TEMPEREDNESS OF $L^2(\Gamma\backslash G)$ AND POSITIVE HARMONIC FUNCTIONS IN HIGHER RANK.

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Abstract. Let $G = \text{SO}^\circ(n,1) \times \text{SO}^\circ(n,1)$ and $X = \mathbb{H}^n \times \mathbb{H}^n$ for $n \geq 2$. For a pair $(\pi_1, \pi_2)$ of non-elementary convex cocompact representations of a finitely generated group $\Sigma$ into $\text{SO}^\circ(n,1)$, let $\Gamma = (\pi_1 \times \pi_2)(\Sigma)$. We show:

1. $L^2(\Gamma \backslash G)$ is tempered and $\lambda_0(\Gamma \backslash X) = \frac{1}{2} (n-1)^2$;
2. There exists no positive harmonic function in $L^2(\Gamma \backslash X)$.

In fact, analogues of (1)-(2) hold for any Anosov subgroup $\Gamma$ in the product of two or three simple algebraic groups of rank one as well as for Hitchin subgroups $\Gamma < \text{PSL}_d(\mathbb{R})$ for $d = 3, 4$. Moreover if $G$ is a semisimple real algebraic group of rank at least 2 and has the trivial opposition involution, then (2) holds for any Anosov subgroup $\Gamma$ of $G$.

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1. Introduction

Motivation and background. Let $(\mathbb{H}^n, d)$, $n \geq 2$, denote the $n$-dimensional hyperbolic space of constant curvature $-1$, and let $G = \text{Isom}^+ (\mathbb{H}^n) \simeq \text{SO}^\circ(n,1)$ denote the group of all orientation preserving isometries of $\mathbb{H}^n$. Let $\Gamma < G$ be a torsion-free\footnote{Edwards was supported by funding from the Heilbronn Institute for Mathematical Research and Oh was supported in part by NSF grants.} discrete subgroup. The critical exponent $0 \leq \delta = \delta_\Gamma \leq n - 1$ is defined as the abscissa of convergence of the Poincare...
series $\sum_{\gamma \in \Gamma} e^{-sd(\alpha, \gamma^o)}$ for $o \in \mathbb{H}^n$. We denote by $\Delta$ the hyperbolic Laplacian and by $\lambda_0 = \lambda_0(\Gamma \setminus \mathbb{H}^n)$ the bottom of the $L^2$-spectrum of the negative Laplace operator $-\Delta$, which is given as

$$\lambda_0 := \inf \left\{ \frac{\int_{\Gamma \setminus \mathbb{H}^n} \|\grad f\|^2 \, d \text{vol}}{\int_{\Gamma \setminus \mathbb{H}^n} |f|^2 \, d \text{vol}} : f \in C^\infty_c(\Gamma \setminus \mathbb{H}^n) \right\} \quad (1.1)$$

(see [47, Theorem 2.2]). In a series of papers, Elstrodt ([14], [15], [16]) and Patterson ([36], [37], [38]) developed the relationship between $\delta$ and $\lambda_0$, proving the following theorem for $n = 2$. The general case is due to Sullivan [47, Theorems 2.21].

**Theorem 1.1** (Generalized Elstrodt-Patterson I). For any discrete subgroup $\Gamma < \text{SO}^\circ(n, 1)$, the following are equivalent:

1. $\delta \le \frac{1}{2} (n - 1)$;
2. $\lambda_0 = \frac{1}{4} (n - 1)^2$.

The right translation action of $G$ on the quotient space $\Gamma \setminus G$ equipped with a $G$-invariant measure gives rise to a unitary representation of $G$ on the Hilbert space $L^2(\Gamma \setminus G)$, called a quasi-regular representation of $G$. If we set $K \simeq \text{SO}(n)$ to be a maximal compact subgroup of $G$ and identify $\mathbb{H}^n$ with $G/K$, the space of $K$-invariant functions of $L^2(\Gamma \setminus G)$ can be identified with $L^2(\Gamma \setminus \mathbb{H}^n)$. The bottom of the $L^2$-spectrum $\lambda_0$ then provides information on which complementary series representation of $G$ can occur in $L^2(\Gamma \setminus G)$. Indeed, it follows from the classification of the unitary dual of $\text{SO}^\circ(n, 1)$ that $\lambda_0 = (n - 1)^2/4$ is equivalent to saying that the quasi-regular representation $L^2(\Gamma \setminus G)$ does not contain any complementary series representation (cf. [47], [13]), which is again equivalent to the *temperedness* of $L^2(\Gamma \setminus G)$. As first introduced by Harish-Chandra [22], a unitary representation $(\pi, \mathcal{H}_\pi)$ of a semisimple real algebraic group $G$ is tempered (Definition 2.6) if and only if $\pi$ is weakly contained in a unitary representation $\sigma$ of $G$ (Definition 2.7).

Therefore Theorem 1.1 can be rephrased as follows:

**Theorem 1.2** (Generalized Elstrodt-Patterson II). For any discrete subgroup $\Gamma < G$, the following are equivalent:

1. $\delta \le \frac{1}{2} (n - 1)$;
2. $L^2(\Gamma \setminus G)$ is tempered.

The size of the critical exponent $\delta$ is also related to the existence of square-integrable positive harmonic function on $\Gamma \setminus \mathbb{H}^n$. A smooth function $f$ on $\Gamma \setminus \mathbb{H}^n$ is called harmonic, or more precisely $\lambda$-harmonic, if

$$-\Delta f = \lambda f.$$
A discrete subgroup $\Gamma < G$ is called convex cocompact if there exists a convex subspace of $\mathbb{H}^n$ on which $\Gamma$ acts co-compactly. For convex cocompact subgroups of $G$ (more generally for geometrically finite subgroups), Patterson and Sullivan showed the following using their theory of conformal measures on the boundary $\partial \mathbb{H}^n$ ([39], [48], [47, Theorem 2.21]):

**Theorem 1.3** (Sullivan). For a convex cocompact subgroup $\Gamma < SO^\circ(n,1)$, the following are equivalent:

1. $\delta \leq \frac{1}{2} \left( (n-1)^2 \right)$;
2. There exists no positive harmonic function in $L^2(\Gamma \setminus \mathbb{H}^n)$.

Since $\lambda_0$ divides the positive spectrum and the $L^2$-spectrum on $\Gamma \setminus \mathbb{H}^n$ by Sullivan’s theorem [47, Theorem 2.1] (see Theorem 4.1), (2) is equivalent to saying that any $\lambda_0$-harmonic function on $\Gamma \setminus \mathbb{H}^n$ is not square-integrable.

**Main results.** The main aim of this article is to discuss analogues of Theorems 1.1, 1.2 and 1.3 for a certain class of discrete subgroups of a connected semisimple real algebraic group of higher rank, i.e., rank at least 2.

We begin by describing a special case of our main theorem when $G = SO^\circ(n_1,1) \times SO^\circ(n_2,1)$ with $n_1, n_2 \geq 2$. Let $X$ be the Riemannian product $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ and $\Delta$ the Laplace-Beltrami operator on $X$. A smooth function $f$ on $\Gamma \setminus X$ is called harmonic (resp. $\lambda$-harmonic) if it is an eigenfunction for $\Delta$ (resp. $-\Delta f = \lambda f$). The number $\lambda_0 = \lambda_0(\Gamma \setminus X)$ is given in the same way as (1.1) replacing $\Gamma \setminus \mathbb{H}^n$ by $\Gamma \setminus X$.

**Theorem 1.4.** Let

$$\Gamma = (\pi_1 \times \pi_2)(\Sigma) = \{ (\pi_1(\sigma), \pi_2(\sigma)) \in G : \sigma \in \Sigma \}$$  \hspace{1cm} (1.2)

where $\pi_i : \Sigma \to SO^\circ(n_i,1)$ is a non-elementary convex cocompact representation of a finitely generated group $\Sigma$ for $i = 1, 2$. Then

1. $L^2(\Gamma \setminus G)$ is tempered and $\lambda_0 = \frac{1}{3}((n_1-1)^2 + (n_2-1)^2)$;
2. There exists no positive harmonic function in $L^2(\Gamma \setminus X)$, or equivalently, any $\lambda_0$-harmonic function is not square-integrable.

Even when $\Sigma$ is a surface group and $\pi_1, \pi_2$ are elements of the Teichmüller space $\mathcal{T}(\Sigma)$, this theorem is new.

**Remark 1.5.** Theorem 1.4 does not hold for a general subgroup $\Gamma < G$ of infinite co-volume. For example, if $\Gamma < SO^\circ(n_1,1) \times SO^\circ(n_2,1)$ is the product of two convex cocompact subgroups, each of which having critical exponent greater than $\frac{1}{2} (n_i - 1)$, then $L^2(\Gamma \setminus G)$ is not tempered and $L^2(\Gamma \setminus X)$ possesses a positive harmonic function.

We now discuss a general setting. Let $G$ be a connected semisimple real algebraic group and $(X,d)$ the associated Riemannian symmetric space with the metric $d$ induced by the Killing form on $\mathfrak{g} = \text{Lie} G$. In the rest of the introduction, we assume that $\Gamma < G$ is a torsion-free Zariski dense discrete subgroup. We let $\psi_T : a \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function
of $\Gamma$ as defined in (2.3), where $a$ is the Lie algebra of a maximal real split torus of $G$. The function $\psi_T$ can be regarded as a higher rank generalization of the critical exponent of $\Gamma$. Let $\rho$ denote the half sum of all positive roots for $(g, a^+)$. Analogous to the fact that $\delta \leq n - 1$ for $\Gamma < SO^\circ(n, 1)$, we have the upper bound $\psi_T \leq 2\rho$ [42].

The following Theorem 1.6 generalizes Theorems 1.1, 1.2, and 1.3 to Anosov subgroups (see Definition 2.4). The norm $\|\rho\|$ is defined via the identification $a^*$ and $a$ using the Killing form on $g$. Denote by $\sigma(\Gamma \setminus X)$ the $L^2$-spectrum of $-\Delta$ on $\Gamma \setminus X$.

**Theorem 1.6.** Let $\Gamma < G$ be a Zariski dense Anosov subgroup of $G$. Then the following (1)-(3) are equivalent, and imply (4):

1. $\psi_T \leq \rho$;
2. $L^2(\Gamma \setminus G)$ is tempered and $\lambda_0 = \|\rho\|^2$;
3. $L^2(G)$ and $L^2(\Gamma \setminus G)$ are weakly contained in each other and $\sigma(\Gamma \setminus X) = \sigma(X) = [\|\rho\|^2, \infty)$;
4. There exists no positive harmonic function in $L^2(\Gamma \setminus X)$.

If we change the norm $\|\cdot\|$ by a constant multiple $c\|\cdot\|$, both $\lambda_0$ and $\|\rho\|^2$ change by $c^2$ and hence the equality $\lambda_0 = \|\rho\|^2$ does not depend on the choice of the norm up to a scalar multiple.

The equivalence (1) $\Leftrightarrow$ (2) is based on the asymptotic behavior of the Haar matrix coefficients for compactly supported continuous functions for Anosov subgroups obtained in [12], using [9]. The equivalence (2) $\Leftrightarrow$ (3) uses the observation that $L^2(G)$ is weakly contained in $L^2(\Gamma \setminus G)$ whenever the injectivity radius of $\Gamma \setminus G$ is infinite, and $\Gamma \setminus G$ has infinite injectivity radius for any Anosov subgroup $\Gamma < G$ except for cocompact lattices of a rank one Lie group. The implication (1) $\Rightarrow$ (4) is based on the study of $\Gamma$-conformal measures and joint eigenfunctions for the whole ring of $G$-invariant differential operators and extension of Sullivan-Thuston’s smearing argument to higher rank groups.

Although the condition $\psi_T \leq \rho$ may appear quite strong, it was verified in a recent work of Kim-Minsky-Oh ([26], Theorem 9.5) for Anosov subgroups in the following setting (see Corollary 9.7 for the treatment of non-Zariski dense groups):

**Theorem 1.7.** Let $\Gamma$ be an Anosov subgroup of the product $G$ of two or three simple real algebraic groups of rank one, or a Hitchin subgroup of $\text{PSL}_d(\mathbb{R})$ for $d = 3, 4$. Then (1)-(4) of Theorem 1.6 hold.

We do not know of any Anosov subgroup of a higher rank semisimple real algebraic group which does not satisfy the condition $\psi_T \leq \rho$.

**Groups with trivial opposition involution.** The opposition involution $i : a \to a$ is defined by

$$i(u) = -\text{Ad}_{w_0}(u),$$

(1.3)
where $w_0$ is a Weyl element such that $\text{Ad}_{w_0}a^+ = -a^+$. The opposition involution is trivial in the product of any rank one simple algebraic groups, as well as in the groups $\text{SO}(p,q)$ and $\text{Sp}(2n,\mathbb{R})$. See Corollary 7.3 for a slightly more general version:

**Theorem 1.8.** Let $G$ be a connected semisimple real algebraic group with rank $G \geq 2$ and trivial opposition involution. For any Zariski dense Anosov subgroup $\Gamma < G$, there exists no positive harmonic function in $L^2(\Gamma \backslash X)$.

The proof of Theorem 1.8 is based on a higher rank version of the smearing argument of Sullivan-Thurston (Theorem 7.5).

**Organization:** In section 2, we review the basic notions and notations which will be used throughout the paper.

In section 3, we show that any positive joint eigenfunction on $\Gamma \backslash X$ (i.e., eigenfunction for the whole ring of $G$-invariant differential operators) arises from a $(\Gamma, \psi)$-conformal density (Proposition 3.7).

In section 4, we compute the Laplace eigenvalue of a positive joint eigenfunction associated to a $(\Gamma, \psi)$-conformal measure (Proposition 4.2).

In section 5, we introduce the notion of subgroups of the second kind. We then construct positive joint eigenfunctions for any $\psi \geq \psi_\Gamma$ for any subgroup of the second kind with $\mathcal{L} \subset \text{int} a^+ \cup \{0\}$ (Theorem 5.2).

In section 6, we compute the $L^2$-spectrum of $X$ (Theorem 6.3) and show that $\lambda_0 = \|\rho\|^2$ if $L^2(\Gamma \backslash G)$ is tempered (Theorem 6.4). We show that a positive harmonic function in $L^2(\Gamma \backslash X)$ is necessarily a joint eigenfunction (Corollary 6.6) and a spherical vector of a unique irreducible subrepresentation of $L^2(\Gamma \backslash G)$ (Theorem 6.8).

In section 7, we use Sullivan-Thurston’s smearing arguments to obtain non-existence theorem of $L^2$-positive harmonic functions.

In section 8, we prove the weak containment $L^2(G) \preceq L^2(\Gamma \backslash G)$ for all Anosov subgroups $\Gamma$ in higher rank groups.

In section 9, we prove the equivalence of the temperedness of $L^2(\Gamma \backslash G)$ and $\psi_\Gamma \leq \rho$ (Theorem 9.4). We also explain how to deduce Theorem 1.6.

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2. Preliminaries and notations

Let $G$ be a connected semisimple real algebraic group, i.e., the identity component of the group of real points of a semisimple algebraic group defined over $\mathbb{R}$. Let $P$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition $P = MAN$ where $A$ is a maximal real split torus of $G$, $M$ is the compact subgroup, which is the centralizer of $A$, and $N$ is the unipotent radical of $P$. We denote by $\mathfrak{g}, \mathfrak{a}, \mathfrak{n}$ respectively the Lie algebras of $G, A, N$. 
We fix a positive Weyl chamber \( \mathfrak{a}^+ \subset \mathfrak{a} \) so that \( \mathfrak{n} \) consists of positive root subspaces. Let \( \Sigma \) (resp. \( \Sigma^+ \)) denote the set of all (resp. positive) roots for \((\mathfrak{g}, \mathfrak{a}^+)\). We also write \( \Pi \subset \Sigma^+ \) for the set of all simple roots. We denote by \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha \) the half sum of the positive roots for \((\mathfrak{g}, \mathfrak{a}^+)\). We denote by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \) the inner product and norm on \( \mathfrak{g} \) respectively, induced from the Killing form: \( B(x, y) = \text{Tr}(\text{ad} x \text{ ad}(y)) \) for \( x, y \in \mathfrak{g} \).

We fix a maximal compact subgroup \( K \) of \( G \) so that the Cartan decomposition \( G = K(\exp \mathfrak{a}^+)K \) holds, that is, for any \( g \in G \), there exists a unique element \( \mu(g) \in \mathfrak{a}^+ \) such that \( g \in K \exp \mu(g) K \). We call the map \( \mu : G \to \mathfrak{a}^+ \) the Cartan projection map.

