PATTERSON-SULLIVAN MEASURES OF ANOSOV GROUPS ARE HAUSDORFF MEASURES

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ABSTRACT. Let $G$ be a connected semisimple real algebraic group. For any Anosov subgroup $\Gamma$ of $G$ with semisimple Zariski closure, we prove that every symmetric Patterson-Sullivan measure on the limit set is exactly the Hausdorff measure of dimension one with respect to the associated conformal premetric. Furthermore, the Hausdorff measure of any ball is proportional to its radius. We discuss several applications of this result.

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1. INTRODUCTION

Let $G$ be a connected semisimple real algebraic group. Let $\Gamma < G$ be a Zariski dense discrete subgroup, or more generally, a discrete subgroup whose Zariski closure is semisimple. Patterson-Sullivan measures are certain families of Borel measures on a generalized flag variety, which are supported on the limit set of $\Gamma$. They were constructed by Quint around 2000 [31] (see also [1, 7]), following Patterson-Sullivan’s construction for Kleinian groups ([27, 33]). Patterson-Sullivan measures play a crucial role in the study of dynamics on the associated locally symmetric space, especially in the counting and equidistribution of $\Gamma$-orbits of various geometric objects.

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Sullivan showed that for convex cocompact Kleinian groups, these measures are exactly the Hausdorff measures on the limit sets, giving a geometric characterization of these measures in terms of the internal metric on the limit sets \[33, \text{Theorem 8}\]. In recent decades, Anosov subgroups have emerged as a higher rank generalization of convex cocompact Kleinian groups. Therefore it is natural to ask when the Patterson-Sullivan measures for Anosov subgroups arise as Hausdorff measures on the limit sets with respect to appropriate metrics. The main goal of this paper is to answer this question.

To state our results precisely, fix a Cartan decomposition \( G = KA^+K \) where \( K \) is a maximal compact subgroup and \( A^+ \) is a positive Weyl chamber of a maximal real split torus \( A \). The rank of \( G \) is defined as the dimension of \( A \). When \( \text{rank} \, G \geq 2 \), we call \( G \) a higher rank Lie group. Let \( g \) and \( a \) denote the Lie algebras of \( G \) and \( A \) respectively, and set \( a^+ = \log A^+ \). Let \( \Pi \) denote the set of all simple roots of \((g, a)\) with respect to the choice of \( a^+ \).

Fix a non-empty subset \( \theta \) of \( \Pi \). Let \( P_\theta \) be the standard parabolic subgroup associated with \( \theta \). Setting \( a_\theta = \bigcap \{ \alpha \in \Pi \setminus \theta \} \ker \alpha \), the centralizer of \( a_\theta \) is a Levi subgroup of \( P_\theta \). For \( \theta = \Pi \), we omit the subscript \( \theta \) from now on. Note that our convention is that \( P = P_\Pi \) is a minimal parabolic subgroup of \( G \). The quotient space \( F_\theta = G/P_\theta \) is called the \( \theta \)-boundary, or a generalized flag variety.

Let \( \Gamma \) be a discrete subgroup of \( G \). We denote by \( \Lambda_\theta \) the limit set of \( \Gamma \) in \( F_\theta \) \[2\]. Let \( a^*_\theta \) denote the set of all linear forms on \( a_\theta \). For a linear form \( \psi \in a^*_\theta \), a \((\Gamma, \psi)\)-Patterson-Sullivan measure is a Borel probability measure \( \nu \) on \( \Lambda_\theta \) which satisfies that for all \( \gamma \in \Gamma \) and \( \xi \in \Lambda_\theta \),

\[
\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(\gamma))}
\]

where \( \beta \) denotes the Busemann map (see \[2.5\]). By a Patterson-Sullivan measure for \( \Gamma \), we mean a \((\Gamma, \psi)\)-Patterson-Sullivan measure for some \( \psi \in a^*_\theta \). Quint constructed Patterson-Sullivan measures for a large class of linear forms \( \psi \) \[31, \text{Theorem 8.4}\].

In this paper, we focus on a special class of discrete subgroups called \( \theta \)-Anosov subgroups. To define them, we denote by \( \mu : G \to a^+ \) the Cartan projection defined by the condition that \( g \in K(\exp \mu(g))K \) for all \( g \in G \).

A finitely generated subgroup \( \Gamma < G \) is called \( \theta \)-Anosov if there exists a constant \( C > 1 \) such that for all \( \alpha \in \theta \),

\[
\alpha(\mu(\gamma)) \geq C^{-1}|\gamma| - C \quad \text{for all } \gamma \in \Gamma
\]

where \( |\cdot| \) is a word metric on \( \Gamma \) with respect to a fixed finite generating set.

In the rest of the introduction, we assume that \( \Gamma \) is a \( \theta \)-Anosov subgroup whose Zariski closure is semisimple. In this case, the space of all Patterson-Sullivan measures for \( \Gamma \) is parameterized by the set \( \mathcal{T}_\Gamma \) of all linear forms
on $a_\theta$ tangent to the $\theta$-growth indicator $\psi_\Gamma^\theta$ (Definition 3.1):

$$T_\Gamma = \{ \psi \in a_\theta^* : \psi \geq \psi_\Gamma^\theta, \psi(u) = \psi_\Gamma^\theta(u) \text{ for some } u \in a_\theta - \{0\} \}.$$ 

More precisely, for any $\psi \in T_\Gamma$, there exists a unique $(\Gamma, \psi)$-Patterson-Sullivan measure $\nu_\psi$ and every Patterson-Sullivan measure of $\Gamma$ on $\Lambda_\theta$ arises in this way (see Theorem 3.2 for a precise statement). Note that $T_\Gamma$ is homeomorphic to a ball of dimension $\#\theta - 1$ when $\Gamma$ is Zariski dense in $G$.

**Hausdorff measures on limit sets.** Anosov subgroups of a rank one Lie group $G$ are precisely convex cocompact subgroups. For $G = SO^\circ(n,1)$ with $n \geq 2$, Sullivan [33] showed that the unique Patterson-Sullivan measure coincides with the Hausdorff measure on its limit set $\Lambda$ with respect to a $K$-invariant Riemannian metric on $S^{n-1}$. In other rank one cases, the unique Patterson-Sullivan measure still coincides with the Hausdorff measure on $\Lambda$, but with respect to a $K$-invariant sub-Riemannian metric (not Riemannian), which can be defined in terms of the Gromov product [36].

The main goal of this paper is to address the question of whether we have an analogous theorem for Anosov subgroups of higher rank Lie groups. It turns out that for Anosov subgroups, we can define a premetric $d_\psi$ on the limit set $\Lambda_\theta$ for each $\psi \in a_\theta^*$ using the Gromov product. More precisely, the $\theta$-Anosov property of $\Gamma$ implies that any two distinct points of $\Lambda_\theta$ are antipodal and hence the following defines a premetric $d_\psi$ on $\Lambda_\theta$:

$$d_\psi(\xi, \eta) = \begin{cases} 
    e^{-\psi(G(\xi, \eta))} & \text{if } \xi \neq \eta \\
    0 & \text{if } \xi = \eta 
\end{cases}$$

where $G$ is the $a$-valued Gromov product (see Definition 2.3).

For $s > 0$, the $s$-dimensional Hausdorff measure $\mathcal{H}_\psi^s$ with respect to $d_\psi$ is defined as follows: for any subset $B \subset \Lambda_\theta$, let

$$\mathcal{H}_\psi^s(B) := \liminf_{\varepsilon \to 0} \left\{ \sum_i (\text{Diam}_\psi U_i)^s : B \subset \bigcup_i U_i, \text{ Diam}_\psi U_i \leq \varepsilon \text{ for all } i \right\}$$

where $\text{Diam}_\psi U = \sup_{\xi, \eta \in U} d_\psi(\xi, \eta)$. This is an outer measure and induces a Borel measure on $\Lambda_\theta$ (see [11, Appendix A]). For $s = 1$, we simply write $\mathcal{H}_\psi$ for $\mathcal{H}_\psi^1$.

When $\Gamma$ is a convex cocompact subgroup of $G = SO^\circ(n,1)$, $\psi \in T_\Gamma$ is simply the multiplication by the critical exponent $\delta_\Gamma$ which is the abscissa of the convergence of the Poincaré series $\sum_{\gamma \in \Gamma} e^{-s d(o, \gamma o)}$ where $d$ is the hyperbolic metric and $o \in \mathbb{H}^n$. In this case, the one-dimensional Hausdorff measure $\mathcal{H}_{\delta_\Gamma}$ is equal to the Hausdorff measure of dimension $\delta_\Gamma$ with respect to the $K$-invariant Riemannian metric on $S^{n-1}$. In higher rank, it turns out

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1On a topological space $X$, a premetric $d$ is a positive definite continuous function $d : X \times X \to \mathbb{R}$ such that $d(x, x) = 0$ for all $x \in X$. 
that whether or not the Patterson-Sullivan measure \( \nu_\psi, \psi \in T_\Gamma \), is given by the Hausdorff measure depends on the symmetricity of \( \psi \): the linear form \( \psi \) is called symmetric if \( \psi \) is invariant under the opposition involution \( i \) of \( a \).

Our main theorem is as follows: here \( B_\psi(\xi, r) = \{ \eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r \} \) denotes the \( d_\psi \)-ball of radius \( r \) with center \( \xi \in \Lambda_\theta \).

**Theorem 1.1.** Let \( \Gamma \) be a \( \theta \)-Anosov subgroup of \( G \) whose Zariski closure is semisimple. Let \( \psi \in T_\Gamma \) be a symmetric linear form. The Patterson-Sullivan measure \( \nu_\psi \) is equal to the one-dimensional Hausdorff measure \( H_\psi \), up to a constant multiple. 

Moreover, for any \( \xi \in \Lambda_\theta \) and \( 0 < r < 1 \),

\[
\nu_\psi(B_\psi(\xi, r)) \asymp r
\]

where the implied constants are independent of \( \xi \) and \( r \).

The symmetric hypothesis on \( \psi \) cannot be removed in this theorem. In fact, we prove the following (Theorem 9.2):

**Theorem 1.2.** Let \( \Gamma \) be a Zariski dense \( \theta \)-Anosov subgroup. If \( \psi \in T_\Gamma \) is not symmetric, then \( \nu_\psi \) is not proportional to \( H_\psi \) for any \( s > 0 \).

We emphasize that one of the novelties of Theorem 1.1 and Theorem 1.2 lies in their treatment of all Patterson-Sullivan measures of \( \Gamma \) associated to symmetric linear forms.

**Critical exponents and Hausdorff dimensions.** We denote by \( L_\theta \subset a_\theta^+ \) the \( \theta \)-limit cone of \( \Gamma \). When \( \psi \in a_\theta^* \) is positive on \( L_\theta - \{ 0 \} \), we denote by \( \delta_\psi \) the abscissa of the convergence of the \( \psi \)-Poincaré series \( s \mapsto \sum_{\gamma \in \Gamma} e^{-s\psi(\mu(\gamma))} \); it is a well-defined positive number which we call the \( \psi \)-critical exponent of \( \Gamma \). We have

\[
\delta_\psi = \delta_\psi(\Gamma) = \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \psi(\mu(\gamma)) \leq T \}}{T}.
\]

The Hausdorff dimension of the premetric space \( (\Lambda_\theta, d_\psi) \) is defined as:

\[
\dim_\psi \Lambda_\theta := \inf \{ s \geq 0 : \mathcal{H}_s^\psi(\Lambda_\theta) < \infty \}.
\]

A natural question is whether \( \dim_\psi \Lambda_\theta \) is equal to \( \delta_\psi \). The following theorem generalizes the classical theorem of Patterson-Sullivan (27, 33) that the Hausdorff dimension of the limit set of a convex cocompact subgroup \( \Gamma \) of \( \text{SO}^\circ(n, 1) \) is equal to the critical exponent \( \delta_\Gamma \) (see also [10] for a general rank one case).

**Theorem 1.3.** For any \( \psi \in a_\theta^+ \) which is positive on \( L_\theta - \{ 0 \} \), we have

\[
\dim_\psi \Lambda_\theta = \delta_\psi
\]

where \( \bar{\psi} = \frac{\psi + \psi_i}{2} \). In particular, if \( \psi \) is symmetric, then \( \dim_\psi \Lambda_\theta = \delta_\psi \).

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1. We write \( f(r) \ll g(r) \) if for some constant \( C > 0 \), \( f(r) \leq Cg(r) \) for all \( r > 0 \). The notation \( f(r) \asymp g(r) \) means that \( f(r) \ll g(r) \) and \( g(r) \ll f(r) \).
Theorem 1.3 implies that \( \dim_\psi \Lambda_\theta \) changes analytically on \( \theta \)-Anosov representations (see Corollary 9.11).

Remark 1.4. If \( \psi \) is not symmetric, we prove in Proposition 9.9 that
\[
\dim_\psi \Lambda_\theta = \delta_\bar{\psi} < \delta_\psi
\]
at least when \( \Gamma \) is Zariski dense.

As an example, let us consider Hitchin subgroups of \( \text{PSL}_d(\mathbb{R}) \): let \( \iota \) be the irreducible representation of \( \text{PSL}_2(\mathbb{R}) \) into \( \text{PSL}_d(\mathbb{R}) \) for \( d \geq 2 \), which is unique up to conjugations. A Hitchin subgroup is the image of a representation \( \pi : \Sigma \to \text{PSL}_d(\mathbb{R}) \) of a uniform lattice \( \Sigma \) belonging to the same connected component as \( \iota|_{\Sigma} \) in the character variety \( \text{Hom}(\Sigma, \text{PSL}_d(\mathbb{R}))/\sim \) where the equivalence is given by conjugations. Hitchin subgroups are \( \Pi \)-Anosov, as was shown by Labourie [22]. For each simple root \( \alpha_k \) of \( \text{diag}(a_1, \ldots, a_d) = a_k - a_{k+1}, 1 \leq k \leq d-1 \), we have \( \delta_\alpha_k = 1 \) by Potrie-Sambarino [28]. So for \( d \) even, Theorem 1.3 implies that
\[
(1.2) \quad \dim_{\alpha_{d/2}} \Lambda_\Pi = 1.
\]

**Hausdorff dimension of \( \Lambda_\theta \) with respect to a Riemannian metric.**

We denote by \( \dim \Lambda_\theta \) the Hausdorff dimension of \( \Lambda_\theta \) for a Riemannian metric on \( F_\theta \); since all Riemannian metrics on \( F_\theta \) are Lipschitz equivalent to each other, this is well-defined. When \( G = \text{SO}^o(n,1) \), the \( K \)-invariant Riemannian metric on \( S^{n-1} \) is equal to the metric given by the Gromov product, up to a fixed power, and hence Theorem 1.3 is equivalent to saying that \( \dim \Lambda = \delta_\Gamma \). Already for other rank one groups, \( \dim \Lambda \) is not in general equal to the critical exponent of \( \Gamma \); note that a Riemannian metric on the geometric boundary is not bi-Lipschitz equivalent to any power of the metric given by the Gromov product.

From Theorem 1.3 we derive an estimate on \( \dim \Lambda_\theta \) in terms of various critical exponents. To give our estimate, for \( \alpha \in \theta \), let \( \chi_\alpha \) denote the highest weight of the Tits representation corresponding to \( \alpha \) as in Theorem 10.1. When \( G \) is split over \( \mathbb{R} \) (e.g., \( G = \text{PSL}_d(\mathbb{R}), \text{Sp}_{2d}(\mathbb{R}), \text{SO}^o(d,d), \text{SO}^o(d,d+1) \), etc), \( \chi_\alpha \) is simply the fundamental weight corresponding to \( \alpha \). The collection \( \{\chi_\alpha : \alpha \in \theta\} \) forms a basis of \( a_\theta^* \). Consider the following cone of \( a_\theta^* \):
\[
E_\theta = \left\{ \psi = \sum_{\alpha \in \theta} \kappa_\alpha \chi_\alpha : \kappa_\alpha \geq 0 \right\}.
\]

For \( \psi = \sum_{\alpha \in \theta} \kappa_\alpha \chi_\alpha \in E_\theta \), we set
\[
\kappa_\psi = \sum_{\alpha \in \theta} \kappa_\alpha.
\]

**Theorem 1.5.** For any \( \theta \)-Anosov subgroup \( \Gamma < G \) whose Zariski closure is semisimple, we have
\[
\dim \Lambda_\theta \geq \frac{1}{2} \sup_{\psi \in E_\theta} \kappa_\psi \delta_\bar{\psi}.
\]
Corollary 1.6. We have
\[
\max_{\alpha \in \Theta} \delta_{\alpha} \geq \dim \Lambda_{\theta} \geq \max_{\alpha \in \Theta} \delta_{\chi_{\alpha} + \chi_{i(\alpha)}}.
\]
Moreover, both the upper and lower bounds are attained by some Anosov subgroups.

In the special case where \( G = \text{PSL}_d(\mathbb{R}) \) and \( \theta = \{\alpha_1\} \), the lower bound was obtained by Dey-Kapovich [11]. See [29] for an upper bound in this special case. For some special class of Anosov subgroups, much sharper bounds are known, see [13], [29]. Recently, Li-Pan-Xu proved that for \( G = \text{PSL}_3(\mathbb{R}) \), \( \dim \Lambda_{\alpha_1} \) coincides with the affinity exponent of \( \Gamma \) [26]. See also [21] which shows that \( \Lambda_{\theta} \) has Lebesgue measure zero in higher rank and [23] for a dimension gap result which shows \( \dim F_{\theta} - \dim \Lambda_{\theta} \) is positive for Zariski dense Anosov subgroups of \( \text{PSL}_d(\mathbb{R}) \), \( d \geq 3 \).

The novelty of Corollary 1.6 is that it applies to all \( \theta \)-Anosov subgroups of any semisimple real algebraic group. Since both upper and lower bounds are realized by some Anosov subgroups, Corollary 1.6 cannot be improved in this generality. In fact, for Hitchin subgroups, we have \( \dim \Lambda_{\theta} = 1 = \delta_{\alpha} \) for all \( \alpha \in \theta \) [28]. The upper bound is also obtained for Anosov subgroups of the product of \( \text{SO}^0(n, 1) \)'s [18].

For the lower bound, let \( \Gamma \) be the image of a uniform lattice \( \Sigma \) of \( \text{PSL}_2(\mathbb{R}) \) under the embedding \( \text{PSL}_2(\mathbb{R}) \to \left( \begin{smallmatrix} \text{PSL}_2(\mathbb{R}) & 0 \\ 0 & I_{d-2} \end{smallmatrix} \right) < \text{PSL}_d(\mathbb{R}) \) where \( I_{d-2} \) is the \((d-2) \times (d-2)\) identity matrix. Then \( \Gamma \) is \( \{\alpha_1\}\)-Anosov. On one hand, the limit set \( \Lambda_{\alpha_1} \) of \( \Gamma \) in \( F_{\alpha_1} = \mathbb{P}(\mathbb{R}^d) \) is the projective line, and hence \( \dim \Lambda_{\alpha_1} = 1 \). On the other hand, since \((\chi_{\alpha_1} + \chi_{i(\alpha_1)})(\text{diag}(a_1, \cdots, a_d)) = a_1 - a_d\), we have
\[
\delta_{\chi_{\alpha_1} + \chi_{i(\alpha_1)}} = \delta_{\Sigma} = 1 = \dim \Lambda_{\alpha_1}.
\]
Therefore the lower bound in Corollary 1.6 is achieved for this example.

