HAUSDORFF DIMENSIONS AND CRITICAL EXPONENTS
FOR PAIRS OF HYPERBOLIC MANIFOLDS

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Abstract. Let $n \geq 2$ and $\rho_1, \rho_2 : \Delta \to \text{SO}^\circ(n, 1)$ be a pair of non-
elementary convex cocompact representations of a finitely generated
group $\Delta$. Let $\Gamma := (\rho_1 \times \rho_2)(\Delta) < \text{SO}^\circ(n, 1) \times \text{SO}^\circ(n, 1)$ and $0 <\delta \leq n - 1$ denote critical exponent of $\Gamma$ with respect to the Riemannian
metric on $\mathbb{H}^n \times \mathbb{H}^n$. We prove the following strong gap property for $\delta$:
$$\delta \leq \frac{n - 1}{\sqrt{2}},$$
moredore if the equality holds, then $\rho_i(\Delta)$ is a lattice for both $i = 1, 2$
(which must be conjugate to each other if $n \geq 3$).

We show that the Hausdorff dimension of the limit set $\Lambda \subset S^{n-1} \times S^{n-1}$ of $\Gamma$ satisfies:
$$\sqrt{2}\delta \leq \dim \Lambda \leq \delta_{\text{min}}$$
where $\delta_{\text{min}}$ denotes the critical exponent of $\Gamma$ with respect to the minimum of the metrics on $\mathbb{H}^n \times \mathbb{H}^n$. We also obtain estimates on the
Hausdorff dimension on the directional conical limit set of $\Gamma$ for any
direction whose slope is between the maximal and minimal geodesic
stretch constants of $\rho_1$ relative to $\rho_2$.

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1. Introduction

Let \( n \geq 2 \) and \((\mathbb{H}^n, d_{\mathbb{H}^n})\) denote the hyperbolic \( n \)-space of constant curvature \(-1\), with geometric boundary identified with the Riemannian sphere \( S^{n-1} \). The Lie group \( \text{SO}^\circ(n, 1) \), the identity component of \( \text{SO}(n, 1) \), is isomorphic to \( \text{Isom}^+(\mathbb{H}^n) \). For a non-elementary convex cocompact subgroup \( \Gamma \leq \text{SO}^\circ(n, 1) \), it is a classical result of Patterson [18] and Sullivan [25] that the critical exponent of \( \Gamma \) coincides with the Hausdorff dimension of its limit set \( \Lambda \). In this paper, we are interested in the analogue of this question for discrete subgroups of the higher rank Lie groups \( \text{SO}^\circ(n, 1) \times \text{SO}^\circ(n, 1) \).

The Lie group \( G = \text{SO}^\circ(n, 1) \times \text{SO}^\circ(n, 1) \) is the identity component of the isometry group of the Riemannian product \( X = (\mathbb{H}^n \times \mathbb{H}^n, d) \) where

\[
d((x_1, x_2), (y_1, y_2)) = \sqrt{d_{\mathbb{H}^n}(x_1, y_1)^2 + d_{\mathbb{H}^n}(x_2, y_2)^2}.
\]

The Riemannian product \( \mathcal{F} = S^{n-1} \times S^{n-1} \) is equal to the Furstenberg boundary of \( G \). We consider a particular class of discrete subgroups of \( G \), constructed as follows. Let \( \Delta \) be a finitely generated group and let \( \rho_1, \rho_2 : \Delta \to \text{SO}^\circ(n, 1) \) be convex cocompact representations with non-elementary images and finite kernels. Let \( \Gamma = (\rho_1 \times \rho_2)(\Delta) \) be the subgroup of \( G \) given by

\[
\Gamma = \{ (\rho_1(\sigma), \rho_2(\sigma)) : \sigma \in \Delta \}.
\]

Such a subgroup \( \Gamma \) is an Anosov subgroup of \( G \) with respect to a minimal parabolic subgroup\(^1\) of \( G \) in the sense of Guichard and Wienhard [12], which are considered as a natural generalization of convex cocompact subgroups of \( \text{SO}^\circ(n, 1) \). Moreover, any such Anosov subgroup of \( G \) arises precisely in this way.

The associated quotient manifold

\[
X_{\rho_1, \rho_2} := \Gamma \backslash (\mathbb{H}^n \times \mathbb{H}^n)
\]

is a locally symmetric rank-2 Riemannian manifold of infinite volume, which projects to the pair of the hyperbolic manifolds \( \rho_i(\Delta) \backslash \mathbb{H}^n \), \( i = 1, 2 \).

We are interested in the limit set and critical exponent associated to this manifold \( X_{\rho_1, \rho_2} \). Fix a basepoint \( o \in X \). The limit set of \( \Gamma \), which we denote by \( \Lambda = \Lambda_\Gamma = \Lambda_{\rho_1, \rho_2} \), is the set of all accumulation points of \( \Gamma o \) in \( \mathcal{F} \). This definition is independent of the choice of \( o \). If we denote by \( \Lambda_{\rho_i} \subset S^{n-1} \) the limit set of \( \rho_i(\Delta) \) for \( i = 1, 2 \), then there exists an equivariant homeomorphism \( \zeta : \Lambda_{\rho_1} \to \Lambda_{\rho_2} \) so that \( \Lambda \) is the graph of \( \Lambda_{\rho_1} \) via \( \zeta \):

\[
\Lambda = \{ (\xi, \zeta(\xi)) : \xi \in \Lambda_{\rho_1} \} \subset \Lambda_{\rho_1} \times \Lambda_{\rho_2}.
\]

For a subset \( S \subset S^{n-1} \) (resp. \( S \subset \mathcal{F} \)), we denote by \( \text{dim} S \) the Hausdorff dimension of \( S \) with respect to the Riemannian metric on \( S^{n-1} \) (resp. on \( \mathcal{F} = S^{n-1} \times S^{n-1} \)).

\(^{1}\)From now on, all Anosov subgroups mentioned in this paper are with respect to a minimal parabolic subgroup.
Hausdorff dimension and critical exponents. The first question may be whether the Hausdorff dimension of $\Lambda$ is equal to the critical exponent $\delta = \delta_\Gamma > 0$ of $\Gamma$, which is the abscissa of convergence of the Poincaré series $P_\Gamma(s) = \sum_{\gamma \in \Gamma} e^{-sd(o,\gamma o)}$. However if $\rho_1 \sim \rho_2$, i.e., if they are conjugate to each other,

$$\dim \Lambda = \dim \Lambda_{\rho_i(\Delta)} = \delta_{\rho_i(\Delta)} = \sqrt{2\delta}$$

for $i = 1, 2$.

Therefore, we cannot expect $\dim \Lambda = \delta$ in our higher rank setting. In the following theorem, let $\delta_{\min}$ denote the critical exponent of $\Gamma$ respect to the minimum of the metrics on $\mathbb{H}^n \times \mathbb{H}^n$.

**Theorem 1.1.** We have

$$\sqrt{2\delta} \leq \dim \Lambda \leq \delta_{\min} \tag{1.1}$$

The inequalities in (1.1) are sharp, since $\sqrt{2\delta} = \delta_{\min}$ for $\rho_1 \sim \rho_2$. We remark that $\delta_{\max} \leq \sqrt{2\delta}$ if $\delta_{\max}$ denotes the critical exponent of $\Gamma$ with respect to the maximum metric. See Theorems 3.1 and 3.2 for a more general statement.

Geodesic stretching constants and $\delta$. According to Thurston [26], the following maximal and minimal geodesic stretching constants of $\rho_1$ relative to $\rho_2$ may be viewed as asymmetric distances between the two hyperbolic manifolds $\rho_i(\Delta) \setminus \mathbb{H}^n$, $i = 1, 2$:

$$d_+(\rho_1, \rho_2) = \sup_{\sigma \in \Delta - \{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)} \quad \text{and} \quad d_-(\rho_1, \rho_2) = \inf_{\sigma \in \Delta - \{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)} \tag{1.2}$$

where $\ell_i(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\rho_i(\Delta) \setminus \mathbb{H}^n$ corresponding to $\rho_i(\sigma)$.

Note that $0 < d_- \leq d_+ < \infty$ and we have $\rho_1 \not\sim \rho_2$ if and only if $d_- < d_+$. The trivial bounds

$$\delta \leq \min(\dim \Lambda_{\rho_1}, \dim \Lambda_{\rho_2}) \leq \max(\dim \Lambda_{\rho_1}, \dim \Lambda_{\rho_2}) \leq \dim \Lambda \tag{1.3}$$

are immediate from the definition of $\delta$, $\delta_{\rho_i(\Delta)} = \dim \Lambda_{\rho_i}$ [25] and the fact that the projection maps $\Lambda \to \Lambda_{\rho_i}$ are Lipschitz.

The following theorem is a more elaborate version of the lower bound of Theorem 1.1 (see Theorem 3.2):

**Theorem 1.2.** Let $\tan \theta_\Gamma \in (d_-, d_+)$ denote the slope of the maximal growth direction of $\Gamma$ as given in (2.4). Then

$$\delta \leq \min(\cos \theta_\Gamma \cdot \dim \Lambda_{\rho_1}, \sin \theta_\Gamma \cdot \dim \Lambda_{\rho_2}) \leq \frac{1}{\sqrt{2}} \dim \Lambda. \tag{1.4}$$

In particular, if $\delta = (\dim \Lambda)/\sqrt{2}$, then $\theta_\Gamma = \pi/4$ and $\dim \Lambda = \dim \Lambda_{\rho_i}$ for $i = 1, 2$.

