Apollonian circle packings: Dynamics and Number theory

Hee Oh

Yale University

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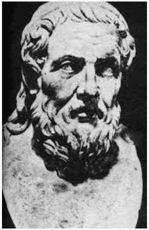
Apollonius of Perga

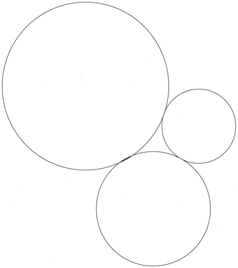


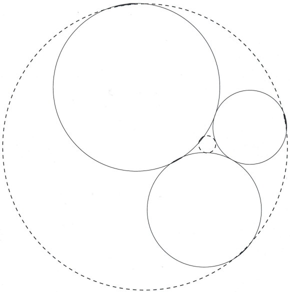
- Lived from about 262 BC to about 190 BC.
- Known as “The Great Geometer”.
- His famous book on Conics introduced the terms parabola, ellipse and hyperbola.

Apollonius' theorem

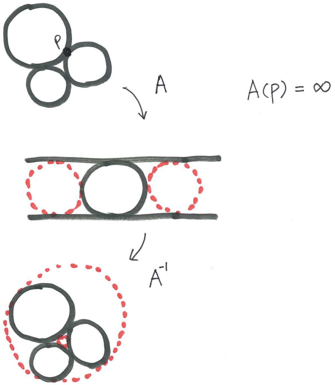
Theorem (Apollonius of Perga)

Given 3 mutually tangent circles, there exist exactly two circles tangent to all three.



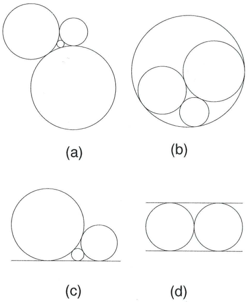


Proof of Apollonius' theorem



4 mutually tangent circles

Four possible configurations



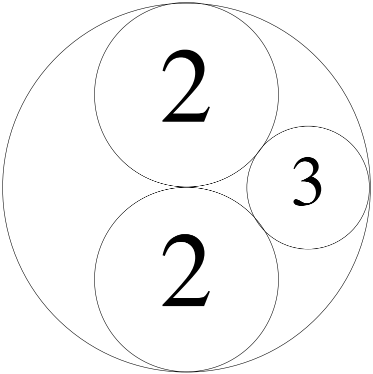
Construction of Apollonian circle packings

Beginning with 4 mutually tangent circles, we can keep adding newer circles tangent to three of the previous circles, provided by the Apollonius theorem. Continuing this process indefinitely, we arrive at an infinite circle packing called an

Apollonian circle packing .

We'll show the first few generations of this process:

Initial stage

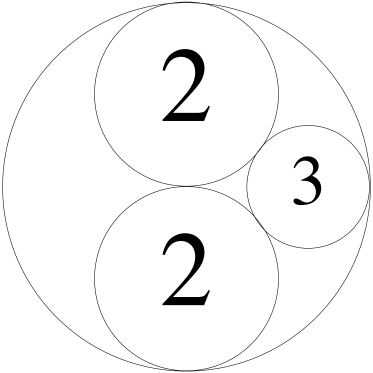


Here each circle C is labelled with its curvature:

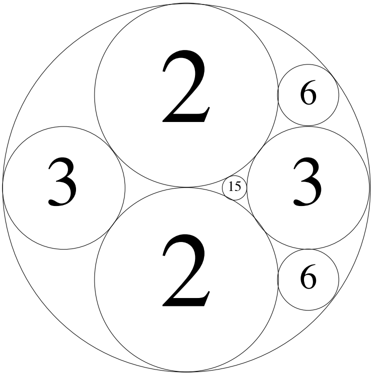
$$\mathrm{curv}(C) = \frac{1}{\mathrm{radius(C)}}.$$

The curvature of the outermost circle is -1 (oriented to have disjoint interiors).