Bounds on growth indicators. The growth indicator \( \psi_{\theta}^{\Gamma} : a_{\theta} \to \mathbb{R} \cup \{-\infty\} \) is a higher rank version of the critical exponent of \( \Gamma \) that captures the growth rate of \( \mu(\Gamma) \) in each direction of \( a_{\theta} \). This was introduced by Quint [30] for \( \theta = \Pi \) and generalized to arbitrary \( \theta \) in [21]. We deduce the following upper bound on the growth indicator from Theorem 1.5.

Corollary 1.7. We have
\[
\psi_{\theta}^{\Gamma} \leq \frac{2}{\# \Theta} \cdot \left( \sum_{\alpha \in \Theta} \chi_{\alpha} \right) \cdot \dim \Lambda_{\theta}.
\]
Especially when \( G \) is split over \( \mathbb{R} \) and \( \Gamma \) is \( \Pi \)-Anosov,
\[
\psi_{\Gamma} \leq \frac{2\rho}{\text{rank } G} \cdot \dim \Lambda
\]
where \( 2\rho \) is the sum of all positive roots of \((\mathfrak{g}, \mathfrak{a})\) counted with multiplicity.
Remark 1.8. The second part follows because $\sum_{\alpha \in \Pi} \lambda_\alpha = \rho$ in the real-split case [3, Proposition 29]. We think this identity may be true for a general semisimple Lie group, but we could not find a reference.

The size of $\psi_\Gamma$ is closely related to the spectral properties of the locally symmetric manifold. Let $\lambda_0(\Gamma \backslash X)$ denote the bottom of the $L^2$-spectrum of the locally symmetric manifold $\Gamma \backslash G/K$ (see (10.2)). As first introduced by Harish-Chandra [5], a unitary representation $(\pi, \mathcal{H}_\pi)$ of a semisimple real algebraic group $G$ is tempered if all of its matrix coefficients belong to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$, or, equivalently, if $\pi$ is weakly contained in the regular representation $L^2(G)$. For any Zariski dense $\Pi$-Anosov subgroup $\Gamma < G$, it was shown in [12] that if $\psi_\Gamma \leq \rho$, then the quasi-regular representation $L^2(\Gamma \backslash G)$ is tempered and $\lambda_0(\Gamma \backslash X) = \|\rho\|^2$.

Therefore we get the following corollary from Corollary 1.7:

**Corollary 1.9.** Let $G$ be split over $\mathbb{R}$ and $\Gamma$ be a Zariski dense $\Pi$-Anosov subgroup of $G$. If

$$\dim \Lambda \leq \frac{\text{rank } G}{2},$$

then $\psi_\Gamma \leq \rho$, and hence $L^2(\Gamma \backslash G)$ is tempered and $\lambda_0(\Gamma \backslash X) = \|\rho\|^2$.

By Remark 1.8 we expect that the split hypothesis in this corollary may not be necessary. Corollary 1.9 immediately applies to many examples of Anosov subgroups with limit sets of low Hausdorff dimensions; for example to all Anosov subgroups of higher rank Lie groups with $\dim \Lambda \leq 1$. These examples include Hitchin subgroups of $\text{PSL}_d(\mathbb{R})$ with $d \geq 3$. Although Corollary 1.9 was already discovered for Hitchin subgroups in [19] and [12] relying on the work of [28], we obtain a different proof of this fact in this paper. As another example, if $\Gamma < \text{PSL}_d(\mathbb{R})$ is a Zariski dense subgroup which arises as the image of a purely hyperbolic Schottky representation of the free group $F_n$ on $n$-generators in the sense of Burelle-Treib [6], then $\Gamma$ is $\Pi$-Anosov and $\dim \Lambda \leq 1$ by [5, Proposition 11.1]. Corollary 1.9 then holds in this case as well.

**On the proof of Theorem 1.1**. The main point of the proof of Theorem 1.1 is to show that the Hausdorff measure $\mathcal{H}_\psi(\Lambda_\theta)$ is positive and finite for any symmetric $\psi \in \mathcal{T}_\Gamma$. The key step is to prove that for all $\xi \in \Lambda_\theta$ and $0 < r < 1$,

$$\nu_\psi(B_\psi(\xi, r)) \asymp r.$$  

Let $X = G/K$ be the associated Riemannian symmetric space and fix $o = [K] \in X$. The $\theta$-Anosov property of $\Gamma$ implies that $\Gamma$ is a hyperbolic group and the orbit map $\gamma \mapsto \gamma o$ extends continuously to a homeomorphism between the Gromov boundary of $\Gamma$ and limit set $\Lambda_\theta$.\footnote{$3\pi$ is weakly contained in a unitary representation $\sigma$ of $G$ if any diagonal matrix coefficients of $\pi$ can be approximated, uniformly on compact sets, by convex combinations of diagonal matrix coefficients of $\sigma$.}
As Sullivan’s original work [33], the notion of shadows plays a key role. For \( R > 0 \) and \( p \in X \), we denote by \( O_R(o, p) \subset F_\theta \) the endpoints of all positive Weyl chambers based at \( o \) passing the Riemannian ball \( B(p, R) \) in \( X \); these are called shadows. The main step in proving (1.3) is to show that for all \( \xi \in \Lambda_\theta \), we can find an element \( \gamma = \gamma(\xi) \in \Gamma \) such that the ball \( B_\psi(\xi, e^{-\psi(\mu(\gamma))}) \) is comparable with the shadow \( O_R(o, \gamma o) \) in \( \Lambda_\theta \), that is, (1.4) \[
B_\psi(\xi, c^{-1}e^{-\psi(\mu(\gamma))}) \subset O_R(o, \gamma o) \cap \Lambda_\theta \subset B_\psi(\xi, ce^{-\psi(\mu(\gamma))})
\]
where \( c, R > 1 \) are uniform constants. A metric-like function \( d_\psi \) on the orbit \( \Gamma o \) plays an important role. Indeed, if we define \( d_\psi \) by (1.5) \[
d_\psi(\gamma_1 o, \gamma_2 o) = \psi(\mu(\gamma_1^{-1}\gamma_2)) \quad \text{for any } \gamma_1, \gamma_2 \in \Gamma,
\]
then it satisfies the coarse triangle inequality: there exists \( D > 0 \) such that for any \( \gamma_1, \gamma_2, \gamma_3 \in \Gamma \), (1.6) \[
d_\psi(\gamma_1 o, \gamma_3 o) \leq d_\psi(\gamma_1 o, \gamma_2 o) + d_\psi(\gamma_2 o, \gamma_3 o) + D.
\]
The fact that there is no multiplicative error in (1.6) is crucial for our purpose.

![Figure 1. The C-diamond with tips x1, x2](image)

The main tool used to establish the coarse triangle inequality (1.6) is the notion of diamonds in the symmetric space \( X \) (Definition 4.5) and a generalization of the classical Morse lemma due to Kapovich-Leeb-Porti [17], which we will henceforth refer to as the KLP Morse lemma. Given a cone \( C \) in \( a^+ \), the \( C \)-diamond \( \diamond_C(x_1, x_2) \) with tips \( x_1, x_2 \in X \) is the intersection of two opposite cones at \( x_1, x_2 \), whose shapes are determined by \( C \) (Figure 1). Thus \( C \)-diamonds can be regarded as thick geodesic segments, whose thickness is given by the cone \( C \). The KLP Morse lemma states that for a suitably chosen cone \( C \), the image of a geodesic \( (\gamma_1, \ldots, \gamma_n) \) under the orbit map \( \Gamma \to \Gamma o \) is contained in a bounded neighborhood of the \( C \)-diamond \( \diamond_C(\gamma_1 o, \gamma_n o) \). This allows us to use the hyperbolic group property of \( \Gamma \) to ensure a good behavior of \( d_\psi \) on \( C \)-diamonds and to prove (1.6).
We again use the KLP Morse lemma to show that the $\Gamma$-orbit $\Gamma o$ has a uniform progression measured by the metric $d_\psi$ (Lemma 6.5). Putting these together, we deduce that if we denote by $[\xi, \eta]$ the image of a bi-infinite geodesic in $\Gamma$ connecting $\xi \neq \eta \in \Lambda_\theta$ under the orbit map, then the $d_\psi$-distance between $o$ and $[\xi, \eta]$ is comparable to the $\psi$-Gromov product of $\xi$ and $\eta$ (Lemma 6.8):

(1.7) \[ \psi(\mathcal{G}(\xi, \eta)) = d_\psi(o, [\xi, \eta]) + O(1), \]

which allows us to interpret the Gromov product as in a Gromov hyperbolic space. Another important ingredient is the continuity of shadows on viewpoints due to Kim-Oh-Wang [20] (see Proposition 7.3). Based on these, we prove that for any $\xi \in \Lambda_\theta$ and $\gamma \in \Gamma$ on a geodesic ray in $\Gamma$ toward $\xi \in \Lambda_\theta \simeq \partial \Gamma$, we have

\[ B_\psi(\xi, e^{-d_\psi(o, \gamma o)} \cup O_R(o, \gamma o) \cap \Lambda_\theta \subset B_\psi(\xi, ce^{-d_\psi(o, \gamma o)}), \]

for uniform constants $c, R > 1$, as in (1.4). Applying the higher rank version of Sullivan’s shadow lemma (Lemma 8.3), it follows that

\[ \nu_\psi(B_\psi(\xi, ce^{-d_\psi(o, \gamma o)})) \asymp e^{-d_\psi(o, \gamma o)}. \]

Finally, we use the uniform progression of $(\Gamma o, d_\psi)$ one more time to replace $ce^{-d_\psi(o, \gamma o)}$ with an arbitrary radius $r > 0$; this gives (1.3) (see Theorem 8.1).

**Organization.**

- In Section 2, we review some basic structures of Lie groups and $\theta$-boundaries. The notations set up in this section will be used throughout the paper.
- In Section 3, we recall the classification of Patterson-Sullivan measures for Anosov subgroups using tangent forms.
- In Section 4, we show that for each $\psi \in a_\theta^+$ positive on $L_\theta - \{0\}$, the composition $\psi \circ \mu$ defines a metric-like function $d_\psi$ on the $\Gamma$-orbit $\Gamma o$. The coarse triangle inequality of $d_\psi$ (Theorem 4.1) is a crucial ingredient of this paper. Its proof makes a heavy use of the notion of diamonds and the Morse lemma due to Kapovich-Leeb-Porti (Theorem 4.9).
- In Section 5, we define a conformal premetric $d_\psi$ on the limit set $\Lambda_\theta$ and discuss its basic properties.
- Sections 6 and 7 are devoted to the proof of the compatibility between shadows and $d_\psi$-balls in the limit set $\Lambda_\theta$ (1.4): this is the main technical result in our paper.
- In Sections 8 and 9, we prove Theorem 1.1. In Section 8, we prove that the Patterson-Sullivan measure of a ball is proportional to its $d_\psi$-radius and in Section 9, we prove that Patterson-Sullivan measures for symmetric linear forms are Hausdorff measures on the limit set, up to a constant multiple. We also prove Theorem 1.3.
In Section 10, we prove Theorem 1.5 on the estimate of the Hausdorff dimension of \( \Lambda^\theta \) with respect to a Riemannian metric. We also discuss its implications on the temperedness of \( L^2(\Gamma \backslash G) \).

2. Basic structure theory of Lie groups and \( \theta \)-boundaries

Throughout the paper, let \( G \) be a connected semisimple real algebraic group. In this section, we review some basic facts about the Lie group structure of \( G \). Let \( A \) be a maximal real split torus of \( G \). Let \( \g \) and \( \a \) respectively denote the Lie algebras of \( G \) and \( A \). Fix a positive Weyl chamber \( \a^+ \subset \a \) and set \( \a^+ = \exp \a^+ \) and a maximal compact subgroup \( K < G \) such that the Cartan decomposition \( G = KA^+K \) holds. Let \( \Phi = \Phi(\g, \a) \) denote the set of all roots and \( \Pi \) the set of all simple roots given by the choice of \( \a^+ \). Denote by \( N_K(A) \) and \( C_K(A) \) the normalizer and centralizer of \( A \) in \( K \) respectively. The Weyl group \( W \) is given by \( N_K(A)/C_K(A) \).

Fix an element \( w_0 \in N_K(A) \) of order 2 representing the longest Weyl element so that \( \text{Ad} w_0 a^+ = -a^+ \). The map \( i = -\text{Ad}_{w_0} : \a \to \a \) is called the opposition involution. It induces an involution of \( \Phi \) preserving \( \Pi \), for which we use the same notation \( i \), so that \( i(\alpha) = \alpha \circ i \) for all \( \alpha \in \Phi \).

Henceforth, we fix a non-empty subset \( \theta \) of \( \Pi \). Let \( a_\theta = \bigcap_{\alpha \in \Pi - \theta} \text{ker} \alpha, \quad a_\theta^+ = a_\theta \cap a^+ \),

\[
A_\theta = \exp a_\theta, \quad \text{and} \quad A_\theta^+ = \exp a_\theta^+.
\]

Let

\[
p_\theta : \a \to a_\theta
\]
denote the projection invariant under \( w \in \mathcal{W} \) fixing \( a_\theta \) pointwise.

Let \( P_\theta \) denote a standard parabolic subgroup of \( G \) corresponding to \( \theta \); that is, \( P_\theta = L_\theta N_\theta \) where \( L_\theta \) is the centralizer of \( A_\theta \) and \( N_\theta \) is the unipotent radical of \( P_\theta \) such that \( \log N_\theta \) is generated by root subgroups associated to all positive roots which are not \( \mathbb{Z} \)-linear combinations of \( \Pi - \theta \).

We set \( M_\theta = K \cap P_\theta = C_K(A_\theta) \). The Levi subgroup \( L_\theta \) can be written as \( L_\theta = A_\theta S_\theta \) where \( S_\theta \) is an almost direct product of a connected semisimple real algebraic subgroup and a compact center. Letting \( B_\theta = S_\theta \cap A \) and \( B_\theta^+ = \{ b \in B_\theta : \alpha(\log b) \geq 0 \text{ for all } \alpha \in \Pi - \theta \} \), we have the Cartan decomposition of \( S_\theta \):

\[
S_\theta = M_\theta B_\theta^+ M_\theta.
\]

Note that \( A = A_\theta B_\theta \) and \( A^+ \subset A_\theta^+ B_\theta^+ \). The space \( a_\theta^* = \text{Hom}(a_\theta, \mathbb{R}) \) can be identified with the subspace of \( a^* \) consisting of \( p_\theta \)-invariant linear forms:

\[
a_\theta^* = \{ \psi \in a^* : \psi \circ p_\theta = \psi \}.
\]

So for \( \theta_1 \subset \theta_2 \), we have

\[
a^*_{\theta_1} \subset a^*_{\theta_2}.
\]
When \( \theta = \Pi \), we will omit the subscript. So \( P = P_{\Pi} \) is a minimal parabolic subgroup and \( P = MAN \).

**Cartan projection.** Recall the Cartan decomposition \( G = KA^+K \), which means that for every \( g \in G \), there exists a unique element \( \mu(g) \in a^+ \) such that \( g \in K \exp \mu(g)K \). The map \( G \to a^+ \) given by \( g \mapsto \mu(g) \) is called the Cartan projection. We have

\[
\mu(g^{-1}) = i(\mu(g)) \quad \text{for all } g \in G.
\]

Let \( X = G/K \) be the associated Riemannian symmetric space, and set \( o = [K] \in X \). Fix a \( K \)-invariant norm \( \| \cdot \| \) on \( g \) and a Riemannian metric \( d \) on \( X \), induced from the Killing form on \( g \). For \( p \in X \) and \( R > 0 \), let \( B(p, R) \) denote the metric ball \( \{ x \in X : d(x, p) < R \} \).

**Lemma 2.1.** [2, Lemma 4.6] For any compact subset \( Q \subset G \), there exists a constant \( C = C(Q) > 0 \) such that for all \( g \in G \),

\[
\sup_{q_1, q_2 \in Q} \| \mu(q_1 g q_2) - \mu(g) \| \leq C.
\]

We then write \( \mu_\theta := p_\theta \circ \mu : G \to a^+_\theta \).

In view of (2.1), we have \( \psi \circ \mu_\theta = \psi \circ \mu \) for all \( \psi \in a^*_{\theta} \).

**The \( \theta \)-boundary \( F_\theta \).** We set

\[
F_\theta = G/P_\theta \quad \text{and} \quad F = G/P.
\]

Let

\[
\pi_\theta : F \to F_\theta
\]

denote the canonical projection map given by \( gP \mapsto gP_\theta \), \( g \in G \). We set

\[
(2.3) \quad \xi_\theta = [P_\theta] \in F_\theta.
\]

By the Iwasawa decomposition \( G = KP = KAN \), the subgroup \( K \) acts transitively on \( F_\theta \), and hence \( F_\theta \simeq K/M_\theta \).

We consider the following notion of convergence of a sequence in \( G \) to an element of \( F_\theta \). For a sequence \( g_i \in G \), we say \( g_i \to \infty \) \( \theta \)-regularly if \( \min_{\alpha \in \theta} \alpha(\mu(g_i)) \to \infty \) as \( i \to \infty \).

**Definition 2.2.** For a sequence \( g_i \in G \) and \( \xi \in F_\theta \), we write \( \lim_{i \to \infty} g_i = \lim_{i \to \infty} g_i o = \xi \) and say \( g_i \) (or \( g_i o \in X \)) converges to \( \xi \) if

- \( g_i \to \infty \) \( \theta \)-regularly; and
- \( \lim_{i \to \infty} \kappa_i \xi_\theta = \xi \) in \( F_\theta \) for some \( \kappa_i \in K \) such that \( g_i \in \kappa_i A^+K \).
Points in general position. Let $P_{\theta}^+$ be the standard parabolic subgroup of $G$ opposite to $P_{\theta}$ such that $P_{\theta} \cap P_{\theta}^+ = L_{\theta}$. We have $P_{\theta}^+ = w_0P_{i(\theta)}w_0^{-1}$ and hence

$$\mathcal{F}_{i(\theta)} = G/P_{\theta}^+.$$  

For $g \in G$, we set

$$g_{\theta}^+ := gP_{\theta} \quad \text{and} \quad g_{\theta}^- := gw_0P_{i(\theta)};$$

as we fix $\theta$ in the entire paper, we write $g^\pm = g_{\theta}^\pm$ for simplicity when there is no room for confusion. Hence for the identity $e \in G$, $(e^+, e^-) = (P_{\theta}, P_{\theta}^+) = (\xi, w_0\xi_{i(\theta)})$. The $G$-orbit of $(e^+, e^-)$ is the unique open $G$-orbit in $G/P_{\theta} \times G/P_{\theta}^+$ under the diagonal $G$-action. We set

$$\mathcal{F}_{i(\theta)}^{(2)} = \{(g_{\theta}^+, g_{\theta}^-) : g \in G\}. \quad (2.4)$$

Two elements $\xi \in \mathcal{F}_{\theta}$ and $\eta \in \mathcal{F}_{i(\theta)}$ are said to be in general position (or antipodal) if $(\xi, \eta) \in \mathcal{F}_{i(\theta)}^{(2)}$.

