Geometric prime number theorems

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Prime number theorem

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\[ \pi(T) := \# \{ p : \text{prime} \leq T \} \]

Theorem (La Vallée-Poussin, Hadamard 1896)

As \( T \to \infty \),

\[ \pi(T) \sim \text{Li}(T) := \int_2^T \frac{dt}{\log t}. \]

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PNT is closely related to the analytic properties (in particular, zeros) of the Riemann zeta function \( \zeta(s) \)

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\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1} \quad \text{analytic for } \text{Re}(s) > 1
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In fact, PNT is equivalent to:

- \( \zeta \) is analytic and non-vanishing on \( \text{Re}(s) \geq 1 \) except for the simple pole at \( s = 1 \)
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In fact, PNT is equivalent to:

$\zeta$ is analytic and non-vanishing on $\Re(s) \geq 1$ except for the simple pole at $s = 1$
Non-vanishing of $\zeta$ on a half-plane $\text{Re}(s) > 1 - \epsilon$? (it would imply $\pi(T) = \text{Li}(T) + O(T^{1-\eta})$ for some $\eta > 0$)

Riemann Hypothesis: $\zeta(s) \neq 0$ on $\text{Re}(s) > 1/2$?
- Non-vanishing of $\zeta$ on a half-plane $\text{Re}(s) > 1 - \epsilon$? (it would imply $\pi(T) = \text{Li}(T) + O(T^{1-\eta})$ for some $\eta > 0$)

- Riemann Hypothesis: $\zeta(s) \neq 0$ on $\text{Re}(s) > 1/2$?
Gaussian primes

For an ideal $P = (a + bi) \subset \mathbb{Z}[i]$, $N(P) = a^2 + b^2$.

$$\pi(T) = \# \{ P : \text{prime ideal in } \mathbb{Z}[i] : N(P) \leq T \}$$
Gaussian primes in sectors

For an ideal $P = (a + bi) \subset \mathbb{Z}[i],$

$$\theta(P) = \text{Arg}(a + bi) \in [0, \pi/2).$$

A natural question is

- Are the **angular components** of Gaussian primes equidistributed?
Theorem (Hecke 1920)

For any $0 \leq \theta_1 < \theta_2 \leq \pi/2$,

$$\#\{ P : \text{prime in } \mathbb{Z}[i] : N(P) \leq T, \ \theta_1 < \theta(P) < \theta_2 \} \sim \left( \frac{\theta_2 - \theta_1}{\pi/2} \right) \text{Li}(T)$$
This result is a consequence of the following analytic properties of the family of Hecke $L$-functions twisted by characters of $S^1$. For each character $\chi$ of $S^1$,

$$L(s, \chi) = \prod_P (1 - \chi(e^{i\theta(P)}N(P)^{-s}))^{-1}$$

is analytic and non-vanishing on $\text{Re}(s) \geq 1$, with the only exception that $L(s, 1)$ has a simple pole at $s = 1$. 
We will now discuss two geometric analogues of the prime number theorem:

- PNT for **geometrically finite** hyperbolic 3-manifolds
- PNT for **hyperbolic** rational maps
Upper half space model of Hyperbolic 3-space

\[ \mathbb{H}^3 = \{ (x_1, x_2, y) : y > 0 \}, \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y} \]

A complete hyperbolic 3-mfld \( M \) is given as

\[ M = \Gamma \backslash \mathbb{H}^3 \]

where \( \Gamma \) is a discrete (torison-free) subgroup of \( \text{Isom}^+(\mathbb{H}^3) \).
Poincaré extension theorem

- $\text{PSL}_2(\mathbb{C})$ acts on $\partial(\mathbb{H}^3) = \hat{\mathbb{C}}$ by Möbius transformations;

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.
\]

- $\text{PSL}_2(\mathbb{C}) = \langle \text{Inv} \mathbb{C} : \text{C circle} \rangle = \langle \text{Inv} \hat{\mathbb{C}} : \text{C circle} \rangle = \text{Isom}^+(\mathbb{H}^3)$
Definition

A Kleinian group $\Gamma$ is a discrete subgp of $\text{PSL}_2(\mathbb{C}) = \text{Isom}^+(\mathbb{H}^3)$.

A complete hyperbolic 3-mfld $M$ is of the form

$$M = \Gamma \backslash \mathbb{H}^3.$$ 

for a Kleinian group $\Gamma$.