The opposition involution \( i : \mathfrak{a} \to \mathfrak{a} \) is defined by \( i(u) = -\text{Ad}_{w_0}(u) \) for all \( u \in \mathfrak{a} \).

The Riemannian symmetric space \((X, \mathcal{d})\) can be identified with the quotient space \( G/K \) with the metric \( \mathcal{d} \) induced from \( \| \cdot \| \). We denote by \( \mathcal{d}\text{vol} \) the Riemannian volume form on \( X \). We also use \( dx \) to denote this volume form as well as the Haar measure on \( G \), or on \( \Gamma \setminus G \). We set \( o = [K] \in X \).

We then have \( \| \mu(g) \| = \mathcal{d}(go, o) \) for \( g \in G \). We do not distinguish a function on \( X \) and a right \( K \)-invariant function on \( G \). Let \( \mathcal{F} := G/P \) denote the Furstenberg boundary of \( G \).

For each \( g \in G \), we define the following visual maps:

\[
g^+ := gP \in \mathcal{F} \quad \text{and} \quad g^- := gw_0P \in \mathcal{F}.
\] (2.1)

The unique open \( G \)-orbit \( \mathcal{F}^{(2)} \) in \( \mathcal{F} \times \mathcal{F} \) under the diagonal \( G \)-action is given by:

\[
\mathcal{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}.
\]

Two points \( \xi, \eta \) in \( \mathcal{F} \) are said to be in general position if \( (\xi, \eta) \in \mathcal{F}^{(2)} \).

**Conformal measures.** Let \( G = KAN \) be the Iwasawa decomposition, \( \kappa : G \to K \) the \( K \)-factor projection of this decomposition, and \( H : G \to \mathfrak{a} \) be the Iwasawa cocycle defined by the relation:

\[
g \in \kappa(g) \exp (H(g)) N.
\]

Note that \( K \) acts transitively on \( \mathcal{F} \) and \( K \cap P = M \), and hence we may identify \( \mathcal{F} \) with \( K/M \). The Iwasawa decomposition can be used to describe both the action of \( G \) on \( \mathcal{F} = K/M \) and the \( \mathfrak{a} \)-valued Busemann map as follows: for all \( g \in G \) and \( [k] \in \mathcal{F} \) with \( k \in K \),

\[
g \cdot [k] = [\kappa(gk)],
\]

and the \( \mathfrak{a} \)-valued Busemann map is defined by

\[
\beta_{[k]}(g(o), h(o)) := H(g^{-1}k) - H(h^{-1}k) \in \mathfrak{a} \quad \text{for all} \ g, h \in G.
\]

**Definition 2.1.** Let \( \psi \in \mathfrak{a}^* \), and let \( \Gamma < G \) be a closed subgroup.
(1) A finite Borel measure $\nu$ on $F = K/M$ is said to be a $(\Gamma, \psi)$-conformal measure (for the basepoint $o$) if for all $\gamma \in \Gamma$ and $\xi = [k] \in K/M$, 
$$
\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{-\psi(\beta_k(\gamma o, o))} = e^{-\psi(H(\gamma^{-1}k))},
$$
or equivalently
$$
d\nu([k]) = e^\psi(H(\gamma k)) \, d\nu(\gamma [k]).
$$

(2) A collection $\{\nu_x : x \in X\}$ of finite Borel measures on $F$ is called a $(\Gamma, \psi)$-conformal density if for all $x, y \in X$, $\xi \in F$ and $\gamma \in \Gamma$,
$$
\frac{d\nu_x}{d\nu_y}(\xi) = e^{-\psi(\beta_k(x, y))} \quad \text{and} \quad d\gamma_* \nu_x = d\nu(\gamma x).
$$

A $(\Gamma, \psi)$-conformal measure $\nu$ defines a $(\Gamma, \psi)$-conformal density $\{\nu_x : x \in X\}$ by the formula:
$$
d\nu_x(\xi) = e^{-\psi(\beta_k(x, o))} \, d\nu(\xi),
$$
and conversely any $(\Gamma, \psi)$-conformal density $\{\nu_x\}$ is uniquely determined by its member $\nu_o$ by (2.2).

**Growth indicator function.** Let $\Gamma < G$ be a Zariski dense closed subgroup. Following Quint [42], let $\psi_\Gamma : a \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$: for any non-zero $v \in a$,
$$
\psi_\Gamma(v) := \|v\| \inf_{v \in \mathcal{C}} \tau_C,
$$
where the infimum is over all open cones $\mathcal{C}$ containing $v$ and $\tau_C$ denotes the abscissa of convergence of the series $\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|}$. For $v = 0$, we let $\psi_\Gamma(0) = 0$. We note that $\psi_\Gamma$ does not change if we replace the norm $\|\cdot\|$ on $g$ by a constant multiple in its definition. For any discrete group $\Gamma$, we have the upper bound $\psi_\Gamma \leq 2\rho$ [42].

**Limit cone and limit set.** The limit cone $\mathcal{L} = \mathcal{L}_\Gamma$ of $\Gamma$ is defined as the asymptotic cone of $\mu(\Gamma)$, i.e.,
$$
\mathcal{L} = \{\lim t_i \mu(\gamma_i) : t_i \to 0, \gamma_i \in \Gamma\}.
$$

For $\Gamma$ Zariski dense, $\mathcal{L}$ is a convex cone with non-empty interior [3]. Quint [42] showed that $\psi_\Gamma$ is a concave and upper-semicontinuous function such that $\psi_\Gamma \geq 0$ on $\mathcal{L}$, $\psi_\Gamma > 0$ on int $\mathcal{L}$ and, $\psi_\Gamma = -\infty$ outside $\mathcal{L}$.

For a sequence $g_i \to G$, we write $g_i \to \infty$ regularly if $\alpha(\mu(g_i)) \to \infty$ for all $\alpha \in \Pi$. For $g \in G$, we write $g = \kappa_1(g) \exp(\mu(g))\kappa_2(g) \in KA^+K$; if $\mu(g) \in \text{int} a^+$, then $[\kappa_1(g)] \in K/M = F$ is well-defined.

**Definition 2.2.** A sequence $p_i \in X$ is said to converge to $\xi \in F$ if there exists $g_i \to \infty$ regularly in $G$ with $p_i = g_i(o)$ and $\lim_{i \to \infty} [\kappa_1(g_i)] = \xi$. 
We denote by $\Lambda \subset \mathcal{F}$ the limit set of $\Gamma$, which is defined as

$$\Lambda = \{ \lim \gamma_i(o) : \gamma_i \in \Gamma \}. \quad (2.4)$$

For $\Gamma < G$ Zariski dense, this is the unique $\Gamma$-minimal subset of $\mathcal{F}$ ([3], [33]).

**Tangent linear forms.** We set

$$D_\Gamma = \{ \psi \in \mathfrak{a}^* : \psi \geq \psi_T \}. \quad (2.5)$$

A linear form $\psi \in \mathfrak{a}^*$ is said to be tangent to $\psi_T$ at $u \in \mathfrak{a}$ if $\psi \in D_\Gamma$ and $\psi(u) = \psi_T(u)$. We denote by $D_\Gamma^\star$ the set of all linear forms tangent to $\psi_T$ at $\mathcal{L} \cap \text{int } \mathfrak{a}^+$, i.e.,

$$D_\Gamma^\star := \{ \psi \in D_\Gamma : \psi(u) = \psi_T(u) \text{ for some } u \in \mathcal{L} \cap \text{int } \mathfrak{a}^+ \}. \quad (2.6)$$

For $\Gamma < SO^o(n,1)$ and $\delta$ its critical exponent, we have $D_\Gamma^\star = \{ \delta \}$ and $D_\Gamma = \{ s \geq \delta \}$.

Extending the construction of Patterson [39] and Sullivan [46], Quint [41] showed the following:

**Theorem 2.3.** For any $\psi \in D_\Gamma^\star$, there exists a $(\Gamma,\psi)$-conformal measure supported on $\Lambda$.

**Anosov subgroups.** A loxodromic element $g \in G$ is of the form $g = hamh^{-1}$ for some $a \in \text{int } \mathfrak{a}^+$, $m \in \mathbb{M}$ and $h \in G$. Then $hP \in \mathcal{F}$ is then called the attracting fixed point of $g$.

**Definition 2.4.** ([21], [24], [19]) A closed subgroup $\Gamma < G$ is called an Anosov subgroup (with respect to $P$) if $\Gamma$ can be realized as the image of an Anosov representation $\pi : \Sigma \to G$ of a finitely generated Gromov hyperbolic group $\Sigma$. Denoting by $\partial \Sigma$ the Gromov boundary of $\Sigma$, an Anosov representation $\pi : \Sigma \to G$ is a representation satisfying the following:

1. $\pi$ induces a continuous equivariant map $\zeta : \partial \Sigma \to \mathcal{F}$ such that $(\zeta(\xi_1), \zeta(\xi_2)) \in \mathcal{F}^{(2)}$ for all $\xi_1 \neq \xi_2$;
2. for all $\sigma \in \Sigma$ of infinite order, $\pi(\sigma)$ is a loxodromic element and $\zeta$ maps the attracting fixed point of $\sigma$ to that of $\pi(\sigma)$;
3. For all $\alpha \in \Pi$,

$$\lim_{\sigma \to \infty} \langle \alpha, \mu(\pi(\sigma)) \rangle = \infty.$$

If $\Gamma$ is Zariski dense, the conditions (2) and (3) are consequences of (1) [21]. We remark that the discreteness of an Anosov subgroup $\Gamma$ is a consequence of the property (3) of an Anosov representation.

Anosov subgroups of $G$ was first introduced by Labourie for surface groups [30], and then extended by Guichard and Wienhard [21] to general word hyperbolic groups. When $G$ has rank one, the class of Anosov subgroups coincides with that of convex cocompact subgroups and when $G$ is a product of two rank one simple algebraic groups, any Anosov subgroup arises in a similar fashion to (1.2). Examples of Anosov subgroups include Schottky groups as well as Hitchin subgroups.
Hitchin subgroups. Let \( \iota_d \) denote the irreducible representation \( \PSL_2(\mathbb{R}) \to \PSL_d(\mathbb{R}) \). A Hitchin subgroup is the image of a representation \( \pi : \Sigma \to \PSL_d(\mathbb{R}) \) of a uniform lattice \( \Sigma < \PSL_2(\mathbb{R}) \), which belongs to the same connected component as \( \iota_d|\Sigma \) in the character variety \( \Hom(\Sigma, \PSL_d(\mathbb{R}))/\sim \) where the equivalence is given by conjugations.

One of the important features of an Anosov subgroup is the following:

**Theorem 2.5.** [40] For any Anosov subgroup \( \Gamma < G \), we have \( \mathcal{L} \subset \text{int} \ a^+ \cup \{0\} \).

Tempered representations. By definition, a unitary representation of \( G \) is a Hilbert space \( H_\pi \) equipped with a strongly continuous homomorphism \( \pi \) from \( G \) to the group of unitary operators on \( H_\pi \). Given two unitary representations \( \pi \) and \( \sigma \) of \( G \), \( \pi \) is said to be weakly contained in \( \sigma \) if any diagonal matrix coefficients of \( \pi \) can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of \( \sigma \). We use the notation \( \pi \propto \sigma \) for the weak containment.

The Harish-Chandra function \( \Xi_G : G \to (0, \infty) \) is a bi-\( K \)-invariant function defined via the formula

\[
\Xi_G(g) = \int_K e^{-\rho(H(gk))} dk \quad \text{for all } g \in G.
\]

The following estimate is well-known, cf. e.g. [27]: for any \( \varepsilon > 0 \), there exist \( C, C_\varepsilon > 0 \) such that for any \( g \in G \),

\[
Ce^{-\rho(\mu(g))} \leq \Xi_G(g) \leq C_\varepsilon e^{-(1-\varepsilon)\rho(\mu(g))}. \tag{2.7}
\]

**Definition 2.6.** A unitary representation \((\pi, \mathcal{H}_\pi)\) of \( G \) is called tempered if for any \( K \)-finite unit vectors \( v, w \in \mathcal{H}_\pi \) and any \( g \in G \)

\[
|\langle \pi(g)v, w \rangle| \leq (\dim(Kv) \dim(Kw))^{1/2} \Xi_G(g)
\]

where \( \langle Kv \rangle \) denotes the linear subspace of \( \mathcal{H}_\pi \) spanned by \( Kv \).

**Proposition 2.7.** ([11], also see [35, Theorem 2.4]) The following are equivalent for a unitary representation \((\pi, \mathcal{H}_\pi)\) of \( G \):

1. \( \pi \) is tempered;
2. \( \pi \propto L^2(G) \);
3. for any vectors \( v, w \in \mathcal{H}_\pi \), the matrix coefficient \( g \mapsto \langle \pi(g)v, w \rangle \) lies in \( L^{2+\varepsilon}(G) \) for any \( \varepsilon > 0 \);
4. for any \( \varepsilon > 0 \), \( \pi \) is strongly \( L^{2+\varepsilon} \), i.e., there exists a dense subset of \( \mathcal{H}_\pi \) whose matrix coefficients all belong to \( L^{2+\varepsilon}(G) \).

3. Positive joint eigenfunctions and conformal densities

Let \( \Gamma < G \) be a Zariski dense discrete subgroup. The main goal of this section is to obtain Proposition 3.7, which explains the relationship between positive joint eigenfunctions on \( \Gamma \backslash X \) and \( \Gamma \)-conformal measures on the Furstenberg boundary of \( G \).
Joint eigenfunctions on $X$. Let $D = D(X)$ denote the ring of all $G$-invariant differential operators on $X$. We call a real valued function on $X$ a joint eigenfunction if it is an eigenfunction for all operators in $D$. For each joint eigenfunction $f$, there exists an associated character $\chi_f : D \to \mathbb{R}$ such that 

$$Df = \chi_f(D)f$$

for all elements $D \in D$. The ring $D$ is generated by rank($G$) elements, and the set of all characters of $D$ is in bijection with the space $a^* = \text{Hom}_\mathbb{R}(a, \mathbb{R})$ modulo the action of the Weyl group, as we now explain. Denote by $Z(g_C)$ the center of the universal enveloping algebra $U(g_C)$ of $g_C$. Recall the well-known fact that the joint eigenfunctions on $X$ can be identified with the right $K$-invariant real-valued $Z(g_C)$-eigenfunctions on $G$ (cf. [23]).

Letting $T$ be a maximal torus in $M$ with Lie algebra $t$, set $\mathfrak{h} = (a \oplus t)$. Then $\mathfrak{h}_C := (a \oplus t)_C$ is a Cartan subalgebra of $g_C$. We let 

$$\imath : Z(g_C) \to S^W(\mathfrak{h}_C)$$

denote the Harish-Chandra isomorphism from $Z(g_C)$ to the Weyl group-invariant elements of the symmetric algebra $S(\mathfrak{h}_C)$ of $\mathfrak{h}$ [27, Theorem 8.18].

For any $\psi \in a^*$, we can extend it to $\mathfrak{h}$ by letting $\psi(J) = 0$ for all $J \in \mathfrak{m}$, and then to $S(\mathfrak{h}_C)$ polynomially. This lets us define a character $\chi_\psi$ on $Z(g_C)$ by

$$\chi_\psi(Z) := \psi(\imath(Z)) \quad (3.1)$$

for all $Z \in Z(g_C)$. Conversely, if $f$ is a right $K$-invariant $Z(g_C)$-eigenfunction, then, since $t$ acts trivially on $f$, the associated character $\chi_f$ must arise as $\psi \circ \imath$ for some $\psi \in a^*$.

Example 3.1.

- Consider the hyperbolic space $\mathbb{H}^n = \{(x_1, \cdots, x_n y) \in \mathbb{R}^{n+1} : y > 0\}$ with the metric $\sqrt{\sum_{i=1}^n dx_i^2 + dy^2}$. The Laplacian $\Delta$ on $\mathbb{H}^n$ is $\Delta = -y^2(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2})$ and the ring of $SO^+(n, 1)$-invariant differential operators is generated by $\Delta$, i.e., a polynomial in $\Delta$. If $\psi \in a^*$ is given by $\psi(v) = \delta v$ for some $\delta \in \mathbb{R}$ under the isomorphism $a = \mathbb{R}$, then $\chi_\psi(-\Delta) = \delta(n - 1 - \delta)$.

