P_C^\theta(s) = \infty\} = \inf\{s \in \mathbb{R} : P_C^\theta(s) < \infty\}.$$  

This definition is independent of the choice of a norm on $a_\theta$. For $\theta = \Pi$, we set $\psi_\theta^{\Gamma} := \psi_\Pi^{\Gamma}$.

The main goal of this section is to establish the following properties of $\psi_\theta^{\Gamma}$ for a general $\theta \subset \Pi$: for $\theta = \Pi$, this theorem is due to Quint [29, Theorem 1.1.1].

Theorem 3.3. Let $\Gamma < G$ be a $\theta$-discrete subgroup.

1. $\psi_\theta^{\Gamma} < \infty$.
2. $\psi_\theta^{\Gamma}$ is a homogeneous, upper semi-continuous and concave function.
3. $L_\theta = \{\psi_\theta^{\Gamma} \geq 0\}$, $\psi_\theta^{\Gamma} = -\infty$ outside $L_\theta$ and $\psi_\theta^{\Gamma} > 0$ on int $L_\theta$.

Here, $L_\theta \subset a_\theta^+$ denotes the $\theta$-limit cone of $\Gamma$, which is the asymptotic cone of $\mu_\theta(\Gamma)$:

$$L_\theta = \{\lim t_i \mu_\theta(\gamma_i) : \gamma_i \in \Gamma, t_i \rightarrow 0\}.  \quad (3.3)$$

We set $L = L_{\Pi}$, which is the usual limit cone. By [3, Sections 1.2, 4.6], $L$ is a convex cone with non-empty interior and $\mu(\Gamma)$ is within a bounded distance from $L$. We have

$$L = \{\psi_\Gamma \geq 0\}, \quad \psi_\Gamma > 0 \text{ on int } L \quad (3.4)$$

and $\psi_\Gamma = -\infty$ outside $L$ [29 Theorem 1.1.1]. Noting that $L_\theta = p_\theta(L)$, we get:

Lemma 3.4. The $\theta$-limit cone $L_\theta$ is a convex cone in $a_\theta^+$ with non-empty interior and $\mu_\theta(\Gamma)$ is within a bounded distance from $L_\theta$.

$\psi_\theta^{\Gamma} < \infty$ and $\theta$-critical exponent. In this subsection, we show Theorem 3.3(1), that is, $\psi_\theta^{\Gamma}$ does not take $+\infty$-value. This will be achieved by proving $\delta_\theta^{\Gamma} < \infty$ (Proposition 3.7) where

$$-\infty \leq \delta_\theta^{\Gamma} \leq \infty$$

denotes the abscissa of convergence of the series $s \mapsto \sum_{\gamma \in \Gamma} e^{-s\|\mu_\theta(\gamma)\|}$. For $\theta = \Pi$, we have $0 < \delta_\Pi = \delta_\Gamma^{\Pi} \leq \infty$ [29 Theorem 4.2.2]. Since $\|\mu_\theta(g)\| \leq \|\mu(g)\|$ for all $g \in G$ and hence $\sum_{\gamma \in \Gamma} e^{-s\|\mu_\theta(\gamma)\|} \leq \sum_{\gamma \in \Gamma} e^{-s\|\mu_\theta(\gamma)\|}$ for all $s \geq 0$, we have

$$0 < \delta_\Gamma \leq \delta_\theta^{\Gamma}. \quad (3.5)$$
Lemma 3.5. If $\Gamma$ is $\theta$-discrete, then
\[
\delta_\Gamma^\theta = \limsup_{t\to\infty} \frac{1}{t} \# \{ \gamma \in \Gamma : \|\mu_\theta(\gamma)\| < t \} \in (0, \infty].
\]

Proof. For $x \in a_\theta$, we denote by $D_x$ the Dirac mass at $x$. Since $\sum_{\gamma \in \Gamma} D_{\mu_\theta(\gamma)}$ is a Radon measure on $a_\theta^+$ and $\delta_\Gamma^\theta > 0$ by (3.5), it follows from [29] Lemma 3.1.1.

For a general discrete subgroup $\Gamma < G$, $\delta_\Gamma^\theta$ may be infinite (e.g., $\Gamma = \Gamma_1 \times \Gamma_2$ where $\Gamma_i$ is an infinite discrete subgroup of $G_i$ for both $i = 1, 2$). Since $r_C^\theta \leq \delta_\Gamma^\theta$ for all cones $C$ in $a_\theta$, we have
\[
\sup_{u \in a_\theta, \|u\|=1} \psi_\Gamma^\theta(u) \leq \delta_\Gamma^\theta.
\]
Hence Theorem [3.3.1] follows once we show the that $\delta_\Gamma^\theta < \infty$ for any $\theta$-discrete subgroup $\Gamma < G$ as in Proposition [3.7].

![Figure 1](image.png)

**Figure 1.** $G = \text{PSL}_3(\mathbb{R})$ and $\theta = \{\alpha_1\}$.

Lemma 3.6. If $p_\theta|_{a^+}$ is a proper map (e.g., $G$ is simple), then
\[
\delta_\Gamma^\theta < \infty
\]
for any discrete subgroup $\Gamma < G$. In particular, if $G$ is simple, any discrete subgroup $\Gamma < G$ is $\theta$-discrete.

Proof. First, observe that if $G$ is simple, then the angle between any two walls of $a^+$ is strictly smaller than $\pi/2$ and hence $p_\theta|_{a^+}$ is a proper map (see Figure 1). Now, if $p_\theta|_{a^+}$ is a proper map, then for some constant $C > 1$, we have
\[
C^{-1}\|u\| \leq \|p_\theta(u)\| \leq C\|u\|
\]
for all $u \in a^+$. Hence $\delta_\Gamma < \infty$ implies that
\[
\delta_\Gamma^\theta < \infty.
\]
\[\square\]
It follows from the definition of $\delta^\theta_\Gamma$ that the finiteness of $\delta^\theta_\Gamma$ implies the $\theta$-discreteness of $\Gamma$. Indeed, the converse holds as well from which Theorem 3.3(1) follows.

**Proposition 3.7.** We have

$$\Gamma \text{ is } \theta\text{-discrete if and only if } \delta^\theta_\Gamma < \infty.$$  

**Proof.** It suffices to show that the $\theta$-discreteness of $\Gamma$ implies $\delta^\theta_\Gamma < \infty$. Write $G = G_1G_2$ as an almost direct product of semisimple real algebraic groups where $G_1$ is the smallest group such that $\theta$ is contained in the set of simple roots for $(\mathfrak{g}_1, \mathfrak{a}_1^+ = \mathfrak{a}^+ \cap \mathfrak{g}_1)$. Then $\mu_\theta(\Gamma) \subseteq \mathfrak{a}_\theta^+ \subseteq \mathfrak{a}_1^+$. Since the kernel of $p_\theta|_{\mu(\Gamma)}$ contains $\mu(\Gamma \cap \{e\} \times G_2)$, the properness hypothesis implies that $\Gamma \cap \{e\} \times G_2$ is finite. By passing to a subgroup of finite index, we may assume that $\Gamma \cap \{e\} \times G_2$ is trivial. The properness of $\mu_\theta|_{\Gamma}$ also implies that the projection of $\Gamma$ to $G_1$ is a discrete subgroup, which we denote by $\Gamma_1$.

Since there exists a unique element, say, $\sigma(\gamma_1) \in G_2$ such that $(\gamma_1, \sigma(\gamma_1)) \in \Gamma$ for each $\gamma_1 \in \Gamma_1$, we get a faithful representation $\sigma : \Gamma_1 \to G_2$, and $\Gamma$ is of the form $\{(\gamma_1, \sigma(\gamma_1)) : \gamma \in \Gamma_1\}$. Since $\mu_\theta(\gamma) = \mu_\theta(\gamma_1)$ for $\gamma = (\gamma_1, \sigma(\gamma_1)) \in \Gamma$, we have

$$\delta^\theta_\Gamma = \delta^\theta_{\Gamma_1}.$$  

Hence we may assume without loss of generality that $\theta$ contains at least one root of each simple factor of $G$. Since the restriction $p_\theta : \mathfrak{a}^+ \cap \text{Lie } G_0 \to \mathfrak{a}_\theta \cap \text{Lie } G_0$ is a proper map for each simple factor $G_0$ of $G$ as mentioned before, it follows that $p_\theta$ is a proper map. Hence the claim $\delta^\theta_\Gamma < \infty$ follows by Lemma 3.6. $$\square$$

**Concavity of $\psi^\theta_\Gamma$.** The growth indicator $\psi^\theta_\Gamma$ is clearly a homogeneous and upper semi-continuous function [29, Lemma 3.1.7]. It is also a concave function, but its proof requires the following lemma, which is proved in [20, Proposition 2.3.1] for $\theta = \Pi$.

**Lemma 3.8.** Suppose that $\Gamma$ is $\theta$-discrete. Then there exists a map $\pi : \Gamma \times \Gamma \to \Gamma$ satisfying the following:

1. there exists $\kappa \geq 0$ such that for every $\gamma_1, \gamma_2 \in \Gamma$,

   $$\|\mu_\theta(\pi(\gamma_1, \gamma_2)) - \mu_\theta(\gamma_1) - \mu_\theta(\gamma_2)\| < \kappa; \text{ and}$$

2. for every $R \geq 0$, there exists a finite subset $H$ of $\Gamma$ such that for $\gamma_1, \gamma_1', \gamma_2, \gamma_2' \in \Gamma$ with $\|\mu_\theta(\gamma_i) - \mu_\theta(\gamma_i')\| \leq R$ for $i = 1, 2$,

   $$\pi(\gamma_1, \gamma_2) = \pi(\gamma_1', \gamma_2') \Rightarrow \gamma_1' \in \gamma_1H \text{ and } \gamma_2' \in \gamma_2H.$$  

**Proof.** Since $p_\theta$ is norm-decreasing, (1) follows from [29, Proposition 2.3.1(1)]. By the proof of [29, Proposition 2.3.1(2)], the claim (2) holds if we set $H$ to be the subset consisting of all elements $\gamma \in \Gamma$ such that $\mu_\theta(\gamma) < R'$ for some $R'>0$ depending only on $R$. Since $\Gamma$ is $\theta$-discrete, this subset $H$ is finite, as desired. $$\square$$

**Proposition 3.9.** If $\Gamma$ is $\theta$-discrete, then $\psi^\theta_\Gamma$ is concave.
Since \( \| \epsilon > 0 \) following, let \( \theta \). For any \( \psi \) tangent to \( \Theta \). For any \( \nu \) we have the following corollary: \( \psi \) growth (see [29, Section 3.2] for details). It follows from [29, Theorem 3.2.1] that \( \psi^\theta \) is concave.

A linear form \( \psi \in a_\theta^* \) is said to be tangent to \( \psi^\theta \) (at \( u \in a_\theta^+ - \{ 0 \} \)) if \( \psi \geq \psi^\theta \) on \( a_\theta^+ \) and \( \psi(u) = \psi^\theta(u) \). By the supporting hyperplane theorem, we have the following corollary:

**Corollary 3.10.** For any \( u \in \text{int} \, L_\theta \), there exists a linear form \( \psi \in a_\theta^* \) tangent to \( \psi^\theta \) at \( u \).

**Positivity of \( \psi^\theta \).** By Lemma 3.4 we have \( \psi^\theta = -\infty \) outside \( L_\theta \). If \( \Theta \supset \theta \), then any \( \theta \)-discrete \( \Gamma \) is \( \Theta \)-discrete as well. The following lemma shows how \( \psi^\theta \) is related to \( \psi^\Theta \) from which Theorem 3.3(3) follows:

**Lemma 3.11.** For \( \Theta \supset \theta \), let \( p_\theta = p_\theta|_{a_\theta} : a_\Theta \to a_\theta \) by abuse of notation. For any \( \theta \)-discrete \( \Gamma < G \), we have

\[
\psi^\theta \circ p_\theta \geq \psi^\Theta \quad \text{on } a_\Theta. \tag{3.6}
\]

In particular,

\[
\psi^\theta_1 \geq 0 \quad \text{on } L_\theta \quad \text{and} \quad \psi^\theta_1 > 0 \quad \text{on } \text{int} \, L_\theta. \tag{3.7}
\]

**Proof.** By (3.4) and the homogeneity, it suffices to prove (3.6) for a unit vector \( u \in L_\theta \). Let \( v \in p_\theta^{-1}(u) \cap a_\Theta \). By (3.4), it suffices to consider the case when \( \psi^\theta(v) \geq 0 \). Let \( C \subset a_\theta \) be an open cone containing \( u \). For each \( \epsilon > 0 \), set

\[
C(v, \epsilon) := \left\{ w \in a_\Theta : p_\theta(w) \neq 0 \quad \text{and} \quad \left| \frac{w}{\|p_\theta(w)\|} - v \right| < \epsilon \right\}. \tag{3.8}
\]

Since \( \|p_\theta(v)\| = \|u\| = 1 \), \( C(v, \epsilon) \) is an open cone containing \( v \). In the following, let \( \epsilon > 0 \) be small enough so that \( C(v, \epsilon) \subset p_\theta^{-1}(C) \).

Then for all \( s \in \mathbb{R} \), we have

\[
\sum_{\gamma \in \Gamma, \nu_\theta(\gamma) \in C(v, \epsilon)} e^{-s\|\nu_\theta(\gamma)\|} \leq \sum_{\gamma \in \Gamma, \nu_\theta(\gamma) \in C(v, \epsilon)} e^{-(s\|v\| - |\epsilon s|)\|\nu_\theta(\gamma)\|} \leq \sum_{\gamma \in \Gamma, \nu_\theta(\gamma) \in C} e^{-(s\|v\| - |\epsilon s|)\|\nu_\theta(\gamma)\|}.
\]

Hence we have

\[
\tau^\Theta_{C(v, \epsilon)} \leq (\|v\| - \epsilon)^{-1} \tau^\Theta_C.
\]

Therefore we have

\[
\psi^\Theta(v) \leq \|v\| \tau^\Theta_{C(v, \epsilon)} \leq \|v\| (\|v\| - \epsilon)^{-1} \tau^\Theta_C.
\]

Taking \( \epsilon \to 0 \) yields that

\[
\psi^\Theta(v) \leq \tau^\Theta_C.
\]

Since \( C \subset a_\Theta \) is an arbitrary open cone in \( a_\theta \) containing \( u \), it follows that

\[
\psi^\Theta(v) \leq \psi^\theta(u),
\]
and hence (3.6) is proved. Last claim follows the from (3.4) and (3.6) applied to Θ = Π.

