TEMPEREDNESS AND POSITIVE HARMONIC FUNCTIONS IN $L^2(\Gamma\backslash G)$ IN HIGHER RANK.

SAM EDWARDS AND HEE OH

Abstract. Let $G = \text{SO}^\circ(n_1, 1) \times \text{SO}^\circ(n_2, 1)$ and $X = \mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ for $n_1, n_2 \geq 2$. Let $\Gamma = (\pi_1 \times \pi_2)(\Sigma)$ where $\pi_i : \Sigma \to G_i$ is a non-elementary convex cocompact representation of a finitely generated group $\Sigma$. We show:

1. $L^2(\Gamma\backslash G)$ is tempered and $\lambda_0(\Gamma\backslash X) = \frac{1}{4}((n_1 - 1)^2 + (n_2 - 1)^2)$;
2. There exists no positive harmonic function in $L^2(\Gamma\backslash X)$.

In fact, analogues of (1)-(2) hold for any Anosov subgroup $\Gamma$ in the product of two or three simple algebraic groups of rank one as well as for Hitchin subgroups $\Gamma < \text{PSL}_d(\mathbb{R})$ for $d = 3, 4$. (2) holds for any discrete subgroup $\Gamma$ of a semisimple algebraic group with trivial opposition involution (e.g., $\text{SO}(p, q), \text{Sp}_{2n}(\mathbb{R})$), provided the limit cone of $\Gamma$ is contained in the interior of the positive Weyl chamber.

Contents

1. Introduction 1
2. Conformal densities and positive eigenfunctions 6
3. Local matrix coefficients and temperedness 10
4. Laplace eigenvalue and positive eigenfunctions in $L^2(\Gamma\backslash X)$ 15
5. Smearing arguments 19
6. Subgroups of the second kind and positive joint eigenfunctions 22
7. Non-existence of positive eigenfunctions in $L^2(\Gamma\backslash X)$ 26
References 29

1. Introduction

Motivation and background. Let $(\mathbb{H}^n, d)$, $n \geq 2$, denote the $n$-dimensional hyperbolic space of constant curvature $-1$, and let $G = \text{Isom}^+(\mathbb{H}^n) \simeq \text{SO}^\circ(n, 1)$. The critical exponent $\delta = \delta_\Gamma$ of a discrete subgroup $\Gamma < G$ is defined as the abscissa of convergence of the Poincare series $\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}$ for $o \in \mathbb{H}^n$. We have $0 \leq \delta \leq n - 1$, and $\delta > 0$ if and only if $\Gamma$ is non-elementary, i.e. it has no abelian subgroup of finite index. Unless mentioned

---

Edwards was supported by funding from the Heilbronn Institute for Mathematical Research and Oh was supported in part by NSF grants.
otherwise, all discrete subgroups of $\text{SO}^\circ(n, 1)$ in this paper are assumed to be non-elementary and torsion-free.

The bottom of the closed $L^2$-spectrum of the negative Laplace operator $-\Delta$ on the hyperbolic orbifold $\Gamma \backslash \mathbb{H}^n$ is given by the following number $\lambda_0 = \lambda_0(\Gamma \backslash \mathbb{H}^n) \geq 0$ [38, Theorem 2.2]:

$$\lambda_0 := \inf \left\{ \frac{\int_{\Gamma \backslash \mathbb{H}^n} \| \nabla f \|^2 \, d\text{vol}}{\int_{\Gamma \backslash \mathbb{H}^n} |f|^2 \, d\text{vol}} : f \in C_c^\infty(\Gamma \backslash \mathbb{H}^n) \right\}. \quad (1.1)$$

In a series of papers, Elstrodt ([11], [12], [13]) and Patterson ([29], [30], [31]) developed the relationship between the critical exponent $\delta_\Gamma$ and $\lambda_0(\Gamma \backslash \mathbb{H}^n)$, proving the following theorem for $n = 2$. The general case is due to Sullivan [38, Theorems 2.21].

**Theorem 1.1** (Generalized Elstrodt-Patterson I). For any discrete subgroup $\Gamma \subset \text{SO}^\circ(n, 1)$, the following are equivalent to each other:

1. $\delta \leq (n - 1)/2$;
2. $\lambda_0 = (n - 1)^2/4$.

The right translation action of $G$ on the quotient space $\Gamma \backslash G$ equipped with the $G$-invariant measure gives rise to a unitary representation $L^2(\Gamma \backslash G)$ on the square-integrable functions on $\Gamma \backslash G$, called the quasi-regular representation. If we set $K \simeq \text{SO}(n)$ to be a maximal compact subgroup of $G$ and identify $\mathbb{H}^n$ with $G/K$, the space of $K$-invariant functions of $L^2(\Gamma \backslash G)$ can be identified with $L^2(\Gamma \backslash \mathbb{H}^n)$. The bottom of the $L^2$-spectrum $\lambda_0$ then gives us the information on which complementary series representation of $G$ can occur in $L^2(\Gamma \backslash G)$. Indeed, it follows from the classification of the unitary dual of $\text{SO}^\circ(n, 1)$ that $\lambda_0 = (n - 1)^2/4$ is equivalent to saying that the quasi-regular representation $L^2(\Gamma \backslash G)$ does not contain any complementary series representation (cf. [38], [10]), which is again equivalent to the temperedness of $L^2(\Gamma \backslash G)$.

The notion of tempered representation (Definition 3.4) was introduced by Harish-Chandra [17]. A unitary representation $(\pi, \mathcal{H}_\pi)$ of a semisimple real algebraic group $G$ is tempered if and only if $\pi$ is weakly contained$^1$ in the regular representation $L^2(G)$; this is again equivalent to the condition that for any $\varepsilon > 0$, all of its matrix coefficients belong to $L^{2+\varepsilon}(G)$, i.e., for any $v, w \in \mathcal{H}_\pi$, the matrix coefficient function $g \mapsto \langle gv, w \rangle$ is an $L^{2+\varepsilon}$-integrable function of $G$ ([8], see Proposition 3.5).

Therefore Theorem 1.1 can be rephrased as follows:

**Theorem 1.2** (Generalized Elstrodt-Patterson II). For any discrete subgroup $\Gamma \subset G$ the following are equivalent to each other.

1. $\delta \leq (n - 1)/2$;
2. The quasi-regular representation $L^2(\Gamma \backslash G)$ is tempered.

$^1$\pi is weakly contained in a unitary representation $\sigma$ of $G$ if any diagonal matrix coefficients of $\pi$ can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of $\sigma$. 

A smooth function $f$ on $\Gamma \backslash \mathbb{H}^n$ is called harmonic, or more precisely $\lambda$-harmonic, if $-\Delta f = \lambda f$. Sullivan [38, Theorem 2.1] showed that unless $\Gamma \backslash \mathbb{H}^n$ is compact, for every $\lambda \leq \lambda_0$, there exists a positive $\lambda$-harmonic function (which are not necessarily unique), and for $\lambda > \lambda_0$, there exists no positive $\lambda$-harmonic function. As $\lambda_0$ is the bottom of the $L^2$-spectrum, the only possible square-integrable harmonic functions are $\lambda_0$-harmonic functions (Theorem 4.5).

A discrete subgroup $\Gamma < G$ is called convex cocompact if there exists a convex subspace of $\mathbb{H}^n$ on which $\Gamma$ acts co-compactly. For convex cocompact subgroups of $G$ (more generally for geometrically finite subgroups), Patterson and Sullivan showed the following using their theory of conformal measures on the boundary $\partial \mathbb{H}^n$ ([32], [39], [38, Theorem 2.21]):

**Theorem 1.3 (Sullivan).** For a convex cocompact subgroup $\Gamma < \text{SO}^\circ(n, 1)$, the following are equivalent to each other.

1. $\delta \leq (n - 1)/2$;
2. There exists no positive harmonic function in $L^2(\Gamma \backslash \mathbb{H}^n)$.

**Main results.** The main aim of this article is to discuss analogues of Theorems 1.1, 1.2 and 1.3 for a certain class of discrete subgroups of a connected semisimple real algebraic group of higher rank, i.e., rank at least 2.

We begin by describing a special case of our main theorem when $G = \text{SO}^\circ(n_1, 1) \times \text{SO}^\circ(n_2, 1)$ with $n_1, n_2 \geq 2$. Let $X = \mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ and $\Delta$ the Laplace-Beltrami operator on $X$. A smooth function $f$ on $\Gamma \backslash X$ is called a harmonic function if it is an eigenfunction for $\Delta$. The number $\lambda_0 = \lambda_0(\Gamma \backslash X)$, the bottom of the $L^2$-spectrum of $\Gamma \backslash X$, is defined in the same way as (1.1) replacing $\Gamma \backslash \mathbb{H}^n$ by $\Gamma \backslash X$.

**Theorem 1.4.** Let

$$\Gamma = (\pi_1 \times \pi_2)(\Sigma) = \{ (\pi_1(\sigma), \pi_2(\sigma)) \in G : \sigma \in \Sigma \} \quad (1.2)$$

where $\pi_i : \Sigma \to \text{SO}^\circ(n_i, 1)$ is a non-elementary convex cocompact representation of a finitely generated group $\Sigma$ for $i = 1, 2$. Then

1. $L^2(\Gamma \backslash G)$ is tempered and $\lambda_0 = \frac{1}{4}((n_1 - 1)^2 + (n_2 - 1)^2)$;
2. There exists no positive harmonic function in $L^2(\Gamma \backslash X)$, or equivalently, $\lambda_0$-harmonic functions are not square-integrable.

Even when $\Sigma = \pi_1(S)$ is a surface group and $\pi_1, \pi_2 : \Sigma \to \text{PSL}_2(\mathbb{R}) \cong \text{SO}^\circ(2, 1)$ are two elements of the Teichmüller space $\mathcal{T}(S)$, this theorem is new.

**Remark 1.5.** Theorem 1.4 does not hold for a general infinite-covolume subgroup $\Gamma$. For example, if $\Gamma < \text{SO}^\circ(n_1, 1) \times \text{SO}^\circ(n_2, 1)$ is the product of two convex cocompact subgroups, each of which having critical exponent greater than $\frac{n_i - 1}{2}$, then $L^2(\Gamma \backslash G)$ is not tempered and $L^2(\Gamma \backslash X)$ possesses a positive harmonic function.
We now discuss a general setting. Let $G$ be a connected semisimple real algebraic group, and let $P < G$ be a minimal parabolic subgroup. Let $P = MAN$ be a Langlands decomposition where $A$ is a maximal real split torus of $G$, $M$ the centralizer of $A$ and $N$ the unipotent radical of $P$. Let $\mathfrak{g} = \text{Lie } G$, $\mathfrak{a} = \text{Lie } A$, and $\mathfrak{a}^+$ denote the Positive Weyl chamber so that $\log N$ is the sum of the all positive root subspaces. Let $K$ be a maximal compact subgroup of $G$ such that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. For $g \in G$, let $\mu(g) \in \mathfrak{a}^+$ denote the Cartan projection, that is, the unique element of $\mathfrak{a}^+$ such that $g \in K \exp \mu(g)K$. Let $(X = G/K, d)$ denote the associated Riemannian symmetric space and $\Delta$ the Laplace operator. Then for $o = [K] \in X$, $d(go, o) = \|\mu(g)\|$ where $\|\cdot\|$ denotes the norm on $\mathfrak{a}$ induced from the Killing form on $\mathfrak{g}$.

Following Quint [35], let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$: for any non-zero $v \in \mathfrak{a}$, $\psi_\Gamma(v) := \inf_{\mathcal{C} \supset v} \tau_\mathcal{C}$, where the infimum is over all open cones $\mathcal{C}$ containing $v$ and $\tau_\mathcal{C}$ denotes the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|}$. For $v = 0$, we let $\psi_\Gamma(0) = 0$. The function $\psi_\Gamma$ can be regarded as a higher rank generalization of the critical exponent of $\Gamma$. Let $\rho$ denote the half sum of all positive roots for $(\mathfrak{g}, \mathfrak{a}^+)$. For example, if $G = SO^0(n_1, 1) \times SO^0(n_2, 1)$, then $\rho(v_1, v_2) = \frac{1}{2}((n_1 - 1)v_1 + (n_2 - 1)v_2)$. Analogous to the fact that $\delta \leq n - 1$ for $\Gamma < SO^0(n, 1)$, the upper bound $\psi_\Gamma \leq 2\rho$ holds for any discrete subgroup $\Gamma < G$ [35].

In the rest of the introduction, we assume that $\Gamma < G$ is a Zariski dense discrete subgroup. Let $\mathcal{F} = G/P$ denote the Furstenberg boundary, and $\mathcal{F}^{(2)}$ the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$; two points $\xi, \eta$ in $\mathcal{F}$ are said to be in general position if $(\xi, \eta) \in \mathcal{F}^{(2)}$.

**Anosov subgroups.** We call $\Gamma < G$ Anosov (with respect $P$) if there exists a finitely generated word hyperbolic group $\Sigma$ such that $\Gamma = \Phi(\Sigma)$ where $\Phi : \Sigma \to G$ is a representation, which induces a continuous equivariant map from the Gromov boundary $\partial_\infty \Sigma$ to $\mathcal{F}$ sending two distinct points to points in general position. The notion of Anosov representations was first introduced by Labourie for surface groups [25], and then extended by Guichard and Wienhard [16] to general word hyperbolic groups. When $G$ has rank one, the class of Anosov subgroups coincides with that of convex cocompact subgroups and when $G$ is a product of two rank one simple algebraic groups, any Anosov subgroup arises as in (1.2).