- Let $G = SO^+(n_1, 1) \times SO^+(n_2, 1)$ and $X$ be the Riemannian product $\mathbb{H}^{n_1} \times \mathbb{H}^{n_2}$ for $n_1, n_2 \geq 2$. Then $D(X)$ is generated by the hyperbolic Laplacians $\Delta_1, \Delta_2$ on each factor $\mathbb{H}^{n_1}$ and $\mathbb{H}^{n_2}$. If we identify $a$ with $\mathbb{R}^2$ and if a linear form $\psi \in a^*$ is given by $\psi(v) = \langle v, (\delta_1, \delta_2) \rangle$ for some vector $(\delta_1, \delta_2) \in \mathbb{R}^2$, then $\chi_\psi(-\Delta_i) = \delta_i(n_i - 1 - \delta_i)$ for $i = 1, 2$.

Joint eigenfunctions on $\Gamma \backslash X$. We now consider joint eigenfunctions on $\Gamma \backslash X$ or, equivalently, right $\Gamma$-invariant joint eigenfunctions on $X$.

Definition 3.2. Let $\psi \in a^*$. Associated to a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, we define the following function $E_\nu$ on $G$: for $g \in G$,

$$E_\nu(g) := |\nu_{g(o)}| = \int_{\mathcal{F}} e^{-\psi(H(g^{-1}k))} \, d\nu(|k|). \quad (3.2)$$
Since $|\nu_{\gamma(x)}| = |\nu_x|$ for all $\gamma \in \Gamma$ and $x \in X$, the left $\Gamma$-invariance and right $K$-invariance of $E_\nu$ are clear. Hence we may consider $E_\nu$ as a $K$-invariant function on $\Gamma \backslash G$, or, equivalently, as a function on $\Gamma \backslash X$.

**Proposition 3.3.** For each $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, $E_\nu$ is a positive joint eigenfunction with character $\chi_\psi - \rho$. Conversely, any positive joint eigenfunction on $\Gamma \backslash X$ arises in this way with $(\psi, \nu)$ uniquely determined.

In order to prove this proposition, we consider the following right $K$-invariant function on $G$ for each $\psi \in a^*$ and $h \in G$:

$$\varphi_{\psi,h}(g) = e^{-\psi(H(g^{-1}h))} \quad (3.3)$$

so that

$$E_\nu(g) = \int_{\mathcal{F}} \varphi_{\psi,k}(g) d\nu([k]).$$

We may also consider $\varphi_{\psi,h}$ as a function on $X$. Hence the first part of Proposition 3.3 is a consequence of the following:

**Lemma 3.4.** ([27, Propositions 8.22 and 9.9]) For any $\psi \in a^*$ and $h \in G$, the function $\varphi_{\psi,h}$ is a joint eigenfunction on $X$ with character $\chi_\psi - \rho$.

**Proof.** While we refer to [27] for the full proof, we outline some of the key points below, as we will use some part of this proof later. Since the elements of $Z(gC)$ commute with translation, we simply need to prove that

$$[Z \varphi_{\psi,e}](e) = \chi_{\psi - \rho}(Z) \varphi_{\psi,e}(e) \quad \text{for any } Z \in Z(gC);$$

the same identity will then hold for the function $g \mapsto \varphi_{\psi,e}(h^{-1}g)$, and thus also for $\varphi_{\psi,h}$ for any $h \in G$. Following [27, Chapter VII], we define the (non-unitary) principal series representation $U^\psi$

$$[U^\psi(g)f](k) := e^{-\psi(H(g^{-1}k))} f(\kappa(g^{-1}k))$$

for all $g \in G$, $k \in K$, and $f \in C(K)$. This extends to a representation $dU^\psi$ of $\mathcal{U}(gC)$ on the right $M$-invariant functions in $C^\infty(K)$ by way of the formula

$$[dU^\psi(X)f](k) = \frac{d}{dt} \Big|_{t=0} [U^\psi(\exp(tX))f](k) \quad \text{for any } X \in g.$$
where $Y \in \mathcal{U}(\mathfrak{h}_C)$, $X_i \in \mathfrak{n}$, and $U_i \in \mathcal{U}(\mathfrak{g}_C)$. Note that in this decomposition, $Y$ is uniquely defined. Now, for arbitrary $Y \in \mathfrak{n}$ and $f$,

\[
[dU^\psi(X)f](e) = \frac{d}{dt} \bigg|_{t=0} [U^\psi(\exp(tX))f](e) = \frac{d}{dt} \bigg|_{t=0} [U^\psi(\exp(tX))](e)
\]

so applying this to the $X_i$ and functions $dU^\psi(U_i)f$ gives

\[
[dU^\psi(X_iU_i)f](e) = [dU^\psi(X_i)(dU^\psi(U_i)f)](e) = 0,
\]

hence $[dU^\psi(Z)f](e) = [dU^\psi(Y)f](e)$. For $L \in \mathfrak{m}$, we have $f(\exp(-L)) = f(e)$, so $[dU^\psi(J)f](e) = 0$ for all $J \in \mathfrak{t}$. Thus, it is only the $a$ component of $Y$ that contributes to $[dU^\psi(Y)f](e)$. Finally, note that for $X \in \mathfrak{a}$, we have

\[
[dU^\psi(X)f](e) = \frac{d}{dt} \bigg|_{t=0} e^{-\psi(H(\exp(-tX)))} f(\kappa(\exp(-tX))) = \frac{d}{dt} \bigg|_{t=0} e^{\psi(X)} f(e) = \psi(X)f(e).
\]

Since the Harish-Chandra isomorphism consists of projection onto $\mathcal{U}(\mathfrak{h}_C)$ and then composition with the “$\delta$-shift” $H \mapsto H + \delta(H)1 = H + \rho(H)1$, this shows that $dU^\psi(Z) = \chi_{\psi-\rho}(Z)$.

Letting $h = kan \in KAN$, we see that for any $g \in G$,

\[
\varphi_{\psi,h}(g) = e^{-\psi(H(g^{-1}h))} = e^{-\psi(H(g^{-1}kan))} = e^{-\psi(H(g^{-1}k))} \cdot e^{-\psi(\log(a))},
\]

i.e., the function $\varphi_{\psi,h}$ is a scalar multiple of $\varphi_{\psi,\kappa(h)}$. In fact, the functions $\varphi_{\psi,k}, k \in K$ form a complete set of minimal positive joint eigenfunctions with character $\chi_{\psi-\rho}$ with $\psi \geq \rho$, in the sense that if $f$ is a positive joint eigenfunction on $X$ with character $\chi_{\psi-\rho}$ such that $f \leq \varphi_{\psi,k}$ for some $k \in K$, then

\[
f = c \cdot \varphi_{\psi,k}
\]

for some $c > 0$ (cf. [18, 25], see also [30, Theorem 1]).

As a consequence, we have the following (cf. [30, Theorem 3]):

**Theorem 3.5.** For any positive joint eigenfunction $f$ on $X$, there exist $\psi \in \mathfrak{a}^*$ with $\psi \geq \rho$ and a Borel measure $\nu$ on $\mathcal{F} = K/M$ such that for all $g \in G$,

\[
f(g) = \int_{\mathcal{F}} \varphi_{\psi,k}(g) \, d\nu([k]).
\]

Moreover, the pair $(\psi, \nu)$ is uniquely determined by $f$.

**Proof of the second part of Proposition 3.3:** Let $f$ be a $\Gamma$-invariant joint eigenfunction on $X$. By Theorem 3.5, there exist unique $\psi \in \mathfrak{a}^*$ and a Borel measure $\nu$ on $\mathcal{F}$ so that for all $g \in G$,

\[
f(g) = \int_{\mathcal{F}} \varphi_{\psi,k}(g) \, d\nu([k]).
\]
Since $f$ is $\Gamma$-invariant, for any $\gamma \in \Gamma$,
\[
    f(g) = f(\gamma g) = \int_{\mathcal{F}} \varphi_{\psi,k}(\gamma g) \, d\nu([k])
    = \int_{\mathcal{F}} \varphi_{\psi,k}(\gamma^{-1}k)(g) e^{-\psi(H(\gamma^{-1}k))} \, d\nu([k])
    = \int_{\mathcal{F}} \varphi_{\psi,k}(\gamma^{-1}k)(g) e^{\psi(H(\gamma^{-1}k))} \, d\nu([k]).
\]
By the uniqueness of $\nu$ in the integral representation of $f$,
\[
    d\nu([k]) = e^{\psi(H(\gamma k))} \, d\nu(\gamma \cdot [k]),
\]
i.e. $\nu$ is a $(\Gamma, \psi)$-conformal measure on $\mathcal{F}$, finishing the proof.

We denote by $\psi_\Gamma : a \to \mathbb{R} \cup \{-\infty\}$ the growth indicator function of $\Gamma$ as defined in (2.3).

**Theorem 3.6.** [41, Theorem 8.1]. Let $\Gamma < G$ be Zariski dense. If there exists a $(\Gamma, \psi)$-conformal measure on $\mathcal{F}$ for some $\psi \in a^*$, then
\[
    \psi \geq \psi_\Gamma.
\]

Therefore Proposition 3.3 and Theorem 3.6 yield the following:

**Proposition 3.7.** Let $\Gamma < G$ be a Zariski dense discrete subgroup. If $\nu$ is a $(\Gamma, \psi)$-conformal measure on $\mathcal{F}$ for some $\psi \in a^*$, then $E_\nu$ is a positive joint eigenfunction on $\Gamma \setminus X$ with character $\chi_{\psi-\rho}$. Conversely, any positive joint eigenfunction on $\Gamma \setminus X$ is of the form $E_\nu$ for some $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$ with $\psi \geq \max(\rho, \psi_\Gamma)$, with $(\psi, \nu)$ uniquely determined.

### 4. Eigenvalues of Positive Harmonic Functions

Let $\Gamma$ be a torsion-free discrete subgroup of a connected semisimple real algebraic group $G$. Let $\Delta$ denote the Laplace-Beltrami operator on $X$ or on $\Gamma \setminus X$. A function $f$ on the Riemannian manifold $\Gamma \setminus X$ is said to be harmonic if it is an eigenfunction of $\Delta$. Since $\Delta$ is an elliptic differential operator, a harmonic function is always smooth. We call a harmonic function $\lambda$-harmonic if
\[
    -\Delta f = \lambda f.
\]

Let $\mathcal{C} \in Z(g_{\mathbb{C}})$ denote the Casimir operator on $C^\infty(G)$ (or on $C^\infty(\Gamma \setminus G)$) whose restriction to $K$-invariant functions coincides with $\Delta$. Then $K$-invariant $\mathcal{C}$-eigenfunctions on $\Gamma \setminus G$ correspond to harmonic functions on $\Gamma \setminus X$. In particular, a joint eigenfunction of $\Gamma \setminus X$ is a harmonic function.

Define the real number $\lambda_0 = \lambda_0(\Gamma \setminus X) \in \mathbb{R}$ as follows:
\[
    \lambda_0 := \inf \left\{ \frac{\int_{\Gamma \setminus X} ||\nabla f||^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |f|^2 \, d\text{vol}} : f \in C^\infty_c(\Gamma \setminus X), \; f \neq 0 \right\}
\]
where $d\text{vol}$ denotes the Riemannian volume form on $\Gamma \setminus X$. 

Positive harmonic functions.

**Theorem 4.1.** [47, Theorem 2.1, 2.2] Suppose that $\Gamma \backslash X$ is not compact.

1. For any $\lambda \leq \lambda_0$, there exists a positive $\lambda$-harmonic function on $\Gamma \backslash X$;

2. For any $\lambda > \lambda_0$, there is no positive $\lambda$-harmonic function on $\Gamma \backslash X$;

We identify $a^*$ with $a$ via the inner product on $a$ induced by the Killing form on $g$. This endows an inner product on $a^*$. More precisely, for each $\psi \in a^*$, there exists a unique $v_\psi \in a$ such that $\psi = \langle v_\psi, \cdot \rangle$. Then $\langle \psi_1, \psi_2 \rangle = \langle v_{\psi_1}, v_{\psi_2} \rangle$. Equivalently, fix an orthonormal basis $\{H_i\}$ of $a$. Then $\langle \psi_1, \psi_2 \rangle = \sum_i \psi_1(H_i) \psi_2(H_i)$.

For $\psi \in a^*$, we set

$$\lambda_\psi := (\|\rho\|^2 - \|\psi - \rho\|^2). \quad (4.2)$$

**Proposition 4.2.**

1. A positive joint eigenfunction on $X$ with character $\chi_{\psi - \rho}$, $\psi \in a^*$, is $\lambda_\psi$-harmonic.

2. A positive harmonic function on $X$ is $\lambda_\psi$-harmonic for some $\psi \in a^*$ with $\psi \geq \rho$.

*Proof.* Let $\psi \in a^*$. Recall the functions $\varphi_{\psi,h}$ in (3.3). By Theorem 3.5, (1) follows if we show that for any $h \in G$,

$$-C\varphi_{\psi,h} = \lambda_\psi \varphi_{\psi,h}. \quad (4.3)$$

Let $\{H_i\}$ be an orthonormal basis of $a$. To each $\alpha \in \Sigma$ corresponds $H_\alpha \in a$ with $\alpha(x) = B(x, H_\alpha) = \langle x, H_\alpha \rangle$ for all $x \in a$. For each $\alpha \in \Sigma$, choose a root vector $E_\alpha \in g$ so that $[x, E_\alpha] = \alpha(x)E_\alpha$ for all $x \in a$. We may write

$$C = \sum_i H_i^2 + \sum_{\alpha \in \Sigma^+} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha) + J,$$

where $J \in U(m_C)$ (cf. [28, Proposition 5.28]). Now using $E_{-\alpha} E_\alpha = E_\alpha E_{-\alpha} - H_\alpha$ gives

$$C = \sum_i H_i^2 - \sum_{\alpha \in \Sigma^+} H_\alpha + \sum_{\alpha \in \Sigma^+} 2E_\alpha E_{-\alpha} + J.$$

As in the proof of Lemma 3.4, $[J\varphi_{\psi,h}](\varepsilon) = 0$, and $[E_\alpha E_{-\alpha} \varphi_{\psi,h}](\varepsilon) = 0$. Applying $-C$ to $\varphi_{\psi,h}$ gives

$$-C\varphi_{\psi,h} = - \left( \sum_i \psi(H_i)^2 - \sum_{\alpha \in \Sigma^+} \psi(H_\alpha) \right) \varphi_{\psi,h}$$

$$= - (\|\psi\|^2 - 2\langle \rho, \psi \rangle) \varphi_{\psi,h}$$

$$= (\|\rho\|^2 - \|\psi - \rho\|^2) \varphi_{\psi,h},$$

proving (4.3). Let $f$ be a positive $\lambda$-harmonic function on $X$, which we consider as a $K$-invariant function on $G$. By [30, Theorem 2], for any $g \in G$,

$$f(g) = \int_{\{\psi \geq \rho, \lambda_\psi = \lambda\} \times K/M} \varphi_{\psi,k}(g) \, d\mu([k], \psi)$$
for some Borel measure \( \mu \) on \( \{ \psi \geq \rho : \lambda_\psi = \lambda \} \times K/M \). By (4.3), this implies (2).

\[ \square \]

Corollary 4.3. For any Zariski dense discrete subgroup \( \Gamma < G \),
\[
\sup \{ \lambda_\psi : \psi \in D^*_\Gamma \} \leq \lambda_0.
\]

Proof. If \( \Gamma \) is cocompact in \( G \), then \( \psi_\Gamma = 2\rho \) and hence \( D^*_\Gamma = \{ 2\rho \} \). Since \( \lambda_0 = 0 = \lambda_{2\rho} \), the claim follows. In general, it follows from Theorem 2.3 and Proposition 3.7 that for any \( \psi \in D^*_\Gamma \), there exists a positive joint eigenfunction on \( \Gamma \backslash X \) with character \( \chi_{\psi-\rho} \). Hence the claim follows from Theorem 4.1 and Proposition 4.2.

\[ \square \]

5. Groups of the second kind and positive joint eigenfunctions

When \( G \) has rank one in which case the Furstenberg boundary is same as the geometric boundary of \( X \), a discrete subgroup \( \Gamma < G \) is said to be of the second kind if \( \Lambda \neq \mathcal{F} \). We extend this definition to higher rank groups as follows:

Definition 5.1. A discrete subgroup \( \Gamma < G \) is of the second kind if there exists \( \xi \in \mathcal{F} \) which is in general position with all points of \( \Lambda \), i.e., \( (\xi, \Lambda) \subset \mathcal{F}(2) \).

