FUCHSIAN GROUPS AND COMPACT HYPERBOLIC SURFACES

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Abstract. We present a topological proof of the following theorem of Benoist-Quint: for a finitely generated non-elementary discrete subgroup Γ_1 of $\mathrm{PSL}(2,\mathbb{R})$ with no parabolics, and for a cocompact lattice Γ_2 of $\mathrm{PSL}(2,\mathbb{R})$, any Γ_1 orbit on $\Gamma_2\backslash \mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.

1. Introduction

Let Γ_1 be a non-elementary finitely generated discrete subgroup with no parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$. Let Γ_2 be a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})$. The following is the first non-trivial case of a theorem of Benoist-Quint [1].

Theorem 1.1. *Any Γ_1-orbit on $\Gamma_2\backslash \mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.*

The proof of Benoist-Quint is quite involved even in the case as simple as above and in particular uses their classification of stationary measures [2]. The aim of this note is to present a short, and rather elementary proof.

We will deduce Theorem 1.1 from the following Theorem 1.2. Let

- $H_1 = H_2 := \mathrm{PSL}(2,\mathbb{R})$ and $G := H_1 \times H_2$;
- $H := \{(h,h) : h \in \mathrm{PSL}_2(\mathbb{R})\}$ and $\Gamma := \Gamma_1 \times \Gamma_2$.

Theorem 1.2. *For any $x \in \Gamma\backslash G$, the orbit xH is either closed or dense.*

Our proof of Theorem 1.2 is purely topological, and inspired by the recent work of McMullen, Mohammadi and Oh [5] where the orbit closures of the $\mathrm{PSL}(2,\mathbb{R})$ action on $\Gamma_0\backslash \mathrm{PSL}(2,\mathbb{C})$ are classified for certain Kleinian subgroups Γ_0 of infinite co-volume. While the proof of Theorem 1.2 follows closely the sections 8-9 of [5], the arguments in this paper are simpler because of the assumption that Γ_2 is cocompact. We remark that the approach of [5] and hence of this paper is somewhat modeled after Margulis's original proof of Oppenheim conjecture [4]. When Γ_1 is cocompact as well, Theorem 1.2 also follows from [6].

Finally we remark that according to [1], both Theorems 1.1 and 1.2 are still true in presence of parabolic elements, more precisely when Γ_1 is any non-elementary discrete subgroup and Γ_2 any lattice in $\mathrm{PSL}(2,\mathbb{R})$. The topological method presented here could also be extended to this case.

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2. Horocyclic flow on convex cocompact surfaces

In this section we prove a few preliminary facts about unipotent dynamics involving only one factor H_1.

The group $\mathrm{PSL}_2(\mathbb{R}) := \mathrm{SL}_2(\mathbb{R})/\{\pm e\}$ is the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. The isometry corresponding to the element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ is $z \mapsto \dfrac{az+b}{cz+d}$. It is implicit in this notation that the matrices g stand for their equivalence class $\pm g$ in $\mathrm{PSL}_2(\mathbb{R})$. This group $\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle $\mathrm{T}^1(\mathbb{H}^2)$ and we choose an identification of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{T}^1(\mathbb{H}^2)$ so that the identity element e corresponds to the upward unit vector at i. We will also identify the boundary of the hyperbolic plane with the extended real line $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ which is topologically a circle.

We recall that Γ_1 is a non-elementary finitely generated discrete subgroup with no parabolic elements of the group $H_1 = \mathrm{PSL}_2(\mathbb{R})$, that is, Γ_1 is a convex cocompact subgroup. Let S_1 denote the hyperbolic orbifold $\Gamma_1\backslash\mathbb{H}^2$, and let $\Lambda_{\Gamma_1} \subset \partial\mathbb{H}^2$ be the limit set of Γ_1. Let A_1 and U_1 be the subgroups of H_1 given by

$$A_1 := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\} \text{ and } U_1 := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}.$$

Let

$$\Omega_{\Gamma_1} = \{x \in \Gamma_1\backslash H_1 : xA_1 \text{ is bounded}\}. \tag{2.1}$$

As Γ_1 is a convex cocompact subgroup, Ω_{Γ_1} is a compact A_1-invariant subset and one has the equality

$$\Omega_{\Gamma_1} = \{[h] \in \Gamma_1\backslash H_1 : h(0), h(\infty) \in \Lambda_{\Gamma_1}\}.$$

In geometric words, seen as a subset of the unit tangent bundle of S_1, the set Ω_{Γ_1} is the union of all the geodesic lines which stays inside the convex core of S_1.

