FUCHSIAN GROUPS AND COMPACT HYPERBOLIC SURFACES

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Abstract. We present a topological proof of the following theorem of Benoist-Quint: for a finitely generated non-elementary discrete subgroup $\Gamma_1$ of $\text{PSL}(2, \mathbb{R})$ with no parabolics, and for a cocompact lattice $\Gamma_2$ of $\text{PSL}(2, \mathbb{R})$, any $\Gamma_1$ orbit on $\Gamma_2 \setminus \text{PSL}(2, \mathbb{R})$ is either finite or dense.

1. Introduction

Let $\Gamma_1$ be a non-elementary finitely generated discrete subgroup with no parabolic elements of $\text{PSL}(2, \mathbb{R})$. Let $\Gamma_2$ be a cocompact lattice in $\text{PSL}(2, \mathbb{R})$. The following is the first non-trivial case of a theorem of Benoist-Quint [1].

Theorem 1.1. Any $\Gamma_1$-orbit on $\Gamma_2 \setminus \text{PSL}(2, \mathbb{R})$ is either finite or dense.

The proof of Benoist-Quint is quite involved even in the case as simple as above and in particular uses their classification of stationary measures [2]. The aim of this note is to present a short, and rather elementary proof.

We will deduce Theorem 1.1 from the following Theorem 1.2. Let

- $H_1 = H_2 := \text{PSL}(2, \mathbb{R})$ and $G := H_1 \times H_2$;
- $H := \{(h, h) : h \in \text{PSL}_2(\mathbb{R})\}$ and $\Gamma := \Gamma_1 \times \Gamma_2$.

Theorem 1.2. For any $x \in \Gamma \setminus G$, the orbit $xH$ is either closed or dense.

Our proof of Theorem 1.2 is purely topological, and inspired by the recent work of McMullen, Mohammadi and Oh [5] where the orbit closures of the $\text{PSL}(2, \mathbb{R})$ action on $\Gamma_0 \setminus \text{PSL}(2, \mathbb{C})$ are classified for certain Kleinian subgroups $\Gamma_0$ of infinite co-volume. While the proof of Theorem 1.2 follows closely the sections 8-9 of [5], the arguments in this paper are simpler because of the assumption that $\Gamma_2$ is cocompact. We remark that the approach of [5] and hence of this paper is somewhat modeled after Margulis’s original proof of Oppenheim conjecture [4]. When $\Gamma_1$ is cocompact as well, Theorem 1.2 also follows from [6].

2. Horocyclic flow on convex cocompact surfaces

In this section we prove a few preliminary facts about unipotent dynamics involving only one factor $H_1$. 

Oh was supported in part by NSF Grant.
We recall that $\Gamma_1$ is a non-elementary finitely generated discrete subgroup with no parabolic elements of the group $H_1 = \text{PSL}_2(\mathbb{R})$, that is, $\Gamma_1$ is a convex cocompact subgroup. We will identify the boundary of the hyperbolic plane $\mathbb{H}^2 := \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ with the extended real line $\partial \mathbb{H}^2 = \mathbb{R} \cup \{ \infty \}$ which is topologically a circle. Let $S_1$ denote the hyperbolic orbifold $\Gamma_1 \backslash \mathbb{H}^2$, and let $\Lambda_{\Gamma_1} \subset \partial \mathbb{H}^2$ be the limit set of $\Gamma_1$. Let $A_1$ and $U_1$ be the subgroups of $H_1$ given by

$$A_1 := \{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \} \quad \text{and} \quad U_1 := \{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \}. $$

The set

$$\Omega_{\Gamma_1} = \{ x \in \Gamma_1 \backslash H_1 : xA_1 \text{ is bounded} \}. \tag{2.1}$$

is called the renormalized frame bundle of $\Gamma_1$. As $\Gamma_1$ is a convex cocompact subgroup, $\Omega_{\Gamma_1}$ is a compact $A_1$-invariant subset and one has the equality

$$\Omega_{\Gamma_1} = \{ [h] \in \Gamma_1 \backslash H_1 : h(0), h(\infty) \in \Lambda_{\Gamma_1} \}. $$

The image of $\Omega_{\Gamma_1}$ in $S_1$ under the map $h \mapsto h(i)$ is equal to the convex core of $S_1$.

