ICERM Lecture 1

( Geodesic planes in 00-vol hyp mflds)

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Geodesic planes in hyperbolic manifolds of infinite volume.

Ref.: Geodesic planes in hyperbolic 3-manifolds (McMullen-Mohammadi-O. - Inventiones 2017)

- Horocycles in hyperbolic 3-manifolds (MMO, GAFA 2016 12p)
- Geodesic planes in the convex core of an acylindrical 3-manifold (MMO - Duke 2022 31p)
- Geodesic planes in geometrically finite acylindrical 3-manifolds (Benoist-O. - ETDS 2022 39p)
- Orbit closures of unipotent flows for hyperbolic manifolds with Fuchsian ends (Minju Lee-O. To appear G & T 101p)

* Dynamics for discrete subgroups of \( SL_2 \mathbb{C} \)
in Dynamics, Geometry, Number theory (Margulis, vol)
geodesic planes in \( H^3 = \{(x_1, x_2, y) | y > 0\} \)

\[ ds = \frac{\text{dEn}_g}{y} \]

\( \text{Isom}^+(H^3) = \text{PSL}_2\mathbb{C} \cong S_0(3,1) \)

\( \Omega H^3 = \mathbb{R}^2 \cup \{\infty\} = S^2 \)

glued planes \( \tilde{\mathcal{P}} = \mathfrak{g}(H^2) \) \( g \in \text{PSL}_2\mathbb{C} \)

circles in \( S^2 \)

\( \mathcal{P} = \mathfrak{g}(\hat{\mathbb{R}}) \)

\( \tilde{\mathcal{P}} = \mathfrak{g}(\hat{\mathbb{R}}) \)

\[ \Gamma \subset \text{PSL}_2\mathbb{C} \text{ t.f. discrete} \]

\( \pi \rightarrow \text{geo.d. planes in } M \)

\[ M = \Gamma \backslash H^3 \]

\( \tilde{\mathcal{P}} \rightarrow \mathcal{P} = \pi(\tilde{\mathcal{P}}) \)
geodesic planes in $H^n = \{(x_1, \ldots, x_n, y) \mid y > 0\}$

$\text{Isom}^+(H^n) \simeq SO(n,1)$

$k \in H^n = \mathbb{R}^{n-1} \cup \{0\}$

$H^k = \{(x_1, \ldots, x_k, 0, \ldots, 0, y) \mid y > 0\}$

$\tilde{P} = g(H^k), \ g \in G$

$\exists \tilde{\mathcal{P}} = g(\mathbb{R}^{k-1})$

$(k-1)$ spheres in $S^{n-1}$

$M = \Gamma \backslash H^n$

$\Gamma \subset G$

$t.f. \text{ discrete}$

$P = \pi(\tilde{P})$

$\tilde{P}$

$\pi$

$\mathcal{P}$

geod. planes in $M$
Question

$P$ geod $k$-plane in $M = \mathbb{R} \setminus \mathbb{H}^n$

$2 \leq k \leq n-1$

Describe $\overline{P}$ !

A Thm (Ratner, Shah ~90)

$M = \mathbb{R} \setminus \mathbb{H}^n$ finite vol

$P \subset M$ geod $k$-plane

$2 \leq k \leq n-1$

$\overline{P}$ is a geod $m$-plane $k \leq m \leq n$

(e.g. if $n=3$, $\overline{P} = P$ or $\overline{P} = M$)

What about $\text{Vol}(M) = \infty$ ?
$M = \mathbb{H}^n \setminus \mathbb{H}^n$ Convex copt

$\text{Int}(\text{Core } M) \neq \emptyset \quad (= \Gamma < \text{SO}(n,1) = G)$

Zariski dense

\[ B \]

**Def.** $M$ has **Fuchsian ends**

$\mathcal{E}(\text{Core } M)$ is totally geodesic

$S^{n-1} - \Lambda = \bigcup_i B_i, \quad \overline{B_i} \cap \overline{B_j} = \emptyset$

round balls

$\text{Hull}(\Lambda)$

$n = 3$
Two types of planes:

- \( P \cap \text{Int}(\text{Core } M) \neq \emptyset \)
- \( P \cap \text{Int}(\text{Core } M) = \emptyset \)

\[ \Rightarrow \bar{P} \subset M - \text{Int}(\text{Core } M) \text{ "Fuchsian ends"} \]

Can use Ratner-Shah to describe \( \bar{P} \).
Thm \( (McMullen - Mohammadi - O. \ n = 3 \ \\
Minju Lee - O. \ n \geq 4 \ )\)

\[ M = P \setminus H^n : \text{hyp mfld with Fuchsian ends} \]

\[ P \subset M \text{ geod } k \text{-plane: } 2 \leq k \leq n - 1 \]

\[ P \cap \text{Int(CoreM)} \neq \emptyset \]

\[ \bar{P} = \text{geod } m \text{-plane } k \leq m \leq n \]