We may identify $\mathfrak{a}^*$ with $\mathfrak{a}$ using the Killing form on $\mathfrak{g}$; this induces an inner product on $\mathfrak{a}^*$ with respect to which $\|\rho\|^2 = \langle \rho, \rho \rangle$ is defined below.

**Theorem 1.6.** Let $\Gamma < G$ be a Zariski dense Anosov subgroup of $G$. Then the following (1) and (2) are equivalent, and implies (3):

1. $\psi_\Gamma \leq \rho$;
(2) $L^2(\Gamma\backslash G)$ is tempered and $\lambda_0 = \|\rho\|^2$

(3) There exists no positive harmonic function in $L^2(\Gamma\backslash X)$.

The equivalence of (1) and (2) is based on the asymptotic behavior of the matrix coefficients for compactly supported continuous functions for Anosov subgroups obtained in [9]. The implication $(1) \Rightarrow (3)$ is based on the study of $\Gamma$-conformal measures and joint eigenfunctions for the whole ring of $G$-invariant differential operators and extension of Sullivan-Thurston’s smearing arguments to higher rank groups.

Let $\Sigma$ be a surface group, realized as a uniform lattice of $\text{PSL}_2(\mathbb{R})$. Let $\iota_d$ denote the irreducible $d$-dimensional representation $\text{PSL}_2(\mathbb{R}) \to \text{PSL}_d(\mathbb{R})$, which is unique up to conjugation. A representation $\pi : \Sigma \to \text{PSL}_d(\mathbb{R})$ is called Hitchin if $\pi$ belongs to the same connected component as $\iota_d|\Sigma$ in the character variety $\text{Hom}(\Sigma, \text{PSL}_d(\mathbb{R}))/\sim$ where the equivalence is given by conjugations. The image of a Hitchin representation is called a Hitchin subgroup.

Although the condition $\psi_\Gamma \leq \rho$ may appear quite strong, it was verified in a recent work of Kim-Minsky-Oh [21] for Anosov subgroups in the following setting:

**Theorem 1.7.** Let $\Gamma$ be a Zariski dense Anosov subgroup of the product $G$ of two or three simple real algebraic groups of rank one, or a Hitchin subgroup of $\text{PSL}_d(\mathbb{R})$ for $d = 3, 4$. Then the quasiregular representation $L^2(\Gamma\backslash G)$ is tempered, $\lambda_0 = \|\rho\|^2$ and $L^2(\Gamma\backslash X)$ has no positive harmonic function.

We do not know any example of an Anosov subgroup which does not satisfy the condition $\psi_\Gamma \leq \rho$.

**Groups with trivial opposition involution.** The opposition involution $i : a \to a$ is defined by

$$i(u) = -\text{Ad}_{w_0}(u),$$

where $w_0$ is a Weyl element such that $\text{Ad}_{w_0}a^+ = -a^+$. The opposition involution is trivial in the product of any rank one simple algebraic groups, as well as in the groups $G = \text{SO}(p, q), \text{Sp}(2n, \mathbb{R})$.

The limit cone $L_\Gamma \subseteq a^+$ is defined as the asymptotic cone of the Cartan projection of $\Gamma$, i.e., $L_\Gamma = \{\lim t_i\mu(\gamma_i) \in a^+ : t_i \to 0, \gamma_i \in \Gamma\}$. It is a well-known property of Anosov subgroups that their limit cones are contained in the interior of $a^+$ [33].

**Theorem 1.8.** Let $G$ be a connected semisimple real algebraic group with trivial opposition involution. For any Zariski dense discrete subgroup $\Gamma < G$ with $L_\Gamma \subseteq \text{int } a^+$, there exists no positive harmonic function in $L^2(\Gamma\backslash X)$.

The proof of Theorem 1.8 is based on a higher rank version of the smearing argument of Sullivan-Thurston (Theorem 5.5).

**Organization:** In section 2, we show that any positive joint eigenfunction on $\Gamma\backslash X$ (i.e., eigenfunction for the whole ring of $G$-invariant differential operators) arises from a $(\Gamma, \psi)$-conformal density (Proposition 2.8). In section
we prove the equivalence of (1) and (2) in Theorem 1.6. In section 4, we study the $\Delta$-eigenvalues of joint eigenfunctions and the relationship between positive harmonic and positive joint eigenfunctions in $L^2(\Gamma\backslash X)$ (Corollary 4.6). In section 5, we use the smearing argument to prove the non-existence of positive square-integrable harmonic function as in Theorem 5.1, which implies Theorem 1.8. In section 6, we prove Theorem 6.13 which restricts the possible characters of positive joint eigenfunctions of $L^2(\Gamma\backslash X)$ for general position subgroups. In the last section 7, we prove the implication $(1) \Rightarrow (3)$ of Theorem 1.6.

Acknowledgement: We would like to thank Marc Burger for bringing the reference [38] to our attention, and Minju Lee for useful discussions. We would also like to thank Dennis Sullivan for insightful comments on the preliminary version, and Dick Canary and Francois Labourie for useful correspondences on Anosov subgroups of the second kind.

2. Conformal densities and positive eigenfunctions

Let $G$ be a connected semisimple real algebraic group and $X$ be the associated Riemannian symmetric space. Let $\Gamma < G$ be a Zariski dense discrete subgroup. The goal of this section is to obtain Proposition 2.8, which explains the relationship between joint eigenfunctions on $\Gamma\backslash X$ and $\Gamma$-conformal measures on the Furstenberg boundary of $G$.

Let $P$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition $P = MAN$ where $A$ is a maximal real split torus of $G$, $M$ is a compact subgroup commuting with $A$ and $N$ is the unipotent radical of $P$. We fix a positive Weyl chamber $a^+ \subset a = \text{Lie} A$ so that $\log N$ consists of positive root subspaces. We also fix a maximal compact subgroup $K$ of $G$ so that the Cartan decomposition $G = K(\exp a^+)K$ holds, that is, for any $g \in G$, there exists a unique element $\mu(g) \in a^+$ such that $g \in K \exp \mu(g)K$. We call the map $\mu : G \to a^+$ the Cartan projection map. We denote by $\rho$ the half sum of the positive roots for $(g, a^+)$. The Riemannian symmetric space $X$ can be identified with the quotient space $G/K$ and set $o = [K] \in X$. We do not distinguish a function on $X$ and a right $K$-invariant function on $G$. Let $\mathcal{F} := G/P$ denote the Furstenberg boundary of $G$.

Joint eigenfunctions on $X$. Let $\mathcal{D} = \mathcal{D}(X)$ denote the ring of all $G$-invariant differential operators on $X$. We call a real valued function on $X$ a joint eigenfunction if it is an eigenfunction for all operators in $\mathcal{D}$. For each joint eigenfunction $f$, there exists an associated character $\chi_f : \mathcal{D} \to \mathbb{R}$ such that

$$Df = \chi_f(D)f$$

for all elements $D \in \mathcal{D}$. The ring $\mathcal{D}$ is generated by $\text{rank}(G)$ elements, and the set of all characters of $\mathcal{D}$ is in bijection with the space $a^* = \text{Hom}_\mathbb{R}(a, \mathbb{R})$ modulo the action of the Weyl group, as we now explain. Denote by $Z(\mathfrak{g}_\mathbb{C})$
the center of the universal enveloping algebra $U(g_C)$ of $g_C$. Recall the well-known fact that the joint eigenfunctions on $X$ can be identified with the right $K$-invariant real-valued $Z(g_C)$-eigenfunctions on $G$ (cf. [18]).

Note that $M$ is equal to the centralizer $Z_K(A)$. Letting $T$ be a maximal torus in $M$ with Lie algebra $\mathfrak{t}$, set $\mathfrak{h} = (\mathfrak{a} \oplus \mathfrak{t})$. Then $\mathfrak{h}_C := (\mathfrak{a} \oplus \mathfrak{t})_C$ is a Cartan subalgebra of $g_C$.

We let

$$\iota: Z(g_C) \rightarrow S^W(\mathfrak{h}_C)$$

denote the Harish-Chandra isomorphism from $Z(g_C)$ to the Weyl group-invariant elements of the symmetric algebra $S(\mathfrak{h}_C)$ of $\mathfrak{h}$ (cf. [22, Theorem 8.18]).

For any $\psi \in \mathfrak{a}^*$, we can extend it to $\mathfrak{h}$ by letting $\psi(J) = 0$ for all $J \in \mathfrak{m}$, and then to $S(\mathfrak{h}_C)$ polynomially. This lets us define a character $\chi_\psi$ on $Z(g_C)$ by

$$\chi_\psi(Z) := \psi(\iota(Z)) \quad (2.1)$$

for all $Z \in Z(g_C)$. Conversely, if $f$ is a right $K$-invariant $Z(g_C)$-eigenfunction, then, since $\mathfrak{t}$ acts trivially on $f$, the associated character $\chi_f$ must arise as $\psi \circ \iota$ for some $\psi \in \mathfrak{a}^*$.

**Example 2.1.**

- Consider the hyperbolic space $\mathbb{H}^n = \{(x_1, \cdots, x_n, y) \in \mathbb{R}^{n+1} : y > 0\}$ with the metric $\sqrt{\sum_{i=1}^n dx_i^2 + dy^2}$. The Laplacian $\Delta$ on $\mathbb{H}^n$ is $\Delta = -y^2(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2})$ and the ring of $SO(n, 1)$-invariant differential operators is generated by $\Delta$, i.e., a polynomial in $\Delta$. If $\psi \in \mathfrak{a}^*$ is given by $\psi(v) = \delta v$ for some $\delta \in \mathbb{R}$ under the isomorphism $\mathfrak{a} = \mathbb{R}$, then $\chi_\psi(-\Delta) = \delta(n - 1 - \delta)$.

- Let $G = SO(n_1, 1) \times SO(n_2, 1)$ and $X$ be the Riemannian product $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ for $n_1, n_2 \geq 2$. Then $\Delta(X)$ is generated by the hyperbolic Laplacians $\Delta_1, \Delta_2$ on each factor $\mathbb{H}^{n_1}$ and $\mathbb{H}^{n_2}$. If we identify $\mathfrak{a}$ with $\mathbb{R}^2$ and if a linear form $\psi \in \mathfrak{a}^*$ is given by $\psi(v) = \langle v, (\delta_1, \delta_2) \rangle$ for some vector $(\delta_1, \delta_2) \in \mathbb{R}^2$, then $\chi_\psi(-\Delta_i) = \delta_i(n_i - 1 - \delta_i)$ for $i = 1, 2$.

**Joint eigenfunctions on $\Gamma \backslash X$.** We now consider joint eigenfunctions on $\Gamma \backslash X$ or, equivalently, right $\Gamma$-invariant joint eigenfunctions on $X$.

Let $G = KAN$ be the Iwasawa decomposition, $\kappa : G \rightarrow K$ the $K$-factor projection of this decomposition, and $H : G \rightarrow \mathfrak{a}$ be the Iwasawa cocycle defined by the relation:

$$g \in \kappa(g) \exp \left(H(g)\right) N.$$ 

Note that $K$ acts transitively on $F$ and $K \cap P = M$, and hence we may identify $F$ with $K/M$. This decomposition can be used to describe both the action of $G$ on $F = K/M$ and the $\mathfrak{a}$-valued Busemann map as follows: for all $g \in G$ and $[k] \in F$ with $k \in K$,

$$g \cdot [k] = [\kappa(gk)],$$
and the \( a \)-valued Busemann map is defined by

\[
\beta_k(g(o), h(o)) := H(g^{-1}k) - H(h^{-1}k) \in a
\]

for all \( g, h \in G \).

**Definition 2.2.** Let \( \psi \in a^* \). 

1. A Borel finite measure \( \nu \) on \( F \) is said to be a \((\Gamma, \psi)\)-conformal measure (for the basepoint \( o \)) if for all \( \gamma \in \Gamma \) and \( \xi = [k] \in F \),

\[
\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{-\psi(\beta_\xi(\gamma o, o))} = e^{-\psi(H(\gamma^{-1}k))},
\]

or equivalently

\[
d\nu([k]) = e^{\psi(H(\gamma k))} d\nu(\gamma [k]).
\]

2. A collection \( \{\nu_x : x \in X\} \) of finite Borel measures on \( F \) is called a \((\Gamma, \psi)\)-conformal density if for all \( x, y \in X \), \( \xi \in F \) and \( \gamma \in \Gamma \),

\[
\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\psi(\beta_\xi(x, y))} \quad \text{and} \quad d\gamma_*\nu_x = d\nu_y(x).
\] (2.2)

A \((\Gamma, \psi)\)-conformal measure \( \nu \) defines a \((\Gamma, \psi)\)-conformal density \( \{\nu_x : x \in X\} \) by the formula:

\[
d\nu_x(\xi) = e^{-\psi(\beta_\xi(x, o))}d\nu(\xi),
\]

and conversely any \((\Gamma, \psi)\)-conformal density \( \{\nu_x\} \) is uniquely determined by its member \( \nu_o \) by (2.2).

**Definition 2.3.** Associated to a \((\Gamma, \psi)\)-conformal measure \( \nu \) on \( F \), we define the following function \( E_\nu \) on \( G \): for \( g \in G \),

\[
E_\nu(g) := |\nu_{g(o)}| = \int_F e^{-\psi(H(g^{-1}k))} d\nu([k]). \tag{2.3}
\]

Since \( |\nu_{\gamma(x)}| = |\nu_x| \) for all \( \gamma \in \Gamma \) and \( x \in X \), the left \( \Gamma \)-invariance and right \( K \)-invariance of \( E_\nu \) are clear. Hence we may consider \( E_\nu \) as a \( K \)-invariant function on \( \Gamma \setminus G \), or, equivalently, as a function on \( \Gamma \setminus X \).