Definition 2.2. *Let $K > 1$. A subset $T \subset \mathbb{R}$ is called K-thick if, for any $t > 0$, T meets $[-Kt, -t] \cup [t, Kt]$.*

Lemma 2.3. *There exists $K > 1$ such that for any $x \in \Omega_{\Gamma_1}$, the subset $T(x) := \{t \in \mathbb{R} : xu_t \in \Omega_{\Gamma_1}\}$ is K-thick.*

Proof. Using an isometry, we may assume without loss of generality that $x = [e]$. Since the element e corresponds to the upward unit vector at i, and since x belongs to Ω_{Γ_1}, both points 0 and ∞ belong to the limit set Λ_{Γ_1}. Since $u_t(\infty) = \infty$ and $u_t(0) = t$, one has the equality

$$T(x) = \{t \in \mathbb{R} : t \in \Lambda_{\Gamma_1}\}.$$

Write $\mathbb{R} - \Lambda_{\Gamma_1}$ as the union $\cup J_\ell$ where J_ℓ's are maximal open intervals. Note that the minimum hyperbolic distance between the convex hulls in $\mathbb{H}^2$

$$\delta := \inf_{\ell \neq m} d(\mathrm{hull}(J_\ell), \mathrm{hull}(J_m))$$

is positive, as 2δ is the length of the shortest closed geodesic of the double of the convex core of S_1. Choose the constant $K > 1$ so that for $t > 0$, one has

$$d(\text{hull}[-Kt, -t], \text{hull}[t, Kt]) = \delta/2.$$

Note that this choice of K is independent of t. If $T(x)$ does not intersect $[-Kt, -t] \cup [t, Kt]$ for some $t > 0$, then the intervals $[-Kt, -t]$ and $[t, Kt]$ must be included in two distinct intervals J_ℓ and J_m, since $0 \in \Lambda_{\Gamma_1}$. This contradicts the choice of K. □

Lemma 2.4. *Let $K > 1$ and let T be a K-thick subset of $\mathbb{R}$. For any sequence h_n in $H_1 \smallsetminus U_1$ converging to e, there exists a sequence $t_n \in T$ such that the sequence $u_{-t_n} h_n u_{t_n}$ has a limit point in $U_1 \smallsetminus \{e\}$.*

Proof. Write $h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. We compute

$$q_n := u_{-t_n} h_n u_{t_n} = \begin{pmatrix} a_n - c_n t_n & (a_n - d_n - c_n t_n) t_n + b_n \\ c_n & d_n + c_n t_n \end{pmatrix}.$$

Since the element h_n does not belong to U_1, it follows that the $(1,2)$-entries $P_n(t_n) := (a_n - d_n - c_n t_n) t_n + b_n$ are non-constant polynomial functions of t_n of degree at most 2 whose coefficients converge to 0. Hence, by Lemma 2.5 below, we can choose $t_n \in T$ going to ∞ so that $k \leq |P_n(t_n)| \leq 1$, for some constant $k > 0$ depending only on K. Since the entry $P_n(t_n)$ is bounded and since h_n converges to e, the product $c_n t_n$ must converge to 0 and the sequence q_n has a limit point in $U_1 - \{e\}$. □

We have used the following basic lemma :

Lemma 2.5. *For every $K > 1$ and $d \geq 1$, there exists $k > 0$ such that, for every non-constant polynomial P of degree d with $|P(0)| \leq k$, and for every K-thick subset T of $\mathbb{R}$, there exists t in T such that $k \leq |P(t)| \leq 1$.*

Proof. Using a suitable homothety in the variable t, we can assume with no loss of generality that P belongs to the set $\mathcal{P}_d$ of polynomials of degree at most d such that $P(1) = \max_{[-1,1]} |P(t)| = 1$.