Busemann maps and Gromov products. The $a$-valued Busemann map $\beta : \mathcal{F} \times G \times G \to a$ is defined as follows: for $\xi \in \mathcal{F}$ and $g, h \in G$,

$$\beta_\xi(g, h) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi)$$

where $\sigma(g^{-1}, \xi) \in a$ is the unique element such that $g^{-1}k \in K \exp(\sigma(g^{-1}, \xi))N$ for any $k \in K$ with $\xi = kP$. For $(\xi, g, h) \in \mathcal{F}_{\theta} \times G \times G$, we define

$$\beta_\xi^{\theta}(g, h) := p_0(\beta_{\xi_0}(g, h)) \quad \text{for } \xi_0 \in \pi_{\theta}^{-1}(\xi); \quad (2.5)$$

this is well-defined independent of the choice of $\xi_0$ [31 Lemma 6.1]. For $p, q \in X$ and $\xi \in \mathcal{F}_{\theta}$, we set $\beta_\xi^{\theta}(p, q) := \beta_\xi^{\theta}(g, h)$ where $g, h \in G$ satisfies $go = p$ and $ho = q$. It is easy to check this is well-defined. The Busemann map has the following properties: for all $\xi \in \mathcal{F}_{\theta}$ and $g_1, g_2, g_3 \in G$,

$$\begin{align*}
\text{(Invariance)} & \quad \beta_\xi^{\theta}(g_1, g_2) = \beta_{g_2}^{\theta}(g_3g_1, g_3g_2) \\
\text{(Cocycle property)} & \quad \beta_\xi^{\theta}(g_1, g_2) = \beta_\xi^{\theta}(g_1, g_3) + \beta_\xi^{\theta}(g_3, g_2).
\end{align*}$$

Definition 2.3. For $(\xi, \eta) \in \mathcal{F}_{i(\theta)}^{(2)}$, we define the $\theta$-Gromov product as

$$G^{\theta}(\xi, \eta) = \frac{1}{2}(\beta_\xi^{\theta}(e, g) + i(\beta_\eta^{i(\theta)}(e, g))$$

where $g \in G$ satisfies $(g^+, g^-) = (\xi, \eta)$. This does not depend on the choice of $g$ [21 Lemma 9.11].

3. Classification of Patterson-Sullivan measures by tangent forms

Let $G$ be a connected semisimple real algebraic group. We fix a non-empty subset $\theta$ of the set $\Pi$ of all simple roots. Throughout this section, let $\Gamma$ be a discrete subgroup of $G$ whose Zariski closure is semisimple. When $\Gamma$ is $\theta$-Anosov, we have a complete classification on all linear forms $\psi \in a_\theta^*$ admitting a $(\Gamma, \psi)$-Patterson-Sullivan measure ([25], [32], [21]). The goal of
this section is to review this classification, in addition to recalling some basic notions such as the limit cone and the growth indicator of $\Gamma$. We refer to \cite{Th2} for more details on this section.

The $\theta$-limit set of $\Gamma$ is defined as follows:

$$\Lambda_\theta = \Lambda_\theta(\Gamma) := \{\lim \gamma_i \in \mathcal{F}_{\theta} : \gamma_i \in \Gamma\}$$

where $\lim \gamma_i$ is defined as in Definition \ref{def:limit}. If $\Gamma < G$ is Zariski dense, then the limit set $\Lambda_\theta$ is the unique $\Gamma$-minimal subset of $\mathcal{F}_{\theta}$ (\cite{Th2}, \cite{Patterson}). Furthermore, if we set $\Lambda = \Lambda_H$, then $\pi_\theta(\Lambda) = \Lambda_\theta$. For $\psi \in a^*_{\theta}$, a Borel probability measure $\nu$ on $\mathcal{F}_{\theta}$ is called $(\Gamma, \psi)$-Patterson-Sullivan measure if for all $\gamma \in \Gamma$ and $\xi \in \mathcal{F}_{\theta}$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_{\theta}^\theta(\xi, \gamma))}$$

where $\gamma_*\nu(B) = \nu(\gamma^{-1}B)$ for any Borel $B \subset \mathcal{F}_{\theta}$. A $(\Gamma, \psi)$-conformal measure is called $(\Gamma, \psi)$-Patterson-Sullivan measure if it is supported on $\Lambda_\theta$.

In order to discuss which linear forms $\psi$ admits a Patterson-Sullivan measure, we need the definitions of the $\theta$-limit cones and growth indicators.

The $\theta$-limit cone $\mathcal{L}_\theta = \mathcal{L}_\theta(\Gamma)$ of $\Gamma$ is defined as the asymptotic cone of $\mu_\theta(\Gamma)$ in $a_\theta$, that is, $u \in \mathcal{L}_\theta$ if and only if $u = \lim t_i \mu_\theta(\gamma_i)$ for some $t_i \rightarrow 0$ and $\gamma_i \in \Gamma$. If $\Gamma$ is Zariski dense, $\mathcal{L}_\theta$ is a convex cone with non-empty interior by \cite{Th2}. Recalling the convention of dropping the subscript $\theta$ when $\theta = H$, we write $\mathcal{L} = \mathcal{L}_H$. We then have $p_\theta(\mathcal{L}) = \mathcal{L}_\theta$, and $\mu_\theta(\Gamma)$ is within bounded distance from $\mathcal{L}_\theta$.

**Growth indicators.** We say that $\Gamma$ is $\theta$-discrete if the restriction $\mu_\theta|_\Gamma : \Gamma \rightarrow a^+_\theta$ is proper. The $\theta$-discreteness of $\Gamma$ implies that $\mu_\theta(\Gamma)$ is a closed discrete subset of $a^+_\theta$. Indeed, $\Gamma$ is $\theta$-discrete if and only if the counting measure on $\mu_\theta(\Gamma)$ weighted with multiplicity is a Radon measure on $a^+_\theta$.

**Definition 3.1 (\theta-growth indicator (\cite{Patterson}, \cite{Th2})).** For a $\theta$-discrete subgroup $\Gamma < G$, the $\theta$-growth indicator $\psi^\theta_\Gamma : a_\theta \rightarrow [-\infty, \infty]$ is defined as follows: if $u \in a_\theta$ is non-zero,

$$\psi^\theta_\Gamma(u) = ||u|| \inf_{u \in \mathcal{L}} \tau^\theta_c$$

where $\mathcal{L} \subset a_\theta$ ranges over all open cones containing $u$, and $\psi^\theta_\Gamma(0) = 0$. Here $-\infty \leq \tau^\theta_c \leq \infty$ denotes the abscissa of convergence of the series $\mathcal{P}^\theta_c(s) = \sum_{\gamma \in \Gamma, \mu_\theta(\gamma) \in \mathcal{C}} e^{-s||\mu_\theta(\gamma)||}$.

This definition is independent of the choice of a norm on $a_\theta$. It was proved in (\cite{Patterson}, \cite{Th2} Theorem 3.3]) that

$$\psi^\theta_\Gamma < \infty, \quad \mathcal{L}_\theta = \{\psi^\theta_\Gamma \geq 0\} \quad \text{and} \quad \psi^\theta_\Gamma > 0 \text{ on int } \mathcal{L}_\theta.$$ 

Moreover, $\psi^\theta_\Gamma$ is upper semi-continuous and concave. We say a linear form $\psi$ is tangent to $\psi^\theta_\Gamma$ (at $u \in a_\theta - \{0\}$) if $\psi \geq \psi^\theta_\Gamma$ and $\psi(u) = \psi^\theta_\Gamma(u)$. For any $u \in \text{int } \mathcal{L}_\theta$, there exists $\psi \in a^*_{\theta}$ tangent to $\psi^\theta_\Gamma$ at $u$. Moreover, for any $\psi \in a^*_{\theta}$ tangent to $\psi^\theta_\Gamma$ at an interior direction of $a^+_\theta$, there exists a $(\Gamma, \psi)$-Patterson-Sullivan measure (\cite{Th2} Theorem 8.4, \cite{Th2} Proposition 5.9)).
For $\theta$-Anosov subgroups, we have a more precise classification of Patterson-Sullivan measures in terms of tangent forms. Recall that $\Gamma$ is $\theta$-Anosov if $\Gamma$ is finitely generated and there exists a constant $C > 1$ such that for all $\alpha \in \theta$ and $\gamma \in \Gamma$, we have

$$\alpha(\mu(\gamma)) \geq C^{-1}|\gamma| - C$$

where $|\cdot|$ denotes a word metric on $\Gamma$ with respect to a fixed finite generating subset.

Define

$$(3.2) \quad \mathcal{T}_\Gamma = \{ \psi \in \mathfrak{a}_\theta^* : \psi \text{ is tangent to } \psi_0^* \}.$$ 

The following theorem was proved in [25] for $\theta = \Pi$ and $\Gamma$ Zariski dense. The general case follows from [32, Theorem A], [21, Theorem 1.11], and [20, Theorem 1.2, Theorem 9.4].

**Theorem 3.2.** Suppose that $\Gamma$ is $\theta$-Anosov. For any $\psi \in \mathcal{T}_\Gamma$, there exists a unique $(\Gamma, \psi)$-Patterson-Sullivan measure on $\Lambda_\theta$ which we denote by $\nu_\psi = \nu_{\psi, \theta}$. The map $\psi \mapsto \nu_\psi$ is a surjection from $\mathcal{T}_\Gamma$ to the space of all $\Gamma$-Patterson-Sullivan measures. If $\Gamma$ is Zariski dense in addition, then the map $\psi \mapsto \nu_\psi$ is bijective. Moreover, if $\psi_1 \neq \psi_2$ in $\mathcal{T}_\Gamma$, then $\nu_{\psi_1}$ and $\nu_{\psi_2}$ are mutually singular to each other.

We also remark that for $\psi \in \mathcal{T}_\Gamma$, $\nu_\psi$ is the unique $(\Gamma, \psi)$-conformal measure on $\mathcal{F}_\theta$, in particular, any $(\Gamma, \psi)$-conformal measure is necessarily supported on $\Lambda_\theta$ [21].

When $\psi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta - \{0\}$, the abscissa of the convergence of the $\psi$-Poincaré series

$$s \mapsto \sum_{\gamma \in \Gamma} e^{-s\psi(\mu(\gamma))}$$

is a well-defined positive number [21, Lemma 4.3]; we denote it by $\delta_\psi$.

**Lemma 3.3.** [21, Lemma 4.3 and 4.5] If $\psi \in \mathfrak{a}_\theta^*$ is positive on $\mathcal{L}_\theta - \{0\}$, then

$$\delta_\psi \psi \in \mathcal{T}_\Gamma.$$ 

In particular, $\psi \in \mathcal{T}_\Gamma$ if and only if $\delta_\psi = 1$.

Since $\mu(g^{-1}) = i(\mu(g))$ for all $g \in G$, we have that $\Gamma$ is $\theta$-Anosov if and only if $\Gamma$ is $\theta \cup i(\theta)$-Anosov. If $\Gamma$ is $\theta$-Anosov, then the canonical projection map $p : \Lambda_{\theta \cup i(\theta)} \to \Lambda_\theta$ is a $\Gamma$-equivariant homeomorphism. Recalling that $\mathfrak{a}_\theta^*$ can be considered as a subset of $\mathfrak{a}_{\theta \cup i(\theta)}^*$ from [2.2], the following observation will be useful.

**Lemma 3.4.** [21, Lemma 9.5] Suppose that $\Gamma$ is $\theta$-Anosov. For any $\psi \in \mathcal{T}_\Gamma$, the measure $\nu_{\psi, \theta}$ coincides with the push-forward of $\nu_{\psi, \theta \cup i(\theta)}$ by $p$. 

4. Metric-like functions on \(\Gamma\)-orbits and diamonds

We fix a non-empty subset \(\theta \subset \Pi\). In this section, we assume \(\theta\) is symmetric, i.e., \(\theta = i(\theta)\). Recall the notation \(X = G/K\) and \(o = [K] \in X\).

For a linear form \(\psi \in a_0^*\), define \(d_\psi : X \times X \to \mathbb{R}\) as follows: for \(g, h \in G\),

\[
d_\psi(go, ho) = \psi(\mu(g^{-1}h)) = \psi(\mu_o(g^{-1}h)).
\]

Since the Cartan projection \(\mu\) is bi-\(K\)-invariant, \(d_\psi\) is a well-defined \(G\)-invariant function.

The main goal of this section is to prove the following theorem saying that when \(\Gamma\) is \(\theta\)-Anosov, \(d_\psi\) behaves like a metric, when restricted to the \(\Gamma\)-orbit \(\Gamma o\) for any \(\psi \in a_0^*\) which is positive on \(L_{\theta} - \{0\}\):

**Theorem 4.1.** Suppose that \(\Gamma\) is a \(\theta\)-Anosov subgroup whose Zariski closure is semisimple. Let \(\psi \in a_0^*\) be such that \(\psi > 0\) on \(L_{\theta} - \{0\}\). Then there exists a constant \(D = D_\psi > 0\) such that for all \(\gamma_1, \gamma_2, \gamma \in \Gamma\),

\[
d_\psi(\gamma_1 o, \gamma_2 o) \leq d_\psi(\gamma_1 o, \gamma o) + d_\psi(\gamma o, \gamma_2 o) + D.
\]

Fixing a finite symmetric generating set of \(\Gamma\), we denote by \(d_\Gamma\) the associated word metric on \(\Gamma\). The \(\theta\)-Anosov property of \(\Gamma\) implies that \((\Gamma, d_\Gamma)\) is a Gromov hyperbolic space and that the orbit map

\[
(\Gamma, d_\Gamma) \to (X, d), \quad \gamma \mapsto \gamma o,
\]

is a quasi-isometric embedding ([16], [14]); recall that a map \(f : (Z, d_Z) \to (Y, d_Y)\) between metric spaces is called a quasi-isometric embedding if there exist \(L \geq 1\) and \(Q \geq 0\) such that for all \(z_1, z_2 \in Z\),

\[
L^{-1} \cdot d_Z(z_1, z_2) - Q \leq d_Y(f(z_1), f(z_2)) \leq L \cdot d_Z(z_1, z_2) + Q.
\]

Indeed, we prove Theorem [14] in a greater generality where the orbit \(\Gamma o\) is replaced by the image of a uniformly regular quasi-isometric embedding of a geodesic metric space into \(X\).

**Coarse triangle inequalities for uniformly regular quasi-isometric embeddings.** We set \(W_\theta = \{w \in W : w(\theta) = \theta\}\), i.e., the stabilizer of \(\theta\) in \(W\). We define a closed cone \(C\) of \(a^+\) to be \(\theta\)-admissible if the following three conditions hold:

1. \(C\) is \(i\)-invariant: \(i(C) = C\);
2. \(W_\theta \cdot C = \bigcup_{w \in W_\theta} \text{Ad}_w C\) is convex;
3. \(C \cap \bigcup_{\alpha \in \theta \ker \alpha} = \{0\}\).

For a \(\theta\)-admissible cone \(C\), we say that an ordered pair \((x_1, x_2)\) of distinct points in \(X\) is \(C\)-regular if for \(g_1, g_2 \in G\) such that \(g_1 o = x_1\) and \(g_2 o = x_2\), we have

\[
\mu(g_1^{-1} g_2) \in C.
\]

In this case, \(x_2 = g_2 o \in g_1 K(\exp C)o\) and hence for some \(g \in g_1 K\), \(x_1 = g_1 o = go\) and \(x_2 \in g(\exp C)o\). Note that if \((x_1, x_2)\) is \(C\)-regular, then \((x_2, x_1)\) is \(i(C)\)-regular and hence \(C\)-regular by the \(i\)-invariance of \(C\).
**Definition 4.2.** Let $(Z, d_Z)$ be a metric space and $f : Z \to X$ be a map. For a $\theta$-admissible cone $C$ and a constant $B \geq 0$, $f$ is called $(C, B)$-regular if the pair $(f(z_1), f(z_2))$ is $C$-regular for all $z_1, z_2 \in Z$ with $d_Z(z_1, z_2) \geq B$.

Theorem 4.1 will be deduced as a special case of the following theorem: we put $C_\theta = p_\theta(C)$.

**Theorem 4.3.** Let $Z$ be a geodesic metric space, $C \subset a^+$ a $\theta$-admissible cone and $B$ a non-negative constant. Let $f : Z \to X$ be a $(C, B)$-regular quasi-isometric embedding. If $\psi \in a^+_\theta$ is positive on $C_\theta - \{0\}$, then there exists a constant $D = D_\psi \geq 0$ such that for all $x_1, x_2, x_3 \in f(Z)$,

$$d_\psi(x_1, x_3) \leq d_\psi(x_1, x_2) + d_\psi(x_2, x_3) + D.$$ 

**Proof of Theorem 4.1 assuming Theorem 4.3.** By the $\theta$-Anosov assumption on $\Gamma$, we have $\rho_\theta - \{0\} \subset \text{int } a^+_\theta$. Since $\theta = \iota(\theta)$ by hypothesis, $\iota|a_\theta$ is an involution preserving $\rho_\theta$. Since $\psi$ is positive on $\rho_\theta - \{0\}$, we can choose a slightly larger closed cone $C_0 \subset \text{int } a^+_\theta \cup \{0\}$ satisfying

1. $\rho_\theta - \{0\} \subset \text{int } C_0$;
2. $\iota(C_0) = C_0$;
3. $\psi > 0$ on $C_0 - \{0\}$.

Define

$$C := p_\theta^{-1}(C_0) \cap a^+.$$

Using the fact that $p_\theta : a \to a_\theta$ is $W_\theta$-equivariant, we have that $W_\theta \cdot C = W_\theta \cdot (p_\theta^{-1}(C_0) \cap a^+) = p_\theta^{-1}(C_0) \cap (W_\theta \cdot a^+)$. Since both $W_\theta \cdot a^+$ and $p_\theta^{-1}(C_0)$ are convex cones, it follows that their intersection $W_\theta \cdot C$ is also convex. After removing some small enough “conical neighborhoods” of $\bigcup_{\alpha \in \theta} \ker \alpha$ from $C$, we can assume that

$$C \cap \bigcup_{\alpha \in \theta} \ker \alpha = \{0\}$$

while keeping the convexity of $W_\theta \cdot C$. Moreover, we can choose those neighborhoods so thin so that $C$ still contains a neighborhood of $\rho_\theta - \{0\}$ in $a^+$. The resulting $C$ is therefore $\theta$-admissible and $\psi > 0$ on $C - \{0\}$.

We claim that there exists $B = B(\Gamma, C, \rho) > 0$ such that the orbit map

$$f : (\Gamma, d_\Gamma) \to (X, d), \quad \gamma \mapsto \gamma \rho,$$

is a $(C, B)$-regular embedding. Suppose not. Then there exist two sequences $\{\gamma_i\}, \{\gamma'_i\} \subset \Gamma$ such that $d_\Gamma(\gamma_i, \gamma'_i) = |\gamma_i^{-1}\gamma'_i| > i$ and $\mu(\gamma_i^{-1}\gamma'_i) \notin C$. Setting $g_i = \gamma_i^{-1}\gamma'_i \in \Gamma$, we then have that $\frac{\mu(g_i)}{\|\mu(g_i)\|} \notin C$ for all $i \geq 1$. Since a neighborhood of $\rho_\theta - \{0\}$ in $a^+$ is contained in $C$, any limit of the sequence $\frac{\mu(g_i)}{\|\mu(g_i)\|}$ cannot lie in $C$. On the other hand, since $|g_i| \to \infty$, we have $\|\mu(g_i)\| \to \infty$ and hence any limit of the sequence $\frac{\mu(g_i)}{\|\mu(g_i)\|}$ must belong to the asymptotic cone of $\mu(\Gamma)$, that is, $L$. This yields a contradiction. Therefore Theorem 4.1 follows from Theorem 4.3. \qed
The rest of this section is now devoted to the proof of Theorem 4.3. We begin by recalling the following theorem; in particular, the metric space $Z$ in Theorem 4.3 is always Gromov hyperbolic.