Sullivan’s theorem 4.1 provides a positive \( \lambda \)-harmonic function for any \( \lambda \leq \lambda_0 \). The following theorem can be viewed as a higher rank generalization of this result. The second-kind hypothesis may be interpreted as an analogue of the hypothesis of Theorem 4.1 that \( \Gamma \backslash X \) is non-compact.

Theorem 5.2. Let \( \Gamma < G \) be of the second kind with \( \mathcal{L}_\Gamma \subset \text{int} \sigma^+ \cup \{0\} \).
For any \( \psi \in D_\Gamma \), there exists a positive joint eigenfunction on \( \Gamma \backslash X \) with character \( \chi_{\psi-\rho} \).

Remark 5.3. (1) Let \( \Gamma_0 < G \) be an Anosov subgroup. Then for any Anosov subgroup \( \Gamma < \Gamma_0 \) with some point \( \xi \in \Lambda_{\Gamma_0} - \Lambda_\Gamma \), \( (\Lambda_\Gamma, \xi) \subset \mathcal{F}(2) \), since any two distinct points of \( \Lambda_{\Gamma_0} \) are in general position by the Anosov assumption on \( \Gamma_0 \). Hence \( \Gamma \) is of the second kind.
(2) If \( \Lambda \subset gN\nu_0 P \) for some \( g \in G \), then \( (\Lambda, g^+) \subset \mathcal{F}(2) \). One can construct many Schottky groups with \( \Lambda \subset N\nu_0 P \), which would then be of the second kind.
(3) Let \( G = \prod_{i=1}^k G_i \) be a product of simple algebraic groups \( G_i \) of rank one. Then \( \mathcal{F} = \prod_i G_i/P_i \), and \( (\xi_i, \eta_i) \in \mathcal{F} \) are in general position if and only if \( \xi_i \neq \eta_i \) for all \( i \). Therefore if there exists \( \xi_i \notin \pi_i(\Lambda) \) where \( \pi_i : \mathcal{F} \to G_i/P_i \) is the canonical projection, then for \( \xi = (\xi_i)_i \), \( (\Lambda, \xi) \subset \mathcal{F}(2) \). Therefore any closed subgroup \( \Gamma < G \) with \( \pi_i(\Lambda) \neq G_i/P_i \) is of the second kind.
(4) The well-known properties of the limit set of a Hitchin subgroup of \( \text{PSL}_d(\mathbb{R}) \) imply that Hitchin groups are not of the second kind for any even \( d \) or \( d = 3 \).
We will use shadow lemma to prove Theorem 5.2. For \( q \in X \) and \( r > 0 \), we set \( B(q, r) = \{ x \in X : d(x, q) < r \} \). For \( p = g(o) \in X \), the shadow of the ball \( B(q, r) \) viewed from \( p \) is defined as

\[
O_r(p, q) := \{ (gk)'^+ \in F : k \in K, \ gk \int A'^+ o \cap B(q, r) \neq \emptyset \}.
\]

Similarly, for \( \xi \in F \), the shadow the ball \( B(q, r) \) viewed from \( \xi \) is defined by

\[
O_r(\xi, q) := \{ h'^+ \in F : h \in G \text{ satisfies } h^- = \xi, \ ho \in B(q, r) \}.
\]

We recall the shadow lemma:

**Lemma 5.4.** [33, Lemma 5.7] There exists \( \kappa > 0 \) such that for any \( g \in G \) and \( r > 0 \),

\[
\sup_{\xi \in O_r(g(o), o)} \| \beta_{\xi}(g(o), o) - \mu(g^{-1}) \| \leq \kappa r.
\]

**Lemma 5.5.** [33, Lemma 5.6] If \( q_i \in X \) converges to \( \eta \in F \) as in Definition 5.1, then for any \( q \in X \), \( r > 0 \) and \( \varepsilon > 0 \),

\[
O_{r - \varepsilon}(q_i, q) \subset O_r(\eta, q) \subset O_{r + \varepsilon}(q_i, q)
\]

for all sufficiently large \( i \).

**Lemma 5.6.** If \( L \subset \text{int } a^+ \cup \{ 0 \} \), then the union \( \Gamma(o) \cup \Lambda \) is compact in the topology given in Definition 2.2.

**Proof.** The hypothesis implies that any sequence \( \gamma_i \to \infty \) in \( \Gamma \) tends to \( \infty \) regularly, and hence has a limit in \( F \). Moreover the limit belongs to \( \Lambda \) by its definition.

**Lemma 5.7.** Suppose that \( L \subset \text{int } a^+ \cup \{ 0 \} \). If \( \xi \in F \) satisfies that \( (\xi, \Lambda) \subset F^{(2)} \), then there exists \( R > 0 \) such that

\[
\xi \in \bigcap_{\gamma \in \Gamma} O_R(\gamma(o), o).
\]

**Proof.** We first claim that \( \xi \in \bigcap_{\eta \in \Lambda} O_R(\eta, o) \) for some \( R > 0 \). Note that

\[
\lim_{R \to \infty} O_R(\eta, o) = \{ z \in F : (z, \eta) \in F^{(2)} \}.
\]

Hence for each \( \eta \in \Lambda \), we have

\[
R_\eta = \inf\{ R + 1 : \xi \in O_R(\eta, o) \} < \infty.
\]

It suffices to show that \( R := \sup_{\eta \in \Lambda} R_\eta < \infty \). Suppose not; then \( R_\eta \to \infty \) for some sequence \( \eta_i \in \Lambda \). By passing to a subsequence, we have \( \eta_i \) converges to some \( \eta \). This follows that \( O_{R_{\eta_i} + 1}(\eta_i, o) \subset O_{R_{\eta_i} + 2}(\eta_i, o) \) for all sufficiently large \( i \). Therefore \( R_\eta \leq R_{\eta_i} + 3 \), yielding a contradiction.

We now claim that \( \xi \in \bigcap_{\gamma \in \Gamma} O_{R'}(\gamma(o), o) \) for some \( R' > 0 \). Suppose not; then there exist sequences \( \gamma_i \to \infty \) in \( \Gamma \) and \( R_i \to \infty \) such that \( \xi \notin O_{R_i}(\gamma_i(o), o) \). By Lemma 5.6, by passing to a subsequence, we may assume that \( \gamma_i(o) \) converges to some \( \eta \in \Lambda \). By the first claim, we have \( \xi \in O_R(\eta, o) \). By Lemma 5.5, we have \( \xi \in O_R(\eta, o) \subset O_{R + 1}(\gamma_i(o), o) \) for all sufficiently large \( i \). This is a contradiction since for \( i \) large enough so that \( R_i > R + 1 \), we have \( \xi \notin O_{R + 1}(\gamma_i(o), o) \). This proves the claim. \( \square \)
As an immediate corollary of Lemmas 5.4 and 5.7, we obtain:

**Corollary 5.8.** If \( \mathcal{L}_\Gamma \subset \text{int} \, \mathfrak{a}^+ \cup \{0\} \) and \( \xi \in \mathcal{F} \) satisfies that \((\xi, \Lambda) \subset \mathcal{F}^{(2)}\),
\[
\sup_{\gamma \in \Gamma} \| \beta_\xi(\gamma^{-1}o, o) - \mu(\gamma) \| < \infty.
\]

**Proof of Theorem 5.2:** If \( \psi \in D_\Gamma^* \), this follows Theorem 2.3. Hence we assume \( \psi \in D_\Gamma - D_\Gamma^* \); this implies that
\[
\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty \tag{5.1}
\]
by [42, Lem. III. 1.3]. As \( \Gamma \) is of the second kind, there exists \( \xi \in \mathcal{F} \) such that \((\xi, \eta) \in \mathcal{F}^{(2)}\) for all \( \eta \in \Lambda \). By Corollary 5.8, \( \| \beta_\xi(\gamma^{-1}o, o) - \mu(\gamma) \| \) is bounded uniformly for all \( \gamma \in \Gamma \). Therefore (5.1) implies that
\[
\sum_{\gamma \in \Gamma} e^{-\psi(\beta_\xi(\gamma^{-1}o, o))} < \infty. \tag{5.2}
\]

For any fixed \( x \in X \), we have \( \beta_\xi(\gamma^{-1}x, o) = \beta_\xi(\gamma^{-1}o, o) + \beta_\xi(x, o) \) and \( \| \beta_\xi(x, o) \| \leq d(x, o) \). Hence \( e^{-\psi(\beta_\xi(\gamma^{-1}o, o))} \triangleq e^{-\psi(\mu(\gamma))} \) with implied constant uniform for all \( \gamma \in \Gamma \).

Therefore, by (5.1) the following function \( F_\psi = F_{\psi, \xi} \) on \( X \) is well-defined: for \( x \in X \),
\[
F_\psi(x) := \sum_{\gamma \in \Gamma} e^{-\psi(\beta_\xi(\gamma^{-1}x, o))}. \tag{5.3}
\]

If we write \( \xi = [k_0] \in K/M = \mathcal{F} \), then for any \( g \in G \),
\[
\beta_\xi(\gamma^{-1}go, o) = \beta_M(k_0^{-1}\gamma^{-1}go, o) = H(g^{-1}\gamma k_0)
\]
and hence \( e^{-\psi(\beta_\xi(\gamma^{-1}go,o))} = \varphi_{\psi, \gamma k_0}(g) \). Therefore \( F_\psi = \sum_{\gamma \in \Gamma} \varphi_{\psi, \gamma k_0} \). It now follows from Lemma 2.2 that \( F_\psi \) is a positive \( \Gamma \)-invariant joint eigenfunction on \( X \) with eigenvalue \( \chi_{\psi-\rho} \). This finishes the proof.

**Remark 5.9.** For \( \psi \in D_\Gamma - D_\Gamma^* \), we have constructed positive joint eigenfunction \( F_{\psi, \xi} \) on \( \Gamma \backslash X \) of eigenvalue \( \chi_{\psi-\rho} \) for any \( \xi \in \mathcal{F} \) with \((\Lambda, \xi) \subset \mathcal{F}^{(2)}\).

Hence we get the strengthened version of Corollary 4.3:

**Corollary 5.10.** For \( \Gamma \) as in Theorem 5.2, we have
\[
\sup \{ \lambda_\psi : \psi \in D_\Gamma \} \leq \lambda_0. \tag{5.4}
\]

**Example 5.11.** Let \( \Gamma < \text{SO}^n(\mathbb{R}) \) be a discrete subgroup with \( \Lambda \neq \partial \mathbb{H}^n \); then \( \Gamma \) satisfies the hypothesis of Proposition 5.2. Since \( \rho = \frac{(n-1)}{2} \) and \( D_\Gamma = \{ s \geq \delta \} \), we have
\[
\sup \{ \| \rho \|^2 - \| \psi - \rho \|^2 : \psi \in D_\Gamma \} = \begin{cases} 
\delta(n-1-\delta) & \text{if } \delta \geq \frac{n-1}{2} \\
(n-1/2)^2 & \text{if } \delta \leq \frac{n-1}{2}.
\end{cases} \tag{5.5}
\]

It then follows from Proposition 3.7 and Theorem 4.1 that we have equality in (5.4) in this case, as was proved by Sullivan [47, Theorem 2.17].
6. The $L^2$-spectrum and uniqueness

Let $\Gamma$ be a torsion-free discrete subgroup of a connected semisimple real algebraic group $G$. The space $L^2(\Gamma \backslash X)$ consists of square-integrable functions together with the inner product $\langle f_1, f_2 \rangle = \int_{\Gamma \backslash X} f_1 f_2 \, d\text{vol}$.

Let $W^1(\Gamma \backslash X) \subset L^2(\Gamma \backslash X)$ denote the closure of $C^\infty_c(\Gamma \backslash X)$ with respect to the norm $\| \cdot \|_{W^1}$ induced by the inner product $\langle f_1, f_2 \rangle_{W^1} = \int_{\Gamma \backslash X} f_1 f_2 \, d\text{vol} + \int_{\Gamma \backslash X} \langle \text{grad } f_1, \text{grad } f_2 \rangle \, d\text{vol}$.

As $\Gamma \backslash X$ is complete, there exists a unique self-adjoint operator on the space $W^1(\Gamma \backslash X)$ extending the Laplacian $\Delta$ on $C^\infty_c(\Gamma \backslash X)$, which we also denote by $\Delta$. The $L^2$-spectrum of $-\Delta$, which we denote by $\sigma(\Gamma \backslash X)$, is the set of $\lambda \in \mathbb{C}$ such that $\Delta + \lambda$ does not have a bounded inverse $(\Delta + \lambda)^{-1} : L^2(\Gamma \backslash X) \to W^1(\Gamma \backslash X)$. The self-adjointness of $\Delta$ and the fact that $\langle -\Delta f, f \rangle = \int_X \| \text{grad } f \|^2 \, d\text{vol}$ for all $f \in C^\infty_c(\Gamma \backslash X)$ imply $\sigma(\Gamma \backslash X) \subset [0, \infty)$.

We will be using Weyl’s criterion (cf. [50, Lemma 2.17]) to determine $\sigma(\Gamma \backslash X)$:

**Theorem 6.1.** For $\lambda \in \mathbb{R}$, $\lambda \in \sigma(\Gamma \backslash X)$ if and only if there exists a sequence of unit vectors $F_n \in W^1(\Gamma \backslash X)$ such that $\| (\Delta + \lambda) F_n \| \to 0$.

The number $\lambda_0 = \lambda_0(\Gamma \backslash X)$ defined in (4.1) is the bottom of the $L^2$-spectrum $\sigma(\Gamma \backslash X)$:

**Theorem 6.2.** [47, Theorem 2.1, 2.2] We have $\lambda_0 \in \sigma(\Gamma \backslash X) \subset [\lambda_0, \infty)$.

Using Harish-Chandra’s Plancherel formula, we can identify $\lambda_0(X)$ and $\sigma(X)$ for the symmetric space $X = G/K$:

**Proposition 6.3.** We have $\lambda_0(X) = \| \rho \|^2$. Moreover, $\sigma(X) = [\| \rho \|^2, \infty)$.

**Proof.** It is shown in [25] that there are no positive harmonic functions on $X$ with eigenvalue strictly bigger than $\| \rho \|^2$; hence the inequality $\lambda_0(X) \leq \| \rho \|^2$ follows from Theorem 4.1 for $\Gamma = \{ e \}$. On the other hand, as seen in the proof of (1), $\varphi_{\rho,h}$ is a positive $\| \rho \|^2$-harmonic function (for any $h \in G$), hence $\lambda_0(X) = \| \rho \|^2$ by Theorem 4.1. We now deduce the second claim $\sigma(X) = [\| \rho \|^2, \infty)$ from Harish-Chandra’s Plancherel theorem (cf. e.g. [44]). For $\psi \in \mathfrak{a}^*$, define $\Phi_\psi \in C^\infty(K\backslash G/K)$ by $\Phi_\psi(g) = \int_K \varphi_{\rho+i\psi,k}(g) \, dk$ where $dk$ denotes the probability Haar measure on $K$. Then the computations proving (1) give $-C\Phi_\psi = -\Delta \Phi_\psi = (\| \rho \|^2 + \| \psi \|^2) \Phi_\psi$. 
Given any \( f \in C_c^\infty(\mathfrak{a}^*) \), we can define a function \( F \in L^2(X) \) by the formula

\[
F(g) = \int_{\mathfrak{a}^*} f(\psi) \Phi_\psi(g) \frac{d\psi}{|c(\psi)|^2};
\]

here \( d\psi \) denotes the Lebesgue measure on \( \mathfrak{a}^* \), \( c(\psi) \) denotes the Harish-Chandra \( c \)-function. The Plancherel formula says

\[
\|F\|_{L^2(X)}^2 = \int_{\mathfrak{a}^*} |f(\psi)|^2 \frac{d\psi}{|c(\psi)|^2}
\]

(see [44]). Let \( \lambda \in [||\rho||^2, \infty) \) be any number. Choose \( \psi_0 \in \mathfrak{a}^* \) so that \( \lambda = ||\rho||^2 + ||\psi_0||^2 \). We then choose a sequence of non-negative functions \( \{f_n\} \subset C_c^\infty(\mathfrak{a}^*) \) with \( \text{supp} f_n \subset B_{1/n}(\psi_0) \) and \( \|F_n\|_{L^2(X)} = 1 \). This gives

\[
\|((\Delta + \lambda)F_n)\|_{L^2(X)}^2 = \int_{\mathfrak{a}^*} |(\lambda - ||\rho||^2 - ||\psi||^2) f(\psi)|^2 \frac{d\psi}{|c(\psi)|^2} \leq \max_{\psi \in B_{1/n}(\psi_0)} ||\psi||^2 - ||\psi||^2 \leq \frac{\max_{\psi \in B_{1/n}(\psi_0)} ||\psi||^2 - ||\psi||^2}{2}.
\]