□

Comparison between \( \psi^\theta \Gamma \) and \( \psi^\Theta \Gamma \). Note that the properness of \( p^\theta|_L \) implies the \( \theta \)-discreteness of \( \Gamma \) as \( \mu(\Gamma) \) is within a bounded distance from \( L \). The following lemma is to appear in [12] in a more general context.

**Lemma 3.12.** If \( p^\theta|_L \) is a proper map (e.g., \( G \) is simple), then for any \( \Theta \supset \theta \) and for any \( u \in a^\theta \),

\[
\psi^\theta \Gamma (u) = \max_{v \in p^{-1}(u)} \psi^\theta \Gamma (v) \tag{3.9}
\]

where \( p^\theta = p^\theta|_{a^\theta} \) by abuse of notation.

**Proof.** Suppose that \( p^\theta|_L : L \to a^\theta \) is a proper map. By Lemma 3.11, it suffices to consider a unit vector \( u \in L^\theta \) with \( \psi^\theta \Gamma (u) > 0 \). Since \( p^{-1}(u) \cap L^\Theta \) is a compact subset and \( \psi^\theta \Gamma \) is upper semi-continuous, we have

\[
\sup_{v \in p^{-1}(u)} \psi^\theta \Gamma (v) = \max_{v \in p^{-1}(u) \cap L^\Theta} \psi^\theta \Gamma (v). \tag{3.10}
\]

For all sufficiently small \( \varepsilon > 0 \) and each \( v \in p^{-1}(u) \), there exists \( 0 < \varepsilon_v < \varepsilon \) such that

\[
\|v\|_{C(v, \varepsilon_v)}^\theta \psi^\theta \Gamma (v) < \psi^\theta \Gamma (v) + \varepsilon
\]

where \( C(v, \varepsilon_v) \) is as defined in (3.8). Since \( p^{-1}(u) \cap L^\Theta \) is compact, there exist \( v_1, \ldots, v_n \in p^{-1}(u) \) such that

\[
p^{-1}(u) \cap L^\Theta \subset \bigcup_{i=1}^n C(v_i, \varepsilon_{v_i}).
\]

Take an open cone \( C \subset a^\theta \) containing \( u \) such that

\[
p^{-1}(u) \cap L^\Theta \subset p^{-1}(C) \cap L^\Theta \subset \bigcup_{i=1}^n C(v_i, \varepsilon_{v_i}).
\]

This is indeed possible; if not, there is a sequence of unit vectors \( u_j \in a^\theta \) converging to \( u \) as \( j \to \infty \) such that for each \( j \), there exists \( w_j \in p^{-1}(u_j) \cap L^\Theta \) that does not belong to \( \bigcup_{i=1}^n C(v_i, \varepsilon_{v_i}) \). Since \( p^\theta|_{L^\Theta} \) is proper and the unit sphere in \( a^\theta \) is compact, we may assume that the sequence \( w_j \) converges to some \( w \in L^\Theta \) after passing to a subsequence. Since \( p^\theta(w_j) = u_j \to u \) as \( j \to \infty \), we have \( p^\theta(w) = u \), and hence \( w \in p^{-1}(u) \cap L^\Theta \). It implies that \( w_j \in \bigcup_{i=1}^n C(v_i, \varepsilon_{v_i}) \) for all large \( j \), contradiction.

Since \( \mu^\theta(\Gamma) \) is within a bounded distance from \( L^\Theta \) (Lemma 3.4), there are only finitely many elements of \( \mu^\theta(\Gamma) \) outside of \( \bigcup_{i=1}^n C(v_i, \varepsilon_{v_i}) \). Hence
for each $s \geq 0$, we have
\[
\sum_{\gamma \in \Gamma} e^{-s\|\mu_\theta(\gamma)\|} \leq \sum_{i=1}^n \sum_{\gamma \in \Gamma} e^{-s\|\mu_\theta(\gamma)\|} = \sum_{i=1}^n \sum_{v_i \in \mathcal{C}(v_{i-1}, \varepsilon_{v_i})} e^{-s(1-\varepsilon_{v_i}) \|\mu_\theta(\gamma)\|/\|v_i\|}. 
\]

Here and afterwards, the notation $f(s) \ll g(s)$ means that for some uniform constant $C \geq 1$, $f(s) \leq Cg(s)$ for all $s$ at hand. Since $\tau^\theta_C \geq \psi^\theta_T(u) > 0$ is positive, it follows that
\[
\tau^\theta_C \leq \max_i \frac{1}{1-\varepsilon_{v_i}} \|v_i\| \max_i \psi^\theta_T(v_i). 
\]

Therefore, together with $0 < \varepsilon_{v_i} < \varepsilon$ and (3.10), we get
\[
\psi^\theta_T(u) \leq \tau^\theta_C \leq \frac{1}{1-\varepsilon} \left( \max_i \psi^\theta_T(v_i) + \varepsilon \right) \leq \frac{1}{1-\varepsilon} \left( \max_{v \in p_{\alpha_{i}^{-1}}(u)} \psi^\theta_T(v) + \varepsilon \right).
\]

Since $0 < \varepsilon < 1$ was arbitrary, this proves the claim by Lemma 3.11. $\square$

**Bounds for $\psi^\theta_T$ for $G = \text{PSL}_d(\mathbb{R})$.** We discuss some explicit upper bounds for $\psi^\theta_T$ when $G = \text{PSL}_d(\mathbb{R})$. Identify $\mathfrak{a}^+ = \{(t_1, \cdots, t_d) : t_1 \geq \cdots \geq t_d, t_1 + \cdots + t_d = 0\}$. Let $\alpha_i(t_1, \cdots, t_d) = t_i - t_{i+1}$ for $i = 1, 2, \cdots, d - 1$. Let
\[
\begin{align*}
\mathbf{w}_i &= \left( \frac{d-i}{d}, \cdots, \frac{d-i}{d}, -\frac{i}{d}, \cdots, -\frac{i}{d} \right), \\
\end{align*}
\]
where the first $i$ coordinates are $\frac{d-i}{d}$’s and the last $d - i$ coordinates are $-\frac{i}{d}$’s, so that $\mathbf{a}_{\alpha_i} = \mathbb{R}\mathbf{w}_i$ and $\alpha_i(w_i) = 1$. We compute that
\[
p_{\alpha_i}(t_1, \cdots, t_d) = \frac{d(t_1 + \cdots + t_i)}{i(d-i)} \mathbf{w}_i
\]
and hence
\[
p_{\alpha_i}^{-1}(w_i) \cap \mathfrak{a}^+ = \{(t_1, \cdots, t_d) \in \mathfrak{a}^+ : d(t_1 + \cdots + t_i) = i(d-i)\}.
\]

For any non-lattice discrete subgroup $\Gamma < \text{PSL}_d(\mathbb{R})$, we have
\[
\psi_T(t_1, \cdots, t_d) \leq \sum_{i<j} (t_i - t_j) - \frac{1}{2} \sum_{i=1}^{[d/2]} (t_i - t_{d+1-i}) \quad (3.11)
\]
by \cite{[32], [25], [24] Theorem 7.1}. By Lemma 3.12, for any discrete non-lattice subgroups, we get
\[
\psi_{\alpha_i}^\theta(w_i) \leq \max_{i<j} (t_i - t_j) - \frac{1}{2} \sum_{i=1}^{[d/2]} (t_i - t_{d+1-i}) \quad (3.12)
\]
where the maximum is taken over all $(t_1, \cdots, t_d) \in \mathfrak{a}^+$ such that $d(t_1 + \cdots + t_i) = i(d-i)$. 

For instance, for \( d = 3 \), the right hand side is always 3 and hence for each \( i = 1, 2 \), \( \psi_{\Gamma}^{\alpha_i} \leq 3\alpha_i \) on \( \mathbb{R}w_i \).

**Hitchin subgroups.** Let \( \iota : \text{PSL}_2(\mathbb{R}) \to \text{PSL}_d(\mathbb{R}) \) be the irreducible representation, which is unique up to conjugations. A Hitchin subgroup is the image of a representation \( \pi : \Sigma \to \text{PSL}_d(\mathbb{R}) \) of a non-elementary geometrically finite subgroup \( \Sigma \subset \text{PSL}_2(\mathbb{R}) \), which belongs to the same connected component as \( \iota|_{\Sigma} \) in the character variety \( \text{Hom}(\Sigma, \text{PSL}_d(\mathbb{R}))/\sim \) where the equivalence is given by conjugations. Hitchin subgroups are \( \Pi \)-transverse, as defined in the introduction, by [5] and hence \( \alpha_i \)-discrete for each \( i = 1, \ldots, d - 1 \). It follows from Lemma 4.5 that if \( \delta_{\alpha_i} \) denotes the abscissa of convergence of \( s \mapsto \sum_{\gamma \in \Gamma} e^{-s\alpha_i(\mu(\gamma))} \), then

\[
\psi_{\Gamma}^{\alpha_i}(w_i) \leq \delta_{\alpha_i} \cdot \alpha_i(w_i) = \delta_{\alpha_i}.
\]

For Hitchin subgroups, it was proved by Potrie and Sambarino [27] for \( \Delta \) cocompact and Canary, Zhang and Zimmer [6] for \( \Delta \) geometrically finite that

\[
\delta_{\alpha_i} \leq 1
\]

for all \( i \) (see also [28]). Hence \( \max_{1 \leq i \leq d - 1} \psi_{\Gamma}^{\alpha_i}(w_i) \leq 1 \). We get a sharper bound in the following:

**Corollary 3.13.** Let \( \Gamma \subset \text{PSL}_d(\mathbb{R}) \) be a Zariski dense Hitchin subgroup. For each \( i = 1, \ldots, d - 1 \),

\[
\psi_{\Gamma}^{\alpha_i} \leq \frac{\max(i, d - i)}{d - 1} \alpha_i \quad \text{on} \quad a_{\alpha_i}.
\]

**Proof.** For a Zariski dense Hitchin subgroup \( \Gamma < G \), it is shown in [19] Corollary 1.10] that

\[
\psi_{\Gamma}(t_1, \ldots, t_d) < \frac{1}{d - 1}(t_1 - t_d) \quad \text{for} \quad (t_1, \ldots, t_d) \in a^+ - \{0\}. \tag{3.13}
\]

Indeed, [19] Corollary 1.10] is stated only for \( \Sigma \) cocompact. However in view of [6] mentioned above, this bound holds for a general Hitchin subgroup. Hence by Lemma 3.12] we get

\[
\psi_{\Gamma}^{\alpha_i}(w_i) < \frac{1}{d - 1} \max(t_1 - t_d) \tag{3.14}
\]

where the maximum is taken over all \( t_1 \geq \cdots \geq t_d \) such that \( d \sum_{j=1}^{i} t_j = i(d - i) \) and \( \sum_{j=1}^{d} t_j = 0 \). Suppose that this maximum is realized at \( (t_1, \ldots, t_d) \). Since \( t_1 - t_d \) does not involve any \( t_j \), \( 2 \leq j \leq d - 1 \), we may assume that \( t_2 = \cdots = t_{i+1} = \cdots = t_{d-1} \), which we denote by \( x \) and \( y \) respectively. Since \( \sum_{j=1}^{i} t_j = \frac{i(i-d)}{d} \) and \( \sum_{j=i+1}^{d} t_j = -\frac{i(d-i)}{d} \), we then have

\[
t_1 = \frac{i(d-i)}{d} - (i-1)x \quad \text{and} \quad t_d = -\frac{i(d-i)}{d} - (d-1-i)y.
\]

Therefore

\[
t_1 - t_d = \frac{2i(d-i)}{d} - ((i-1)x - (d-1-i)y) \tag{3.15}
\]
where \( \frac{d-1}{d} \geq x \geq y \geq -\frac{j}{d} \). It follows from \( t_j \geq t_{j+1} \) for all \( j \) that \( \frac{d-1}{d} \geq x \geq y \geq -\frac{j}{d} \). Therefore, for each fixed \( x \), the maximum in (3.15) is obtained when \( y = x \). Hence we have

\[
\psi^{\alpha_i}(w_i) < \frac{1}{d-1} \max_{x \in [-i/(d-1) \times d]} \frac{2i(d-i)}{d} - (2i-d)x
\]

\[
= \frac{1}{d-1} \max(i, d-i).
\]

\( \Box \)

**Remark 3.14.** We remark that the \( \theta \)-discreteness of \( \Gamma \) does not necessarily imply that the map \( p_\theta|_\mathcal{L} \) is a proper map. For example, let \( \Gamma_0 \) be a non-elementary convex cocompact subgroup of \( \mathrm{SO}^\circ(k, 1) \), \( k \geq 2 \), and let \( \sigma : \Gamma_0 \to \mathrm{SO}^\circ(k, 1) \) be a discrete faithful representation such that \( \sigma(\Gamma_0) \) is not convex cocompact. Consider \( \Gamma = \{(g, \sigma(g)) : g \in \Gamma_0\} \) and \( G = \mathrm{SO}^\circ(k, 1) \times \mathrm{SO}^\circ(k, 1) \). We may identify \( a = \{(x_1, x_2) \in \mathbb{R}^2\} \) and \( a^+ = \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \). Then the limit cone of \( \Gamma \) is a convex cone of \( a^+ \) containing the \( x_1 \)-axis; otherwise, \( \sigma \) must be convex cocompact. Hence for \( \theta = \{\alpha_2\} \) where \( \alpha_2(x_1, x_2) = x_2 \), \( p_{\theta}^{-1}(0) \) is the whole \( x_1 \)-axis, and hence \( p_\theta|_\mathcal{L} \) is not proper. On the other hand, the discreteness of \( \sigma(\Gamma_0) \) is same as \( \theta \)-discreteness of \( \Gamma \).

4. On the proper and critical linear forms

Let \( \Gamma \) be a \( \theta \)-discrete subgroup of \( G \).

**Definition 4.1.** A linear form \( \psi \in a_\theta^* \) is called \((\Gamma, \theta)\)-proper if \( \text{Im}(\psi \circ \mu_\theta) \subset [-\varepsilon, \infty) \) and \( \psi \circ \mu_\theta : \Gamma \to [-\varepsilon, \infty) \) is proper for some \( \varepsilon > 0 \).