**Theorem 4.4.** ([17, Theorem 1.4]) Let $Z$ and $f : Z \to X$ be as in Theorem 4.3. Then $Z$ is Gromov hyperbolic. If $Z$ is locally compact in addition, then $f$ continuously extends to

$$f : \bar{Z} \to X \cup F$$

where $\bar{Z} = Z \cup \partial Z$ is the Gromov compactification and $f$ maps distinct points in $\partial Z$ to points in general position.

**Diamonds.** The notion of diamonds in $X$ plays a key role in the proof of Theorem 4.3. We recall the notion of diamonds in the symmetric space $X$, following Kapovich-Leeb-Porti: we refer to [16, Section 2.5] for details. We fix a $\theta$-admissible cone $C \subset a^+$ in the following. For a $C$-regular pair $(x_1, x_2)$ of points in $X$, define the $C$-cone with the tip at $x_1$ containing $x_2$ to be

$$V_C(x_1, x_2) = g M_\theta(\exp C) o,$$

where $g = g(x_1, x_2) \in G$ is any element such that $x_1 = go$ and $x_2 \in g(\exp C) o$; it is easy to check such $g$ always exist and this definition is independent of the choice of $g$.

**Definition 4.5 (Diamonds).** For a $C$-regular pair $(x_1, x_2)$ of points in $X$, the $C$-diamond with tips at $x_1$ and $x_2$ is defined as

$$\Diamond_C(x_1, x_2) = V_C(x_1, x_2) \cap V_C(x_2, x_1).$$

The $C$-cones and $C$-diamonds are convex subsets of $X$, see [16, Proposition 2.10, Proposition 2.13].

**Example 4.6.** For $a \in \exp C$, the pair $(o, ao)$ is $C$-regular, and the diamond $\Diamond_C(o, ao)$ can be explicitly described as follows. First note that as we can take $g(o, ao) = e$, we have $V_C(o, ao) = M_\theta(\exp C) o$. Recalling that $i = - \Ad_{w_0}$ for the longest Weyl element $w_0 \in K$, we also have $ao = aw_0 o$ and $o = (aw_0)(w_0^{-1} a^{-1} w_0) o \in aw_0(\exp C) o$. So we can take $g(ao, o) = aw_0$. Since $w_0 M_\theta w_0^{-1} = M_\theta$ and $w_0(\exp C) w_0^{-1} = \exp(-C)$, we have $V_C(ao, o) = aw_0 M_\theta(\exp C) o = aM_\theta \exp(-C) o$. Therefore

$$\Diamond_C(o, ao) = M_\theta(\exp C) o \cap aM_\theta \exp(-C) o.$$

See Figure 2.

**Lemma 4.7 (Simultaneously nesting property).** If $(x_1, x_2)$ is $C$-regular, then for any $x \in \Diamond_C(x_1, x_2)$, there exists $g \in G$ and $a \in \exp C$ such that

$$x_1 = go, \quad x = gao, \quad x_2 \in ga(M_\theta \exp C) o.$$
We may assume that $x_1 = o$, $x_2 \in M_\theta(\exp C) o$ and $x = ao$ for some $a \in \exp C$. Then it suffices to show that $x_2 = aka' o$ for some $k \in M_\theta$ and $a' \in \exp C$. We write $x_2 = ma_0 o$ for $m \in M_\theta$ and $a_0 \in \exp C$. Since $o \in V_C(x_2, o)$, we have $o = ma_0 k_0 M_\theta(\exp C) o$ for some $k_0 \in K$. Hence we have

$$k^{-1}_0 w^{-1}_0 (w_0 a_0^{-1} w_0^{-1}) \in M_\theta(\exp C) K.$$

This implies $k^{-1}_0 \in M_\theta w_0$ and hence $k_0 \in w_0 M_\theta$. Since $ao \in V_C(x_2, o)$ as well, we now have $ao = ma_0 w_0 M_\theta(\exp C) o$. Then for some $k \in K$, we have

$$ak \in ma_0 w_0 M_\theta \exp Cw_0^{-1} = ma_0 M_\theta \exp (-C).$$

Hence for some $a' \in \exp C$, we have

$$aka' \in ma_0 M_\theta.$$

Looking at $G/P_\theta$, we have $kP_\theta = a^{-1} ma_0 M_\theta a^{-1} P_\theta = P_\theta$. Therefore $k \in M_\theta$. Since $x_2 = ma_0 o = aka' o$, the claim follows. \hfill \Box

We first prove that $d_\psi$ is additive on each diamond for any $\psi \in a^*_\theta$.

**Lemma 4.8 (Additivity of $d_\psi$ on diamonds).** Let $\psi \in a^*_\theta$. For any $C$-regular pair $(x_1, x_2)$ and for any $x \in \Diamond_C(x_1, x_2)$, we have

$$d_\psi(x_1, x) + d_\psi(x, x_2) = d_\psi(x_1, x_2).$$

**Proof.** By Lemma 4.7, we may assume that $x_1 = o$, $x = ao$ and $x_2 = aka' o$, for some $a, a' \in \exp C \subset A^+$ and $k \in M_\theta$. Under this hypothesis, $V_C(x_1, x_2) = M_\theta(\exp C) o$ and hence $x_2 = k\tilde{a} o$, for some $\tilde{a} \in \exp C$ and $\tilde{k} \in M_\theta$. We may write

$$a = a_1 a_2 \in A^+_\theta B^+_\theta \quad \text{and} \quad a' = a'_1 a'_2 \in A^+_\theta B^+_\theta.$$  

We then have

$$aka' = a_2 ka'_2 (a_1 a'_1).$$
Since $a_2 k a'_2 \in S_\theta$, we can write its Cartan decomposition $a_2 k a'_2 = m b m' \in M_\theta B_\theta^+ M_\theta$, and hence
\[
aka' = m (b a_1 a'_1) m'.
\]
Let $w \in W$ be a Weyl element such that $b a_1 a'_1 \in w A^+ w^{-1}$. Noting that $\tilde{a} = \exp \mu(aka')$, we must have $b a_1 a'_1 = w \tilde{a} w^{-1}$. Hence we have
\[
x_2 = aka' o = mw \tilde{a} o.
\]
On the other hand, we also have
\[
x_2 = \tilde{k} \tilde{a} o \quad \text{where } \tilde{k} \in M_\theta.
\]
This implies $mw \in M_\theta$; in particular, $w \in M_\theta$. Therefore $\tilde{a} = w^{-1} b a_1 a'_1 w = (w^{-1} bw)(a_1 a'_1) \in B_\theta A_\theta^+$, which implies
\[
p_\theta(\log \tilde{a}) = \log a_1 + \log a'_1 = p_\theta(\log a) + p_\theta(\log a').
\]
Since
\[
d_\psi(x_1, x) = d_\psi(o, ao) = \psi(p_\theta(\log a)),
\]
\[
d_\psi(x, x_2) = d_\psi(a o, aka'o) = \psi(p_\theta(\log a')),
\]
\[
d_\psi(x_1, x_2) = d_\psi(o, \tilde{k} ao) = \psi(p_\theta(\log \tilde{a}))
\]
applying $\psi$ to (4.2) finishes the proof.

\textbf{Morse lemma of Kapovich-Leeb-Porti.} We now fix an auxiliary $\theta$-admissible cone $C' \subset a^+$ which contains a neighborhood of $C - \{0\}$ in $a^+$ (Figure 3).

\textbf{Figure 3.} Choice of $C'$ viewed on the unit sphere of $a^+$

The Morse lemma due to Kapovich-Leeb-Porti, which we will call the KLP Morse lemma, is stated as follows [17, Theorem 5.16, Corollary 5.28]: the image of an interval in $\mathbb{R}$ under an $(L, Q)$-quasi-isometry is called an $(L, Q)$-quasi-geodesic.

\textbf{Theorem 4.9} (KLP Morse lemma). \textit{Given $L \geq 1$ and $Q, B \geq 0$, there exists a constant $D_0 = D_0(L, Q, B, C, C') \geq 0$ so that the following holds: let $I \subset \mathbb{R}$ be an interval and $c : I \to X$ a $(\bar{C}, \bar{B})$-regular $(L, Q)$-quasi-geodesic.}

1. \textit{If $I = [a, b]$ with $b - a \geq B$, then the image $c(I)$ is contained in the $D_0$-neighborhood of the diamond $\Diamond_C(c(a), c(b))$.}
2. \textit{If $I = [a, \infty)$ for some $a \in \mathbb{R}$, then $c(I)$ is contained in the $D_0$-neighborhood of the cone $g M_\theta(\exp C') o$ where $g \in G$ be such that $g o = c(a)$ and $g^+ = c(\infty) \in F_\theta$.}
Lemma 4.10. Let $D = D_1$ and $X$ lies in the $d$-neighborhood of the point $z$ of $Q$. We choose $z$ to be the point closest to $t$ in the Riemannian metric $d$ on $X$. By Lemma 4.8,

\[ d_\psi(c(a), x_t) + d_\psi(x_t, c(b)) = d_\psi(c(a), c(b)). \]

Consider a compact subset $S := \{ q \in G : d(go, o) \leq D_0 \}$ and $C(S)$ given by Lemma 2.1. Then it follows from $d(x_t, c(t)) \leq D_0$ that we have

\[ |d_\psi(c(a), x_t) - d_\psi(c(a), c(t))| < C(S)||\psi|| \text{ and } |d_\psi(x_t, c(b)) - d_\psi(c(t), c(b))| \leq C(S)||\psi||. \]

Therefore we set $D_1 = 2C(S)||\psi||$ and completes the proof. \qed

We are ready to give:

Proof of Theorem 4.3. Let $f : Z \to X$ be as in Theorem 4.3. Let $\psi \in a_0^*$ be such that $\psi > 0$ on $C_0 - \{0\}$. Taking $C'$ small enough, we may assume that $\psi$ is positive on $C' - \{0\}$ as well. Let $x_1, x_2, x_3 \in f(Z)$.

We choose $z_1, z_2, z_3 \in Z$ such that $x_i = f(z_i)$ for $i = 1, 2, 3$ and choose geodesics $c_1 : [0, b_1] \to Z$ and $c_2 : [0, b_2] \to Z$ connecting $z_1$ to $z_2$ and $z_2$ to $z_3$ respectively. By Theorem 4.4, $(Z, d_Z)$ is Gromov hyperbolic, and hence there exist $a_1 \in [0, b_1], a_2 \in [0, b_2]$, and a uniform constant $q > 0$ such that the concatenation of the paths

\[ c_1|_{[0, a_1]} \text{ and } c_2|_{[a_2, b_2]} \]

is a $(1, q)$-quasi-geodesic in $Z$ connecting $z_1$ to $z_3$. After a reparametrization of $c_2$, we can assume that $a_1 = a_2$. Applying Lemma 4.10 to the image of this quasi-geodesic under $f$, we get

\[ d_\psi(x_1, x_3) \leq d_\psi(x_1, (f \circ c_1)(a_1)) + d_\psi((f \circ c_1)(a_1), x_3) + D_1 \]

\[ \leq d_\psi(x_1, (f \circ c_1)(a_1)) + d_\psi((f \circ c_2)(a_2), x_3) + D_1', \]

where $D_1'$ is a constant dependent on $D_1$ and $d_\psi((f \circ c_1)(a_1), (f \circ c_2)(a_2))$ which is uniformly bounded.

Again, applying Lemma 4.10 to the quasi-geodesic $f \circ c_1$, we have

\[ d_\psi(x_1, (f \circ c_1)(a_1)) \leq d_\psi(x_1, (f \circ c_1)(a_1)) + d_\psi((f \circ c_1)(a_1), x_2) + D_2 \]

\[ \leq d_\psi(x_1, x_2) + D_1 + D_2 \]
for some $D_2 > 0$ depending on $f$ and $\psi$, which is for the case when $(f \circ c_1)(a_1)$ and $x_2$ are so close that $d_\psi((f \circ c_1)(a_1), x_2) < 0$. Similarly, we also get

$$d_\psi((f \circ c_2)(a_2), x_3) \leq d_\psi(x_2, x_3) + D_1 + D_2.$$ Combining the above inequalities, we obtain

$$d_\psi(x_1, x_3) \leq d_\psi(x_1, x_2) + d_\psi(x_2, x_3) + D'_1 + 2(D_1 + D_2).$$

This completes the proof of Theorem 4.3.

5. Conformal premetrics on limit sets

Let $\Gamma$ be a $\theta$-Anosov subgroup of $G$ with semisimple Zariski closure. Suppose $\theta = i(\theta)$. Fix a linear form $\psi \in a_\theta^*$ positive on $L_\theta - \{0\}$. The goal of this section is to define a premetric $d_\psi$ on the limit set $\Lambda_\theta$, which is conformal, almost symmetric, and satisfies almost triangle inequality with bounded multiplicative error. We also discuss how this definition can be extended to non-symmetric $\theta$ at the end of the section.

Recall the definition of the Gromov product from Definition 2.3. The $\theta$-Anosov property of $\Gamma$ implies that any two distinct points in $\Lambda_\theta$ are in general position: if $\xi \neq \eta$ in $\Lambda_\theta$, then $(\xi, \eta) \in F_\theta^{(2)}$. Therefore the following premetric on $\Lambda_\theta$ is well-defined:

**Definition 5.1.** For $\xi, \eta \in \Lambda_\theta$, we set

$$d_\psi(\xi, \eta) = \begin{cases} e^{-\psi(G^\theta(\xi, \eta))} & \text{if } \xi \neq \eta, \\
0 & \text{if } \xi = \eta. \end{cases}$$

We first observe the following $\Gamma$-conformal property of $d_\psi$:

**Lemma 5.2.** For $\gamma \in \Gamma$ and $\xi, \eta \in \Lambda_\theta$, we have

$$d_\psi(\gamma^{-1} \xi, \gamma^{-1} \eta) = e^{\frac{1}{2}\psi(\beta_\theta^\gamma(e, \gamma^{-1} g) + i(\beta_\theta^\gamma(e, \gamma^{-1} g)) - d_\psi(\xi, \eta).}$$

**Proof.** Let $\xi \neq \eta$, and $g \in G$ be such that $g^+ = \xi$ and $g^- = \eta$. Then for any $\gamma \in \Gamma$,

$$2G^\theta(\gamma^{-1} \xi, \gamma^{-1} \eta) = \beta_\theta^\gamma(e, \gamma^{-1} g) + i(\beta_\theta^\gamma(e, \gamma^{-1} g))$$

$$= 2G^\theta(\xi, \eta) + \beta_\theta^\gamma(e, \gamma) + i(\beta_\theta^\gamma(e, \gamma)).$$

Now the claim follows from the definition of $d_\psi$. \qed

Recall that $G^\theta(\xi, \eta) = i(G^\theta(\eta, \xi))$ for all $\xi, \eta \in \Lambda_\theta$. Hence if $\psi$ is $i$-invariant, then $d_\psi$ is symmetric. We have the following in general:

**Proposition 5.3** (Metric-like properties of $d_\psi$).

1. There exists $R = R(\psi) > 1$ such that for all $\xi, \eta \in \Lambda_\theta$,

$$R^{-1}d_\psi(\eta, \xi) \leq d_\psi(\xi, \eta) \leq R d_\psi(\eta, \xi).$$

2. There exists $N = N(\psi) > 0$ such that for all $\xi_1, \xi_2, \xi_3 \in \Lambda_\theta$,

$$d_\psi(\xi_1, \xi_3) \leq N(d_\psi(\xi_1, \xi_2) + d_\psi(\xi_2, \xi_3)).$$
The second property was obtained in [25, Lemma 6.11] and the same proof can be repeated for a general $\theta$ in verbatim.

The first property follows from Lemma 5.4 below. For $x \neq y$ in the Gromov boundary $\partial \Gamma$, denote by $\gamma_{x,y} \in \Gamma$ the nearest projection of the identity $e$ to the the bi-infinite geodesic $[x,y]$ in $(\Gamma, d_\Gamma)$, that is, $d_\Gamma(e, \gamma_{x,y}) = \inf\{d_\Gamma(e, g) : g \in [x,y]\}$. The following was proved in [25, Lemma 6.6] for $\theta = \Pi$ and the same proof works for a general $\theta$:

**Lemma 5.4.** There exists $C_1 > 0$ such that for any $x \neq y \in \partial \Gamma$,
\[
\left\| G^\theta(f(x), f(y)) - \frac{\mu(\gamma_{x,y}) + \iota(\gamma_{x,y})}{2} \right\| < C_1.
\]
In particular, for $\xi \neq \eta \in \Lambda_\theta$, we have
\[
\| G^\theta(\xi, \eta) - G^\theta(\eta, \xi) \| < 2C_1.
\]

**Symmetrization.** Consider the following symmetrization of $\psi$:
\[
\tilde{\psi} := \frac{\psi + \psi \circ \iota}{2}.
\]
Since $\mathcal{L}_\theta$ is i-invariant, we have $\tilde{\psi} > 0$ on $\mathcal{L}_\theta - \{0\}$. Lemma 5.4 implies that $d_{\tilde{\psi}}$ and $d_\psi$ are bi-Lipschitz equivalent:

**Proposition 5.5.** There exists $R > 1$ such that for any $\xi, \eta \in \Lambda_\theta$, we have
\[
R^{-1}d_\psi(\xi, \eta) \leq d_{\tilde{\psi}}(\xi, \eta) \leq Rd_\psi(\xi, \eta).
\]

**Proof.** Since $G^\theta(\eta, \xi) = \iota(G^\theta(\xi, \eta))$ for all $\eta \neq \xi$ in $\Lambda_\theta$, it follows from Lemma 5.4 that
\[
|\psi(G^\theta(\xi, \eta)) - \tilde{\psi}(G^\theta(\xi, \eta))| = \frac{1}{2} |\psi(G^\theta(\xi, \eta) - G^\theta(\eta, \xi))| < \|\psi\|C_1
\]
where $C_1$ is given in Lemma 5.4 and $\|\psi\|$ is the operator norm of $\psi$. It suffices to set $R = e\|\psi\|C_1$ to finish the proof. \hfill $\Box$

We also record the following Vitali covering type lemma for a later use: this is a standard consequence of Proposition 5.3(2) (cf. [25]): here $B_\psi(\xi, r)$ denotes the $d_\psi$-ball $\{ \eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r \}$.

**Lemma 5.6.** [25, Lemma 6.12] There exists $N_0 = N_0(\psi) \geq 1$ satisfying the following: for any finite collection $B_\psi(\xi_1, r_1), \ldots, B_\psi(\xi_n, r_n)$, where $\xi_i \in \Lambda_\theta$ and $r_i > 0$ for $i = 1, \ldots, n$, there exists a disjoint subcollection $B_\psi(\xi_{i_1}, r_{i_1}), \ldots, B_\psi(\xi_{i_k}, r_{i_k})$ such that
\[
\bigcup_{i=1}^n B_\psi(\xi_i, r_i) \subset \bigcup_{j=1}^k B_\psi(\xi_{i_j}, N_0r_{i_j}).
\]

**Remark 5.7.** Recall that the canonical projection $p : \Lambda_{\theta \cup \iota(\theta)} \to \Lambda_\theta$ is a $\Gamma$-equivariant homeomorphism and that $\mathfrak{a}^*_\theta \subset \mathfrak{a}^*_{\theta \cup \iota(\theta)}$. Using this homeomorphism, we can also define a function $d_\psi$ on $\Lambda_\theta$ even when $\theta$ is not symmetric,
so that $p : (\Lambda_{\theta, i(\theta)}, d_\psi) \to (\Lambda_{\theta}, d_\psi)$ is an isometry:

$$d_\psi(\xi, \eta) := d_\psi(p^{-1}(\xi), p^{-1}(\eta))$$

for all $\xi, \eta \in \Lambda_{\theta}$. In this regard, the above discussion is still valid without the symmetric hypothesis on $\theta$.

