LOCAL MIXING AND INVARIANT MEASURES FOR HOROSPHERICAL SUBGROUPS ON ABELIAN COVERS

HEE OH AND WENYU PAN

ABSTRACT. Abelian covers of hyperbolic 3-manifolds are ubiquitous. We prove the local mixing theorem of the frame flow for abelian covers of closed hyperbolic 3-manifolds. We obtain a classification theorem for measures invariant under the horospherical subgroup. We also describe applications to the prime geodesic theorem as well as to other counting and equidistribution problems. Our results are proved for any abelian cover of a homogeneous space $\Gamma_0\backslash G$ where G is a rank one simple Lie group and $\Gamma_0 < G$ is a convex cocompact Zariski dense subgroup.

1. INTRODUCTION

1.1. **Motivation.** Let $\mathcal{M}$ be a closed hyperbolic 3-manifold. We can present $\mathcal{M}$ as the quotient $\Gamma_0\backslash\mathbb{H}^3$ of the hyperbolic 3-space $\mathbb{H}^3$ for some co-compact lattice Γ_0 of $G = \mathrm{PSL}_2(\mathbb{C})$. The frame bundle $\mathcal{F}(\mathcal{M})$ is isomorphic to the homogeneous space $\Gamma_0\backslash G$ and the frame flow on $\mathcal{F}(\mathcal{M})$ corresponds to the right multiplication of $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ on $\Gamma_0\backslash G$.

The strong mixing property of the frame flow [24] is well-known:

Theorem 1.1 (Strong mixing)**.** *For any $\psi_1, \psi_2 \in L^2(\Gamma_0\backslash G)$,*

$$\lim_{t\to\infty} \int \psi_1(xa_t)\psi_2(x)\, dx = \int \psi_1 dx \cdot \int \psi_2\, dx$$

where dx denotes the G-invariant probability measure on $\Gamma_0\backslash G$.

This mixing theorem and its effective refinements are of fundamental importance in homogeneous dynamics, and have many applications in various problems in geometry and number theory.

One recent spectacular application was found in the resolution of the surface subgroup conjecture by Kahn-Markovic [25]. Based on their work, as well as Wise's, Agol settled the virtually infinite betti number conjecture [1]:

Theorem 1.2. *Any closed hyperbolic 3-manifold $\mathcal{M} = \Gamma_0\backslash\mathbb{H}^3$ virtually has a $\mathbb{Z}^d$-cover for any $d \geq 1$.*

That is, after passing to a subgroup of finite index, Γ_0 contains a normal subgroup Γ with $\Gamma\backslash\Gamma_0 \simeq \mathbb{Z}^d$, and hence the hyperbolic 3-manifold $\Gamma\backslash\mathbb{H}^3$ is a regular cover of $\mathcal{M}$ whose deck transformation group is isomorphic to $\mathbb{Z}^d$.

Oh was supported in part by NSF Grant #1361673.

On the frame bundle of the $\mathbb{Z}^d$-cover $\Gamma\backslash\mathbb{H}^3$, the strong mixing property fails [24], because for any $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$,

$$\lim_{t\to\infty} \int_{\Gamma\backslash G} \psi_1(xa_t)\psi_2(x)\, dx = 0.$$

The aim of this paper is to formulate and prove the *local mixing* property of the frame flow for any abelian cover of a closed hyperbolic 3-manifold, or more generally for any abelian cover of a convex cocompact rank one locally symmetric space. The local mixing property is an appropriate substitute of the strong mixing property in the setting of an infinite volume homogeneous space, and has similar applications. We will establish a classification theorem for measures invariant under the horospherical subgroup, extending the work of Babillot, Ledrappier and Sarig. We will also describe applications to the prime geodesic theorem as well as other counting and equidistribution problems.

1.2. **Local limit theorem.** In order to motivate our definition of the local mixing property, we recall the classical local limit theorem on the Euclidean space $\mathbb{R}^d$ [11]:

Theorem 1.3 (Local limit theorem)**.** *For any absolutely continuous compactly supported probability measure μ on $\mathbb{R}^d$, and any continuous function ψ on $\mathbb{R}^d$ with compact support,*

$$\lim_{n\to+\infty} n^{d/2} \int \psi \, d\mu^{*n} = c(\mu) \int_{\mathbb{R}^d} \psi(x)dx$$

where μ^{*n} denotes the n-th convolution of μ and $c(\mu) > 0$ is a constant depending only on μ.

The virtue of this theorem is that although the sequence μ^{*n} weakly converges to zero, the re-normalized measure $n^{d/2}\mu^{*n}$ converges to a non-trivial locally finite measure on $\mathbb{R}^d$, which is the Lebesgue measure in this case.

1.3. **Local mixing theorem.** Let G be a connected semisimple linear Lie group and $\Gamma < G$ a discrete subgroup. Let $\{a_t : t \in \mathbb{R}\}$ be a one-parameter diagonalizable subgroup of G, acting on $\Gamma\backslash G$ by right translations. Denote by $C_c(\Gamma\backslash G)$ the space of all continuous functions with compact support. For a compactly supported probability measure μ on $\Gamma\backslash G$, we consider the following family $\{\mu_t\}$ of probability measures on $\Gamma\backslash G$ translated by the flow a_t: for $\psi \in C_c(\Gamma\backslash G)$,

$$\mu_t(\psi) := \int_{\Gamma\backslash G} \psi(xa_t)d\mu(x).$$

We formulate the following notion, which is analogous to the local limit theorem for $\mathbb{R}^d$:

Definition 1.4. A probability measure μ on $\Gamma\backslash G$ has the local mixing property for $\{a_t\}$ if there exist a positive function α on $\mathbb{R}_{>0}$ and a non-trivial Radon measure m on $\Gamma\backslash G$ such that for any $\psi \in C_c(\Gamma\backslash G)$,

$$\lim_{t\to+\infty} \alpha(t) \int \psi(x)\, d\mu_t(x) = \int \psi \, d\mathsf{m}(x).$$

If $\Gamma < G$ is a lattice, then any absolutely continuous probability measure μ has the local mixing property for $\{a_t\}$ [24]. If $\Gamma = \{e\}$, no probability measure has the local mixing for $\{a_t\}$.

FIGURE 1. Z-covers of convex cocompact surfaces

We focus on the rank one situation in which case the action of a_t induces the geodesic flow on the corresponding locally symmetric space. Throughout the introduction, suppose that G is a connected simple Lie group of real rank one, that is, G is the group of orientation preserving isometries of a simply connected Riemmanian space $\tilde{X}$ of rank one. Let $\Gamma_0 < G$ be a Zariski dense and convex cocompact subgroup of G, i.e. its convex core, which is the quotient of the convex hull of the limit set of Γ_0 by Γ_0, is compact. For instance, Γ_0 can be a cocompact lattice of G. Let

$$\Gamma < \Gamma_0$$

be a normal subgroup with $\mathbb{Z}^d$-quotient. Then $X := \Gamma\backslash\tilde{X}$ is a regular cover of $X_0 := \Gamma_0\backslash\tilde{X}$ whose group of deck transformations is isomorphic to $\mathbb{Z}^d$. Let $\{a_t \in G : t \in \mathbb{R}\}$ be a one-parameter subgroup which is the lift of the geodesic flow $\mathcal{G}^t$ on the unit tangent bundle $\mathrm{T}^1(X)$. When $\tilde{X}$ is a real hyperbolic space, the a_t action on $\Gamma\backslash G$ corresponds to the frame flow on the oriented frame bundle of X.

Denote by $\mathcal{P}_{\mathsf{acc}}(\Gamma\backslash G)$ the space of compactly supported absolutely continuous probability measures on $\Gamma\backslash G$ with continuous densities.

Theorem 1.5 (Local mixing theorem)**.** *Any $\mu \in \mathcal{P}_{\mathsf{acc}}(\Gamma\backslash G)$ has the local mixing property for $\{a_t\}$: for $\psi \in C_c(\Gamma\backslash G)$,*

$$\lim_{t\to+\infty} t^{d/2} e^{(D-\delta)t} \int \psi \, d\mu_t = c(\mu) \cdot \int \psi \, d\mathsf{m}^{\mathrm{BR}_+}$$

where D is the volume entropy of $\tilde{X}$, δ is the critical exponent of Γ_0, $\mathsf{m}^{\mathrm{BR}_+}$ is the Burger-Roblin measure on $\Gamma\backslash G$ for the expanding horospherical subgroup, and $c(\mu) > 0$ is a constant depending on μ.

We remark that the critical exponent δ of Γ_0 is same as that of Γ by [16].

Denote by dx a G-invariant measure on $\Gamma\backslash G$. Theorem 1.5 is deduced from the following theorem, which describes the precise asymptotic of the correlation functions.

Theorem 1.6. *For $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$,*

$$\lim_{t\to+\infty} t^{d/2} e^{(D-\delta)t} \int_{\Gamma\backslash G} \psi_1(xa_t)\psi_2(x) \, dx = \frac{\mathsf{m}^{\mathrm{BR}_+}(\psi_1)\mathsf{m}^{\mathrm{BR}_-}(\psi_2)}{(2\pi\sigma)^{d/2}\mathsf{m}^{\mathrm{BMS}}(\Gamma_0\backslash G)}.$$

where $\mathsf{m}^{\mathrm{BMS}}$ is the Bowen-Margulis-Sullivan measure on $\Gamma_0\backslash G$, $\mathsf{m}^{\mathrm{BR}_-}$ is the Burger-Roblin measure on $\Gamma\backslash G$ for the contracting horospherical subgroup, and $\sigma = \sigma_\Gamma > 0$ is a constant given in (3.16).

For the trivial cover, i.e. when $d = 0$, this theorem was obtained by Winter [65], based on the earlier work of Babillot [3].

If $\Gamma_0 < G$ is cocompact, then $D = \delta$ and the measures $\mathsf{m}^{\mathrm{BR}_+}$, $\mathsf{m}^{\mathrm{BR}_-}$ and $\mathsf{m}^{\mathrm{BMS}}$ are all proportional to the invariant measure dx. Hence the following is a special case of Theorem 1.6:

Theorem 1.7. *Let $\Gamma\backslash G$ be a $\mathbb{Z}^d$-cover of a compact rank one space $\Gamma_0\backslash G$. For $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$, we have*

$$\lim_{t\to+\infty} t^{d/2} \int \psi_1(xa_t)\psi_2(x)\, dx = \frac{1}{(2\pi\sigma)^{d/2}} \int \psi_1 dx \int \psi_2 dx.$$

Remark 1.8.
1. Let S_g be a compact hyperbolic surface of genus g and $\Gamma_0 < \mathrm{PSL}_2(\mathbb{R})$ be a realization of the surface group $\pi_1(S_g)$. Then $\Gamma := [\Gamma_0, \Gamma_0]$ is a normal subgroup of Γ_0 with $\mathbb{Z}^{2g}$-quotient, and $\Gamma\backslash\mathbb{H}^2$ is the homology cover of S_g. Theorem 1.7 is already new in this case.
2. For any $n \geq 2$, there is a congruence lattice of $\mathrm{SO}(n, 1)$ admitting co-abelian subgroups of infinite index ([39], [35], [36]). Moreover, such a congruence lattice can be found in any arithmetic subgroup of $\mathrm{SO}(n, 1)$ if $n \neq 3, 7$.
3. An infinite abelian cover of a compact quotient $\Gamma_0\backslash G$ may exist only when G is $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$, since other rank one groups have Kazhdan's property T, which forces the vanishing of the first Betti number of any lattice in G. On the other hand, there are plenty of normal subgroups of a convex cocompact subgroup Γ_0 of any G with $\mathbb{Z}^d$-quotients; for instance, if Γ_0 is a Schottky group generated by g-elements, we have a $\mathbb{Z}^d$-cover of $\Gamma_0\backslash G$ for any $1 \leq d \leq g$.
4. All of our results can be generalized to any co-abelian subgroup Γ of Γ_0 as there exists a co-finite subgroup $\Gamma_1 < \Gamma_0$, which is necessarily convex cocompact and such that $\Gamma\backslash\Gamma_1$ is isomorphic to $\mathbb{Z}^d$ for some $d \geq 0$.