Consequently, \( \lim_{n \to \infty} ||((\Delta + \lambda)F_n)\|_{L^2(X)} = 0 \). By Weyl’s criterion (Theorem 6.1), this implies that \( \lambda \in \sigma(X) \). This proves the claim. \( \square \)

**Theorem 6.4.** If \( L^2(\Gamma \backslash G) \) is tempered, then \( \lambda_0(\Gamma \backslash X) = ||\rho||^2 \).

**Proof.** Note that \( \lambda_0 = \lambda_0(\Gamma \backslash X) \leq \lambda_0(X) = ||\rho||^2 \) by Proposition 4.2(3). Assume that \( \lambda_0 < ||\rho||^2 \). By Theorem 6.1, we can then find a \( K \)-invariant unit vector \( f \in L^2(\Gamma \backslash G)_K \) such that

\[
||(\Delta - \lambda_0)f|| < \frac{||\rho||^2 - \lambda_0}{2}.
\]

This gives

\[
||Cf|| = ||\Delta f|| \leq ||(\Delta - \lambda_0)f|| + \lambda_0 < \frac{||\rho||^2 + \lambda_0}{2} < ||\rho||^2.
\]

On the other hand, consider the direct integral representation of \( L^2(\Gamma \backslash G) = \int_{\mathbb{R}} (\pi_\zeta, H_\zeta) d\mu(\zeta) \) into irreducible unitary representations of \( G \) which are tempered, by the hypothesis on the temperedness of \( L^2(\Gamma \backslash G) \). Hence

\[
||Cf||^2 = \int_{\mathbb{R}} \|d\pi_\zeta(C)f\|_2^2 d\mu(\zeta) \geq \min_{\pi \text{ spherical tempered}} |d\pi(C)|^2
\]

where \( d\pi \) denotes the differential of \( \pi \). However, since all tempered representations are weakly contained in \( L^2(G) \); appearing either discrete series or Lebesgue integrals over \( \mathfrak{a}^* \) or principal series representations. Thus for any spherical tempered representation \( (\pi, \mathcal{H}) \) we have \( d\pi(C) \in \sigma(X) \) and hence, by Proposition 6.3,

\[
\min_{\pi \text{ spherical tempered}} |d\pi(C)| \geq ||\rho||^2,
\]

giving a contradiction. \( \square \)

**Theorem 6.5.** [47, Theorem 2.8 and Corollary 2.9]
(1) Any positive harmonic function in \( L^2(\Gamma \setminus X) \) is \( \lambda_0 \)-harmonic.
(2) If there exists a \( \lambda_0 \)-harmonic function in \( L^2(\Gamma \setminus X) \), then the space of \( \lambda_0 \)-harmonic functions in \( \Gamma \setminus X \) is one-dimensional and generated by a positive function.

Proof. Sullivan’s proof in [47] uses the heat operator and superharmonic functions. We provide a more direct proof here.

Note that if \( f \in L^2(\Gamma \setminus X) \), then \( f \in W^1(\Gamma \setminus X) \), since

\[
\int_{\Gamma \setminus X} \| \nabla f \|^2 \, d\text{vol} = -\int_{\Gamma \setminus X} f \Delta f \, d\text{vol} = \lambda \int_{\Gamma \setminus X} f^2 \, d\text{vol}.
\]

The key fact for us is that \( \lambda_0 \) may also be expressed as an infimum over functions in \( W^1(\Gamma \setminus X) \); for \( f \neq 0 \) in \( W^1(\Gamma \setminus X) \), define \( R(f) \) by

\[
R(f) = \frac{\|f\|^2_{W^1}}{\|f\|^2} - 1 \geq 0
\]

where \( \| \cdot \| \) denotes the \( L^2(\Gamma \setminus X) \) norm. For any \( f \neq 0 \in W^1(\Gamma \setminus X) \), and all \( \varphi \) with \( \| f - \varphi \|_{W^1} \) small enough, we have

\[
\frac{\|\varphi\|_{W^1} - \| f - \varphi \|_{W^1}}{\|\varphi\| + \| f - \varphi \|_{W^1}} - 1 \leq R(f) \leq \frac{\|\varphi\|_{W^1} + \| f - \varphi \|_{W^1}}{\|\varphi\| - \| f - \varphi \|_{W^1}} - 1,
\]

i.e. \( f \mapsto R(f) \) is continuous at each \( f \neq 0 \in W^1(\Gamma \setminus X) \). The density of \( C_0^\infty(\Gamma \setminus X) \) in \( W^1(\Gamma \setminus X) \) then gives

\[
\lambda_0 = \inf_{f \in C_0^\infty(\Gamma \setminus X) \setminus \{0\}} R(f) = \inf_{f \in W^1(\Gamma \setminus X) \setminus \{0\}} R(f).
\]

Now suppose that \( \phi \in L^2(\Gamma \setminus X) \) is a positive \( \lambda \)-harmonic function; so \( \phi \in W^1(\Gamma \setminus X) \). We claim that \( \lambda = \lambda_0 \). By Green’s identity we have

\[
\lambda_0 \leq R(\phi) = \frac{\int_{\Gamma \setminus X} \| \nabla \phi \|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\phi|^2 \, d\text{vol}} = \frac{\int_{\Gamma \setminus X} \phi (-\Delta \phi) \, d\text{vol}}{\int_{\Gamma \setminus X} |\phi|^2 \, d\text{vol}} = \lambda
\]

(cf. Proposition 4.2). On the other hand, for any \( \varphi \in C_0^\infty(\Gamma \setminus X) \),

\[
\frac{\int_{\Gamma \setminus X} \| \nabla \varphi \|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\varphi|^2 \, d\text{vol}} = \frac{\int_{\Gamma \setminus X} \| \nabla (\phi \cdot \frac{\varphi}{\phi}) \|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\varphi|^2 \, d\text{vol}}.
\]

By Barta’s identity [1],

\[
\int_{\Gamma \setminus X} \| \nabla (\phi \cdot \frac{\varphi}{\phi}) \|^2 \, d\text{vol} = \int_{\Gamma \setminus X} \phi^2 \| \nabla \frac{\varphi}{\phi} \|^2 \, d\text{vol} - \int_{\Gamma \setminus X} \left( \frac{\varphi}{\phi} \right)^2 \phi \Delta \phi \, d\text{vol},
\]

so

\[
\int_{\Gamma \setminus X} \| \nabla \varphi \|^2 \, d\text{vol} \geq \int_{\Gamma \setminus X} \left( \frac{\varphi}{\phi} \right)^2 \phi (-\Delta \phi) \, d\text{vol} = \lambda \int_{\Gamma \setminus X} \varphi^2 \, d\text{vol},
\]

i.e.

\[
\lambda \leq \frac{\int_{\Gamma \setminus X} \| \nabla \varphi \|^2 \, d\text{vol}}{\int_{\Gamma \setminus X} |\varphi|^2 \, d\text{vol}}.
\]
for any \( \varphi \in C_c^\infty(\Gamma\setminus X) \), showing that \( \lambda_0 \geq \lambda \). Hence \( \lambda = \lambda_0 \).

In order to prove (2), we first claim that \( f \in W^1(\Gamma\setminus X) \) satisfies \(-\Delta f = \lambda_0 f \) if and only if \( R(f) = \lambda_0 \).

Suppose that \( R(f) = \lambda_0 \). We will then show that for any \( \varphi \in C_c^\infty(\Gamma\setminus X) \),

\[
\langle f, -\Delta \varphi \rangle = \lambda_0 \langle f, \varphi \rangle;
\]

(6.1)

this implies \( f \) is \( \lambda_0 \)-harmonic. Let \( \varphi \in C_c^\infty(\Gamma\setminus X) \). Since \( R(f) = \lambda_0 \), \( f \) minimizes \( R \), so for any \( \varphi \in C_c^\infty(\Gamma\setminus X) \), the function \( F : \mathbb{R} \to \mathbb{R}_{\geq 0} \) defined by \( F(x) = R(f + x\varphi) \) has a local minimum at \( x = 0 \), hence \( F'(0) = 0 \). Now computing \( F'(0) \) gives

\[
F'(0) = \frac{2\langle f, \varphi \rangle W^1 \|f\|^2 - 2\langle f, \varphi \rangle \|f\|^2_{W^1}}{\|f\|^4} = 0.
\]

From \( R(f) = \lambda_0 \) we obtain \( \|f\|^2_{W^1} = (\lambda_0 + 1)\|f\|^2 \), which, when entered into the identity above, gives

\[
\langle f, \varphi \rangle_{W^1} = (\lambda_0 + 1)\langle f, \varphi \rangle.
\]

(6.2)

Letting \( \{f_i\}_{i \in \mathbb{N}} \subset C_c^\infty(\Gamma\setminus X) \) be a sequence converging to \( f \) in \( W^1(\Gamma\setminus X) \), Green’s identity again gives

\[
\langle f, \varphi \rangle_{W^1} = \lim_{i \to \infty} \langle f_i, \varphi \rangle_{W^1} = \lim_{i \to \infty} \int_{\Gamma\setminus X} f_i \varphi + (\text{grad } f_i, \text{grad } \varphi) \, d\text{vol}
\]

\[
= \lim_{i \to \infty} \int_{\Gamma\setminus X} f_i \varphi + f_i(-\Delta \varphi) \, d\text{vol} = \langle f, \varphi \rangle + \langle f, -\Delta \varphi \rangle.
\]

(6.3)

Combined with (6.2), this gives \( \langle f, -\Delta \varphi \rangle = \lambda_0 \langle f, \varphi \rangle \) as in (6.1).

Conversely, if \( f \in W^1(\Gamma\setminus X) \) satisfies \(-\Delta f = \lambda_0 f \), then for any \( \varphi \in C_c^\infty(\Gamma\setminus X) \), we have (as in (6.3))

\[
\langle f, \varphi \rangle_{W^1} = \langle f, \varphi \rangle + \langle f, -\Delta \varphi \rangle = (\lambda_0 + 1)\langle f, \varphi \rangle,
\]

hence

\[
\|f\|^2_{W^1} = \sup_{\varphi \in C_c^\infty(\Gamma\setminus X)} \langle f, \varphi \rangle_{W^1} = \sup_{\varphi \in C_c^\infty(\Gamma\setminus X)} (\lambda_0 + 1)\langle f, \varphi \rangle = (\lambda_0 + 1)\|f\|^2,
\]

giving \( R(f) = \lambda_0 \). This proves the claim.

Let \( f \in W^1(\Gamma\setminus X) \cap C^\infty(\Gamma\setminus X) \) now be a \( \lambda_0 \)-harmonic function. Then \( \|f\| \in W^1(\Gamma\setminus X) \) and \( R(\|f\|) = \lambda_0 \). As shown above, \( \|f\| \) is also a \( \lambda_0 \)-harmonic function. Hence either \( f \) is a constant multiple of \( |f| \) or \( f \) must change sign at some point \( x_0 \), hence \( |f(x)| \geq |f(x_0)| = 0 \) for all \( x \in \Gamma\setminus X \). This means that \( |f| \) attains its minimum and thus violates the strong minimum principle which says that \( \phi(x) > \inf_y \phi(y) \) for all \( x \in \Gamma\setminus X \) and any \( \Delta \) eigenfunction \( \phi \). We therefore conclude that any \( \lambda_0 \)-harmonic function in \( L^2(\Gamma\setminus X) \) is a constant multiple of a positive function. This then implies that the space of \( \lambda_0 \)-harmonic functions must be one-dimensional as two positive functions cannot be orthogonal to each other. \( \square \)
The uniqueness in the above theorem has the following implications on joint eigenfunctions:

**Corollary 6.6.**

1. There exists at most one positive joint eigenfunction in $L^2(\Gamma \setminus X)$ up to a constant multiple.
2. If there exists a positive joint eigenfunction in $L^2(\Gamma \setminus X)$ with character $\chi_{\psi - \rho}$, $\psi \in a^*$, then
   \[ \lambda_0 = \lambda_\psi. \]
3. There exists a positive harmonic function in $L^2(\Gamma \setminus X)$ if and only if there exists a positive joint eigenfunction in $L^2(\Gamma \setminus X)$ of character $\chi_{\psi - \rho}$ with $\lambda_\psi = \lambda_0$.

**Proof.** We only need to verify the third claim. Suppose that $\phi \in L^2(\Gamma \setminus X)$ is a positive harmonic function. Via the identification $L^2(\Gamma \setminus X) = L^2(\Gamma \setminus G)_K$, we may consider $\phi \in L^2(\Gamma \setminus G)_K$ as a positive $C$-eigenfunction for the Casimir operator $C$. By Theorem 6.5, $C \phi = -\lambda_0 \phi$. Let $D \in Z(g_C)$. Then $C \circ D \phi = D \circ C \phi = -\lambda_0 D \phi$. By the uniqueness in Theorem 6.5, it follows that $D \phi$ is a constant multiple of $\phi$; and hence $\phi$ is an eigenfunction for $D$ as well. Therefore $\phi$ is a joint eigenfunction. \[\square\]

**Spherical unitary representations contained in $L^2(\Gamma \setminus G)$**. We let $C_c(G//K)$ denote the Hecke algebra of $G$, i.e.

\[ C_c(G//K) = \{ f \in C_c(G) : f(k_1 g k_2) = f(g) \quad \text{for all } g \in G, k_1, k_2 \in K \}. \]

Each element of $C_c(G//K)$ acts on $C(G)$ via right convolution $\ast$.

**Lemma 6.7.** A positive $K$-invariant joint eigenfunction on $G$ is an eigenfunction for the action of the Hecke algebra. More precisely, if

\[ \phi(g) = \int_F \varphi_{\psi,k}(g) d\nu([k]), \quad g \in G, \quad (6.4) \]

for some $\psi \in a^*$ and a $(\Gamma, \psi)$-conformal measure $\nu$ on $F = K/M$, then for all $f \in C_c(G//K)$,

\[ (\phi \ast f)(g) = \left( \int_G f(h) e^{-\psi(H(h))} dh \right) \phi(g). \]

**Proof.** Given $f \in C_c(G//K)$, we have

\[
(\phi \ast f)(g) = \int_G \phi(gh^{-1}) f(h) dh = \int_G \int_F \varphi_{\psi,k}(gh^{-1}) f(h) d\nu([k]) dh \\
= \int_F \int_G f(h) e^{-\psi(H(hg^{-1}k))} dh d\nu([k]).
\]
Now using $H(hg^{-1}k) = H(hκ(g^{-1}k)) + H(g^{-1}k)$ and then the change of variables $h' = hκ(g^{-1}k)$ gives

$$(\phi * f)(g) = \int_F \left( \int_G f(hκ(g^{-1}k)^{-1})e^{-\psi(H(h))}dh \right) e^{-\psi(H(g^{-1}k))}d\nu([k])$$

$$= \int_F \left( \int_G f(h)e^{-\psi(H(h))}dh \right) e^{-\psi(H(g^{-1}k))}d\nu([k])$$

$$= \left( \int_G f(h)e^{-\psi(H(h))}dh \right) \phi(g),$$

since $f \in C(G//K)$, and is thus right $K$-invariant. In total, we have shown that $\phi$ is an eigenfunction of the $f$-action, with eigenvalue $\int_G f(h)e^{-\psi(H(h))}dh$. 

\[\square\]

**Theorem 6.8.** If $\phi \in L^2(\Gamma\backslash G)_K$ is a positive harmonic function of norm one, there exists a unique irreducible spherical unitary subrepresentation $(\pi, H_\phi)$ of $L^2(\Gamma\backslash G)$, and $\phi$ is the unique $K$-invariant unit vector in $H_\phi$.