Consider the series \( P_\psi = P_{\Gamma, \psi} \) given by

\[
P_\psi(s) = \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))},
\]

(4.1)

The abscissa of convergence of \( P_\psi \) is well-defined for a \((\Gamma, \theta)\)-proper linear form:

**Lemma 4.2.** If \( \psi \) is \((\Gamma, \theta)\)-proper, the following \( \delta_\psi = \delta_\psi(\Gamma) \) is well-defined and positive (possibly \( +\infty \)):

\[
\delta_\psi := \sup \{s \in \mathbb{R} : P_\psi(s) = \infty\} = \inf \{s \in \mathbb{R} : P_\psi(s) < \infty\} \in (0, \infty].
\]

Moreover,

\[
\delta_\psi = \limsup_{t \to \infty} \frac{\log \# \{\gamma \in \Gamma : \psi(\mu_\theta(\gamma)) \leq t\}}{t}.
\]

**Proof.** Since \( \psi \) is \((\Gamma, \theta)\)-proper, \( \psi(\mu_\theta(\gamma)) > 0 \) for all but finitely many \( \gamma \in \Gamma \). Hence we may replace \( P_\psi(s) \) by the series \( P_\psi^+(s) = \sum_{\gamma \in \Gamma, \psi(\mu_\theta(\gamma)) > 0} e^{-s\psi(\mu_\theta(\gamma))} \) in proving this claim. Since \( P_\psi^+(s) \) is a decreasing function of \( s \in \mathbb{R} \), \( I_1 := \{P_\psi(s) = \infty\} \) and \( I_2 := \{P_\psi(s) < \infty\} \) are disjoint intervals. Since \( \Gamma \) is infinite, \( 0 \in I_1 \), and hence \( \delta_\psi = \overline{I}_1 \cap \overline{I}_2 \in [0, \infty] \) is well-defined. To show
\( \delta_\psi > 0 \), fix \( u \in \text{int} \mathcal{L}_\theta \). Then \( \psi(u) > 0 \) by Lemma 4.3. Since \( \psi^{\theta}_1(u) > 0 \) as well by Theorem 3.3(3), we have \( s_0 \psi(u) < \psi^{\theta}_1(u) \) for some \( 0 < s_0 < \infty \).

By [29, Lemma 3.1.3], we have \( \mathcal{P}_\psi(s_0) = \infty \), and therefore \( \delta_\psi \geq s_0 > 0 \). The last claim follows by [29, Lemma 3.1.1] since the counting measure on \( \psi(\mu_\theta(\Gamma)) \) is locally finite and \( \delta_\psi > 0 \). \( \square \)

Hence for a \((\Gamma, \theta)\)-proper form \( \psi \), \( 0 < \delta_\psi \leq \infty \) is the abscissa of convergence of \( \mathcal{P}_\psi(s) \).

**Lemma 4.3.**

1. If \( \psi > 0 \) on \( \mathcal{L}_\theta - \{0\} \), then \( \psi \) is \((\Gamma, \theta)\)-proper and \( \delta_\psi < \infty \).
2. If \( \psi \) is \((\Gamma, \theta)\)-proper, then \( \psi \geq 0 \) on \( \mathcal{L}_\theta \) and \( \psi > 0 \) on \( \text{int} \mathcal{L}_\theta \).

**Proof.** If \( \psi \) is positive on \( \mathcal{L}_\theta - \{0\} \), then \( \psi > 0 \) on some open cone \( C \) containing \( \mathcal{L}_\theta - \{0\} \). Then for some \( c > 1 \), \( c^{-1}\|u\| \leq \psi(u) \leq c\|u\| \) for all \( u \in C \).

Since there can be only finitely many points of \( \mu_\theta(\Gamma) \) outside \( C \) by Lemma 3.4, this implies that \( \psi \) is \((\Gamma, \theta)\)-proper. Since \( \delta_\psi < \infty \) by Proposition 3.7, we also have \( \delta_\psi < \infty \).

To prove (2), suppose to the contrary that \( \psi(u) < 0 \) for some \( u \in \mathcal{L}_\theta \). Then there exists an open cone \( C \subset \mathcal{L}_\theta \) so that \( \psi < 0 \) on \( C \). In particular, there are infinitely many \( \gamma_i \in \Gamma \) such that \( \psi(\mu_\theta(\gamma_i)) < 0 \), which contradicts \((\Gamma, \theta)\)-properness of \( \psi \). Therefore, \( \psi \geq 0 \) on \( \mathcal{L}_\theta \). Since \( \ker \psi \) is a hyperplane in \( a_\theta \), it follows \( \psi > 0 \) on \( \text{int} \mathcal{L}_\theta \). \( \square \)

**Critical forms.** Analogous to the critical exponent of a discrete subgroup of a rank one Lie group, we define:

**Definition 4.4.** A linear form \( \psi \in a_\theta^* \) is \((\Gamma, \theta)\)-critical if it is tangent to \( \psi^\theta_1 \).

The following lemma can be proved by adapting the proof of [19, Theorem 2.5] replacing \( \psi_1 \) by \( \psi^\theta_1 \).

**Lemma 4.5.** If a \((\Gamma, \theta)\)-proper \( \psi \in a_\theta^* \) satisfies \( \delta_\psi < \infty \), then \( \delta_\psi \psi \) is \((\Gamma, \theta)\)-critical; in particular,

\[
\psi^\theta_\Gamma \leq \delta_\psi \psi.
\]

**Proof.** Suppose that \( \delta_\psi < \infty \). By Lemma 4.2, \( \delta_\psi > 0 \). We first claim

\[
\psi^\theta_\Gamma(v) \leq \delta_\psi \psi(v) \quad \text{for all } v \in \text{int} \mathcal{L}_\theta.
\]  

(4.3)

Fix \( v \in \text{int} \mathcal{L}_\theta \) and \( \varepsilon > 0 \). Since \( \psi \) is \((\Gamma, \theta)\)-proper, \( \psi(v) > 0 \) by Lemma 4.3.

We then consider

\[
\mathcal{C}_\varepsilon(v) = \left\{ w \in a_\theta : \psi(w) > 0 \text{ and } \left| \frac{\|w\|}{\psi(w)} - \frac{\|v\|}{\psi(v)} \right| < \varepsilon \right\};
\]

since \( \psi(v) > 0 \), this is a well-defined open cone containing \( v \). Therefore by the definition of \( \psi^\theta_\Gamma \), we have

\[
\psi^\theta_\Gamma(v) \leq \|v\|_{C_\varepsilon(v)}.
\]  

(4.4)
Observe that for any \( s \geq 0 \),
\[
\sum_{\gamma \in \Gamma, \mu\theta(\gamma) \in \mathcal{L}_\theta(v)} e^{-s\|\mu\theta\|} \leq \sum_{\gamma \in \Gamma, \mu\theta(\gamma) \in \mathcal{L}_\theta(v)} e^{-s\psi(\mu\theta(\gamma))(\frac{\|v\|}{\psi(v)} - \varepsilon)} \\
\leq \sum_{\gamma \in \Gamma} e^{-s\psi(\mu\theta(\gamma))(\frac{\|v\|}{\psi(v)} - \varepsilon)}.
\]
It follows from the definitions of \( \tau^{\theta}_{\mathcal{L}_\theta(v)} \) and \( \delta_\psi \) that
\[
\tau^{\theta}_{\mathcal{L}_\theta(v)} \leq \frac{\delta_\psi}{\|v\|\psi(v)^{-1} - \varepsilon} = \frac{\delta_\psi \psi(v)}{\|v\| - \varepsilon \psi(v)},
\]
and hence
\[
\psi_T^\theta(v) \leq \|v\| \frac{\delta_\psi \psi(v)}{\|v\| - \varepsilon \psi(v)}.
\]
Since \( \varepsilon > 0 \) is arbitrary, we get \( \psi_T^\theta(v) \leq \delta_\psi \psi(v) \), proving the claim (4.3).

We now claim that the inequality (4.3) also holds for any \( v \) in the boundary \( \partial \mathcal{L}_\theta \). Choose any \( v_0 \in \text{int} \mathcal{L}_\theta \). From the concavity of \( \psi_T^\theta \) (Theorem 3.3), we have
\[
tv_T^\theta(v_0) + (1 - t)\psi_T^\theta(t v_0 + (1 - t) v) \leq \psi_T^\theta(t v_0 + (1 - t) v) \quad \text{for all } 0 < t < 1.
\]
Since \( \mathcal{L}_\theta \) is convex, \( tv_0 + (1 - t)v \in \text{int} \mathcal{L}_\theta \) for all \( 0 < t < 1 \). As we have already shown \( \psi_T^\theta \leq \delta_\psi \psi \) on \( \text{int} \mathcal{L}_\theta \), we get
\[
tv_T^\theta(v_0) + (1 - t)\psi_T^\theta(v) \leq \delta_\psi \psi(t v_0 + (1 - t) v) \quad \text{for all } 0 < t < 1.
\]
By sending \( t \to 0^+ \), we get
\[
\psi_T^\theta(v) \leq \delta_\psi \cdot \psi(v).
\]
Since \( \psi_T^\theta = -\infty \) outside \( \mathcal{L}_\theta \), we have established \( \psi_T^\theta \leq \delta_\psi \psi \). Suppose that \( \psi_T^\theta < \delta_\psi \psi \) on \( a - \{0\} \). Then the abscissa of convergence of the series \( s \mapsto \sum_{\gamma \in \Gamma} e^{-s\delta_\psi \psi(\mu\theta(\gamma))} \) is strictly less than 1 by [29, Lemma 3.1.3]. However the abscissa of convergence of this series is equal to 1 by the definition of \( \delta_\psi \). Therefore \( \delta_\psi \psi \) is tangent to \( \psi_T^\theta \), finishing the proof.

**Corollary 4.6.** A \((\Gamma, \theta)\)-proper linear form \( \psi \in \mathfrak{a}_\theta^* \) with \( \delta_\psi = 1 \) is \((\Gamma, \theta)\)-critical. Moreover, if \( \psi > 0 \) on \( \mathcal{L}_\theta \), then \( \psi \) is \((\Gamma, \theta)\)-critical if and only if \( \delta_\psi = 1 \).

Via the identification \( \mathfrak{a}_\theta^* = \{ \psi \in \mathfrak{a}^* : \psi = \psi \circ p_\theta \} \), Lemma 3.12 implies the following identity:

**Corollary 4.7.** If \( p_\theta |_{\mathcal{L}} \) is a proper map, then
\[
\{ \psi \in \mathfrak{a}_\theta^* : \psi \text{ is } (\Gamma, \theta)\text{-critical} \} = \{ \psi \in \mathfrak{a}^* : \psi = \psi \circ p_\theta, \psi \text{ is } (\Gamma, \Pi)\text{-critical} \}.
\]
**Proof.** To show the inclusion \( \subseteq \), suppose \( \psi = \psi \circ p_\theta \) and \( \psi \) is \((\Gamma, \Pi)\)-critical. Then for any \( u \in \mathfrak{a}_\theta \) and any \( v' \in p_\theta^{-1}(u) \), \( \psi(u) = \psi(v') \geq \psi_T(v') \) and hence \( \psi(u) \geq \psi_T^\theta(u) \) by Lemma 3.12. Moreover, if \( \psi(v) = \psi_T(v) \), then for \( u = p_\theta(v) \), \( \psi(u) \geq \psi_T^\theta(u) \geq \psi_T(v) = \psi(v) \) and hence \( \psi(u) = \psi_T^\theta(u) \).
proving \( \psi \) is \((\Gamma, \theta)\)-critical. For the other inclusion \( \subset \), suppose that \( \psi \geq \psi^0_\Gamma \) on \( a^+_\theta \) and \( \psi(u) = \psi^0_\Gamma(u) \) for some \( u \in a^+_\theta \). Then for any \( v \in a^+ \), \( \psi(v) = \psi(p_\theta(v)) \geq \psi^0_\Gamma(p_\theta(v)) \geq \psi_\Gamma(v) \) by Lemma 3.11. Let \( v \in p_\theta^{-1}(u) \) be such that \( \psi^0_\Gamma(u) = \psi_\Gamma(v) \) given by Lemma 3.12. Then \( \psi(v) = \psi(u) = \psi^0_\Gamma(u) = \psi_\Gamma(v) \); so \( \psi \) is \((\Gamma, \Pi)\)-critical. \( \square \)

5. Limit set, \( \theta \)-conical set, and conformal measures

Let \( \Gamma < G \) be a discrete subgroup.

**Definition 5.1** \((\theta\)-limit set\). We define the \( \theta \)-limit set of \( \Gamma \) as follows:

\[
\Lambda_\theta = \Lambda_\theta(\Gamma) := \{ \lim \gamma_i \in F_\theta : \gamma_i \in \Gamma \}
\]

where \( \lim \gamma_i \) is defined as in Definition 2.2

This is a \( \Gamma \)-invariant closed subset of \( F_\theta \), which may be empty in general. Set \( \Lambda = \Lambda_\Pi \). Denote by \( \text{Leb}_\theta \) the \( K \)-invariant probability measure on \( F_\theta \). This definition of \( \Lambda_\theta \) coincides with that of Benoist:

**Lemma 5.2** ([3], [30], Corollary 5.2, Lemma 6.3, Theorem 7.2), [23], Lemma 2.13\]. If \( \Gamma \) is Zariski dense in \( G \), we have the following:

1. \( \Lambda_\theta = \{ \xi \in F_\theta : (\gamma_i)_* \text{Leb}_\theta \to D_\xi \ \text{for some infinite sequence } \gamma_i \in \Gamma \} \), where \( D_\xi \) is the Dirac measure at \( \xi \);
2. \( \Lambda_\theta = \pi_\theta(\Lambda) \);
3. \( \Lambda_\theta \) is the unique \( \Gamma \)-minimal subset of \( F_\theta \).