Definition

$M = \Gamma \backslash \mathbb{H}^3$ is geometrically finite if $\exists$ a finite-sided fund. domain.

$$\left\{ \text{finite-volume hyp. 3-mflds} \right\} \subset \left\{ \text{geometrically finite hyp. 3-mflds} \right\}$$
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Ex of g.f. hyperbolic 3 mflds

For circles $C_1, \cdots, C_n$ with mutually disj. interiors, set

$$\Gamma := \langle \text{Inversions w.r.t } C_i : i = 1, \cdots, n \rangle \lt PSL_2^\pm(\mathbb{C})$$

Poincare theorem says:

- $\Gamma$ is discrete;
- common exterior of the hemispheres above $C_i$’s is a fund. domain for $\Gamma$.

$$\rightsquigarrow \Gamma \backslash \mathbb{H}^3 \text{ is a g.f mfld with } Vol(\Gamma \backslash \mathbb{H}^3) = \infty.$$
Geometrically finite hyperbolic 3 mflds

- Finite volume $M = \text{compact} \cup \{\text{f.m. cusps}\}$
- Geom. finite $M = \text{compact} \cup \{\text{f.m. cusps}\} \cup \{\text{f.m. flares}\}$
Mostow rigidity theorem implies there are only countably many hyperbolic 3-mflds of finite volume, up to isometry.

**Density conjecture (Bers, Sullivan, Thurston)**

Geometrically finite groups are open and dense in the space of all fin. generated Kleinian groups

Proved by Namazi-Suoto and Ohshika around 2011, using the tameness theorem (Agol, Calegari-Gabai) and the ending lamination theorem (Minsky-Canary-Brock).
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PNT for g.f. hyperbolic 3-mfld $M$

- Analogue of a prime: **primitive closed geodesic** $C$
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PNT for g.f. hyperbolic 3-mfld $M$

- Analogue of a prime: primitive closed geodesic $C$
- Size of a prime: $\exp(\ell(C))$
- Angular component of a prime: holonomy $\theta(C) \in S^1$
Closed geodesics in hyperbolic 3-mfld $M = \Gamma \backslash \mathbb{H}^3$

- $C \Leftrightarrow \Gamma$-conj. class of $\gamma_C \sim \begin{pmatrix} a + bi & 0 \\ 0 & (a + bi)^{-1} \end{pmatrix}$.
- $\exp \ell(C) = N(a + bi) = a^2 + b^2 > 1$
- $\theta(C) = \text{Arg}(a + bi)$. 

![Diagram of a closed geodesic in hyperbolic 3-manifold](image)
Set

\[ \mathcal{P}_T := \{ C : \text{prim. closed geodesic in } M \text{ of } e^{\ell(C)} < T \}. \]

For g.f. \( M \), \( \#\mathcal{P}_T < \infty \)

**Question**

- What is the asymptotic of \( \#\mathcal{P}_T \)?
- Is \( \{ \theta_C : C \in \mathcal{P}_T \} \) equidistributed in \( S^1 \)?
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▷ What is the asymptotic of \( \#P_T \)?
▷ Is \( \{ \theta_C : C \in P_T \} \) equidistributed in \( S^1 \)?
Definition (Limit set of $\Gamma$)

$\Lambda_{\Gamma}$: the set of all accum. pts of $\Gamma(z)$ for any $z \in \hat{\mathbb{C}}$

- If $\text{Vol}(M) < \infty$, $\Lambda_{\Gamma} = \hat{\mathbb{C}}$.
- If $M$ is g.f. and $\text{Vol}(M) = \infty$, then $\dim_H(\Lambda_{\Gamma}) < 2$. 
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Theorem (Roblin 2003, Margulis-Mohammadi-O. 2014)

Let $M = \Gamma \backslash \mathbb{H}^3$ be geom. finite and let $\delta := \text{dim}_H(\Lambda \Gamma)$.

1. As $T \to \infty$, 
   
   \[ \# \mathcal{P}_T \sim \text{Li}(T^\delta) \]

2. If $M$ is non-fuchsian (i.e. $\Lambda \not\subset S^1$), then for any $0 \leq \theta_1 < \theta_2 \leq \pi$

   \[ \# \{ C \in \mathcal{P}_T : \theta_1 < \theta_C < \theta_2 \} \sim \left( \frac{\theta_2 - \theta_1}{\pi} \right) \text{Li}(T^\delta). \]

► For $\text{Vol}(M) < \infty$, these are due to Selberg, Margulis (1970) and Sarnak-Wakayama (1999) resp.
PNT for GF manifolds

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Theorem (Stoyanov 2011, M-M-O 2014, Sarkar-Winter 2021)

Let $M = \Gamma \backslash \mathbb{H}^3$ be geom. finite with no cusps.