**Definition 2.2.** Let $K > 1$. A subset $I \subset \mathbb{R}$ is called $K$-thick if, for any $t > 0$, $I$ meets $[-Kt, -t] \cup [t, Kt]$.

**Lemma 2.3.** There exists $K > 1$ such that for any $x \in \Omega_{\Gamma_1}$, the subset $I(x) := \{ t \in \mathbb{R} : xu_t \in \Omega_{\Gamma_1} \}$ is $K$-thick.

**Proof.** Using an isometry, we may assume without loss of generality that $x = [e]$ where $e$ corresponds to a downward unit vector at $i$ in the identification of $\text{PSL}_2(\mathbb{R})$ and $T^1(\mathbb{H}^2)$. As $x \in \Omega_{\Gamma_1}$, both points $0$ and $\infty$ belong to the limit set $\Lambda_{\Gamma_1}$. Since $u_t(\infty) = \infty$ and $u_t(0) = t$, one has the equality $I(x) = \{ t \in \mathbb{R} : t \in \Lambda_{\Gamma_1} \}$. Write $\mathbb{R} - \Lambda_{\Gamma_1}$ as the union $\cup J_i$ where $J_i$'s are maximal open intervals. Note that the minimum distance between the convex hulls

$$\delta := \inf_{t \neq m} d(\text{conv}(J_i), \text{conv}(J_m))$$

is positive, as $2\delta$ is the length of the shortest closed geodesic of the double of the core of $S_1$. Choose the constant $K > 1$ so that for $t > 0$, one has

$$d(\text{conv}[-Kt, -t], \text{conv}[t, Kt]) = \delta/2.$$ 

Note that this choice of $K$ is independent of $t$. If $I(x)$ does not intersect $[-Kt, -t] \cup [t, Kt]$ for some $t > 0$, then the intervals $[-Kt, -t]$ and $[t, Kt]$ must belong to two distinct intervals $J_i$ and $J_m$, since $0 \in \Lambda_{\Gamma_1}$. This contradicts to the choice of $K$. $\square$

**Lemma 2.4.** Let $K > 1$ and let $I$ be a $K$-thick subset of $\mathbb{R}$. For any sequence $h_n$ in $H_1 \setminus U_1$ converging to $e$, there exists a sequence $t_n \in I$ such that the sequence $u_{-t_n} h_n u_{t_n}$ has a non-trivial limit point in $U_1$. 

Proof. Write \( h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \). We compute
\[
q_n := u_{-t_n} h_n u_{t_n} = \begin{pmatrix} a_n - c_n t_n & (a_n - d_n - c_n t_n) t_n + b_n \\ c_n & d_n + c_n t_n \end{pmatrix}.
\]
Since \( h_n \) does not belong to \( U_1 \), it follows that the \((1,2)\)-entries \( P_n := (a_n - d_n - c_n t_n) t_n + b_n \) are non-constant polynomials in \( t_n \) of degree at most 2 whose coefficients converge to 0. Hence we can choose \( t_n \in I \) going to \( \infty \) so that \( 1 \leq |P_n| \leq k \), for some positive constant \( k \) depending only on \( K \). Then the product \( c_n t_n \) must converge to 0 and the sequence \( q_n \) has a limit point in \( U_1 - \{e\} \). \( \square \)

**Lemma 2.5.** Let \( U_1^+ \) be the semigroup \( \{u_t : t \geq 0\} \). If \( \Gamma_1 \) is cocompact, any \( U_1^+ \)-orbit is dense in \( \Gamma_1 \backslash H_1 \).