**Definition 5.3** \((\theta\)-conical set\). We define the \( \theta \)-conical set \( \Lambda^\text{con}_\theta \subset F_\theta \) as

\[
\Lambda^\text{con}_\theta = \{ gP_\theta \in F_\theta : \lim \sup \Gamma gA^+ \neq \emptyset \}.
\]

That is, \( g\xi_\theta \in \Lambda^\text{con}_\theta \) if and only if for some \( p \in P_\theta \), \( \gamma_i gP_{a_i} \) converges for some infinite sequence \( \gamma_i \in \Gamma \) and \( a_i \in A^+ \). If we set \( \Lambda^\text{con} = \Lambda^\text{con}_\Pi \), then

\[
\Lambda^\text{con}_\theta = \pi_\theta(\Lambda^\text{con}). \quad (5.1)
\]

Note that the conical set is not contained in the limit set \( \Lambda \) in general even for \( \theta = \Pi \). For example, if \( G = \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \) and \( \Gamma = \Gamma_1 \times \Gamma_2 \) is a product of two convex cocompact subgroups, then \( \Lambda = \Lambda(\Gamma_1) \times \Lambda(\Gamma_2) \) while \( \Lambda^\text{con} = (\Lambda(\Gamma_1) \times S^1) \cup (S^1 \times \Lambda(\Gamma_2)) \).

\( \theta \)-shadows. For \( q \in X \) and \( R > 0 \), let \( B(q, R) = \{ x \in X : d(x, q) \leq R \} \). For \( p \in X \), the \( \theta \)-shadow \( O^\theta_R(p, q) \subset F_\theta \) of \( B(q, R) \) viewed from \( p \) is

\[
O^\theta_R(p, q) = \{ gP_\theta \in F_\theta : g \in G, \ g = p, \ gA^+ \cap B(q, R) \neq \emptyset \}. \quad (5.2)
\]

Clearly, for \( O_R(p, q) = O^\Pi_R(p, q) \), we have

\[
O^\theta_R(p, q) := \pi_\theta(O_R(p, q)).
\]

It is immediate that

\[
\Lambda^\text{con}_\theta = \bigcup_{N=1}^{\infty} \Lambda^N_\theta.
\]
where
\[ \Lambda^N_{\theta} := \left\{ \xi \in \mathcal{F}_{\theta} : \text{there exists } \gamma_i \to \infty \text{ in } \Gamma \text{ such that } \xi \in \bigcap_i O^\theta_N(o, \gamma_i o) \right\}. \]

Hence \( \xi \in \Lambda^\text{con}_{\theta} \) if and only if \( \xi \in \bigcap_i O^\theta_N(o, \gamma_i o) \) for some \( N > 0 \) and an infinite sequence \( \gamma_i \in \Gamma \).

**Definition 5.4.** We say that \( \Gamma \) is \( \theta \)-regular if for any sequence \( \gamma_i \to \infty \) in \( \Gamma \), we have
\[ \min_{\alpha \in \theta} \alpha(\mu(\gamma_i)) \to \infty. \]

Observe that \( \theta \)-regularity is same as \( \theta \cup \iota(\theta) \)-regularity by (2.1) and that \( \theta \)-regular subgroups are \( \theta \)-discrete, but not conversely.

**Proposition 5.5.** If \( \Gamma \) is \( \theta \)-regular, then
1. \( \Lambda^\text{con}_{\theta} \subset \Lambda_{\theta} \);
2. for any compact subset \( Q \subset G \), the union \( \Gamma Q \cup \Lambda_{\theta} \) is compact; that is, any infinite sequence has a limit.

**Proof.** To show (1), let \( \xi \in \Lambda^\text{con}_{\theta} \). Then there exist \( g \in G \), a sequence \( \gamma_i \in \Gamma \) and \( a_i \in A^+ \) such that \( \xi = g\xi_{\theta} \) and \( d(ga_i o, \gamma_i o) \) is uniformly bounded. Since \( \mu(\gamma_i) - \log a_i \) is uniformly bounded by Lemma 2.1 and \( \min_{\alpha \in \theta} \alpha(\mu(\gamma_i)) \to \infty \) by the \( \theta \)-regularity, we have \( \min_{\alpha \in \theta} \alpha(\log a_i) \to \infty \) as \( i \to \infty \). Therefore \( ga_i o \to g\xi_{\theta} \) by Lemma 2.7. This implies that \( \gamma_i o \to g\xi_{\theta} \) by Lemma 2.6. Hence \( \xi \in \Lambda_{\theta} \). For (2), if \( \gamma_i \in \Gamma \) is an infinite sequence and \( q_i \in Q \), then \( \min_{\alpha \in \theta} \alpha(\mu(\gamma_i q_i)) \to \infty \) by the \( \theta \)-regularity of \( \Gamma \) and Lemma 2.1. Hence the claim is now immediate from Definition 2.2 and Lemma 2.6. \( \square \)

**Conformal measures.** The \( a \)-valued Busemann map \( \beta : \mathcal{F} \times G \times G \to a \) is defined as follows: for \( \xi \in \mathcal{F} \) and \( g, h \in G \),
\[ \beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi) \]
where \( \sigma(g^{-1}, \xi) \in a \) is the unique element such that we have the Iwasawa decomposition \( g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N \) for any \( k \in K \) with \( \xi = kP \). We define the \( a_\theta \)-valued Busemann map \( \beta^\theta : \mathcal{F}_\theta \times G \times G \to a_\theta \) as follows: for \( (\xi, g, h) \in \mathcal{F}_\theta \times G \times G \), we set
\[ \beta^\theta_\xi(g, h) := p_{\theta}(\beta_{\xi_0}(g, h)) \quad \text{for } \xi_0 \in \pi^{-1}_\theta(\xi); \quad (5.3) \]
this is well-defined independent of the choice of \( \xi_0 \) [30 Lemma 6.1].

**Definition 5.6** (Conformal measures). For a linear form \( \psi \in a^*_\theta \) and a closed subgroup \( \Gamma < G \), a Borel probability measure \( \nu \) on \( \mathcal{F}_{\theta} \) is called a \( (\Gamma, \psi) \)-conformal measure if
\[ \frac{d\gamma_{\psi, \nu}}{d\nu}(\xi) = e^{\psi(\beta^\theta_\xi(\gamma, \gamma))} \quad \text{for all } \gamma \in \Gamma \text{ and } \xi \in \mathcal{F}_{\theta}. \]
Following Patterson-Sullivan ([20], [36]) and Quint [30], we have the following construction of conformal measures (see also [20 Section 2], [35 Section 5], [7]).

**Proposition 5.5.** Suppose that \( \Gamma \) is \( \theta \)-regular. For any \( \psi \in a_\theta^* \) such that \( \delta_\psi = 1 \) and \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty \), there exists a \((\Gamma, \psi)\)-conformal measure supported on \( \Lambda_\theta \).

**Proof.** By Proposition 5.5, \( \Gamma \cup \Lambda_\theta \) is a compact space. Recall that \( \mathcal{P}_\psi(s) = \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))} \). As \( \delta_\psi = 1 \), \( \mathcal{P}_\psi(s) < \infty \) for \( s > 1 \). and hence we may consider the probability measure on \( \Gamma \cup \Lambda_\theta \) given by

\[
\nu_{\psi,s} := \frac{1}{\mathcal{P}_\psi(s)} \sum_{\gamma \in \Gamma} e^{-s\psi(\mu_\theta(\gamma))}.
\]  

(5.4)

Since the space of probability measures on a compact metric space is weak* compact, by passing to a subsequence, as \( s \to 1 \), \( \nu_{\psi,s} \) weakly converges to a probability measure, say \( \nu_\psi \), on \( \Gamma \cup \Lambda_\theta \). Since \( \mathcal{P}_\psi(1) = \infty \), \( \nu_\psi \) is supported on \( \Lambda_\theta \). It is standard to check that \( \nu_\psi \) is a \((\Gamma, \psi)\)-conformal measure. \( \square \)

6. **Transverse subgroups and multiplicity of \( \theta \)-shadows**

We say that a discrete subgroup \( \Gamma \leq G \) is \( \theta \)-antipodal if any two distinct points \( \xi \neq \eta \) in \( \Lambda_{\theta, \iota}(\theta) \) are in general position, i.e., \( \xi = gp_{\theta, \iota}(\theta) \) and \( \eta = gw_0p_{\theta, \iota}(\theta) \) for some \( g \in G \). Recall that a Zariski dense discrete subgroup \( \Gamma \leq G \) is called \( \theta \)-transverse if \( \Gamma \) is both \( \theta \)-regular and \( \theta \)-antipodal. Note that for \( \theta_1 \subset \theta_2 \), \( \theta_2 \)-transverse implies \( \theta_1 \)-transverse.

**Remark 6.1.** We may try to define \( \Gamma \) to be \( \theta \)-Antipodal if for any \( (\xi, \eta) \in \Lambda_{\theta, \iota}(\theta) \) such that \( \pi_{\iota}^{-1}(\xi) \cap \pi_{\iota}^{-1}(\eta) = \emptyset \), \( (\xi, \eta) \) is in general position, i.e., \( \xi = gp_{\theta, \iota}(\theta) \) and \( \eta = gw_0p_{\theta, \iota}(\theta) \) for some \( g \in G \). While the \( \theta \)-antipodality implies the \( \theta \)-Antipodality, the converse direction is not true in general; for instance, any lattice of \( \text{PSL}_3(\mathbb{R}) \) is \( \{\alpha_1\}\)-Antipodal but not \( \{\alpha_1, \alpha_2\}\)-Antipodal, i.e., not \( \{\alpha_1\}\)-antipodal, where \( \alpha_i(\text{diag}(u_1, u_2, u_3)) = u_i - u_{i+1} \) for \( i = 1, 2 \).

The main aim of this section is to prove the following proposition, which is the essential reason why the main results of this paper are proved for \( \theta \)-transverse subgroups.

**Proposition 6.2** (Bounded multiplicity of \( \theta \)-shadows). Assume that \( \Gamma \) is \( \theta \)-transverse. Let \( \phi \in a_\theta^* \) be a \((\Gamma, \theta)\)-proper linear form. Then for any \( R, D > 0 \), there exists \( q = q(\phi, R, D) > 0 \) such that for any \( T > 0 \), the collection of shadows

\[
\{O^\theta_R(\phi, \gamma) \subset F_{\theta} : T \leq \phi(\mu_\theta(\gamma)) \leq T + D \}
\]

have multiplicity at most \( q \).

The following lemma is a key ingredient of the proof of Proposition 6.2.
Lemma 6.3. Assume that $\Gamma$ is $\theta$-transverse. For any compact subset $Q$ of $G$, there exists $C_0 = C_0(Q) > 0$ such that if $\gamma_1, \gamma_2 \in \Gamma$ are such that $Q \cap \gamma_1Qa^{-1} \cap \gamma_2Qb^{-1}m^{-1} \neq \emptyset$ for some $a, b \in A^+$ and $m \in M_\theta$, then
\[
\min\{\|\mu_\theta(\gamma_2) - \mu_\theta(\gamma_1) - \mu_\theta(\gamma_1^{-1}\gamma_2)\|, \|\mu_\theta(\gamma_1) - \mu_\theta(\gamma_2) - \mu_\theta(\gamma_2^{-1}\gamma_1)\|\} \leq C_0.
\]
(6.1)

In order to prove this lemma, we need the following property of $\theta$-Cartan projection map:

Lemma 6.4. For any $C > 0$, there exists $C' > 0$ (depending only on $C$) such that if $a, b \in A^+$ and $m \in M_\theta$ satisfy $a^{-1}mb \in A_\theta^+S_\theta A_C$, then
\[
\|\mu_\theta(a^{-1}mb) + \mu_\theta(a) - \mu_\theta(b)\| \leq C',
\]
where $A_C = \{a \in A : \|\log a\| \leq C\}$.

Proof. Since $A^+ \subset A_\theta^+B_\theta$, we may write $a = a_1a_2$ and $b = b_1b_2$ where $a_1, b_1 \in A_\theta^+$ and $a_2, b_2 \in B_\theta$. Since $M_\theta$ commutes with $A_\theta$, we have $a^{-1}mb = a_1^{-1}b_1(a_2^{-1}mb_2)$. Since $a^{-1}mb \in A_\theta^+S_\theta A_C$ and $a_2^{-1}mb_2 \in S_\theta$, we have $d \in A_C \cap A_\theta$ such that $da_1^{-1}b_1 \in A_\theta^+$. Since $a_2^{-1}mb_2 \in S_\theta = M_\theta B_\theta^+$, we can write $a_2^{-1}mb_2 = m_1a'm_2$ where $m_1, m_2 \in M_\theta$ and $a' \in B_\theta^+$. Using the commutativity between $M_\theta$ and $A_\theta$ one more time, we have
\[
a^{-1}mb = m_1(a_1^{-1}b_1a')m_2.
\]
Hence $da_1^{-1}mb = m_1(da_1^{-1}b_1a')m_2$. Note that
\[
\mu_\theta(m_1(da_1^{-1}b_1a')m_2) = (p_\theta \circ \mu)(da_1^{-1}b_1a') = \log(da_1^{-1}b_1).
\]
Therefore by Lemma 2.1
\[
\|\mu_\theta(a^{-1}mb) - \log(a_1^{-1}b_1)\| \leq C'
\]
for some $C' > 0$ depending only on $C$. Since $\log a_1 = \mu_\theta(a)$ and $\log b_1 = \mu_\theta(b)$, it completes the proof. \qed

Proof of Lemma 6.3. Since $\|p_\theta(u)\| \leq \|p_{\theta,i}(u)\|$ for all $u \in a$, it suffices to prove the lemma for $\theta \cup i(\theta)$ in place of $\theta$. Therefore we may assume without loss of generality that $i(\theta) = \theta$ by replacing $\theta$ with $\theta \cup i(\theta)$.