1. As $T \to \infty$,

   $$\#\mathcal{P}_T = Li(T^\delta) + O(T^{\delta-\epsilon})$$

2. If $M$ is non-fuchsian,

   $$\#\{C \in \mathcal{P}_T : \theta_1 < \theta_C < \theta_2\} = \left(\frac{\theta_2-\theta_1}{\pi}\right) Li(T^\delta) + O(T^{\delta-\epsilon}).$$

▶ For $\text{Vol}(M) < \infty$, Selberg (1970) and Sarnak-Wakayama (1999)
A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0}$$

for relative prime poly. $p$ and $q$.

Fundamental problem in the dynamics of rational maps is to understand the behavior of successive iterates $f, f^2, f^3, \ldots$ where

$$f^k = \underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}.$$
Iterated Rational maps

A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is

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Succesive iterations

Example

\[ f_c(z) = z^2 + c, \]

\[ f_c^2(z) = (z^2 + c)^2 + c = z^4 + 2cz^2 + c^2 + c, \]
\[ f_c^3(z) = z^8 + 4cz^6 + 2c(3c + 1)z^4 + 4c^2(c + 1)z^2 + c^2(c + 1)^2 \]
\[ \vdots \]
\[ f_c^k(z) = z^{2^k} + \cdots \]

Definition

\( f, g \): conjugate if \( g = h \circ f \circ h^{-1} \) for some \( h \in \text{Mob}(\hat{\mathbb{C}}) \)
For $f = p/q$, $\text{deg}(f) = \max\{\text{deg } p, \text{deg } q\}$

- If $\text{deg}(f) = 1$, the dynamics of $f^k$ is very simple.
- If $\text{deg}(f) \geq 2$, $\text{deg}(f^k) = d^k$ grows exp. fast which makes the dynamics of iterations quite complicated.

Ex: Any quad. poly. is conjugate to unique

$$f_c(z) = z^2 + c$$

so the quadratic parameter space is $\mathbb{C}$. Already for quad. poly, many fund. problems remain open.
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Both Kleinian gps and Iterated rational maps define dynamical system on $\hat{\mathbb{C}}$. Around 1980, Sullivan proposed a dictionary between the two:

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In view of this dictionary, it is natural to ask if we have an analogue of PNT for hyperbolic rational maps.
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PNT \hspace{2cm} PNT?

In view of this dictionary, it is natural to ask if we have an analogue of PNT for hyperbolic rational maps.
Assume $\deg(f) \geq 2$.

**Definition**

$J_f$ is the set of $z \in \hat{\mathbb{C}}$ around which $\{f^k : k = 1, 2, \cdots\}$ is chaotic (=not normal)

Ex: For $f(z) = z^2$, $f^k(z) = z^{2^k}$.

- If $|z| < 1$, $f^k \to 0$ uniformly in a nbd of $z$;
- If $|z| > 1$, $f^k \to \infty$ uniformly in a nbd of $z$.

So

$$J_f = \{|z| = 1\}$$
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So

\[ J_f = \{ |z| = 1 \} \]
Julia set of $z^2 - 1$: Basilica
Julia sets of $f_c(z) = z^2 + c$: totally disconnected or connected
Hyperbolic rational map

**Definition**

$f$ is hyperbolic if for each $z$ with $f'(z) = 0$,

$$\{f^k(z) : k = 1, 2, \ldots\} \cap J_f = \emptyset.$$

Equivalently, $f^k : J_f \to J_f$ is unif. expanding for all $k$.

Ex: $f = z^2$ has a uniq. critical pt 0. Since $f^k(0) = 0 \ \forall k$ and $J_f = \{|z| = 1\}$, $f$ is hyp.

Alternatively, note that $|(f^k)'| = 2^k$ on $J_f$. 
Hyperbolic rational map

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Generic hyperbolicity conjecture

Hyperbolic rational maps are \textbf{dense} in \text{Rat}.