**Proposition 2.4.** For each \((\Gamma, \psi)\)-conformal measure \( \nu \) on \( F \), \( E_\nu \) is a positive joint eigenfunction with character \( \chi_{\psi - \rho} \). Conversely, any positive joint eigenfunction on \( \Gamma \setminus X \) arises in this way.

For each \( \psi \in a^* \) and \( h \in G \), consider the following right \( K \)-invariant function on \( G \):

\[
\varphi_{\psi, h}(g) = e^{-\psi(H(g^{-1}h))}. \tag{2.4}
\]

We may also consider \( \varphi_{\psi, h} \) as a function on \( X \). For a \((\Gamma, \psi)\)-conformal measure \( \nu \), we have \( E_\nu(g) = \int_F \varphi_{\psi, k}(g) d\nu([k]) \) for \( x = g(o) \), and hence the first part of Proposition 2.4 is a consequence of the following:

**Lemma 2.5.** ([22, Propositions 8.22 and 9.9]) For any \( \psi \in a^* \) and \( h \in G \), the function \( \varphi_{\psi, h} \) is a joint eigenfunction on \( X \) with character \( \chi_{\psi - \rho} \).
Letting $h = kan \in KAN$, we see that for any $g \in G$,

$$
\varphi_{\psi,h}(g) = e^{-\psi(H(g^{-1}h))} = e^{-\psi(H(g^{-1}kan))} = e^{-\psi(H(g^{-1}k))} \cdot e^{-\psi(\log(a))},
$$

i.e., the function $\varphi_{\psi,h}$ is a scalar multiple of $\varphi_{\psi,\kappa(h)}$. In fact, the functions $\varphi_{\psi,k}$, $k \in K$ form a complete set of minimal positive joint eigenfunctions with character $\chi_{\psi-\rho}$ with $\psi \geq \rho$, in the sense that if $f$ is a positive joint eigenfunction on $X$ with character $\chi_{\psi-\rho}$ such that $f \leq \varphi_{\psi,k}$ for some $k \in K$, then

$$
f = c \cdot \varphi_{\psi,k}
$$

for some $c > 0$ (cf. [14, 20], see also [25, Theorem 1]).

As a consequence, we have the following (cf. [25, Theorem 3]):

**Theorem 2.6.** For any positive joint eigenfunction $f$ on $X$, there exist $\psi \in \mathfrak{a}^*$ with $\psi \geq \rho$ and a Borel measure $\nu$ on $F = K/M$ such that for all $g \in G$,

$$
f(g) = \int_F \varphi_{\psi,k}(g) \, d\nu([k]).
$$

Moreover, the pair $(\psi, \nu)$ is uniquely determined by $f$.

**Proof of the second part of Proposition 2.4:** Let $f$ be a $\Gamma$-invariant joint eigenfunction on $X$. By Theorem 2.6, there exist unique $\psi \in \mathfrak{a}^*$ and a Borel measure $\nu$ on $F = K/M$ so that for all $g \in G$,

$$
f(g) = \int_F \varphi_{\psi,k}(g) \, d\nu([k]), \quad g \in G.
$$

Since $f$ is $\Gamma$-invariant, for any $\gamma \in \Gamma$,

$$
\int_F \varphi_{\psi,k}(g) \, d\nu([k]) = f(g) = f(\gamma g) = \int_F \varphi_{\psi,k}(\gamma g) \, d\nu([k])
$$

$$
= \int_F \varphi_{\psi,k(\gamma^{-1}k)}(g) \, d\nu([k]) = \int_F \varphi_{\psi,k}(g) \, e^{\psi(H(\gamma^{-1}k))} \, d\nu(\gamma \cdot [k]).
$$

By the uniqueness of $\nu$ in the integral representation of $f$,

$$
d\nu([k]) = e^{\psi(H(\gamma))} \, d\nu(\gamma \cdot [k]),
$$

i.e. $\nu$ is a $(\Gamma, \psi)$-conformal measure on $F$. This finishes the proof of Proposition 2.4.

We denote by $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ the growth indicator function of $\Gamma$ as defined in (1.3).

**Theorem 2.7.** [34, Theorem 8.1]. Let $\Gamma < G$ be Zariski dense. If there exists a $(\Gamma, \psi)$-conformal measure on $F$ for some $\psi \in \mathfrak{a}^*$, then

$$
\psi \geq \psi_\Gamma.
$$

Therefore Proposition 2.4 and Theorem 2.7 yield the following:
Proposition 2.8. Let $\Gamma < G$ be a Zariski dense discrete subgroup. If $\nu$ is a $(\Gamma, \psi)$-conformal measure on $F$ for some $\psi \in a^*$, then $E_\nu$ is a positive joint eigenfunction on $\Gamma \setminus X$ with character $\chi_{\psi - \rho}$. Conversely, any positive joint eigenfunction on $\Gamma \setminus X$ is of the form $E_\nu$ for some $(\Gamma, \psi)$-conformal measure $\nu$ on $F$ with $\psi \geq \max(\rho, \psi_{\Gamma})$.

3. LOCAL MATRIX COEFFICIENTS AND TEMPEREDNESS

Let $G$ be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. The goal of this section is to prove the equivalence of (1) and (2) of Theorem 1.6.

**Limit cone and limit set.** Let $L = L_\Gamma \subset a^+$ denote the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$, i.e., $L = \{\lim t_i \mu(\gamma_i) : t_i \to 0, \gamma_i \in \Gamma\}$. Quint showed that $\psi_{\Gamma} \geq 0$ on $L$, $\psi_{\Gamma} > 0$ on $\text{int} L$, and $\psi_{\Gamma} = -\infty$ outside $L$.

**Definition 3.1.** A sequence $p_i \in X$ is said to converge to $\xi \in F$ if there exists $g_i \to \infty$ regularly in $G$ with $p_i = g_i(o)$ and $\lim_{i \to \infty} [\kappa_1(g_i)] = \xi$.

We denote by $\Lambda \subset F$ the limit set of $\Gamma$, which is defined as

$$\Lambda = \{\lim \gamma_i(o) : \gamma_i \in \Gamma\}; \quad \text{(3.1)}$$

this is the unique $\Gamma$-minimal subset of $F$ ([2], [27]).

**Tangent linear forms.** We set

$$D_\Gamma = \{\psi \in a^* : \psi \geq \psi_{\Gamma}\}.$$

A linear form $\psi \in a^*$ is said to be tangent to $\psi_{\Gamma}$ at $u \in a$ if $\psi \in D_\Gamma$ and $\psi(u) = \psi_{\Gamma}(u)$. We denote by $D_\Gamma^+$ the set of all linear forms tangent to $\psi_{\Gamma}$ at $\mathcal{L} \cap \text{int} a^+$, i.e.,

$$D_\Gamma^+ := \{\psi \in D_\Gamma : \psi(u) = \psi_{\Gamma}(u) \text{ for some } u \in \mathcal{L} \cap \text{int} a^+\}.$$

**Example 3.2.** For $\Gamma < \text{SO}^\circ(n, 1)$, $D_\Gamma^+ = \{\delta\}$ and $D_\Gamma = \{s \geq \delta\}$.

For each $\psi \in D_\Gamma^+$, Quint [34] constructed a $(\Gamma, \psi)$-conformal measure supported on the limit set $\Lambda$, extending the construction of Patterson and Sullivan in the rank one case.

**Burger-Roblin measures.** Denote by $w_0 \in K$ a representative of the unique element of the Weyl group $N_K(A)/M$ such that $\text{Ad}_{w_0} a^+ = -a^+$. The opposition involution $i : a \to a$ is defined by

$$i(u) = -\text{Ad}_{w_0}(u).$$
We set $N^+ = w_0Nw_0^{-1}$ and $N^- = N$. For each $g \in G$, we define the following visual maps:

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.$$ \hfill (3.2)

For a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, we denote by $m_{\nu}^{BR}$ and $m_{\nu}^{BR^*}$ the associated $N^+$ and $N^-$-invariant Burger-Roblin measures on $\Gamma \backslash G$ respectively, as defined in [9]. By [9, Lem. 4.9], it can also be defined as follows: for any $f \in C_c(\Gamma \backslash G)$,

$$m_{\nu}^{BR}(f) = \int_{[k]} f(km(\exp a)n)e^{-\psi_0(a)} \, d\nu(k^-) \, dm \, da \, dn$$

and

$$m_{\nu}^{BR^*}(f) = \int_{[k]} f(km(\exp a)n)e^{\psi(a)} \, d\nu(k^+) \, dm \, da \, dn$$

where $dm, da, dn$ are Haar measures on $M, a, N$ respectively.

We denote by $dx$ the $G$-invariant measure on $\Gamma \backslash G$ which is defined using the $(G, 2\rho)$-conformal measure, that is, the $K$-invariant probability measure on $\mathcal{F}$ (see [9, (3.11)]). For functions $f_1, f_2$ on $\Gamma \backslash G$, we write

$$\langle f_1, f_2 \rangle = \int_{\Gamma \backslash G} f_1(x)f_2(x) \, dx$$

whenever the integral converges. We write $C_c(\Gamma \backslash G)_K$ for the space of $K$-invariant compactly supported continuous functions on $\Gamma \backslash G$.

**Lemma 3.3.** For a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$ and any $f \in C_c(\Gamma \backslash G)_K$, we have

$$m_{\nu}^{BR}(f) = \langle f, E_{\nu} \rangle = m_{\nu}^{BR^*}(f).$$

**Proof.** If $g = (\exp b)nk \in AN^+K$, then $\beta_{e^-}(go,o) = \beta_{e^+}(\exp(-i(b)), o) = i(b)$. Hence

$$m_{\nu}^{BR}(f) = \int_{AN^+K} \int_K f(k \exp b nk_0)e^{-\psi_0(b)} \, d\nu(k^-) \, dbdn$$

$$= \int_K \int_G f(gk)e^{-\psi(\beta_{e^-}(go,o))} \, d\nu(k^-) \, dg$$

$$= \int_K \int_G f(g) e^{-\psi(\beta_{e^-}(go,o))} \, d\nu(k^-) \, dg = \langle f, E_{\nu} \rangle$$

If $g = (\exp b)nk \in ANK$, then $\beta_{e^+}(go,o) = -b$ and using this, the second identity can be proved similarly. \hfill \Box

**Tempered representations.** The Harish-Chandra function $\Xi_G : G \to (0, \infty)$ is a bi-$K$-invariant function defined via the formula

$$\Xi_G(g) = \int_K e^{-\rho(H(gk))} \, dk \quad \text{for all} \quad g \in G.$$
The following estimate is well-known, cf. e.g. [22]: for any \( \varepsilon > 0 \), there exist \( C, \varepsilon > 0 \) such that for any \( g \in G \),
\[
Ce^{-\rho(\mu(g))} \leq \Xi_G(g) \leq Ce^{-(1-\varepsilon)\rho(\mu(g))}.
\]

**Definition 3.4.** A unitary representation \( (\pi, \mathcal{H}_\pi) \) of \( G \) is called tempered if for any \( K \)-finite unit vectors \( v, w \in \mathcal{H}_\pi \) and any \( g \in G \)
\[
|\langle (\pi(g)v, w) \rangle| \leq (\dim(Kv) \dim(Kw))^{1/2}\Xi_G(g).
\]

**Proposition 3.5.** ([8], also see [28, Theorem 2.4]) The following are equivalent for a unitary representation \( (\pi, \mathcal{H}_\pi) \):

1. \( \pi \) is tempered;
2. \( \pi \) is weakly contained in the regular representation \( L^2(G) \);
3. for any vectors \( v, w \in \mathcal{H}_\pi \), the matrix coefficient \( g \mapsto \langle \pi(g)v, w \rangle \) lies in \( L^{2+\varepsilon}(G) \) for any \( \varepsilon > 0 \);
4. for any \( \varepsilon > 0 \), \( \pi \) is strongly \( L^{2+\varepsilon} \), i.e., there exists a dense subset of \( \mathcal{H}_\pi \) whose matrix coefficients all belong to \( L^{2+\varepsilon}(G) \).

**Local matrix coefficients for Anosov subgroups.**

**Lemma 3.6.** For any \( \psi \in D_\Gamma \), there exists \( 0 < c \leq 1 \) such that \( c \cdot \psi \in D_\Gamma \).

If \( \Gamma \) is Anosov, then there exists a unique unit vector \( v \in \mathfrak{a}^+ \) such that \( c\psi(u) = \psi_T(u) \) and \( u \in \text{int} \mathcal{L}_\Gamma \).

**Proof.** Since \( \psi_T \) is concave, there exists \( 0 < c \leq 1 \) such that \( c\psi(u) = \psi_T(u) \) for some \( u \in \mathcal{L}_\Gamma \). Suppose now that \( \Gamma \) is Anosov. Then there is no linear form tangent to \( \psi_T \) at \( \partial \mathcal{L}_\Gamma \) [33], and hence \( u \in \text{int} \mathcal{L}_\Gamma \). Since \( \psi_T \) is even strictly concave [33, Propositions 4.6, 4.11], \( u \) is determined uniquely up to a scalar multiple. \( \square \)

In the rest of this section, we assume that
\[
\Gamma < G \text{ is an Anosov subgroup with respect to } P
\]
as defined in the introduction.