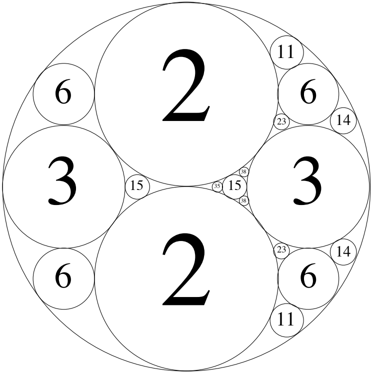
First generation



Second generation

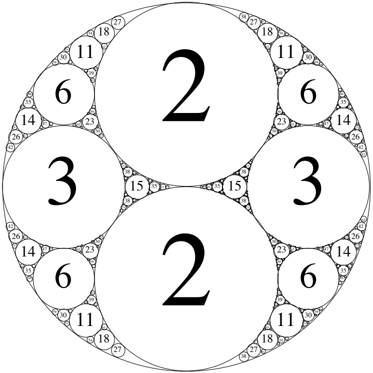


Third generation



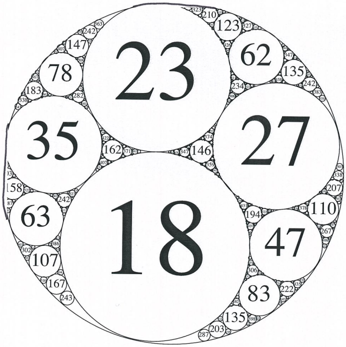
Example of bounded Apollonian circle packing

The outermost circle has curvature -1.

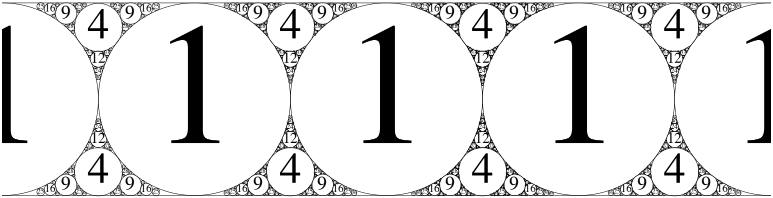


Example of bounded Apollonian circle packing

The outermost circle has curvature -10.



Example of unbounded Apollonian circle packing



There are also other **unbounded** Apollonian packings containing either only one line or no line at all. Since circles will get enormously large, it is hard to draw them.

Circle-counting question

For a bounded Apollonian packing $\mathcal{P}$, there are only finitely many circles of radius bigger than a given number.

For each $T > 0$, we set

$$N_{\mathcal{P}}(T) := \#\{C \in \mathcal{P} : \mathrm{curv}(C) < T\} < \infty.$$

Clearly, $N_{\mathcal{P}}(T) \to \infty$ as $T \to \infty$.

Question

- Is there an asymptotic of $N_{\mathcal{P}}(T)$ as $T \to \infty$?
- If so, can we compute?

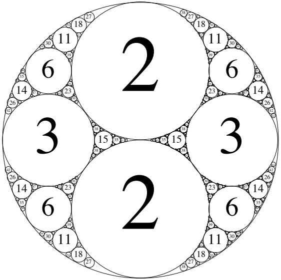
The study of this question involves notions related to metric properties of the underlying fractal set called residual set.

Residual set

Definition (Residual set of $\mathcal{P}$)

$$\mathrm{Res}(\mathcal{P}) := \overline{\cup_{C \in \mathcal{P}} C}.$$

Equivalently, the residual set of $\mathcal{P}$ is the fractal set which is left in the plane after removing all the open disks enclosed by circles in $\mathcal{P}$.



Residual dimension

The Hausdorff dimension of the residual set of $\mathcal{P}$ is called the Residual dimension of $\mathcal{P}$, which we denote by α.

Hausdorff dim. (Hausdorff and Carathéodory (1914))

Definition

Let $s \geq 0$. $F \subset \mathbb{R}^n$. The s-dim. Hausdorff meas. of F is def. by:

$$\mathcal{H}^s(F) := \lim_{\epsilon \to 0} \left(\inf\{\sum d(B_i)^s : F \subset \cup_i B_i, d(B_i) < \epsilon\} \right)$$

where $d(B_i)$ is the diameter of B_i.