The work of Guéritaud, Guichard, Kassel and Wienhard [10, Thm. 1.14] (see also [11]) implies that $1 \not\in [d_-, d_+]$ if and only if $\Gamma$ acts properly discontinuously on the quotient space $\text{SO}^o(n, 1) \times \text{SO}^o(n, 1)/\text{diag}(\text{SO}^o(n, 1))$. 


In this case, \( \delta_{\min} = \max(\dim \Lambda_{\rho_1}, \dim \Lambda_{\rho_2}) \). Hence examples of \( \Gamma \) satisfying the following corollary of Theorems 1.1 and 1.2 include those arising from compact anti-de Sitter 3-manifolds, which are of the form \( \Gamma \backslash (\SO^o(2, 1) \times \SO^o(2, 1))/\diag(\SO^o(2, 1)) \).

**Corollary 1.3.** If \( \Gamma \) acts on \( (\SO^o(n, 1) \times \SO^o(n, 1))/\diag(\SO^o(n, 1)) \) properly discontinuously, then

\[
\delta < \frac{1}{\sqrt{2}} \dim \Lambda \quad \text{and} \quad \dim \Lambda = \delta_{\min} = \max(\dim \Lambda_{\rho_1}, \dim \Lambda_{\rho_2}).
\]

We remark that the second identity also follows from \([11]\).

**Gap property.** In the group \( \SO^o(n, 1) \), the critical exponent of a lattice is equal to \( n - 1 \), which is the volume entropy of \( \mathbb{H}^n \), and there are convex cocompact (non-lattice) subgroups of \( \SO^o(n, 1) \) whose critical exponents are arbitrarily close to \( n - 1 \), constructed by McMullen \([15, \text{Sec.6}]\). The critical exponent of a discrete subgroup of \( G \) is at most \( 2(n - 1) \), which is the volume entropy of \( \mathbb{H}_n \times \mathbb{H}^n \). For Anosov subgroups, note that \( \delta \leq n - 1 \) by \((1.3)\).

In contrast to the case of convex cocompact subgroups of \( \SO^o(n, 1) \), the following consequence of Theorem 1.2 presents a strong gap property for critical exponents of Anosov subgroups of the product group \( \SO^o(n, 1) \times \SO^o(n, 1) \).

**Theorem 1.4** (Gap property). For any Anosov subgroup \( \Gamma \subset \SO^o(n, 1) \times \SO^o(n, 1) \), we have

\[
\delta \leq \frac{n - 1}{\sqrt{2}};
\]

moreover if \( \delta = (n - 1)/\sqrt{2} \), then \( 1 \in (d_-(\rho_1, \rho_2), d_+(\rho_1, \rho_2)) \) and \( \rho_i(\Delta) \) must be a lattice in \( \SO^o(n, 1) \) for both \( i = 1, 2 \), which are moreover conjugate for \( n \geq 3 \).

Since the equality is achieved when \( \rho_1 \sim \rho_2 \) with \( \rho_i(\Delta) \) a lattice, the above bound is sharp.

In view of special interests in low dimensional hyperbolic manifolds which come with huge deformation spaces, we also formulate the following corollary of Theorem 1.2, using the isomorphisms \( \PSL_2(\mathbb{C}) \simeq \SO^o(3, 1) \) and \( \PSL_2(\mathbb{R}) \simeq \SO^o(2, 1) \):

**Corollary 1.5.** We have the following:

1. If \( \rho_1(\Delta), \rho_2(\Delta) < \PSL_2(\mathbb{C}) \), then \( \delta < \sqrt{2} \), with the equality if and only if \( \rho_1(\Delta) \) and \( \rho_2(\Delta) \) are lattices which are conjugate to each other.
2. If \( \rho_1(\Delta) < \PSL_2(\mathbb{R}) \) and \( \rho_2(\Delta) < \PSL_2(\mathbb{C}) \), then \( \delta < 2/\sqrt{5} \).
3. If \( \rho_1(\Delta), \rho_2(\Delta) < \PSL_2(\mathbb{R}) \), then \( \delta < 1/\sqrt{2} \).

See Theorem 3.2 and Corollary 3.3 for more general versions. Some cases of Corollary 1.5(3) when \( \Delta \) is a surface group were also considered in \([19, \text{Thm. 1.8}]\).
Extra symmetries. As usual, $\text{Out} \Delta$ denotes the outer automorphism group of $\Delta$, i.e., the group of automorphisms of $\Delta$ modulo the inner automorphisms. Note that the conjugacy class of $\rho_1 \circ \iota$ does not depend on the choice of a representative of $\iota \in \text{Out} \Delta$.

**Corollary 1.6.** If $\rho_1 \sim \rho_2 \circ \iota$ for some involution $\iota \in \text{Out} \Delta$, then $\theta_\Gamma = \pi/4$ and hence

$$
\delta_{\text{max}} = \sqrt{2} \delta \leq \dim \Lambda_{\rho_1} = \dim \Lambda_{\rho_2} \leq \dim \Lambda.
$$

For instance, when $\Delta = \pi_1(S)$ is a surface group, there always exists an involution $\iota \in \text{Out} \Delta$ and a Fuchsian representation $\rho$ of $\Delta$ which is not a fixed point of $\iota$ in the Teichmüller space $T(S)$, i.e., $\rho \circ \iota \neq \rho$. See section 6 for more examples including 3-manifold groups.

![Figure 1. Example of an involution](image)

**Directional conical limit sets.** We also obtain estimates on the Hausdorff dimension of directional conical limit sets of $\Gamma$. Let $a = \mathbb{R}^2$ and set $a^+=\{(u_1,u_2) \in \mathbb{R}^2 : u_1 \geq 0, u_2 \geq 0 \}$. Given $u=(u_1,u_2) \in a^+$, a point $(\xi_1,\xi_2) \in \mathcal{F} = S^{a-1} \times S^{a-1}$ is called a $u$-directional conical limit point of $\Gamma$ if the geodesic ray

$$
\{(\xi_1(u_1 t),\xi_2(u_2 t)) : t \geq 0 \}
$$

accumulates on $\Gamma \backslash X$ under the canonical projection $X \to \Gamma \backslash X$, where $\{\xi(t) : t \geq 0 \}$ denotes a unit speed geodesic in $\mathbb{H}^a$ toward $\xi \in S^{a-1}$.

We denote by

$$
\Lambda_u \subset \Lambda
$$

the set of all $u$-directional conical limit points of $\Gamma$; note that $\Lambda_u$ depends only on the slope $u_2/u_1$ of the line $\mathbb{R}u$. It is not hard to observe that $\Lambda_u = \emptyset$ if $u_2/u_1 \notin [d_-,d_+]$. On the other hand, the non-triviality of $\Lambda_u$ is much more subtle to decide.

**Theorem 1.7.** Assume that $\rho_1 \not\sim \rho_2$. If $u_2/u_1 \in (d_-,d_+)$, then

$$
\frac{\delta_u}{\max(u_1,u_2)} \leq \dim \Lambda_u \leq \frac{\delta_u}{\min(u_1,u_2)}
$$

where $\delta_u > 0$ denotes the value of the growth indicator function $\psi_\Gamma$ (see Def. (2.3)) at $u$. 
We remark that
\[ \delta_{\text{max}} = \sup_{u \in [d_-, d_+]} \frac{\delta_u}{\max(u_1, u_2)} \] and \[ \delta_{\text{min}} \geq \sup_{u \in [d_-, d_+]} \frac{\delta_u}{\min(u_1, u_2)} \]
(see Lemma 2.5), which indicates that Theorem 1.1 does not (at least directly) follow from Theorem 1.7.

**On the proofs.** Our approach in this paper is heavily based on the fact that \( \Gamma \) is an Anosov subgroup of \( G \). Indeed, in the rest of the paper, we discuss Anosov subgroups of a general product group \( \prod_{i=1}^{k} \SO^n(n, 1) \); while analogous upper bounds in Theorems 1.1 and 1.7 hold for any \( k \geq 1 \), our proof of the lower bounds works only for \( k = 2, 3 \). This is because our proof of the lower bound is based on a recent theorem of Burger, Landesberg, Lee and Oh [6] that \( \Lambda_u \) has positive measure with respect to the unique Patterson-Sullivan measure \( \nu_u \) on \( \Lambda \) associated to \( u \) if and only if \( k \leq 3 \) (see Theorem 2.4).

In section 2, we review basic notions and state known results about Anosov subgroups of \( \prod_{i=1}^{k} \SO^n(n, 1) \). In section 3, we prove Theorem 1.1 with some constraints on \( k \). In section 4, we discuss a trivial vector bundle associated to the dynamics of one-dimensional diagonal flow, and prove a result that the associated vector-valued coordinate map grows with speed \( o(t) \) under the time \( t \)-flow (Theorem 4.4). The result in this section is based on the reparameterization theorem of Bridgeman, Canary, Labourie and Sambarino [4] and the work of Sambarino [22]. In section 5, we prove Theorem 1.7. We also obtain the local behavior of the measures \( \nu_u \) in Theorems 5.4 and 5.7. In the last section 6, we prove Corollary 1.6 and discuss some geometric examples with extra symmetries to which the corollary applies.