For each \( v \in \text{int} \mathcal{L}_\Gamma \), there exists a unique \( \psi_v \in D_\Gamma \) such that \( \psi_v(v) = \psi_T(v) \) and a unique \( (\Gamma, \psi_v) \)-conformal probability measure, say, \( \nu_v \) supported on \( \Lambda \) [9, Corollary 7.8 and Theorem 7.9].

Hence [9, Theorem 7.12], together with Lemma 3.3, implies (let \( r = \text{rank}(G) \)):

**Theorem 3.7.** For any \( v \in \text{int} \mathcal{L}_\Gamma \), there exists \( \kappa_v > 0 \) such that for all \( f_1, f_2 \in C_c(\Gamma \backslash G)_K \) and any \( w \in \ker \psi_v \),
\[
\lim_{t \to +\infty} t^{(r-1)/2} e^{t(2\rho - \psi_v)(tv + \sqrt{tw})} \langle \exp(tv + \sqrt{tw})f_1, f_2 \rangle
= \kappa_v e^{-I(w)} \cdot \langle f_1, E_{\nu_v} \rangle \cdot \langle f_2, E_{\nu_v} \rangle
\]
where \( I(w) \in \mathbb{R} \) is given as in [9, 7.5]. Moreover, the left-hand side is uniformly bounded over all \( (t, w) \in (0, \infty) \times \ker \psi_v \) such that \( tv + \sqrt{tw} \in \mathfrak{a}^+ \)
Proof. Suppose that $\psi_T \leq \rho$. In order to show that $L^2(\Gamma \setminus G)$ is tempered, by Proposition 3.5, it suffices to show that the matrix coefficients $g \mapsto \langle g \cdot f_1, f_2 \rangle$ are in $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$ and for all $f_1, f_2 \in C_c(\Gamma \setminus G)$, since $C_c(\Gamma \setminus G)$ is dense in $L^2(\Gamma \setminus G)$. Without loss of generality, we may just consider nonnegative functions $f_1, f_2 \in C_c(\Gamma \setminus G)$. Fix any $\varepsilon > 0$. Then using the Cartan decomposition $G = KA^+K$, we have

$$
\int_G (g \cdot f_1, f_2)^{2+\varepsilon} \, dg = \int_K \int_{\mathfrak{a}^+} \int_K \langle k \exp(v)k_2 \cdot f_1, f_2 \rangle^{2+\varepsilon} \Xi(v) \, dk_1 \, dv \, dk_2,
$$

where $\Xi(v) \approx e^{2\rho(v)}$ (here and henceforth, $f(v) \asymp g(v)$ means that the ratio $f(v)/g(v)$ is bounded uniformly between two positive constants). Denoting $F_i = \int_K k \cdot f_1, f_2 \in C_c(\Gamma \setminus G)_K$, we then have

$$
\int_G (g \cdot f_1, f_2)^{2+\varepsilon} \, dg \asymp \int_{\mathfrak{a}^+} (\exp(v) \cdot F_1, F_2)^{2+\varepsilon} e^{2\rho(v)} \, dv.
$$

Since $\psi_T \leq \rho$, we have $\rho \in D_T^\circ$. By Lemma 3.6, there exists $0 < c \leq 1$ such that $cp \in D_T^\circ$ and a unit vector $u_0 \in \text{int } \mathcal{L}_T$ such that $\psi_T(u_0) = cp(u_0)$.

We now parameterize $\mathfrak{a}^+$ as follows: for each $v \in \ker \rho$, define

$$
t_v := \min \{ t \in \mathbb{R}_{>0} : tu_0 + \sqrt{t}v \in \mathfrak{a}^+ \}.
$$

Substituting $u = tu_0 + \sqrt{t}v$ for $t \geq 0$ and $v \in \mathfrak{b} \cap \ker \rho$ gives $du = s \cdot t^{r-1} \, dt \, dv$ for some constant $s > 0$. Then (letting $r = \dim(\mathfrak{a})$)

$$
\int_{\mathfrak{a}^+} (\exp(u) \cdot F_1, F_2)^{2+\varepsilon} e^{2\rho(u)} \, du \asymp \int_{\ker \rho} \int_{t_v}^{\infty} (\exp(tu_0 + \sqrt{t}v) \cdot F_1, F_2)^{2+\varepsilon} e^{2\rho(u)} t^{(r-1)/2} \, dt \, dv.
$$

By Theorem 3.7 ([9, Theorem 7.19 (1)]), there exists $C = C(F_1, F_2) > 0$ such that

$$
t^{r-1/2} e^{(2-\eta)(2\rho(u))} (\exp(tu_0 + \sqrt{t}v) \cdot F_1, F_2) \leq C
$$

for all $(v, t) \in \ker \rho \times [t_v, \infty)$.

Combining this with the trivial bound

$$
\langle g \cdot F_1, F_2 \rangle \leq \|F_1\| \cdot \|F_2\|,
$$

we have (again, for all $(v, t) \in \ker \rho \times [t_v, \infty)$,

$$
\langle \exp(tu_0 + \sqrt{t}v) \cdot F_1, F_2 \rangle^{2+\varepsilon} \leq (C + \|F_1\| \cdot \|F_2\|)^{2+\varepsilon} \left( \min \left\{ 1, t^{-(r-1)/2} e^{-(2-\eta)(2\rho(u_0))} \right\} \right)^{2+\varepsilon}
$$

$$
\ll \min \left\{ 1, e^{-\eta\rho(u_0)} \right\} \leq e^{-\eta\rho(u_0)},
$$

We now show the following main theorem of this section, which is Theorem 1.6 of the introduction.

**Theorem 3.8.** We have $L^2(\Gamma \setminus G)$ is tempered if and only if $\psi_T \leq \rho$. 
where \( \eta = (2 - c)(2 + \varepsilon) > 2 \). This gives
\[
\int_G \langle g \cdot f_1, f_2 \rangle^2 + \varepsilon \, dg \ll \int_{v \in \ker \rho} e^{-\eta\rho(u_0)} e^{2t\rho(u_0)} t^{(r-1)/2} dt \, dv
\ll \int a^+ e^{-(\eta-2)\rho(u)} \, du < \infty.
\]

Therefore \( L^2(\Gamma \backslash G) \) is tempered.

To show the converse, suppose now that \( L^2(\Gamma \backslash G) \) is tempered. Then by the definition of temperedness and the estimate of \( \Xi_G(g) \) in (3.3), it follows that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that for any \( f_1, f_2 \in L^2(\Gamma \backslash G)_K \) and \( u \in a^+ \),
\[
|\langle \exp(u) \cdot f_1, f_2 \rangle| \leq C_\varepsilon \|f_1\| \|f_2\| e^{-(1-\varepsilon)\rho(u)}.
\]

Given any unit vector \( v \in \text{int} \, \mathcal{L}_\Gamma \), let \( \psi_v \) be the linear form tangent to \( \psi_\Gamma \) at \( v \). We then choose two non-negative functions \( f_1, f_2 \in C_c(\Gamma \backslash G)_K \) such that \( \langle f_1, E_{\psi(v)} \rangle \cdot \langle f_2, E_{\psi(v)} \rangle > 0 \). Then by Theorem 3.7, there exists \( \kappa_v > 0 \) such that
\[
\lim_{t \to +\infty} t^{(r-1)/2} e^{(2\rho-\psi_v)(tv)} (\exp(tv)f_1, f_2) = \kappa_v \langle f_1, E_{\psi(v)} \rangle \cdot \langle f_2, E_{\psi(v)} \rangle.
\]

Combining the above with (3.4) gives
\[
0 < \lim_{t \to +\infty} t^{(r-1)/2} e^{(2\rho-\psi_v)(tv)} (\exp(tv)f_1, f_2)
\leq \liminf_{t \to +\infty} C_\varepsilon \|f_1\| \|f_2\| t^{(r-1)/2} e^{(2\rho-\psi_v)(tv)} e^{-(1-\varepsilon)\rho(tv)}.
\]

Hence \( (1 + \varepsilon)\rho(v) \geq \psi_v(v) = \psi_\Gamma(v) \). Since \( \varepsilon > 0 \) is arbitrary, we have
\[
\rho(v) \geq \psi_\Gamma(v) \quad \text{for all } v \in \text{int} \, \mathcal{L}_\Gamma.
\]

Note now that since \( \psi_\Gamma \) is an upper semi-continuous concave function, \( \psi_\Gamma \) is continuous on any line segment connecting a point in \( \text{int} \, \mathcal{L}_\Gamma \) and a point on the boundary of \( \mathcal{L}_\Gamma \) (cf. [21, Lem. 3.11]). This implies that \( \rho(v) \geq \psi_\Gamma(v) \) for all \( v \in \mathcal{L}_\Gamma \). By definition, \( \psi_\Gamma = -\infty \) outside \( \mathcal{L}_\Gamma \); it then follows that \( \rho \geq \psi_\Gamma \). \( \square \)

Now recall the following recent theorem of Kim, Minsky, and Oh:

**Theorem 3.9.** [21] Let \( \Gamma \) be an Anosov subgroup of the product of two or three simple real algebraic groups or \( \Gamma < \text{PSL}_d(\mathbb{R}) \) be the image of a Hitchin representation with \( d = 3, 4 \). Then
\[
\psi_\Gamma \leq \rho.
\]

Therefore Theorem 3.7(1) and the first part of Corollary 1.7 follow from Theorems 3.8 and 3.9.
4. LAPLACE EIGENVALUE AND POSITIVE EIGENFUNCTIONS IN $L^2(\Gamma \backslash X)$

Let $\Gamma < G$ be a torsion-free discrete subgroup. Let $\Delta$ denote the Laplace-Beltrami operator on $X$ or on $\Gamma \backslash X$. A smooth function $f$ on a Riemannian manifold $\Gamma \backslash X$ is said to be harmonic if it is an eigenfunction of $\Delta$. We call a harmonic function $\lambda$-harmonic if $-\Delta f = \lambda f$.

A positive joint eigenfunction of $\Gamma \backslash X$ is clearly a harmonic function. Let $C \in \mathbb{Z} \left( g_\mathbb{C} \right)$ denote the Casimir operator on $C^\infty(\mathbb{C} \Gamma \backslash X)$ (or on $C^\infty(\mathbb{C} \Gamma \backslash G)$) whose restriction to $K$-invariant functions coincides with $\Delta$. Then $K$-invariant $C$-eigenfunctions on $\Gamma \backslash G$ correspond to harmonic functions on $\Gamma \backslash X$.

In this section, we compute the Laplace eigenvalue of a positive joint eigenfunction of $\Gamma \backslash X$ (Lemma 4.2), and recall that there exists at most one positive harmonic function, and hence at most one positive joint eigenfunction, in $L^2(\Gamma \backslash X)$ up to a constant multiple (Theorem 4.5 and Corollary 4.6). We also show that a positive joint eigenfunction in $L^2(\Gamma \backslash X)$ determines a unique irreducible spherical unitary representation of $G$ contained in $L^2(\Gamma \backslash G)$ (Theorem 4.7).

**Bottom of the $L^2$-spectrum.** Define the real number $\lambda_0 = \lambda_0(\Gamma \backslash X) \in [0, \infty)$ as follows:

$$\lambda_0 := \inf \left\{ \frac{\int_{\Gamma \backslash X} \|\text{grad } f\|^2 \, d\text{vol}}{\int_{\Gamma \backslash X} |f|^2 \, d\text{vol}} : f \in C^\infty_c(\Gamma \backslash X), \ f \neq 0 \right\}. \hspace{1cm} (4.1)$$

As $\Gamma \backslash X$ is complete, there exists a unique self-adjoint operator on $L^2(\Gamma \backslash X)$ extending the Laplacian on $C^\infty_c(\Gamma \backslash X)$, which we also denote by $\Delta$.

**Theorem 4.1.** [38, Theorem 2.1, 2.2] Suppose that $\Gamma \backslash G$ is not compact. For each $\lambda \leq \lambda_0$, there exists a positive $\lambda$-harmonic function on $\Gamma \backslash X$ and for each $\lambda > \lambda_0$, there exists no positive $\lambda$-harmonic function on $\Gamma \backslash X$. The closed $L^2$-spectrum of $-\Delta$ on $L^2(\Gamma \backslash X)$ contains $\lambda_0$ and is contained in $[\lambda_0, \infty)$.

We identify $\mathfrak{a}^*$ with $\mathfrak{a}$ via the inner product on $\mathfrak{a}$ induced by the Killing form on $\mathfrak{g}$. This endows an inner product on $\mathfrak{a}^*$. More precisely, for each $\psi \in \mathfrak{a}^*$, there exist a unique $v_\psi \in \mathfrak{a}$ such that $\psi = \langle v_\psi, \cdot \rangle$. Then $\langle \psi_1, \psi_2 \rangle = \langle v_{\psi_1}, v_{\psi_2} \rangle$. Equivalently, fix an orthonormal basis $\{H_i\}$ of $\mathfrak{a}$ with respect to the inner product induced by the Killing form on $\mathfrak{g}$. Then $\langle \psi_1, \psi_2 \rangle = \sum_i \psi_1(H_i)\psi_2(H_i)$.

For $\psi \in \mathfrak{a}^*$, we set

$$\lambda_\psi := (\|\rho\|^2 - \|\psi - \rho\|^2). \hspace{1cm} (4.2)$$

**Lemma 4.2.** (1) If $f$ is a positive joint eigenfunction on $X$ with character $\chi_\psi - \rho$, $\psi \in \mathfrak{a}^*$, it is $\lambda_\psi$-harmonic.