Remark 1.9. We expect Theorem 1.5 to hold in a greater generality where the quotient group $N := \Gamma\backslash\Gamma_0$ is a finitely generated nilpotent group, and the exponent d is the polynomial growth of N [7]: for a finite generating set S of N, there is $\beta > 1$ such that $\beta^{-1} n^d \leq \# S^n \leq \beta n^d$ for all $n \geq 1$.

1.4. **Ergodicity of the frame flow on abelian covers.** We also establish the following ergodic property of the $A := \{a_t\}$-action on $\Gamma\backslash G$.

Theorem 1.10. *The Bowen-Margulis-Sullivan measure $\mathsf{m}^{\mathrm{BMS}}$ on $\Gamma\backslash G$ is ergodic for the A-action if and only if $d \leq 2$.*

This strengthens the previous works of Rees [56] and Yue [66] on the equivalence of the ergodicity of geodesic flow and the condition $d \leq 2$

1.5. **Measure classification for a horospherical subgroup action.** Let N denote the contracting horospherical subgroup of G:

$$N = \{g \in G : a_{-t} g a_t \to e \text{ as } t \to +\infty\}.$$

Let M be the compact subgroup which is the centralizer of A, so that G/M is isomorphic to the unit tangent bundle $\mathrm{T}^1(\tilde{X})$.

Let $(\Gamma\backslash\Gamma_0)^*$ denote the group of characters of the abelian group $\Gamma\backslash\Gamma_0 \simeq \mathbb{Z}^d$. Babillot and Ledrappier constructed a family $\{\mathsf{m}_\chi : \chi \in (\Gamma\backslash\Gamma_0)^*\}$ of NM-invariant

Radon measures on $\Gamma\backslash G$ ([4], [6]); see Def. 6.7 for a precise definition. Each m_χ is known to be NM-ergodic ([6], [51], [13]).

We show that m_χ is N-ergodic and deduce the following theorem from the classification result of NM-ergodic invariant measures due to Sarig and Ledrappier ([59], [33]):

Theorem 1.11. *Let $\Gamma_0 < G$ be cocompact and $\Gamma\backslash\Gamma_0 \simeq \mathbb{Z}^d$. Any N-invariant ergodic Radon measure on $\Gamma\backslash G$ is proportional to m_χ for some $\chi \in (\Gamma\backslash\Gamma_0)^*$.*

1.6. **Prime geodesic theorems and holonomies.** For $T > 0$, let $\mathcal{P}_T$ be the collection of all primitive closed geodesics in $\mathrm{T}^1(X)$ of length at most T. To each closed geodesic C, we can associate a conjugacy class h_C in M, called the holonomy class of C.

We normalize $\mathsf{m}^{\mathrm{BMS}}$ so that $\mathsf{m}^{\mathrm{BMS}}(\Gamma_0\backslash G) = 1$. We write $f(T) \sim g(T)$ if $\lim_{T\to\infty} f(T)/g(T) = 1$.

Theorem 1.12. *Let $\Omega \subset \mathrm{T}^1(X)$ be a compact subset with BMS-negligible boundary and $\xi \in C(M)$ a class function. Then as $T \to \infty$,*

$$\sum_{C\in\mathcal{P}_T} \frac{\ell(C\cap\Omega)}{\ell(C)}\xi(h_C) \sim \frac{e^{\delta T}}{(2\pi\sigma)^{d/2}\delta T^{d/2+1}}\mathsf{m}^{\mathrm{BMS}}(\Omega)\int_M \xi dm$$

where dm is the probability Haar measure on M.

For $d = 0$, the above theorem was proved earlier in [38] (also [58] for Γ lattice). Indeed, given Theorem 1.6, the proof of [38] applies in the same way. Another formulation of the prime geodesic theorem in this setting would be studying the distribution of closed geodesics in $\mathrm{T}^1(X_0)$ satisfying some homological constraints. An explicit main term in this setting was first described by Phillips and Sarnak [49]. See also ([27], [28], [31], [47], [2]) for subsequent works.

1.7. **Distribution of a discrete Γ-orbit in $H\backslash G$.** Let H be either the trivial, a horospherical or a symmetric subgroup of G (that is, H is the group of fixed points of an involution of G). Another application of the local mixing result can be found in the study of the distribution of a discrete Γ-orbit on the quotient space $H\backslash G$. That this question can be approached by a mixing type result has been understood first by Margulis [37] at least when H is compact. It was further developed by Duke, Rudnick and Sarnak [17], and Eskin and McMullen [19] when Γ is a lattice. See [57], [44], [40], [42] for generalizations to geometrically finite groups Γ. The following theorem extends especially the works of [44] and [40] to geometrically infinite groups which are co-abelian subgroups of convex cocompact groups. We give an explicit formula for a Borel measure $\mathcal{M} = \mathcal{M}_\Gamma$ (see Def. 7.5) on $H\backslash G$ for which the following holds:

Theorem 1.13. *Suppose that $[e]\Gamma_0 \subset H\backslash G$ is discrete and that $[H\cap\Gamma_0 : H\cap\Gamma] < \infty$. If B_T is a well-rounded sequence of compact subsets in $H\backslash G$ with respect to $\mathcal{M}$ (see Def. 7.6), then*

$$\#[e]\Gamma \cap B_T \sim \mathcal{M}(B_T).$$

In the case where H is compact, Γ_0 is cocompact, and B_T is the Riemannian ball in $G/K = \tilde{X}$, this result implies that for any $o \in \tilde{X}$,

$$\#\{\gamma(o) \in \Gamma(o) : d(\gamma(o), o) < T\} \sim c\frac{e^{DT}}{T^{d/2}}.$$

This was earlier obtained by Pollitcott and Sharp [52] (also see [18]). The novelty of Theorem 1.13 lies in the treatment the homogeneous space $H\backslash G$ with non-compact H and of sequence B_T of very general shape (e.g. sectors).

We present a concrete example: let $Q = Q(x_1, \cdots, x_{n+1})$ be a quadratic form of signature $(n, 1)$ for $n \geq 2$, and let $G = \mathrm{SO}(Q)$ be the special orthogonal group preserving Q. Let Γ be a normal subgroup of a Zariski dense convex cocompact subgroup Γ_0 with $\mathbb{Z}^d$-quotient.

Corollary 1.14. *Let $w_0 \in \mathbb{R}^{n+1}$ be a non-zero vector such that $w_0\Gamma_0$ is discrete, and $[\mathrm{Stab}_{\Gamma_0}(w_0) : \mathrm{Stab}_{\Gamma}(w_0)] < \infty$. Then for any norm $\|\cdot\|$ on $\mathbb{R}^{n+1}$,*

$$\#\{v \in w_0\Gamma : \|v\| \leq T\} \sim c\frac{T^\delta}{(\log T)^{d/2}}$$

where $c > 0$ depends only on Γ and $\|\cdot\|$.

In this case, the G-orbit w_0G is isomorphic to $H\backslash G$ where H is either $\mathrm{SO}(n-1, 1)$, $\mathrm{SO}(n)$ or MN according as $Q(w) > 0$, $Q(w) < 0$, or $Q(w) = 0$. Under this isomorphism, the norm balls give rise to a well-rounded family of compact subsets, say B_T and the explicit computation of the $\mathcal{M}$-measure of $B_T \subset H\backslash G$ gives the above asymptotic.

1.8. **Discussion of the proof.** The proof of Theorem 1.6 is based on extending the symbolic dynamics approach of studying the geodesic flow on $\mathrm{T}^1(X_0)$ as the suspension flow on $\Sigma \times \mathbb{R}/\sim$ for a subshift of finite type (Σ, σ). The a_t flow on a $\mathbb{Z}^d$-cover $\Gamma\backslash G$ can be studied via the suspension flow on

$$\tilde{\Sigma} := \Sigma \times \mathbb{Z}^d \times M \times \mathbb{R}/\sim$$

where the equivalence is defined via the shift map $\sigma : \Sigma \to \Sigma$, the first return time map $\tau : \Sigma \to \mathbb{R}$, the $\mathbb{Z}^d$-coordinate map $f : \Sigma \to \mathbb{Z}^d$ and the holonomy map $\theta : \Sigma \to M$. The asymptotic behavior of the correlation function of the suspension flow on $\tilde{\Sigma}$ with respect to the BMS measure can then be investigated using analytic properties of the associated Ruelle transfer operators $L_{s,v,\mu}$ of three parameters $s \in \mathbb{C}, v \in \hat{\mathbb{Z}}^d, \mu \in \hat{M}$ where $\hat{\mathbb{Z}}^d$ and $\hat{M}$ denote the unitary dual of $\mathbb{Z}^d$ and of M respectively (see Def. 3.2). The key ingredient is to show that on the plane $\Re(s) \geq \delta$, the map

$$s \mapsto (1 - L_{s,v,\mu})^{-1}$$

is holomorphic except for a simple pole at $s = \delta$, which occurs only when both v and μ are trivial. To each element $\gamma \in \Gamma_0$ we can associate the length $\ell(\gamma) \in \mathbb{R}$ and the Frobenius element $f(\gamma) \in \mathbb{Z}^d$ and the holonomy representation $\theta(\gamma) \in M$. Our proof of the desired analytic properties of $(1 - L_{s,v,\mu})^{-1}$ is based on the study of the generalized length spectrum of Γ_0 relative to Γ:

$$\mathcal{G}L(\Gamma_0, \Gamma) := \{(\ell(\gamma), f(\gamma), \theta(\gamma)) \in \mathbb{R} \times \mathbb{Z}^d \times M : \gamma \in \Gamma_0\}.$$

The correlation function for the BMS measure can then be expressed in terms of the operator $(1 - L_{s,v,\mu})^{-1}$ via an appropriate Laplace/Fourier transform. We then perform the necessary Fourier analysis to extract the main term coming from the residue. Finally we can deduce the precise asymptotic of the correlation function for the Haar measure from that for the BMS measure using ideas originated in Roblin's work (see Theorem 4.10). In order to prove Theorem 1.11, we first deduce from Theorem 1.6 and the closing lemma that the group generated by the generalized length spectrum $\mathcal{G}L(\Gamma_0,\Gamma)$ is dense in $\mathbb{R}\times\mathbb{Z}^d\times M$. Using this, we show that for any *generalized* BMS measure m_Γ on $\Gamma\backslash G$, any N-invariant measurable function on $\Gamma\backslash G$ is invariant by $\Gamma\backslash\Gamma_0$ on the left and by AM on the right. Then the AM-ergodicity of m_Γ, considered as a measure on $\Gamma_0\backslash G$, implies Theorem 1.11, since the transverse measure of a Babillot-Ledrappier measure m_χ equals to the transverse measure of some generalized BMS measure.

1.9. **Organization.** The paper is organized as follows. In section 2, we introduce the suspension model for the frame flow on abelian covers. In section 3, we investigate the analytic properties of the Ruelle transfer operators $L_{s,v,\mu}$. In section 4, we deduce the asymptotic behavior of the correlation functions of the suspension flow with respect to the BMS measure from the study of the transfer operators made in section 3, and prove Theorem 1.6. In section 5, we study the ergodicity of the frame flow with respect to generalized BMS-measures. In section 6, we discuss the ergodicity of the Babillot-Ledrappier measures for the horospherical subgroup action, and deduce a measure classification invariant under the horospherical subgroup. Applications to the prime geodesic theorem, and to other counting theorems 1.12 and 1.13 are discussed in the final section 7.

Acknowlegment. We would like to thank Curt McMullen for useful comments.