It can be shown that as s increases, the s-dim Haus measure of F will be ∞ up to a certain value and then jumps to 0.

Definition

The Hausdorff dim of F is this critical value of s:

$$\dim_{\mathcal{H}}(F) = \sup\{s : \mathcal{H}^s(F) = \infty\} = \inf\{s : \mathcal{H}^s(F) = 0\}.$$

$\alpha = \dim_{\mathcal{H}}(\text{Res}(\mathcal{P}))$: Residual dim

We observe

- $1 \leq \alpha \leq 2$
- α is independent of $\mathcal{P}$: any two Apollonian packings are equivalent to each other by a Mobius transformation.
- The precise value of α is unknown, but approximately, $\alpha = 1.30568(8)$ (McMullen 1998)

In particular, Res($\mathcal{P}$) is much bigger than a c'ble union of circles of $\mathcal{P}$, but not too big in the sense that its Leb. area (=2 dimensional Haus. measure) is zero.

Confirming Wilker's prediction based on computer experiments, Boyd showed: ($N_{\mathcal{P}}(T) := \#\{C \in \mathcal{P} : \mathrm{curv}(C) < T\}$)

Theorem (Boyd 1982)

$$\lim_{T\to\infty} \frac{\log N_{\mathcal{P}}(T)}{\log T} = \alpha.$$

Boyd asked whether $N_{\mathcal{P}}(T) \sim cT^{\alpha}$ as $T \to \infty$, and wrote that his numerical experiments suggest this may be false and perhaps

$$N_{\mathcal{P}}(T) \sim c \cdot T^{\alpha}(\log T)^{\beta}$$

might be more appropriate.

Theorem (Kontorovich-O. 2009)

For a bounded Apollonian packing $\mathcal{P}$*, there exists a constant* $c_{\mathcal{P}} > 0$ *such that*

$$N_{\mathcal{P}}(T) \sim c_{\mathcal{P}} \cdot T^{\alpha}$$

where $\alpha = 1.30568(8)$ *is the residual dimension of* $\mathcal{P}$*.*

Theorem (Lee-O. 2012)

There exists $\eta > 0$ *such that for any bounded Apollonian packing* $\mathcal{P}$*,*

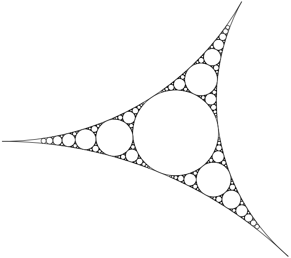
$$N_{\mathcal{P}}(T) = c_{\mathcal{P}} \cdot T^{\alpha} + O(T^{\alpha-\eta})$$

where $\alpha = 1.30568(8)$ *is the residual dimension of* $\mathcal{P}$*.*

Counting inside Triangle

For unbounded Apollonian packing $\mathcal{P}$, $N_{\mathcal{P}}(T) = \infty$.

Consider a curvilinear triangle $\mathcal{R}$ whose sides are given by three mutually tangent circles in **any** Apollonian packing (either bounded or unbounded):



Set

$$N_{\mathcal{R}}(T) := \#\{C \in \mathcal{R} : \text{curv}(C) < T\} < \infty.$$

Theorem (O.-Shah 10)

For a curvilinear triangle $\mathcal{R}$ of any Apollonian packing $\mathcal{P}$,

$$N_{\mathcal{R}}(T) \sim c_{\mathcal{R}} \cdot T^{\alpha}.$$

Distribution of circles in Apollonian packing

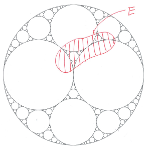
Question

Can we describe the asymptotic distribution of circles in $\mathcal{P}$ of curvature at most T as $T \to \infty$?

For a bounded Borel subset E, set

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : \mathrm{curv}(C) < T, C \cap E \neq \emptyset\}.$$

As we vary $E \subset \mathbb{C}$, how does $N_T(\mathcal{P}, E)$ depend on E?