6. Compatibility of shadows and $d_\psi$-balls

As before, let $\Gamma$ be a $\theta$-Anosov subgroup with semisimple Zariski closure. Fix a linear form $\psi \in a_\theta^*$ which is positive on $L_{\theta} - \{0\}$ and $\psi = \psi \circ i$. In particular, the conformal premetric $d_\psi$ on $\Lambda_{\theta}$, defined by (5.1), is symmetric.

Shadows play a basic role in studying the metric property of $(\Lambda_{\theta}, d_\psi)$ in relation with the geometry of the symmetric space $X$, as in the original work of Sullivan.

We recall the definition of shadows in $F_\theta$. For $p, q \in X$, the shadow $O_{\theta}^R(p, q)$ of the ball $B(q, R)$ viewed from $p$ is defined as

$$O_{\theta}^R(p, q) = \{ gP_\theta \in F_\theta : g \in G, go = p, gA^+ o \cap B(g, R) \neq \emptyset \}$$

We refer to [20] and [21] for basic properties of these shadows. In particular, we have the following property of $\theta$-Anosov subgroups:

Lemma 6.1. [21, Lemma 5.4] For $\xi \in F_\theta$, we have $\xi \in \Lambda_{\theta}$ if and only if there exists $R > 0$ and an infinite sequence $\gamma_k \in \Gamma$ such that $\xi \in O_{\theta}^R(o, \gamma_k o)$ for all $k \geq 1$.

By Theorem 4.4, we have:

Lemma 6.2. The orbit map $\Gamma \to X$, $\gamma \mapsto \gamma o$, extends to the $\Gamma$-equivariant homeomorphism $f : \Gamma \cup \partial \Gamma \to \Gamma o \cup \Lambda_{\theta}$ where $\Gamma \cup \partial \Gamma$ is given the topology of the Gromov compactification.

We will henceforth identity $\partial \Gamma$ and $\Lambda_{\theta}$ using this boundary map $f$. For $\xi, \eta \in \Gamma \cup \partial \Gamma$, $[\xi, \eta]$ denotes a geodesic in $(\Gamma, d_\Gamma)$ connecting $\xi$ to $\eta$, which is possibly be an infinite geodesic ray or a bi-infinite geodesic, and $[\xi, \eta]o$ denotes the image of $[\xi, \eta]$ under the orbit map.

The main technical ingredient of this paper is the following theorem which says that shadows in $\Lambda_{\theta}$ are comparable with $d_\psi$-balls.

Theorem 6.3. Let $\psi \in a_\theta^*$ be such that $\psi > 0$ on $L_{\theta} - \{0\}$ and $\psi = \psi \circ i$. Then there exist constants $c, R_0 > 0$ such that for any $R > R_0$, there exists $c' = c'(R) > 0$ so that the following holds: for any $\xi \in \Lambda_{\theta}$ and any element $g \in \Gamma$ on a geodesic ray $[e, \xi]$ in $\Gamma$, we have

$$B_\psi(\xi, ce^{-d_\psi(o, go)}) \subset O_{\theta}^R(o, go) \cap \Lambda_{\theta} \subset B_\psi(\xi, c' e^{-d_\psi(o, go)}).$$

Since the proof of this theorem is quite lengthy, we will prove the first inclusion in this section and the second inclusion in the next section. The rest of this section is devoted to the proof of the first inclusion. In view of Remark 5.7, we assume that $\theta = i(\theta)$ in the rest of the section. Strictly speaking, $d_\psi$ is not a metric on the $\Gamma$-orbit $\Gamma o$. Nevertheless, we will still
employ terminologies for the metric space on \((\Gamma o, d_\psi)\) for convenience. For instance, for a subset \(B \subset \Gamma o\), \(d_\psi(go, B) = \inf_{ho \in B} d_\psi(go, ho)\) and the \(R\)-neighborhood of \(B\) is given by \(\{go \in \Gamma o : d_\psi(go, B) < R\}\), etc.

Two main ingredients of the proof of the first inclusion of (6.1) are
1. \((\Gamma o, d_\psi)\) satisfies the coarse triangle inequality (Theorem 4.1).
2. the \(\psi\)-Gromov product \(\psi(G(\xi, \eta))\) behaves like the premetric \(d_\psi(o, [\xi, \eta]o)\) up to an additive error (see Lemma 6.8 for a precise statement).

In the rank one case, the property (2) is a well-known consequence of a uniform thin triangle property and the Morse lemma of the rank one symmetric space. In our setting, we will also use a uniform thin triangle property for \((\Gamma o, d_\psi)\) and the KLP Morse lemma using diamonds to obtain (2).

We begin with the following:

Proposition 6.4. The orbit map \((\Gamma, d_\Gamma) \rightarrow (\Gamma o, d_\psi), \gamma \mapsto \gamma o\), is a quasi-isometry\(^4\). In particular, the images of geodesic triangles in \(\Gamma\) under the orbit map are uniformly thin, that is, there exists \(R > 0\) such that for any \(\xi_1, \xi_2, \xi_3 \in \Gamma \cup \partial \Gamma\), the image \([\xi_1, \xi_3]o\) is contained in the \(R\)-neighborhood of \([\xi_1, \xi_2] \cup [\xi_2, \xi_3])o\) with respect to \(d_\psi\).

Proof. Since \(\psi > 0\) on \(L_\theta - \{0\}\), we also have \(\psi > 0\) on \(L - \{0\}\), and hence \((u + \ker \psi) \cap L\) is compact for any \(u \in L\). Hence there exists a constant \(C > 1\) such that for any \(\gamma \in \Gamma\),

\[C^{-1}\|\mu(\gamma)\| \leq \psi(\mu(\gamma)) \leq C\|\mu(\gamma)\|.
\]

In particular, for any \(\gamma_1, \gamma_2 \in \Gamma\), we have

\[C^{-1}d(\gamma_1o, \gamma_2o) \leq d_\psi(\gamma_1o, \gamma_2o) \leq Cd(\gamma_1o, \gamma_2o).
\]

Combining this with the fact that the orbit map \((\Gamma, d_\Gamma) \rightarrow (\Gamma o, d)\) is a quasi-isometry \((16, 14)\), the first claim follows. The second claim follows since \(\Gamma\) is a hyperbolic group, and hence it has a uniform thin triangle property which is a quasi-isometry invariance.

We use the KLP Morse lemma (Theorem 4.9) once again to obtain that the image of a geodesic ray under the orbit map has a uniform progression:

Lemma 6.5 (Uniform progression lemma). For any \(r > 0\), there exists \(n(r) > 0\) such that for any geodesic ray \(\{e, \gamma_1, \gamma_2, \cdots\}\) in \((\Gamma, d_\Gamma)\),

\[d_\psi(o, \gamma_{i+n}o) \geq d_\psi(o, \gamma_i o) + r
\]

for all \(i \in \mathbb{N}\) and all \(n \geq n(r)\).

Proof. Fix \(r > 0\). Let \(x_i := \gamma_i o\) for \(i \geq 0\). By Theorem 4.9 (KLP Morse lemma), for all \(n \geq B\) and \(i \geq 0\), the sequence \(x_0, \cdots, x_{i+n}\) is contained in the \(D_0\)-neighborhood of to the diamond \(Q_{C^r}(x_0, x_{i+n})\) in the symmetric space

\[L \geq 1, Q \geq 0\) such that for any \(\gamma_1, \gamma_2 \in \Gamma, L^{-1}d_\Gamma(\gamma_1, \gamma_2) - Q \leq d_\psi(\gamma_1o, \gamma_2o) \leq Ld_\Gamma(\gamma_1, \gamma_2) + Q.
\]

\[^4\]That is, there exists \(L \geq 1, Q \geq 0\) such that for any \(\gamma_1, \gamma_2 \in \Gamma, L^{-1}d_\Gamma(\gamma_1, \gamma_2) - Q \leq d_\psi(\gamma_1o, \gamma_2o) \leq Ld_\Gamma(\gamma_1, \gamma_2) + Q.
\]
(X, d). Let \( \bar{x}_i \) denote the nearest-point projection of \( x_i \) to \( \Diamond_{C'}(x_0, x_{i+n}) \).

Applying Lemma 4.8, we obtain that
\[
(6.2) \quad d_\psi(x_0, \bar{x}_i) + d_\psi(\bar{x}_i, x_{i+n}) = d_\psi(x_0, x_{i+n}).
\]

Since the map \([0, i+n] \cap \mathbb{Z} \rightarrow (X, d), j \rightarrow x_j, \) is an \((L, Q)\)-quasi-isometric embedding for some \( L \geq 1 \) and \( Q \geq 0, \) we get that for all \( i \geq 0, \)
\[
(6.3) \quad d_\psi(x_i, x_{i+n}) \geq C \cdot (L^{-1}n - Q),
\]
where \( C = \inf \{ \psi(v) : \|v\| = 1, v \in C' \} > 0. \) Moreover, we also have
\[
(6.4) \quad |d_\psi(x_0, \bar{x}_i) - d_\psi(x_0, x_i)| \leq D_0\|\psi\| \quad \text{and} \quad |d_\psi(\bar{x}_i, x_{i+n}) - d_\psi(x_i, x_{i+n})| \leq D_0\|\psi\|.
\]

Therefore by (6.2), (6.3), and (6.4), we get
\[
d_\psi(x_0, x_{i+n}) = d_\psi(x_0, \bar{x}_i) + d_\psi(\bar{x}_i, x_{i+n})
\geq d_\psi(x_0, x_i) + d_\psi(x_i, x_{i+n}) - 2D_0\|\psi\|
\geq d_\psi(x_0, x_i) + C \cdot (L^{-1}n - Q) - 2D_0\|\psi\|.
\]

Thus it is sufficient to set \( n(r) = B + \frac{L(r+2D_0\|\psi\|+CQ)}{C} \) to get the desired conclusion. \( \square \)

**Lemma 6.6** (Small inscribed triangle). **There exists** \( C > 0 \) **satisfying the following property: Let** \([\xi, \eta]\) **be a bi-infinite geodesic** \((\Gamma, d_T)\). **If** \( \gamma \in [\xi, \eta] \) **is such that** \( d_\psi(\gamma, \gamma) = d_\psi(\gamma, [\xi, \eta]_0) \), **then there exist** \( u \in [\xi, \xi] \) **and** \( v \in [\xi, \eta] \) **so that** \( \{u_0, v_0, \gamma_0\} \) **has** \( d_\psi\)-**diameter less than** \( C \).

![Figure 4. Small inscribed triangle](image)

**Proof.** Let \( \delta > 0 \) be a constant so that every triangle in \( \Gamma_0 \), obtained as an image of the geodesic triangle in \((\Gamma, d_T)\) under the orbit map, is \( \delta \)-thin in \( d_\psi \)-metric. We write \([e, \xi] = \{u_i\}_{i \geq 0}, [e, \eta] = \{v_i\}_{i \geq 0} \) and \([\xi, \eta] = \{\gamma_i\}_{i \in \mathbb{Z}}. \) By the \( \delta \)-thinness, we have either \( d_\psi(\{v_i\}_{i \geq 0}, \gamma_0) \leq \delta \) or \( d_\psi(\{u_i\}_{i \geq 0}, \gamma_0) \leq \delta. \) We will assume the former case; the other case can be proved similarly. Choose \( j \) so that \( j = \min \{i \geq 0 : d_\psi(\gamma_0, v_i) \leq \delta + D\} \) where \( D \) is given in
Theorem 4.1 Let $n_1$ be the number $n(3\delta+3D)$ from Lemma 6.5. We may assume that $j > n_1$; otherwise, we can simply take $u = v = e$ to finish the proof. We claim that

$$d_\psi(v_j - n_1 o, \{\gamma_i o\}_{i \in \mathbb{Z}}) > \delta.$$  

Indeed, otherwise, $d_\psi(v_j - n_1 o, \gamma_i o) \leq \delta$ for some $i \in \mathbb{Z}$, and hence we have

$$d_\psi(o, \gamma_i o) \leq d_\psi(o, v_j o) - \delta - D - D \leq d_\psi(o, v_j o) - \delta.$$  

where the first and the last inequalities follow from Theorem 4.1 and the second is from Lemma 6.5. This yields a contradiction to the minimality of $d_\psi(o, \gamma_o o)$, proving the claim.

\[\text{Figure 5. Choice of } \gamma_o, v_j o, v_j - n_1 o \text{ and } u_k o\]

Since the triangle consisting of the sides $\{\gamma_i o\}_{i \in \mathbb{Z}}, \{v_i o\}_{i \geq 0}$ and $\{u_i o\}_{i \geq 0}$ is $\delta$-thin, the above claim implies that $v_j - n_1 o$ lies in the $\delta$-neighborhood of $\{u_i o\}_{i \geq 0}$. Hence there exists $u_k$ such that $d_\psi(v_j - n_1 o, u_k o) \leq \delta$ (see Figure 5). Note that $d_\psi(v_{\ell}, v_{\ell - 1}) \leq Q_0$ for all $\ell \geq 1$ where $Q_0$ depends only on the quasi-isometry constant of the orbit map $(\Gamma, d_\Gamma) \to (\Gamma_0, d_\psi)$. Hence we have so far obtained

- $d_\psi(\gamma_o, v_j o) \leq \delta + D$;
- $d_\psi(v_j o, u_k o) \leq n_1(Q_0 + D) + \delta$;
- $d_\psi(u_k o, \gamma_o) \leq n_1(Q_0 + D) + 2D + 2\delta$.

Therefore the claim follows by setting $C = n_1 Q_0 + (n_1 + 2)D + 2\delta$. □

The following was shown for $\theta = \Pi$ in [25, Lemma 5.7] which directly implies the statement for general $\theta$:
Lemma 6.7. There exists $\kappa > 0$ such that for any $g, h \in G$ and $R > 0$, we have
\[
\sup_{\xi \in \partial_G^\theta(o)} \left\| \beta_\xi^\theta(g \circ o) - \mu_\theta(g^{-1} h) \right\| \leq \kappa R.
\]

We now prove that the $\psi$-Gromov product $\psi(G(\xi, \eta))$ behaves like the distance $d(\psi(o, \gamma o) \circ [\xi, \eta] o)$ up to an additive error, which is an analogue of Lemma 5.4 in $(\Gamma, d)$:

Lemma 6.8. There exists $C_1 > 0$ such that for any $\xi \neq \eta \in \Lambda_\theta = \partial \Gamma$, we have
\[
|\psi(G^\theta(\xi, \eta)) - d(\psi(o, \gamma o)| < C_1
\]
where $\gamma = \gamma_\psi \in \Gamma$ is such that $d(\psi(o, \gamma o) = d(\psi(o, [\xi, \eta] o)$.

Proof. We write $[e, \xi] = \{u_i\}_{i \geq 0}, [e, \eta] = \{v_i\}_{i \geq 0}$ and $[\xi, \eta] = \{\gamma_i\}_{i \in \mathbb{Z}}$. Let $k, \ell \in K$ and $h \in G$ be such that $k P_\theta = \xi$, $\ell P_\theta = \eta$ and $(h P_\theta, h w_0 P_\theta) = (\xi, \eta)$.

By Theorem 4.9, we have for some uniform $D_0 > 0$ that
\[
\sup_{i \geq 0} d(u_i o, k M_\theta A^+ o) < D_0
\]
\[
\sup_{i \geq 0} d(v_i o, \ell M_\theta A^+ o) < D_0
\]
\[
\sup_{i \in \mathbb{Z}} d(\gamma_i o, h M_\theta A o) < D_0.
\]

By replacing $h$ with an element of $h M_\theta A$, we may assume that $h = h^\psi$ satisfies that $d(h o, \gamma o) < D_0$. Since $G^\theta(\xi, \eta) = \frac{1}{2}(\beta_\xi^\theta(o, h o) + i(\beta_\eta^\theta(o, h o)))$ and $\psi = \psi \circ i$, Lemma 5.1 and Lemma 6.7 imply that it suffices to show that
\[
\xi, \eta \in O^\theta_R(o, \gamma o)
\]
for some uniform $R > 0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{The dotted triangle is of diameter less than $C$ and the gray ball has radius $R$.}
\end{figure}
Applying Lemma 6.6 let $C > 0$ be the constant given by Lemma 6.6 and choose $u_k \in \{v_i\}_{i \geq 0}$ and $v_j \in \{v_i\}_{j \geq 0}$ so that $\{u_k, o, v_j, o\}$ has the $d_\psi$-diameter smaller than $C$ (see Figure 6). Hence, for some constant $C'$ determined by $C$ and the quasi-isometry constant of the identity map $(\Gamma o, d_\psi) \to (\Gamma o, d)$, the Riemannian diameter of $\{u_k o, v_j o, o\}$ is less than $C'$. Hence it follows from (6.5) that

$$d(\gamma o, kM_\theta A^+ o) < D_0 + C'$$

Since $kP_\theta = \xi$ and $\ell P_\theta = \eta$, we obtain by setting $R = D_0 + C'$ that

$$\xi, \eta \in O^0_R(o, \gamma o),$$

as desired. \hfill \square

We are now ready to prove the first inclusion in Theorem 6.3 which we formulate again as follows:

**Proposition 6.9.** There exist constants $c, R_0 > 0$ such that for any $\xi \in A_\theta$ and $g \in [e, \xi]$ in $\Gamma$, we have

(6.6) $$B_\psi(\xi, ce^{-d_\psi(o, go)}) \subset O^0_{R_0}(o, go) \cap A_\theta.$$

**Proof.** Let $C_1, D > 0$ be the constants given by Lemma 6.8 and Theorem 4.1 respectively. We also let $\delta > 0$ be a constant so that every triangle of $\Gamma o$, obtained as an image of the geodesic triangle in $(\Gamma, d_\Gamma)$ under the orbit map, is $\delta$-thin in $d_\psi$-metric (Proposition 6.4). We now claim that (6.6) holds with $c := e^{-(2\delta + C_1 + D)}$. Let $\eta \in B_\psi(\xi, ce^{-d_\psi(o, go)})$. So

$$\psi(G^0(\xi, \eta)) > d_\psi(o, go) + 2\delta + C_1 + D.$$

Let $\gamma \in [\xi, \eta]$ be chosen so that $d_\psi(o, \gamma o) = d_\psi(o, [\xi, \eta]o)$ as in Lemma 6.8. By Lemma 6.8 we have

$$d_\psi(o, \gamma o) \geq \psi(G^0(\xi, \eta)) - C_1 > d_\psi(o, go) + 2\delta + D.$$

By Theorem 4.1, we also have

$$d_\psi(o, \gamma o) \leq d_\psi(o, go) + d_\psi(go, [\xi, \eta]o) + D,$$

which implies $d_\psi(go, [\xi, \eta]o) > 2\delta$.