2. Suspension model for the frame flow on abelian covers

2.1. **Set-up and Notations.** Unless mentioned otherwise, we use the notation and assumptions made in this section throughout the paper. Let G be a connected simple linear Lie group of real rank one and K a maximal compact subgroup of G. Let $\tilde{X}=G/K$ be the associated simply connected Riemmanian symmetric space and let $\partial_\infty(\tilde{X})$ be its geometric boundary. Choosing a unit tangent vector v_o at $o:=[K]\in G/K$, the unit tangent bundle $\mathrm{T}^1(\tilde{X})$ can be identified with G/M where M is the stabilizer subgroup of v_o. We let d denote the right K-invariant and left G-invariant distance function on G which induces the Riemannian metric on $\tilde{X}$ which we will also denote by d. Let $A=\{a_t\}<G$ be the one-parameter subgroup of semisimple elements whose right translation action on G/M gives the geodesic flow. Then M equals to the centralizer of A in K. We denote by N^+ and N^- the expanding and the contracting horospherical subgroups of G for the action of a_t:

$$N^{\pm}=\{g\in G: a_t g a_t^{-1}\to e \text{ as } t\to\pm\infty\}.$$

We denote by $D>0$ the volume entropy of $\tilde{X}$, i.e.

$$D:=\lim_{T\to\infty}\frac{\log \mathrm{Vol}(B(o,T))}{T}$$

where $B(o,T)=\{x\in\tilde{X}: d(o,x)\le T\}$. For instance, if $\tilde{X}=\mathbb{H}^n$, then $D=n-1$.

For a discrete subgroup $\Gamma_0<G$, we denote by $\Lambda(\Gamma_0)$ the limit set of Γ_0, which is the set of all accumulation points in $\tilde{X}\cup\partial(\tilde{X})$ of an orbit of Γ_0 in $\tilde{X}$.

Let Γ_0 be a Zariski dense and convex cocompact subgroup of G; this means that the convex hull of $\Lambda(\Gamma)$ is compact modulo Γ.

Let $\Gamma<\Gamma_0$ be a normal subgroup with

$$\Gamma\backslash\Gamma_0\simeq\mathbb{Z}^d.$$

Set $X_0=\Gamma_0\backslash\tilde{X}$ and $X=\Gamma\backslash\tilde{X}$. So we may identify $\mathrm{T}^1(X_0)=\Gamma_0\backslash G/M$ and $\mathrm{T}^1(X)=\Gamma\backslash G/M$. The critical exponents of Γ and Γ_0 coincide [16], which we will denote by δ. Then $0 < \delta \le D$, and $\delta = D$ if and only if Γ_0 is co-compact in G by Sullivan [64]. As Γ is normal, we have $\Lambda(\Gamma) = \Lambda(\Gamma_0)$.

We recall the construction of the Bowen-Margulis-Sullivan measure $\mathsf{m}^{\mathrm{BMS}}$ and Burger-Roblin measures $\mathsf{m}^{\mathrm{BR}_\pm}$ on $\Gamma_0\backslash G$.

We let $\{\mathsf{m}_x : x \in \tilde{X}\}$ and $\{\nu_x : x \in \tilde{X}\}$ be Γ_0-invariant conformal densities of dimensions D and δ respectively, unique up to scalings. They are called the Lebesgue density and the Patterson-Sullivan density, respectively.

The notation $\beta_\xi(x, y)$ denotes the Busemann function for $\xi \in \partial(\tilde{X})$, and $x, y \in \tilde{X}$. The Hopf parametrization of $\mathrm{T}^1(\tilde{X})$ as $(\partial^2(\tilde{X}) - \text{Diagonal}) \times \mathbb{R}$ is given by

$$u \mapsto (u^+, u^-, s = \beta_{u^-}(o, u))$$

where $u^\pm \in \partial(\tilde{X})$ are the forward and the backward end points of the geodesic determined by u and $\beta_{u^-}(o, u) = \beta_{u^-}(o, \pi(u))$ for the canonical projection $\pi : \mathrm{T}^1(\tilde{X}) \to \tilde{X}$.

Using this parametrization, the following defines locally finite Borel measures on $\mathrm{T}^1(\tilde{X})$:

$$\begin{aligned}
d\tilde{\mathsf{m}}^{\mathrm{BMS}}(u) &= e^{\delta\beta_{u^+}(o,u)+\delta\beta_{u^-}(0,u)} d\nu_o(u^+) d\nu_o(u^-) ds;\\
d\tilde{\mathsf{m}}^{\mathrm{BR}_+}(u) &= e^{D\beta_{u^+}(o,u)+\delta\beta_{u^-}(0,u)} d\mathsf{m}_o(u^+) d\nu_o(u^-) ds;\\
d\tilde{\mathsf{m}}^{\mathrm{BR}_-}(u) &= e^{\delta\beta_{u^+}(o,u)+D\beta_{u^-}(0,u)} d\nu_o(u^+) d\mathsf{m}_o(u^-) ds;\\
d\tilde{\mathsf{m}}^{\mathrm{Haar}}(u) &= e^{D\beta_{u^+}(o,u)+D\beta_{u^-}(0,u)} d\mathsf{m}_o(u^+) d\mathsf{m}_o(u^-) ds;
\end{aligned}$$

They are left Γ_0-invariant measures on $\mathrm{T}^1(\tilde{X}) = G/M$. We will use the same notation for their M-invariant lifts to G, which are, respectively, right AM, N^+M, N^-M and G-invariant. By abuse of notation, the induced measures on $\Gamma_0\backslash G$ will be denoted by $\mathsf{m}^{\mathrm{BMS}}, \mathsf{m}^{\mathrm{BR}_\pm}, \mathsf{m}^{\mathrm{Haar}}$ respectively. If Γ_0 is cocompact in G, these measures are all equal to each other, being simply the Haar measure. In general, only $\mathsf{m}^{\mathrm{BMS}}$ is a finite measure on $\Gamma_0\backslash G$. An important feature of $\mathsf{m}^{\mathrm{BMS}}$ is that it is the unique measure of maximal entropy (which is δ) as a measure on $\mathrm{T}^1(X_0)$.

Since the measures $\tilde{\mathsf{m}}^{\mathrm{BMS}}, \tilde{\mathsf{m}}^{\mathrm{BR}_\pm}, \tilde{\mathsf{m}}^{\mathrm{Haar}}$ are all Γ-invariant as $\Gamma < \Gamma_0$, they also induce measures on $\Gamma\backslash G$ for which we will use the same notation $\mathsf{m}^{\mathrm{BMS}}, \mathsf{m}^{\mathrm{BR}_\pm}, \mathsf{m}^{\mathrm{Haar}}$ respectively.

We will normalize $\mathsf{m}^{\mathrm{BMS}}$ so that

$$\mathsf{m}^{\mathrm{BMS}}(\Gamma_0\backslash G) = 1$$

which can be done by rescaling ν_o.

Remark 2.1. We remark that $\{\nu_x\}$ is also the unique Γ-invariant conformal density of dimension δ whose support is $\Lambda(\Gamma)$, up to a constant multiple; this can be deduced from [48, Prop. 11.10, Thm. 11.17]). Therefore the BMS-measures and BR measures on $\Gamma\backslash G$ can be defined canonically without the reference to Γ_0.

2.2. **Markov sections and suspension space Σ^τ.** Denote by Ω_0 the non-wandering set of the geodesic flow $\{a_t\}$ in $\Gamma_0\backslash G/M$, i.e.

$$\{x : \text{for any neighborhood } U \text{ of } x, U a_{t_i} \cap U \neq \emptyset \ \text{ for some } t_i \to \infty\}.$$

The set Ω_0 coincides with the support of m^{BMS} and is a hyperbolic set for the geodesic flow. In this subsection, we recall the well-known construction of a subshift of finite type and its suspension space which gives a symbolic space model for (Ω_0, a_t) (see [47], [9], [23], [54], [52] for a general reference). For each $z \in \Omega_0$, the strong unstable manifold $W^{su}(z)$, the strong stable manifold $W^{ss}(z)$, the weak unstable manifold $W^u(z)$ and the weak stable manifold $W^s(z)$ are respectively given by the sets zN^+, zN^-, zN^+A, and zN^-A in $\Gamma_0\backslash G/M$ respectively.

Consider a finite set $z_1,\ldots,z_k$ in Ω_0 and choose small compact neighborhoods U_i and S_i of z_i in $W^{su}(z_i)\cap\Omega_0$ and $W^{ss}(z_i)\cap\Omega_0$ respectively such that $U_i=\overline{\mathrm{int}^u(U_i)}$ and $S_i=\overline{\mathrm{int}^s(S_i)}$. Here $\mathrm{int}^u(U_i)$ denotes the interior of U_i in the set $W^{su}(z_i)\cap\Omega_0$ and $\mathrm{int}^s(S_i)$ is defined similarly. For $x\in U_i$ and $y\in S_i$, we write $[x,y]$ for the unique local intersection of $W^{ss}(x)$ and $W^u(y)$. We call the following sets *rectangles*:

$$R_i=[U_i,S_i]:=\{[x,y]:x\in U_i,y\in S_i\}$$

and denote their interiors by $\mathrm{int}(R_i)=[\mathrm{int}^u(U_i),\mathrm{int}^s(S_i)]$. Note that $U_i=[U_i,z_i]\subset R_i$.

Given a disjoint union $R=\cup_i R_i$ of rectangles such that $RA=\Omega_0$, the first return time $\tau:R\to\mathbb{R}_{>0}$ and the first return map $\mathcal{P}:R\to R$ are given by

$$\tau(x):=\inf\{t>0:xa_t\in R\}\quad\text{and}\quad\mathcal{P}(x):=xa_{\tau(x)}.$$

The associated transition matrix A is the $k\times k$ matrix given by

$$\mathsf{A}_{lm}=\begin{cases}1 & \text{if } \mathrm{int}(R_l)\cap\mathcal{P}^{-1}\mathrm{int}(R_m)\neq\emptyset\\ 0 & \text{otherwise.}\end{cases}$$

Fix $\epsilon>0$ much smaller than the injectivity radius of $\Gamma_0\backslash G$. By Ratner [54] and Bowen [9], we have a Markov section for the flow a_t of size ϵ, that is, a family $\mathcal{R}=\{R_1,\ldots,R_k\}$ of disjoint rectangles satisfying the following:

(1) $\Omega_0=\cup_1^k R_i a_{[0,\epsilon]}$
(2) the diameter of each R_i is at most ϵ, and
(3) for any $i\neq j$, at least one of $R_i\cap R_j a_{[0,\epsilon]}$ or $R_j\cap R_i a_{[0,\epsilon]}$ is empty.
(4) $\mathcal{P}([\mathrm{int}^u U_i,x])\supset[\mathrm{Int}^u U_j,\mathcal{P}(x)]$ and $\mathcal{P}([x,\mathrm{Int}^s S_i]))\subset[\mathcal{P}(x),\mathrm{Int}^s S_j]$ for any $x\in\mathrm{int}(R_i)\cap\mathcal{P}^{-1}(\mathrm{int}(R_j))$.
(5) A is aperiodic, i.e. for some $N\geq 1$, all the entries of A^N are positive.

Set $R:=\cup R_i$ and $U:=\cup U_i$.

Let Σ be the space of bi-infinite sequences $x\in\{1,\ldots,k\}^{\mathbb{Z}}$ such that $\mathsf{A}_{x_l x_{l+1}}=1$ for all l. We denote by Σ^+ the space of one sided sequences

$$\Sigma^+=\{(x_i)_{i\geq 0}:\mathsf{A}_{x_i,x_{i+1}}=1\text{ for all }i\geq 0\}.$$

Non-negative coordinates of $x\in\Sigma$ will be referred to as future coordinates of x. A function on Σ which depends only on future coordinates can be regarded as a function on Σ^+.

We will write $\sigma:\Sigma\to\Sigma$ for the shift map $(\sigma x)_i=x_{i+1}$. By abuse of notation we will also denote by σ the shift map acting on Σ^+.

For $\beta\in(0,1)$, we can give a metric d_β on Σ (resp. on Σ^+) by

$$d_\beta(x,x')=\beta^{\inf\{|j|:x_j\neq x'_j\}}.$$

Definition 2.2 (The map $\zeta:\Sigma\to R$)**.** Let $\hat{R}$ be the set of $x\in R$ such that $\mathcal{P}^m x\in\mathrm{int}(R)$ for all $m\in\mathbb{Z}$. For $x\in\hat{R}$, we obtain a sequence $\omega=\omega(x)\in\Sigma$ such that $\mathcal{P}^k x\in R_{\omega_k}$ for all $k\in\mathbb{Z}$. The set $\hat{\Sigma}:=\{\omega(x):x\in\hat{R}\}$ is a residual set in Σ. This map $x\mapsto\omega(x)$ is injective on $\hat{R}$ as the distinct pair of geodesics diverge from each other either in positive or negative time. We can extend $\omega^{-1}:\hat{\Sigma}\to\hat{R}$ a continuous surjective function $\zeta:\Sigma\to R$, which intertwines σ and $\mathcal{P}$.