Assume by contradiction that there exists a sequence P_n of polynomials in $\mathcal{P}_d$ and a sequence of K-thick subsets T_n of $\mathbb{R}$ such that $\sup_{T_n \cap [-1,1]} |P_n(t)|$ converge to 0. After extraction, the sequence T_n converges to a K-thick subset T_∞ and the sequence P_n converges to a polynomial $P_\infty \in \mathcal{P}_d$ which is equal to 0 on the set $T_\infty \cap [-1,1]$. This is not possible since this set is infinite. □

We record also, for further use, the following classical lemma :

Lemma 2.6. *Let U_1^+ be the semigroup $\{u_t : t \geq 0\}$. If the quotient space $X_1 := \Gamma_1 \backslash H_1$ is compact, any U_1^+-orbit is dense in X_1.*

Proof. For $x \in X_1$, set $x_n := xu_n$. We then have $x_n u_{-n} U_1^+ = xU_1^+$. Hence if z is a limit point of the sequence x_n in X_1, we have $zU \subset \overline{xU_1^+}$. By Hedlund's theorem [3], zU is dense. Hence the orbit xU_1^+ is also dense. □

3. Proof of Theorems 1.1 and 1.2

In this section, using minimal sets and unipotent dynamics on the product space $\Gamma\backslash G$, we provide a proof of Theorem 1.2.

3.1. **Unipotent dynamics.** We recall the notation $G := \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ and $\Gamma := \Gamma_1 \times \Gamma_2$. Set

- $H_1 = \{(h, e)\}$, $H_2 = \{(e, h)\}$, $H = \{(h, h)\}$;
- $U_1 = \{(u_t, e)\}$, $U_2 = \{(e, u_t)\}$, $U = \{(u_t, u_t)\}$;
- $A_1 = \{(a_t, e)\}$, $A_2 = \{(e, a_t)\}$, $A = \{(a_t, a_t)\}$;
- $X_1 = \Gamma_1\backslash H_1$, $X_2 = \Gamma_2\backslash H_2$, $X = \Gamma\backslash G = X_1 \times X_2$.

Recall that Γ_1 is a non-elementary finitely generated discrete subgroup of H_1 with no parabolic elements and that Γ_2 is a cocompact lattice in H_2.

For simplicity, we write $\widetilde{u}_t$ for (u_t, u_t) and $\widetilde{a}_t$ for (a_t, a_t). Note that the normalizer of U in G is AU_1U_2.

Lemma 3.1. *Let g_n be a sequence in $G \smallsetminus AU_1U_2$ converging to e, and let T be a K-thick subset of $\mathbb{R}$ for some $K > 1$. Then for any neighborhood G_0 of e in G, there exist sequences $s_n \in T$ and $t_n \in \mathbb{R}$ such that the sequence $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ has a limit point $q \neq e$ in $AU_2 \cap G_0$.*

Proof. Fix $0 < \varepsilon \le 1$. Write $g_n = (g_n^{(1)}, g_n^{(2)})$ with $g_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}$. Then the products $q_n := \widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ are given by

$$q_n^{(i)} = u_{-s_n} g_n^{(i)} u_{t_n} = \begin{pmatrix} a_n^{(i)} - c_n^{(i)} s_n & (b_n^{(i)} - d_n^{(i)} s_n) - t_n (c_n^{(i)} s_n - a_n^{(i)}) \\ c_n^{(i)} & d_n^{(i)} + c_n^{(i)} t_n \end{pmatrix}.$$

Set

$$t_n = \tfrac{b_n^{(1)} - d_n^{(1)} s_n}{c_n^{(1)} s_n - a_n^{(1)}}.$$

The differences $q_n - e$ are now rational functions in s_n of the form

$$q_n - e = \tfrac{1}{c_n^{(1)} s_n - a_n^{(1)}} P_n(s_n),$$

where $P_n(s)$ is a polynomial function of s of degree at most 2 with values in $M_2(\mathbb{R}) \times M_2(\mathbb{R})$. Since the elements g_n do not belong to AU_1U_2, these polynomials P_n are non-constants. In particular, the real valued polynomial functions $s \mapsto \|P_n(s)\|^2$ are non-constant of degree at most 4.