\textbf{Ex: Mandelbrot set}
\[ M := \{ c \in \mathbb{C} : \text{Julia set of } f_c(z) = z^2 + c \text{ is connected} \} \]
\begin{itemize}
  \item For \( c \notin M \), \( f_c \) is hyp.;
  \item For \( c \in \partial(M) \), \( f_c \) is not hyp.;
  \item For \( c \in M \), \( f_c \) is hyp. iff \( c \in \text{Int}(M) \) (GHC).
\end{itemize}
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PNT for a rational map \( f \) with \( d \geq 2 \)

Analogue of a prime: **primal periodic orbit**

\( \hat{z} = \{ z \mapsto f(z) \mapsto \cdots \mapsto f^{n-1}(z) \} \) of minimal period \( n \).

**Definition**

For a primal periodic orbit \( \hat{z} \) of period \( n \), the multiplier of \( \hat{z} \) is

\[ \lambda(\hat{z}) := (f^n)'(z) \in \mathbb{C}. \]

- size of the periodic orbit \( \hat{z} \): \( |\lambda(\hat{z})| \)
- angular component (holonomy): \( \theta(\hat{z}) = \text{Arg} \lambda(\hat{z}) \in [0, 2\pi) \)
PNT for a rational map $f$ with $d \geq 2$

Analogue of a prime: **primitive periodic orbit**

$\hat{z} = \{ z \mapsto f(z) \mapsto \cdots \mapsto f^{n-1}(z) \}$ of minimal period $n$.

**Definition**

For a prim. periodic orbit $\hat{z}$ of period $n$, the multiplier of $\hat{z}$ is

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Periodic orbits of $f_c$

$f_c(z) = z^2 + c = z$ gives $z = \frac{1 \pm \sqrt{1 - 4c}}{2}$: two periodic orbits of period 1 with multipliers

$$f'_c(z) = 1 \pm \sqrt{1 - 4c}$$

$f_c^2(z) = z^4 + 2cz^2 + c^2 + c = z$ gives one primitive periodic orbit of period 2: $z = \frac{1 \pm \sqrt{-3 - 4c}}{2}$ with multiplier

$$(f_c^2)'(z) = 2 - 4c$$

...
PNT for rational maps

\[ \mathcal{P}_T := \{ \hat{z} \text{ prim. periodic orbit of } f : |\lambda(\hat{z})| < T \}. \]

- If \( f \) is hyperbolic, \( \# \mathcal{P}_T < \infty \).

Theorem (O.-Winter, 2016)

\( f \) is hyperbolic rational map of \( d \geq 2 \).

1. If \( f \not\sim z^\pm d \),

\[ \# \mathcal{P}_T = \text{Li}(T^\delta) + O(T^{\delta - \epsilon}) \]

where \( 0 < \delta = \text{dim}_H(J) < 2 \).

2. If \( J \not\subset S^1 \), for any \( 0 \leq \theta_1 < \theta_2 \leq 2\pi \),

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   \[ \#\{ \hat{z} \in \mathcal{P}_T : \theta_1 < \theta(\hat{z}) < \theta_2 \} = \text{Li}(T^{\delta}) \cdot \frac{\theta_2 - \theta_1}{2\pi} + O(T^{\delta-\epsilon}) \]
This theorem is obtained by establishing a zero-free half plane of the associated zeta functions beyond the line \( \text{Re}(s) = \delta \):

Define

\[
\zeta(s) = \prod_{\hat{z} \in \mathcal{P}} (1 - |\lambda(\hat{z})|^{-s})^{-1}
\]

and for a character \( \chi \) of \( S^1 \),

\[
\zeta(s, \chi) = \prod_{\hat{z} \in \mathcal{P}} (1 - \chi(e^{i\theta(\hat{z})})|\lambda(\hat{z})|^{-s})^{-1};
\]

they are analytic on \( \text{Re}(s) > \delta \).
### Theorem (O.-Winter, 2016)

1. If $f \not\sim z^{\pm \delta}$, $\exists \epsilon > 0$ s.t. $\zeta$ is non-vanishing on $\text{Re}(s) \geq \delta - \epsilon$ except for the simple pole $s = \delta$;

2. If $J \not\subset S^1$, then $\exists \epsilon > 0$ s.t. for any non-trivial character $\chi$ of $S^1$, $\zeta(s, \chi)$ is non-vanishing on $\text{Re}(s) \geq \delta - \epsilon$. 

![Diagram of a plane with a shaded region and a vertical line at Re(s) = \delta - \epsilon]