Note that the triangle $[e, \xi]o \cup [\xi, \eta]o \cup [e, \eta]o$ is $\delta$-thin. Hence $go$ is contained in the $\delta$-neighborhood of $[\xi, \eta]o \cup [e, \eta]o$ in $d_\psi$. Since $d_\psi(go, [\xi, \eta]o) > 2\delta$, we must have $d_\psi(go, [e, \eta]o) \leq \delta$ (see Figure 7). For some constant $\delta' > 0$ determined by $\delta$ and the quasi-isometry constant of the identity map $(\Gamma o, d_\psi) \to (\Gamma o, d)$, we have

$$d(go, [e, \eta]o) \leq \delta'.$$

By Theorem 4.9, there exist $\ell \in K$ and a uniform constant $D_0 > 0$ so that $\ell P_\theta = \eta$ and $[e, \eta]o$ is contained in the $D_0$-neighborhood of $\ell M_\theta A^+ o$. This implies

$$\eta \in O^0_{\delta' + D_0}(o, go).$$
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Since it is clear from definition that $B_{\psi}(\xi, ce^{-d_{\psi}(o,go)}) \subset \Lambda_\theta$, setting $R_0 = \delta' + D_0$ finishes the proof. □

7. SHADOWS INSIDE BALLS: THE SECOND INCLUSION IN THEOREM 6.3

We continue the notations from Section 6. Hence $i(\theta) = \theta$ and $\psi \in a^*_\theta$ is a linear form such that $\psi > 0$ on $L_\theta - \{0\}$ and $\psi = \psi \circ i$. In this section, we prove the second inclusion of Theorem 6.3, which can be stated as follows:

**Proposition 7.1.** For $r > 0$, there exists $c' = c'(r) > 0$ such that for any $\xi \in \Lambda_\theta = \partial \Gamma$ and any $g \in [e, \xi]$ in $\Gamma$, we have

$$O^\theta_{\psi'}(o, go) \cap \Lambda_\theta \subset B_{\psi}(\xi, c'e^{-d_{\psi}(o,go)}).$$

In addition to the coarse triangle inequality of $d_{\psi}$ (Theorem 4.1) and the uniform progression lemma (Lemma 6.5), we will use the property that the shadows in $(\Gamma, d_{\Gamma})$ are comparable to shadows in $\Lambda_\theta$ (Proposition 7.2).

In a Gromov hyperbolic space $\Gamma$, for $R > 0$ and $\gamma_1, \gamma_2 \in \Gamma$, the shadow $O^F_R(\gamma_1, \gamma_2)$ can be defined as the set of all $\xi \in \partial \Gamma$ such that a geodesic ray $[\gamma_1, \xi]$ intersects the $R$-ball centered at $\gamma_2$.

The following lemma states that shadows in $\partial \Gamma$ and shadows in $\mathcal{F}_\theta$ are compatible via the boundary map $f : \partial \Gamma \to \Lambda_\theta$: let $(L, Q)$ be the quasiconformal constants of the orbit map $(\Gamma, d_{\Gamma}) \to (\Gamma o, d)$ and $R_0 = Q + D_0 + 1$ where $D_0$ is given in Theorem 4.9.

**Proposition 7.2.** For all $R > R_0$, there exists $R_1, R_2 > 0$ such that for any $\gamma_1, \gamma_2 \in \Gamma$,

$$f(O^F_{R_1}(\gamma_1, \gamma_2)) \subset O^\theta_R(\gamma_1 o, \gamma_2 o) \cap \Lambda_\theta \subset f(O^F_{R_2}(\gamma_1, \gamma_2)).$$

In proving this proposition, we will also need to consider shadows whose viewpoints are on the boundary $\mathcal{F}_\theta$. For $\eta \in \mathcal{F}_\theta$, $p \in X$, and $R > 0$, the

![Figure 7](image.png)

**Figure 7.** $go$ is far from $[\xi, \eta]o$ and hence close to $[e, \eta]o$; so $\eta$ lies in the shadow $O^\theta_{\delta' + D_0}(o, go)$.
\(\theta\)-shadow \(O^0_R(\eta, p)\) is defined as follows:
\[
O^0_R(\eta, p) = \{ gP_\theta \in F_\theta : g \in G, \ gw_0P_\theta = \eta, \ go \in B(p, R) \}.
\]
We will need the following proposition on continuity of shadows:

**Proposition 7.3** (Continuity of shadows on viewpoints, [20, Proposition 3.4]). Let \(p \in X, \ \eta \in F_\theta\) and \(r > 0\). If a sequence \(q_i \in X\) converges to \(\eta\) as \(i \to \infty\), then for any \(0 < \varepsilon < r\), we have
\[
O^0_{r-\varepsilon}(\eta, p) \subset O^0_r(q_i, p) \subset O^0_{r+\varepsilon}(\eta, p) \text{ for all large } i \geq 1.
\]

**Proof of Proposition 7.2** Let \(R > R_0\). By the \(\Gamma\)-equivariance of \(f\) as well as of shadows, we may assume \(\gamma_1 = e\) and write \(\gamma_2 = \gamma\). By applying Theorem 4.9(2), we get that for any \(\xi \in \partial \Gamma\) and \(k \in K\) with \(kP_\theta = f(\xi)\), the image \([e, \xi]\) is contained in the \(D_0\)-neighborhood of \(kM_0(\exp C')o \subset kM_0A^+o\) in the symmetric space \((X, d)\). Hence if \(\xi \in O^\Gamma_{R_1}(e, \gamma)\), and hence \([e, \xi]\) intersects the ball \(\{ g \in \Gamma : d_{\Gamma}(\gamma, g) < R \}\), then \(kM_0A^+o\) intersects \(LR_1 + Q + D_0\)-neighborhood of \(\gamma o\). Choosing \(R_1 > 0\) so that \(LR_1 + Q + D_0 < R\), we get \(f(\xi) \in O^\theta_R(o, \gamma o)\). This shows the first inclusion.

We now prove the second. Let \(R > 0\) and suppose that the claim does not hold for \(R\). Then for each \(i \geq 1\), there exists \(\gamma_i \in \Gamma\) such that
\[
O^\theta_R(o, \gamma_i o) \cap A_\theta \not\subset f(O^\Gamma_{r}(e, \gamma_i)).
\]
Hence for each \(i \geq 1\), we can choose \(x_i \in \partial \Gamma - O^\Gamma_{r}(e, \gamma_i)\) such that \(f(x_i) \in O^\theta_R(o, \gamma_i o)\). By the \(\Gamma\)-equivariance of \(f\), it follows that
\[
\gamma_i^{-1}x_i \notin O^\Gamma_{r}(\gamma_i^{-1}e) \text{ and } f(\gamma_i^{-1}x_i) \in O^\theta_R(\gamma_i^{-1}o, o) \text{ for all } i \geq 1.
\]
After passing to a subsequence, we may assume that \(\gamma_i^{-1} \to y \in \partial \Gamma\) and \(\gamma_i^{-1}x_i \to x\) as \(i \to \infty\). By Lemma 6.2, we deduce \(\gamma_i^{-1}o \to f(y)\) as \(i \to \infty\). Applying Proposition 7.3 to \(q_i = \gamma_i^{-1}o, \ p = o\) and \(\eta = f(y)\), we have for some \(\varepsilon > 0\) that
\[
O^\theta_R(\gamma_i^{-1}o, o) \subset O^\theta_R(\gamma_i^{-1}o, o) \subset O^\theta_R(\gamma_i^{-1}o, o) \subset O^\theta_{R+\varepsilon/2}(f(y), o) \text{ for all } i \geq 1.
\]
Since \(f(\gamma_i^{-1}x_i) \in O^\theta_R(\gamma_i^{-1}o, o)\) for all \(i \geq 1\) and \(f(\gamma_i^{-1}x_i) \to f(x)\) as \(i \to \infty\), we have
\[
f(x) \in O^\theta_R(\gamma_i^{-1}o, o).
\]
This implies that \(f(x)\) is in general position with \(f(y)\), i.e., \((f(x), f(y)) \in \mathcal{F}_{(2)}\), and in particular \(f(x) \neq f(y)\). On the other hand, since \(\gamma_i^{-1}x_i \notin O^\Gamma_{r}(\gamma_i^{-1}e)\) for all \(i \geq 1\), the sequence of geodesics \([\gamma_i^{-1}x_i, \gamma_i^{-1}]\) escapes any large ball centered at \(e\). This implies that two sequences \(\gamma_i^{-1}x_i\) and \(\gamma_i^{-1}\) must have the same limit, and hence \(x = y\) which is a contradiction. Therefore the claim follows.

We recall the Gromov product in \((\Gamma, d_{\Gamma})\): for \(\alpha, \beta, \gamma \in \Gamma\),
\[
(\alpha, \beta)_\gamma = \frac{1}{2}(d_{\Gamma}(\alpha, \gamma) + d_{\Gamma}(\beta, \gamma) - d_{\Gamma}(\alpha, \beta))
\]
and for \(x, y \in \partial \Gamma\),
\[
(x, y)_{\gamma} = \sup_{i,j} \lim_{n \to \infty} (x_i, y_j)_{\gamma}
\]
where the supremum is taken over all sequences \(\{x_i\}, \{y_j\}\) in \(\Gamma\) such that \(\lim_{n \to \infty} x_i = x\) and \(\lim_{n \to \infty} y_j = y\). The Gromov product for a pair of a point in \(\Gamma\) and a point in \(\partial \Gamma\) is defined similarly. As \(\Gamma\) is a hyperbolic group, the Gromov product \((x, y)_{\gamma}\) is known to measure a crude distance from \(\gamma\) and the geodesic \([x, y]\).

**Lemma 7.4.** For \(R > 0\), there exists \(r_0 = r_0(R) > 0\) such that for any \(x \in \partial \Gamma, \gamma \in [e, x]\) and \(y \in O_{R}^{e}(e, \gamma)\), we have
\[
d_{\Gamma}(\gamma_{x,y}, [\gamma, x]) \leq r_0
\]
where \(\gamma_{x,y} \in \Gamma\) is such that \(d_{\Gamma}(e, \gamma_{x,y}) = d_{\Gamma}(e, [x, y]) = \inf_{g \in [x, y]} d_{\Gamma}(e, g)\).

**Proof.** Let \([e, x] = \{\gamma_i\}_{i \geq 0}\). We fix \(\gamma := \gamma_i\) and \(y \in O_{R}^{e}(e, \gamma)\). It suffices to show that there exists a uniform constant \(r_0 = r_0(R) > 0\) such that \(d_{\Gamma}(\gamma_{x,y}, \gamma_j) \leq r_0\) for some \(j \geq i\). Since \(\Gamma\) is a hyperbolic group, an equivalent definition of a shadow in terms of the Gromov product implies that \((e, y)_{\gamma} < R + \delta/2\) for some uniform \(\delta > 0\). On the other hand, the hyperbolicity of \(\Gamma\) also implies that we can take \(\delta\) large enough so that
\[
(e, y)_{\gamma} \geq \min\{(e, \gamma_{x,y})_{\gamma}, (\gamma_{x,y}, y)_{\gamma}\} - \delta/2
\]
and every geodesic triangle in \(\Gamma \cup \partial \Gamma\) is \(\delta\)-thin (see [5] for basic properties of Gromov hyperbolic spaces). Therefore
\[
\min\{(e, \gamma_{x,y})_{\gamma}, (\gamma_{x,y}, y)_{\gamma}\} < R + \delta.
\]

First consider the case when \((\gamma_{x,y}, y)_{\gamma} < R + \delta\). Then for some constant \(\delta'\) depending on \(R + \delta\), there exists \(\gamma' \in [\gamma_{x,y}, y]\) such that \(d_{\Gamma}(\gamma', \gamma) < \delta'\). Consider the geodesic triangle with vertices \(x, \gamma, \gamma'\). Since this triangle is \(\delta\)-thin and \(\gamma_{x,y} \in [x, \gamma]\), the \(\delta\)-neighborhood of \(\gamma_{x,y}\) intersects \([x, \gamma] \cup [\gamma, \gamma']\). Hence it follows from \(d_{\Gamma}(\gamma, \gamma') < \delta'\) that the \((\delta + \delta')\)-neighborhood of \(\gamma_{x,y}\) intersects \([x, \gamma]\). Hence the claim follows in this case.

Now consider the case that \((e, \gamma_{x,y})_{\gamma} < R + \delta\). Since \(\gamma_{x,y}\) is the projection of \(e\) to \([x, y]\), for some uniform constant \(\bar{\delta}\), the \(\bar{\delta}\)-neighborhood of \(\gamma_{x,y}\) intersects both geodesic rays \([e, x]\) and \([e, y]\). In particular, there exists \(\gamma_{j_0} \in [e, x]\) such that \(d_{\Gamma}(\gamma_{x,y}, \gamma_{j_0}) < \bar{\delta}\). This implies
\[
(e, \gamma_{j_0})_{\gamma} \leq (e, \gamma_{x,y})_{\gamma} + d_{\Gamma}(\gamma_{x,y}, \gamma_{j_0}) < R + \delta + \bar{\delta}.
\]
Since both \(\gamma = \gamma_i\) and \(\gamma_{j_0}\) lie on \([e, x]\), this implies \(j_0 \geq i - (R + \delta + \bar{\delta})\). We then set \(j = j_0 + \lceil R + \delta + \bar{\delta} \rceil\) to have \(d_{\Gamma}(\gamma_{x,y}, \gamma_j) \leq R + \delta + 2\bar{\delta} + 1\).

Therefore it remains to set \(r_0 = R + \delta + \delta' + 2\bar{\delta} + 1\). \(\square\)

Now we are ready to prove:
Proof of Proposition 7.1. Let $\xi \in \Lambda_\theta = \partial \Gamma$ and $g \in [e, \xi]$ in $\Gamma$. Fix $r > 0$, and let $\eta \in O^\theta_\Gamma(o, go) \cap \Lambda_\theta$. We will continue to use the convention of identifying $\Lambda_\theta$ and $\partial \Gamma$ in this proof. As in Lemma 7.4, we let $\gamma := \gamma_{\xi, \eta}$ be the projection of $e$ to the bi-infinite geodesic $[\xi, \eta]$ in $(\Gamma, d_\Gamma)$. By Proposition 7.2 there exists $R > 0$, depending only on $r$, such that $\eta \in O^\theta_\Gamma(e, g)$. Write the geodesic ray $[e, \xi]$ as $\{g_k\}_{k \geq 0}$ with $g_0 = e$. Since $g \in [e, \xi]$ by the hypothesis, we have $g_i = g$ for some $i \geq 0$. Then for $r_0 = r_0(R) > 0$ given in Lemma 7.4 there exists $j \geq i$ such that $d_\Gamma(\gamma, g_j) \leq r_0$. It follows from Lemma 6.5 (uniform progression lemma) and Theorem 4.1 (coarse triangle inequality) that

$$d_\psi(o, g_j o) \geq d_\psi(o, g_{i-n(1)} o) + 1$$

$$\geq d_\psi(o, go) - d_\psi(g_{i-n(1)} o, go) - D + 1$$

where $n(1) \geq 0$ and $D \geq 0$ are given in Lemma 6.5 and Theorem 4.1 respectively. Since $g = g_i$, $d_\psi(g_{i-n(1)} o, go) = d_\psi(g_{i-1}^{-1} g_{i-n(1)} o, o)$ is bounded above by $\max\{d_\psi(o, \gamma o) : |\gamma| \leq n(1)\}$. Hence there exists a uniform constant $D' > 0$ so that

$$d_\psi(o, g_j o) \geq d_\psi(o, go) - D'.$$

On the other hand, applying the coarse triangle inequality (Theorem 4.1) again, we have

$$d_\psi(o, g_j o) \leq d_\psi(o, \gamma o) + d_\psi(\gamma o, g_j o) + D.$$ 

Since $d_\Gamma(\gamma, g_j) \leq r_0$, we have $d_\psi(\gamma o, g_j o) < r_1$ for some constant $r_1$ determined by $r_0$ and the quasi-isometry constant of the orbit map $(\Gamma, d_\Gamma) \to (\Gamma_0, d_\psi)$. We then have

$$d_\psi(o, \gamma o) \geq d_\psi(o, go) - D' - r_1 - D.$$ 

Since we have $|\psi(G^\theta(\xi, \eta)) - d_\psi(o, \gamma o)| < \|\psi\| C_1$ with $C_1$ given by Lemma 5.4

$$\psi(G^\theta(\xi, \eta)) \geq d_\psi(o, go) - D' - r_1 - D - \|\psi\| C_1.$$ 

Setting $c' = e^{D' + r_1 + D + \|\psi\| C_1}$, we have

$$d_\psi(\xi, \eta) \leq c' e^{-d_\psi(o, go)}.$$ 

Hence $\eta \in B_\psi(\xi, c' e^{-d_\psi(o, go)})$ as desired. \hfill \qedsymbol

8. Local size of Patterson-Sullivan measures

As before, let $\Gamma$ be a $\theta$-Anosov subgroup with semisimple Zariski closure. Recall from Theorem 3.2 that the space of $\Gamma$-Patterson-Sullivan measures on $\Lambda_\theta$ is parameterized by the set

$$\mathcal{H}_\Gamma = \{\psi \in a^\theta_\Gamma : \psi \text{ is tangent to } \psi^\theta_\Gamma\}.$$ 

We continue to use the notation $\nu_\psi$ for the unique $(\Gamma, \psi)$-Patterson-Sullivan measure. Recall that $d_\psi$ is the premetric on $\Lambda_\theta$ defined by $d_\psi(\xi, \eta) = e^{-\psi(G(\xi, \eta))}$ for all $\xi \neq \eta$ in $\Lambda_\theta$ and $B_\psi(\xi, r) = \{\eta \in \Lambda_\theta : d_\psi(\xi, \eta) < r\}$.
The goal of this section is to deduce the local behavior of \( \nu_\psi \) on \( d_\psi \)-balls from Theorem 6.3.

**Theorem 8.1.** For any symmetric \( \psi \in \mathcal{R}_\Gamma \), there exists a constant \( c_1 > 1 \) such that for all \( \xi \in \Lambda_\theta \) and \( 0 < r < 1 \)

\[
c_1^{-1} r \leq \nu_\psi(B_\psi(\xi, r)) \leq c_1 r.
\]

An immediate consequence is that \( \nu_\psi \) is exact dimensional, that is,

\[
\lim_{r \to 0} \frac{\log \nu_\psi(B_\psi(\xi, r))}{\log r} = 1 \quad \text{for all} \ \xi \in \Lambda_\theta.
\]

**Remark 8.2.** When \( \Gamma \) is a convex cocompact subgroup of \( G = \text{SO}^\theta(n, 1) \), \( \mathcal{R}_\Gamma \) is a singleton consisting of the critical exponent \( \delta_\Gamma \) (more precisely, the multiplication by \( \delta_\Gamma \)), and the metric \( d_\delta_\Gamma \) is the \( \delta_\Gamma \)-power of a \( K \)-invariant Riemannian metric on \( \mathbb{S}^{n-1} \). Hence Theorem 8.1 is equivalent to Sullivan’s theorem [33] that the Patterson-Sullivan measure of a Riemannian ball of radius \( r \) is proportional to \( r^{\delta_\Gamma} \).