Definition 2.3 (The map $\zeta^+ : \Sigma^+ \to U$)**.** Let $\hat{U}$ be the set of $u \in U$ such that $\hat{\sigma}^m u \in \mathrm{int}^u(U)$ for all $m \in \mathbb{N} \cup \{0\}$. Similarly to the above, we can define an injective map $\hat{U} \to \Sigma^+$, and then a continuous surjection $\zeta^+ : \Sigma^+ \to U$.

For β sufficiently close to 1, the embeddings ζ and ζ^+ are Lipschitz. We fix such a β once and for all. The space $C_\beta(\Sigma)$ (resp. $C_\beta(\Sigma^+)$) of d_β-Lipschitz functions on Σ (resp. on Σ^+) is a Banach space with the usual Lipschitz norm

$$||\psi||_{d_\beta} = \sup |\psi| + \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{d_\beta(x, y)}. \tag{2.4}$$

We denote $\tau \circ \zeta \in C_\beta(\Sigma)$ by τ by abusing the notation.

We form the suspension

$$\Sigma^\tau := \Sigma \times \mathbb{R}/(x, s) \sim (\sigma x, s - \tau(x))$$

and the suspension flow on Σ^τ is given by $[(x, s)] \mapsto [(x, t + s)]$. The map $[(x, t)] \to \zeta(x)a_t$ is a semi-conjugacy $\Sigma^\tau \to \Omega_0$ intertwining the suspension flow and the geodesic flow a_t.

For $g \in C_\beta(\Sigma)$, called the potential function, the pressure of g is defined as

$$\mathrm{Pr}_\sigma(g) := \sup_\mu \left(\int g d\mu + \mathrm{entropy}_\mu(\sigma) \right)$$

over all σ-invariant Borel probability measures μ on Σ, where $\mathrm{entropy}_\mu(\sigma)$ denotes the measure theoretic entropy of σ with respect to μ. The critical exponent δ is the unique positive number such that $\mathrm{Pr}(-\delta\tau) = 0$. Let ν denote the unique equilibrium measure for $-\delta\tau$, i.e. $\delta \int \tau d\nu = \mathrm{entropy}_\nu(\sigma)$.

The BMS measure $\mathsf{m}^{\mathrm{BMS}}$ on $\Omega_0 \simeq \Sigma^\tau$ being the unique probability measure of maximal entropy for the geodesic flow, corresponds to the measure locally given by $\frac{1}{\int \tau d\nu}(d\nu \times ds)$ where ds is the Lebesgue measure on $\mathbb{R}$.

For a map g on Σ or on Σ^+ and $n \geq 1$, we write

$$g_n(x) = g(x) + g(\sigma(x)) + \cdots + g(\sigma^{(n-1)}(x)).$$

2.3. $\mathbb{Z}^d \times \mathbb{R}$**-suspension space** $\Sigma^{f,\tau}$**.** Note that X is a regular $\mathbb{Z}^d$-cover of the convex cocompact manifold X_0. Let p denote the canonical projection map $\mathrm{T}^1(X) \to \mathrm{T}^1(X_0)$. Then

$$\Omega_X := p^{-1}(\Omega_0) \simeq \mathbb{Z}^d \times \Omega_0$$

is the support of the BMS measure in $\mathrm{T}^1(X)$. We enumerate the group of deck transformation $\Gamma\backslash\Gamma_0$ for the covering map $X \to X_0$ as $\{D_\xi : \xi \in \mathbb{Z}^d\}$ so that $D_{\xi_1} \circ D_{\xi_2} = D_{\xi_1 + \xi_2}$. Note D_ξ acts on $\Gamma\backslash G/M$ as well as on $\Gamma\backslash G$.

Definition 2.5. Fix a precompact and connected fundamental domain $\mathcal{F} \subset \Gamma\backslash G/M$ for the $\mathbb{Z}^d$-action on $\mathrm{T}^1(X)$.

(1) We define the $\mathbb{Z}^d$-coordinate of $x \in \mathrm{T}^1(X)$ relative to $\mathcal{F}$ to be the unique $\xi \in \mathbb{Z}^d$ such that $x \in D_\xi(\mathcal{F})$, and write $\xi(x : t)$ for the $\mathbb{Z}^d$-coordinate of xa_t.

(2) Choose a continuous section, say s, from R into $\mathcal{F}$. Define $f : \Sigma \to \mathbb{Z}^d$ as follows: for $x \in \Sigma$,

$$f(x) = \xi(\mathsf{s} \circ \zeta(x) : \tau(x)),$$

that is, the $\mathbb{Z}^d$-coordinate of $\mathsf{s}(\zeta(x))a_{\tau(x)}$.

Note that $f(x)$ depends only on the two coordinates x_0 and x_1, and if $\sigma^n(x) = x$, then $f_n(x)$ is the Frobenius element of the closed geodesic in $\mathrm{T}^1(X_0)$ given by x.

Consider the suspension space

$$\Sigma^{f,\tau} := \Sigma \times \mathbb{Z}^d \times \mathbb{R}/(x, \xi, s) \sim (\sigma(x), \xi + f(x), s - \tau(x))$$

with the suspension flow $[(x, \xi, s)] \to [(x, \xi, s + t)]$. The map $\Sigma^{f,\tau} \to \Omega_X$ given by

$$[(x, \xi, t)] \mapsto D_\xi(\mathsf{s} \circ \zeta(x))a_t$$

is a Lipschitz surjective map intertwining the suspension flow and the geodesic flow.

If $\sigma^n(x) = x$, then

$$(x, \xi, s + \tau_n(x)) \sim (\sigma^n(x), \xi + f_n(x), s) = (x, \xi + f_n(x), s).$$

Hence $[(x, \xi, s)]$ gives rise to a periodic orbit if and only if $\sigma^n(x) = x$ and $f_n(x) = 0$ for some $n \in \mathbb{N}$.

2.4. **$\mathbb{Z}^d \times M \times \mathbb{R}$-suspension space $\Sigma^{f,\theta,\tau}$.** The homogeneous space $\Gamma\backslash G$ is a principal M-bundle over $\mathrm{T}^1(X) = \Gamma\backslash G/M$. Now take a smooth section $\mathsf{S} : \mathsf{s}(R) \to \Gamma\backslash G$ by trivializing the bundle locally.

Definition 2.6. Define $\theta : \Sigma \to M$ as follows: for $x \in \Sigma$, $\theta(x) \in M$ is the unique element satisfying

$$(\mathsf{S} \circ \mathsf{s} \circ \zeta)(x)a_{\tau(x)} = D_{f(x)}(\mathsf{S} \circ \mathsf{s} \circ \zeta)(\sigma(x))\theta(x)^{-1}.$$

We choose the section S a bit more carefully so that the resulting holonomy map θ depends only on future coordinates: first trivialize the bundle over each $\mathsf{s}(U_j)$ and extend the trivialization to $\mathsf{s}(R_j)$ by requiring $(\mathsf{S} \circ \mathsf{s})([u, s_1])$ and $(\mathsf{S} \circ \mathsf{s})([u, s_2])$ be forward asymptotic for all $u \in U_j$ and $s_1, s_2 \in S_j$.

If $x \in \Sigma$ has period n, then $\theta_n(x)^{-1}$ is in the same conjugacy class as the holonomy associated to the closed geodesic $\zeta(x)A$.

We set Ω to be the preimage of Ω_X under the projection $\Gamma\backslash G \to \Gamma\backslash G/M$; this is precisely the support of $\mathsf{m}^{\mathrm{BMS}}$ defined as a measure on $\Gamma\backslash G$ in the previous section.

Consider the suspension space

$$\Sigma^{f,\theta,\tau} := \Sigma \times \mathbb{Z}^d \times M \times \mathbb{R}/(x, \xi, m, s) \sim (\sigma x, \xi + f(x), \theta^{-1}(x)m, s - \tau(x))$$

with the suspension flow $[(x, \xi, m, s)] \mapsto [(x, \xi, m, s+t)]$. Now the map $\pi : \Sigma^{f,\theta,\tau} \to \Omega$ given by

$$[(x, \xi, m, t)] \mapsto D_\xi(\mathsf{S} \circ \mathsf{s} \circ \zeta(x))ma_t$$

is a Lipschitz surjective map intertwining the suspension flow and the a_t-flow.

The BMS measure $\mathsf{m}^{\mathrm{BMS}}$ on Ω corresponds to the measure locally given by the product of ν and the Haar measure on $\mathbb{Z}^d \times M \times \mathbb{R}$:

$$d\mathsf{m}^{\mathrm{BMS}} = \frac{1}{\int \tau d\nu}\pi_*(d\nu d\xi dm ds).$$

3. Analytic properties of Ruelle operators $L_{z,v,\mu}$

We continue the setup and notations from section 2 for $G, \Gamma, \Gamma_0, \Sigma, M, \tau, \theta, \delta$ etc. In particular, Γ is a normal subgroup of a Zariski dense convex cocompact subgroup Γ_0 with $\Gamma\backslash\Gamma_0 \simeq \mathbb{Z}^d$ for some $d \geq 0$. We note that the functions $\tau : \Sigma \to \mathbb{R}_{>0}$ and $\theta : \Sigma \to M$ depend only on future coordinates, and hence we may regard them as functions on Σ^+.

For $\psi \in C_\beta(\Sigma^+)$, the Ruelle operator $L_\psi : C_\beta(\Sigma^+) \to C_\beta(\Sigma^+)$ is defined by

$$L_\psi(g)(x) = \sum_{\sigma(y)=x} e^{-\psi(y)} g(y).$$

The Ruelle-Perron-Frobenius theorem implies the following (cf. [47]):

Theorem 3.1. *(1) 1 is the unique eigenvalue of the maximum modulus of $L_{\delta\tau}$, and the corresponding eigenfunction $h \in C_\beta(\Sigma^+)$ is positive.*

(2) The remainder of the spectrum of $L_{\delta\tau}$ is contained in a disc of radius strictly smaller than 1.

(3) There exists a unique probability measure ρ on Σ^+ such that $L^*_{\delta\tau}(\rho) = \rho$, i.e. $\int L_{\delta\tau}\psi d\rho = \int \psi d\rho$, and $h d\rho = d\nu$.

3.1. **Three-parameter Ruelle operators on vector-valued functions.** Denote by $\hat{M}$ the unitary dual of M, i.e. the space of all irreducible unitary representations (μ, W) of M up to isomorphism. As M is compact, they are precisely irreducible finite dimensional representations of M. We write $\mu = 1$ for the trivial representation. Similarly, $\hat{\mathbb{Z}}^d$ denotes the unitary dual of $\mathbb{Z}^d$. We identify $\hat{\mathbb{Z}}^d$ with $\mathbf{T}^d := (\mathbb{R}/(2\pi\mathbb{Z}))^d$ via the isomorphism $\mathbf{T}^d \to \hat{\mathbb{Z}}^d$ given by $\chi_v(\xi) = e^{i\langle v, \xi\rangle}$. In our study of the correlation function of the suspension flow on $\Omega = \Sigma \times \mathbb{Z}^d \times M \times \mathbb{R}/\sim$ with respect to the BMS measure, an understanding of the spectrum of *three* parameter Ruelle operators indexed by triples $(z, v, \mu) \in \mathbb{C} \times \mathbf{T}^d \times \hat{M}$ will play a crucial role.

Definition 3.2. For each triple $(z, v, (\mu, W)) \in \mathbb{C} \times \mathbf{T}^d \times \hat{M}$, define the transfer operator

$$L_{z,v,\mu} : C_\beta(\Sigma^+, W) \to C_\beta(\Sigma^+, W) \tag{3.3}$$

by

$$L_{z,v,\mu}(g)(x) = \sum_{\sigma(y)=x} e^{-z\tau(y)+i\langle v, f(y)\rangle} \mu(\theta(y)) g(y),$$

where $C_\beta(\Sigma^+, W)$ denotes the Banach space of W-valued Lipschitz maps with Lipschitz norm defined analogously as (2.4) using a Hermitian norm on W.