(2) If $f$ is a positive $\lambda$-harmonic function on $X$, then $\lambda = \lambda_\psi$ for some $\psi \in \mathfrak{a}^*$ with $\psi \geq \rho$. 

(3) $\lambda_0(X) = \|\rho\|^2$.

Proof. Recall the functions $\varphi_{\psi,h}$ in (2.4). By Theorem 2.6, (1) follows if we show that for any $h \in G$,

$$-C\varphi_{\psi,h} = \lambda_\psi \varphi_{\psi,h}.$$  \hspace{1cm} (4.3)

We may write

$$C = \sum_i H_i^2 + \sum_{\alpha \in \Sigma^+} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) + J,$$

where $J \in U(m_C)$ and each $X_\alpha$ is a unit root vector (cf. [23, Prop. 5.28]). Now using $X_{-\alpha} X_\alpha = X_\alpha X_{-\alpha} - H_\alpha$ gives

$$C = \sum_i H_i^2 - \sum_{\alpha \in \Sigma^+} H_\alpha + \sum_{\alpha \in \Sigma^+} 2X_\alpha X_{-\alpha} + J.$$

As in the proof of Proposition 2.5, $[J\varphi_{\psi,h}] (e) = 0$, and $[X_\alpha X_{-\alpha} \varphi_{\psi,h}] (e) = 0$. Applying $-C$ to $\varphi_{\psi,h}$ gives

$$-C\varphi_{\psi,h} = - \left( \sum_i \psi(H_i)^2 - \sum_{\alpha \in \Sigma^+} \psi(H_\alpha) \right) \varphi_{\psi,h}$$

$$= - (\|\psi\|^2 - 2\langle \rho, \psi \rangle) \varphi_{\psi,h}$$

$$= (\|\rho\|^2 - \|\psi - \rho\|^2) \varphi_{\psi,h}.$$

This shows (1). Let $f$ be a positive $\lambda$-harmonic function. By [25, Theorem 2],

$$f(g) = \int_{\{\psi \geq \rho : \lambda_{\psi} = \chi_\psi - \rho\} \times K/M} \varphi_{\psi,k}(g) d\mu([k], \psi)$$

for some Borel measure $\mu$ on $\{\psi \geq \rho : \lambda_{\psi} = \chi_\psi - \rho\} \times K/M$. This implies (2) by (4.3). It is shown in [20] that there are no positive harmonic functions on $X$ with eigenvalue greater than $\|\rho\|^2$; hence the inequality $\lambda_0(X) \leq \|\rho\|^2$ follows from Theorem 4.1 for $\Gamma = \{e\}$. On the other hand, there exists a positive $\|\rho\|^2$-harmonic function on $X$ by [25]. Hence $\lambda_0(X) = \|\rho\|^2$. \hfill $\square$

**Corollary 4.3.** Let $\Gamma < G$ be a Zariski dense discrete subgroup. Then

$$\sup \{\lambda_\psi : \psi \in D^*_\Gamma\} \leq \lambda_0 \leq \|\rho\|^2.$$

Proof. The right inequality follows from Lemma 4.2(3), since $\lambda_0(\Gamma \setminus X) \leq \lambda_0(X)$ by Theorem 4.1. If $\Gamma$ is cocompact in $G$, then $\psi_T = 2\rho$ and hence $D^*_\Gamma = \{2\rho\}$, and hence the first two terms are equal to 0. In general, for each $\psi \in D^*_\Gamma$, Quint [34] constructed a $(\Gamma, \psi)$-conformal measure supported on $\Lambda$. It then follows from Proposition 2.8 that for any $\psi \in D^*_\Gamma$, there exists a positive joint eigenfunction on $\Gamma \setminus X$ with character $\chi_{\psi - \rho}$. Hence the claim follows from Theorem 4.1 and Lemma 4.2. \hfill $\square$

**Theorem 4.4.** If $L^2(\Gamma \setminus G)$ is tempered, then $\lambda_0 = \|\rho\|^2$. 

Proof. Assume that $\lambda_0 < \|\rho\|^2$. By Weyl’s criterion, we can then find a $K$-invariant unit vector $f \in L^2(\Gamma \backslash G)_K$ such that

$$\|(\Delta - \lambda_0)f\| < \frac{\|\rho\|^2 - \lambda_0}{2}.$$  

This gives

$$\|Cf\| = \|\Delta f\| \leq \|(\Delta - \lambda_0)f\| + \lambda_0 < \frac{\|\rho\|^2 + \lambda_0}{2} < \|\rho\|^2.$$  

On the other hand, using the direct integral representation of $L^2(\Gamma \backslash G) = \int_Z^\oplus (\pi(z), H_z) d\mu(z)$ into irreducible unitary representations of $G$ gives

$$\|Cf\|^2 = \int_Z \|d\pi(C)f_z\|^2 d\mu(z) \geq \left( \min_{\pi \text{ tempered}} |d\pi(C)|^2 \right),$$

since all the representations $(\pi(z), H_z)$ are tempered. However, since all tempered spherical representations are weakly contained in $L^2(G/K)$ and $\lambda_0(G/K) = \|\rho\|^2$,

$$\min_{\pi \text{ tempered}} |d\pi(C)| = \|\rho\|^2,$$

giving a contradiction. \qed

**Theorem 4.5.** [38, Theorem 2.8 and Corollary 2.9]

1. If there exists a positive harmonic function in $L^2(\Gamma \backslash X)$, then it is $\lambda_0$-harmonic.

2. If there exists a $\lambda_0$-harmonic function $\phi$ in $L^2(\Gamma \backslash X)$, then the space of $\lambda_0$-harmonic functions is one-dimensional and generated by a positive function.

**Corollary 4.6.**

1. There exists at most one positive joint eigenfunction in $L^2(\Gamma \backslash X)$ up to a constant multiple.

2. If there exists a positive joint eigenfunction in $L^2(\Gamma \backslash X)$ with character $\chi_{\psi - \rho}$, then

$$\lambda_0 = \lambda_\psi.$$  

3. There exists a positive harmonic function in $L^2(\Gamma \backslash X)$ if and only if there exists a positive joint eigenfunction in $L^2(\Gamma \backslash X)$ of character $\chi_{\psi - \rho}$ with $\lambda_\psi = \lambda_0$.

**Proof.** We only need to verify the third claim. Suppose that $\phi \in L^2(\Gamma \backslash X)$ is a positive harmonic function. Via $L^2(\Gamma \backslash X) = L^2(\Gamma \backslash G)_K$, we may consider $\phi \in L^2(\Gamma \backslash G)_K$ as a positive $\mathcal{C}$-eigenfunction for the Casimir operator $\mathcal{C}$. By Theorem 4.5, $\mathcal{C}\phi = -\lambda_0\phi$ where $\lambda_0$ is defined as in (4.1). Let $D \in Z(\mathfrak{g}_C)$. Then $\mathcal{C} \circ D\phi = D \circ \mathcal{C}\phi = -\lambda_0 D\phi$. By the uniqueness in Theorem 4.5, it follows that $D\phi$ is a constant multiple of $\phi$; and hence $\phi$ is an eigenfunction for $D$ as well. Therefore $\phi$ is a joint eigenfunction. \qed
Spherical unitary representations contained in $L^2(\Gamma \backslash G)$. We let $C_c(G//K)$ denote the Hecke algebra of $G$, i.e.

$$C_c(G//K) = \{ f \in C_c(G) : f(k_1 g k_2) = f(g) \text{ for all } g \in G, k_1, k_2 \in K \}.$$ 

Each element of $C_c(G//K)$ acts on $C(G)$ via right convolution $\ast$. In fact each positive $K$-invariant joint eigenfunction is an eigenfunction for the action of the Hecke algebra. This is seen by using Theorem 2.6 as follows: let $\phi \in C^\infty(G)$ be a positive joint eigenfunction, with integral representation

$$\phi(g) = \int_F \varphi_{\psi,k}(g) \, d\nu([k]).$$

Given $f \in C_c(G//K)$, we then have

$$(\phi \ast f)(g) = \int_G \phi(g h^{-1}) f(h) \, dh = \int_G \int_F \varphi_{\psi,k}(g h^{-1}) f(h) \, d\nu([k]) \, dh$$

$$= \int_F \int_G f(h) e^{-\psi(H(h g^{-1} k))} \, dh \, d\nu([k]).$$

Now using $H(h g^{-1} k) = H(h \kappa(g^{-1} k)) + H(g^{-1} k)$ and then the change of variables $h' = h \kappa(g^{-1} k)$ gives

$$(\phi \ast f)(g) = \int_F \left( \int_G f(h \kappa(g^{-1} k)^{-1}) e^{-\psi(H(h))} \, dh \right) e^{-\psi(H(g^{-1} k))} \, d\nu([k])$$

$$= \int_F \left( \int_G f(h) e^{-\psi(H(h))} \, dh \right) e^{-\psi(H(g^{-1} k))} \, d\nu([k])$$

$$= \left( \int_G f(h) e^{-\psi(H(h))} \, dh \right) \phi(g),$$

since $f \in C(G//K)$, and is thus right $K$-invariant. In total, we have shown that $\phi$ is an eigenfunction of the $f$-action, with eigenvalue $\int_G f(h) e^{-\psi(H(h))} \, dh$.

**Theorem 4.7.** If $\phi \in L^2(\Gamma \backslash G)_K$ is a positive joint eigenfunction of norm one, then there exists a unique irreducible spherical unitary subrepresentation $(\pi, \mathcal{H}_\phi)$ of $L^2(\Gamma \backslash G)$, and $\phi$ is the unique $K$-invariant unit vector in $\mathcal{H}_\phi$.

**Proof.** Let $\phi$ be as in the statement. Define $\Phi : G \to \mathbb{C}$ by

$$\Phi(g) := \langle g.\phi, \phi \rangle$$

for all $g \in G$ where the $g$ action on $L^2(\Gamma \backslash G)$ is via the translation action of $G$ on $\Gamma \backslash G$ from the right.
Given $f \in C_c(G//K)$, we then have

\[
(\Phi * f)(g) = \int_G \Phi(gh^{-1})f(h)\,dh = \int_G \langle (gh^{-1})\cdot \phi, \phi \rangle f(h)\,dh
= \int_G \langle f(h)h^{-1}\cdot \phi, g^{-1}\cdot \phi \rangle\,dh
= \left(\int_G f(h)e^{-\psi(H(h))}\,dh\right)\Phi(g),
\]

i.e. $\Phi$ is also a $C_c(G//K)$-eigenfunction. Also note that $\Phi(e) = 1$, and since $\phi$ is right $K$-invariant, $\Phi$ is bi-$K$-invariant. Moreover, being the matrix coefficient of a unitary representation, $\Phi$ is also positive definite, i.e., for any $g_1, \cdots, g_n \in G$ and $z_1, \cdots, z_n \in \mathbb{C}$,

\[
\sum_{1 \leq i,j \leq n} z_i \bar{z}_j \Phi(g_j^{-1}g_i) \geq 0.
\]

We have thus shown that $\Phi$ is a positive definite spherical function. Letting $H_\phi$ denote the closure of $\text{span}\{g.\phi : g \in G\}$ in $L^2(\Gamma \setminus G)$, by [26, Chapter IV§5, Corollary of Theorem 9], $H_\phi$ is an irreducible (spherical) unitary subrepresentation of the quasi-regular representation $L^2(\Gamma \setminus G)$. The uniqueness follows from Corollary 4.6. □

5. Smearing arguments

Let $\Gamma$ be a Zariski dense discrete subgroup of a connected semisimple real algebraic group $G$. Recall the notation $i$ for the opposition involution of $G$. The goal of this section is to prove:

**Theorem 5.1.** Let $\psi \in \mathfrak{a}^*$ be stabilized by $i$, i.e., $\psi \circ i = \psi$. Suppose either $\psi \approx \psi_\Gamma$ or that $L_\Gamma \subset \text{int} \mathfrak{a}^+ \cup \{0\}$. Then no positive joint eigenfunction of character $\chi_{\psi-\rho}$ belongs to $L^2(\Gamma \setminus X)$.

Every Anosov subgroup $\Gamma \subset G$ satisfies $L_\Gamma \subset \text{int} \mathfrak{a}^+ \cup \{0\}$ [33]. Hence the following corollary shows the implication (1) $\Rightarrow$ (3) in Theorem 1.6.

**Corollary 5.2.** If $L_\Gamma \subset \text{int} \mathfrak{a}^+ \cup \{0\}$ and $\lambda_0(\Gamma \setminus X) = \|\rho\|^2$, then there exists no positive harmonic function in $L^2(\Gamma \setminus X)$.

**Proof.** Suppose that there exists a positive harmonic function in $L^2(\Gamma \setminus X)$ By Corollary 4.6, there exists a positive joint eigenfunction $\phi$ in $L^2(\Gamma \setminus X)$ of character $\chi_{\psi-\rho}$ for some $\psi \in \mathfrak{a}^*$ satisfying $\lambda_\phi = \lambda_0$. Since $\lambda_0 = \|\rho\|^2 = \lambda_\psi = \|\rho\|^2 - \|\rho - \psi\|^2$, it follows that $\psi = \rho$. Since $\rho$ is invariant under $i$, Theorem 5.1 implies the $\phi$ cannot belong to $L^2(\Gamma \setminus X)$, yielding a contradiction. □

Similarly, we deduce the following, which implies Theorem 1.8 of the introduction:

**Corollary 5.3.** Suppose that $i$ is trivial. For any Zariski dense discrete subgroup $\Gamma \subset G$ with $L_\Gamma \subset \text{int} \mathfrak{a}^+ \cup \{0\}$, there exists no positive harmonic function in $L^2(\Gamma \setminus X)$. 