We use the higher rank version of Sullivan’s shadow lemma. The following is a special case of [21, Lemma 7.2]:

**Lemma 8.3** (Shadow lemma). Let \( \Gamma < G \) be a \( \theta \)-Anosov subgroup such that \( \Lambda_\theta \) is infinite. There exists \( c_0 = c_0(\psi) \geq 1 \) such that for all large enough \( R \) and all \( r \in \Gamma \),

\[
c_0^{-1} e^{-\psi(\mu(\gamma))} \leq \nu_\psi(O^\theta_R(o, \gamma o)) \leq c_0 e^{-\psi(\mu(\gamma))}.
\]

**Proof of Theorem 8.1.** For simplicity, we write \( \nu = \nu_\psi \). By Lemma 3.4 it suffices to consider the case of \( \theta = i(\theta) \). Let \( e \) and \( R_0 \) be the constants as in Theorem 6.3. Fix \( \xi \in \Lambda_\theta \) and \( r > 0 \). Write the geodesic ray \([e, \xi]\) as \( \{\gamma_t\}_{t \geq 0} \) in \((\Gamma, d_\Gamma)\). Setting

\[
i_r := \max\{i : r \leq ce^{-d_\psi(o, \gamma_i o)}\},
\]

Theorem 6.3 implies that for any \( R > R_0 \),

\[
B_\psi(\xi, r) \subset B_\psi(\xi, ce^{-d_\psi(o, \gamma_i o)}) \subset O^\theta_R(o, \gamma_i o).
\]

By Lemma 8.3, we get

\[
\nu(B_\psi(\xi, r)) \leq c_0 e^{-d_\psi(o, \gamma_i o)}.
\]

By the coarse triangle inequality of \( d_\psi \) (Theorem 4.1), we have

\[
d_\psi(o, \gamma_{i+1} o) \leq d_\psi(o, \gamma_i o) + d_\psi(o, \gamma_i^{-1} \gamma_{i+1} o) + D
\]

where \( D \) is as in loc. cit. Since \( d_\psi(o, \gamma_i^{-1} \gamma_{i+1} o) \) is bounded from above by \( \max\{d_\psi(o, \gamma) : |\gamma| = 1\} \), we have

\[
d_\psi(o, \gamma_{i+1} o) \leq d_\psi(o, \gamma_i o) + D'
\]

where \( D' = D + \max\{d_\psi(o, \gamma) : |\gamma| = 1\} \). This implies

\[
ce^{-D'} e^{-d_\psi(o, \gamma_i o)} \leq ce^{-d_\psi(o, \gamma_i+1 o)} < r
\]
where the last inequality follows from the definition of \(i_r\) in (8.1). Hence we deduce from (8.2) that
\[
\nu(B_\psi(\xi, r)) \leq (c_0 e^{D'}/c) \cdot r.
\]

Now let \(c' = c'(R) > 0\) be given by Theorem 6.3 and set
\[
(8.3) \quad j_r := \min\{j : c' e^{-d_\psi(o, \gamma_j o)} \leq r\}.
\]
By Theorem 6.3, we have
\[
O^0_R(o, \gamma_j o) \cap \Lambda_\theta \subset B_\psi(\xi, c' e^{-d_\psi(o, \gamma_j o)} \subset B_\psi(\xi, r).
\]
By Lemma 8.3,
\[
c_0^{-1} e^{-d_\psi(o, \gamma_j o)} \leq \nu(B_\psi(\xi, r)).
\]
By the minimality of \(j_r\) as defined in (8.3) and the coarse triangle inequality of \(d_\psi\) (Theorem 4.1), we have
\[
r < c' e^{-d_\psi(o, \gamma_{j_r} o)} \leq c' e^{D} e^{-d_\psi(o, \gamma_{j_r} o)} + d_\psi(o, \gamma_{j_r}^{-1} \gamma_j o).
\]
Recalling that \(D' = D + \max\{d_\psi(o, \gamma) : |\gamma| = 1\}\), we have
\[
r < c' e^{D'} e^{-d_\psi(o, \gamma_{j_r} o)}
\]
and hence
\[
(c_0 c' e^{D'})^{-1} \cdot r \leq \nu(B_\psi(\xi, r)).
\]
Therefore the theorem is proved with \(c_1 = \max(c_0 e^{D'} c^{-1}, c_0 c' e^{D'}).\)

9. Hausdorff measures on limit sets

Let \(\Gamma < G\) be a \(\theta\)-Anosov subgroup with semisimple Zariski closure. For a linear form \(\psi \in a_\theta^*\) which is positive on \(L_\theta - \{0\}\), consider the associated conformal premetric on \(d_\psi\) on \(\Lambda_\theta\). For \(s > 0\), we define the \(s\)-dimensional Hausdorff measure \(H^s_\psi\) on \(\Lambda_\theta\) as follows: For \(B \subset \Lambda_\theta\) and \(\varepsilon > 0\), set
\[
H^s_\psi,\varepsilon(B) := \inf \left\{ \sum_{i \in \mathbb{N}} (\text{Diam}_\psi U_i)^s : B \subset \bigcup_{i \in \mathbb{N}} U_i, \sup_{i \in \mathbb{N}} \text{Diam}_\psi U_i \leq \varepsilon \right\}
\]
where \(\text{Diam}_\psi U = \sup\{d_\psi(\xi, \eta) : \xi, \eta \in U\}\) for \(U \subset \Lambda_\theta\). The \(s\)-dimensional Hausdorff measure is then defined as
\[
H^s_\psi(B) = \lim_{\varepsilon \to 0} H^s_\psi,\varepsilon(B) = \sup_{\varepsilon > 0} H^s_\psi,\varepsilon(B).
\]
This is indeed an outer measure and every Borel subset of \(\Lambda_\theta\) is \(H^s_\psi\)-measurable \(\text{[11] Appendix A}\). For \(s = 1\), we simply write \(H_\psi = H^1_\psi\).

The main goal of this section is to prove the following two theorems (Theorem 1.1, Theorem 1.2). We also obtain an identity between the Hausdorff dimension and critical exponent as stated in Theorem 1.3 as an immediate corollary: recall that \(\psi \in a_\theta^*\) is symmetric if \(\psi = \psi \circ i\).
Theorem 9.1. For any symmetric $\psi \in \mathcal{T}_\Gamma$, the associated Patterson-Sullivan measure $\nu_\psi$ coincides with the one-dimensional Hausdorff measure $\mathcal{H}_\psi$, up to a constant multiple. In other words, $\mathcal{H}_\psi$ is the unique $(\Gamma, \psi)$-conformal measure on $\Lambda_\theta$, up to a constant multiple.

We also show that the symmetric hypothesis is necessary in the above theorem:

Theorem 9.2. If $\psi \in \mathcal{T}_\Gamma$ is not symmetric and $\Gamma$ is Zariski dense, then $\nu_\psi$ is not proportional to $\mathcal{H}_s^\psi$ for any $s > 0$.

Remark 9.3. If $\psi \in a^*_\theta$ is positive on $L_\theta - \{0\}$, then $\delta_\psi \psi \in \mathcal{T}_\Gamma$. Since $\mathcal{H}_\delta_\psi \psi = \mathcal{H}_\delta_\psi \psi$, Theorem 9.1 says that for $\psi$ symmetric,

$$(9.1) \quad \mathcal{H}_\delta_\psi \psi = \nu_\delta_\psi \psi$$

up to a constant multiple.

Remark 9.4. For a special class of symmetric $\psi$ whose gradient lies in the interior of $a^*_\theta$, Dey-Kapovich [11, Corollary 4.8] showed that $(\Gamma_0, d_\psi)$ is a Gromov hyperbolic space and they proved Theorem 9.1 relying upon the work of Coornaert [9] which gives the positivity and finiteness of $\mathcal{H}_\psi$ for the Gromov hyperbolic space. In our generality, $(\Gamma_0, d_\psi)$ is not even a metric space, and hence their approach cannot be extended.

The main work is to establish the positivity and the finiteness of $\mathcal{H}_\psi$ and the key ingredient is the precise local behavior of $\nu_\psi$ obtained in Theorem 8.1.

Proposition 9.5 (Positivity). For any symmetric $\psi \in \mathcal{T}_\Gamma$, we have

$$\mathcal{H}_\psi(\Lambda_\theta) > 0.$$ 

Proof. Fix $\varepsilon > 0$ and a countable cover $\{U_i\}_{i \in \mathbb{N}}$ such that $\text{Diam}_\psi U_i \leq \varepsilon$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, we choose $\xi_i \in U_i$. By Theorem 8.1, we have

$$\sum_{i \in \mathbb{N}} \text{Diam}_\psi U_i \gg \sum_{i \in \mathbb{N}} \nu_\psi(B_\psi(\xi_i, \text{Diam}_\psi U_i))$$

where the implied constant depends only on $\psi$. Since $\Lambda_\theta \subset \bigcup_{i \in \mathbb{N}} U_i \subset \bigcup_{i \in \mathbb{N}} B_\psi(\xi_i, \text{Diam}_\psi U_i)$, it follows that

$$\sum_{i \in \mathbb{N}} \text{Diam}_\psi U_i \gg \nu_\psi(\Lambda_\theta) = 1.$$ 

Since $\{U_i\}_{i \in \mathbb{N}}$ is an arbitrary countable cover, it follows that $\mathcal{H}_\psi, \varepsilon(\Lambda_\theta) \gg 1$ with the implied constant depending only on $\psi$. Since $\varepsilon > 0$ is arbitrary, we have $\mathcal{H}_\psi(\Lambda_\theta) > 0$. \qed

Proposition 9.6 (Finiteness). For any symmetric $\psi \in \mathcal{T}_\Gamma$, we have

$$\mathcal{H}_\psi(\Lambda_\theta) < \infty.$$
Proof. Let $N = N(\psi)$ and $N_0 = N_0(\psi)$ be the constants given in Proposition 5.3 and Lemma 5.6 respectively. Fix $\varepsilon > 0$. Since $\Lambda_\theta$ is compact, we have a finite cover $\Lambda_\theta$ by $\bigcup_{i=1}^n B_\psi(\xi_i, \frac{\varepsilon}{2N N_0})$ for some finite set $\xi_1, \cdots, \xi_n \in \Lambda_\theta$. Applying the Vitali covering type lemma (Lemma 5.6), there exists a disjoint subcollection $B_\psi(\xi_{i_1}, \frac{\varepsilon}{2N N_0}), \cdots, B_\psi(\xi_{i_k}, \frac{\varepsilon}{2N N_0})$ such that

$$\Lambda_\theta \subset \bigcup_{j=1}^k B_\psi(\xi_{i_j}, \frac{\varepsilon}{2N}).$$

Since $\text{Diam}_\psi B_\psi(\xi_{i_j}, \frac{\varepsilon}{2N}) \leq \varepsilon$ for each $1 \leq j \leq k$ by Proposition 5.3(2), we have

$$\mathcal{H}_{\psi, \varepsilon}(\Lambda_\theta) \leq \sum_{j=1}^k \text{Diam}_\psi B_\psi(\xi_{i_j}, \frac{\varepsilon}{2N}) \leq k \cdot \varepsilon.$$ 

Applying Theorem 8.1, we obtain

$$k \cdot \varepsilon \ll \sum_{j=1}^k \nu_\psi(B_\psi(\xi_{i_j}, \frac{\varepsilon}{2N N_0})) = \nu_\psi\left(\bigcup_{j=1}^k B_\psi(\xi_{i_j}, \frac{\varepsilon}{2N N_0})\right) \leq \nu_\psi(\Lambda_\theta) = 1$$

where the equality follows from the disjointness. This implies $\mathcal{H}_{\psi, \varepsilon}(\Lambda_\theta) \ll 1$. Since $\varepsilon$ is arbitrary, we have $\mathcal{H}_{\psi}(\Lambda_\theta) \ll 1$. \hfill \(\square\)

Hence $\mathcal{H}_\psi$ is a non-trivial measure on $\Lambda_\theta$. It is also $\Gamma$-conformal:

**Lemma 9.7** (Conformality). For any symmetric $\psi \in \mathcal{R}_\Gamma$, we have

$$\frac{d\gamma_*\mathcal{H}_\psi}{d\mathcal{H}_\psi}(\xi) = e^{\psi(\beta_\theta(\varepsilon, \gamma))}$$

for all $\gamma \in \Gamma$ and $\xi \in \Lambda_\theta$.

**Proof.** Since $d_\psi$ is invariant under the $\Gamma$-equivariant homeomorphism $p : \Lambda_{\theta, \beta_i(\theta)} \to \Lambda_\theta$ by the definition of $d_\psi$, the measure $(\mathcal{H}_\psi, \Lambda_\theta)$ is the push-forward of the Hausdorff measure $(\mathcal{H}_\psi, \Lambda_{\theta, \beta_i(\theta)})$ via $p$. Therefore it suffices to prove this lemma assuming that $\theta = i(\theta)$. We simply write $\beta^\theta = \beta$ in this proof to ease the notation.

Fix $\gamma \in \Gamma$ and $\xi \in \Lambda_\theta$. Let $U \subset \Lambda_\theta$ be a small open neighborhood of $\xi$. To estimate $\gamma_*\mathcal{H}_\psi(U)$ in terms of $\mathcal{H}_\psi(U)$, we fix $\varepsilon > 0$ and take any cover $\{U_i\}_{i \in \mathbb{N}}$ of $U$ such that $\sup_i \text{Diam}_\psi U_i \leq \varepsilon$ and that $U \cap U_i \neq \emptyset$ for all $i \in \mathbb{N}$.

By Lemma 5.2 and Proposition 5.3 with $N = N(\psi) > 0$ therein, we have that for each $i \geq 1$

$$\text{Diam}_\psi \gamma^{-1} U_i \leq e^{\sup_{\eta \in U_i} \psi(\beta_\eta(\varepsilon, \gamma))} \text{Diam}_\psi U_i \leq e^{\sup_{\eta \in B_\psi(\xi, R_\varepsilon)} \psi(\beta_\eta(\varepsilon, \gamma))} \text{Diam}_\psi U_i$$

where $R_\varepsilon = R_\varepsilon(U) = N(Diam_\psi U + \varepsilon)$. Setting $\bar{\varepsilon} := e^{\sup_{\eta \in B_\psi(\xi, R_\varepsilon)} \psi(\beta_\eta(\varepsilon, \gamma))}$, we then have

$$\mathcal{H}_{\psi, \varepsilon}(\gamma^{-1} U) \leq \sum_{i \in \mathbb{N}} \text{Diam}_\psi \gamma^{-1} U_i \leq e^{\sup_{\eta \in B_\psi(\xi, R_\varepsilon)} \psi(\beta_\eta(\varepsilon, \gamma))} \sum_{i \in \mathbb{N}} \text{Diam}_\psi U_i.$$
Since \( \{U_i\}_{i \in \mathbb{N}} \) is an arbitrary countable open cover of \( U \), the above inequality implies
\[
\mathcal{H}_\psi(\gamma^{-1}U) \leq e^{\sup_{\eta \in B_\psi(\xi, R_{\gamma^{-1}}) \psi(\beta_\eta(e, \gamma^{-1}))}} \mathcal{H}_\psi(U).
\]
Taking \( \varepsilon \to 0 \), we have \( \bar{\varepsilon} \to 0 \) and \( R_{\bar{\varepsilon}} \to R_U := N \cdot \text{Diam}_\psi U \). Therefore
\[
(9.2) \quad \mathcal{H}_\psi(\gamma^{-1}U) \leq e^{\sup_{\eta \in B_\psi(\xi, R_U) \psi(\beta_\eta(e, \gamma^{-1}))}} \mathcal{H}_\psi(U).
\]
Applying (9.2) after replacing \( U \) with \( \gamma^{-1}U \), and \( \gamma \) by \( \gamma^{-1} \), we have
\[
\mathcal{H}_\psi(U) = \mathcal{H}_\psi(\gamma(\gamma^{-1}U)) \leq e^{\sup_{\eta \in B_\psi(\gamma^{-1}\xi, R_{\gamma^{-1}}U) \psi(\beta_\eta(e, \gamma^{-1}))}} \mathcal{H}_\psi(\gamma^{-1}U).
\]
If we set \( c = \sup_{\xi \in \Lambda_\theta} e^{\psi(\beta_\xi(e, \gamma^{-1}))} \), then for any \( \eta \in B_\psi(\gamma^{-1}\xi, R_{\gamma^{-1}}U) \), it follows from Lemma 5.2 that
\[
d(\psi, \eta) \leq cd(\psi, \gamma^{-1} \xi, \eta) \leq eR_U.
\]
Hence we have
\[
\mathcal{H}_\psi(U) \leq e^{\sup_{\eta \in B_\psi(\xi, cR_U) \psi(\beta_\eta(e, \gamma^{-1}))}} \mathcal{H}_\psi(\gamma^{-1}U) = e^{-\inf_{\eta \in B_\psi(\xi, cR_U) \psi(\beta_\eta(e, \gamma))}} \mathcal{H}_\psi(\gamma^{-1}U).
\]
Together with (9.2), we deduce
\[
e^{\inf_{\eta \in B_\psi(\xi, cR_U) \psi(\beta_\eta(e, \gamma))}} \frac{\gamma_s \mathcal{H}_\psi(U)}{\mathcal{H}_\psi(U)} \leq e^{\sup_{\eta \in B_\psi(\xi, R_U) \psi(\beta_\eta(e, \gamma))}}.
\]
Now shrinking \( U \to \xi \), we have \( R_U \to 0 \) and hence the both sides in the above inequality converges to \( e^{\psi(\beta_\xi(e, \gamma))} \), by the continuity of the Busemann map \( \beta_\eta(e, \gamma) \) on the \( \eta \)-variable. Therefore
\[
\frac{d\gamma_s \mathcal{H}_\psi(\xi)}{d\mathcal{H}_\psi} = e^{\psi(\beta_\xi(e, \gamma))}
\]
as desired. \( \square \)

**Proof of Theorem 9.1** By Proposition 9.5 and Proposition 9.6, we have \( \mathcal{H}_\psi(\Lambda_\theta) \in (0, \infty) \). Moreover, it follows from Lemma 9.7 that \( \frac{1}{\mathcal{H}_\psi(\Lambda_\theta)} \mathcal{H}_\psi \) is a \((\Gamma, \psi)\)-Patterson-Sullivan measure. Since there exists unique \((\Gamma, \psi)\)-Patterson-Sullivan measure on \( \Lambda_\theta \) (Theorem 3.2), this completes the proof. \( \square \)

**Proof of Theorem 9.2** By Lemma 3.4, we may assume \( \theta = i(\theta) \). Since \( \psi \neq \psi \circ i \), \( \psi \) and \( \tilde{\psi} \) are not proportional. Since \( d_\psi \) and \( d_{\tilde{\psi}} \) are bi-Lipschitz by Proposition 5.5, \( \mathcal{H}_\psi^s \) is in the same measure class as \( \mathcal{H}_{\tilde{\psi}}^s \) for all \( s > 0 \). Hence it follows from Theorem 9.1 that \( \mathcal{H}_\psi^s(\Lambda_\theta) = 0 \) or \( \infty \) if \( s \neq \delta_{\tilde{\psi}} \). Now it suffices to show that \( \nu_{\psi} \) is not proportional to \( \mathcal{H}_\psi^\delta_{\tilde{\psi}} \). Since \( \psi \) and \( \tilde{\psi} \) are not proportional, \( \psi \) and \( \delta_{\tilde{\psi}} \psi \) are two different forms tangent to \( \psi_\theta \). By Theorem 3.2 it follows that \( \nu_{\psi} \) is mutually singular to \( \nu_{\delta_{\tilde{\psi}} \psi} \). Since the latter is proportional to \( \mathcal{H}_\psi^\delta_{\tilde{\psi}} \),
by Theorem 9.1, it follows that \( \nu_{\psi} \) is not proportional to \( \mathcal{H}^{\delta_{\bar{\psi}}} \), finishing the proof. □

**Critical exponents and Hausdorff dimensions.** The Hausdorff dimension of \( \Lambda_\theta \) with respect to \( d_\psi \) is defined as

\[
\dim_\psi \Lambda_\theta = \inf \{ s > 0 : \mathcal{H}^s_\psi(\Lambda_\theta) = 0 \} = \sup \{ s > 0 : \mathcal{H}^s_\psi(\Lambda_\theta) = \infty \}.
\]

As a corollary of Theorem 9.1, we obtain the following (Theorem 1.3):

**Corollary 9.8.** For any \( \psi \in a_\theta^* \) positive on \( L_\theta - \{0\} \), we have

\[
\delta_{\bar{\psi}} = \dim_\psi \Lambda_\theta
\]

where \( \bar{\psi} = \frac{\psi + \psi_{\text{oi}}}{2} \).