**Proof.** By Corollary 6.6, $\phi$ is given by (6.4) for some $\psi \in a^*$. Define $\Phi : G \to \mathbb{C}$ by

$$\Phi(g) := \langle g, \phi, \phi \rangle$$

for all $g \in G$ where the $g$ action on $L^2(\Gamma\backslash G)$ is via the translation action of $G$ on $\Gamma\backslash G$ from the right. Given $f \in C_c(G//K)$, we then have, using Lemma 6.7,

$$(\Phi \ast f)(g) = \int_G \Phi(g h^{-1}) f(h) dh = \int_G \langle (g h^{-1}), \phi, \phi \rangle f(h) dh$$

$$= \int_G \langle f(h) h^{-1}, \phi, g^{-1}, \phi \rangle dh = \langle \phi \ast f, g^{-1}, \phi \rangle$$

$$= \left( \int_G f(h)e^{-\psi(H(h))}dh \right) \Phi(g),$$

i.e. $\Phi$ is also a $C_c(G//K)$-eigenfunction. Also note that $\Phi(e) = 1$, and since $\phi$ is right $K$-invariant, $\Phi$ is bi-$K$-invariant. Moreover, being the matrix coefficient of a unitary representation, $\Phi$ is also positive definite, i.e., for any $g_1, \cdots, g_n \in G$ and $z_1, \cdots, z_n \in \mathbb{C},$

$$\sum_{1 \leq i,j \leq n} z_i \bar{z}_j \Phi(g_j^{-1} g_i) \geq 0.$$

We have thus shown that $\Phi$ is a positive definite spherical function. Letting $\mathcal{H}_\phi$ denote the closure of span$\{g.\phi : g \in G\}$ in $L^2(\Gamma\backslash G)$, by [31, Chapter IV§5, Corollary of Theorem 9], $\mathcal{H}_\phi$ is an irreducible (spherical) unitary subrepresentation of the quasi-regular representation $L^2(\Gamma\backslash G)$. The uniqueness follows from Corollary 6.6. 

\[\square\]

**Lemma 6.9.** Let $\psi \geq \rho$ and $\psi \notin \mathbb{R}\rho$. Denote by $\psi'$ be the element of the line $\mathbb{R}\psi$ closest to $\rho$. Then $\psi' \not\geq \rho$. 


Proof. Let $\phi := \psi - \rho$. Note that $\phi \geq 0$ on $\mathfrak{a}$ by the hypothesis. Then
\[
\psi' = \frac{\langle \psi, \rho \rangle}{\|\psi\|^2} \psi = \frac{\langle \rho + \phi, \rho \rangle}{\|\rho + \phi\|^2} \psi = \left(1 - \frac{\|\phi\|^2}{\|\rho + \phi\|^2}\right)\psi,
\]
i.e. $\psi' = t\psi$ with $0 < t < 1$. Now, if $\psi' \geq \rho$, we could repeat the process with $\psi'$ in place of $\psi$ to find another, different, closest vector in $\mathbb{R}\psi$ to $\rho$, which is not possible.

Theorem 6.10. Let $\Gamma < G$ be of the second kind with $L_\Gamma \subset \text{int} \mathfrak{a}^+ \cup \{0\}$. If there exists a $\lambda_0$-harmonic function in $L^2(\Gamma \backslash X)$, then $\lambda_0 = \lambda_\psi$ for some $\psi \in D^*_\Gamma \cup \{\rho\}$.

Proof. Suppose that $\psi \in D_\Gamma \setminus (\{\rho\} \cup D^*_\Gamma)$ and that $\psi \geq \rho$. Assume that there exists a positive joint eigenfunction $\phi \in L^2(\Gamma \backslash X)$ with character $\chi_{\psi - \rho}$. By Corollary 6.6,
\[
\lambda_0 = \lambda_\psi = \|\rho\|^2 - \|\psi - \rho\|^2. \quad (6.5)
\]
Since $\psi_\Gamma$ is concave, there exists $0 < c \leq 1$ such that $c\psi(u) = \psi_\Gamma(u)$ for some $u \in L_\Gamma$. So $\psi_0 := c\psi \in D^*_\Gamma$. Since $\psi \notin D^*_\Gamma$, we have $0 < c < 1$. There exists a unique $s_0 \in \mathbb{R}$ such that
\[
\|s_0\psi_0 - \rho\| = \min\{\|s\psi - \rho\| : s \in \mathbb{R}\}, \quad (6.6)
\]
that is, $s_0\psi_0$ be the element on the line $\mathbb{R}\psi$ that is closest to $\rho$.

We claim that $s_0c \leq 1$; since $0 < c < 1$, this implies that $\max\{1, s_0\} < c^{-1}$. If $\psi \in \mathbb{R}\rho$, then $s_0\psi_0 = \rho$. Since $\psi_0 = c\psi$, we get $s_0c\psi = \rho$. By the hypothesis $\rho \leq \psi$, $s_0c \leq 1$. Now suppose $\psi \notin \mathbb{R}\rho$. Assume that $s_0c > 1$. Then $s_0\psi_0 = s_0c\psi > \psi$. Hence $s_0c\psi \in D_\Gamma$. By Corollary 5.10 and (6.5), we get $\|s_0c\psi - \rho\| \geq \|\psi - \rho\|$. By the choice of $s_0$ in (6.6), it follows that $\|s_0c\psi - \rho\| = \|\psi - \rho\|$. Since $s_0c\psi \geq \psi \geq \rho$, this yields a contradiction. Therefore the claim $s_0c \leq 1$ follows.

We now choose $t$ so that $\max\{1, s_0\} < t < c^{-1}$. Since $t > 1$ and $\psi_0 \in D^*_\Gamma$, $t\psi_0 \in D^*_\Gamma$. Note also that $s \mapsto \lambda_{s\psi_0}$ is strictly decreasing on the interval $[s_0, \infty)$. Since $s_0 < t < c^{-1}$ and $c^{-1}\psi_0 = \psi$, we get
\[
\lambda_0 = \lambda_\psi < \lambda_{t\psi_0}.
\]
This contradicts Corollary 5.10. This implies the claim by Corollary 6.6.

If we use the norm on $\text{Lie}(\text{SO}\,^0(n, 1))$ which endows the constant curvature $-1$ metric on $\mathbb{H}^n$, then for any non-elementary discrete subgroup $\Gamma < G$, $D^*_\Gamma = \{\delta\}$ and hence the above theorem says that if a $\lambda_0$-harmonic function belongs to $L^2(\Gamma \backslash X)$, then $\lambda_0 = \delta(n - 1 - \delta)$ or $\frac{1}{4}(n - 1)^2$.

7. Smearing arguments

Let $\Gamma$ be a discrete subgroup of a connected semisimple real algebraic group $G$. Recall the notation $i$ for the opposition involution of $G$. The goal of this section is to prove the following, which implies Theorem 1.8 by Theorem 2.5.
Theorem 7.1. Let $\psi \in D_Γ$ be stabilized by $i$, i.e., $\psi \circ i = \psi$. If $\mathcal{L}_Γ \subset \text{int } a^+ \cup \{0\}$, then no positive joint eigenfunction of character $\chi_{\psi-\rho}$ belongs to $L^2(Γ\backslash X)$.

Corollary 7.2. If $\mathcal{L}_Γ \subset \text{int } a^+ \cup \{0\}$ and $\lambda_0(Γ\backslash X) = \|\rho\|^2$, then there exists no positive harmonic function in $L^2(Γ\backslash X)$.

Proof. Suppose that there exists a positive harmonic function $\phi$ in $L^2(Γ\backslash X)$ by Corollary 6.6, $\phi$ is a joint eigenfunction in $L^2(Γ\backslash X)$ of character $\chi_{\psi-\rho}$ for some $\psi \in a^*$ satisfying $\lambda_\psi = \lambda_0$. Since $\lambda_0 = \|\rho\|^2 = \lambda_\psi = \|\rho\|^2 - \|\rho - \psi\|^2$, it follows that $\psi = \rho$. Since $\rho$ is invariant under $i$, Theorem 7.1 implies the $\phi$ cannot belong to $L^2(Γ\backslash X)$, yielding a contradiction. \hfill $\Box$

Corollary 7.3. Suppose that $i$ is trivial. For any discrete subgroup $Γ \subset G$ with $\mathcal{L}_Γ \subset \text{int } a^+ \cup \{0\}$, there exists no positive harmonic function in $L^2(Γ\backslash X)$.

Theorem 7.1 will be deduced from Theorem 7.5 whose proof is based on the smearing argument of Thuston and Sullivan (see [48] and also [49] for historical remarks).

In the rest of this section, we fix $\psi \in D_Γ$. For each $x \in X$, define the following analogue $d_x = d_{\psi,x}$ of the visual metric on $\mathcal{F}$: for any $(\xi, \eta) \in \mathcal{F}^{(2)}$,

$$d_x(\xi, \eta) = e^{-\psi(\beta_g(x,go))}$$

where $g \in G$ is any element such that $g^+ = \xi$ and $g^- = \eta$; this definition is independent of the choice of such $g$. The following $G$-equivariance property follows from that of the Busemann function: for any $h \in G$,

$$d_x(\xi, \eta) = d_{hx}(h\xi, h\eta).$$

Definition 7.4 (Hopf parameterization). The homeomorphism $G/M \rightarrow \mathcal{F}^{(2)} \times a$ given by $gM \mapsto (g^+, g^-, b = \beta_g^-(e, g))$ is called the Hopf parameterization of $G/M$.

Fix $\{\nu_x : x \in X\}$ and $\{\tilde{\nu}_x : x \in X\}$ be respectively $(Γ, \psi)$ and $(Γ, \psi \circ i)$-conformal densities on $\mathcal{F}$. Using the Hopf parametrization 7.4, define the following locally finite Borel measure $\tilde{m}_{\nu,\tilde{\nu}}$ on $G/M$: for $(\xi, \eta, v) \in \mathcal{F}^{(2)} \times a$,

$$d\tilde{m}_{\nu,\tilde{\nu}}(\xi, \eta, v) = \frac{1}{d_x(\xi, \eta)} dv_x(\xi) d\tilde{v}_x(\eta) dv \quad (7.1)$$

where $dv$ is the Lebesgue measure on $a$ and $x \in X$ is any element; it follows from the $Γ$-conformality of $\{\nu_x\}$ and $\{\tilde{\nu}_x\}$ that this definition is independent of $x \in X$. The measure $\tilde{m}_{\nu,\tilde{\nu}}$ is left $Γ$-invariant and right $A$-invariant. We denote by $m_{\nu,\tilde{\nu}}$ the $AM$-invariant Borel measure on $Γ\backslash G$ induced by $\tilde{m}_{\nu,\tilde{\nu}}$; this measure is called the Bowen-Margulis-Sullivan measure associated to the pair $(\nu, \tilde{\nu})$ [12].
Theorem 7.5. For any pair $(\nu, \bar{\nu})$ of $(\Gamma, \psi)$ and $(\Gamma, \psi \circ i)$-conformal measures on $\mathcal{F}$ respectively, we have

$$m_{\nu, \bar{\nu}}(\Gamma \setminus G) \ll \int_{\Gamma \setminus X} E_\nu(x)E_{\bar{\nu}}(x) \, d\text{vol}.$$ 

Proof. We extend the smearing argument due to Sullivan and Thurston ([48], [10]). Let $Z = G/K \times \mathcal{F}^{(2)}$. For any $(\xi, \eta) \in \mathcal{F}^{(2)}$, we write $[\xi, \eta] = g_Ao \subset X$ for any $g \in G$ such that $g^+ = \xi$ and $g^- = \eta$; $[\xi, \eta]$ is a maximal flat in $X$ defined independently of the choice of $g \in G$. We also denote by $W_{\xi, \eta} \subset X$ the one neighborhood of $[\xi, \eta]$. Consider the following locally finite Borel measure $\alpha$ on $Z$ defined as follows: for any $f \in C_c(Z)$,

$$\alpha(f) = \int_{(\xi,\eta) \in \mathcal{F}^{(2)}} \int_{z \in W_{\xi,\eta}} f(z, \xi, \eta) \, dz \, dm(\xi, \eta)$$

where $dz$ is the $G$-invariant measure on $X$, and $dm(\xi, \eta) = \frac{1}{d_\lambda(\xi,\eta)}d\nu_x(\xi)d\bar{\nu}_x(\eta)$ (independent of $x$); in other words,

$$d\alpha(z, \xi, \eta) = d\lambda_{\xi,\eta}(z)dm(\xi, \eta)$$

where $\lambda_{\xi,\eta}$ is the restriction of $\lambda$ to $W_{\xi,\eta}$. Consider natural diagonal action of $\Gamma$ on $Z$. Since $dz$ and $dm$ are both left $\Gamma$-invariant, $\alpha$ is also left $\Gamma$-invariant and hence induces a measure the quotient space $\Gamma \setminus Z$, which we also denote by $\alpha$ by abuse of notation.

Define the projection $\pi' : Z \to G/M$ as follows: for $(x, \xi, \eta) \in X \times \mathcal{F}^{(2)}$, choose $g \in G$ so that $g^+ = \xi$ and $g^- = \eta$. Then there exists a unique element $a \in A$ such that $d(x, gao) = d(x, gA_o) = \inf_{b \in A} d(x, gbo)$; this follows from [7, Proposition 2.4] since $X$ is a CAT(0) space and $gA(o)$ is a convex complete subspace of $X$. In other words, the point $gao$ is the orthogonal projection of $x$ to the flat $[\xi, \eta] = gA_o$. We then set

$$\pi'(x, \xi, \eta) = gaM \in G/M;$$

this is well-defined independent of the choice of $g \in G$.

Noting that $\pi'$ is $\Gamma$-equivariant, we denote by

$$\pi : \text{supp}(\alpha) \subset \Gamma \setminus Z \to \text{supp} \, (m_{\nu, \bar{\nu}}) \subset \Gamma \setminus G/M$$

the map induced by $\pi'$.

Fixing $[ga] \in \Gamma \setminus G/M$, the fiber $\pi^{-1}[ga]$ is of the form $[(gaD_0, g^+, g^-)]$ where

$$D_0 = \{ s \in X : d(s, o) \leq 1, \text{ the geodesic connecting } s \text{ and } o \text{ is orthogonal to } Ao \text{ at } o \}.$$ 

Since each fiber $\pi^{-1}(v)$, $v \in \text{supp} \, m_{\nu, \bar{\nu}}$, is isometric to $D_0$, we have for any Borel subset $S \subset \text{supp} \, m_{\nu, \bar{\nu}}$, we have

$$\alpha(\pi^{-1}(S)) = c \cdot m_{\nu, \bar{\nu}}(S)$$

where $c = \text{Vol}(D_0)$; the volume of $D_0$ being computed with respect to the volume form induced by the $G$-invariant measure on $X$. 

Consider now the map \( p : \text{supp}(\alpha) \to \Gamma \setminus X \) defined by \( p([(z, \xi, \eta)]) = [z] \) for any \((z, \xi, \eta) \in \text{supp}(\alpha)\).

Let \( F = \pi^{-1}(\text{supp} \, m_{\mu, \nu}) \subset \text{supp}(\alpha) \). We write

\[
\alpha(F) = \int_{\Gamma \setminus X} \alpha_x(p^{-1}(x) \cap F) \, dx,
\]

where \( \alpha_x \) is a conditional measure on the fiber \( p^{-1}(x) \).

We claim that for each \( x \in \Gamma \setminus X \),

\[
\alpha_x(p^{-1}(x)) \ll E_\nu(x) \cdot E_\varphi(x).
\]

This implies that \( \alpha(F) \ll \int_{\Gamma \setminus X} E_\nu(x)E_\varphi(x) \, dx \), which then finishes the proof by (7.2).