We write $L_{z,v}$ for $L_{z,v,1}$ and L_z for $L_{z,0,1}$ for simplicity.

Denoting the center of M by $Z(M)$, the following is well-known:

Lemma 3.4. *The group $Z(M)$ is at most 1-dimensional and*

$$M = Z(M)[M, M].$$

Hence we may identify $Z(M)$ with $M/[M, M]$. We denote by $[m] \in Z(M) = M/[M, M]$ for the projection of $m \in M$. If μ is one-dimensional, then μ is determined by $\mu|_{Z(M)}$. If $Z(M)$ is non-trivial, then $Z(M) \simeq \mathbb{R}/(2\pi\mathbb{Z})$, which we may identify with $[0, 2\pi)$, and hence any one dimensional unitary representation μ is of the form $\chi_p(m) = e^{ip[m]}$ for some integer $p \in \mathbb{Z}$. In this case, we write $L_{z,v,p}$ for $L_{z,v,\mu}$.

3.2. **Spectrum of Ruelle operators.** The aim of this subsection is to prove Theorem 3.14 on analytic properties of $L_{\delta+it,v,\mu}$. We denote by $\mathrm{Fix}(\sigma^n)$ the set of $y \in \Sigma^+$ fixed by σ^n. The following proposition is a key ingredient in understanding the spectrum of $L_{\delta+it,v,\mu}$'s.

Proposition 3.5. *(1) There exists $y \in \Sigma^+$ such that $\{(\sigma^n(y), \theta_n(y)) \in \Sigma^+ \times M : n \in \mathbb{N}\}$ is dense in $\Sigma^+ \times M$.*

(2) There exists $y \in \mathrm{Fix}(\sigma^n)$ for some n with $f_n(y) = 0$ such that $[\theta_n(y)]$ generates a dense subgroup in $Z(M)$.

Proof. Claim (1) follows from the existence of a dense A^+ orbit in $\Omega_0 \subset \mathrm{T}^1(X_0)$ which is a consequence of the A-ergodicity of $\mathsf{m}^{\mathrm{BMS}}$ on Ω_0 [65] (cf. Appendix of this paper). The claim (2) is non-trivial only when $Z(M)$ is non-trivial; in this case, $Z(M) = \mathrm{SO}(2)$ by Lemma 3.4. Applying the work of Prasad and Rapinchuk [53] to Γ, we obtain a hyperbolic element $\gamma \in \Gamma$ that is conjugate to $a_\gamma m_\gamma \in AM$ and m_γ generates a dense subset in $Z(M) = M/[M,M]$. The element γ defines a closed geodesic in Ω_0 which again yields an element $y \in \mathrm{Fix}(\sigma^n)$, $f_n(y) = 0$ and $[\theta_n(y)] = [m_\gamma]$ for some n. This implies the claim. □

Lemma 3.6. *The subgroup generated by $\cup_{n\ge1}\{(\tau_n(y), f_n(y)) \in \mathbb{R}\times\mathbb{Z}^d : y \in \mathrm{Fix}(\sigma^n)\}$ is dense in $\mathbb{R}\times\mathbb{Z}^d$.*

Proof. Denote by H the subgroup in concern. The projection of H to $\mathbb{Z}^d$ is surjective by the construction of f. Therefore it suffices to show that $H \cap (\mathbb{R}\times\{0\})$ is $\mathbb{R}$. This follows because the length spectrum of Γ is non-arithmetic [29]. □

We will denote by $\sigma_0(L_{z,v,\mu})$ the spectral radius of the operator $L_{z,v,\mu}$ on $C_\beta(\Sigma^+, W)$.

Proposition 3.7. *Let $(\mu, W) \in \hat{M}$, and $(t, v) \in \mathbb{R}\times\mathbf{T}^d$.*

(1) We have $\sigma_0(L_{\delta+it,v,\mu}) \le 1$.

(2) If $\sigma_0(L_{\delta+it,v,\mu}) = 1$, then $L_{\delta+it,v,\mu}$ has a simple eigenvalue of modulus one and μ is 1-dimensional.

Proof. (1) and the first part of (2) follow from Theorems 8.1 and 8.3 of [47].

Suppose $\sigma_0(L_{\delta+it,v,\mu}) = 1$. Then for some $w \in C_\beta(\Sigma^+, W)$ and $b \in \mathbb{R}$,

$$L_{\delta+it,v,\mu} w = e^{ib} w.$$

Using the convexity argument (as in p.54 of [47]), it follows that

$$e^{i(-t\cdot r(y) + \langle v, f(y)\rangle)}\mu(\theta(y))w(y) = e^{ib} w(\sigma(y))$$

for all $y \in \Sigma^+$. In other words,

$$e^{i(-t\cdot r(y) + \langle v, f(y)\rangle - b)} w(y) = \mu(\theta(y))^{-1} w(\sigma(y)).$$

Consider the function g on $\Sigma^+ \times M$:

$$g(y, m) = \mu(m)^{-1} w(y).$$

Then

$$\begin{aligned} g(\sigma(y), \theta(y)m) &= \mu(m)^{-1}\mu(\theta(y)^{-1}) w(\sigma(y)) \\ &= \mu(m^{-1}) e^{i(-t\cdot r(y)+\langle v, f(y)\rangle - b)} w(y) = e^{i(-t\cdot r(y)+\langle v, f(y)\rangle - b)} g(y, m). \end{aligned}$$

Writing $w_0 := g(y, e)$, we have that for all n, $g(\sigma^n(y), \theta_n(y))$ lies in the compact set $\{e^{ia} w_0 : a \in \mathbb{R}\}$.

Let y be an element such that the set $\{(\sigma^n(y), \theta_n(y)) : n \in \mathbb{N}\}$ is dense in $\Sigma^+ \times M$ given by Proposition 3.5. It follows that $g(\Sigma^+ \times M) \subset \{e^{ia} w_0\}$. This implies that μ is 1-dimensional. □

We will use the following simple observation by considering the reversing the orientation of a closed geodesic in $\mathrm{T}^1(X_0)$:

Lemma 3.8. *For any $y \in \mathrm{Fix}(\sigma^n)$, there exists $y' \in \mathrm{Fix}(\sigma^n)$ such that $\tau_n(y) = \tau_n(y')$, $f_n(y) = -f_n(y')$ and $[\theta_n(y)] = [\theta_n(y')]$.*

We will repeatedly use the following result of Pollicott [50, Prop. 2]: let $\psi = \mathsf{u} + i\mathsf{v} \in C_\beta(\Sigma^+, \mathbb{C})$ and consider the complex Ruelle operator $\mathcal{L}_\psi$ given by $\mathcal{L}_\psi(h)(x) = \sum_{\sigma(y)=x} e^{\psi(y)} h(y)$. Suppose that $\mathcal{L}_{\mathsf{u}} 1 = 1$.

Lemma 3.9. *For $0 \le a < 2\pi$, $\mathcal{L}_\psi$ has an eigenvalue $e^{ia + \mathrm{Pr}_\sigma(\mathsf{u})}$ if and only if there exists $\omega \in C(\Sigma^+)$ such that*

$$\mathsf{v} - a = \omega - \omega \circ \sigma + L$$

where $L : \Sigma^+ \to 2\pi\mathbb{Z}$ is a lattice function.

Proposition 3.10. *If $L_{\delta+it,v,\mu}$ has an eigenvalue e^{ia} for some $(v, \mu) \in \mathbf{T}^d \times \hat{M}$, then there exists some integer $p \in \mathbb{Z}$ such that $\mu(m) = e^{ip[m]}$ for all $m \in M$, and $\bigcup_{n \in \mathbb{Z}} \{t\tau_n(y) - p[\theta_n(y)] + na : y \in \mathrm{Fix}(\sigma^n)\} \subset \pi\mathbb{Z}$.*

Proof. Assume that $L_{\delta+it,v,\mu}$ has eigenvalue e^{ia}. By Proposition 3.7, μ is 1-dimensional, i.e. $\mu(m) = e^{ip[m]}$ for some integer $p \in \mathbb{Z}$. Therefore for $g \in C_\beta(\Sigma^+, \mathbb{C})$,

$$L_{\delta+it,v,\mu}(g)(x) = \sum_{\sigma(y)=x} e^{-(\delta+it)\tau(y) + i\langle v, f(y)\rangle + ip[\theta(y)]} g(y).$$

By Lemma 3.9, the function

$$-t \cdot \tau(y) + \langle v, f(y)\rangle + p[\theta(y)]$$

is cohomologous to a function $a + L(y)$ where $L : \Sigma^+ \to 2\pi\mathbb{Z}$ is a lattice function. Fixing any $y \in \mathrm{Fix}(\sigma^n)$, we have

$$-t \cdot \tau_n(y) + \langle v, f_n(y)\rangle + p[\theta_n(y)] - na \in 2\pi\mathbb{Z}.$$

By Lemma 3.8, we have $y' \in \mathrm{Fix}(\sigma^n)$ with $f_n(y') = -f_n(y)$, $\tau_n(y) = \tau_n(y')$ and $[\theta_n(y)] = [\theta_n(y')]$.

$$-t \cdot \tau_n(y) - \langle v, f_n(y)\rangle + p[\theta_n(y)] - na \in 2\pi\mathbb{Z}.$$

Adding the above two terms, we get $-2t\tau_n(y) + 2p[\theta_n(y)] - 2na \in 2\pi\mathbb{Z}$. This proves the claim. $\square$

Theorem 3.11. *Let $(t, v, \mu) \in \mathbb{R} \times \mathbf{T}^d \times \hat{M}$.*

1. *If $L_{\delta,v,\mu}$ has an eigenvalue e^{ia}, then $v = 0$ mod $\pi\mathbb{Z}^d$. Furthermore, if $\mu = 1$ and $v \neq 0$, then $e^{ia} = -1$; if $\mu \neq 1$, then a is an irrational multiple of π.*
2. *Let $t \neq 0$. If $L_{\delta+it,v,\mu}$ has an eigenvalue e^{ia}, then $v = 0$ mod $\pi\mathbb{Z}^d$ and a is an irrational multiple of π.*

In each case, e^{ia} is a maximal simple eigenvalue.

Proof. Suppose $e^{2\pi ai}$ is an eigenvalue of $L_{\delta+it,v,\mu}$ for some $a \in \mathbb{R}$. Since $|e^{2\pi ai}| = 1 = e^{\mathrm{Pr}_\sigma(-\delta\tau)}$, the eigenvalue is maximal simple by the complex RPF theorem (see [47, Thm. 4.5]). First note that μ is one-dimensional; $\mu(m) = e^{ip[m]}$ for some integer $p \in \mathbb{Z}$. By Lemma 3.9, we have for any $y \in \mathrm{Fix}(\sigma^n)$,

$$\langle v, f_n(y)\rangle + p[\theta_n(y)] - na \in 2\pi\mathbb{Z}. \tag{3.12}$$

Using Lemma 3.8, we get

$$-\langle v, f_n(y)\rangle + p[\theta_n(y)] - na \in 2\pi\mathbb{Z}.$$

By subtracting one from the other, we get $\langle v, f_n(y)\rangle \subset \pi\mathbb{Z}$. As $\cup_n\{f_n(y) : y \in \mathrm{Fix}(\sigma^n)\}$ generates $\mathbb{Z}^d$, it follows that $v = 0 \bmod \pi\mathbb{Z}^d$.

Now we prove the rest of (1). Suppose $p = 0$ and $v \neq 0$. It follows from from (3.12) that

$$na \in \pi\mathbb{Z} \text{ for all } n \in \mathbb{N} \text{ with } \mathrm{Fix}(\sigma^n) \neq \emptyset.$$

Since the transition matrix A is aperiodic, $\{n \in \mathbb{N} : \mathrm{Fix}(\sigma^n) \neq \emptyset\}$ contains all sufficiently large integers. Therefore $a = 0$ or π. However $a = 0$ implies $v = 0$. Hence $a = \pi$.