**Proof.** Consider \( xU_1^+ \) for \( x \in \Gamma_1 \backslash H_1 \). Set \( x_n := xu_n \). We then have \( x_n u_{-n} U_1^+ \subset xU_1^+ \). Hence if \( z \) is a limit point of the sequence \( x_n \), we have \( zU \subset xU_1^+ \). By Hedlund’s theorem [3], \( zU \) is dense, proving the claim. \( \square \)

3. **Proof of Theorems 1.1 and 1.2**

In this section, using minimal sets and unipotent dynamics on the product space \( \Gamma \backslash G \), we provide a proof of Theorem 1.2.

3.1. **Unipotent dynamics.** We recall the notation \( G := \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \) and \( \Gamma := \Gamma_1 \times \Gamma_2 \). Set
- \( H_1 = \{(h,e)\}, H_2 = \{(e,h)\}, H = \{(h,h)\} \);
- \( U_1 = \{(u_t,e)\}, U_2 = \{(e,u_t)\}, U = \{(u_t,u_t)\} \);
- \( A_1 = \{(a_t,e)\}, A_2 = \{(e,a_t)\}, A = \{(a_t,a_t)\} \);
- \( X_1 = \Gamma \backslash H_1, X_2 = \Gamma \backslash H_2, X = \Gamma \backslash G = X_1 \times X_2 \).

Recall that \( \Gamma_1 \) is a non-elementary finitely generated discrete subgroup of \( H_1 \) with no parabolic elements and that \( \Gamma_2 \) is a cocompact lattice in \( H_2 \).

For simplicity, we write \( \tilde{u}_t \) for \((u_t,u_t)\) and \( \tilde{a}_t \) for \((a_t,a_t)\). Note that the normalizer of \( U \) in \( G \) is \( AU_1U_2 \).

**Lemma 3.1.** Let \( g_n \) be a sequence in \( G \times AU_1U_2 \) converging to \( e \), and let \( I \) be a \( K \)-thick subset of \( \mathbb{R} \) for some \( K > 1 \). Then for any neighborhood \( G_0 \) of \( e \) in \( G \), there exist sequences \( s_n \in I \) and \( t_n \in \mathbb{R} \) such that the sequence \( \tilde{u}_{-s_n} g_n \tilde{u}_t \) has a non-trivial limit point \( q \in AU_2 \cap G_0 \).

**Proof.** Fix \( \varepsilon > 0 \). Write \( g_n = (g_n^{(1)}, g_n^{(2)}) \) with \( g_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix} \). Then the products \( q_n := \tilde{u}_{-s_n} g_n \tilde{u}_{t_n} \) are given by
\[
q_n^{(i)} = \begin{pmatrix} a_n^{(i)} - c_n^{(i)} s_n & (b_n^{(i)} - d_n^{(i)} s_n) - t_n (c_n^{(i)} s_n - a_n^{(i)}) \\ c_n^{(i)} & d_n^{(i)} + c_n^{(i)} t_n \end{pmatrix}.
\]
Set $t_n = \frac{b_n^{(1)} - a_n^{(1)} s_n}{c_n^{(1)} s_n - a_n^{(1)}}$. The differences $q_n - e$ are now rational functions in $s_n$ of the form $q_n - e = \frac{1}{c_n^{(1)} s_n - a_n^{(1)}} P_n$, where $P_n$ is a polynomial in $s_n$ of degree at most 2 with values in $M_2(\mathbb{R}) \times M_2(\mathbb{R})$. Since the elements $g_n$ do not belong to $AU_1U_2$, these polynomials $P_n$ are non-constants. Hence we can choose $s_n \in I$ going to infinity so that $\varepsilon \leq \|P_n\| \leq k\varepsilon$ for some constant $k > 1$ depending only on $K$. We can also simultaneously impose that the denominators satisfy $1/2 \leq |c_n^{(1)} s_n - a_n^{(1)}| \leq k$ so that $\varepsilon/k \leq \|q_n - e\| \leq 2k\varepsilon$. By construction, when $\varepsilon$ is small enough, the sequence $q_n$ has a non-trivial limit point $q$ in $A_1A_2U_2 \cap G_0$.