Note that \( V_{h(o)} := \{ (\xi, \eta) \in \mathcal{F}(2) : [\xi, \eta] \cap B(h(o), 1) \neq \emptyset \} \) is a compact subset of \( \mathcal{F}(2) \); if \( g_i \in G \) such that \( d(g_i a_i o, h(o)) \leq 1 \) for some \( a_i \in A \), then \( g_i a_i \) converges to some \( g_0 \in G \) by passing to a subsequence. This implies \( (g_i^+, g_i^-) \to (g_0^+, g_0^-) \in \mathcal{F}(2) \), from which the compactness of \( V_{h(o)} \) follows. It follows that

\[
\kappa := \inf \{ d_o(\xi, \eta) : (\xi, \eta) \in V_o \} > 0.
\]

By the equivariance \( d_{h(o)}(\xi, \eta) = d_o(h^{-1}\xi, h^{-1}\eta) \), we have for any \( h \in G \),

\[
\kappa = \inf \{ d_{h(o)}(\xi, \eta) : (\xi, \eta) \in V_{h(o)} \}.
\]

Note that if \( x = [h(o)] \in X \) for \( h \in G \), then

\[
p^{-1}(x) = \{ [(x, \xi, \eta)] \in \text{supp}(\alpha) : [\xi, \eta] \cap B(h(o), 1) \neq \emptyset \} \simeq V_{h(o)}.
\]

Now

\[
\alpha_x(p^{-1}(x)) = \alpha_x(V_{h(o)})
\]

\[
= \int_{(\xi, \eta) \in V_{h(o)}} \frac{1}{d_{h(o)}(\xi, \eta)} \, d\nu_{h(o)}(\xi) \, d\bar{\nu}_{h(o)}(\eta)
\]

\[
\leq \frac{1}{\kappa} \int_{(\xi, \eta) \in V_{h(o)}} \, d\nu_{h(o)}(\xi) \, d\bar{\nu}_{h(o)}(\eta)
\]

\[
\leq \frac{1}{\kappa} |\nu_{h(o)}| \cdot |\bar{\nu}_{h(o)}| = \frac{1}{\kappa} E_\nu(x) \cdot E_\varphi(x).
\]

\[\square\]

**Proof of Theorem 7.1.** Suppose that \( \phi \in L^2(\Gamma \setminus X) \) is a positive joint eigenfunction with character \( \chi_{\psi^{-1}} \). By Proposition 3.7, \( \phi = E_\psi \) for some \((\Gamma, \psi)\)-conformal measure \( \nu \). Since \( \psi \circ i = \psi \), we may form the measure \( m_{\psi, \nu} \) and apply Theorem 7.5. Since \( E_\psi \in L^2(\Gamma \setminus G) \), it follows that \( m_{\psi, \nu} \) is a finite \( MA \)-invariant Borel measure on \( \Gamma \setminus G \). Since \( m_{\psi, \nu} \) is finite, it is conservative for any one-parameter subgroup of \( A \). In particular, for any non-zero \( v \in \mathfrak{a}^+ \), there exist sequences \( t_i \to +\infty \) and \( \gamma_i \in \Gamma \) such that the sequence \( \gamma_i \exp(t_i v) \) is convergent. This implies that \( t_i^{-1} \mu(\gamma_i) \) converges to \( v \), and hence \( v \in \mathcal{L}_\Gamma \). Therefore \( \mathcal{L}_\Gamma = \mathfrak{a}^+ \). This finishes the proof.
Remark 7.6. Suppose that $\Gamma < G$ is Zariski dense and that $\psi > \psi_\Gamma$. Then, by [42, Lem. III. 1.3], we have

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty.$$  

On the other hand, by Theorem 1.4 of [8], the finiteness of $m_{\nu,\nu}$ implies that $\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty$, yielding a contradiction. Therefore the same conclusion of Theorem 7.1 holds in this setting.

8. Injectivity radius and $L^2(G) \varpropto L^2(\Gamma \backslash G)$

As before let $G$ be a connected semisimple real algebraic group. Recall from Proposition 6.3 that $\sigma(X) = [\|\rho\|^2, \infty)$. In this section, we prove the following:

Theorem 8.1. Let $\Gamma < G$ be an Anosov subgroup. We suppose that $\Gamma$ is not a cocompact lattice in a rank one group $G$. Then

$$L^2(G) \varpropto L^2(\Gamma \backslash G) \quad \text{and} \quad \sigma(X) = [\|\rho\|^2, \infty) \subset \sigma(\Gamma \backslash X).$$

We will need the following lemma: when $G$ is of rank one, we may write $A$ is a one-parameter subgroup $A = \{a_t : t \in \mathbb{R}\}$. A loxodromic element $g \in G$ is of the form $g = h a_t m h^{-1}$ for some $t \neq 0$, $m \in M$ and $h \in G$. The translation axis of $g$ is then given by $hA(o)$.

Lemma 8.2. Let $G$ be a simple real algebraic group of rank one. For any loxodromic element $g \in G$ with translation axis $L$, we have $d(x, gx) \to \infty$ as $d(x, L) \to \infty$.

Proof. Write $A = \{a_t : t \in \mathbb{R}\}$. Without loss of generality, we may assume $g = m^{-1} a_{-s_0} \in MA$ with $s_0 \neq 0$ so that $L = A(o)$. Let $x_i \in X$ be a sequence such that $d(x_i, A(o)) \to \infty$. Write $x_i = n_i a_{-t_i}(o)$ with $n_i \in N$ and $t_i \in \mathbb{R}$.

We may then write

$$d(gx_i, x_i) = d(a_{t_i} h_i n_i a_{-t_i}, a^{-1} o).$$

where $h_i = ma_{s_0} n_i^{-1} a_{-s_0} m^{-1} \in N$. As $d(x_i, A(o)) \to \infty$, we have $a_{t_i} n_i a_{-t_i} \to \infty$. It suffices to show $a_{t_i} h_i n_i a_{-t_i} \to \infty$.

By the assumption that $G$ has rank one, there is only one simple root, say $\alpha$ and $n$ is the sum of two root subspaces $n = n_\alpha + n_{2\alpha}$ where $[n, n] \subset n_{2\alpha}$. Since $n$ is a two-step nilpotent, we have that for any $X, Y \in n$,

$$\log \left( \exp(X) \exp(Y) \right) = X + Y + \frac{1}{2} [X, Y]. \quad (8.1)$$

Write $\log n_i = Y_i + Z_i$ with $Y_i \in n_\alpha$ and $Z_i \in n_{2\alpha}$. Since $\text{Ad}_m$ preserves $n_\alpha$ and $n_{2\alpha}$, we have

$$\log h_i = -\text{Ad}_{m a_{s_0}} \log n_i = -e^\alpha(s_0) \text{Ad}_m Y_i - e^{2\alpha(s_0)} \text{Ad}_m Z_i.$$
Therefore by (8.1), we get
\[ \log h_i n_i = (1 - e^{\alpha(s_0)} \text{Ad}_m) Y_i + (1 - e^{2\alpha(s_0)} \text{Ad}_m) Z_i - \frac{1}{2} [e^{\alpha(s_0)} \text{Ad}_m Y_i, Y_i]. \]

Hence
\[ \text{Ad}_{a_i} \log h_i n_i = (1 - e^{\alpha(s_0)} \text{Ad}_m) e^{\alpha(t_i)} Y_i + (1 - e^{2\alpha(s_0)} \text{Ad}_m) e^{2\alpha(t_i)} Z_i - [e^{\alpha(s_0)} \text{Ad}_m e^{\alpha(t_i)} Y_i, e^{\alpha(t_i)} Y_i]. \]

Now suppose that \( a_i, h_i n_i a_{-t_i} \) does not go to infinity as \( i \to \infty \). By passing to a subsequence, we may assume that \( \text{Ad}_{a_i} \log h_i n_i \) is uniformly bounded. It follows that both sequences \( (1 - e^{\alpha(s_0)} \text{Ad}_m) e^{\alpha(t_i)} Y_i \) and \( (1 - e^{2\alpha(s_0)} \text{Ad}_m) e^{2\alpha(t_i)} Z_i - [e^{\alpha(s_0)} \text{Ad}_m e^{\alpha(t_i)} Y_i, e^{\alpha(t_i)} Y_i] \) are uniformly bounded. Since \( \alpha(s_0) \neq 0 \), we have \( e^{\alpha(t_i)} Y_i \) is uniformly bounded, which then implies that \( e^{2\alpha(t_i)} Z_i \) is uniformly bounded.

This implies that \( \text{Ad}_{a_i} \log n_i = e^{\alpha(t_i)} Y_i + e^{2\alpha(t_i)} Z_i \) is uniformly bounded, contradicting the hypothesis \( d(a_i n_i a_{-t_i}) \to \infty \). This proves the claim.

Let \( \Gamma < G \) be a discrete subgroup. For \( x = [g] \in \Gamma \backslash G \), the injectivity radius \( \text{inj} x \) is defined as the supremum \( r > 0 \) such that the ball \( B_r(g) = \{ h \in G \mid d(h, g) < r \} \) injects to \( \Gamma \backslash G \) under the canonical quotient map \( G \to \Gamma \backslash G \). The injectivity radius of \( \Gamma \backslash G \) is defined as \( \text{inj}(\Gamma \backslash G) = \sup_{x \in \Gamma \backslash G} \text{inj}(x) \).

**Proposition 8.3.** If any Anosov subgroup \( \Gamma < G \) which is not a cocompact lattice in a rank one group \( G \), \( \text{inj}(\Gamma \backslash G) = \infty \).

**Proof.** If \( G \) has rank one, \( \Gamma \) is a convex cocompact subgroup which is not a cocompact lattice. In this case, take any \( \xi \in \partial X \) which is not a limit point, and any \( g_i \in G \) such that \( g_i(o) \to \xi \). Then \( \text{inj}(g_i) \to \infty \). Now suppose \( \text{rank} G \geq 2 \). Then \( \text{Vol}(\Gamma \backslash G) = \infty \); otherwise, \( \Gamma < G \) is a co-compact lattice as Anosov subgroups consists only of loxodromic elements. Since \( \Gamma \) is a Gromov hyperbolic group as an abstract group, it follow that \( G \) is a Gromov hyperbolic space and hence must be of rank one, which contradicts the hypothesis.

Hence, if every simple factor of \( G \) has rank at most 2, the claim follows from a more general result of Fraczyk and Gelander [17] which applies to all discrete subgroups of infinite co-volume.

Therefore it suffices to consider the case where \( G = G_1 \times G_2 \) where \( G_1 \) and \( G_2 \) are respectively semisimple real algebraic subgroups of rank at least one and precisely one. Let \( \Sigma \) be a Gromov hyperbolic group and \( \pi : \Sigma \to G \) be an Anosov representation with \( \Gamma = \pi(\Sigma) \) as in the definition 2.4. Let \( \pi_i : \Sigma \to G_i \) be the composition of \( \pi \) and the projection \( G \to G_i \) for each \( i \). It follows from the property (3) in the definition 2.4 that \( \pi_i(\Sigma) \) is a discrete subgroup of \( G_i \) for each \( i = 1, 2 \).

Let \( X_i \) denote the rank one symmetric space associated to \( G_i \) and set \( X \) denote the Riemannian product \( X = X_1 \times X_2 \). Let \( R > 0 \) be an arbitrary number. We will find a point \( x \in X \) with \( \text{inj}(x) \geq R \), i.e., \( d(x, \gamma x) > R \).
for all non-trivial $\gamma \in \Gamma$; this implies the claim. Choose any $x_1 \in X_1$. By
the discreteness of $\pi_1(\Sigma)$, the set $\{\sigma \in \Sigma \mid 0 < d_1(\pi_1(\sigma)x_1, x_1) < R\}$ is
finite, which we write as $\{\sigma_1, \ldots, \sigma_m\}$. For each $\sigma \in \Sigma \setminus \{e\}$, define a subset
$T_2(\sigma) \subset X_2$ by

$$T_2(\sigma) = \{z \in X_2 : d_2(\pi_2(\sigma)z, z) < R\}.$$  

Note that $\pi_2(\sigma)$ is a loxodromic element of $G_2$ and $T_2(\sigma)$ is contained in
a bounded neighborhood of the translation axis of $\pi_2(\sigma)$ by Lemma 8.2.
In particular, the symmetric space $X_2$ is not covered by the finite union
$\bigcup_{j=1}^m T_2(\sigma_j)$. Hence we may choose $x_2 \in X_2$ outside of $\bigcup_{j=1}^m T_2(\sigma_j)$. We
now claim that the injectivity radius at $(x_1, x_2)$ is at least $\hat{R}$; suppose not.
Then for some $\sigma \in \Sigma \setminus \{e\}$, $d((\pi_i(\sigma)x_i), x) < R$. In particular, for $i = 1, 2$,
$d_i(\pi_i(\sigma)x_i, x) < R$. It follows that $\sigma = \sigma_j$ for some $1 \leq j \leq m$ and
$x_2 \in T_2(\sigma_j)$, contradicting the choice of $x_2$. This proves the claim. \hfill \square

Theorem 8.1 follows from Proposition 8.3 and the following proposition,
which was suggested by C. McMullen.

**Proposition 8.4.** Let $\Gamma < G$ be a discrete subgroup with inj($\Gamma \setminus G$) = $\infty$.
Then

$$L^2(G) \propto L^2(\Gamma \setminus G) \quad \text{and} \quad \sigma(X) \subset \sigma(\Gamma \setminus X).$$

**Proof.** Let $v \in L^2(G)$. We may choose a sequence $f_i \in C_c(G)$ such that
$f_i$ vanishes outside $B_{R_i}(e)$ and $\|v - f_i\| \rightarrow 0$ as $i \rightarrow \infty$. Since the matrix
coefficients $(g_{f_i} f_i)$ converges to $(gv, v)$ uniformly on compact subsets.

For each $i$, consider the function $F_i \in C_c(\Gamma \setminus G)$ given by

$$F_i(x) = \sum_{\gamma \in \Gamma} g_i^{-1} f_i(\gamma h) \quad \text{for any } x = |h| \in \Gamma \setminus G.$$  

Since $B_{R_i}(g_i)$ injects to $\Gamma \setminus G$, we have for any $g \in G$,

$$\langle g.F_i, F_i \rangle_{L^2(\Gamma \setminus G)} = \int_{\Gamma \setminus G} F_i(xg) F_i(x) dx$$

\[= \int_{\Gamma \setminus G} F_i(\Gamma h g) \left( \sum_{\gamma \in \Gamma} f_i(\gamma h) \right) d(\Gamma h) = \int_{\Gamma \setminus G} F_i(\Gamma h g) f_i(h) dh \]

\[= \int_{h \in B_{R_i}(g_i)} \left( \sum_{\gamma \in \Gamma} f_i(\gamma h g_i^{-1}) \right) f_i(h) dh = \int_{B_{R_i}(g_i)} g_i^{-1} f_i(h g_i) f_i(h) dh \]

\[= \langle g_i^{-1} f_i, f_i \rangle_{L^2(G)}. \]

Therefore the diagonal matrix coefficient $g \mapsto \langle gv, v \rangle$ can be approximated
by the matrix coefficients in $L^2(\Gamma \setminus G)$ uniformly on compact subsets. This
implies the first claim.

In order to prove the second claim, let $W^1(\Gamma \setminus X) \subset L^2(\Gamma \setminus X)$ be as defined
in the proof of Theorem 6.5. Let $\lambda \in \sigma(X)$. By Weyl’s criterion (Theorem


6.1), there exists a sequence of $L^2(X)$-unit vectors $\{u_n\}_{n \in \mathbb{N}} \subset W^1(X)$ such that
\[
\lim_{n \to \infty} \|(\Delta + \lambda)u_n\|_{L^2(X)} = 0.
\]
Since $C_c^\infty(X)$ is dense in $W^1(X)$ with respect to $\| \cdot \|_{W^1(X)}$, we may assume that $\{u_n\}_{n \in \mathbb{N}} \subset C_c^\infty(X)$. Denoting the support of $u_n$ by $B_n$, since $\Gamma \setminus X$ has infinite injectivity radius, for each $n \in \mathbb{N}$ we can find $g_n \in G$ so that $g_n B_n$ injects to $\Gamma \setminus G$. We may therefore define $\{v_n\}_{n \in \mathbb{N}} \subset W^1(\Gamma \setminus X)$ by
\[
v_n(\Gamma g_n x) = \begin{cases} u_n(x) & \text{if } x \in B_n \\ 0 & \text{otherwise.} \end{cases}
\]
The $G$-invariance of $\Delta$ then gives
\[
\lim_{n \to \infty} \|(\Delta + \lambda)v_n\|_{L^2(\Gamma \setminus X)} = \lim_{n \to \infty} \|(\Delta + \lambda)u_n\|_{L^2(X)} = 0;
\]
and so using Weyl’s criterion again yields $\lambda \in \sigma(\Gamma \setminus X)$, hence $\sigma(X) \subset \sigma(\Gamma \setminus X)$, as claimed. \hfill $\Box$

9. Temperedness of $L^2(\Gamma \setminus G)$

Let $G$ be a connected semisimple real algebraic group and $\Gamma < G$ be a Zariski dense discrete subgroup. The goal of this section is to prove Theorem 9.4 and Corollary 9.7.