We prove by contradiction. Suppose to the contrary that there exist sequences $q_{0,i}, q_{1,i}, q_{2,i} \in Q$, $a_i, b_i \in A^+$, $m_i \in M_\theta$ and $\gamma_{1,i}, \gamma_{2,i} \in \Gamma$ such that
\[
q_{0,i} = \gamma_{1,i}q_{1,i}a_i^{-1} = \gamma_{2,i}q_{2,i}b_i^{-1}m_i^{-1};
\]
\[
\|\mu_\theta(\gamma_{2,i}) - \mu_\theta(\gamma_{1,i}) - \mu_\theta(\gamma_{1,i}^{-1}\gamma_{2,i})\| \to \infty; \tag{6.3}
\]
\[
\|\mu_\theta(\gamma_{1,i}) - \mu_\theta(\gamma_{2,i}) - \mu_\theta(\gamma_{2,i}^{-1}\gamma_{1,i})\| \to \infty. \tag{6.4}
\]
By Lemma 2.1, it follows that all sequences $\gamma_{1,i}, \gamma_{2,i}, \gamma_{1,i}^{-1}\gamma_{2,i}$ and $\gamma_{2,i}^{-1}\gamma_{1,i}$ are unbounded. Without loss of generality, we assume that each of these...
sequences tends to infinity. By (6.2) and Lemma 2.1, there exists $C' = C'(Q) > 1$ such that
\[
\sup_i \left\{ \| \mu_\theta(\gamma_{1,i}) - \mu_\theta(a_i) \|, \| \mu_\theta(\gamma_{2,i}) - \mu_\theta(b_i) \| \right\} \leq C'
\]
(6.5)

As $\Gamma$ is $\theta$-regular, as $i \to \infty$,
\[
\alpha_{\in \theta}(\log a_i), \alpha_{\in \theta}(\log b_i) \to \infty.
\]

Note that $\alpha(\log w_0^{-1}a^{-1}w_0) = \alpha(i(\log a)) = i(\alpha)(\log a)$ for all $a \in A$ and all $\alpha \in \Phi$. Since $\theta$ is symmetric, it follows that
\[
\min_{\alpha \in \theta}(\log(w_0^{-1}a_i^{-1}w_0)), \min_{\alpha \in \theta}(\log(w_0^{-1}b_i^{-1}w_0)) \to \infty. \quad (6.6)
\]

Passing to a subsequence, we may assume that $q_{1,i}$ converges to some $q_1 \in Q$. We claim that
\[
q_1w_0\xi_\theta \in \Lambda_\theta \quad \text{and} \quad q_1m_1w_0\xi_\theta \in \Lambda_\theta \quad (6.7)
\]
for some $m_1 \in M_\theta$ and $w \in N_K(A)$. By Lemma 5.5, we may also assume that $\gamma_{1,i}q_0, o$ converges to some $\xi \in \Lambda_\theta$ as $i \to \infty$. Since $\gamma_{1,i}q_0, o = q_1,a_{i}^{-1}o = q_{1,i}w_0(w_0^{-1}a_i^{-1}w_0)\xi_\theta$, it follows from Lemma 2.7 and (6.6) that $\xi = q_1w_0\xi_\theta$. Therefore
\[
q_1w_0\xi_\theta \in \Lambda_\theta.
\]

Since $A = A_\theta B_\theta$, we may write $a_i = a_{1,i}a_{2,i} \in A_\theta B_\theta$ and $b_i = b_{1,i}b_{2,i} \in A_\theta B_\theta$. Using $S_\theta = M_\theta B_\theta^+ M_\theta$, write
\[
a_{2,i}^{-1}m_ib_{2,i} = m_{1,i}c_im_{2,i} \in M_\theta B_\theta^+ M_\theta.
\]

Then
\[
\gamma_{1,i}^{-1}\gamma_{2,i}q_{2,i} = q_{1,i}a_{i}^{-1}m_ib_i
\]
\[
= q_{1,i}(a_{1,i}^{-1}b_{1,i})(a_{2,i}^{-1}m_ib_{2,i}) = q_{1,i}m_{1,i}(a_{1,i}^{-1}b_{1,i}c_i)m_{2,i}.
\]

By passing to a subsequence, we have $w \in N_K(A)$ such that for all $i \geq 1$,
\[
d_i := w^{-1}a_{1,i}b_{1,i}c_iw \in A^+.
\]

Then we have the following:
\[
\gamma_{1,i}^{-1}\gamma_{2,i}q_{2,i} = q_{1,i}(m_{1,i}w)d_i(w^{-1}m_{2,i}) \in q_{1,i}KA^+ K. \quad (6.8)
\]

Since $\gamma_{1,i}^{-1}\gamma_{2,i} \to \infty$, by the $\theta$-regularity of $\Gamma$, we have $\min_{\alpha \in \theta}(\log d_i) \to \infty$. We may assume that $m_{1,i} \to m_1 \in M_\theta$. By Lemma 5.5 and Lemma 2.7 we get
\[
\lim_{i \to \infty} \gamma_{1,i}^{-1}\gamma_{2,i}q_{2,i} = q_1m_1w_0\xi_\theta \in \Lambda_\theta
\]
by passing to a subsequence. Hence (6.7) is proved. By the $\theta$-antipodal property of $\Gamma$, two distinct points of $\Lambda_\theta$ must be in general position; hence we must have either
\[
w_0\xi_\theta = m_1w_0\xi_\theta \quad \text{or} \quad m_1w_0\xi_\theta \in N_\theta^+ \xi_\theta.
First consider the case where \( w_0 \xi_0 = m_1 w \xi_0 \). In this case, \( m' := w_0^{-1} m_1 w \in P_\theta \cap K = M_\theta \). In particular, we have \( w = m_1^{-1} w_0 m' \). Since \( i(\theta) = \theta \), note that
\[
    w_0 A_\theta^+ w_0^{-1} = (A_\theta^+)^{-1} \quad \text{and} \quad w_0 S_\theta w_0^{-1} = S_\theta.
\]
Hence we get
\[
    a_i^{-1} m_i b_i = (a_i^{-1} b_i, c_i)(c_i^{-1} m_i, c_i m_2, i) \in (w A^+ w^{-1}) S_\theta
\]
\[
    \subset (m_1^{-1} w_0 m' A_\theta^+ m'^{-1} w_0^{-1} m_1) S_\theta \subset (A_\theta^+)^{-1} S_\theta.
\]
Therefore \( b_i^{-1} m_i^{-1} a_i \in A_\theta^+ S_\theta \). By Lemma \([6.4]\) and \([6.5]\), this implies that \( \| \mu_\theta(\gamma_1, i) - \mu_\theta(\gamma_2, i) - \mu_\theta(\gamma_2^{-1}, 1, i) \| \) is uniformly bounded, contradicting \([6.4]\).

Now the second case to consider is when \( (m_1 w) \xi_0 \in N_\theta^+ \xi_0 \). By Corollary \([2.5]\), this implies that \( w \in M_\theta \). As above,
\[
    a_i^{-1} m_i b_i = (a_i^{-1} b_i, c_i)(c_i^{-1} m_i, c_i m_2, i) \in (w A^+ w^{-1}) S_\theta \subset A_\theta^+ S_\theta.
\]

Again, by Lemma \([6.4]\) and \([6.5]\), this implies that \( \| \mu_\theta(\gamma_2, i) - \mu_\theta(\gamma_1, i) - \mu_\theta(\gamma_1^{-1}, \gamma_2, i) \| \) is uniformly bounded, contradicting \([6.3]\). This finishes the proof.

**Proof of Proposition \([6.2]\)** Suppose that there exists \( \xi \in \bigcap_{i=1}^n O_R^\theta(o, \gamma_i o) \) and \( T \leq \phi(\mu_\theta(\gamma_i)) \leq T + D \) for some distinct \( \gamma_i \in \Gamma, \ i = 1, \cdots, n \). Setting \( Q = K A R K \), let \( C_0 = C_0(Q) \) be as in Lemma \([6.3]\). Set
\[
    D' = D'(\phi, Q, D) := \| \phi \| C_0 + D
\]
where \( \| \phi \| \) is the operator norm of \( \phi : a_\theta \rightarrow \mathbb{R} \). Then the following number
\[
    q := \# \{ \gamma \in \Gamma : \phi(\mu_\theta(\gamma)) \leq D' \}
\]
is finite by the \((\Gamma, \theta)\)-properness of \( \phi \). We claim that
\[
    n \leq 2q;
\]
this proves the proposition. It suffices to show that
\[
    \max_i \min_i \{ \phi(\mu_\theta(\gamma_1^1 \gamma_i)), \phi(\mu_\theta(\gamma_i^{-1} \gamma_1)) \} \leq D', \quad (6.9)
\]
as this implies that
\[
    n = \# \{ \gamma_1, \cdots, \gamma_n \} \leq \# \{ \gamma_1 \gamma, \gamma_1^{-1} \gamma : \gamma \in \Gamma, \phi(\mu_\theta(\gamma)) \leq D' \} \leq 2q.
\]
To prove \((6.9)\), for each \( i = 1, \cdots, n \), there exist \( k_i \in K \) and \( a_i \in A^+ \) such that \( \xi = k_i \xi_0 \) and \( d(k_i a_i o, \gamma_i o) < R \). Then \( k_i = k_i m_i \) for some \( m_i \in K \cap P_\theta = M_\theta \). Hence we have \( d(\gamma_1^{-1} k_i a_i o, o) < R \) and \( d(\gamma_i^{-1} k_i m_i a_i o, o) < R \), which implies
\[
    k_i \in Q \cap \gamma_1 Q a_i^{-1} \cap \gamma_i Q a_i^{-1} m_i^{-1}.
\]
By Lemma \([6.3]\) we have
\[
    \| \mu_\theta(\gamma_i) - \mu_\theta(\gamma_1) - \mu_\theta(\gamma_1^{-1} \gamma_i) \| \leq C_0 \quad \text{or} \quad \| \mu_\theta(\gamma_1) - \mu_\theta(\gamma_i) - \mu_\theta(\gamma_1^{-1} \gamma_i) \| \leq C_0.
\]
Suppose first that \( \|\mu_{\theta}(\gamma_i) - \mu_{\theta}(\gamma_1) - \mu_{\theta}(\gamma_i^{-1}\gamma_i)\| \leq C_0 \). Now we have
\[
\phi(\mu_{\theta}(\gamma_i^{-1}\gamma_i)) = \phi(\mu_{\theta}(\gamma_i^{-1}\gamma_i) - (\mu_{\theta}(\gamma_i) - \mu_{\theta}(\gamma_1))) + \phi(\mu_{\theta}(\gamma_i) - \mu_{\theta}(\gamma_1))
\]
\[
\leq \|\phi\|C_0 + |\phi(\mu_{\theta}(\gamma_i)) - \phi(\mu_{\theta}(\gamma_1))|
\]
\[
\leq \|\phi\|C_0 + D = D'
\]
where the last inequality follows from \( \phi(\mu_{\theta}(\gamma_1)), \phi(\mu_{\theta}(\gamma_i)) \in [T, T + D] \).
When \( \|\mu_{\theta}(\gamma_1) - \mu_{\theta}(\gamma_i) - \mu_{\theta}(\gamma_i^{-1}\gamma_1)\| \leq C_0 \), similarly, we have
\[
\phi(\mu_{\theta}(\gamma_i^{-1}\gamma_1)) \leq \|\phi\|C_0 + D = D'.
\]
Therefore (6.9) follows.

7. Dimensions of conformal measures and growth indicators

For a general Zariski dense discrete subgroup \( \Gamma < G \), Quint [30, Theorem 8.1] showed that if there exists a \((\Gamma, \psi)\)-conformal measure on \( \mathcal{F}_\Pi \) for \( \psi \in a^\ast \), then
\[
\psi \geq \psi^\ast_\Gamma.
\]
The main aim of this section is to prove the following analogous inequality
for \( \theta \)-transverse subgroups, using Lemma [7.3] whose key ingredient is the control on multiplicity of shadows obtained in Proposition [6.2].

**Theorem 7.1.** Let \( \Gamma \) be a \( \theta \)-transverse subgroup of \( G \). If there exists a \((\Gamma, \psi)\)-conformal measure \( \nu \) on \( \mathcal{F}_\theta \) for a \((\Gamma, \theta)\)-proper \( \psi \) \( \in a_{\theta}^\ast \), then
\[
\psi \geq \psi^\ast_\theta.
\]
Moreover if \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_{\theta}(\gamma))} = \infty \) in addition, then \( \delta_\psi = 1 \) and \( \psi \) is \((\Gamma, \theta)\)-critical.

The following lemma was proved in [23, Lemma 7.8] for \( \theta = \Pi \), and a general case can be proved verbatim, just replacing \( P \) and \( N \) by \( P_\theta \) and \( N_\theta \) respectively and noting that the projection \( p_{\theta} : a \to a_{\theta} \) is a Lipschitz map.

**Lemma 7.2** (\( \theta \)-shadow lemma). Let \( \Gamma < G \) be a Zariski dense discrete subgroup. Let \( \nu \) be a \((\Gamma, \psi)\)-conformal measure on \( \mathcal{F}_\theta \) for \( \psi \in a_{\theta}^\ast \). Then for all sufficiently large \( R > 1 \), there exists \( C > 1 \) (depending on \( R \)) such that
\[
C^{-1}e^{-\psi(\mu_{\theta}(\gamma))} \leq \nu(O^\theta_R(o, \gamma o)) \leq Ce^{-\psi(\mu_{\theta}(\gamma))}
\]
for all \( \gamma \in \Gamma \).

**Lemma 7.3.** Let \( \Gamma \) be a \( \theta \)-transverse subgroup of \( G \). If there exists a \((\Gamma, \psi)\)-conformal measure \( \nu \) on \( \mathcal{F}_\theta \) for a \((\Gamma, \theta)\)-proper \( \psi \) \( \in a_{\theta}^\ast \), then
\[
\delta_\psi \leq 1.
\]

**Proof.** For each \( n \in \mathbb{Z} \), we set
\[
\Gamma_{\psi, n} := \{ \gamma \in \Gamma : n \leq \psi(\mu_{\theta}(\gamma)) < n + 1 \}.
\]
Since \( \psi \) is \((\Gamma, \theta)\)-proper, \( \bigcup_{n \geq 0} \Gamma_{\psi, n} \) is a finite subset, and hence can be ignored in the arguments below. Let \( \nu \) be a \((\Gamma, \psi)\)-conformal measure. We fix a sufficiently large \( R > 0 \) satisfying the conclusion of Lemma [7.2] for \( \nu \).
Since \( \psi \) is a \((\Gamma, \theta)\)-proper linear form, by Proposition 6.2, we have that for all \( n \in \mathbb{N} \),
\[
1 \gg \sum_{\gamma \in \Gamma_{\psi, n}} \nu(O_{R}^{\theta}(o, \gamma o)) \gg \sum_{\gamma \in \Gamma_{\psi, n}} e^{-\psi(\mu_{\theta}(\gamma))} \geq e^{-(n+1)\#\Gamma_{\psi, n}}
\]
where the implied constants do not depend on \( n \). It implies
\[
\#\Gamma_{\psi, n} \ll e^{n+1} \quad \text{for each } n \geq 0.
\]
Therefore, we have (cf. [29, Lemma 3.1.1])
\[
\delta_{\psi} \leq \limsup_{N \to \infty} \frac{\log \#\{ \gamma \in \Gamma : \psi(\mu_{\theta}(\gamma)) < N \}}{N} \leq \limsup_{N \to \infty} \frac{1}{N} \log \sum_{0 \leq n < N} e^{n+1} = 1.
\]
Hence the claim follows. \( \square \)

**Proof of Theorem 7.1.** By Lemmas 4.5 and 7.3, we have that \( \delta_{\psi} \leq 1 \) and \( \delta_{\psi} \psi \) is tangent to \( \psi_{\Gamma}^{0} \), and therefore we have
\[
\delta_{\psi} \psi \geq \psi_{\Gamma}^{0}.
\]
Since \( \psi \) is \((\Gamma, \theta)\)-proper, \( \psi \geq 0 \) on \( \mathcal{L}_{\theta} \) by Lemma 4.3 and hence \( \psi \geq \delta_{\psi} \psi \) on \( \mathcal{L}_{\theta} \). Therefore \( \psi \geq \psi_{\Gamma}^{0} \) on \( \mathcal{L}_{\theta} \). Since \( \psi_{\Gamma}^{0} = -\infty \) outside of \( \mathcal{L}_{\theta} \), \( \psi \geq \psi_{\Gamma}^{0} \) on \( a_{\theta} \). If \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_{\theta}(\gamma))} = \infty \) in addition, then \( \delta_{\psi} \geq 1 \) and hence \( \delta_{\psi} = 1 \). In particular, \( \psi = \delta_{\psi} \psi \) is tangent to \( \psi_{\Gamma}^{0} \). Therefore this finishes the proof.