We claim that this limit $q = (q^{(1)}, q^{(2)})$ belongs to the group $AU_2$. It suffices to check that the diagonal entries of $q^{(1)}$ and $q^{(2)}$ are equal. If not, the two sequences $c_n^{(i)} s_n$ converge to real numbers $c^{(i)}$ with $c^{(1)} \neq c^{(2)}$, and a simple calculation shows that the $(1, 2)$-entries of $q^{(2)}_n$ are comparable to $\frac{c^{(2)} - c^{(1)}}{1 - c^{(1)}} s_n$ which tends to $\infty$. Contradiction. Hence $q$ belongs to $AU_2$. \hfill $\Box$

3.2. $H$-minimal and $U$-minimal subsets. Let

\[ \Omega := \Omega_{\Gamma_1} \times X_2 \]

where $\Omega_{\Gamma_1}$ is the renormalized frame bundle of $\Gamma_1$ as in (2.1). Note that, since $\Gamma_2$ is cocompact, the renormalized frame bundle of $\Gamma_2$ is $\Omega_{\Gamma_2} = X_2$.

Let $x = (x_1, x_2) \in \Gamma \backslash G$ and consider the orbit $xH$. Note that $xH$ intersects $\Omega$ non-trivially. Let $Y$ be an $H$-minimal subset of the closure $\overline{xH}$ with respect to $\Omega$, i.e., $Y$ is a closed $H$-invariant subset of $\overline{xH}$ such that $Y \cap \Omega \neq \emptyset$ and the orbit $yH$ is dense in $Y$ for any $y \in Y \cap \Omega$. Since any $H$ orbit intersects $\Omega$, it follows that $yH$ is dense in $Y$ for any $y \in Y$. Let $Z$ be a $U$-minimal subset of $Y$ with respect to $\Omega$. Since $\Omega$ is compact, such minimal sets $Y$ and $Z$ exist. Set

\[ Y^* = Y \cap \Omega \quad \text{and} \quad Z^* = Z \cap \Omega. \]

In the following, we assume that

the orbit $xH$ is not closed

and aim to show that $xH$ is dense in $X$.

\textbf{Lemma 3.2.} For any $y \in Y$, the identity element $e$ is an accumulation point of the set $\{g \in G \setminus H : yg \in \overline{xH}\}$.

\textbf{Proof.} If $y$ does not belong to $xH$, there exists a sequence $h_n \in H$ such that $xh_n$ converges to $y$. Hence there exists a sequence $g_n \in G$ converging to $e$ such that $xh_n = yg_n$. These elements $g_n$ do not belong to $H$; hence proving the claim.

Suppose now that $y$ belongs to $xH$. If the claim does not hold, then for a sufficiently small neighborhood $G_0$ of $e$ in $G$, the set $yG_0 \cap Y$ is included in the orbit $yH$. This implies that the orbit $yH$ is an open subset of $Y$. The minimality of $Y$ implies that $Y = yH$, contradicting the assumption that the orbit $yH$ is not closed. \hfill $\Box$
Lemma 3.3. There exists a non-trivial element \( v \in U_2 \) such that \( Zv \subset \bar{xH} \).

Proof. Choose a point \( z = (z_1, z_2) \in Z^* \). By Lemma 3.2, there exists a sequence \( g_n \) in \( G \setminus H \) converging to \( e \) such that \( zg_n \in \bar{xH} \). We may assume without loss of generality that \( g_n \) belongs to \( H_2 \). If \( g_n \) belongs to \( U_2 \) for some \( n \), the Lemma follows. Suppose that \( g_n \) does not belong to \( U_2 \). Then, since the set \( I(z_1) \) is \( K \)-thick for some \( K > 1 \) by Lemma 2.3, it follows from Lemma 2.4 that there exist a sequence \( t_n \to \infty \) in \( I(z_2) \) such that, after extraction, the products \( \tilde{u}_{-t_n}g_n\tilde{u}_{t_n} \) converge to a non-trivial element \( v \in U_2 \).