Suppose $p \neq 0$. Then

$$p[\theta_n(y)] - na \in \pi\mathbb{Z}.$$

Since $\{[\theta_n(y)] : y \in \mathrm{Fix}(\sigma^n)\}$ generates $Z(M)$ by [22, Thm. 1.9], a must be an irrational multiple of π. This shows (1).

In order to show (2), if a were a rational multiple of π and $t \neq 0$, it follows from Proposition 3.10 that for some integer p, the union $\cup_{n\geq 1}\{t\tau_n(y) - p[\theta_n(y)] : y \in \mathrm{Fix}(\sigma^n)\}$ would be contained in $q\pi\mathbb{Z}$ for some $q \in \mathbb{Q}$. If $p = 0$ or $Z(M) = \{e\}$, this contradicts Lemma 3.6. Otherwise, we get $\cup_{n\geq 1}\{e^{it\tau_n(y)-ip[\theta_n(y)]} : y \in \mathrm{Fix}(\sigma^n)\} \subset F$ for some finite subgroup F of $\{e^{i\theta} : \theta \in [0, 2\pi)\}$. This contradicts [22, Thm. 1.9]. □

The following result from the analytic perturbation theory of bounded linear operators is an important ingredient in our subsequent analysis.

Theorem 3.13 (Perturbation theorem)**.** [26] *Let $B(V)$ be the Banach algebra of bounded linear operators on a complex Banach space V. If $L_0 \in B(V)$ has a simple isolated eigenvalue λ_0 with a corresponding eigenvector v_0, then for any $\epsilon > 0$, there exists $\eta > 0$ such that if $L \in B(V)$ with $\|L - L_0\| < \eta$ then L has a simple isolated eigenvalue $\lambda(L)$ and corresponding eigenvector $v(L)$ with $\lambda(L_0) = \lambda_0$, $v(L_0) = v_0$ and such that*

(1) $L \mapsto \lambda(L)$, $L \mapsto v(L)$ are analytic for $\|L - L_0\| < \eta$;
(2) for $\|L - L_0\| < \eta$, $|\lambda(L) - \lambda_0| < \epsilon$ and $\mathrm{spec}(L) - \lambda(L) \subset \{z \in \mathbb{C} : |z - \lambda_0| > \epsilon\}$. Moreover if $\mathrm{spec}(L_0) - \{\lambda_0\}$ is contained in the interior of a circle C centered at 0 and $\eta > 0$ is sufficiently small, then $\mathrm{spec}(L) - \lambda(L)$ is also contained in the interior of C.

Finally we are ready to prove:

Theorem 3.14. *Let $\mu \in \hat{M}$. Consider the map*

$$(s, w) \in \mathbb{R} \times \mathbb{R}^d \mapsto \sum_{n=0}^{\infty} L^n_{\delta+is,w,\mu} = (1 - L_{\delta+is,w,\mu})^{-1}.$$

(1) Let $\mu \neq 1$. For any $(t, v) \in \mathbb{R} \times \mathbf{T}^d$, there exists a neighborhood $\mathcal{O} \subset \mathbb{R} \times \mathbf{T}^d$ of (t, v) and an analytic map

$$(s, w) \in \mathcal{O} \mapsto H_{s,w} \in \mathrm{Hom}(C_\beta(\Sigma^+, \mathbb{C}), C_\beta(\Sigma^+, \mathbb{C}))$$

such that $(1 - L_{\delta+is,w,\mu})^{-1}$ agrees with $H_{s,w}$ on $\mathcal{O} \setminus \{(s, w) : w = 0 \bmod \pi\mathbb{Z}^d\}$.

(2) *Let $\mu = 1$. For any $(t, v) \in \mathbb{R}\backslash\{0\} \times \mathbf{T}^d$, there exists a neighborhood $\mathcal{O} \subset \mathbb{R} \times \mathbf{T}^d$ of (t, v) and an analytic map*
$$(s, w) \in \mathcal{O} \mapsto H_{s,w} \in \mathrm{Hom}(C_\beta(\Sigma^+, \mathbb{C}), C_\beta(\Sigma^+, \mathbb{C}))$$
such that $(1-L_{\delta+is,w,\mu})^{-1}$ agrees with $H_{s,w}$ on $\mathcal{O}\backslash\{(s, w) : w = 0 \bmod \pi\mathbb{Z}^d\}$.

(3) *Let $\mu = 1$. There exists a neighborhood $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^d$ of $(0, 0)$ such that for all non-zero $(s, w) \in \mathcal{O}$, we have*
$$(1 - L_{\delta+is,w})^{-1} = \frac{P_{s,w}}{1 - \lambda_{\delta+is,w}} + Q_{s,w}$$
where $\lambda_{\delta+is,w}$ is the unique eigenvalue of $L_{\delta+is,w}$ of maximum modulus obtained by the perturbation theorem 3.13, $P_{s,w}$ and $Q_{s,w}$ are analytic maps from $\mathcal{O}$ to $\mathrm{Hom}(C_\beta(\Sigma^+, \mathbb{C}), C_\beta(\Sigma^+, \mathbb{C}))$.

Proof. If $\sigma_0(L_{\delta+it,v,\mu}) < 1$, then, by the perturbation theorem 3.13, there is a neighborhood $\mathcal{O}$ of (t, v) in $\mathbb{R} \times \mathbb{R}^d$ such that $\sigma_0(L_{\delta+is,w,\mu}) < 1$ for any $(s, w) \in \mathcal{O}$. This implies that $\sum_{n=0}^{\infty} L^n_{\delta+is,w,\mu}$ converges absolutely and hence analytic on $\mathcal{O}$.

Now suppose $\sigma_0(L_{\delta+it,v,\mu}) = 1$. In any of the following three case (1) $\mu \neq 1$, (2) $\mu = 1$ and $t \neq 0$ or (3) $\mu = 1, t = 0, v \in \mathbf{T}^d\backslash\{0\}$, by Theorem 3.11, $L_{\delta+it,v,\mu}$ has a simple eigenvalue e^{ia} of maximum modulus with a some irrational multiple π or π. By Theorem 3.13, there exists a neighborhood $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^d$ of (t, v) such that for any $(s, w) \in \mathcal{O}$, $L_{\delta+is,w,\mu}$ can be written as
$$L_{\delta+is,w,\mu} = \lambda_{\delta+is,w,\mu} P_{\delta+is,w,\mu} + N_{\delta+is,w,\mu}$$
where $\lambda_{\delta+is,w\mu}$ is the simple maximal eigenvalue of $L_{\delta+is,w,\mu}$, $P_{\delta+is,w}$ is the eigenprojection to the eigenspace associated to $\lambda_{\delta+is,w,\mu}$ and $\sigma_0(N_{\delta+is,w,\mu}) < 1$. Moreover, $\lambda_{\delta+is,w\mu}$, $P_{\delta+is,w,\mu}$, $N_{\delta+is,w,\mu}$ are analytic on $\mathcal{O}$. Hence for all $n \in \mathbb{N}$,
$$L^n_{\delta+is,w,\mu} = \lambda^n_{\delta+is,w,\mu} P_{\delta+is,w,\mu} + N^n_{\delta+is,w,\mu}.$$
By choosing $\mathcal{O}$ sufficiently small, we have $\sum_n N^n_{\delta+is,w,\mu}$ converges absolutely on $\mathcal{O}$, and the map
$$\frac{P_{\delta+is,w,\mu}}{1 - \lambda_{\delta+is,w,\mu}} + \sum_n N^n_{\delta+is,w,\mu} \tag{3.15}$$
is analytic on $\mathcal{O}$. Note that on $\mathcal{O}\backslash\{(s, w) : w = 0 \bmod \pi\mathbb{Z}^d\}$, by Proposition 3.7 and Theorem (3.11), the spectral radius of $L_{\delta+is,w,\mu}$ is strictly less than 1. Hence $\sum_n L^n_{\delta+is,w,\mu}$ converges absolutely on this set. We have (3.15) agrees with $(1 - L_{\delta+is,w,\mu})^{-1}$ on $\mathcal{O}\backslash\{(s, w) : w = 0 \bmod \pi\mathbb{Z}^d\}$.

Suppose $(t, v, \mu) = (0, 0, 1)$. Then the map $\sum \lambda^n_{\delta+is,w,\mu} = (1 - \lambda_{\delta+is,w,\mu})^{-1}$ is analytic on $\mathcal{O} - \{(0, 0)\}$ and hence (3.15) is analytic on $\mathcal{O} - \{(0, 0)\}$. This finishes the proof. □

3.3. **Asymptotic expansion.** For each $u \in \mathbb{R}^d$ close to 0, there exists a unique
$$P(u) \in \mathbb{R}$$
such that the pressure of the function $x \mapsto -P(u)\tau(x) + \langle u, f(x)\rangle$ on Σ is 0. Moreover $P(0) = \delta$, $\nabla P(0) = 0$, the map $u \to P(u)$ is analytic, and the matrix $\nabla^2 P(0) = (\frac{\partial^2 P}{\partial u_i \partial u_j}(0))_{d\times d}$ is a positive definite matrix (cf. [52, Lem. 8]).

Set
$$\sigma = \det(\nabla^2 P(0))^{1/d}. \tag{3.16}$$

Let

$$C(0) = \int_{\mathbb{R}^d} e^{-\frac{1}{2}w^t \nabla^2 P(0) w} dw = \left(\frac{2\pi}{\sigma}\right)^{d/2}.$$

Remark 3.17. It follows from ([28] and [55]) that, for X_0 compact, the distribution $\frac{\xi(x:t)}{\sqrt{t}}$ as x ranges over the image of $\mathcal{F}$ in $\mathrm{T}^1(X_0)$ converges to the distribution of a multivariable Gaussian random variable N on $\mathbb{R}^d$ with a positive definite covariance matrix $\mathrm{Cov}(N) = \nabla^2 P(0)$.

Definition 3.18. Let $\mathcal{L}$ be the family of functions on $\Sigma^+ \times \mathbb{R}$ which are of the form $\Phi \otimes u$ where $\Phi \in C_\beta(\Sigma^+)$ and $u \in C_c(\mathbb{R})$.

In the rest of this subsection, we fix $(\mu, W) \in \hat{M}$ and $w \in W$.

For each $T > 1$ and $\Phi \otimes u \in \mathcal{L}$, define the W-valued functions

$$\mathsf{Q}^{(n)}_{\mu,w,T}(\Phi \otimes u) \quad \text{and} \quad \mathsf{Q}_{\mu,w,T}(\Phi \otimes u)$$

on $\Sigma^+ \times \mathbb{Z}^d \times M$ as follows: for $(x, \xi, m) \in \Sigma^+ \times \mathbb{Z}^d \times M$,

$$\begin{aligned}&\mathsf{Q}^{(n)}_{\mu,w,T}(\Phi \otimes u)(x, \xi, m)\\ &\qquad = \frac{1}{2\pi} \int_{t \in \mathbb{R}} e^{-iTt} \hat{u}(t) \left(\int_{v \in \mathbf{T}^d} e^{i\langle v, \xi\rangle} L^n_{\delta - it, v, \mu}(\Phi h \mu(m) w)(x) dv \right) dt\end{aligned}$$

and

$$\begin{aligned}&\mathsf{Q}_{\mu,w,T}(\Phi \otimes u)(x, \xi, m)\\ &\qquad = \frac{1}{2\pi} \int_{(t,v) \in \mathbb{R} \times \mathbf{T}^d} e^{-iTt + \langle v, \xi\rangle} \hat{u}(t) \cdot \left(\sum_{n \geq 0} L^n_{\delta - it, v, \mu}(\Phi h \mu(m) w)(x) \right) dv dt.\end{aligned}$$

Here $\hat{u}(t) = \int_{\mathbb{R}} e^{-ist} u(s) ds$. We set $\mathsf{Q}_T = \mathsf{Q}_{1,1,T}$ and $\mathsf{Q}^{(n)}_T = \mathsf{Q}^{(n)}_{1,1,T}$.