8. **Divergence of Poincaré series, conical sets and Ergodicity**

Let \( \psi \in a_{\theta}^{*} \). Denote by \( M_{\psi}^{\theta} = M_{\Gamma, \psi}^{\theta} \) the collection of all \((\Gamma, \psi)\)-conformal (probability) measures on \( \mathcal{F}_{\theta} \). We suppose that
\[
M_{\psi}^{\theta} \neq \emptyset.
\]
The main goal of this section is to prove the following theorem and discuss its applications.

**Theorem 8.1.** Let \( \Gamma < G \) be a \( \theta \)-transverse subgroup. If \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_{\theta}(\gamma))} = \infty \) (resp. \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_{\theta}(\gamma))} < \infty \)), then \( \nu(\Lambda_{\theta}^{\text{con}}) = 1 \) (resp. \( \nu(\Lambda_{\theta}^{\text{con}}) = 0 \)) for all \( \nu \in M_{\psi}^{\theta} \).

We make the following simple observation:

**Lemma 8.2.** Suppose that \( \nu(\Lambda_{\theta}^{\text{con}}) > 0 \) for all \( \nu \in M_{\psi}^{\theta} \). Then
\[
\nu(\Lambda_{\theta}^{\text{con}}) = 1 \quad \text{for all } \nu \in M_{\psi}^{\theta}.
\]

**Proof.** If \( \nu(\Lambda_{\theta}^{\text{con}}) < 1 \) for some \( \nu \in M_{\psi}^{\theta} \), then \( \nu_{F} := \frac{1}{\nu(F)}\nu|_{F} \), for \( F = \mathcal{F}_{\theta} - \Lambda_{\theta}^{\text{con}} \), belongs to \( M_{\psi}^{\theta} \) and \( \nu_{F}(\Lambda_{\theta}^{\text{con}}) = 0 \). \( \square \)

We will use the following:
Lemma 8.3 (Kochen–Stone Lemma [21]). Let \((Z, \nu)\) be a finite measure space. If \(\{A_n\}\) is a sequence of measurable subsets of \(Z\) such that

\[
\sum_{n=1}^{\infty} \nu(A_n) = \infty \quad \text{and} \quad \liminf_{N \to \infty} \frac{\sum_{m=1}^{N} \sum_{n=1}^{N} \nu(A_n \cap A_m)}{(\sum_{n=1}^{N} \nu(A_n))^2} < \infty, \tag{8.1}
\]

then \(\nu(\limsup_n A_n) > 0\).

Proof of Theorem 8.1. Suppose that \(\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty\). By Lemma 8.2, it suffices to show that \(\nu(A_n^\text{con}) > 0\) for all \(\nu \in M_\psi\). Let \(\nu \in M_\psi\).

We fix \(\alpha \in \theta\). Since \(\Gamma\) is \(\theta\)-regular, \(\alpha \in \theta\) is \((\Gamma, \theta)\)-proper; in particular, \(\alpha(\mu_\theta(\Gamma))\) is a discrete closed subset of \([0, \infty)\). Therefore we may enumerate \(\Gamma = \{\gamma_1, \gamma_2, \cdots\}\) so that \(\alpha(\mu_\theta(\gamma_n)) \leq \alpha(\mu_\theta(\gamma_{n+1}))\) for all \(n \in \mathbb{N}\). Fix a sufficiently large \(R\) which satisfies the conclusion of Lemma 7.2. Setting \(A_n := O_R(\alpha, \gamma_n\alpha)\), we then have

\[
\sum_{n=1}^{\infty} \nu(A_n) \gg \sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty
\]

where the implied constant depends only on \(R\). Since \(\limsup_n A_n \subset A_n^\text{con}\), by Lemma 8.3 it suffices to show that

\[
\liminf_{N \to \infty} \frac{\sum_{m=1}^{N} \sum_{n=1}^{N} \nu(A_n \cap A_m)}{(\sum_{n=1}^{N} \nu(A_n))^2} < \infty. \tag{8.2}
\]

Set \(Q := KA_n^{\text{con}} \alpha K\) where \(A_n^\alpha = \{a \in A^\alpha : \|a\| \leq R\}\) and \(C_0 = C_0(Q)\) be as in Lemma 6.3. Define

\[T_N := \max\{n \in \mathbb{N} : \alpha(\mu_\theta(\gamma_n)) \leq \alpha(\mu_\theta(\gamma_N)) + \|a\| C_0\}\]

for each \(N \geq 1\). Clearly, \(N \leq T_N\). Unless mentioned otherwise, all implied constants in this proof are independent of \(N\). Since \(\Gamma\) is \(\theta\)-regular, \(\alpha|_{a_n}\) is \((\Gamma, \theta)\)-proper. Proposition 6.2 implies that the collection \(A_n, N \leq n \leq T_N\), has multiplicity at most \(q = q(\alpha, R, \|a\| C_0)\), and hence

\[
\sum_{N \leq n \leq T_N} \nu(A_n) \leq q \cdot \nu(\mathcal{F}_\theta).
\]

Therefore by Lemma 7.2 we have that for all \(N \geq 1,\)

\[
\sum_{n=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))} - \sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))} \ll \sum_{n=N+1}^{T_N} \nu(A_n)
\]

\[
\ll \nu(\mathcal{F}_\theta) = e^{\psi(\mu_\theta(\gamma_1))} e^{-\psi(\mu_\theta(\gamma_1))} \leq e^{\psi(\mu_\theta(\gamma_1))} \sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))}
\]

with all implied constants independent of \(N\). Therefore we have:

\[
\sum_{n=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))} \ll \sum_{n=1}^{N} e^{-\psi(\mu_\theta(\gamma_n))}. \tag{8.3}
\]
Fix $N \in \mathbb{N}$. If $A_n \cap A_m \neq \emptyset$ for some $n, m \leq N$, then there exist $k \in K$ and $m_\theta \in M_\theta$ such that $d(kA^+o, \gamma_n o) < R$ and $d(km_\theta A^+o, \gamma_m o) < R$. Since $K \subset Q$, it follows that

$$Q \cap \gamma_n Qa_n^{-1} \cap \gamma_m Qa_m^{-1} m_\theta^{-1} \neq \emptyset$$

for some $a_n, a_m \in A^+$. Hence, setting

$$E_1 = \{(n, m) : n, m \leq N \text{ and } \|\mu_\theta(\gamma_n) - (\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n))\| \leq C_0\},$$

$$E_2 = \{(n, m) : n, m \leq N \text{ and } \|\mu_\theta(\gamma_n) - (\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n))\| \leq C_0\},$$

we get from Lemma [6.3] that

$$\sum_{n,m \leq N} \nu(A_n \cap A_m) \leq \sum_{(n,m) \in E_1} \nu(A_n) + \sum_{(n,m) \in E_2} \nu(A_m). \quad (8.4)$$

For all $(n, m) \in E_1$, we have

$$\alpha(\mu_\theta(\gamma_m^{-1}\gamma_n)) \leq \alpha(\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n))$$

$$= \alpha(\mu_\theta(\gamma_m) + \mu_\theta(\gamma_m^{-1}\gamma_n) - \mu_\theta(\gamma_n)) + \alpha(\mu_\theta(\gamma_n)) \quad (8.5)$$

$$\leq \|\alpha\|C_0 + \alpha(\mu_\theta(\gamma_n)).$$

Therefore, by Lemma [7.2]

$$\sum_{(n,m) \in E_1} \nu(A_n) \leq \sum_{(n,m) \in E_1} e^{-\psi(\mu_\theta(\gamma_n))}$$

$$\leq \sum_{(n,m) \in E_1} e^{-\psi(\mu_\theta(\gamma_m))} e^{-\psi(\mu_\theta(\gamma_m^{-1}\gamma_n))} \quad (8.6)$$

$$\leq \sum_{m=1}^N \sum_{j=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_m))} e^{-\psi(\mu_\theta(\gamma_j))};$$

the last inequality follows because, for each fixed $1 \leq m \leq N$, the correspondence $n \leftrightarrow \gamma_m^{-1}\gamma_n$ is one-to-one and when $(n, m) \in E_1$, $\gamma_j = \gamma_m^{-1}\gamma_n$ for some $j \leq T_n \leq T_N$ by [8.5]. Similarly, we have

$$\sum_{(n,m) \in E_2} \nu(A_m) \leq \sum_{n=1}^N \sum_{j=1}^{T_N} e^{-\psi(\mu_\theta(\gamma_n))} e^{-\psi(\mu_\theta(\gamma_j))}.$$
where we have applied (8.3) for the second last inequality and Lemma 7.2 for the last inequality. Hence (8.2) is verified, completing the proof of the first statement.

We now suppose that \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty \). Consider the following increasing sequence

\[
\Lambda_{\theta}^{N} = \limsup_{\gamma \in \Gamma} O_{\theta}^{N}(o, \gamma o), \quad N \geq 1.
\]

Since \( \Lambda_{\theta}^{con} = \bigcup_{N} \Lambda_{\theta}^{N} \), it suffices to show \( \nu(\Lambda_{\theta}^{N}) = 0 \) for all sufficiently large \( N \geq 1 \). Since

\[
\Lambda_{\theta}^{N} \subset \bigcup_{\gamma \in \Gamma, \|\mu(\gamma)\| > t} O_{\theta}^{N}(o, \gamma o)
\]

for any \( t > 0 \), we get from Lemma 7.2 that for all \( t > 0 \),

\[
\nu(\Lambda_{\theta}^{N}) \ll \sum_{\gamma \in \Gamma, \|\mu(\gamma)\| > t} e^{-\psi(\mu(\gamma))}
\]

where the implied constant depends only on \( N \). Since \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty \) implies that \( \lim_{t \to \infty} \sum_{\gamma \in \Gamma, \|\mu(\gamma)\| > t} e^{-\psi(\mu(\gamma))} = 0 \), we have \( \nu(\Lambda_{\theta}^{N}) = 0 \), finishing the proof.

**Comparing with** \( \psi_{\Gamma} \). Quint showed that for a discrete subgroup \( \Gamma < G \), the existence of a \( (\Gamma, \psi) \)-conformal measure on \( F_{\theta} \) for \( \psi \in a_{\theta}^{*} \) implies the inequality

\[
\psi \circ p_{\theta} + 2\rho_{\Pi - \theta} \geq \psi_{\Gamma} \quad \text{on} \quad a,
\]

(8.7)

where \( 2\rho_{\Pi - \theta} \) is the sum of all positive roots which can be written as \( \mathbb{Z} \)-linear combinations of elements of \( \Pi - \theta \) (counted with multiplicity) [30, Theorem 8.1]. For \( \theta \)-transverse subgroups, Theorem 1.5 and (1.2) imply that the term \( 2\rho_{\Pi - \theta} \) turns out to be redundant:

**Corollary 8.4.** Let \( \Gamma < G \) be a \( \theta \)-transverse subgroup and \( \psi \in a_{\theta}^{*} \) be \( (\Gamma, \theta) \)-proper. If there exists a \( (\Gamma, \psi) \)-conformal measure \( \nu \) on \( F_{\theta} \), then

\[
\psi \circ p_{\theta} \geq \psi_{\Gamma} \quad \text{on} \quad a.
\]

(8.8)

Moreover, if \( \nu(\Lambda_{\theta}^{con}) > 0 \), then \( \psi \circ p_{\theta} \) is tangent to \( \psi_{\Gamma} \).

**Proof.** The first statement follows from Theorem 7.1 and Lemma 3.11. For the second claim, if \( \nu(\Lambda_{\theta}^{con}) > 0 \), then we have \( \sum_{\gamma \in \Gamma} e^{-(\psi \circ p_{\theta})(\mu(\gamma))} = \infty \) by Theorem 8.1. Together with \( \psi \circ p_{\theta} \geq \psi_{\Gamma} \), the second claim follows by [29, Lemma 3.1.3].

**Remark 8.5.** In the special case when \( \Gamma \) is a \( \theta \)-Anosov subgroup and \( \nu \) is supported on \( \Lambda_{\theta} \), the inequality (8.8) also follows from [28, Theorem 5.14].
Ergodicity and Conservativity. A linear form $\psi \in a^*_\theta$ can be considered as a linear form on $a$ which is $p_\theta$-invariant and hence $\psi \circ i$ is a linear form on $a$ which is $p_{i(\theta)}$-invariant. Therefore $\psi \circ i \in a^*_i(\theta)$.

We extend the dichotomy as follows. In the following, we denote by $\mathcal{F}^{(2)}_\theta$ the unique open $G$-orbit in $\mathcal{F}_\theta \times \mathcal{F}_{i(\theta)}$, that is,

$$
\mathcal{F}^{(2)}_\theta = \{ (\xi, \eta) : \xi, \eta \text{ are in general position} \}
$$

and set $\Lambda^{(2)}(\theta) = (\Lambda_\theta \times \Lambda_{i(\theta)}) \cap \mathcal{F}^{(2)}_\theta$.