Finally we remark that our approach works for any Anosov subgroup of a semisimple algebraic group of rank at most 3, provided the Hausdorff dimension of the limit set is computed with respect to a well-chosen metric on the Furstenberg boundary. The reason we have chosen to write this paper only for the product of \( \SO(n, 1) \)'s is because \( \mathcal{F} \) in this case is simply the Riemannian product of the spheres \( \mathbb{S}^{n-1} \) and hence is equipped with a natural Riemannian metric. We think that our approach should give a stronger lower bound for the Hausdorff dimension of the limit set considered in [9] for Hitchin representations in \( \text{PSL}_n(\mathbb{R}) \) for \( n \leq 4 \).

**Acknowledgements.** Our work has been largely inspired by a pioneering paper of Marc Burger [5] on a higher rank Patterson-Sullivan theory. In particular, the upper bound of Theorem 1.7 was already noted in [5, Thm. 2]. We would like to dedicate this paper to him on the occasion of his sixtieth birthday with affection and admiration.

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2. Preliminaries

Fixing $k \geq 1$, let $G = \prod_{i=1}^{k} \text{SO}^0(n, 1)$, and consider the Riemannian symmetric space $(X = \prod_{i=1}^{k} \mathbb{H}^n, d)$ where $d((x_i), (y_i)) = \sqrt{\sum_{i=1}^{n} d_{\mathbb{H}^n}(x_i, y_i)^2}$. Consider the Furstenberg boundary $\mathcal{F} = \prod_{i=1}^{k} S^{n-1}$ equipped with the Riemannian metric. Let $\Delta$ be a finitely generated group and let $\rho_i : \Delta \to \text{SO}^0(n, 1)$ be a convex cocompact representation with non-elementary image and finite kernel for each $1 \leq i \leq k$. Fix $\Gamma$ be the subgroup of $G$ defined as

$$
\Gamma = \{(\rho_1(\sigma), \ldots, \rho_k(\sigma)) \in G : \sigma \in \Delta\}.
$$

We remark that the class of these groups is precisely the class of Anosov subgroups of $G$ with respect to a minimal parabolic subgroup of $G$ in the sense of Guichard and Wienhard [12]. In the setting of our product group $G$, a discrete subgroup $\Gamma < G$ is Anosov with respect to a minimal parabolic subgroup of $G$, if there exist a Gromov hyperbolic group $\Delta$ and a representation $\rho = (\rho_1, \ldots, \rho_k) : \Delta \to G$ with $\Gamma = \rho(\Delta)$ satisfying that for each $1 \leq i \leq k$, $\rho_i : \Delta \to \text{SO}^0(n, 1)$ is a proper map which induces an equivariant continuous embedding of the Gromov boundary $\partial_\infty \Delta$ into $\partial \mathbb{H}^n$. Such $\rho_i$ is a convex cocompact representation with finite kernel.

We will assume that no two $\rho_i$ are conjugate to each other. This assumption implies that if $H$ denotes the identity component of the Zariski closure of $\Gamma$, then $H$ is isomorphic to $\prod_{i=1}^{k} \text{SO}^0(n_i, 1)$ for some $2 \leq n_i \leq n$. Hence $\Gamma$ is a Zariski dense Anosov subgroup of a semisimple real algebraic group $H$ of rank $k = \text{rank } G$, which enables us to use the general theory developed for such groups. Fix a basepoint $o \in X$. By abuse of notation, we write $o = (o, \ldots, o)$. Let $a = \mathbb{R}^k$ and $a^+ = \{(u_1, \ldots, u_k) \in \mathbb{R}^k : u_i \geq 0 \text{ for all } i\}$. We denote by $\| \cdot \|$ the Euclidean norm on $a$.

The limit set of $\Gamma$, which we denote by $\Lambda = \Lambda_\Gamma$, is defined as the set of all accumulation points of $\Gamma o$ in the Furstenberg boundary $\mathcal{F}$. It is a $\Gamma$-minimal subset of $\mathcal{F}$ ([2], [13, Lem. 2.3]).

For each $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{F}$ and $(t_1, \ldots, t_k) \in a$, we write

$$
\xi(t_1, \ldots, t_k) = (\xi_1(t_1), \ldots, \xi_k(t_k))
$$

where $\{\xi_i(t) : t \geq 0\}$ denotes the unit speed geodesic from $o$ to $\xi_i$ in $\mathbb{H}^n$.

Set

$$
\xi(a^+) := \{\xi(t_1, \ldots, t_k) \in X : t_i \geq 0 \text{ for all } i\}. \tag{2.1}
$$

Recall that $\xi \in \mathcal{F}$ is called a conical limit point if there exists a sequence $\gamma_j \in \Gamma$ such that

$$
\sup_j d(\xi(a^+), \gamma_j o) < \infty.
$$

If $\Lambda_c$ denotes the set of all conical limit points, then it is a well-known property of an Anosov subgroup (cf. [13, Prop. 7.4]) that

$$
\Lambda = \Lambda_c. \tag{2.2}
$$
The Cartan projection of $g = (g_i)_{i=1}^k \in G$ is given by
$$\mu(g) = (d_{g_{-1}}(g_1o, o), \ldots, d_{g_{-1}}(g_ko, o)) \in a^+.$$ In particular, $d(go, o) = \|\mu(g)\|$. We denote by $L_\Gamma \subset a^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$. It is a convex cone with non-empty interior [2]. For $k = 2$, $L_\Gamma$ coincides with $\{(u_1, u_2) \in a^+: u_2/u_1 \in [d_-, d_+]\} \cup \{(0, 0)\}$ where $d_\pm$ is defined as in (1.2). Let $\delta$ denote the critical exponent of $\Gamma$, which is the abscissa of convergence of the Poincaré series
$$P_\Gamma(s) = \sum_{\gamma \in \Gamma} e^{-s\|\mu(\gamma)\|}.$$ It follows from the non-elementary assumption on $\rho_1(\Delta)$ that $\delta > 0$.

Let $\psi_\Gamma: a^+ \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$ following Quint [20]: for any non-zero $u \in a^+$,
$$\psi_\Gamma(u) := \inf_{\text{open cones } C \subset a^+, u \in C} \tau_C$$ (2.3)
where $\tau_C$ is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu(\gamma) \in C} e^{-s\|\mu(\gamma)\|}$, and $\psi_\Gamma(0) = 0$. Quint also showed that $\psi_\Gamma$ is a concave function such that $\psi_\Gamma > 0$ on int $L_\Gamma$,
$$\delta = \sup_{\|u\|=1} \psi_\Gamma(u)$$ and that there exists a unique unit vector
$$u_\Gamma = (\cos \theta_\Gamma, \sin \theta_\Gamma) \in L_\Gamma$$ (2.4)
such that $\delta = \psi_\Gamma(u_\Gamma)$ [20].

The following proposition follows from the fact that $\Gamma$ is a Zariski dense Anosov subgroup of $H \simeq \prod_{i=1}^k SO(n_i, 1)$ for $2 \leq n_i \leq k$ by the works of Sambarino [22, Lem. 4.8] and Sambarino-Potrie [24, Prop. 4.6 and 4.11].

**Theorem 2.1.** We have the following:
1. $L_\Gamma \subset \text{int } a^+ \cup \{0\}$;
2. $\psi_\Gamma$ is strictly concave on int $L_\Gamma$;
3. $u_\Gamma \in \text{int } L_\Gamma$.

For $x = (x_1, \ldots, x_k), y = (y_1, \ldots, y_k) \in X$, and $\xi = (\xi_1, \ldots, \xi_k) \in F$, the $a$-valued Busemann function is given as
$$\beta_\xi(x, y) = (\beta_{\xi_1}(x_1, y_1), \ldots, \beta_{\xi_k}(x_k, y_k)) \in a$$ where $\beta_{\xi_i}(x_i, y_i) = \lim_{t \to +\infty} d_{\mathbb{H}^n}(\xi_i(t), x_i) - d_{\mathbb{H}^n}(\xi_i(t), y_i)$ is the Busemann function on $\mathbb{H}^{n-1} \times \mathbb{H}^n \times \mathbb{H}^n$.

**Definition 2.2.** For $\psi \in a^*$, a Borel probability measure $\nu$ on $\Lambda$ is called a $(\Gamma, \psi)$-Patterson-Sullivan measure if the following holds: for any $\xi \in \Lambda$ and $\gamma \in \Gamma$,
$$d_{\gamma \ast \nu}(\xi) = e^{-\psi(\beta_\xi(\gamma, o))}$$ where $\gamma \ast \nu(W) = \nu(\gamma^{-1}W)$ for any Borel subset $W \subset \Lambda$.

**Theorem 2.3.** ([8, Thm. 7.7 and Cor. 7.8] [13, Cor. 7.12]) Let $u \in \text{int } L_\Gamma$. 
(1) There exists a unique $\psi_u \in \mathfrak{a}^*$ such that $\psi_u \geq \psi_T$ and $\psi_u(u) = \psi_T(u)$.
(2) There exists a unique $(\Gamma, \psi_u)$-Patterson-Sullivan measure, say, $\nu_u$.
(3) The abscissa of convergence of the series $P_u(s) := \sum_{\gamma \in \Gamma} e^{-s\psi_u(\mu(\gamma))}$ is equal to 1 and $P_u(1) = \infty$.