Distribution of circles

Question

Does there exist a measure $\omega_{\mathcal{P}}$ on $\mathbb{C}$ such that for any bdd Borel $E \subset \mathbb{C}$,

$$\lim_{T \to \infty} \frac{N_T(\mathcal{P}, E)}{T^\alpha} = \omega_{\mathcal{P}}(E)?$$

Note that all the circles in $\mathcal{P}$ lie on the residual set of $\mathcal{P}$.

Hence any measure describing the asymptotic distribution of circles of $\mathcal{P}$ must be supported on the residual set of $\mathcal{P}$.

What measure could that be?

We show that the α-dim. Hausd. measure $\mathcal{H}^\alpha$ on Res$(\mathcal{P})$ does the job.

Theorem (O.-Shah, 10)

For any bdd. Borel $E \subset \mathbb{C}$ with smooth bdry,

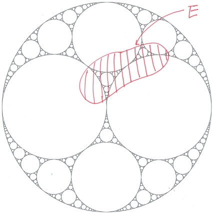
$$N_T(\mathcal{P}, E) \sim c_A \cdot \mathcal{H}^\alpha(E \cap \mathrm{Res}(\mathcal{P})) \cdot T^\alpha$$

where $0 < c_A < \infty$ is an absolute constant independent of $\mathcal{P}$.

Distribution of circles

Thm says that circles in an Apollonian packing are uniformly distributed w.r.t the α-dim. Hausdorff meas. on its residual set:

$$\frac{N_T(\mathcal{P}, E_1)}{N_T(\mathcal{P}, E_2)} \sim \frac{\mathcal{H}^\alpha(E_1 \cap \mathrm{Res}(\mathcal{P}))}{\mathcal{H}^\alpha(E_2 \cap \mathrm{Res}(\mathcal{P}))}.$$



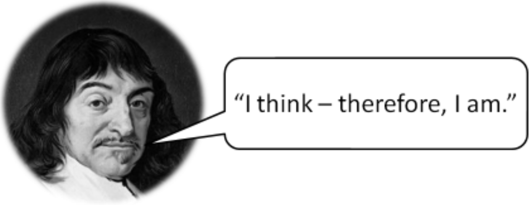
Integral Apollonian circle packings

We call an Apollonian packing $\mathcal{P}$ integral if every circle in $\mathcal{P}$ has integral curvature.

Does there exist any integral $\mathcal{P}$?

The answer is positive thanks to the following beautiful thm of Descartes:

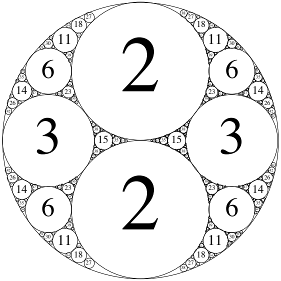
Descartes circle theorem, 1643



Theorem (in a letter to Princess Elisabeth of Bohemia)

A quadruple (a, b, c, d) is the curvatures of four mutually tangent circles if and only if it satisfies the quadratic equation:

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2.$$



E.g: $2((-1)^2 + 2^2 + 2^2 + 3^2) = 36 = (-1 + 2 + 2 + 3)^2$

E.g. $2(2^2 + 6^2 + 3^2 + 23^2) = 1156 = (2 + 6 + 3 + 23)^2$

Given three mutually tangent circles of curvatures a, b, c, if we denote by d and d' for the curvatures of the two circles tangent to all three, then

$$2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$$

and

$$2(a^2 + b^2 + c^2 + (d')^2) = (a + b + c + d')^2.$$

By subtracting one from the other, we obtain

$$d + d' = 2(a + b + c).$$

So, if a, b, c, d are integers, so is d'.