Since the points \( z\tilde{u}_{t_n} \) belong to \( \Omega \), this sequence has a limit point \( z' \in Z^* \). Since one has the equality

\[
z'v = \lim_{n \to \infty} z\tilde{u}_{t_n}(\tilde{u}_{-t_n}g_n\tilde{u}_{t_n})
\]

the point \( z'v \) belongs to \( \bar{xH} \). Since \( v \) commutes with \( U \) and \( Z \) is \( U \)-minimal with respect to \( \Omega \), one has the equality \( Zv = z'vU \), hence the set \( Zv \) is included in \( \bar{xH} \).

Lemma 3.4. For any \( z \in Z^* \), there exists a sequence \( g_n \) in \( G \setminus U \) converging to \( e \) such that \( zg_n \in Z \).

Proof. Since the group \( \Gamma_2 \) is cocompact, it does not contain unipotent elements and hence the orbit \( zU \) is not compact. Since the orbit \( zU \) is recurrent in \( Z^* \), the set \( Z^* \setminus zU \) contains at least one point. Call it \( z' \). Since the orbit \( z'U \) is dense in \( Z \), there exists a sequence \( \tilde{u}_{t_n} \in U \) such that \( z = \lim z\tilde{u}_{t_n} \). Hence one can write \( z\tilde{u}_{t_n} = zg_n \) with \( g_n \) in \( G \setminus U \) converging to \( e \). \( \square \)

Proposition 3.5. There exists a one-parameter semi-group \( L^+ \subset AU_2 \) such that \( ZL^+ \subset Z \).

Proof. It suffices to find, for any neighborhood \( G_0 \) of \( e \), a non-trivial element \( q \) in \( AU_2 \cap G_0 \) such that the set \( Zq \) is included in \( Z \); then writing \( q = \exp w \) for an element \( w \) of the Lie algebra of \( G \), we can take \( L^+ \) to be the semigroup \( \{ \exp(sw) : s \geq 0 \} \) where \( w \) is a limit point of the elements \( \frac{w}{\|w\|} \) when the diameter of \( G_0 \) shrinks to 0.

Fix a point \( z = (z_1, z_2) \in Z^* \). According to Lemma 3.4 there exists a sequence \( g_n \in G \setminus U \) converging to \( e \) such that \( zg_n \in Z \).

Suppose first that \( g_n \) belongs to \( AU_1U_2 \) for infinitely many \( n \); then one can find \( \tilde{u}_{t_n} \in U \) such that the product \( q_n := g_n\tilde{u}_{t_n} \) belongs to \( AU_2 \) and is non-trivial, and \( zg_n \) belongs to \( Z \). Hence, since \( g_n \) normalizes \( U \) and since \( Z \) is \( U \)-minimal with respect to \( \Omega \), the set \( Zg_n \) is included in \( Z \).

Now suppose that \( g_n \) is not in \( AU_1U_2 \). By Lemmas 2.3 and 3.1, there exist sequences \( s_n \in I(z_1) \) and \( t_n \in \mathbb{R} \) such that, after passing to a subsequence, the products \( \tilde{u}_{-s_n}g_n\tilde{u}_{t_n} \) converge to a non-trivial element \( q \in AU_2 \cap G_0 \). Since the elements \( z\tilde{u}_{t_n} \) belong to \( Z^* \), they have a limit point \( z' \in Z^* \). Since we have

\[
z'q = \lim_{n \to \infty} z\tilde{u}_{s_n}(\tilde{u}_{-s_n}g_n\tilde{u}_{t_n})
\]
the element $z'q$ belongs to $Z$. As $q$ normalizes $U$, it follows that $Zq$ is contained in $Z$.

**Proposition 3.6.** There exist an element $z \in \overline{xH}$ and a one-parameter semigroup $U_2^+ \subset U_2$ such that $zU_2^+ \subset \overline{xH}$.

*Proof.* By Proposition 3.5 there exists a one-parameter semigroup $L^+ \subset AU_2$ such that $ZL^+ \subset Z$. This semigroup $L^+$ is equal to one of the following: $U_2^+$, $A^+$ or $v_0^{-1}A^+v_0$ for some non-trivial element $v_0 \in U_2$, where $U_2^+$ and $A^+$ are one-parameter semigroups of $U_2$ and $A$ respectively.