Theorem 5.1 will be deduced from Theorem 5.5 whose proof is based on the smearing argument of Sullivan. For each \( g \in G \), recall \( g^+, g^- \in G/P \) from (3.2). Observe that \((gm)^\pm = g^\pm\) for all \( g \in G, m \in M \). For the identity element \( e \in G, e^+ = [P], e^- = [w_0P] \) and \( g^\pm = g(e^\pm) \) for any \( g \in G \). The unique open \( G \)-orbit \( F^{(2)} \) in \( F \times F \) is given by:

\[
F^{(2)} = G.(e^+, e^-) = \{(g^+, g^-) \in F \times F : g \in G\}.
\]

For each \( x \in X \), and \((\xi, \eta) \in F^{(2)} \), define

\[
d_x(\xi, \eta) = e^{-\psi(h(\xi, \eta) + i\beta(x, g))}
\]

where \( g \in G \) is any element such that \( g^+ = \xi \) and \( g^- = \eta \); this definition is independent of the choice of such \( g \). The following \( G \)-equivariance property follows from that of the Busemann function: for any \( h \in G \),

\[
d_x(\xi, \eta) = d_{hx}(h\xi, h\eta).
\]

**Definition 5.4** (Hopf parameterization). The homeomorphism \( G/M \to F^{(2)} \times a \) given by \( gM \mapsto (g^+, g^-, b = \beta_g(e, g)) \) is called the Hopf parameterization of \( G/M \).

Let \( \psi \in a^* \), and fix \( \{\nu_x : x \in X\} \) and \( \{\tilde{\nu}_x : x \in X\} \) be respectively \((\Gamma, \psi)(\Gamma, \psi \circ i)\)-conformal densities on \( F \). Using the Hopf parametrization 5.4, define the following locally finite Borel measure \( \bar{m}_{\nu, \tilde{\nu}} \) on \( G/M \): for \((\xi, \eta, v) \in F^{(2)} \times a \),

\[
d\bar{m}_{\nu, \tilde{\nu}}(\xi, \eta, v) = \frac{1}{d_x(\xi, \eta)} d\nu_x(\xi) d\tilde{\nu}_x(\eta) dv \tag{5.1}
\]

where \( dv \) is the Lebesgue measure on \( a \) and \( x \in X \) is any element; it follows from the \( \Gamma \)-conformality of \( \{\nu_x\} \) and \( \{\tilde{\nu}_x\} \) that this definition is independent of \( x \in X \). The measure \( \bar{m}_{\nu, \tilde{\nu}} \) is left \( \Gamma \)-invariant and right \( A \)-invariant. We denote by \( m_{\nu, \tilde{\nu}} \) the \( AM \)-invariant Borel measure on \( \Gamma \backslash G \) induced by \( \bar{m}_{\nu, \tilde{\nu}} \); this measure is called the Bowen-Margulis-Sullivan measure associated to the pair \((\nu, \tilde{\nu})\) [9].

**Theorem 5.5.** For any \((\Gamma, \psi)\)-conformal measure \( \nu \) and \((\Gamma, \psi \circ i)\)-conformal measure \( \tilde{\nu} \) on \( F \), we have

\[
m_{\nu, \tilde{\nu}}(\Gamma \backslash G) \ll \int_{\Gamma \backslash G} E_\nu(x) E_{\tilde{\nu}}(x) dx.
\]

**Proof.** We extend the smearing argument due to Sullivan and Thurston ([39], [7]). Let \( Z = G/K \times F^{(2)} \). For any \((\xi, \eta) \in F^{(2)} \), we write \([\xi, \eta] = gA_0 \subset X\) for any \( g \in G \) such that \( g^+ = \xi \) and \( g^- = \eta \); \([\xi, \eta]\) is a maximal flat in \( X \) defined independently of the choice of \( g \in G \). We also denote by \( W_{\xi, \eta} \subset X \) the one neighborhood of \([\xi, \eta]\). Consider the following locally finite Borel measure \( \alpha \) on \( Z \) defined as follows: for any \( f \in C_c(Z) \),

\[
\alpha(f) = \int_{([\xi, \eta]) \in F^{(2)}} \int_{z \in W_{\xi, \eta}} f(z, \xi, \eta) dz \, dm(\xi, \eta)
\]
where $dz$ is the $G$-invariant measure on $X$, and $dm(\xi, \eta) = \frac{1}{d_x(\xi, \eta)}d\nu_x(\xi)d\bar{\nu}_x(\eta)$ (independent of $x$); in other words,
\[
d\alpha(z, \xi, \eta) = d\lambda_{\xi, \eta}(z)dm(\xi, \eta)
\]
where $\lambda_{\xi, \eta}$ is the restriction of $\lambda$ to $W_{\xi, \eta}$. Consider natural diagonal action of $\Gamma$ on $Z$. Since $dz$ and $dm$ are both left $\Gamma$-invariant, $\alpha$ is also left $\Gamma$-invariant and hence induces a measure the quotient space $\Gamma \backslash Z$, which we also denote by $\alpha$ by abuse of notation.

Define the projection $\pi' : Z \to G/M$ as follows: for $(x, \xi, \eta) \in X \times F^{(2)}$, choose $g \in G$ so that $g^+ = \xi$ and $g^- = \eta$. Then there exists a unique element $a \in A$ such that $d(x, gao) = d(x, gAo) = \inf_{b \in A} d(x, gbo)$; this follows from [4, Proposition 2.4] since $X$ is a CAT(0) space and $gA(o)$ is a convex complete subspace of $X$. In other words, the point $gao$ is the orthogonal projection of $x$ to the flat $[\xi, \eta] = gAo$. We then set
\[
\pi'(x, \xi, \eta) = gaoM \in G/M;
\]
this is well-defined independent of the choice of $g \in G$.

Noting that $\pi'$ is $\Gamma$-equivariant, we denote by $\pi : \text{supp}(\alpha) \subset \Gamma \backslash Z \to \text{supp} (m_{\nu, \bar{\nu}}) \subset \Gamma \backslash G/M$ the map induced by $\pi'$.

Fixing $[ga] \in \Gamma \backslash G/M$, the fiber $\pi^{-1}[ga]$ is of the form $[(gaD_0, g^+, g^-)]$ where
\[
D_0 = \{s \in X : d(s, o) \leq 1, \text{ the geodesic connecting } s \text{ and } o \text{ is orthogonal to } Ao \text{ at } o\}.
\]

Since each fiber $\pi^{-1}(v)$, $v \in \text{supp} m_{\nu, \bar{\nu}}$, is isometric to $D_0$, we have for any Borel subset $S \subset \text{supp} m_{\nu, \bar{\nu}}$, we have
\[
\alpha(\pi^{-1}(S)) = c \cdot m_{\nu, \bar{\nu}}(S) \tag{5.2}
\]
where $c = \text{Vol}(D_0)$; the volume of $D_0$ being computed with respect to the volume form induced by the $G$-invariant measure on $X$.

Consider now the map $p : \text{supp}(\alpha) \to \Gamma \backslash X$ defined by $p([z, \xi, \eta]) = [z]$ for any $(z, \xi, \eta) \in \text{supp}(\alpha)$.

Let $F = \pi^{-1}(\text{supp} m_{\nu, \bar{\nu}}) \subset \text{supp}(\alpha)$. We write
\[
\alpha(F) = \int_{\Gamma \backslash X} \alpha_x(p^{-1}(x) \cap F) dx,
\]
where $\alpha_x$ is a conditional measure on the fiber $p^{-1}(x)$.

We claim that for each $x \in \Gamma \backslash X$,
\[
\alpha_x(p^{-1}(x)) \ll E_{\nu}(x) \cdot E_{\bar{\nu}}(x) \tag{5.3}
\]
This implies that $\alpha(F) \ll \int_{\Gamma \backslash X} E_{\nu}(x)E_{\bar{\nu}}(x) dx$, which then finishes the proof by (5.2).
Note that $V_{h(o)} := \{(\xi, \eta) \in F^{(2)} : [\xi, \eta] \cap B(h(o), 1) \neq \emptyset\}$ is a compact subset of $F^{(2)}$; if $g_i \in G$ such that $d(g_i a_i, h(o)) \leq 1$ for some $a_i \in A$, then $g_i a_i$ converges to some $g_0 \in G$ by passing to a subsequence. This implies $(g_i^+ g_i^-) \to (g_0^+ g_0^-) \in F^{(2)}$, from which the compactness of $V_{h(o)}$ follows. It follows that

$$\kappa := \inf \{d_{h(o)}(\xi, \eta) : (\xi, \eta) \in V_o\} > 0.$$ 

By the equivariance $d_{h(o)}(\xi, \eta) = d_{h^{-1}}(h^{-1}\xi, h^{-1}\eta)$, we have for any $h \in G$,

$$\kappa = \inf \{d_{h(o)}(\xi, \eta) : (\xi, \eta) \in V_{h(o)}\}.$$ 

Note that if $x = [h(o)] \in X$ for $h \in G$, then

$$p^{-1}(x) = \{[(x, \xi, \eta)] \in \supp(\alpha) : [\xi, \eta] \cap B(h(o), 1) \neq \emptyset\} \simeq V_{h(o)}.$$ 

Now

$$\alpha_x(p^{-1}(x)) = \alpha_x(V_{h(o)}) = \int_{(\xi, \eta) \in V_{h(o)}} \frac{1}{d_{h(o)}(\xi, \eta)} d\nu_{h(o)}(\xi) d\bar{\nu}_{h(o)}(\eta) \leq \frac{1}{\kappa} \int_{(\xi, \eta) \in V_{h(o)}} d\nu_{h(o)}(\xi) d\bar{\nu}_{h(o)}(\eta) \leq \frac{1}{\kappa} |\nu_{h(o)}| \cdot |\bar{\nu}_{h(o)}| = \frac{1}{\kappa} E_\nu(x) \cdot E_{\bar{\nu}}(x).$$

\[\square\]

**Proof of Theorem 5.1.** Suppose that $\phi \in L^2(\Gamma\backslash X)$ is a positive joint eigenfunction with character $\chi_{\psi - \rho}$. By Proposition 2.8, $\phi = E_\nu$ for some $(\Gamma, \psi)$-conformal measure $\nu$. Since $\psi \circ i = \psi$, we may form the measure $m_{\nu, \nu}$ and apply Theorem 5.5. Since $E_\nu \in L^2(\Gamma\backslash G)$, it follows that $m_{\nu, \nu}$ is a finite $\mathcal{MA}$-invariant Borel measure on $\Gamma\backslash G$. Since $m_{\nu, \nu}$ is finite, it is conservative for any one-parameter subgroup of $A$. In particular, for any non-zero $v \in \mathfrak{a}^+$, there exist sequences $t_i \to +\infty$ and $\gamma_i \in \Gamma$ such that the sequence $\gamma_i \exp(t_i v)$ is convergent. This implies that $t_i^{-1} \mu(\gamma_i)$ converges to $v$, and hence $v \in \mathcal{L}_\Gamma$. Therefore $\mathcal{L}_\Gamma = \mathfrak{a}^+$. Suppose that $\psi > \psi_\Gamma$. Then, by [35, Lem. III. 1.3], we have

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty.$$ 

On the other hand, by Theorem 1.4 of [5], the finiteness of $m_{\nu, \nu}$ implies that $\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty$, yielding a contradiction. This finishes the proof.

6. **SUBGROUPS OF THE SECOND KIND AND POSITIVE JOINT EIGENFUNCTIONS**

Let $\Gamma$ be a Zariski dense discrete subgroup of a connected semisimple real algebraic group $G$. Let $\Lambda \subset F$ denote the limit set of $\Gamma$ (see (3.1)), which is the unique $\Gamma$-minimal subset of $F$. When $G$ has rank one, a discrete
A subgroup $\Gamma < G$ is said to be of the second kind if $\Lambda \neq \partial_\infty X$. We extend this definition:

**Definition 6.1.** A discrete subgroup $\Gamma < G$ is of the second kind if there exists $\xi \in \mathcal{F}$ which is in general position with all points of $\Lambda$, i.e., $(\xi, \Lambda) \subset \mathcal{F}^{(2)}$.

**Remark 6.2.** For instance, $\Gamma$ is of the second kind in the following cases.

1. Let $\Gamma_0 < G$ be Anosov. Then for any Anosov subgroup $\Gamma < \Gamma_0$ with some point $\xi \in \Lambda_{\Gamma_0} - \Lambda_{\Gamma}$, $(\Lambda, \xi) \subset \mathcal{F}^{(2)}$, since any two distinct points of $\Lambda_{\Gamma_0}$ are in general position by the Anosov assumption on $\Gamma_0$. Hence $\Gamma$ is of the second kind.
2. If $\Lambda \subset g N w_0 P$ for some $g \in G$, then $(\Lambda, g + \eta) \subset \mathcal{F}^{(2)}$. One can construct many Schottky groups with $\Lambda \subset N w_0 P$.
3. Let $G = \prod_i G_i$ be a product of simple algebraic groups $G_i$ of rank one. Then $\mathcal{F} = \prod_i G_i / P_i$, and $(\xi_i, \eta_i) \in \mathcal{F}$ are in general position if and only if $\xi_i \neq \eta_i$ for all $i$. Therefore if there exists $\xi_i \in G_i / P_i \in \pi_i(\Lambda)$ where $\pi_i: \mathcal{F} \rightarrow G_i / P_i$ is the canonical projection, then for $\xi = (\xi_i)_i$, $(\Lambda, \xi) \subset \mathcal{F}^{(2)}$.