Construction of $\nu_u$. Fix $u \in \text{int} L_\Gamma$. By Theorem 2.1(1), $\Gamma_0 \cup \Lambda$ is a compact space. For $s > 1$, consider the probability measure on $\Gamma_0 \cup \Lambda$ given by
$$\nu_{u,s} := \frac{1}{P_u(s)} \sum_{\gamma \in \Gamma} e^{-s\psi_u(\mu(\gamma))} D_{\gamma_0}$$
(2.5)
where $D_{\gamma_0}$ denotes the Dirac measure on $\gamma_0$; this is a probability measure by Theorem 2.3(3). Note that the space of probability measures on $\Gamma_0 \cup \Lambda$ is a weak$^*$ compact space. Therefore, by passing to a subsequence, it weakly converges to a probability measure, say $\nu_u$, on $\Gamma_0 \cup \Lambda$. Since $P_u(1) = \infty$ by Theorem 2.1, $\nu_u$ is supported on $\Lambda$. Now the uniqueness of $(\Gamma, \psi_u)$-Patterson-Sullivan measure (Theorem 2.3) implies that as $s \to +1$, $\nu_{u,s}$ weakly converges to $\nu_u$.

The directional conical limit sets $\Lambda_u$. For each $u = (u_1, \cdots, u_k) \in \mathfrak{a}^+$, the $u$-directional conical limit set $\Lambda_u \subset \Lambda_c$ is defined as
$$\Lambda_u := \{ \xi \in \mathcal{F} : \liminf_{t \to +\infty} d(\xi(u_1 t, \cdots, u_k t), \Gamma_0) < \infty \}.$$  

Theorem 2.4. [6, Thm. 1.6, Thm. 6.3] Let $k \leq 3$ and $u \in \text{int} L_\Gamma$. Then
(1) we have $\nu_u(\Lambda_u) = 1$;
(2) for all sufficiently large $R > 0$, the abscissa of convergence of the series
$$\sum_{\gamma \in \Gamma, \|\mu(\gamma)\| \leq R} e^{-s\psi_u(\mu(\gamma))}$$
is equal to 1.

Critical exponents $\delta_{\text{max}}$ and $\delta_{\text{min}}$. Let $\delta_{\text{max}}$ and $\delta_{\text{min}}$ respectively denote the abscissa of convergence of the series $\sum_{\gamma \in \Gamma} e^{-sd_{\text{max}}(0,\gamma_0)}$ and $\sum_{\gamma \in \Gamma} e^{-sd_{\text{min}}(0,\gamma_0)}$ where $d_{\text{max}}$ and $d_{\text{min}}$ denote the maximum and minimum of the Riemannian metrics on $\mathbb{H}^n \times \mathbb{H}^n$ respectively. Let $\delta > 0$ denote the critical exponent of $\Gamma$ with respect to the Riemannian metric. Since $d \geq d_{\text{max}}$ and $\sqrt{2}d_{\text{min}} \leq d \leq \sqrt{2}d_{\text{max}}$, we have $\delta \leq \delta_{\text{max}} \leq \sqrt{2}\delta \leq \delta_{\text{min}}$. Then by [20, Lemma 3.1.1], we have
$$\delta = \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \|\mu(\gamma)\| < T \}}{T};$$
$$\delta^{\boldsymbol{\star}} = \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \|\mu(\gamma)\| < T \}}{T}$$
where $\boldsymbol{\star} = \text{max}$ or $\text{min}$ and $\max \mu(\gamma)$ and $\min \mu(\gamma)$ denote respectively the maximum and minimum of the entries of the vector $\mu(\gamma) \in \mathfrak{a}^+$. Although
Lemma 2.5. We have

\[ M_u := \max_{1 \leq i \leq k} u_i, \quad m_u := \min_{1 \leq i \leq k} u_i, \quad \delta_u := \psi_T(u) > 0. \] (2.8)

Lemma 2.5. We have

\[ \delta_{\text{max}} = \sup_{u \in \mathcal{L}_\Gamma} \delta_u / M_u \quad \text{and} \quad \delta_{\text{min}} \geq \sup_{u \in \mathcal{L}_\Gamma} \delta_u / m_u. \]

Moreover, there exists a unique vector \( w_T \in \mathcal{L}_\Gamma \) such that \( \|w_T\|_{\text{max}} = 1 \) and \( \delta_{\text{max}} = \psi_T(w_T) \) and \( \delta_{\text{max}} > \delta \).

Proof. Note that

\[ \sup_{u \in \mathcal{L}_\Gamma} \frac{\delta_u}{M_u} = \sup_{u \in a^+, \|u\| = 1} \psi_T(u), \]

and this is equal to \( \delta_{\text{max}} \) by [20, Cor. 3.14]. By the strict concavity of \( \psi_T \) on \( \text{int} \mathcal{L}_\Gamma \) (see Theorem 2.1), there exists a unique \( w_T \in \mathcal{L}_\Gamma \) as claimed in the statement. Since \( \mathcal{L}_\Gamma \subset \text{int} a^+ \) by Theorem 2.1(1), for any \( v \in \mathcal{L}_\Gamma \), we have \( \|v\| < \|v\|_{\text{max}} \) from which \( \delta_{\text{max}} > \delta \) follows.

To show the second claim, let \( u \in \mathcal{L}_\Gamma \subset \text{int} a^+ \). For \( \varepsilon > 0 \), set \( C_{\varepsilon} = \{ v = (v_i) \in \text{int} a^+ : \max_{1 \leq i \leq k} \| v_i - u_i \| < \varepsilon \} \). For all sufficiently small \( \varepsilon > 0 \), if \( v = (v_i) \in C_{\varepsilon} \) and \( \min v_i = v_j \), then \( m_u = u_j \). It follows that if \( \mu(\gamma) \in C_{\varepsilon} \) and \( \|\mu(\gamma)\| < T/m_u \), then \( \min \mu(\gamma) \leq T/(1 - \varepsilon m_u) \). Hence we deduce from the definition of \( \psi_T(u) \) that for all small \( \varepsilon > 0 \),

\[
\delta_u/m_u \leq \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \mu(\gamma) \in C_{\varepsilon}, \|\mu(\gamma)\| < T/m_u \} }{T} \\
\leq \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \min \mu(\gamma) < T/(1 - \varepsilon m_u) \} }{T} \\
= \delta_{\text{min}}/(1 - \varepsilon m_u)
\]

by (2.7). As \( \varepsilon > 0 \) is arbitrary, we get \( \delta_{\text{min}} \geq \delta_u/m_u \) for all \( u \in \mathcal{L}_\Gamma \). \( \square \)

3. Hausdorff dimension of \( \Lambda \)

In this section, we prove the main theorems on the Hausdorff dimension of \( \Lambda \) stated in the introduction. For a subset \( S \subset \mathcal{F} \), the Hausdorff dimension of \( S \), denoted by \( \dim S \), is the infimum of those \( s > 0 \) such that \( S \) admits coverings \( \bigcup B_i \) with \( \sum_i (\text{diam } B_i)^s \to 0 \) where \( \text{diam } B_i \) denotes the diameter of \( B_i \) with respect to the Riemannian metric on \( \mathcal{F} = S^{n-1} \times S^{n-1} \). We refer to [3] for general facts on Hausdorff dimension. It is convenient to use the upper half-space model of \( \mathbb{H}^n \) so that \( \partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{ \infty \} \). Let \( o = (0, \cdots, 0, 1) \in \mathbb{H}^n \) and \( v_o = (0, \cdots, 0, -1) \in T^1 \mathbb{H}^n \) denote the downward normal vector based at \( o \). By abuse of notation, we set \( o = (o, \cdots, o) \in \prod_{i=1}^n \mathbb{H}^n \) and \( v_o = (v_o, \cdots, v_o) \in \prod_{i=1}^n T^1 \mathbb{H}^n \). For \( \xi \in \mathcal{F} \cap \prod_{i=1}^k \mathbb{H}^{n-1} \) and \( r > 0 \), let \( B(\xi, r) \) denote the ball centered at \( \xi \) of radius \( r \).
Let $K_0 \simeq \text{SO}(n)$ be the maximal compact subgroup of $\text{SO}^0(n, 1)$ given as the stabilizer of $o \in \mathbb{H}^n$. Fix a unit tangent vector $v_0$ at $o$ and let $M_0 := \text{Stab}(v_0)$. We then have the following identification: $\text{SO}^0(n, 1)/K_0 = \mathbb{H}^n$ and $\text{SO}^0(n, 1)/M_0 = T^1 \mathbb{H}^n$. Let $A_0 = \{a_t : t \in \mathbb{R}\} < \text{SO}^0(n, 1)$ denote the one-parameter subgroup of semisimple elements whose right translation action on $\text{SO}^0(n, 1)/M_0$ corresponds to the geodesic flow on $T^1 \mathbb{H}^n$. Set $K = \prod K_0 < G$, $M = \prod M_0 < G$, and $A := \prod A_0$. Then $X = G/K$.