**Proof.** By Proposition 5.5, we have \( \dim_\psi \Lambda_\theta = \dim_{\bar{\psi}} \Lambda_\theta \). Applying Theorem 9.1 to \( \delta_{\bar{\psi}} \bar{\psi} \) (see Remark 9.3), we have \( \mathcal{H}^{\delta_{\bar{\psi}}}_{\bar{\psi}}(\Lambda_\theta) \in (0, \infty) \), which implies \( \dim_{\bar{\psi}} \Lambda_\theta = \delta_{\bar{\psi}} \). This shows the claim. □

For \( \psi \) non-symmetric, \( \dim_\psi \Lambda_\theta \) is not in general equal to \( \delta_{\psi} \):

**Proposition 9.9.** Suppose that \( \Gamma \) is Zariski dense. For \( \psi \in a_\theta^* \) positive on \( L_\theta - \{0\} \), we have

\[
\delta_{\bar{\psi}} \leq \delta_{\psi}
\]

and the equality holds if and only if \( \psi = \psi \circ i \).

**Proof.** It suffices to show that if \( \psi \neq \psi \circ i \), then \( \delta_{\bar{\psi}} < \delta_{\psi} \). As before, we may assume \( \theta = i(\theta) \). Suppose that \( \psi \neq \psi \circ i \). Note that \( \delta_{\psi} = \delta_{\psi_{\text{oi}}} \) and hence \( \delta_{\psi} \psi, \delta_{\psi}(\psi \circ i) \) are tangent to the \( \theta \)-growth indicator \( \psi^{\theta}_T \) ([19, 21, Lemma 4.5]). We then have \( \psi^{\theta}_T \leq \delta_{\psi} \psi \) and \( \psi^{\theta}_T \leq \delta_{\psi}(\psi \circ i) \), and for some unique unit \( u \in \text{int}L_\theta, \psi^{\theta}_T(u) = \delta_{\psi} \psi(u) \) and \( \psi^{\theta}_T(i(u)) = \delta_{\psi}(\psi \circ i)(i(u)) \) by the strict concavity of \( \psi^{\theta}_T \). Since the tangent linear forms are in one-to-one correspondence with directions in \( \text{int}L_\theta \) as tangent directions [21, Theorem 12.2] (see also [32]) and \( \psi \neq \psi \circ i \), we have \( u \neq i(u) \). Hence the inequality \( \psi^{\theta}_T \leq \delta_{\psi} \psi \) and \( \psi^{\theta}_T \leq \delta_{\psi}(\psi \circ i) \) cannot become equalities simultaneously at the same vector. This implies that \( \psi^{\theta}_T < \delta_{\bar{\psi}} \). On the other hand, \( \delta_{\bar{\psi}} \psi \) is tangent to \( \psi^{\theta}_T \). This proves that \( \delta_{\bar{\psi}} < \delta_{\psi} \). □

**Remark 9.10.** The inequality \( \delta_{\bar{\psi}} \leq \delta_{\psi} \) holds for general Zariski dense discrete subgroups. The Anosov property was used in the proof to ensure the one-to-one correspondence with directions in \( \text{int}L_\theta \) and tangent linear forms, from which we deduce that the equality holds only when the given linear form is symmetric.

For a hyperbolic group \( \Sigma \), a representation \( \rho : \Sigma \to G \) is called \( \theta \)-Anosov if \( \rho \) has a finite kernel and its image \( \rho(\Sigma) \) is a \( \theta \)-Anosov subgroup of \( G \). It was shown in [1] that for a given \( \psi \in a_\theta^* \) which is non-negative on \( a_\theta^+ \), the \( \psi \)-critical exponents \( \delta_{\psi}(\rho(\Sigma)) \) vary analytically on analytic families of
θ-Anosov representations ρ in the variety Hom(Σ, G). Hence the following is an immediate consequence of [4] and Corollary 9.8.

**Corollary 9.11.** Let Σ be a hyperbolic group and ψ ∈ θ+ non-negative on a+θ. Let D ⊂ Hom(Σ, G) be an analytic family of θ-Anosov representations whose images have semisimple Zariski closures. Then ρ ↦ dimψΛθ(ρ(Σ)) is analytic on D.

10. Hausdorff dimensions with respect to Riemannian metrics

Let G be a connected semisimple real algebraic group. As before, let θ be a non-empty subset of the set Π of simple roots of (g, a). In this final section, we present an estimate on the Hausdorff dimensions of limit sets with respect to Riemannian metrics which we deduce from Corollary 9.8. We denote by dRiem the Riemannian metric on Fθ and dim Λθ the Hausdorff dimension on Λθ with respect to dRiem.

**Tits representations and Riemannian metrics.** We use the following result of Tits:

**Theorem 10.1** (Tits, [35, Theorem 7.2]). There exists a family of irreducible representations \{(ρα, Vα) : α ∈ Π\} of G such that

1. the highest weight χα of ρα is a positive integral multiple of the fundamental weight ωα of α;
2. the highest weight space of ρα is one-dimensional;
3. all other weights of ρα are of the form χα − α − ∑β∈Π nββ for some non-negative integers nβ, for each β ∈ Π.

When G is split over \(\mathbb{R}\), we have χα = ωα for all α ∈ Π. The subset \{χα : α ∈ θ\} is a basis of a+θ. We will call ρα’s Tits representations of G. For each α ∈ Π, we denote by Vα+ the highest weight space of ρα and by Vα− its unique complementary A-invariant subspace in Vα. Then the map g ∈ G ↦ (ρα(g)Vα+)α∈θ factors through a proper immersion

\[ Fθ → \prod_{α∈θ} P(Vα). \]

Let \(\langle \cdot , \cdot \rangle_α\) be a K-invariant inner product on Vα with respect to which A is symmetric, so that Vα+ is perpendicular to Vα−. We denote by \(\| \cdot \|_α\) the norm on Vα induced by \(\langle \cdot , \cdot \rangle_α\). We also use the notation \(\| \cdot \|_α\) for a bi-ρα(K)-invariant norm on GL(Vα). The angle between a line E and a subspace F is defined as minimum of all angles between all non-zero \(v \in E\) and non-zero \(w \in F\). Recall the notation: for \(g \in G\), we set \(g^+ = gP ∈ F\) and \(g^- = gw_0P ∈ F\).

**Lemma 10.2** ([31, Lemma 6.4], [25, Lemma 3.11]). For all α ∈ Π and \(g ∈ G\), we have

\[ 2χ_α(G(g^+, g^-)) = −\log \sin \angle(gVα+, gVα−). \]
Note that \( w_0 V^+_{\alpha} \subseteq V^-_{\alpha} \); to see this, first note that \( V^-_{\alpha} \) is the sum of all weight subspaces of \( V_{\alpha} \) whose weight is not equal to \( \chi_{\alpha} \). On the other hand, \( w_0 V^+_{\alpha} \) is a weight space with the weight given by \( \chi_{\alpha} \circ \text{Ad} w_0 = -\chi_{\alpha} \circ i \). Since \( -\chi_{\alpha} \circ i(a^+) \leq 0 \) while \( \chi_{\alpha}(a^+) \geq 0 \), \( \chi \circ \text{Ad} w_0 \neq \chi_{\alpha} \), which shows \( w_0 V^+_{\alpha} \subseteq V^-_{\alpha} \).

For \( g_1, g_2 \in G \), we have up to Lipschitz equivalence, 

\[
d_{\text{Riem}}(g_1 P_\theta, g_2 P_\theta) = \left( \sum_{\alpha \in \theta} \sin^2 \angle (g_1 V^+_{\alpha}, g_2 V^+_{\alpha}) \right)^{1/2}.
\]

Hence we simply use the above identification as the definition of \( d_{\text{Riem}} \) by abusing the notation. It follows from Lemma 10.2 that for \( g \in G \),

(10.1)

\[
d_{\text{Riem}}(g^+, g^-) = \left( \sum_{\alpha \in \theta} \sin^2 \angle (g V^+_{\alpha}, g w_0 V^+_{\alpha}) \right)^{1/2} \geq \left( \sum_{\alpha \in \theta} e^{-4\chi_{\alpha}(g^+, g^-)} \right)^{1/2} = \left( \sum_{\alpha \in \theta} d_{2\chi_{\alpha}}(g^+, g^-) \right)^{1/2}.
\]

**Lower bounds.** For a linear form \( \psi \in a^*_\theta \), we set

\[
\kappa_{\psi} := \sum_{\alpha \in \theta} \kappa_{\alpha}
\]

where \( \psi = \sum_{\alpha \in \theta} \kappa_{\alpha} \chi_{\alpha} \) is the \( \mathbb{R} \)-linear combination of the basis \( \{ \chi_{\alpha} : \alpha \in \theta \} \) of \( a^*_\theta \). We consider a set \( E_{\theta} \) of linear forms which are non-negative linear combination of \( \{ \chi_{\alpha} : \alpha \in \theta \} \). That is,

\[
E_{\theta} := \{ \psi \in a^*_\theta : \kappa_{\alpha} \geq 0 \text{ for all } \alpha \in \theta \}.
\]

In the rest of this section, we assume that

\( \Gamma \) is a \( \theta \)-Anosov subgroup with semisimple Zariski closure.

**Lemma 10.3.** For any \( \psi \in E_{\theta} \), the identity map

\( (\Lambda_{\theta}, d_{\text{Riem}}) \to (\Lambda_{\theta}, d_{\psi} \kappa_{\psi}) \)

is a Lipschitz map.

**Proof.** As in (10.1), we have for each \( \alpha \in \theta \),

\[
d_{\text{Riem}}(g^+, g^-) \geq \left( \sum_{\alpha \in \theta} d_{2\chi_{\alpha}}(g^+, g^-) \right)^{1/2} \geq d_{2\chi_{\alpha}}(g^+, g^-).
\]

This implies

\[
d_{\psi}(g^+, g^-) = \prod_{\alpha \in \theta} d_{2\chi_{\alpha}}(g^+, g^-)^{\frac{\kappa_{\alpha}}{2}} \leq \prod_{\alpha \in \theta} d_{\text{Riem}}(g^+, g^-)^{\frac{\kappa_{\alpha}}{2}} = d_{\text{Riem}}(g^+, g^-)^{\frac{\kappa_{\psi}}{2}}.
\]

Hence the claim follows. \( \square \)
Remark 10.4. Since $d_\psi$ and $d_{\bar{\psi}}$ are bi-Lipschitz (Proposition 5.5), Proposition 6.9 and the above lemma imply that there exist $c, R > 0$ such that for any $\xi \in \Lambda_\theta$ and $g \in [e,\xi]$ in $\Gamma$, the shadow $O_R^\theta(o,go) \cap \Lambda_\theta$ contains the Riemannian ball of center $\xi$ and of radius $ce^{-\frac{\delta_{\bar{\psi}}}{2\kappa_\psi}(o,go)}$.

Theorem 10.5. For any $\psi \in E_\theta$, we have

$$\dim \Lambda_\theta = \frac{\kappa_\psi}{2} \dim_{\psi} \Lambda_\theta = \frac{\kappa_\psi}{2} \delta_{\bar{\psi}}.$$ 

Proof. It follows from Lemma 10.3 that we get

$$\dim \Lambda_\theta \geq \frac{\kappa_\psi}{2} \dim_{\psi} \Lambda_\theta.$$ 

Since $\dim_{\psi} \Lambda_\theta = \delta_{\bar{\psi}}$ by Corollary 9.8, the claim follows. \hfill $\square$

For each $\alpha \in \theta$, $\psi = \chi_\alpha \in E_\theta$ with $\kappa_\psi = 1$, and hence we obtain:

Corollary 10.6. We have

$$\dim \Lambda_\theta \geq \max_{\alpha \in \theta} \delta_{\chi_\alpha + \chi_\alpha}.$$ 

Example 10.7. For $G = \text{PSL}_d(\mathbb{R})$, we have $\Pi = \{\alpha_1, \cdots, \alpha_d-1\}$ where $\alpha_i : \text{diag}(a_1, \cdots, a_d) \mapsto a_i - a_{i+1}$.

Since $\chi_{\alpha_p}$ is equal to the fundamental weight $\omega_p : \text{diag}(a_1, \cdots, a_d) \mapsto a_1 + \cdots + a_p$, we deduce from Corollary 10.6 that for an $\alpha_p$-Anosov subgroup $\Gamma$ of $\text{PSL}_d(\mathbb{R})$ with semisimple Zariski closure for some $1 \leq p \leq d-1$, we have

$$\dim \Lambda_{\alpha_p} \geq \delta_{\omega_p + \omega_{d-p}}.$$ 

When $p = 1$, this lower bound is obtained in [11].

The following upper bound is more or less known, for instance, see ([8], [29]) for $G = \text{PSL}_d(\mathbb{R})$. We include it for the sake of completeness.

Proposition 10.8. We have

$$\dim \Lambda_\theta \leq \max_{\alpha \in \theta} \delta_\alpha.$$ 

Proof. Via the proper immersion of $\mathcal{F}_\theta$, into $\prod_{\alpha \in \theta} \mathbb{P}(V_\alpha)$, we may consider the following metric on $\mathcal{F}_\theta$: for $g_1, g_2 \in G$, 

$$d_{\mathcal{F}_\theta}(g_1P_\theta, g_2P_\theta) = \max_{\alpha \in \theta} \sin \angle(g_1V_\alpha^+, g_2V_\alpha^+),$$ 

which is Lipschitz equivalent to $d_{\text{Riem}}$.

Let $R > 0$ and set $\Lambda_R := \limsup_{\gamma \in \Gamma} O_R^\theta(o, \gamma o)$, that is, $\xi \in \Lambda_R$ if and only if there exists an infinite sequence $\gamma_k \in \Gamma$ such that $\xi \in O_R^\theta(o, \gamma_k o)$. By Lemma 6.1, we have $\Lambda_\theta = \bigcup_{i \geq 1} \Lambda_i$.

We denote by $\alpha_{1,\rho}$ the first simple root $\alpha_1$ of $\text{PSL}(V_\alpha)$. As in [29], for each $\gamma \in \Gamma$ and $\alpha \in \theta$, the image of $O_R^\theta(o, \gamma o)$ in $\mathbb{P}(V_\alpha)$ is contained in a ball of radius $C_1 e^{-\alpha_{1,\rho} \mu(\rho_\alpha(\gamma))}$ for some uniform constant $C_1 > 0$. Since $\alpha_{1,\rho}$ is the
the difference between the highest weight \( \chi_\alpha \) and the second highest weight, which is of the form \( \chi_\alpha - n_\beta \beta \) for some \( n_\beta \in \mathbb{N} \cup \{0\} \), we have

\[
\alpha_1, \rho(\mu(\rho_\alpha(\gamma))) = \left( \alpha + \sum_{\beta \in \Pi} n_\beta \beta \right) (\mu(\gamma)) \geq \alpha(\mu(\gamma)).
\]

Hence the \( d_{F_\theta} \)-diameter of \( O_\theta^H(o, \gamma o) \) is less than \( C_2 e^{-\min_{\alpha \in \theta} \alpha(\mu(\gamma))} \) for some uniform constant \( C_2 > 0 \). This yields the inequality \( \dim \Lambda_R \leq \min_{\alpha \in \theta} \alpha(\rho_\alpha(\mu(\gamma))) \) for all \( R > 0 \). Since \( \Lambda_\theta = \bigcup_{i \geq 1} \Lambda_i \), the claim follows.

\[\text{Corollary 1.6}\] is a combination of Corollary 10.6 and Proposition 10.8

**Growth indicator bounds.** Consider

\[
\rho_{\theta} := \sum_{\alpha \in \theta} \chi_\alpha.
\]

Then \( \kappa_{\rho_{\theta}} = \#\theta \). By Theorem 10.5 we have:

**Corollary 10.9.** We have

\[
\dim \Lambda_\theta \geq \frac{\# \theta}{2} \delta_{\rho_{\theta}}.
\]

In particular, when \( \theta = i(\theta) \),

\[
\dim \Lambda_\theta \geq \frac{\# \theta}{2} \delta_{\rho_{\theta}}.
\]

Observing that \( \delta_{\psi, \psi} \) is tangent to the growth indicator for any linear form \( \psi \) \((19, 21 \text{ Lemma 4.5})\), we deduce the following (Corollary 1.7):

**Corollary 10.10.** We have

\[
\psi_\Gamma^\theta \leq \frac{2}{\#\theta} \dim \Lambda_\theta \cdot \sum_{\alpha \in \theta} \chi_\alpha.
\]

We now consider the case \( \theta = \Pi \) and that \( G \) is split over \( \mathbb{R} \). In this case, \( \rho_\Pi = \sum_{\alpha \in \Pi} \chi_\alpha \) coincides with the half-sum of all positive roots for \( (g, a) \), which we denote by \( \rho \) \([3]\). Hence Corollary 10.10 is written in this case as

\[
\psi_\Gamma \leq \frac{2\rho}{\text{rank} G} \dim \Lambda.
\]

Define the real number \( \lambda_0(\Gamma \backslash X) \in [0, \infty) \) as follows:

\[
(10.2) \quad \lambda_0(\Gamma \backslash X) := \inf \left\{ \frac{\int_{\Gamma \backslash X} \| \text{grad} f \|^2 \text{dVol}}{\int_{\Gamma \backslash X} |f|^2 \text{dVol}} : f \in C^\infty_c(\Gamma \backslash X), \ f \neq 0 \right\},
\]

this is equal to the bottom of the \( L^2 \)-spectrum of \( \Gamma \backslash X \) of the Laplace-Beltrami operator \([34]\).

**Theorem 10.11.** \([12 \text{ Theorem 1.6}]\) If \( \Gamma < G \) is a Zariski dense \( \Pi \)-Anosov subgroup of \( G \) with \( \psi_\Gamma \leq \rho \), then \( L^2(\Gamma \backslash G) \) is tempered and \( \lambda_0(\Gamma \backslash X) = \| \rho \|^2 \).
Applying this, we obtain a criterion on the temperedness of \( L^2(\Gamma \backslash G) \) in terms of \( \dim \Lambda \) (Corollary 10.12):

**Corollary 10.12.** Let \( G \) be split over \( \mathbb{R} \) and \( \Gamma \) be Zariski dense \( \Pi \)-Anosov subgroup of \( G \). If \( \dim \Lambda \leq \frac{\text{rank} G}{2} \), then \( L^2(\Gamma \backslash G) \) is tempered and \( \lambda_0(\Gamma \backslash X) = \| \rho \|^2 \).

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