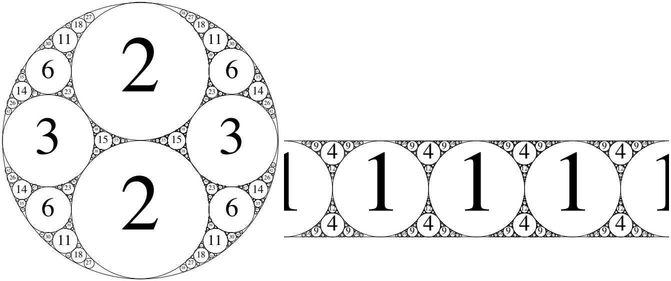
Theorem (Soddy 1936)

If the initial 4 circles in an Apollonian packing $\mathcal{P}$ have integral curvatures, $\mathcal{P}$ is integral.

Therefore, for any integral solution of $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$, $\exists$ an integral Apollonian packing!

Integral Apollonian circle packings

Any integral Apollonian packing is either bounded or lies between two parallel lines:



Diophantine questions

For a given integral Apollonian packing $\mathcal{P}$, it is natural to inquire about its the Diophantine properties such as

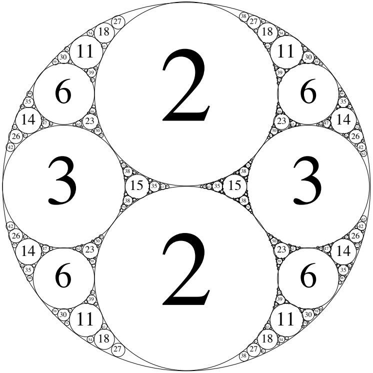
Question

- Are there infinitely many circles with prime curvatures?
- Which integers appear as curvatures?

Assume that $\mathcal{P}$ is primitive, i.e.,$\operatorname{g.c.d}_{C \in \mathcal{P}}(\operatorname{curv}(C)) = 1$.

Definition

1. A circle is prime if its curvature is a prime number.
2. A pair of tangent prime circles is a tangent prime.



prime circles: 2,3, 11, 23,... Tangent prime circles: (2,3), (2,11), (3, 23), ...

Infinitude of prime circles

Theorem (Sarnak 07)

In any primitive integral $\mathcal{P}$, there are infinitely many prime circles as well as tangent prime circles.

Set

$$\Pi_T(\mathcal{P}) := \#\{\text{prime } C \in \mathcal{P} : \text{curv}(C) < T\}$$

and

$$\Pi_T^{(2)}(\mathcal{P}) := \#\{\text{tangent primes } C_1, C_2 \in \mathcal{P} : \text{curv}(C_i) < T\}.$$

Analogue of Prime number theorem?

Using the sieve method based on heuristics on the randomness of the Mobius function, Fuchs and Sanden conjectured:

Conjecture (Fuchs-Sanden)

$$\Pi_T(\mathcal{P}) \sim c_1 \frac{N_T(\mathcal{P})}{\log T}; \quad \Pi_T^{(2)}(\mathcal{P}) \sim c_2 \frac{N_T(\mathcal{P})}{(\log T)^2}$$

where c_1 and c_2 can be given explicitly.

Using the breakthrough work of Bourgain, Gamburd, Sarnak on expanders together with Selberg's upper bound sieve, we obtain upper bounds of true order of magnitude:

Theorem (Kontorovich-O. 09)

1. $\Pi_T(\mathcal{P}) \ll \frac{N_T(\mathcal{P})}{\log T}$
2. $\Pi_T^{(2)}(\mathcal{P}) \ll \frac{N_T(\mathcal{P})}{(\log T)^2}$.

The lower bounds are still open and seem very challenging.

Question

For a primitive integral $\mathcal{P}$, how many integers appear as curvatures of circles in $\mathcal{P}$?

I.e., how big is $\#\{\text{curv}\,(C) \leq T, C \in \mathcal{P}\}$ compared to T?