**Burger-Roblin measures.** We set $N^+ = u_0 N u_0^{-1}$ and $N^- = N$.

For a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$, we denote by $m^\text{BR}_\nu$ and $m^\text{BR}_\nu^+$ the associated $N^+$ and $N^-$-invariant Burger-Roblin measures on $\Gamma \setminus G$ respectively, as defined in [12]. By [12, Lem. 4.9], it can also be defined as follows: for any $f \in C_c(\Gamma \setminus G)$,
\[
m^\text{BR}_\nu(f) = \int [k]m(\exp a)n_{K/M \times MAN^+} f([k]m(\exp a)n)e^{-\psi(a)} d\nu(k^-) dm da dn
\]
and
\[
m^\text{BR}_\nu^+(f) = \int [k]m(\exp a)n_{K/M \times MAN^+} f([k]m(\exp a)n)e^{\psi(a)} d\nu(k^+) dm da dn
\]
where $dm, da, dn$ are Haar measures on $M, a, N$ respectively.

We denote by $dx$ the $G$-invariant measure on $\Gamma \setminus G$ which is defined using the $(G, 2\rho)$-conformal measure, that is, the $K$-invariant probability measure on $\mathcal{F}$ (see [12, (3.11)])). For functions $f_1, f_2$ on $\Gamma \setminus G$, we write
\[
\langle f_1, f_2 \rangle = \int_{\Gamma \setminus G} f_1(x)f_2(x) dx
\]
whenever the integral converges. We write $C_c(\Gamma \setminus G)_K$ for the space of $K$-invariant compactly supported continuous functions on $\Gamma \setminus G$.

**Lemma 9.1.** For a $(\Gamma, \psi)$-conformal measure $\nu$ on $\mathcal{F}$ and any $f \in C_c(\Gamma \setminus G)_K$, we have
\[
m^\text{BR}_\nu(f) = \langle f, E_\nu \rangle = m^\text{BR}_\nu^+(f).
\]
Proof. If \( g = (\exp b)nk \in AN^+K \), then \( \beta_{e^-}(go, o) = \beta_{e^+}(\exp(-i(b)), o) = i(b) \). Hence
\[
m^\nu_BR(f) = \int_{KAN^+} \int_K f(k \exp b nk_0)e^{-\psi_{i(b)}} dk_0 dv(k^-) dbdn
= \int_G \int_K f(k) e^{-\psi_{(\beta_{e^-}(go, o))}} dv(k^-) dg
= \int_G f(g) \int_K e^{-\psi_{(\beta_{e^-}(go, o))}} dv(k^-) dg = \langle f, E_\nu \rangle
\]
If \( g = (\exp b)nk \in ANK \), then \( \beta_{e^+}(go, o) = -b \) and using this, the second identity can be proved similarly. □

Local matrix coefficients for Anosov subgroups. In the rest of this section, we assume that
\[
\Gamma < G \text{ is a Zariski dense Anosov subgroup with respect to } P.
\]

Lemma 9.2. For any \( \psi \in D_\Gamma \), there exists a unique unit vector \( u \in a^+ \) and \( 0 < c \leq 1 \) such that \( cv(u) = \psi_T(u) \) and \( u \in \text{int } L_\Gamma \).

Proof. Since \( \psi_T \) is strictly concave \([40, \text{Propositions~4.6, 4.11}]\), there exists \( 0 < c \leq 1 \) and unique \( u \in L \) such that \( cv(u) = \psi_T(u) \). Moreover there is no linear form tangent to \( \psi_T \) at \( \partial L_\Gamma \) \([40]\), and hence \( u \in \text{int } L_\Gamma \). □

For each \( v \in \text{int } L_\Gamma \), there exists a unique \( \psi_v \in D^*_\Gamma \) such that \( \psi_v(v) = \psi_T(v) \) and a unique \( (\Gamma, \psi_v) \)-conformal probability measure, say, \( \nu_v \) supported on \( \Lambda \) \([12, \text{Corollary~7.8 and Theorem~7.9}]\).

Hence \([12, \text{Theorem~7.12}]\), together with Lemma 9.1, implies (let \( r = \text{rank } G \)):

Theorem 9.3. For any \( v \in \text{int } L_\Gamma \), there exists \( \kappa_v > 0 \) such that for all \( f_1, f_2 \in C_c(\Gamma \backslash G)_K \) and any \( w \in \text{ker } \psi_v \),
\[
\lim_{t \to +\infty} t^{(r-1)/2} e^{t(2c-\psi_v)(tv + \sqrt{tw})} (\exp(tv + \sqrt{tw})f_1, f_2) = \kappa_v e^{-I(w)} \cdot \langle f_1, E_{\nu_v} \rangle \cdot \langle f_2, E_{\nu_v} \rangle
\]
where \( I(w) \in \mathbb{R} \) is given as in \([12, 7.5]\). Moreover, the left-hand side is uniformly bounded over all \( (t, w) \in (0, \infty) \times \text{ker } \psi_v \) such that \( tv + \sqrt{tw} \in a^+ \).

Theorem 9.4. (1) We have \( L^2(\Gamma \backslash G) \) is tempered if and only if \( \psi_T \leq \rho \).
(2) If \( L^2(\Gamma \backslash G) \) is tempered, then \( \lambda = \|\rho\|^2 \) and \( \sigma(\Gamma \backslash X) = \|\rho^2, \infty) \).

Proof. The second claim follows from Theorems 6.4 and 8.1. Suppose that \( \psi_T \leq \rho \). In order to show that \( L^2(\Gamma \backslash G) \) is tempered, by Proposition 2.7, it suffices to show that the matrix coefficients \( g \mapsto \langle g \cdot f_1, f_2 \rangle \) are in \( L^{2+\varepsilon}(\Gamma \backslash G) \) for all \( \varepsilon > 0 \) and for all \( f_1, f_2 \in C_c(\Gamma \backslash G) \), since \( C_c(\Gamma \backslash G) \) is dense in \( L^2(\Gamma \backslash G) \). Without loss of generality, we may just consider non-negative functions.
\[ f_1, f_2 \in C_c(\Gamma \setminus G). \] Fix any \( \varepsilon > 0. \) Then using the Cartan decomposition \( G = KA^+K, \) we have
\[
\int_G \langle g \cdot f_1, f_2 \rangle^{2+\varepsilon} \, dg = \int_K \int_{a^+} \int_K \langle k_1 \exp(v)k_2 \cdot f_1, f_2 \rangle^{2+\varepsilon} \Xi(v) \, dk_1 \, dv \, dk_2,
\]
where \( \Xi(v) \leq e^{2\rho(v)} \) (here and henceforth \( f(v) \propto g(v) \) means that the ratio \( f(v)/g(v) \) is bounded uniformly between two positive constants, and \( f \ll g \) means that \( |f| \leq c|g| \) for some \( c > 0). \) Denoting \( F_i(\Gamma g) = \max_{k \in K} f_i(\Gamma gk) \in C_c(\Gamma \setminus G)_K, \) we then have
\[
\int_G \langle g \cdot f_1, f_2 \rangle^{2+\varepsilon} \, dg \ll \int_{a^+} \langle \exp(v) \cdot F_1, F_2 \rangle^{2+\varepsilon} e^{2\rho(v)} \, dv.
\]
Since \( \psi_T \leq \rho, \) we have \( \rho \in D_T. \) By Lemma 9.2, there exists \( 0 < c \leq 1 \) such that \( c \rho \in D_T^+ \) and a unit vector \( u_0 \in \text{int} \mathcal{L}_T \) such that
\[
\psi_T(u_0) = c \rho(u_0).
\]
We now parameterize \( a^+ \) as follows: for each \( v \in \ker \rho, \) define
\[
t_v := \min\{t \in \mathbb{R}_{>0} : tu_0 + \sqrt{tv} \in a^+ \}.
\]
Substituting \( u = tu_0 + \sqrt{tv} \) for \( t \geq 0 \) and \( v \in b \cap \ker \rho \) gives \( du = s \cdot t^{\varepsilon - 1} \, dt \, dv \) for some constant \( s > 0. \) Then (letting \( r = \text{dim}(a) \))
\[
\int_{a^+} \langle \exp(u) \cdot F_1, F_2 \rangle^{2+\varepsilon} e^{2\rho(u)} \, du \ll \int_{\ker \rho} \int_{t_v}^\infty \langle \exp(tu_0 + \sqrt{tv}) \cdot F_1, F_2 \rangle^{2+\varepsilon} e^{2t\rho(u_0)} t^{(r-1)/2} \, dt \, dv.
\]
By Theorem 9.3 ([12, Theorem 7.19 (1)]), there exists \( C = C(F_1, F_2) > 0 \) such that
\[
t^{(r-1)/2} e^{(2-c\rho(u_0))} \langle \exp(tu_0 + \sqrt{tv}) \cdot F_1, F_2 \rangle \leq C
\]
for all \( (v, t) \in \ker \rho \times [t_v, \infty). \)

Combining this with the trivial bound
\[
\langle g \cdot F_1, F_2 \rangle \leq \|F_1\|\|F_2\|,
\]
we have (again, for all \( (v, t) \in \ker \rho \times [t_v, \infty)\)),
\[
\langle \exp(tu_0 + \sqrt{tv}) \cdot F_1, F_2 \rangle^{2+\varepsilon} \leq (C + \|F_1\|\|F_2\|)^{2+\varepsilon} \left( \min \left\{ 1, t^{-(r-1)/2} e^{-(2-c)\rho(u_0)} \right\} \right)^{2+\varepsilon} \\
\ll \min\{1, e^{-\eta\rho(u_0)} \} \leq e^{-\eta\rho(u_0)},
\]
where \( \eta = (2-c)(2+\varepsilon) > 2. \) This gives
\[
\int_G \langle g \cdot f_1, f_2 \rangle^{2+\varepsilon} \, dg \ll \int_{v \in \ker \rho} \int_{t_v}^\infty e^{-\eta\rho(u_0)} e^{2t\rho(u_0)} t^{(r-1)/2} \, dt \, dv \\
\ll \int_{a^+} e^{-(\eta-2)\rho(u)} \, du < \infty.
\]
Therefore \( L^2(\Gamma \setminus G) \) is tempered.
To show the converse, suppose now that \( L^2(\Gamma \backslash G) \) is tempered. Then by the definition of temperedness and the estimate of \( \Xi_G \) in (2.7), it follows that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that for any \( f_1, f_2 \in L^2(\Gamma \backslash G)_K \) and \( u \in a^+ \),

\[
\langle \exp(u) \cdot f_1, f_2 \rangle \leq C_\varepsilon \| f_1 \| \| f_2 \| e^{-(1-\varepsilon)\rho(u)}.
\]  

(9.1)

Given any unit vector \( v \in \text{int} \mathcal{L}_\Gamma \), let \( \psi_v \) be the linear form tangent to \( \psi_\Gamma \) at \( v \). We then choose two non-negative functions \( f_1, f_2 \in C_c(\Gamma \backslash G)_K \) such that \( \langle f_1, E_{\nu_1(v)} \rangle \cdot \langle f_2, E_{\nu_2} \rangle > 0 \). Then by Theorem 9.3, \( \kappa_v > 0 \) such that

\[
\lim_{t \to +\infty} t^{(r-1)/2} e^{(2\rho - \psi_v)(tv)} \langle \exp(tv) f_1, f_2 \rangle = \kappa_v \langle f_1, E_{\nu_1(v)} \rangle \cdot \langle f_2, E_{\nu_2} \rangle.
\]

Combining the above with (9.1) gives

\[
0 < \lim_{t \to \infty} t^{(r-1)/2} e^{(2\rho - \psi_v)(tv)} \langle \exp(tv) f_1, f_2 \rangle \leq \lim_{t \to \infty} C_\varepsilon \| f_1 \| \| f_2 \| t^{(r-1)/2} e^{(2\rho - \psi_v)(tv)} e^{-(1-\varepsilon)\rho(tv)}.
\]

Hence \( (1 + \varepsilon)\rho(v) \geq \psi_v(v) = \psi_\Gamma(v) \). Since \( \varepsilon > 0 \) is arbitrary, we have

\[
\rho(v) \geq \psi_\Gamma(v) \quad \text{for all } v \in \text{int} \mathcal{L}_\Gamma.
\]

Note now that since \( \psi_\Gamma \) is an upper semi-continuous concave function, \( \psi_\Gamma \) is continuous on any line segment connecting a point in \( \text{int} \mathcal{L}_\Gamma \) and a point on the boundary of \( \mathcal{L}_\Gamma \) (cf. [26, Lem. 3.11]). This implies that \( \rho(v) \geq \psi_\Gamma(v) \) for all \( v \in \mathcal{L}_\Gamma \). By definition, \( \psi_\Gamma = -\infty \) outside \( \mathcal{L}_\Gamma \); it then follows that \( \rho \geq \psi_\Gamma \).

Now recall the following recent theorem of Kim, Minsky, and Oh [26, Theorem 4.1 and Corollary 4.2]:

**Theorem 9.5.** [26] Let \( \Gamma \) be an Anosov subgroup of the product \( G \) of two or three simple real algebraic groups or \( \Gamma < G = \text{PSL}_d(\mathbb{R}) \) be the image of a Hitchin representation with \( d = 3, 4 \). Then

\[
\psi_\Gamma \leq \rho.
\]

Moreover, if \( H \) denotes the Zariski closure of \( \Gamma \) (which is known to be semisimple) and rank \( H \geq 2 \), then \( \psi_\Gamma \leq \rho_H \) still holds on \( a \cap \text{Lie}(H) \) when \( \rho_H \) is the half-sum of the positive roots for \( (\text{Lie}(H), a \cap \text{Lie}(H)) \).

For any unimodular closed subgroup \( H \), the quotient space \( H \backslash G \) admits a \( G \)-invariant Radon measure, and hence \( G \) has a unitary action on \( L^2(H \backslash G) \) via the right translations. We will use the following proposition to obtain the temperedness of \( L^2(\Gamma \backslash G) \) when \( \Gamma \) is not Zariski dense in \( G \) in Corollary 9.7.

**Proposition 9.6.** Let \( \Gamma_0 < H \) be unimodular closed subgroups of \( G \). If either \( L^2(\Gamma_0 \backslash H) \) or \( L^2(H \backslash G) \) is tempered, then \( L^2(\Gamma_0 \backslash G) \) is tempered.
Proof. If $L^2(\Gamma_0 \backslash H)$ is tempered, then $L^2(\Gamma_0 \backslash H)$ is weakly contained in $L^2(H)$ by Proposition 2.7. Then by [51, Proposition 7.3.7], $\text{Ind}_H^G(L^2(\Gamma_0 \backslash H))$ is weakly contained in $\text{Ind}_H^G(L^2(H))$, where Ind is the unitary induction operator of a unitary representation of $H$ to that of $G$ (cf. [34, 5.2]). Since $\text{Ind}_H^G(L^2(\Gamma_0 \backslash H)) = L^2(\Gamma_0 \backslash G)$ and $\text{Ind}_H^G(L^2(H)) = L^2(G)$ [34, 5.2.1], it follows that $L^2(\Gamma_0 \backslash G)$ is weakly contained in $L^2(G)$. Now suppose that $L^2(H \backslash G)$ is tempered. The “Herz majorization principle” [2, Ch. 6] implies that $\text{Ind}_H^G(L^2(\Gamma_0 \backslash H)) \simeq L^2(\Gamma \backslash G)$ is tempered. □

Corollary 9.7. Let $\Gamma < G$ be as in Theorem 9.5. Then $L^2(\Gamma \backslash G)$ is tempered.

Proof. Let $H$ denote the Zariski closure of $\Gamma$, which is a semisimple real algebraic subgroup of $G$. We use the classification of possible $H$ from ([20], [45]). Suppose rank $H \geq 2$. Then Theorems 9.4 and 9.5 imply that $L^2(\Gamma \backslash H)$ is tempered. Hence by Proposition 9.6, $L^2(\Gamma \backslash G)$ is tempered. Now suppose rank $H = 1$. When $G = \prod_i G_i$ is a product of rank one groups, $H \cap G_i = \{e\}$ and hence $L^2((H \cap G_i) \backslash G_i) = L^2(G_i)$ is $G_i$-tempered for each $i$. This implies that $L^2(H \backslash G)$ is tempered by [5, Proposition 8.4].

In the Hitchin cases, we have $H \simeq SO(2,1) < G = \text{PSL}_3(\mathbb{R})$ or $H \simeq SO(2,1) < G = \text{PSL}_4(\mathbb{R})$ ([20], [45]). In each of these cases, we check that $L^2(H \backslash G)$ is tempered using [4, Example 5.9]. Therefore the claim follows from Proposition 9.6. □

Proofs of Theorem 1.6. The equivalence $(1) \iff (2)$ is proved in Theorem 9.4. The equivalence $(2) \iff (3)$ is proved in Theorem 8.1. The implication $(1) + (2) \Rightarrow (4)$ follows from Corollary 7.2.

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