For \( n = 3 \), \( \bar{P} = P \) or \( \bar{P} = M \).

\[ C_\wedge = \left\{ C \subset S^{n-1} \mid (k-1) \text{ sphere} \right\} \]

\[ C \in C_\wedge \middle\mid |C \cap \wedge| \geq 2 \]

\[ \bar{C} = \left\{ D \in C_\wedge \mid D \subset \partial S^3 \right\} \text{ for some (n-1) sphere} \]

For \( n = 3 \), \( \bar{C} = \bar{C} \) or \( \bar{C} = C_\wedge \).
\[ M = \Gamma \setminus \mathbb{H}^3 \quad \text{Convex Cocpt} \]

**Def.** \( M \) has quasi-Fuchsian ends

\[ M \sim M_0 = \mathbb{H}^3 \quad \text{CC with Fuchsian ends} \]

\[ \Gamma \text{ is a Quasi-conf. def of } \Gamma_0 \]

\[ S^2 - \Lambda = \bigcup B_i \quad \bar{B}_i \cap \bar{B}_j = \emptyset \]

\[ \exists \text{ Core } M_0 = \bigcup_{i=1}^k S_i \]

Q. I. (\( M_0 \)) = Q. C. (\( \Gamma_0 \)) = \frac{\kappa}{\zeta} \text{ Teich}(S_i) \approx \frac{\kappa}{\zeta} \mathbb{R}^{6g-6} \]
Thm (McMullen-Mohammadi-O.)

\[ M = \mathbb{P} \setminus \mathbb{H}^3 \quad \text{quasi-Fuchsian ends} \]

\[ P \subset M \quad \text{geod plane with } P \cap \text{Int}(\text{Core}M) \neq \emptyset \]

\[ \Rightarrow P \text{ is either closed or dense in } \text{Int}(\text{Core}M) \]

\[ \text{i.e., } \overline{P} \cap \text{Int}(\text{Core}M) = \{ P \cap \text{Int}(\text{Core}M) \} \cup \text{Int}(\text{Core}M) \]

Thm (Yongquan Zhang)

\[ \exists P \subset M \text{ s.t} \]

\[ P \text{ is closed in } \text{Int}(\text{Core}M) \]

but not closed in \( M \).

\[ \text{Rmk } \overline{P} \text{ for } P \subset C \text{ Ends of } M \]

is not completely understood.
Rmk Analogous results for some geom. finite Hyp 3-mflds were obtained by Benoist-0.

Open problem

\[ \varphi : \text{Apollonian circle packing} \]

\[ \Gamma = \{ g \in \mathrm{PSL}_2 \mathbb{C} \mid g \varphi = \varphi \} \]

\[ \overline{\Gamma C} = \{ \Gamma C \} \]

\[ |C \cap \Lambda| \geq 2 \]
Homogeneous dynamics.

\[ G = \text{Isom}^+(\mathbb{H}^n) = \text{SO}_0(Q) \quad Q = 2x_1x_{n+1} + \sum_{i=2}^{n} x_i^2 \]

\[ A = \{ a_t = (e^{t_1}, \ldots, e^{t_n}) \mid t \in \mathbb{R} \} \]

\[ \text{Frame flow} \]

\[ \text{gSO(n,1) : SO(n,1) orbits} \]

\[ \text{gIH}^k : \text{geodesic } \mathbb{R}-\text{planes} \]
\[ F(M) = G \left\langle a_t \right\rangle \]

\[ T^1(M) = \frac{G}{\sqrt{SO(n-1)}} \left\langle a_t \right\rangle \]

\[ M = \frac{G}{\sqrt{SO(n)}} \]

\[ \pi(G/H^k) : \text{geodesic } \mathbb{R} \text{-planes} \]

Describe \([g]SO(k,1)\) in \(G\).
Let $M = \mathbb{H}^n$ have Fuchsian ends.

**Thm** (McMullen-Mohammadi-O. Lee-O.)

$W < G$ conn subgp gen. by unipotent elts ($\&$ normalized by $A$)

Any $W$-orbit closure is relative homogeneous in $\Omega$

For $x \in \Omega$, $\overline{xW \cap \Omega} = xL \cap \Omega$

where $W < L < G$ $\&$ $xL$ closed.
Moreover, \( 2 \leq k \leq n-1 \)

\[
x SO(k, 1) \cap \Omega = x SO(m, 1) C \cap \Omega
\]

where \( C \subset \text{Centralizer} SO(m, 1) \)

\[
\text{closed subgp} = SO(n-m)
\]

\[
x SO(k, 1) = x SO(m, 1) C \cap \Omega + SO(k, 1)
\]

where \( \Omega_+ = \{ g \mid g^+ e \in \mathcal{V} \} \)

If \( M \) has empty Fuchsian ends (i.e, \( M \) cpt),

\[
\Omega = \frac{G}{\Gamma} \quad \text{and this is Ratner & Shah.}
\]
Thank you!