Since $\|P_n(0)\| \to 0$ as $n \to \infty$, it follows from Lemma 2.5 that for any $0 < \epsilon$, we can choose $s_n \in T$ going to ∞ so that $k\varepsilon \le \|P_n(s_n)\| \le \varepsilon$ for some constant $k > 0$ depending only on K. Moreover we can deduce

$1/2 \le |c_n^{(1)} s_n - a_n^{(1)}| \le 2$ from the condition $\|P_n(s_n)\| \le \varepsilon$ by looking at the $(1,1)$ and $(2,2)$ entries of the first component of $P_n(s_n)$.

Therefore

$$k\varepsilon/2 \le \|q_n - e\| \le 2\varepsilon.$$

By construction, when ε is small enough, the sequence q_n has a limit point $q \neq e$ in $A_1 A_2 U_2 \cap G_0$.

We claim that this limit $q = (q^{(1)}, q^{(2)})$ belongs to the group AU_2. It suffices to check that the diagonal entries of $q^{(1)}$ and $q^{(2)}$ are equal. If not, the two sequences $c_n^{(i)} s_n$ converge to real numbers $c^{(i)}$ with $c^{(1)} \neq c^{(2)}$, and a simple calculation shows that the $(1,2)$- entries of $q_n^{(2)}$ are comparable to $\frac{c^{(2)} - c^{(1)}}{c^{(1)} - 1} s_n$ which tends to ∞, yielding a contradiction. Hence q belongs to AU_2. □

3.2. ***H*-minimal and *U*-minimal subsets.** Let

$$\Omega := \Omega_{\Gamma_1} \times X_2$$

where $\Omega_{\Gamma_1} \subset X_1$ is defined in (2.1). Note that, since Γ_2 is cocompact, one has the equality $\Omega_{\Gamma_2} = X_2$.

Let $x = (x_1, x_2) \in \Gamma\backslash G$ and consider the orbit xH. Note that xH intersects Ω non-trivially. Let Y be an H-minimal subset of the closure $\overline{xH}$ with respect to Ω, i.e., Y is a closed H-invariant subset of $\overline{xH}$ such that $Y \cap \Omega \neq \emptyset$ and the orbit yH is dense in Y for any $y \in Y \cap \Omega$. Since any H orbit intersects Ω, it follows that yH is dense in Y for any $y \in Y$. Let Z be a U-minimal subset of Y with respect to Ω. Since Ω is compact, such minimal sets Y and Z exist. Set

$$Y^* = Y \cap \Omega \quad \text{and} \quad Z^* = Z \cap \Omega.$$

In the following, we assume that

$$\text{the orbit } xH \text{ is not closed}$$

and aim to show that xH is dense in X.

Lemma 3.2. *For any $y \in Y$, the identity element e is an accumulation point of the set $\{g \in G \smallsetminus H : yg \in \overline{xH}\}$.*

Proof. If y does not belong to xH, there exists a sequence $h_n \in H$ such that xh_n converges to y. Hence there exists a sequence $g_n \in G$ converging to e such that $xh_n = yg_n$. These elements g_n do not belong to H; hence proving the claim.

Suppose now that y belongs to xH. If the claim does not hold, then for a sufficiently small neighborhood G_0 of e in G, the set $yG_0 \cap Y$ is included in the orbit yH. This implies that the orbit yH is an open subset of Y. The minimality of Y implies that $Y = yH$, contradicting the assumption that the orbit $yH = xH$ is not closed. □

Lemma 3.3. *There exists an element $v \in U_2 \smallsetminus \{e\}$ such that $Zv \subset \overline{xH}$.*

Proof. Choose a point $z = (z_1, z_2) \in Z^*$. By Lemma 3.2, there exists a sequence g_n in $G \smallsetminus H$ converging to e such that $zg_n \in \overline{xH}$. We may assume without loss of generality that g_n belongs to H_2.