**Theorem 3.1.** For any $k \geq 1$, we have

$$\dim \Lambda \leq \delta_{\min}. \quad (3.1)$$

**Proof.** For $R > 0$ and $x \in X$, the shadow $O_R(o, x)$ is defined as

$$O_R(o, x) = \{\eta \in \mathcal{F} : \exists k \in K \text{ s.t. } k^+ = \eta, a \in A^+, \text{ and } d(kao, x) \leq R\}. \quad (3.2)$$

For each $N \in \mathbb{N}$, let $\Lambda_N := \Lambda \cap \limsup_{\gamma \in \Gamma} O_N(o, \gamma o)$, that is,

$$\Lambda_N = \{\xi \in \Lambda : \exists \gamma_i \to \infty \text{ in } \Gamma \text{ such that } \xi \in O_N(o, \gamma_i o) \text{ for all } i \geq 1\}.$$

There exists a constant $c_N > 0$ such that for any $\gamma \in \Gamma$, the shadow $O_N(o, \gamma o)$ is contained in a ball $B(\xi_\gamma, c_N e^{-\min \mu(\gamma)})$ for some $\xi_\gamma \in \mathcal{F}$.

From Theorem 2.3 (1), $\gamma_i \to \infty$ if and only if $\min \mu(\gamma_i) \to \infty$. Hence, for any fixed $t > 0$, we have

$$\Lambda_N \subset \bigcup_{\gamma \in \Gamma, \min \mu(\gamma) > t} O_N(o, \gamma o) \subset \bigcup_{\gamma \in \Gamma, \min \mu(\gamma) > t} B(\xi_\gamma, c_N e^{-\min \mu(\gamma)}).$$

Let $s > \delta_{\min}$. Since $\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)} < \infty$,

$$\lim_{t \to \infty} \sum_{\gamma \in \Gamma, \min \mu(\gamma) > t} e^{-s \min \mu(\gamma)} = 0.$$

This implies that the $s$-dimensional Hausdorff measure of $\Lambda_N$ is equal to zero; so $\dim \Lambda_N \leq \delta_{\min}$. Since $\Lambda$ is equal to the conical limit set $\Lambda_{\varepsilon}$ by (2.2), we have $\Lambda = \cup_{N \in \mathbb{N}} \Lambda_N$. Consequently,

$$\dim \Lambda \leq \sup_{N \in \mathbb{N}} \dim \Lambda_N \leq \delta_{\min}.$$

\qed

The following theorem implies Theorem 1.2:

**Theorem 3.2.** For $k \leq 3$, we have

$$\delta \leq \min_{1 \leq i \leq k} (u_i \dim \Lambda_{\rho_i}) \leq \left(\min_{1 \leq i \leq k} u_i\right) \dim \Lambda$$

where $u_\Gamma = (u_1, \cdots, u_k)$ is the unit vector of the maximal growth of $\Gamma$. In particular,

$$\delta < \min_{1 \leq i \leq k} \dim \Lambda_{\rho_i} \text{ and } \delta \leq \frac{1}{\sqrt{k}} \dim \Lambda.$$

Moreover, if $\delta = \frac{1}{\sqrt{k}} \dim \Lambda$, then $u_\Gamma = \frac{1}{\sqrt{k}}(1, \cdots, 1)$ and $\dim \Lambda = \dim \Lambda_{\rho_i}$ for all $1 \leq i \leq k$.
Proof. Let \( S_R := \{ g \in G : \| \mu(g) - \Re \mu \| < R \} \) for \( R > 0 \). Note that
\[
\psi_{u_\Gamma}(v) = \delta_{u_\Gamma}(u_\Gamma, v)
\]
for \( v \in a; \) here \( \langle , \rangle \) denotes the Euclidean inner product (cf. [8, Lem. 2.24]). By Theorem 2.4, there exists \( R > 0 \) such that the abscissa of the convergence of \( \mathcal{D}_R(s) = \sum_{\gamma \in \Gamma \cap S_R} e^{-s(\Re \mu(\gamma))} \) is equal to \( \delta \).

Fix \( 1 \leq i \leq k \). We claim that for any \( s > u_i \delta_{\rho_i(\Delta)} \), we have \( \mathcal{D}_R(s) \ll \infty \); this implies that \( \delta = \delta_{u_\Gamma} \leq u_i \dim \Lambda_{\rho_i} \) since \( \delta_{\rho_i(\Delta)} = \dim \Lambda_{\rho_i} \).

Write \( \rho = \prod \rho_j \) so that an element of \( \Gamma = \rho(\Delta) \) is of the form \( \rho(\sigma) = (\rho_1(\sigma), \cdots, \rho_k(\sigma)) \). Since \( \| u_\Gamma \| = 1 \) and for any \( \rho(\sigma) \in S_R \),
\[
\left\| \mu(\rho(\sigma)) - \frac{\mu(\rho_i(\sigma))}{u_i} u_\Gamma \right\|_{\text{max}} \leq \frac{R}{u_i},
\]
we deduce
\[
\mathcal{D}_R(s) = \sum_{\gamma \in \Gamma \cap S_R} e^{-s(\Re \mu(\gamma))} \ll \sum_{\gamma \in \Gamma} e^{-s u_i^{-1} \mu(\rho_i(\sigma))} \tag{3.3}
\]
which is finite when \( s > u_i \delta_{\rho_i(\Delta)} \). This finishes the claim, and hence the first inequality. The second inequality follows since \( \max_i \dim \Lambda_{\rho_i} \leq \dim \Lambda \). The rest of the claim follows easily from these.

The following is an immediate consequence since any convex cocompact subgroup of \( \text{SO}^0(n, 1) \) with critical exponent \( n - 1 \) is necessarily a lattice [25].

**Corollary 3.3.** Let \( k \leq 3 \). If \( \Gamma \) is a Zariski dense Anosov subgroup of \( \prod_{i=1}^{k} \text{SO}^0(n_i, 1) \), \( 2 \leq n_i \leq n \) with respect to a minimal parabolic subgroup, then
\[
\delta < \min(n_1 - 1, \cdots, n_k - 1).
\]
If \( n_i = n \) for all \( 1 \leq i \leq k \), then
\[
\delta \leq \frac{\dim \Lambda}{\sqrt{k}} \leq \frac{n - 1}{\sqrt{k}}.
\]
Moreover, if \( \delta = (n - 1)/\sqrt{k} \), then \( \rho_i(\Delta) \) is a lattice in \( \text{SO}^0(n, 1) \) for all \( 1 \leq i \leq k \).

4. Fibered dynamical systems and \( \ker \psi_u \)-coordinate map

We continue to use notation \( M_0, K_0, M, K, A \) from the last section. For \( [g] \in \text{SO}^0(n, 1)/M_0 \), we denote by \( g^\pm = g(v_\pm) \) the forward and backward endpoints of the geodesic determined by the tangent vector \( [g] \in T^1 \mathbb{H}^n \). Now the map \( [g] \to (g^+, g^-, \beta g^-(o, go)) \) gives an \( \text{SO}^0(n, 1) \)-equivariant homeomorphism between the space \( \text{SO}^0(n, 1)/M_0 \) and \( \{ (\xi, \eta) \in \mathbb{H}^n \times \mathbb{H}^n : \xi \neq \eta \} \times \mathbb{R} \), where the \( \text{SO}^0(n, 1) \)-action on the latter space is given by \( g.(\xi, \eta, s) = (g\xi, g\eta, s + \beta \xi(g^{-1}o, o)) \). This homeomorphism is called the Hopf-parametrization of \( \text{SO}^0(n, 1)/M_0 \) under which the right \( A \)-action on \( \text{SO}^0(n, 1)/M_0 \) corresponds to the translation flow on \( \mathbb{R} \).
For $\xi \in \mathcal{F} = \prod_{i=1}^{k} S^{n-1}$, we write $\xi_i$ for its $i$-th component. We set $\mathcal{F}^{(2)} = \{(\xi, \eta) \in \mathcal{F} \times \mathcal{F} : \xi_i \neq \eta_i \text{ for all } i\}$. Then the Hopf parametrization of $\text{SO}^0(n,1)/M_0$ extends to the Hopf-parametrization of $G/M$ componentwise, and gives the $G$-equivariant homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times \mathfrak{a}$ given by

$$[g] \mapsto (g^+, g^-, \beta_g^+(o, go))$$

where $g^\pm = (g^+_i)$. Then $\Lambda^{(2)} = \mathcal{F}^{(2)} \cap (\Lambda \times \Lambda)$. Then $\Omega := \Gamma \backslash \Lambda^{(2)} \times \mathfrak{a}$ is identified with the closed subspace $\{[g] \in \Gamma \backslash G/M : g^\pm \in \Lambda\}$ of $\Gamma \backslash G/M$ via the Hopf parameterization.