**Theorem 8.6.** Let $\Gamma < G$ be a $\theta$-transverse subgroup. Let $\psi \in a^*_\theta$ be $(\Gamma, \theta)$-proper. Let $\nu$ and $\nu_i$ be a $(\Gamma, \psi)$ and $(\Gamma, \psi \circ i)$-conformal measures on $\mathcal{F}_\theta$ respectively. Then we have (1) $\iff$ (2) $\iff$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). If $\nu$ and $\nu_i$ are supported on $\Lambda_\theta$ and $\Lambda_i(\theta)$ respectively, then (5) $\Rightarrow$ (1) and hence all are equivalent to each other.

1. $\max \{ \nu(\Lambda^{\text{con}}_\theta), \nu_i(\Lambda^{\text{con}}_{i(\theta)}) \} > 0$;
2. $\nu(\Lambda^{\text{con}}_\theta) = \nu_i(\Lambda^{\text{con}}_{i(\theta)}) = 1$;
3. $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\gamma(\gamma))} = \infty$;
4. We have $\nu \times \nu_i$ is atomless and $(\nu \times \nu_i)(\mathcal{F}^{(2)}_\theta) = 1$. The $\Gamma$-action on $(\mathcal{F}^{(2)}_\theta, \nu \times \nu_i)$ is ergodic;
5. The $\Gamma$-action on $(\mathcal{F}^{(2)}_\theta, \nu \times \nu_i)$ is completely conservative.

That the $\Gamma$-action on $(\Lambda^{(2)}_\theta, \nu \times \nu_i)$ is completely conservative is equivalent to saying that for almost all $x \in \Lambda^{(2)}_\theta$, there exists a compact subset $\Omega_x \subset \Lambda^{(2)}_\theta$ such that $\# \{ \gamma \in \Gamma : \gamma x \in \Omega_x \} = \infty$, which we take it as a definition (cf. [21, Lemma 8.8]).

For the implication $(3) \Rightarrow (4)$, we will use the following recent theorem:

**Theorem 8.7** (Canary-Zhang-Zimmer, [21]). Suppose that $\theta = i(\theta)$. Let $\Gamma$ be $\theta$-transverse. Suppose that $\psi \in a^*_\theta$ satisfies $\delta_\psi = 1$ and $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\gamma(\gamma))} = \infty$. Then there exists a unique $(\Gamma, \psi)$-conformal measure $\nu$ on $\Lambda_\theta$ and a unique $(\Gamma, \psi \circ i)$-conformal measure $\nu_i$ on $\Lambda_i(\theta)$ such that the $\Gamma$-action on $(\Lambda^{(2)}_\theta, \nu \times \nu_i)$ is ergodic.

The following observation is useful to transfer theorems from $\theta$-symmetric to general $\theta$.

**Lemma 8.8.** If $\Gamma$ be $\theta$-antipodal, then for any $\theta_0 \subset \theta$, the projection map $p : \Lambda_{\theta_0, i(\theta)} \to \Lambda_{\theta_0}$ given by $gP_{\theta_0, i(\theta)} \to gP_{\theta_0}$ is a $\Gamma$-equivariant homeomorphism.

**Proof.** Suppose that $\xi \neq \eta \in \Lambda_{\theta_0, i(\theta)}$. By the $\theta$-antipodality of $\Gamma$, $\xi = gP_{\theta_0, i(\theta)}$ and $\eta = g\omega P_{\theta_0, i(\theta)}$ for some $g \in G$. Then $p(\xi) = gP_{\theta_0}$ and $p(\eta) = g\omega P_{\theta_0}$, and hence $p(\xi) \neq p(\eta)$, showing that $p$ is injective. □

By the above lemma, for a given $\psi \in a^*_\theta$, the study of $(\Gamma, \psi)$-conformal measures on $\Lambda_{\theta_0}$ is reduced to the study of $(\Gamma, \psi)$-conformal measures on
\( \Lambda_{\theta,i(\theta)} \), as we may consider \( \psi \) as a linear form on \( a_{\theta,i(\theta)} \) via the projection \( a_{\theta,i(\theta)} \to a_{\theta} \).

**Equivalent definition of conical sets for \( \theta \)-transverse groups.** For the implication \((5) \Rightarrow (1)\), we will need an alternative description of \( \Lambda^\text{con}_{\theta} \) for \( \theta \)-transverse groups. The action of a \( \theta \)-transverse subgroup \( \Gamma \) on \( \Lambda_{\theta} \) is known to be a convergence group action, that is, for any infinite sequence \( \theta \) for the implication \((5) \Rightarrow (1)\), we will need an alternative description of \( \Lambda^\text{con}_{\theta} \) for \( \theta \)-transverse groups. The action of a \( \theta \)-transverse subgroup \( \Gamma \) on \( \Lambda_{\theta} \) is known to be a convergence group action, that is, for any infinite sequence \( \gamma \in \Gamma \), there exist \( a, b \in \Lambda_{\theta} \) and a subsequence \( \gamma_{n_k} \) such that \( \gamma_{n_k} x \) converges to \( a \) for all \( x \in \Lambda_{\theta} - \{ b \} \) uniformly on compact subsets (see [18, Proposition 5.38]). Let \( \Lambda^\theta_{\theta, \psi} \) denote the set of all \( x \in \Lambda_{\theta} \) such that there exist distinct \( a, b \in \Lambda_{\theta} \) and a sequence \( \gamma_n \in \Gamma \) so that \( \gamma_n x \to a \) and \( \gamma_n y \to b \) for all \( y \in \Lambda_{\theta} - \{ x \} \) uniformly on compact subsets; this is a conical set in the sense of convergence group action.

**Theorem 8.9** ([Kapovich-Leeb-Porti, [18] Proposition 6.8]). For a \( \theta \)-transverse subgroup \( \Gamma < G \), we have

\[
\Lambda^\theta_{\theta, \psi} = \Lambda^\text{con}_{\theta}.
\]

**Lemma 8.10.** Let \( \Gamma < G \) be a \( \theta \)-antipodal subgroup and let \( \nu \) and \( \nu_i \) be measures on \( \Lambda_{\theta} \) and \( \Lambda_{i(\theta)} \). If at least one of \( \nu \) and \( \nu_i \) is atomless, then \( \nu \times \nu_i \) is supported on \( F_{\theta}^{(2)} \).

**Proof.** Replacing \( \theta \) by \( i(\theta) \) if necessary, we may assume that \( \nu_i \) is atomless. Since \( \Gamma \) is \( \theta \)-antipodal, for each \( \xi \in \Lambda_{\theta} \), there exists at most one \( \eta \in \Lambda_{i(\theta)} \) such that \( (\xi, \eta) \in F_{\theta} - F_{\theta}^{(2)} \). Since \( \nu_i \) is atomless, we have

\[
(\nu \times \nu_i) \left( \Lambda_{\theta} \times \Lambda_{i(\theta)} - F_{\theta}^{(2)} \right) = \int_{\Lambda_{\theta}} \nu_i \left( \{ \eta \in \Lambda_{i(\theta)} : (\xi, \eta) \notin F_{\theta}^{(2)} \} \right) d\nu(\xi) = 0.
\]

We are now ready to give:

**Proof of Theorem 8.6.** Note that \( a_{\theta} \) can be regarded as a subspace of \( a_{\theta, i(\theta)}^* \) and that \( \psi \in a_{\theta}^* \) is \( (\Gamma, \theta) \)-proper if and only if \( \psi \circ i \) is \( (\Gamma, i(\theta)) \)-proper. Moreover, for all \( \gamma \in \Gamma \), \( \psi(\mu_{\theta}(\gamma)) = (\psi \circ i)(\mu_{\theta}^{-1}(\gamma)) \). Therefore the equivalence \((1) \iff (2) \iff (3) \) follows from Theorem 8.1. To show \((3) \Rightarrow (4)\), suppose that \((3) \) holds. Since \( \psi \) is \( (\Gamma, \theta) \)-proper, \( \psi \circ i \) is \( (\Gamma, \theta) \)-proper and \( \delta_{\psi} = \delta_{\psi i} \leq 1 \) by Lemma 7.3. Moreover, \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_{\theta}(\gamma))} = \sum_{\gamma \in \Gamma} e^{-\psi i(\mu_{\theta}(\gamma))} = \infty \).

Since any conical point \( \xi \in \Lambda^\text{con}_{\theta} \) is contained in the intersection \( \bigcap_{R>0} O_R(o, \gamma o) \) for some \( R > 0 \) and an infinite sequence \( \gamma_i \in \Gamma \), The \( \theta \)-shadow lemma (Lemma 7.2) implies that \( \nu \) has no atom. Therefore, Lemma 8.10 implies that the product measure \( \nu \times \nu_i \) is supported on \( F_{\theta}^{(2)} \). When \( \theta = i(\theta) \), the \( \Gamma \)-ergodicity on \( F_{\theta}^{(2)} \) follows from Theorem 8.7. For a general \( \theta \), by Lemma 8.8 \( p^*\nu \) and \( p^*\nu \) define \( (\Gamma, \psi) \) and \( (\Gamma, \psi \circ i) \)-conformal measures on \( \Lambda_{\theta, i(\theta)} \). By Lemma 8.8, the statements for the pull-back measures \( p^*\nu \) and \( p^*\nu \) imply those for \( \nu \) and \( \nu_i \). Therefore \((3) \) implies \((4) \) for a general \( \theta \).

The implication \((4) \Rightarrow (5) \) follows from [15] Theorem 1.4.
Now supposing that \( \nu \) and \( \nu_i \) are supported on \( \Lambda_{\theta} \) and \( \Lambda_{i(\theta)} \) respectively, assume (5) holds, i.e., the \( \Gamma \)-action on \( (\Lambda_{\theta}^{(2)}, \nu \times \nu_i) \) is completely conservative. Then for \( \nu \times \nu_i \)-a.e. \( (\xi, \eta) \in \Lambda_{\theta}^{(2)} \), there exists a compact subset \( \Omega_{(\xi,\eta)} \subset \Lambda_{\theta}^{(2)} \) such that \( \gamma_i(\xi, \eta) \in \Omega_{(\xi,\eta)} \) for infinitely many \( \gamma_i \in \Gamma \). Hence passing to a subsequence, we may assume that \( \gamma_i \xi \to x \) and \( \gamma_i \eta \to y \) for some \( (x, y) \in \Omega_{(\xi,\eta)} \). In particular, \( x \neq y \).

Since the action of \( \Gamma \) on \( \Lambda_{\theta} \) is a convergence group action, after passing to a subsequence, there exist \( a, b \in \Lambda_{\theta} \) such that \( \gamma_i z \to a \) for all \( z \neq b \). It implies that \( b \) is either \( \xi \) or \( \eta \). Suppose that \( b = \xi \). Then \( y = a \) and for all \( z \neq \xi \), \( \gamma_i z \to y \). Since \( x \neq y \) as well, this implies that \( \xi \in \Lambda_{\theta,c} \) and hence \( \xi \in \Lambda_{\theta}^{\text{con}} \) by Theorem 8.9. Similarly, if \( b = \eta \), then \( \eta \in \Lambda_{\theta}^{\text{con}} \). Therefore, we have for \( \nu \times \nu_i \)-a.e. \( (\xi, \eta) \), we have \( \xi \in \Lambda_{\theta}^{\text{con}} \) or \( \eta \in \Lambda_{\theta}^{\text{con}} \). Hence \( \max \{ \nu(\Lambda_{\theta}^{\text{con}}), \nu_i(\Lambda_{\theta}^{\text{con}}) \} > 0 \), proving (1). This finishes the proof. □

**Proof of Theorem 1.5** Theorem 7.1 implies Theorem 1.5(1). Theorem 1.5(2) follows from Theorem 8.1 and the following corollary.

**Corollary 8.11.** Let \( \Gamma \) be \( \theta \)-transverse. If \( \psi \in \mathfrak{a}_{\theta}^0 \) is \( (\Gamma, \theta) \)-proper and \( \sum_{\gamma \in \Gamma} e^{-\psi(\mu_{\theta}(\gamma))} = \infty \), then \( \# \mathcal{M}_\psi = 1 \).

**Proof.** By Theorem 7.1 and the hypothesis on \( \psi \), we have \( \delta_{\psi} = 1 \). By Proposition 5.7 for a \( \theta \)-transverse subgroup \( \Gamma \), there exists a \( (\Gamma, \psi) \)-conformal measure on \( \mathcal{F}_{\theta, j(\theta)} \), and is supported on \( \Lambda_{\theta,j(\theta)} \). Moreover it is unique by Theorem 8.7. It then follows from Lemma 8.8 that there exists a unique \( (\Gamma, \psi) \)-conformal measure on \( \mathcal{F}_{\theta} \) as well. □

**Lebesgue measure of conical sets.**

**Corollary 8.12.** If \( \Gamma < G \) is \( \theta \)-transverse, then

\[
\Lambda_{\theta} = \mathcal{F}_{\theta} \quad \text{or} \quad \text{Leb}_{\theta}(\Lambda_{\theta}^{\text{con}}) = 0.
\]

Moreover, in the former case, \( \theta \) is the simple root of a rank one factor of \( G \).

**Proof.** Note that \( \text{Leb}_{\theta} \) is a \( (\Gamma, 2\rho \circ p_{\theta}) \)-conformal measure where \( \rho \) is the half sum of all positive roots of \( (g, \mathfrak{a}^+) \) [30, Lemma 6.3]. If \( \Lambda_{\theta} \neq \mathcal{F}_{\theta} \), \( \text{Leb}_{\theta}(\Lambda_{\theta}^{\text{con}}) \leq \text{Leb}_{\theta}(\Lambda_{\theta}) < 1 \) as \( \mathcal{F}_{\theta} - \Lambda_{\theta} \) is a non-empty open subset. Therefore \( \text{Leb}_{\theta}(\Lambda_{\theta}^{\text{con}}) = 0 \) by Theorem 8.1. The second claim follows from Proposition 8.13 below. □

**Proposition 8.13.** Suppose that \( \Gamma \) is \( \theta \)-antipodal and \( \Lambda_{\theta} = \mathcal{F}_{\theta} \). Then \( \theta \) must be the simple root of a rank one factor of \( G \).