**Construction of positive joint eigenfunctions.**

**Proposition 6.3.** Let $\Gamma < G$ be the of second kind with $L_\Gamma \subset \text{int } a^+ \cup \{0\}$. For any $\psi \in a^*$ with $\psi \geq \psi_\Gamma$, there exists a positive joint eigenfunction on $\Gamma \setminus X$ with character $\chi_{\psi - \rho}$.

We will use shadow lemma to prove this proposition. For $q \in X$ and $r > 0$, we set $B(q, r) = \{x \in X : d(x, q) < r\}$. For $p = g(o) \in X$, the shadow of the ball $B(q, r)$ viewed from $p$ is defined as

$$O_r(p, q) := \{ (gk)^+ \in \mathcal{F} : k \in K, \text{ } gk \text{ int } A^+ o \cap B(q, r) \neq \emptyset \}.$$

Similarly, for $\xi \in \mathcal{F}$, the shadow the ball $B(q, r)$ viewed from $\xi$ is defined by

$$O_r(\xi, q) := \{ h^+ \in \mathcal{F} : h \in G \text{ satisfies } h^- = \xi, ho \in B(q, r) \}.$$

We recall the shadow lemma:

**Lemma 6.4.** [27, Lemma 5.7] There exists $\kappa > 0$ such that for any $g \in G$ and $r > 0$,

$$\sup_{\xi \in O_r(g(o), o)} \| \beta_\xi(g(o), o) - \mu(g^{-1}) \| \leq \kappa r.$$

**Lemma 6.5.** [27, Lemma 5.6] If $q_i \in X$ converges to $\eta \in \mathcal{F}$ as in Definition 6.1, then for any $q \in X$, $r > 0$ and $\varepsilon > 0$,

$$O_{r-\varepsilon}(q_i, q) \subset O_r(\eta, q) \subset O_{r+\varepsilon}(q_i, q)$$

for all sufficiently large $i$.

**Lemma 6.6.** If $L_\Gamma \subset \text{int } a^+ \cup \{0\}$, then the union $\Gamma(o) \cup \Lambda$ is compact in the topology given in Definition 3.1.
Proof. The hypothesis implies that any sequence $\gamma_i \to \infty$ in $\Gamma$ tends to $\infty$ regularly, and hence has a limit in $\mathcal{F}$. Moreover the limit belongs to $\Lambda$ by its definition. \hfill \Box

Lemma 6.7. Suppose that $\mathcal{L}_{\Gamma} \subset \text{int} \ a^+ \cup \{0\}$. If $\xi \in \mathcal{F}$ satisfies that $(\xi, \Lambda) \subset \mathcal{F}^{(2)}$, then there exists $R > 0$ such that

$$\xi \in \bigcap_{\gamma \in \Gamma} O_R(\gamma(o), o).$$

Proof. We first claim that $\xi \in \bigcap_{\eta \in \Lambda} O_R(\eta, o)$ for some $R > 0$. Note that $\lim_{R \to \infty} O_R(\eta, o) = \{z \in \mathcal{F} : (z, \eta) \in \mathcal{F}^{(2)}\}$. Hence for each $\eta \in \Lambda$, we have

$$R_\eta = \inf\{R + 1 : \xi \in O_R(\eta, o)\} < \infty.$$ It suffices to show that $R := \sup_{\eta \in \Lambda} R_\eta < \infty$. Suppose not; then $R_\eta_i \to \infty$ for some sequence $\eta_i \in \Lambda$. By passing to a subsequence, we have $\eta_i$ converges to some $\eta$. This follows that $O_{R_\eta_i + 1}(\eta, o) \subset O_{R_\eta_i + 2}(\eta_i, o)$ for all sufficiently large $i$. Therefore $R_\eta_i \leq R_\eta + 3$, yielding a contradiction.

We now claim that $\xi \in \bigcap_{\gamma \in \Gamma} O_{R'}(\gamma(o), o)$ for some $R' > 0$. Suppose not; then there exist sequences $\gamma_i \rightarrow \infty$ in $\Gamma$ and $R_i \rightarrow \infty$ such that $\xi \notin O_{R_i}(\gamma_i(o), o)$. By Lemma 6.6, by passing to a subsequence, we may assume that $\gamma_i(o)$ converges to some $\eta \in \Lambda$. By the first claim, we have $\xi \in O_{R}(\eta, o)$. By Lemma 6.5, we have $\xi \in O_{R}(\eta, o) \subset O_{R+1}(\gamma_i(o), o)$ for all sufficiently large $i$. This is a contradiction since for $i$ large enough so that $R_i > R + 1$, we have $\xi \notin O_{R+1}(\gamma_i(o), o)$. This proves the claim. \hfill \Box

As an immediate corollary of Lemmas 6.5 and 6.7, we obtain:

Corollary 6.8. If $\mathcal{L}_{\Gamma} \subset \text{int} \ a^+ \cup \{0\}$ and $\xi \in \mathcal{F}$ satisfies that $(\xi, \Lambda) \subset \mathcal{F}^{(2)}$,

$$\sup_{\gamma \in \Gamma} \|\beta_\xi(\gamma^{-1}o, o) - \mu(\gamma)\| < \infty.$$  

Proof of Proposition 6.3: If $\psi \in D_{\Gamma}^*$, this follows from the work of Quint [34]. Hence we assume $\psi \in D_{\Gamma} - D_{\Gamma}^*$; this implies that

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} \leq \infty \quad (6.1)$$

by [35, Lem. III. 1.3]. As $\Gamma$ is of the second kind, there exists $\xi \in \mathcal{F}$ such that $(\xi, \eta) \in \mathcal{F}^{(2)}$ for all $\eta \in \Lambda$. By Corollary 6.8, $\|\beta_\xi(\gamma^{-1}o, o) - \mu(\gamma)\|$ is bounded uniformly for all $\gamma \in \Gamma$. Therefore (6.1) implies that

$$\sum_{\gamma \in \Gamma} e^{-\psi(\beta_\xi(\gamma^{-1}o,o))} < \infty. \quad (6.2)$$

For any fixed $x \in X$, we have $\beta_\xi(\gamma^{-1}x, o) = \beta_\xi(\gamma^{-1}o, o) + \beta_\xi(x, o)$ and $\|\beta_\xi(x, o)\| \leq d(x, o)$. Hence $e^{-\psi(\beta_\xi(\gamma^{-1}o,o))} \leq e^{-\psi(\mu(\gamma))}$ with implied constant uniform for all $\gamma \in \Gamma$. 


Therefore, by (6.1) the following function $F_\psi = F_{\psi, \xi}$ on $X$ is well-defined: for $x \in X$,

$$F_\psi(x) := \sum_{\gamma \in \Gamma} e^{-\psi(\beta_\xi(\gamma^{-1}x, o))}.$$  \hfill (6.3)

If we write $\xi = [k_0] \in K/M = \mathcal{F}$, then for any $g \in G$,

$$\beta_\xi(\gamma^{-1}go, o) = \beta_M(k_0^{-1}\gamma^{-1}go, o) = H(g^{-1}\gamma k_0)$$

and hence $e^{-\psi(\beta_\xi(\gamma^{-1}go, o))} = \varphi_{\psi, \gamma k_0}(g)$. Therefore $F_\psi = \sum_{\gamma \in \Gamma} \varphi_{\psi, \gamma k_0}$. It now follows from Lemma 2.2 that $F_\psi$ is a positive $\Gamma$-invariant joint eigenfunction on $X$ with eigenvalue $\chi_\psi - \rho$. This finishes the proof.

**Remark 6.9.** For $\psi \in D_\Gamma - D_\star \Gamma$, we have constructed positive joint eigenfunction $F_{\psi, \xi}$ on $\Gamma \setminus X$ of eigenvalue $\chi_\psi - \rho$ for any $\xi \in \mathcal{F}$ with $(\Lambda, \xi) \subset F^{(2)}$.

We obtain the following strengthening of Lemma 4.3 by Theorem 4.1 and Lemma 4.2.

**Corollary 6.10.** For $\Gamma$ as in Proposition 6.3, we have

$$\sup\{\|\rho\|^2 - \|\psi - \rho\|^2 : \psi \in D_\Gamma\} \leq \lambda_0 \leq \|\rho\|^2.$$ \hfill (6.4)

**Example 6.11.** Let $\Gamma < SO^\circ(n, 1)$ be a discrete subgroup with $\Lambda \neq \partial \mathbb{H}^n$; then $\Gamma$ satisfies the hypothesis of Proposition 6.3. Since $\rho = \frac{(n-1)}{2}$ and $D_\Gamma = \{s \geq \delta\}$, we have

$$\sup\{\|\rho\|^2 - \|\psi - \rho\|^2 : \psi \in D_\Gamma\} = \begin{cases} \delta(n-1-\delta) & \text{if } \delta \geq \frac{n-1}{2} \\ \frac{(n-1)^2}{4} & \text{if } \delta \leq \frac{n-1}{2}. \end{cases} \hfill (6.5)$$

It then follows from Proposition 2.8 and Theorem 4.1 that we have equality in (6.4) in this case, as was proved by Sullivan [38, Theorem 2.17].

We require the following lemma in the proof of Theorem 6.13:

**Lemma 6.12.** Let $\psi \geq \rho$ and $\psi \notin \mathbb{R}\rho$. Denote by $\psi'$ be the element of the line $\mathbb{R}\psi$ closest to $\rho$. Then $\psi' \geq \rho$.

**Proof.** Let $\phi := \psi - \rho$. Note that $\phi \geq 0$ on $a$ by the hypothesis. Then

$$\psi' = \frac{\langle \psi, \rho \rangle}{\|\psi\|^2} \psi = \frac{\langle \rho + \phi, \rho \rangle}{\|\rho + \phi\|^2} \psi = \left(1 - \frac{\|\phi\|^2}{\|\rho + \phi\|^2}\right) \psi,$$

i.e. $\psi' = t\psi$ with $0 < t < 1$. Now, if $\psi' \geq \rho$, we could repeat the process with $\psi'$ in place of $\psi$ to find another, different, closest vector in $\mathbb{R}\psi$ to $\rho$, which is not possible. \hfill $\Box$

**Theorem 6.13.** Let $\Gamma < G$ be of the second kind with $\mathcal{L}_\Gamma \subset \text{int } a^+ \cup \{0\}$. Let $\psi \geq \max(\psi_T, \rho)$ and $\psi \neq \rho$. If there exists a positive joint eigenfunction in $L^2(\Gamma \setminus X)$ with character $\chi_{\psi - \rho}$, then $\psi \in D_\Gamma^*$. 
Proof. Suppose that $\psi \in D_\Gamma \setminus (\{\rho\} \cup D_\Gamma^*)$ and that $\psi \geq \rho$. Assume that there exists a positive joint eigenfunction $\phi \in L^2(\Gamma \setminus X)$ with character $\chi_{\psi - \rho}$. By Corollary 4.6,
\[
\lambda_0 = \lambda_\psi = \|\rho\|^2 - \|\psi - \rho\|^2.
\] (6.6)
Let $0 < c \leq 1$ be so that $\psi_0 := c\psi \in D_\Gamma^*$, as provided by Lemma 3.6. Since $\psi^0 \not\in D_\Gamma^*$, we have $0 < c < 1$. There exists a unique $s_0 \in \mathbb{R}$ such that
\[
\|s_0\psi_0 - \rho\| = \min\{\|s\psi - \rho\| : s \in \mathbb{R}\},
\] (6.7)
that is, $s_0\psi_0$ be the element on the line $\mathbb{R}\psi$ that is closest to $\rho$.

We claim that $s_0c \leq 1$; since $0 < c < 1$, this implies that $\max\{1, s_0\} < c^{-1}$. If $\psi \in \mathbb{R}\rho$, then $s_0\psi_0 = \rho$. Since $\psi_0 = c\psi$, we get $s_0c\psi = \rho$. By the hypothesis $\rho \leq \psi$, $s_0c \leq 1$. Now suppose $\psi \notin \mathbb{R}\rho$. Assume that $s_0c > 1$. Then $s_0\psi_0 = s_0c\psi > \psi$. Hence $s_0c\psi \in D_\Gamma$. By Corollary 6.10 and (6.6), we get $\|s_0c\psi - \rho\| \geq \|\psi - \rho\|$. By the choice of $s_0$ in (6.7), it follows that $\|s_0\psi_0 - \rho\| = \|\psi - \rho\|$. Since $s_0c\psi > \psi \geq \rho$, this yields a contradiction. Therefore the claim $s_0c \leq 1$ follows.

We now choose $t$ so that $\max\{1, s_0\} < t < c^{-1}$. Since $t > 1$ and $\psi_0 \in D_\Gamma^*$, $t\psi_0 \in D_\Gamma$. Note also that $s \mapsto \lambda_s\psi_0$ is strictly decreasing on the interval $[s_0, \infty)$. Since $s_0 < t < c^{-1}$ and $c^{-1}\psi_0 = \psi$, we get
\[
\lambda_0 = \lambda_\psi < \lambda_{t\psi_0}.
\]
This contradicts Corollary 6.10, finishing the proof. \qed

7. Non-existence of Positive Eigenfunctions in $L^2(\Gamma \setminus X)$

In this section, we give a different proof of the following theorem:

**Theorem 7.1.** Let $\Gamma < G$ be Anosov. Suppose that $\psi_\Gamma \leq \rho$. Then $L^2(\Gamma \setminus X)$ contains no positive joint eigenfunction with character $\chi_{\psi - \rho}$, $\psi \in D_\Gamma^*$.