**Trivial ker $\psi_u$-vector bundle.** We fix a unit vector $u \in \mathfrak{L}_\Gamma$ in the rest of this section. Consider the $\Gamma$-action on the space $\Lambda^{(2)} \times \mathbb{R}$ by

$$\gamma.(\xi, \eta, s) = (\gamma \xi, \gamma \eta, s + \psi_u(\beta_\xi(\gamma^{-1} o, o))).$$

The reparametrization theorems for Anosov groups ([4, Prop. 4.1], [7, Thm. 4.15]) imply that $\Gamma$ acts properly discontinuously and cocompactly on $\Lambda^{(2)} \times \mathbb{R}$. Hence $Z := \Gamma \backslash (\Lambda^{(2)} \times \mathbb{R})$ is a compact space. Now the $\Gamma$-equivariant projection $\Lambda^{(2)} \times \mathfrak{a} \to \Lambda^{(2)} \times \mathbb{R}$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi_u(v))$ induces an affine bundle with fiber ker($\psi_u$):

$$\pi : \Omega = \Gamma \backslash \Lambda^{(2)} \times \mathfrak{a} \to Z = \Gamma \backslash (\Lambda^{(2)} \times \mathbb{R}).$$

It is well known that such a bundle is indeed a trivial vector bundle, and hence we can choose a continuous global section

$$s : Z \to \Omega$$

so that $\pi \circ s = \text{id}_Z$. Denote by $\{\tau_t : t \in \mathbb{R}\}$ the flow on $Z$ given by translations on $\mathbb{R}$. For $v = (v_1, \cdots, v_k) \in \mathfrak{a}$, we write

$$a_v = (a_{v_1}, \cdots, a_{v_k}) \in \mathfrak{A}.$$

**Definition 4.1** (ker $\psi_u$-coordinate map). We define the following continuous ker $\psi_u$-valued map:

$$\hat{K}_u : Z \times \mathbb{R} \to \ker \psi_u$$

defined as follows: for $z \in Z$ and $t \in \mathbb{R}$,

$$s(z) a_{tu} = s(z \tau_t) a_{K_u(z,t)}. \quad (4.1)$$

Fix a compact subset $D \subset G$ such that $s(Z) = \Gamma \backslash G D$, and for each $z \in Z$, write $s(z) = \Gamma \hat{s}(z)$ for some $\hat{s}(z) \in D$. Hence

$$\Lambda^{(2)} \times \mathfrak{a} = \Gamma D a_{\ker \psi_u}. \quad (4.2)$$

**Lemma 4.2.** For any $g \in G$ with $g^\pm \in \Lambda$, there exist $z_g \in Z$ and $w_g \in \ker \psi_u$ such that for all $t \in \mathbb{R}$, there exists $\gamma_{g,t} \in \Gamma$ satisfying

$$\gamma_{g,t} g a_{tu} = \hat{s}(z_g \tau_t) a_{K_u(z_g,t) + w_g}. \quad (4.3)$$
Proof. By (4.2), there exist $\gamma \in \Gamma$, $z \in Z$ and $w \in \ker \psi_u$ such that $\gamma g = \tilde{s}(z) a_w$, and hence
\[
\gamma g a_{tu} = \tilde{s}(z) a_{tu+w}.
\]
On the other hand, by (4.1), there exists $\gamma_{z,t} \in \Gamma$ such that
\[
\gamma_{z,t} \tilde{s}(z) a_{tu} = \tilde{s}(z \tau_t) a_{K_u(z,t)}.
\]
Therefore,
\[
\gamma_{z,t} \gamma g a_{tu} = \tilde{s}(z \tau_t) a_{K_u(z,t)+w}.
\]
It remains to set $\gamma_{g,t} = \gamma_{z,t} \gamma$. \hfill \Box

We also set $K^1_u(g,t) := K_u(z_g, t) + w_g \in \ker \psi_u$, so that for all $t \in \mathbb{R}$,
\[
\gamma_{g,t} g a_{tu} \in D a_{K^1_u(g,t)}.
\]

Lemma 4.3. For any $g \in G$ with $g^+ \in \Lambda$, we have
\[
Q(g) := \sup_{t>0} \inf_{\gamma \in \Gamma} \psi_u(\mu(\gamma g a_{tu})) < \infty.
\]
Moreover, $\sup_{g \in G, g^+ \in \Lambda} Q(g) < \infty$.

Proof. If $g, h \in G$ satisfy $g^+ = h^+$, then $\sup_{t>0} d(g a_{tu}, h a_{tu}) < \infty$. It follows that $Q(g) < \infty$ if and only if $Q(h) < \infty$. Therefore, by replacing $g$ with $h \in G$ satisfying $g^+ = h^+$ and $h^\pm \in \Lambda$, we may assume without loss of generality that $g^\pm \in \Lambda$. By (4.4), we have that for all $t > 0$,
\[
\gamma_{g,t} g a_{tu} \in D \cdot a_{\ker \psi_u},
\]
and hence
\[
\mu(\gamma_{g,t} g a_{tu}) \in \mu(D \cdot a_{\ker \psi_u}) \subset C + \ker \psi_u,
\]
where $C$ is a fixed compact subset of $a$ depending only on $D$. Therefore
\[
Q(g) \leq \sup_{t>0} \psi_u(\mu(\gamma_{g,t} g a_{tu})) \leq \max_{c \in C} \psi_u(c) < \infty.
\]
It is clear from the above proof that $\sup_{g \in G, g^+ \in \Lambda} Q(g) \leq \max_{c \in C} \psi_u(c)$. \hfill \Box

Theorem 4.4. For $\nu_u$-a.e. $\xi \in \Lambda$, we have
\[
\lim_{t \to \infty} \frac{1}{t} \hat{K}_u(z,t) = 0
\]
whenever $z = [\xi, \eta, s]$ for some $\eta \in \Lambda$ and $s \in \mathbb{R}$.

Proof. Let $m_u$ denote the $\psi_u$-Bowen-Margulis-Sullivan measure on $Z$; that is, $m_u$ is the unique probability measure on $Z$ which is locally equivalent to $\nu_u \otimes \nu_u \otimes ds$. It follows from [4] that $m_u$ is the measure of maximal entropy and in particular ergodic for the $\tau_t$-flow. Combining the reparametrization theorem [4, Prop. 4.1], and [23, Prop. 3.5], we deduce that there exists a Hölder continuous function $F : Z \to \ker \psi_u$ with $\int_Z F dm_u = 0$ such that for all $z \in Z$ and $t \in \mathbb{R}$,
\[
\hat{K}_u(z,t) = \int_0^t F(z \tau_s) \, ds + E(z) - E(z \tau_t) \tag{4.5}
\]
for some bounded measurable function $E : Z \to \ker \psi_u$. The Birkhoff ergodic theorem for the $\tau_s$ flow on $(Z, m_u)$ implies that for $m_u$-almost all $z \in Z$, we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t F(z\tau_s) \, ds = \int_Z F \, dm_u = 0;
\]
hence
\[
\lim_{t \to \infty} \frac{1}{t} K_u(z, t) = 0
\]
since $E$ is bounded. This proves the claim. \hfill $\Box$

**Corollary 4.5.** For $\nu_u$-a.e. $\xi \in \Lambda$, we have the following: for any $g \in G$ with $g^+ = \xi$ and $g^- \in \Lambda$,
\[
\lim_{t \to \infty} \frac{1}{t} K_u^+(g, t) = 0.
\]

**Proof.** Let $z = [(\xi_1, \eta_1, s)] \in Z$ be such that $\gamma_{g, 0}g = \tilde{s}(z) a_{K_0^+(g, 0)}$ as given by (4.4). It follows that $g^+ = \xi \in \Gamma \xi_1$. Hence if $\xi_1 \in \Lambda$ satisfies Theorem 4.4, so does $g^+ = \xi$. Since $\sup_{t>0} \| K_u^+(g, t) - \tilde{K}_u(z, t) \| < \infty$ by their definitions, the claim now follows from Theorem 4.4. \hfill $\Box$

Let $m_u^{\text{BMS}}$ denote the Bowen-Margulis-Sullivan measure on $\Omega \subset \Gamma \setminus G$ given by $m_u^{\text{BMS}} = m_u \otimes \text{Leb}_{\ker \psi_u}$; this is an $\Lambda$-invariant ergodic (infinite) Radon measure, as shown in [14]. We also remark that by [6], $m_u^{\text{BMS}} = \{ a_{tu} : t \in \mathbb{R} \}$-ergodic if and only if $k \leq 3$. In terms of this measure, Corollary 4.5 can be formulated as the following which may be regarded as an analogue of Sullivan’s result [25, Coro. 19], which predates his logarithm law.

**Theorem 4.6.** For $m_u^{\text{BMS}}$-a.e. $x \in \Gamma \setminus G$, we have
\[
\lim_{t \to \infty} \frac{d(x a_{tu} o, o)}{t} = 0.
\]

Since $D$ is compact, this theorem follows from Corollary 4.5 in view of (4.4).

**5. Hausdorff Dimension of $\Lambda_u$ and Local Behavior of $\nu_u$**

In this section, we obtain estimates on $\dim \Lambda_u$ and local estimates on $\nu_u$ for $u \in \mathrm{int} \, \mathcal{L}_\Gamma$. As in section 3, we use the upper half-space model of $\mathbb{H}^n$ so that $\partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{ \infty \}$. For $r > 0$, let $B(r)$ denote the ball in $\mathbb{R}^{n-1}$ centered at 0 of radius $r$. Note that for $w = (w_1, \ldots, w_k) \in \mathfrak{a}$, the set $\{ a_{tw}v_o = (a_{uw_1}v_o, \ldots, a_{uw_k}v_o) : t > 0 \}$ represents the geodesic ray emanating from $(o, \cdots, o)$ pointing toward $(0, \cdots, 0) \in \prod_{i=1}^k \mathbb{R}^{n-1}$, with the speed controlled by $w$. By abuse of notation, we write $d(o, a_{uw}v_o)$ for the distance between $o$ and the basepoint of $a_{uw}v_o$ in $X$.