Theorem 3.19. *Let $\Phi \otimes u \in \mathcal{L}$ and $(x, \xi, m) \in \Sigma^+ \times \mathbb{Z}^d \times M$.*

(1) For each $T > 0$,

$$\sum_n \mathsf{Q}^{(n)}_{\mu,w,T}(\Phi \otimes u)(x, \xi, m) = \mathsf{Q}_{\mu,w,T}(\Phi \otimes u)(x, \xi, m)$$

where the convergence is uniform on compact subsets.

(2) We have

$$\lim_{T \to +\infty} T^{d/2} \mathsf{Q}_T(\Phi \otimes u)(x, \xi, m) = \frac{\hat{u}(0) C(0)}{\int \tau d\nu} \rho(\Phi h) h(x)$$

where the convergence is uniform on compact subsets.

(3) For any non-trivial $(\mu, W) \in \hat{M}$ and $w, w' \in W$, we have

$$\lim_{T \to +\infty} \langle T^{d/2} \mathsf{Q}_{\mu,w,T}(\Phi \otimes u)(x, \xi, m), w' \rangle = 0$$

where the convergence is uniform on compact subsets.

Proof. In proving this theorem, we may assume that the Fourier transform $\hat{u}$ belongs to $C^N_c(\mathbb{R})$ for some $N \geq \frac{d}{2} + 2$ (see [5, Lemma 2.4]). For (1), it is sufficient to show that $\|\hat{u}(t) \sum_{n \geq N} L^n_{\delta - it, v, \mu}(\Phi h \mu(m) w)(x)\|$ is dominated by a single absolutely integrable function of (t, v) almost everywhere.

We have

$$\begin{aligned}(3.20)\qquad &\|\hat{u}(t)\sum_{n\geq N} L^n_{\delta-it,v,\mu}(\Phi h\mu(m))(x)\| \\ \leq&|\hat{u}(t)|\cdot\|L^N_{\delta-it,v,\mu}\|\cdot\|\sum_{n=0}^{\infty} L^n_{\delta-it,v,\mu}(\Phi h\mu(m)w)(x)\|.\end{aligned}$$

When μ is nontrivial, $\sum_{n=0}^{\infty} L^n_{\delta-it,v,\mu}$ agrees almost everywhere with the operator $H_{\delta-it,v,\mu}$ which is described in Theorem 3.14. Noting that $\|H_{\delta-it,v,\mu}\|$ is bounded on compact sets (e.g. $\mathrm{supp}(\hat{u})\times \mathrm{T}^d$), we have

$$(3.20) \ll \|(\Phi h\mu(m)w)x\|\cdot|\hat{u}(t)|,$$

verifying (1) for the case when μ is nontrivial. We refer to Step 7 of [32, Appendix] for the proof of (1) for μ trivial.

To prove (2), consider the function

$$F(t,v) = e^{i\langle v,\xi\rangle}\sum_{n\geq 0} L^n_{\delta-it,v}(\Phi h)(x)$$

so that

$$\mathsf{Q}_T(\Phi\otimes u)(x,\xi,m) = \frac{1}{2\pi}\int_{(v,t)\in\mathbf{T}^d\times\mathbb{R}} e^{-iTt}\hat{u}(t)F(t,v)dvdt.$$

Let $\mathcal{O}\subset\mathbb{R}\times\mathbb{R}^d$ be a neighborhood of $(0,0)$ as in Theorem 3.14 (3) and choose any C^∞-function $\kappa(t,v)=\kappa_1(t)\kappa_2(v)$ supported in $\mathcal{O}$.

Since $F(t,v)$ is analytic outside $\mathcal{O}$ and $\hat{u}\in C^N(\mathbb{R})$, the following value of the Fourier transform is at most $O(T^{-N})$:

$$(3.21)\qquad \int_{t\in\mathbb{R}} e^{-iTt}\left(\int_{v\in\mathbf{T}^d}\hat{u}(t)(1-\kappa(v,t))F(t,v)dv\right)dt = O(T^{-N}).$$

We now need to estimate

$$\int_{t\in\mathbb{R}} e^{-iTt}\left(\int_{v\in\mathbf{T}^d}\hat{u}(t)\kappa(v,t)F(t,v)dv\right)dt.$$

This can be done almost identically to Step 5 in the appendix of [32]; we give a brief sketch of their arguments here for readers' convenience. On $\mathcal{O}$, we can write

$$F(t,v) = e^{i\langle v,\xi\rangle}\frac{P_{t,v}(\Phi h)(x)}{1-\lambda_{\delta-it,v}} + Q_{t,v}(\Phi h)(x)$$

where $\lambda_{\delta-it,v}$, $P_{t,v}$ and $Q_{t,v}$ are as described in Theorem 3.14 (3).

Applying Weierstrass preparation theorem to $1-\lambda_{\delta-it,v}$, we have that for $(t,v)\in\mathcal{O}$,

$$(3.22)\qquad 1-\lambda_{\delta-it,v} = A(t,v)(\delta-it-P(v)),$$

where A is non-vanishing and analytic in $\mathcal{O}$, by replacing $\mathcal{O}$ by a smaller neighborhood if necessary.

We have $P(0)=\delta$, $P(v)=P(0)-\frac{1}{2}v^t\nabla^2P(0)v+o(\|v\|^2)$ for v small, and $A(0,0)=-\frac{d\lambda(s,0)}{ds}|_{s=\delta}=\int\tau d\nu$. Set $R(v)=\delta-P(v)$.

For $(t,v)\in\mathcal{O}$, set

$$(3.23)\qquad a(t,v) := \frac{-\kappa_1(t)\kappa_2(v)\hat{u}(t)P_{t,v}(\Phi h)(x)}{A(t,v)}.$$

Suppose for now that $a(t,v)$ is of the form $c_x(t)b(v)$. Using $1/z = -\int_0^\infty e^{T'z}dT'$ for $\Re(z) < 0$, we get

$$\begin{aligned}
&\int_{t\in\mathbb{R}} e^{-iTt}\left(\int_{v\in\mathbf{T}^d} \hat{u}(t)\kappa(t,v)F(t,v)dv\right)dt\\
=&\int_{t\in\mathbb{R}} e^{-iTt}\left(\int_{v\in\mathbf{T}^d} \frac{a(t,v)e^{i\langle\xi,v\rangle}}{it-R(v)}dv\right)dt + O(T^{-N})\\
=&-\int_{t\in\mathbb{R}} e^{-iTt}\left(\int_{v\in\mathbf{T}^d} a(t,v)e^{i\langle\xi,v\rangle}\int_0^\infty e^{(it-R(v))T'}dT'dv\right)dt + O(T^{-N})\\
=&-\int_0^\infty\int_{t\in\mathbb{R}} e^{-i(T-T')t}\left(\int_{v\in\mathbf{T}^d} a(t,v)e^{i\langle\xi,v\rangle-R(v)T'}dv\right)dtdT' + O(T^{-N})\\
=&-\int_{T/2}^\infty \hat{c}_x(T-T')\int_{v\in\mathbf{T}^d} b(v)e^{i\langle\xi,v\rangle-R(v)T'}dvdT' + O(T^{-N}) = -\int_{-\infty}^{T/2}\tfrac{\hat{c}_x(T'')}{(T-T'')^{2/d}}\\
&\int_{v\in\sqrt{T-T''}\mathbf{T}^d} b(\tfrac{v}{\sqrt{T-T''}})e^{i\langle\xi,\frac{v}{\sqrt{T-T''}}\rangle-R(\frac{v}{\sqrt{T-T''}})(T-T'')}dvdT'' + O(T^{-N}).
\end{aligned}$$

Using $R(v) = \frac{1}{2}v^t\nabla^2P(0)v + o(\|v\|^2)$, and $C(0) = \int_{\mathbb{R}^d} e^{-\frac{1}{2}v^t\nabla^2P(0)v}dv$, the above is asymptotic to

$$\int_{-\infty}^{T/2}(T-T'')^{-2/d}\hat{c}_x(T'')C(0)b(0)dT'' = T^{-d/2}(2\pi c_x(0)C(0)b(0) + o(1)).$$

By approximating $a(t,v)$ by a sum of functions of the form $c_x(0)b(v)$ using Taylor series expansion, one obtains the following estimation:

(3.24)
$$\lim_{T\to\infty} T^{d/2}\int_{t\in\mathbb{R}} e^{-iTt}\left(\int_{v\in\mathbf{T}^d}\hat{u}(t)\kappa(t,v)F(t,v)dv\right) = \frac{2\pi\hat{u}(0)C(0)}{\int\tau d\nu}\rho(\Phi h)h(x).$$

Therefore, putting (3.21) and (3.24) together, we deduce

$$\lim_{T\to\infty} T^{d/2}\mathsf{Q}_T(\Phi\otimes u)(x,\xi,m) = \frac{\hat{u}(0)C(0)}{\int\tau d\nu}\rho(\Phi h)h(x),$$

verifying (2).

For (3), we have

$$\begin{aligned}
&\langle \mathsf{Q}_{\mu,w,T}(x,\xi,m)(\Phi\otimes u), w'\rangle\\
=&\frac{1}{2\pi}\int_{t\in\mathbb{R}} e^{-iTt}\hat{u}(t)\int_{v\in\mathbf{T}^d} e^{i\langle v,\xi\rangle}\langle (I-L_{\delta-it,v,\mu})^{-1}(\Phi h\mu(m)w)(x), w'\rangle dvdt.
\end{aligned}$$

Hence by Theorem 3.14, and the assumption that $\hat{u}$ is of class C^N $(N \geq d/2+2)$, the Fourier transform decays as:

$$\langle \mathsf{Q}_{\mu,w,T}(x,\xi,m)(\Phi\otimes u), w'\rangle = O(T^{-N})$$

which implies (3). $\square$

4. Local mixing and matrix coefficients for local functions

We retain the assumptions and notations from section 3. Recall the BMS measure $\mathsf{m}^{\mathrm{BMS}}$ on $\Gamma\backslash G$, and its support $\Omega \subset \Gamma\backslash G$. In this section, we study the asymptotic behavior of the correlation functions

$$\langle a_t\psi_1, \psi_2\rangle_{\mathsf{m}^{\mathrm{BMS}}} := \int_\Omega \psi_1(xa_t)\psi_2(x)d\mathsf{m}^{\mathrm{BMS}}(x)$$

and

$$\langle a_t\psi_1, \psi_2\rangle := \int_{\Gamma\backslash G} \psi_1(xa_t)\psi_2(x)d\mathsf{m}^{\mathrm{Haar}}(x)$$

for $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$.

4.1. **Correlation functions for $(\Omega, a_t, \mathsf{m}^{\mathrm{BMS}})$.** We use the suspension flow model for $(\Omega, a_t, \mathsf{m}^{\mathrm{BMS}})$ which was constructed in Section 2. That is, we identify the right translation action of a_t on Ω with the suspension flow on

$$\Sigma^{f,\theta,\tau} := \Sigma \times \mathbb{Z}^d \times M \times \mathbb{R}/\sim$$

where $\sim$ is given by $\zeta(x,\xi,m,s) = (\sigma x,,\xi + f(x), \theta^{-1}(x)m, s-\tau(x))$.

We write

$$\tilde{\Omega} := \Sigma \times \mathbb{Z}^d \times M \times \mathbb{R}, \quad \tilde{\Omega}^+ := \Sigma^+ \times \mathbb{Z}^d \times M \times \mathbb{R}$$

and

$$\Omega^+ := \Sigma^+ \times \mathbb{Z}^d \times M \times \mathbb{R}/\sim .$$

Consider the product measure on $\tilde{\Omega}$:

$$d\tilde{\mathsf{M}} := \frac{1}{\int \tau d\nu}(d\nu d\xi dm ds).$$

Recall that the BMS measure $\mathsf{m}^{\mathrm{BMS}}$ on Ω corresponds to the measure M on $\Sigma^{f,\theta,\tau}$ induced by $\tilde{\mathsf{M}}$.