Our counting result for circles says

$$\#\{\text{curv}\,(C) \leq T \text{ counted with multiplicity} : C \in \mathcal{P}\} \sim c \cdot T^{1.305\ldots}$$

So we may hope that a positive density (=proportion) of integers arise as curvatures, conjectured by Graham, Lagarias, Mallows, Wilkes, Yan (Positive density conjecture):

(Strong) Positive density conjecture

$\mathcal{P}$: primitive integral Apollonian packing

Theorem (Bourgain-Fuchs 10)

$$\#\{curv\,(C) < T : C \in \mathcal{P}\} \gg T.$$

Theorem (Bourgain-Kontorovich 12)

$$\#\{curv\,(C) < T : C \in \mathcal{P}\} \sim \frac{\kappa(\mathcal{P})}{24} \cdot T$$

where $\kappa(\mathcal{P}) > 0$ is the number of residue classes mod 24 of curvatures of $\mathcal{P}$.

- There are congruence restriction: modulo 24, not all residue classes appear.

Improving Sarnak's result on the infinitude of prime circles, Bourgain showed that a positive fraction of prime numbers appear as curvatures in $\mathcal{P}$.

Theorem (Bourgain, 2011)

$$\#\{\textit{prime curv } (C) \leq T : C \in \mathcal{P}\} \gg \frac{T}{\log T}.$$

Hidden symmetries

Question

How are we able to count circles in an Apollonian packing?

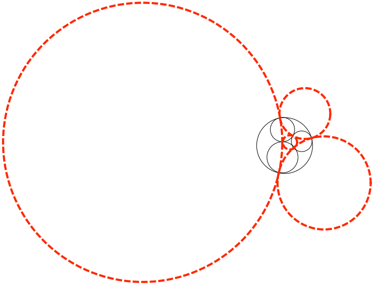
We exploit the fact that

an Apollonian packing has lots of hidden symmetries.

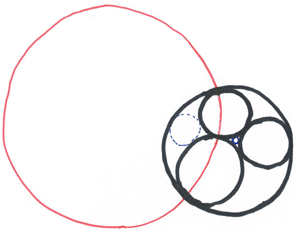
Explaining these hidden symmetries will lead us to explain the relevance of the packing with a Kleinian group, called the Apollonian group.

Symmetry group of $\mathcal{P}$

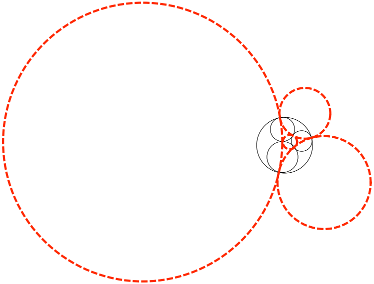
Fixing 4 mutually tangent (black) circles in $\mathcal{P}$, we obtain four dual circles, each passing through 3 tangent points.



Inverting w.r.t a dual circle fixes the three circles that it meets perpendicularly and interchanges the two circles which are tangent to the three circles; indeed, it preserves $\mathcal{P}$.



Apollonian group



The Apollonian group $\mathcal{A}$ associated to $\mathcal{P}$ is generated by 4 inversions w.r.t those dual circles:

$$\mathcal{A} \quad = \quad \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle < \mathrm{Mob}(\hat{\mathbb{C}})$$

where $\mathrm{Mob}(\hat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C})^{\pm}$ the gp of Mobius transf. of $\hat{\mathbb{C}}$.

The Apollonian group $\mathcal{A}$ is a Kleinian group (= discrete subgroup of $\mathrm{PSL}_2^{\pm}(\mathbb{C})$) and satisfies

$$\mathcal{P} = \cup_{i=1}^{4} \mathcal{A}(C_i),$$

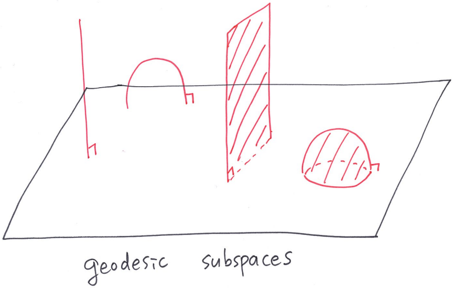
i.e., inverting the initial four (black) circles in $\mathcal{P}$ w.r.t the (red) dual circles generate the whole packing $\mathcal{P}$.