In the whole section, we fix a unit vector
\[ u = (u_1, \ldots, u_k) \in \mathrm{int} \, \mathcal{L}_\Gamma. \]
Recall the notation $M_u = \max_{1 \leq i \leq k} u_i$, $m_u = \min_{1 \leq i \leq k} u_i$ and $\delta_u := \psi_u(u) = \psi_\Gamma(u) > 0$. 


Upper bound for dimension.

**Theorem 5.1.** For any $k \geq 1$, we have

$$\dim \Lambda_u \leq \frac{\delta_u}{m_u}. \quad (5.1)$$

**Proof.** For any $N \in \mathbb{N}$, set

$$\Gamma_N(u) := \{ \gamma \in \Gamma : \|\mu(\gamma) - t_u\gamma\| \leq N \text{ for some } t_u > 0 \}$$

and

$$\Lambda_N^*(u) := \limsup_{t \to \infty} \{ O_N(o, \gamma) : \gamma \in \Gamma_N(u), \|\mu(\gamma)\| \geq t \},$$

where $O_N(o, \gamma)$ is defined as in (3.2).

There exists $d_N > 0$ such that for any $\gamma \in \Gamma_N(u)$, the shadow $O_N(o, \gamma)$ is contained in a ball $B(\xi_\gamma, d_N e^{-t_u m_u})$ for some $\xi_\gamma \in \mathcal{F}$. Since $\|\mu(\gamma) - t_u\gamma\| \leq N$, by applying $\psi_u$, we get

$$|\psi_u(\mu(\gamma))\delta_u^{-1} - t_u| \leq N\delta_u^{-1}\|\psi_u\|_{\text{op}}$$

where $\| \cdot \|_{\text{op}}$ denotes the operator norm of $\psi_u$. Fix $N \in \mathbb{N}$. For any $t > 1$,

$$\Lambda_N^*(u) \subset \bigcup_{\gamma \in \Gamma_N(u), \|\mu(\gamma)\| \geq t} O_N(o, \gamma) \subset \bigcup_{\gamma \in \Gamma_N(u), \|\mu(\gamma)\| \geq t} B(\xi_\gamma, d_N e^{-t_u m_u})$$

$$\subset \bigcup_{\gamma \in \Gamma, \|\mu(\gamma)\| \geq t} B(\xi_\gamma, d_N' e^{-t_u m_u})$$

for some constant $d_N' \geq 1$. On the other hand, by Theorem 2.3(3), for any $s > \delta_u/m_u$, we have

$$\lim_{t \to \infty} \sum_{\gamma \in \Gamma, \|\mu(\gamma)\| \geq t} e^{-s u \delta_u^{-1} \psi_u(\mu(\gamma))} = 0.$$ 

It follows that the $s$-dimensional Hausdorff measure of $\Lambda_N^*(u)$ is zero. Hence $\dim \Lambda_N^*(u) \leq \frac{\delta_u}{m_u}$. Since

$$\Lambda_u \subset \bigcup_{N \in \mathbb{N}} \Lambda_N^*(u),$$

it follows that $\dim \Lambda_u \leq \frac{\delta_u}{m_u}$. \qed

**Remark 5.2.** In the case when $m_u(n - 1)(k - 1) < \delta_u$, the upper bound in (5.1) can be improved to $\frac{\delta_u + (M_u - m_u)(n - 1)(k - 1)}{M_u}$ by replacing $B(\xi_\gamma, d_N e^{-t_u m_u})$ with $e^{(M_u - m_u) t_u}$ number of balls of radius $d_N e^{-t_u m_u}$.

Lower bound for dimension.

**Lemma 5.3.** [13, Lem. 10.6] There exists a compact subset $S \subset G$ such that for any $\eta \in \Lambda$, there exists $g \in S$ such that $g^+ = \eta$ and $g^- \in \Lambda$.
Theorem 5.4. For any $k \geq 1$, there exists $C > 0$, depending only on $\psi_u$, such that for any $\xi = (\xi_1, \cdots , \xi_k) \in \Lambda_u$ and for all $t > 0$,

$$
\nu_u \left( \prod_{i=1}^{k} B(\xi_i, e^{-u_i t}) \right) \leq C e^{-\delta_u t}.
$$

Proof. Without loss of generality, we may assume $\xi = (0, \cdots , 0) \in \Lambda_u$ by translating $\xi$ to 0. By Lemma 5.3, we can choose $g \in \mathcal{S}$ so that $g^+ = \xi$ and $g^- \in \Lambda$. By Lemma 4.3, there exists $c > 0$ (depending only on $\Gamma$) such that for any $t > 0$, there exists $\gamma_{g,t} \in \Gamma$ such that $|\psi_u(\mu(\gamma_{g,t} g a_{t u}))| \leq c$.

Let $R_t$ denote the semiball centered at 0 of radius $e^{-t}$. Using the explicit construction of $\nu_u = \lim_{s \to 1} \nu_{u,s}$ with $\nu_{u,s}$ given in (2.5), we get

$$
\nu_u \left( \prod_{i=1}^{k} B(e^{-u_i t}) \right) = \lim_{s \to 1} \frac{\sum_{\gamma \in \Gamma} e^{-\psi_u(\mu(\gamma))}}{\sum_{\gamma \in \Gamma} e^{-s \psi_u(\mu(\gamma))}}. \tag{5.2}
$$

Let $\gamma_0 \in \prod R_{u,t}$ (such $\gamma$ exists since $\xi \in \Lambda_u$) and write $\gamma = (\rho_1(\sigma), \cdots, \rho_k(\sigma))$ for $\sigma \in \Delta$. By the hyperbolic geometry on $\mathbb{H}^n$, there exists a constant $C > 0$ (depending only on $\mathcal{S}$) such that for all $1 \leq i \leq k$ and $t > 1$,

$$
u_i t + d(g a_{t u} v_0, \rho_i(\sigma) o) - C \leq d(\rho_i(\sigma) o, o) \leq u_i t + d(g a_{t u} v_0, \rho_i(\sigma) o). \tag{5.3}
$$

This implies that $u t + \mu(a_{-t}^{-1} g^{-1} \gamma) - \mu(\gamma) \in Q_0$ for some compact subset $Q_0 \subset \mathfrak{a}$. Therefore

$$
c_0 := \sup_{t \geq 0} ||u t + \mu(\gamma_{g,t} \gamma) - \mu(\gamma_{g,t} g a_{t u}) - \mu(\gamma)|| < \infty.
$$

By applying $\psi_u$ and using $|\psi_u(\mu(\gamma_{g,t} g a_{t u}))| \leq c$, we then get

$$
c_1 := \sup_{t > 0} |\delta_u t + \psi_u(\mu(\gamma_{g,t} \gamma)) - \psi_u(\mu(\gamma))| < \infty,
$$

where $c_1$ depends only on $\mathcal{S}$ and $\psi_u$.

Hence for all $t > 0$ and $s > 1$,

$$
\sum_{\gamma \in \Gamma} e^{-s \psi_u(\mu(\gamma))} \leq e^{s c_1} e^{-s \delta_u t} \sum_{\gamma \in \mathcal{S}} e^{-s \psi_u(\mu(\gamma))}.
$$

By (5.2), it follows that for all $t > 0$,

$$
\nu_u \left( \prod_{i=1}^{k} B(e^{-u_i t}) \right) \leq e^{c_1} e^{-\delta_u t}.
$$

This finishes the proof. \qed

Corollary 5.5. For $k \leq 3$, we have

$$
\dim \Lambda_u \geq \frac{\delta_u}{M_u}.
$$
Proof. By Proposition 5.4, there exists $C > 0$ such that for all $0 < r < 1$,

$$\nu_u(B_{\text{max}}(\xi, r)) \leq C_1 \delta_u / M_u$$

where $B_{\text{max}}(\xi, r)$ is a radius $r > 0$ ball centered at $\xi$ in $\mathcal{F} = \prod_{i=1}^{k} S_{n-1}$ with respect to the maximum metric. Since $k \leq 3$, we have $\nu_u(\Lambda_u) = 1$ by Theorem 2.4. Since the Riemannian metric and the maximum metric on $\mathcal{F}$ are Lipschitz equivalent to each other, the well-known mass distribution principle (cf. [3, Lemma 1.2.8]) now implies that

$$\dim \Lambda_u \geq \frac{\delta_u}{M_u}.$$

□

Remark 5.6. Since $\delta_{\text{max}} = \sup \delta_u / M_u$ by Lemma 2.5, Theorem 3.2 implies that the Hausdorff dimension of $\Lambda$ is strictly bigger than that of any $\Lambda_u$ unless $\delta_{\text{max}} = \sqrt{2} \delta$.

Lower bound on the local size of $\nu_u$. Let $\Lambda_u^\ast$ be a subset of $\Lambda_u$ defined as follows: $\xi \in \Lambda_u^\ast$ if and only if for any $g \in G$ with $g^+ = \xi$ and $g^- \in \Lambda$,

$$\limsup_{t \to \infty} \frac{1}{t} K_u^\dagger(g, t) = 0. \quad (5.4)$$

By Corollary 4.5, $\nu_u(\Lambda_u^\ast) = 1$.