Suppose first that at least one g_n belongs to U_2. Set $v = g_n$ be one of those belonging to U_2, so that the point zv belongs to $\overline{xH}$. Since v commutes with U and Z is U-minimal with respect to Ω, one has the equality $Zv = \overline{zvU}$, hence the set Zv is included in $\overline{xH}$.

Now suppose that g_n does not belong to U_2. Then, since the set $T(z_1)$ is K-thick for some $K > 1$ by Lemma 2.3, it follows from Lemma 2.4 that there exist a sequence $t_n \to \infty$ in $T(z_1)$ such that, after extraction, the products $\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}$ converge to an element $v \in U_2 \smallsetminus \{e\}$.

Since the points $z\widetilde{u}_{t_n}$ belong to Ω, this sequence has a limit point $z' \in Z^*$. Since one has the equality

$$z'v = \lim_{n\to\infty} z\widetilde{u}_{t_n}(\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}) = \lim_{n\to\infty} (zg_n)\widetilde{u}_{t_n},$$

the point $z'v$ belongs to $\overline{xH}$. We conclude as in the first case that the set $Zv = \overline{z'vU}$ is included in $\overline{xH}$. □

Lemma 3.4. *For any $z \in Z^*$, there exists a sequence g_n in $G \smallsetminus U$ converging to e such that $zg_n \in Z$ for all n.*

Proof. Since the group Γ_2 is cocompact, it does not contain unipotent elements and hence the orbit zU is not compact. By Lemma 2.3, the orbit zU is recurrent in Z^*, hence the set $Z^* \smallsetminus zU$ contains at least one point. Call it z'. Since the orbit $z'U$ is dense in Z, there exists a sequence $\widetilde{u}_{t_n} \in U$ such that $z = \lim z'\widetilde{u}_{t_n}$. Hence one can write $z'\widetilde{u}_{t_n} = zg_n$ with g_n in $G \smallsetminus U$ converging to e. □

Proposition 3.5. *There exists a one-parameter semi-group $L^+ \subset AU_2$ such that $ZL^+ \subset Z$.*

Proof. It suffices to find, for any neighborhood G_0 of e, an element $q \neq e$ in $AU_2 \cap G_0$ such that the set Zq is included in Z; then writing $q = \exp w$ for an element w of the Lie algebra of G, we can take L^+ to be the semigroup $\{\exp(sw_\infty) : s \geq 0\}$ where w_∞ is a limit point of the elements $\frac{w}{\|w\|}$ when the diameter of G_0 shrinks to 0.

Fix a point $z = (z_1, z_2) \in Z^*$. According to Lemma 3.4 there exists a sequence $g_n \in G \smallsetminus U$ converging to e such that $zg_n \in Z$.

Suppose first that g_n belongs to AU_1U_2 for infinitely many n; then one can find $\widetilde{u}_{t_n} \in U$ such that the product $q_n := g_n\widetilde{u}_{t_n}$ belongs to $AU_2 \smallsetminus \{e\}$ and zq_n belongs to Z. Since q_n normalizes U and since Z is U-minimal with respect to Ω, one has the equality $Zq_n = \overline{zU}q_n = \overline{zq_nU}$, hence the set Zq_n is included in Z.

Now suppose that g_n is not in AU_1U_2. By Lemmas 2.3 and 3.1, there exist sequences $s_n \in T(z_1)$ and $t_n \in \mathbb{R}$ such that, after passing to a subsequence, the products $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ converge to an element $q \neq e$ in $AU_2 \cap G_0$. Since the elements $z\widetilde{u}_{s_n}$ belong to Z^*, they have a limit point $z' \in Z^*$. Since we have

$$z'q = \lim_{n\to\infty} z\widetilde{u}_{s_n}(\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}) = \lim_{n\to\infty}(zg_n)\widetilde{u}_{t_n},$$

the element $z'q$ belongs to Z. We conclude as in the first case that the set $Zq = \overline{z'qU}$ is included in Z. □

Proposition 3.6. *There exist an element $z \in \overline{xH}$ and a one-parameter semi-group $U_2^+ \subset U_2$ such that $zU_2^+ \subset \overline{xH}$.*

Proof. By Proposition 3.5 there exists a one-parameter semigroup $L^+ \subset AU_2$ such that $ZL^+ \subset Z$. This semigroup L^+ is equal to one of the following: U_2^+, A^+ or $v_0^{-1}A^+v_0$ for some element $v_0 \in U_2 \smallsetminus \{e\}$, where U_2^+ and A^+ are one-parameter semigroups of U_2 and A respectively.