3 dim. hyp. geometry

The upper-half space model for hyp. 3 space $\mathbb{H}^3$:

$$\mathbb{H}^3 = \{(x_1, x_2, y) : y > 0\} \text{ with } \quad ds = \frac{\sqrt{dx_1^2 + dx_2^2 + dy^2}}{y}$$

and $\partial_\infty(\mathbb{H}^3) = \hat{\mathbb{C}}$.



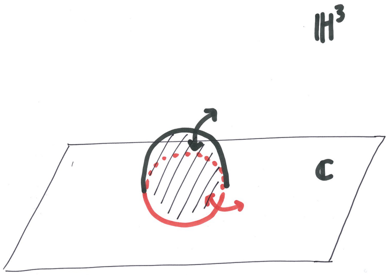
For a circle C in $\hat{\mathbb{C}}$, put

$$\hat{C} := \text{ the vertical hemisphere in } \mathbb{H}^3 \text{ above } C.$$

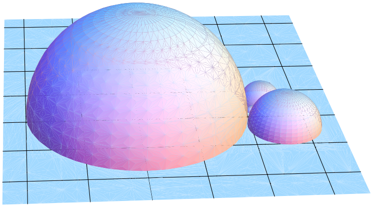
Theorem (Poincare extension thm)

The map "Inversion w.r.t. $C \mapsto$ Inversion w.r.t. $\hat{C}$" extends to an isomorphism

$$\mathrm{PSL}_2(\mathbb{C})^{\pm} = \mathrm{Isom}(\mathbb{H}^3)$$



The Apollonian gp $\mathcal{A}$ (now considered as a discrete subgp of Isom($\mathbb{H}^3$)) has a fund. domain in $\mathbb{H}^3$, given by the exterior of the hemispheres above the dual circles to $\mathcal{P}$:



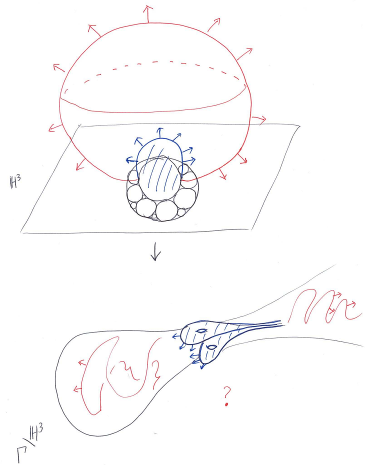
In particular, $\mathcal{A}\backslash\mathbb{H}^3$ is an infinite vol. hyperbolic 3-mfld and has finitely many sided fund. domain.

Orthogonal translates of geodesic surface

Counting circles of curvature at most T is same as counting vertical hemispheres of height at least $1/T$.

Noting that vertical hemispheres in $\mathbb{H}^3$ are totally geodesic subspaces, we relate the circle-counting problem with the equidistribution of translates of a closed totally geodesic surface in $\mathcal{A}\backslash\mathbb{H}^3$.

For a tot. geo. surface S of $\mathrm{T}^1(\mathcal{A}\backslash\mathbb{H}^3)$, what is the asymptotic dist. of its orthogonal translates $g^t(S)$ as $t \to \infty$?



Difficulties lie in the fact that the Apollonian mfld is of infinite volume, as the dynamics of flows in inf. volume hyp. mflds are very little understood.

We show that this distribution in $\mathrm{T}^1(\mathcal{A}\backslash\mathbb{H}^3)$ is described by a singular measure, called the Burger-Roblin measure, whose conditional measures on horizontal planes turn out to be equal to the α-dim'l Haus. measures in this case, and this is why we have the α-dim'l Haus. measure in our counting thm.

Main ingredients of our proofs include

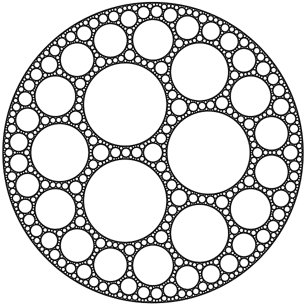
- the Lax-Phillips spectral theory for the Laplacian on $\mathcal{A}\backslash\mathbb{H}^3$;
- Ergodic properties of flows on $\mathrm{T}^1(\mathcal{A}\backslash\mathbb{H}^3)$ based on the Patterson-Sullivan theory and the work of Burger-Roblin.