Theorem 5.7. Let $k \geq 1$. There exists $C > 0$ such that for any $\xi = (\xi_1, \ldots, \xi_k) \in \Lambda_u^\ast$, and for any sufficiently small $\varepsilon > 0$, there exists $t_0 = t_{\varepsilon, \xi} > 0$ such that for all $t \geq t_0$,

$$C \cdot e^{-\delta_u(1+\varepsilon)t} \leq \nu_u \left( \prod_{i=1}^{k} B(\xi_i, e^{-u_i t}) \right).$$

Proof. Without loss of generality, we may assume $\xi = 0$ and choose $g \in S$ such that $g^+ = \xi = 0$ and $g^- \in \Lambda$ where $S$ is a compact subset of $G$ given in Lemma 5.3. Let $\varepsilon > 0$. Since $0 = g^+ \in \Lambda_u^\ast$, there exists $t_0 = t_{\varepsilon, g} > 0$ such that for each $1 \leq i \leq k$, absolute of the $i$-th component of $K_u^\dagger(g, t) \in a = \mathbb{R}^{k}$ is at most $\frac{\delta_u}{t}$ for all $t > t_0$.

Recall the definition of $\gamma_{g,t}$ from (4.4): $\gamma_{g,t} g a_{tu} = d_t a_{K_u^\dagger(g,t)}$ where $d_t \in D$. Therefore $\gamma_{g,t}^{-1} = a_{t u - K_u^\dagger(g,t)} d_t^{-1}$. Let $q$ be the diameter of $D^{-1} a$.

Note that there exists $c_0 > 0$ such that for all $t > t_0$,

$$O_1(a, \gamma_{g,t}^{-1} a) \subset O_{q+1}(a, a_{t u - K_u^\dagger(g,t)} u_0) \subset \prod_{i=1}^{k} B(c_0 e^{-u_i (1-\varepsilon/4)t}).$$

On the other hand, the shadow lemma (cf. [13, Lem. 7.8]) says that for any $R > 0$, there exists $c = c(\psi_u, R) \geq 1$ such that for any $\gamma \in \Gamma$,

$$c^{-1} e^{-\psi_u(\mu(\gamma))} \leq \nu_u(O_R(a, \gamma a)) \leq ce^{-\psi_u(\mu(\gamma))}.$$
Hence we deduce that for all $t > \max(t_0, 2 \log c_0/(u_i \varepsilon))$,
\[
\beta \cdot e^{-\delta u t} \leq \nu_u \left( \prod_{i=1}^{k} B(\xi_i, c_0 e^{-u_i(1-\varepsilon/4)t}) \right) \leq \nu_u \left( \prod_{i=1}^{k} B(\xi_i, e^{-u_i(1-\varepsilon/2)t}) \right)
\]
for some constant $\beta = \beta(\psi_u, D) > 0$. By reparameterizing $(1 - \varepsilon/2)t = s$, this implies the claim. \hfill \Box

6. Examples with extra symmetries

Lemma 6.1. Let $k \geq 1$. Let $i \in \text{Out} \Delta$ be of order $k$ and $\rho_i = \rho_1 \circ i^{-1}$ for $1 \leq i \leq k$. Let $\Gamma_i := (\prod_{i=1}^{k} \rho_i)(\Delta)$. Then $\psi_{\Gamma_i}(u)/\|u\|$ and $\psi_{\Gamma_i}(u)/\|u\|_{\max}$ achieve their maximums at $(1, \cdots, 1)$. In particular,
\[
\delta_{\Gamma_i, \max} = \sqrt{k} \delta_{\Gamma_i}.
\]

Proof. For each $1 \leq n \leq k$, let $\Gamma^{(n)} = (\prod_{i=1}^{k} \rho_i)(\Delta)$. Since $i^k = 1$ in $\text{Out} \Delta$, $\Gamma^{(n)}$ can be regarded as a group obtained by permuting coordinates in a cyclic way. Hence, if we denote the cyclic permutation $(x_1, \cdots, x_k) \mapsto (x_2, \cdots, x_k, x_1)$ in $\mathbb{R}^k$ by $\theta$, we have
\[
\mathcal{L}_{\Gamma^{(n)}} = \theta(\mathcal{L}_{\Gamma^{(n-1)}}) \quad \text{and} \quad \psi_{\Gamma^{(n)}} = \psi_{\Gamma^{(n-1)}} \circ \theta^{-1}.
\]

However, $\Gamma^{(n)} = \Gamma_i$ for all $n$; since applying an automorphism to all coordinates does not change the group. Hence, (6.1) implies that $\mathcal{L}_{\Gamma_i}$ and $\psi_{\Gamma_i}$ are invariant under the cyclic permutation $\theta$ of coordinates.

Now let $u_{\Gamma_i} = (u_1, \cdots, u_k) \in \mathcal{L}_{\Gamma_i}$ be the maximal growth direction for the maximum norm, that is, $\|u_{\Gamma_i}\|_{\max} = 1$ and
\[
\psi_{\Gamma_i}(u_{\Gamma_i}) = \sup_{\|u\|_{\max} = 1} \psi_{\Gamma_i}(u) = \delta_{\Gamma_i, \max}.
\]

Noting that $\|\theta(\cdot)\|_{\max} = \|\cdot\|_{\max}$, the symmetry (6.1) implies further that $(\psi_{\Gamma_i} \circ \theta)(u_{\Gamma_i}) = \psi_{\Gamma_i}(u_{\Gamma_i})$ so $\theta^n(u_{\Gamma_i})$ is also the maximal growth direction for each $n$. By the uniqueness of the maximal growth direction, we have $\theta^n u_{\Gamma_i} = u_{\Gamma_i}$, in particular,
\[
u_{\Gamma_i} = (1, \cdots, 1).
\]

Repeating same argument replacing $\|\cdot\|_{\max}$ with the Euclidean norm $\|\cdot\|$, we can deduce that $\psi_{\Gamma_i}(u)/\|u\|$ also achieves its maximum at $u_{\Gamma_i}$. Since $\|(1, \cdots, 1)\| = \sqrt{k}$, the last claim follows. \hfill \Box

Hence Corollary 5.5 implies:

Corollary 6.2. Let $\Gamma_i$ be as in the above lemma. For $k = 2, 3$, we have
\[
\dim \Lambda_{\Gamma_i} \geq \sqrt{k} \delta_{\Gamma_i}.
\]
Examples in $\mathbb{H}^2 \times \mathbb{H}^2$. Let us describe some examples to which Lemma 6.1 and Corollary 6.2 can be applied. We begin in dimension 2.

For a closed surface $S$ of genus $g \geq 2$, one can obtain homeomorphisms $\iota : S \to S$ of order 2 in a number of ways. Figure 2 indicates how this can be done: Arrange the surface in $\mathbb{R}^3$ so that it is symmetric by a $180^\circ$ rotation. There are several possibilities distinguished by the number of intersection points of the surface with the rotation axis, which yield fixed points of $\iota$.

![Figure 2](image.png)

**Figure 2.** Examples of involutions $\iota \in \text{Out}\pi_1(S_3)$. Indicated curves are mapped to each other by $\iota$.

In order for the example $(\rho, \rho \circ \iota)$ not to be trivial, we need the groups not to be conjugate in $\text{SO}^\circ(2,1)$. That is, $\rho$ should not represent a point of Teichmüller space $\mathcal{T}(S)$ which is fixed by $\iota$. This is always possible when $g \geq 3$; to see this, note that there are disjoint, non-homotopic simple closed curves exchanged by $\iota$ in each case. They can be assigned different lengths by a hyperbolic structure, which would then not be fixed by $\iota$.

In genus 2, one just needs to avoid the hyperelliptic involution – the one with 6 fixed points – which fixes every point in $\mathcal{T}(S)$. All other rotations will do.

Examples in $\mathbb{H}^3 \times \mathbb{H}^3$. Examples involving 3-manifolds are also plentiful. Consider for example a “book of $I$-bundles” constructed as follows (see Anderson-Canary [1]). Let $S_1, \ldots, S_\ell$ be $\ell$ copies of a surface of genus $g \geq 1$ with one boundary component and let $Y$ be the 2-complex obtained by identifying all the boundary circles to one. A choice of cyclic order $c$ on the $\ell$ surfaces determines a thickening of $Y$ to a 3-manifold $N_c$: form $S_i \times [-1, 1]$ for each $i$, and identify the annulus $\partial S_i \times [0, 1]$ with $\partial S_j \times [-1, 0]$ whenever $j$ follows $i$ in the order $c$ (the identification should take $[0, 1] \to [-1, 0]$ by an orientation-reversing homeomorphism, and should respect the original identification of the boundary circles). See Figure 3.

The result $N_c$ is homotopy-equivalent to $Y$, and has $\ell$ boundary components of genus $2g$. It admits many convex cocompact hyperbolic structures: it is easy to construct one “by hand” by attaching Fuchsian structures along the common boundary using the Klein-Maskit combination theorem [17].
The Ahlfors-Bers theory parametrizes all convex cocompact representations as the Teichmüller space of \( \partial N_c \) (cf. [16]).

A permutation of \((1, \cdots, \ell)\) induces a homeomorphism of \(Y\) which extends to a homotopy-equivalence of \(\bar{N}\) which, if the permutation does not respect the cyclic order, will not correspond to a homeomorphism. Selecting such a permutation of order 2, we have an automorphism that cannot be an isometry for any hyperbolic structure on \(N_c\). (Even if it does correspond to a homeomorphism one can choose the hyperbolic structure on \(N_c\) using a point in \(\mathcal{T}(\partial N_c)\) that is not symmetric with respect to the involution).

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