When $L^+ = U_2^+$, our claim is proved.

Suppose now $L^+ = A^+$. By Lemma 3.3 there exists a non-trivial element $v \in U_2$ such that $Zv \subset \overline{xH}$. Then one has the inclusions

$$ZA^+vA \subset ZvA \subset \overline{xH}A \subset \overline{xH}.$$ Choose a point $z' \in Z^+$ and a sequence $\tilde{a}_{t_n} \in A^+$ going to $\infty$. Since $z'\tilde{a}_{t_n}$ belong to $\Omega$, after passing to a subsequence, the sequence $z'\tilde{a}_{t_n}$ converges to a point $z \in \overline{xH} \cap \Omega$. Moreover, since the Hausdorff limit of the sets $\tilde{a}_{-t_n}A^+$ is $A$, one has the inclusions

$$zAvA \subset \lim_{n \to \infty} z'\tilde{a}_{t_n}(\tilde{a}_{-t_n}A^+)vA = z'A^+vA \subset \overline{xH}.$$ Now by a simple computation, we can check that the set $AvA$ contains a one-parameter semigroup $U_2^+$ of $U_2$, and hence the orbit $zU_2^+$ is included in $\overline{xH}$ as desired.

Suppose finally $L^+ = v_0^{-1}A^+v_0$ for some $v_0 \in U_2$. We can assume without loss of generality that $A^+ = \{\tilde{a}_{et} : t \geq 0\}$ where $\varepsilon = \pm 1$ and that $v_0 = (e, u_1)$. A simple computation shows that the set $v_0^{-1}A^+v_0A$ contains the set $U_2^+ := \{(e, u_{et}) : 0 \leq t \leq 1\}$. Hence one has the inclusions

$$ZU_2^+ \subset Zv_0^{-1}A^+v_0A \subset ZA \subset \overline{xH}.$$ Choose a point $z' \in Z^*$ and let $z \in \overline{xH}$ be a limit of a sequence $z'\tilde{a}_{-t_n}$ with $t_n$ going to $+\infty$. Since the Hausdorff limit of the sets $\tilde{a}_{t_n}U_2^+\tilde{a}_{-t_n}$ is the semigroup $U_2^+ := \{(e, u_{et}) : t \geq 0\}$, one has the inclusions

$$zU_2^+ \subset \lim_{n \to \infty} (z'\tilde{a}_{-t_n})\tilde{a}_{t_n}U_2^+\tilde{a}_{-t_n} \subset \overline{ZU_2^+A} \subset \overline{xH}.$$  

**3.3. Conclusion.**

*Proof of Theorem 1.2.* Suppose that the orbit $xH$ is not closed. By Proposition 3.6, the orbit closure $\overline{xH}$ contains an orbit $zU_2^+$ of a one-parameter subsemigroup of $U_2$. Since $\Gamma_2$ is cocompact in $H_2$, by Lemma 2.5, this orbit $zU_2^+$ is dense in $zH_2$. Hence we have the inclusions

$$X = zG = zH_2H \subset HzU_2^+ \subset \overline{xH}.$$ This proves the claim.
Proof of Theorem 1.1. Let $x = [g]$ be a point of $X_2 = \Gamma_2 \backslash H_2$. By replacing $\Gamma_1$ by $g^{-1} \Gamma_1 g$, we may assume without loss of generality that $g = e$. One deduces Theorem 1.1 from Theorem 1.2 thanks to the following equivalences:

The orbit $[e]H$ is closed (resp. dense) in $\Gamma \backslash G \iff$

The orbit $\Gamma [e]$ is closed (resp. dense) in $G / H \iff$

The product $\Gamma_2 \Gamma_1$ is closed (resp. dense) in $\text{PSL}_2(\mathbb{R}) \iff$

The orbit $[e] \Gamma_1$ is closed (resp. dense) in $\Gamma_2 \backslash \text{PSL}_2(\mathbb{R})$. □

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