When $L^+ = U_2^+$, our claim is proved.

Suppose now $L^+ = A^+$. By Lemma 3.3 there exists an element $v \in U_2 \smallsetminus \{e\}$ such that $Zv \subset \overline{xH}$. Then one has the inclusions

$$ZA^+vA \subset ZvA \subset \overline{xH}A \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and a sequence $\widetilde{a}_{t_n} \in A^+$ going to ∞. Since $z'\widetilde{a}_{t_n}$ belong to Ω, after passing to a subsequence, the sequence $z'\widetilde{a}_{t_n}$ converges to a point $z \in \overline{xH} \cap \Omega$. Moreover, since the Hausdorff limit of the sets $\widetilde{a}_{-t_n}A^+$ is A, one has the inclusions

$$zAvA \subset \lim_{n\to\infty} z'\widetilde{a}_{t_n}(\widetilde{a}_{-t_n}A^+)vA = z'A^+vA \subset \overline{xH}.$$

Now by a simple computation, we can check that the set AvA contains a one-parameter semigroup U_2^+ of U_2, and hence the orbit zU_2^+ is included in $\overline{xH}$ as desired.

Suppose finally $L^+ = v_0^{-1}A^+v_0$ for some v_0 in $U_2 \smallsetminus \{e\}$. We can write $A^+ = \{\widetilde{a}_{\varepsilon t} : t \geq 0\}$ with $\varepsilon = \pm 1$ and $v_0 = (e, u_s)$ with $s \neq 0$. A simple computation shows that the set $U_2' := \{(e, u_{\varepsilon s t}) : 0 \leq t \leq 1\}$ is included in $v_0^{-1}A^+v_0A$. Hence one has the inclusions

$$ZU_2' \subset Zv_0^{-1}A^+v_0A \subset ZA \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and let $z \in \overline{xH}$ be a limit of a sequence $z'\widetilde{a}_{-t_n}$ with t_n going to $+\infty$. Since the Hausdorff limit of the sets $\widetilde{a}_{t_n}U_2'\widetilde{a}_{-t_n}$ is the semigroup $U_2^+ := \{(e, u_{\varepsilon s t}) : t \geq 0\}$, one has the inclusions

$$zU_2^+ \subset \lim_{n\to\infty}(z'\widetilde{a}_{-t_n})\widetilde{a}_{t_n}U_2'\widetilde{a}_{-t_n} \subset \overline{ZU_2'A} \subset \overline{xH}.$$ □

3.3. **Conclusion.**

Proof of Theorem 1.2. Suppose that the orbit xH is not closed. By Proposition 3.6, the orbit closure $\overline{xH}$ contains an orbit zU_2^+ of a one-parameter subsemigroup of U_2. Since Γ_2 is cocompact in H_2, by Lemma 2.6, this orbit zU_2^+ is dense in zH_2. Hence we have the inclusions

$$X = zG = zH_2H \subset \overline{zU_2^+}H \subset \overline{xH}.$$

This proves the claim. □

Proof of Theorem 1.1. Let $x = [g]$ be a point of $X_2 = \Gamma_2 \backslash H_2$. By replacing Γ_1 by $g^{-1}\Gamma_1 g$, we may assume without loss of generality that $g = e$. One deduces Theorem 1.1 from Theorem 1.2 thanks to the following equivalences:
The orbit $[e]H$ is closed (resp. dense) in $\Gamma\backslash G \iff$
The orbit $\Gamma[e]$ is closed (resp. dense) in $G/H \iff$
The product $\Gamma_2\Gamma_1$ is closed (resp. dense) in $\mathrm{PSL}_2(\mathbb{R}) \iff$
The orbit $[e]\Gamma_1$ is closed (resp. dense) in $\Gamma_2\backslash \mathrm{PSL}_2(\mathbb{R})$. □

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