More circle packings

This viewpoint via the study of Kleinian groups allows us to deal with more general circle packings, provided they are invariant under a finitely generated Kleinian group Γ.

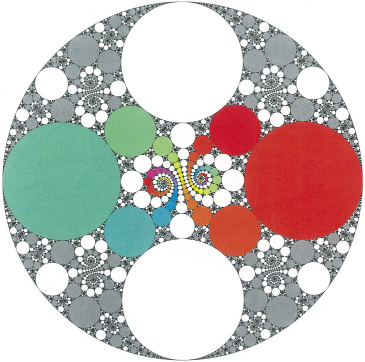
Here are some other pictures of circle packings for which we can count circles of bounded curvature:

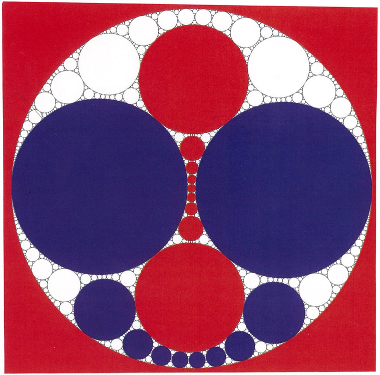
Ex. of Sierpinski curve (McMullen)

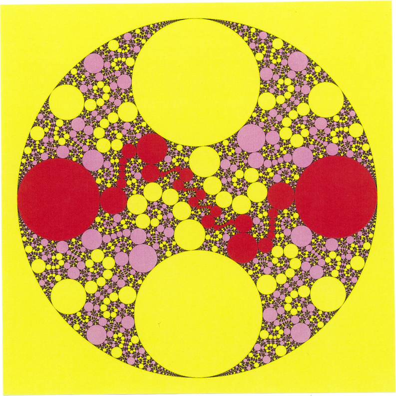


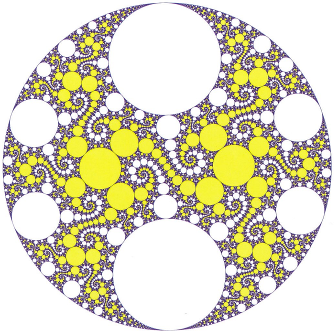
Here the symmetry group is
π_1(cpt. hyp. 3-mfd with tot. geod. bdry).

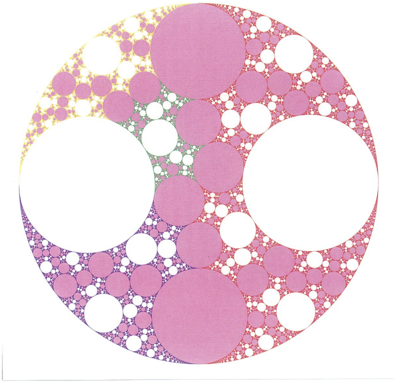
The next pictures are reproduced from the book ”Indra’s pearls” by Mumford, Series and Wright (Cambridge Univ. Press 2002).

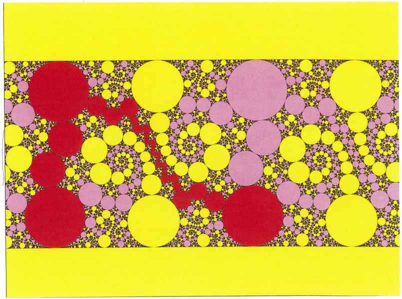












For these circle packings, we have:

Theorem (O.-Shah)

Let $\delta := \dim_{\mathcal{H}}(\mathrm{Res}(\mathcal{P}))$. *For any bdd. Borel* $E \subset \mathbb{C}$ *with nice boundary,*

$$N_T(\mathcal{P}, E) \sim c \cdot \mathcal{H}^\delta(E \cap \mathrm{Res}(\mathcal{P})) \cdot T^\delta$$

for some absolute constant $c > 0$.