Together with Theorems 5.1 and 6.13 and Corollary 4.6, we obtain the following, which shows the implication (1) $\Rightarrow$ (3) of Theorem 1.6.

**Corollary 7.2.** Let $\Gamma < G$ be an Anosov subgroup of the second kind satisfying $\psi_\Gamma \leq \rho$. Then $L^2(\Gamma \setminus X)$ contains no positive harmonic function.

The rest of this section is devoted to a proof of Theorem 7.1. We fix a left $G$-invariant and right $K$-invariant Riemannian metric $d$ on $G$. For a subset $S \subset G$ and $\varepsilon > 0$, the notation $S_\varepsilon$ means $\{s \in S : d(s, e) \leq \varepsilon\}$.

We need the following lemma:

**Lemma 7.3.** There exists $C > 1$ satisfying the following for all $\varepsilon > 0$:

1. For any $a \in A$, $G_\varepsilon a \subset K_{C\varepsilon} a A_{C\varepsilon} N$;
2. For any $a \in A^+$, $a G_\varepsilon \subset K_{C\varepsilon} a A_{C\varepsilon} N$;
3. For any $a \in A^+$, $G_\varepsilon a K_\varepsilon \subset K_{C\varepsilon} a A_{C\varepsilon} N$. 


Proof. Recall that the product map $K \times A \times N \to G$ is a diffeomorphism, and the product map $N^+ \times A \times M \times N \to G$ is a diffeomorphism onto its image which is a Zariski open neighborhood of $e$. It follows that there exists $c_1 > 1$ such that for all $\varepsilon > 0$,

$$N_{c_1^{-1} \varepsilon}^+ A_{c_1^{-1} \varepsilon} M_{c_1^{-1} \varepsilon} N_{c_1^{-1} \varepsilon} \subset G_\varepsilon \subset N_{c_1 \varepsilon}^+ A_{c_1 \varepsilon} M_{c_1 \varepsilon} N_{c_1 \varepsilon}$$

and

$$K_{c_1^{-1} \varepsilon} A_{c_1^{-1} \varepsilon} N_{c_1^{-1} \varepsilon} \subset G_\varepsilon \subset K_{c_1 \varepsilon} A_{c_1 \varepsilon} N_{c_1 \varepsilon}.$$ 

Therefore for $a \in A$, we get

$$G_\varepsilon^a \subset N_{c_1 \varepsilon}^+ a A_{c_1 \varepsilon} M_{c_1 \varepsilon} N_{c_1 \varepsilon}$$

$$\subset K_{c_1 \varepsilon} A_{c_1 \varepsilon} N_{c_1 \varepsilon} a A_{c_1 \varepsilon} M_{c_1 \varepsilon} N_{c_1 \varepsilon}$$

$$\subset K_{(c_1^2 + c_1)} a A_{(c_1^2 + c_1)} N_{c_1 \varepsilon}$$

showing (1). For (2), let $a \in A^+$. Then for all $\varepsilon > 0$, $aN_{\varepsilon}^+ a^{-1} \subset N_{\varepsilon}^+$ and $aA_{\varepsilon} M_{\varepsilon} N_{\varepsilon} \subset A_{\varepsilon} M_{\varepsilon} a N$. Therefore

$$aG_\varepsilon \subset a N_{c_1 \varepsilon}^+ A_{c_1 \varepsilon} M_{c_1 \varepsilon} N_{c_1 \varepsilon}$$

$$\subset N_{c_1 \varepsilon}^+ a A_{c_1 \varepsilon} M_{c_1 \varepsilon} N_{c_1 \varepsilon}$$

$$\subset N_{c_1 \varepsilon}^+ A_{c_1 \varepsilon} M_{c_1 \varepsilon} a N$$

$$\subset G_{c_1 \varepsilon}^a a N$$

$$\subset K_{c_1 \varepsilon} A_{c_1 \varepsilon} N_{c_1 \varepsilon} a N$$

$$\subset K_{c_1 \varepsilon} a A_{c_1 \varepsilon} N,$$

(7.1)

proving (2).

For (3), observe that

$$G_\varepsilon aK_\varepsilon \subset K_{c_1 \varepsilon} A_{c_1 \varepsilon} N_{c_1 \varepsilon} a K_{c_1 \varepsilon}$$

$$\subset K_{c_1 \varepsilon} a A_{c_1 \varepsilon} N_{c_1 \varepsilon} K_{c_1 \varepsilon}$$

$$\subset K_{c_1 \varepsilon} a G_{c_1 \varepsilon}$$

$$\subset K_{c_1 \varepsilon} K_{c_1 \varepsilon} a A_{c_1 \varepsilon} N$$

$$\subset K_{(c_1 + c_1^2)} a A_{c_1^2 \varepsilon} N$$

where the second last inclusion uses (7.1).

For $x \in \Gamma \setminus G/M$, the injectivity radius $\text{inj}(x)$ at $x$ is defined as the supremum $r > 0$ such that for $g \in G$ with $x = [g]$, the $r$-ball $\{ hM : d(hM, gM) < r \}$ around $gM$ injects to $\Gamma \setminus G/M$ under the canonical projection $G/M \to \Gamma \setminus G/M$.

Lemma 7.4. If $\Gamma < G$ is Anosov, then

$$\varepsilon_0 := \inf\{ \text{inj}(x) : x \in \Gamma \setminus G/M \} > 0.$$
Proof. If not, there exist sequences \( g_i \in G \) and \( \gamma_i \in \Gamma \) both tending to \( \infty \) such that \( g_i \gamma_i g_i^{-1} \) tends to an element of \( M \). It follows that the Jordan projection \( \lambda(\gamma_i) \) of \( \gamma_i \) tends to \( e \) as \( i \to \infty \). This is a contradiction to [15, Theorem 1.7] (also see [19]) which implies \( \lambda(\gamma_i) \to \infty \) as \( \gamma_i \to \infty \) in Anosov subgroups.

In the rest of this section, we assume that \( \Gamma < G \) is Anosov and let \( \varepsilon_0 \) be as in Lemma 7.4. We set \( \Lambda(2) := F(2) \cap (\Lambda \times \Lambda) \). We will use a special structure of the quotient \( \Gamma \backslash (\Lambda(2) \times a) \subset \Gamma \backslash (F(2) \times a) = \Gamma \backslash G/M \) for Anosov subgroups.

Let \( \psi \in D_\Gamma^\ast \), and \( u = u_\psi \in \text{int} \mathcal{L}_\Gamma \) be the unique vector such that \( \psi(u) = \psi_\Gamma(u) = 1 \).

**Proposition 7.5.** Let \( g \in G \) satisfy \( g^+, g^- \in \Lambda \). Then for any fixed \( t \in \mathbb{R} \), the subset \( g \exp(tu)(\ker \psi)G_{\varepsilon_0} \) injects to \( \Gamma \backslash G/M \) under the projection \( G/M \to \Gamma \backslash G/M \).

**Proof.** Consider the action of \( \Gamma \) on \( \Lambda(2) \times \mathbb{R} \) given as follows: for \( \gamma \in \Gamma \) and \( (\xi, \eta, t) \in \Lambda(2) \times \mathbb{R} \),

\[
\gamma.(\xi, \eta, tu) = (\gamma \xi, \gamma \eta, tu + \psi(\beta_\xi(\gamma^{-1}, e))u).
\]

The reparametrization theorems for Anosov groups [3, Proposition 4.1] imply that \( \Gamma \) acts properly discontinuously and cocompactly on \( \Lambda(2) \times \mathbb{R}u \). Hence \( Z := \Gamma \backslash (\Lambda(2) \times \mathbb{R}u) \) is a compact space. Now the \( \Gamma \)-equivariant projection \( \Lambda(2) \times a \to \Lambda(2) \times \mathbb{R}u \) given by \( (\xi, \eta, v) \mapsto (\xi, \eta, \psi(v)u) \) induces a principal \( \ker \psi \)-bundle with a global section:

\[
\pi : \Omega = \Gamma \backslash (\Lambda(2) \times a) \to Z = \Gamma \backslash (\Lambda(2) \times \mathbb{R}u).
\]

Hence it is a trivial vector bundle so that we have a \( \ker \psi \)-equivariant homeomorphism:

\[
\Gamma \backslash (\Lambda(2) \times a) \simeq \Gamma \backslash (\Lambda(2) \times \mathbb{R}u) \times \ker \psi.
\] (7.2)

Therefore, the claim follows together with Lemma 7.4. \( \square \)

**Proposition 7.6.** Let \( \psi \in D_\Gamma \) and \( u = u_\psi \). If \( \psi_\Gamma(u) \leq \rho(u) \), then there is no positive joint eigenfunction in \( L^2(\Gamma \backslash X) \) of character \( \chi_{\psi - \rho} \).

**Proof.** Let \( \nu \) be a \( (\Gamma, \psi) \)-conformal measure on \( \mathcal{F} \). Without loss of generality, we may assume that \( e^\pm \in \Lambda \). For a subset \( I \subset \mathbb{R} \), we set \( a_I := \{a_s : s \in I\} \). Let \( \varepsilon := \varepsilon_0/2 \) where \( \varepsilon_0 \) is defined as in Lemma 7.4. Then by Proposition 7.5, for each fixed \( t \in \mathbb{R} \), \( a_{[t-\varepsilon, t+\varepsilon]} \exp(\ker \psi)G_{\varepsilon} \) injects to \( \Gamma \backslash G \).

For any \( t > 0, b \in A \) with \( \log b \in \ker \psi \cap (tu - a^+) \) and \( h \in G_{\varepsilon} \), we claim that

\[
E_\nu(\exp(-tu)bh) \gg \nu([K_\varepsilon])e^{-\psi(tu)}
\] (7.3)

where the implied constant is independent of \( t, b, \varepsilon \). Using Lemma 7.3(3) and the continuity of the Iwasawa projection \( H : G \to a \), we obtain that for
any fixed $\varepsilon > 0$ and for all $a \in A$ with $\log a \in -a^+$,
\[
\inf_{h \in G_x} E_{\nu}(ah) \geq \int_{[k] \in [K_x]} e^{-\psi(H^{-1}a^{-1}(k))} d\nu([k]) \gg \nu([K_x]) e^{\psi(\log a)} \quad (7.4)
\]
where the implied constant is independent of $\varepsilon$ and $a$. If $\log b \in \ker \psi \cap (tu - a^+)$ and $t > 0$, then $\log(\exp(-tu)b) \in -a^+$ and hence the claim $(7.3)$ follows from $(7.4)$.

We now use $(7.3)$ to obtain the following for each $t > \varepsilon_0$,
\[
\int_{\Gamma \setminus G} E^2_{\nu}(g) \, dg \geq \int_{g=a_{-\varepsilon}bh \in a_{-t-\varepsilon,-t+\varepsilon}} \exp(\ker \psi \cap (su-a^+)) G_x E^2_{\nu}(a_{-\varepsilon}bh) \, dg
\]
\[
\gg \nu[K_x] e^{-2\psi(tu)} \int_{g=a_{-\varepsilon}bh \in a_{-t-\varepsilon,-t+\varepsilon}} \exp(\ker \psi \cap (su-a^+)) G_x \, dg
\]
\[
\gg \nu[K_x] e^{-2\psi(tu)} \int_{t+\varepsilon}^{t-\varepsilon} \int_{w \in \ker \psi \cap (a^+ - su)} e^{2\rho(su+w)} \, dw \, ds
\]
\[
\gg \nu[K_x] \int_{w \in \ker \psi \cap (a^+ - tu)} e^{2\rho(w)} \, dw,
\]
since $\rho(u) = \psi(u)$. By the hypothesis $e^+ \in \Lambda$, $\nu([K_x]) > 0$.

On other hand, since $u \in a^+$, for any $v \in a$, we have $v + tu \in a^+$ for all sufficiently large $t \gg 1$. Therefore, in the Gromov-Hausdorff topology on closed subsets of $a$, $a^+ - tu$ converges to $a$ and hence $\ker \psi \cap (a^+ - tu)$ converges to $\ker \psi$ as $i \to \infty$. Since $\rho \geq 0$ on $a^+$, we get
\[
\int_{\Gamma \setminus G} E^2_{\nu}(x) \, dx \gg \limsup_{t \to \infty} \text{Vol}(\ker \psi \cap (a^+ - tu)) = \text{Vol}(\ker \psi) = \infty,
\]
where the volume is computed with respect to the Lebesgue measure on the vector subspace $\ker \psi$. Hence $\int_{\Gamma \setminus G} E^2_{\nu}(x) \, dx = \infty$. \qed

**Proof of Theorem 7.1:** Suppose that there exists a positive harmonic function in $L^2(\Gamma \setminus X)$. By Corollary 4.6, $L^2(\Gamma \setminus X)$ contains a positive joint eigenfunction, which must be of the form $E_{\nu}$ for some $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$ where $\psi \in D^+_F$ satisfies $\psi \geq \max(\psi_T, \rho)$ by Proposition 2.8. For $u = u_\psi$, we get $\psi_T(u) \leq \rho(u)$ by the hypothesis. Therefore the claim follows from Proposition 7.6.

**References**


School of Mathematics, University of Bristol, BS8 1QU, Bristol

Mathematics department, Yale university, New Haven, CT 06520

*Email address: samuel.edwards@bristol.ac.uk*

*Email address: hee.oh@yale.edu*