**Proof.** We write \( G \) as the almost direct product of simple real algebraic groups \( G = G_1 \cdots G_m \). Let \( n \) be an index such that \( \theta \) contains a simple root of \( G_n \). Denoting by \( \pi_n : G \to G_n \) the canonical projection, \( \pi_n(P_{\theta}) \) is a proper parabolic subgroup of \( G_n \) and the limit set of \( \pi_n(\Gamma) \) in \( G_n/\pi_n(P_{\theta}) \) is equal to all of \( G_n/\pi_n(P_{\theta}) \). Suppose that the rank of \( G_n \) is at least 2. Fix \( kP_{\theta,j(\theta)} \in \Lambda_{\theta,j(\theta)} \) for some \( k \in K \). Let \( w \) be a Weyl element given by Lemma
below such that $w \notin w_0 N^+_\theta P_\theta \cup P_\theta$. Noting that $w_0 N^+_{\theta,i(\theta)} P_{\theta,i(\theta)} M_\theta \subseteq w_0 P^+_\theta P_\theta = w_0 N^+_\theta P_\theta$, we have

$$w \notin w_0 N^+_{\theta,i(\theta)} P_{\theta,i(\theta)} M_\theta \cup P_{\theta,i(\theta)} M_\theta.$$  \hspace{1cm} (8.9)

Since $\mathcal{F} = K/M$ and $kM_\theta \in \mathcal{F}_\theta = K/M_\theta = \Lambda_\theta$, we may choose $m \in M_\theta$ such that $kwmP \in \Lambda_{\Pi}$, and hence $kwmP_{\theta,i(\theta)} \in \Lambda_{\theta,i(\theta)}$. Then by (8.9),

$$w \notin w_0 N^+_{\theta,i(\theta)} P_{\theta,i(\theta)} \cup P_{\theta,i(\theta)}.$$  \hspace{1cm}  (8.14)

The condition that $w \notin P_{\theta,i(\theta)}$ implies that $kwm P_{\theta,i(\theta)} \cap kP_{\theta,i(\theta)} = \emptyset$.

Also, by Corollary 2.5, the condition that $w \notin w_0 N^+_{\theta,i(\theta)} P_{\theta,i(\theta)}$ implies that $(kwm P_{\theta,i(\theta)}) \notin G(P_{\theta,i(\theta)}, w_0 P_{\theta,i(\theta)})$, that is, $kwm P_{\theta,i(\theta)}$ is not in general position with $P_{\theta,i(\theta)}$. This yields a contradiction to the $\theta,i(\theta)$-antipodal property of $\Gamma$. Therefore for any $n$ such that $\theta$ contains a simple root of $G_n$, the rank of $G_n$ must be one. If there are $n \neq n'$ with this property, the map $\gamma \to (\pi_n(\gamma), \pi_{n'}(\gamma))$ must be a discrete subgroup of $G_n G_{n'}$ (because of the $\theta$-regularity property) with full limit set $G_n/\pi_n(P_\theta) \times G_{n'}/\pi_{n'}(P_\theta)$.

However this yields a contradiction to the $\theta$-antipodal property, because the product of two rank one geometric boundaries does not have the antipodal property. Therefore $\theta$ must be a singleton, proving the claim.

We now prove the following which was used in the above proof.

**Lemma 8.14.** If $G$ has a connected normal subgroup $G_n$ of rank at least 2 and $\theta \subset \Pi$ contains a simple root of $G_n$, then we can find a representative of a Weyl element $w \in N_K(A)$ such that $w \notin w_0 N^+_{\theta} P_\theta \cup P_\theta$.

**Proof.** By replacing $\theta$ with the intersection of $\theta$ and the set of simple roots of $G_n$, we may assume without loss of generality that $G = G_n$. Since the rank of $G$ is at least 2, we can find a representative $w \in N_K(A)$ of a Weyl element such that $\text{Ad}_w(a^+_\theta) = a^+_{\theta}$.

If $w$ were contained in $P_\theta \cap K = M_\theta$, $w$ would commute with $a_\theta$ and hence $\text{Ad}_w(a^+_{\theta}) = a^+_{\theta}$. Therefore $w \notin P_\theta$. On the other hand, if $w \in w_0 N^+_{\theta} P_\theta$, then $w^{-1} w \in M_\theta$ by Corollary 2.5, and hence $\text{Ad}_w(a^+_{\theta}) = \text{Ad}_{w_0}(a^+_{\theta}) = -a^+_{i(\theta)}$, which contradicts our choice of $w$. Hence $w \notin w_0 N^+_{\theta} P_\theta$.

**Disjoint dimensions and entropy drop.** Note that any subgroup of a $\theta$-transverse subgroup of $G$ is again a $\theta$-transverse subgroup.

**Proof of Corollary 1.8** Let $\psi \in \mathcal{D}_\theta$. By Proposition 5.7, there exists a $(\Gamma, \psi)$-conformal measure $\nu$ on $\Lambda_{\theta}(\Gamma))$. By Theorem 1.5, $\nu(\Lambda^\text{con}_{\theta}(\Gamma)) = 1$. While $\nu$ is also a $(\Gamma_0, \psi)$-conformal measure, since $\Lambda_{\theta}(\Gamma_0) \neq \Lambda_{\theta}(\Gamma)$ and hence $\Lambda_{\theta}(\Gamma) - \Lambda_{\theta}(\Gamma_0)$ is a non-empty open subset of $\Lambda_{\theta}(\Gamma)$, we have $\nu(\Lambda^\text{con}_{\theta}(\Gamma_0)) < 1$. Again by Theorem 1.5, $\sum_{\gamma \in \Gamma_0} e^{-\psi(\mu_{\theta}(\gamma))} < \infty$. Hence $\psi \notin \mathcal{D}^\theta_{\Gamma_0}$, finishing the proof.

This turns out to be equivalent to the entropy drop phenomenon which is proved by Canary-Zhang-Zimmer [7, Theorem 4.1] for $\theta = i(\theta)$:
Corollary 8.15 (Entropy drop). Let $\Gamma < G$ be a Zariski dense $\theta$-transverse subgroup. Let $\Gamma_0 < \Gamma$ be a Zariski dense subgroup such that $\Lambda_\theta(\Gamma_0) \neq \Lambda_\theta(\Gamma)$. If $\psi \in a_\theta^*$ with $\delta_\psi(\Gamma) < \infty$ and $\sum_{\gamma \in \Gamma_0} e^{-\delta_\psi(\Gamma_0)\psi(\mu_\theta(\gamma)))} = \infty$, then $\delta_\psi(\Gamma_0) < \delta_\psi(\Gamma)$.

Proof. Suppose that $\delta_\psi(\Gamma_0) = \delta_\psi(\Gamma)$. If we set $\phi = \delta_\psi(\Gamma) \cdot \psi = \delta_\psi(\Gamma_0) \cdot \psi$, then $\delta_\phi(\Gamma) = \delta_\phi(\Gamma_0) = 1$. Since $\infty = \sum_{\gamma \in \Gamma_0} e^{-\phi(\mu_\theta(\gamma)))} \leq \sum_{\gamma \in \Gamma} e^{-\phi(\mu_\theta(\gamma)))}$, we have $\phi \in D_\theta^\Gamma \cap D_\theta^\Gamma_0$, contradicting Corollary 1.8. □

9. Conformal measures for $\theta$-Anosov subgroups

Recall the definition of a $\theta$-Anosov subgroup given in the introduction.

The notion of a $\theta$-conical set in [17] is equal to the one we use here for $\theta$-Anosov subgroups, by the Morse property of $\theta$-Anosov subgroups obtained in loc. cit. Note that $\Gamma$ is $\theta$-Anosov if and only if $\Gamma$ is $\theta \cup i(\theta)$-Anosov by (2.1).

Proposition 9.1 ([14], [17, Theorem 1.1]). If $\Gamma$ is $\theta$-Anosov, then

1. $\Gamma$ is $\theta$-regular;
2. $L_\theta - \{0\} \subset \text{int} a_\theta^+$;
3. $\theta$-antipodal;
4. $\Lambda_\theta = \Lambda_\theta^\text{con}$.

Therefore a $\theta$-Anosov subgroup is $\theta$-transverse. We remark that a stronger antipodality is known for $\theta$-Anosov subgroups: if $\Gamma$ is $\theta$-Anosov and $\partial \Gamma$ denotes the Gromov boundary of $\Gamma$, then there exists a pair of $\Gamma$-equivariant homeomorphisms $f_\theta : \partial \Gamma \to \Lambda_\theta$ and $f_{i(\theta)} : \partial \Gamma \to \Lambda_{i(\theta)}$ such that if $\xi \neq \eta$, then $f_\theta(\xi)$ and $f_{i(\theta)}(\eta)$ are in general position. Our definition of $\theta$-antipodality does not require existence of such homeomorphisms.

Sambarino [35, Theorem A] showed that if $\Gamma$ is $\theta$-Anosov, then the set $\{\psi \in a_\theta^* : \delta_\psi = 1\}$ is analytic and is equal to the boundary of a strictly convex subset $\{0 < \delta_\psi < 1\}$. By the duality lemma ([31, Section 4], [34, Lemma 4.8]), we then deduce the following property of the $\theta$-growth indicator:

Theorem 9.2. If $\Gamma$ is $\theta$-Anosov, then $\psi_\Gamma^\theta$ is strictly concave and vertically tangent.

The vertical tangency means that if $\psi_\Gamma^\theta(u) = \psi(u)$ for some $(\Gamma, \theta)$-critical form $\psi$ and $u \neq 0$, then $u \in \text{int} L_\theta$. Recall from the introduction that

$D_\theta^\Gamma = \{\psi \in a_\theta^* : (\Gamma, \theta)$-proper, $\delta_\psi = 1, \mathcal{P}_\psi(1) = \infty\}$

and

$T_\theta^\Gamma := \{\psi \in a_\theta^* : \psi \text{ is } (\Gamma, \theta)$-critical\}.

Lemma 9.3. If $\Gamma$ is $\theta$-Anosov, then

$T_\theta^\Gamma = \{\psi \in a_\theta^* : (\Gamma, \theta)$-proper, $\delta_\psi = 1\} = D_\theta^\Gamma$. 

Proof. The second identity is proved in [35 Section 5.9]. It suffices to prove the inclusion $\subset$ in the first equality due to Corollary 4.6. Suppose that $\psi \in a_\theta^r$ is tangent to $\psi_\Gamma^\theta$. By the vertical tangency property of $\psi_\Gamma^\theta$ of a $\theta$-Anosov subgroup (Theorem 9.2), $\psi > \psi_\Gamma^\theta$ on $\partial \mathcal{L}_\theta$. It follows that $\psi > 0$ on $\mathcal{L}_\theta$. Hence by the second claim in Corollary 4.6, $\delta_\psi = 1$. □

Lemma 9.4. If $\Gamma$ is $\theta$-Anosov and there exists a $(\Gamma, \psi)$-conformal measure on $\mathcal{F}_\theta$ for $\psi \in a_\theta^r$, then $\psi$ is $(\Gamma, \theta)$-proper.

Proof. If $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} < \infty$, then it implies that $\# \{ \gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < T \}$ is finite for any $T > 0$. Therefore $\psi$ is $(\Gamma, \theta)$-proper. If $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, then $\nu(\Lambda_\theta) = 1$ by Theorem 8.1. This implies that $\limsup \frac{1}{T} \log \# \{ \gamma \in \Gamma : \psi(\mu_\theta(\gamma)) < T \} < \infty$ by [35 Theorem A]. Therefore, $\psi$ is $(\Gamma, \theta)$-proper in either case. □

Proof of Theorem 1.9. Let $\Gamma$ be $\theta$-Anosov. Note that a $\theta$-Anosov group is $\theta$-transverse. Hence (1) follows from Theorem 7.1 since $\psi$ is $(\Gamma, \theta)$-proper by Lemma 9.4.

Since $\Lambda_\theta = \Lambda_\theta^\con$ (Proposition 9.1), (a) $\Leftrightarrow$ (b) in (2) follows from Theorem 8.1. The equivalence (b) $\Leftrightarrow$ (c) follows from Lemma 9.3 and Sambarino’s parametrization of the space of all conformal measures on $\Lambda_\theta$ as $\{ \delta_\psi = 1 \}$, together with (1) shown above. For (3), let $\psi$ be a $(\Gamma, \theta)$-critical form. By Lemma 9.3 and Proposition 5.7, there exists a $(\Gamma, \psi)$-conformal measure on $\Lambda_\theta$, which is the unique $(\Gamma, \psi)$-conformal measure on $\Lambda_\theta$ by loc. cit. Since $\sum_{\gamma \in \Gamma} e^{-\psi(\mu_\theta(\gamma))} = \infty$, by Theorem 1.5 any $(\Gamma, \psi)$-conformal measure on $\mathcal{F}_\theta$ is supported on $\Lambda_\theta$. This finishes the proof.

Proof of Corollary 1.10. Since a $\theta$-Anosov subgroup is $\theta$-transverse and $\Lambda_\theta = \Lambda_\theta^\con$ (Proposition 9.1), we deduce from Corollary 8.12 that either $\Lambda_\theta = \mathcal{F}_\theta$ or $\text{Leb}_\theta(\Lambda_\theta) = 0$. In the former case, $\theta$ is the simple root of a rank one factor $G_0$ of $G$ with $\mathcal{F}_\theta = \Lambda_\theta$ by Proposition 8.13, the projection of $\Gamma$ to $G_0$ is a convex cocompact subgroup with full limit set, and hence a cocompact lattice of $G_0$.

Corollary 1.11 follows from Theorem 1.5 and Lemma 9.3.

Proof of Corollary 1.12. By Corollary 1.8 it remains to prove the second part. Since $\Gamma_0 < \Gamma$, we have $\psi_{\Gamma_0}^\theta \leq \psi_\Gamma^\theta$. Suppose that $\psi_{\Gamma_0}^\theta(u) = \psi_\Gamma^\theta(u)$ for some $u$ in the interior of $\mathcal{L}_\theta(\Gamma)$. Then there exists a tangent form $\psi$ to $\psi_{\Gamma_0}^\theta$ at $u$ by Corollary 3.10. Since $\psi_{\Gamma_0}^\theta \leq \psi_\Gamma^\theta$ and $\psi_{\Gamma_0}^\theta(u) = \psi_\Gamma^\theta(u)$, $\psi$ is also tangent to $\psi_{\Gamma_0}^\theta$ at $u$. Hence $\psi \in \mathcal{T}_{\Gamma_0}^\theta \cap \mathcal{T}_{\Gamma}^\theta$, contradicting the first part.

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