Definition 4.1. Let $\mathcal{F}_0$ be the family of functions on $\tilde{\Omega}^+$ which are of the form

$$\Psi(x,\xi,m,s) = \Phi(x)\delta_{\xi_0}(\xi)u(s)\langle\mu(m)w_1, w_2\rangle$$

where $\Phi \in C_\beta(\Sigma^+)$, $u \in C_c(\mathbb{R})$, $\xi_0 \in \mathbb{Z}^d$, $(\mu, W) \in \hat{M}$ and $w_1, w_2 \in W$ are unit vectors. We will write $\Psi = \Phi \otimes \delta_{\xi_0} \otimes u \otimes \langle\mu(\cdot)w_1, w_2\rangle$.

For $\Psi_1, \Psi_2 \in C_c(\tilde{\Omega}^+)$, define

$$I_t(\Psi_1,\Psi_2) := \sum_{n=0}^{\infty} \int_{\tilde{\Omega}} \Psi_1 \circ \zeta_n(x,\xi,m,s+t)\cdot \Psi_2(x,\xi,m,s)\, d\tilde{\mathsf{M}}(x,\xi,m,s)$$

where $\zeta_n(x,\xi,m,s) = (\sigma^n(x), \xi + f_n(x), \theta_n^{-1}(x)m, s - \tau_n(x))$.

Lemma 4.2. *Let $\Psi_2 = \Phi\otimes\delta_{\xi_0}\otimes u\otimes\langle\mu(\cdot)w_1, w_2\rangle \in \mathcal{F}_0$. Then for any $\Psi_1 \in C_c(\tilde{\Omega}^+)$,*

$$I_t(\Psi_1,\Psi_2) =$$

$$\frac{1}{(2\pi)^d\int\tau d\nu}\int \Psi_1(x,\xi_0-\xi,m,s)\cdot\langle \mathsf{Q}_{\mu,w_1,t-s}(\Phi\otimes u)(x,\xi,m), w_2\rangle d\xi d\rho(x) ds dm.$$

Proof. Since $d\nu(x) = h(x)d\rho(x)$, we have

$$\int \tau d\nu \cdot I_t(\Psi_1, \Psi_2) =$$
$$\sum_{n=0}^{\infty} \int \Psi_1(\sigma^n x, \xi + f_n(x), \theta_n^{-1}(x)m, s - \tau_n(x))\, \Psi_2(x, \xi, m, s - t)h(x)d\rho(x)dsdmd\xi.$$

Since $d\rho$ is an eigenmeasure of L_δ with eigenvalue 1,

$$\int_{\Sigma^+} (L_\delta^n F)(x)d\rho(x) = \int_{\Sigma^+} F(x)d\rho(x).$$

Using this, the above is equal to

$$\sum_{n=0}^{\infty} \int_{\tilde{\Omega}^+} \Psi_1(x, \xi, m, s) \sum_{\sigma^n y = x} e^{-\delta\tau_n(y)}(\Phi \cdot h)(y)\delta_{\xi_0}(\xi - f_n(y))$$
$$u(s - t + \tau_n(y))\langle \mu(\theta_n(y)m)w_1, w_2\rangle d\rho(x)d\xi dsdm.$$

Using the identity $\delta_\xi(f_n(y)) = \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} e^{i\langle w, \xi - f_n(y)\rangle}dw$ and the Fourier inversion formula of u: $u(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ist}\hat{u}(s)ds$, the above is again equal to

$$\frac{1}{(2\pi)^d} \int \Psi_1(x, \xi_0 - \xi, m, s) \cdot \langle \mathsf{Q}_{\mu, w_1, t-s}(\Phi \otimes u)(x, \xi, m), w_2\rangle d\xi d\rho(x)dsdm.$$

This proves the claim. $\square$

Proposition 4.3. *For $\Psi_1, \Psi_2 \in C_c(\tilde{\Omega}^+)$, we have*

$$\lim_{t \to +\infty} t^{d/2} I_t(\Psi_1, \Psi_2) = \frac{1}{(2\pi\sigma)^{d/2}} \tilde{\mathsf{M}}(\Psi_1)\tilde{\mathsf{M}}(\Psi_2). \tag{4.4}$$

Proof. Let $\Psi_1 \in C_c(\tilde{\Omega}^+)$.
Step 1: Let $\mathcal{F}$ be the space of functions which are finite linear combinations of functions from $\mathcal{F}_0$. We first show (4.4) holds for any $\Psi_2 \in \mathcal{F}$. It suffices to consider the case where

$$\Psi_2(x, \xi, m, s) = \Phi(x) \otimes \delta_{\xi_0}(\xi) \otimes u(s) \cdot \langle \mu(m)w_1, w_2\rangle \in \mathcal{F}_0.$$

Let

$$F_t(x, \xi, m, s) := \tfrac{t^{d/2}}{\int \tau d\nu \cdot (2\pi)^d} \Psi_1(x, \xi_0 - \xi, m, s)\langle \mathsf{Q}_{\mu, w_1, t-s}(\Phi \otimes u)(x, \xi, m), w_2\rangle.$$

Then Lemma 4.2 gives

$$t^{d/2} I_t(\Psi_1, \Psi_2) = \int F_t(x, \xi, \theta, s)d\rho(x)dsd\xi dm.$$

We consider two cases. First suppose $\mu = 1$. Then Theorem 3.19(2) implies that $F_t(x, \xi, m, s)$ is dominated by a constant multiple of Ψ_1 and converges pointwise to an L^1-integrable function on $\tilde{\Omega}$:

$$\tfrac{1}{(\int \tau dv)^2 (2\pi)^d} C(0)\hat{u}(0)\rho(\Phi h)h(x)\Psi_1(x, \xi_0 - \xi, m, s).$$

Hence by the dominated convergence theorem,

$$\begin{aligned}&\lim_{t\to\infty} t^{d/2} I_t(\Psi_1,\Psi_2)\\&=\frac{1}{(2\pi)^d(\int \tau dv)^2}C(0)\hat{u}(0)\rho(\Phi h)\int \Psi_1(x,\xi_0-\xi,m,s)h(x)d\rho(x)dsd\xi dm\\&=\frac{C(0)}{(2\pi)^d}\int \Psi_1 d\tilde{\mathsf{M}}\cdot\int \Psi_2 d\tilde{\mathsf{M}}.\end{aligned}$$

Plugging $C(0)=(2\pi/\sigma)^{d/2}$ in the above, we get

$$t^{d/2}I_t(\Psi_1,\Psi_2)\sim\frac{1}{(2\pi\sigma)^{d/2}}\int \Psi_1 d\tilde{\mathsf{M}}\cdot\int \Psi_2 d\tilde{\mathsf{M}}.$$

Now suppose μ is non-trivial. Then $d\tilde{\mathsf{M}}(\Psi_2)=0$. On the other hand, Theorem 3.19(3) implies that F_T converges to 0 pointwise, and is dominated by Ψ_1. Therefore, by the dominated convergence theorem, we get

$$\lim_{T\to\infty} t^{d/2}I_t(\Psi_1,\Psi_2)=0$$

proving the claim.

As a consequence, we have

$$\limsup_t t^{d/2}|I_t(\Psi_1,\Psi_2)|<\infty \tag{4.5}$$

for any $\Psi_1\in C_c(\tilde{\Omega})$ and $\Psi_2\in\mathcal{F}$.

Step 2: Let $\Psi_2\in C_c(\tilde{\Omega}^+)$ be a general function. For any $\epsilon>0$, there exist $F_2,\omega_2\in\mathcal{F}$ such that for any $(x,\xi,m,s)\in\tilde{\Omega}^+$,

$$|\Psi_2(x,\xi,m,s)-F_2(x,\xi,m,s)|\le\epsilon\cdot\omega_2(x,\xi,m,s). \tag{4.6}$$

First to find F_2, the Peter-Weyl theorem implies that $\Psi_2(x,\xi,m,s)$ can be approximated by a linear combination of functions of form

$$\kappa(x,\xi,s)\langle\mu(m)w_1,w_2\rangle$$

for $\kappa\in C_c(\Sigma^+\times\mathbb{Z}^d\times\mathbb{R})$ and $(\mu,W)\in\hat{M}$ and $w_1,w_2\in W$ unit vectors. As $\mathbb{Z}^d$ is discrete, κ can be approximated by linear combinations of functions of form $c(x,s)\delta_{\xi_0}$ with $c(x,s)\in C_c(\Sigma^+\times\mathbb{R})$. Now $c(x,s)$ can be approximated by linear combinations of functions of form $\Phi(x)u(s)$ with $\Phi\in C_\beta(\Sigma^+)$ and $u\in C_c^\infty(\mathbb{R})$ by the Stone-Weierstrauss theorem. This gives that for any $\epsilon>0$, we can find $F_2\in\mathcal{F}$ such that

$$\sup|\Psi_2(x,\xi,m,s)-F_2(x,\xi,m,s)|\le\epsilon.$$

Now let $\mathcal{O}$ be the union of the supports of F_2 and Ψ_2, $\mathcal{O}'$ be the 1-neighborhood of $\mathcal{O}$, and let $\kappa:=\|F_2\|_\infty+\|\Psi_2\|_\infty+1$. We can then find $\omega_2\in\mathcal{F}$ such that $\omega_2=\kappa$ on Ω and $\omega_2=0$ outside Ω'. Then for any $(x,\xi,m,s)\in\tilde{\Omega}^+$,

$$|\Psi_2(x,\xi,m,s)-F_2(x,\xi,m,s)|\le\epsilon\omega_2(x,\xi,m,s)$$

as required in (4.6).

Step 3: By Step (1) and (2), we have

$$\limsup|t^{d/2}I_t(\Psi_1,\Psi_2)-t^{d/2}I_t(\Psi_1,F_2)|\le\epsilon\limsup t^{d/2}|I_t(\Psi_1,\omega_2)|\le\epsilon c_0,$$

where $c_0:=\limsup_t t^{d/2}|I_t(\Psi_1,\omega_2)|<\infty$.

Hence

$$\begin{aligned}&\lim_t t^{d/2} I_t(\Psi_1,\Psi_2)\\ =&\lim_t t^{d/2} I_t(\Psi_1,F_2)+O(\epsilon)\\ =&\frac{1}{(2\pi\sigma)^{d/2}}\tilde{\mathsf{M}}(\Psi_1)\tilde{\mathsf{M}}(F_2)+O(\epsilon)\\ =&\frac{1}{(2\pi\sigma)^{d/2}}\tilde{\mathsf{M}}(\Psi_1)\tilde{\mathsf{M}}(\Psi_2)+O(\epsilon),\end{aligned}$$

since $\tilde{\mathsf{M}}(F_2)=\tilde{\mathsf{M}}(\Psi_2)+O(\epsilon)$.

As $\epsilon>0$ is arbitrary, this proves the claim for any $\Psi_2\in C_c(\tilde{\Omega}^+)$. □

Theorem 4.7. *Let $\psi_1,\psi_2\in C_c(\Omega)$. Then*

$$\lim_{t\to\infty} t^{d/2}\int \psi_1(ga_t)\psi_2(g)d\mathsf{m}^{\mathrm{BMS}}(g)=\frac{1}{(2\pi\sigma)^{d/2}}\cdot \mathsf{m}^{\mathrm{BMS}}(\psi_1)\mathsf{m}^{\mathrm{BMS}}(\psi_2).$$

Proof. For each $i=1,2$, let $\Psi_i\in C_c(\tilde{\Omega})$ be the lift of ψ_i to $\tilde{\Omega}$ so that

$$\psi_i[(x,\xi,m,s)]=\sum_{n\in\mathbb{Z}}\Psi_i\circ\zeta^n(x,\xi,m,s).$$

We assume that the support of ψ_2 (and hence of Ψ_2) is small enough so that $\Psi_2(\zeta^n(x,\xi,m,s))=0$ for all $n\neq 0$ if $(x,\xi,m,s)\in\mathrm{supp}(\Psi_2)$.

We first claim that for all large $t\gg 1$,

$$\int_\Omega \psi_1(ga_t)\psi_2(g)d\mathsf{m}^{\mathrm{BMS}}(g)=I_t(\Psi_1,\Psi_2). \tag{4.8}$$

Using the unfolding,

$$\begin{aligned}&\int_\Omega \psi_1(ga_t)\psi_2(g)d\mathsf{m}^{\mathrm{BMS}}(g)\\ =&\sum_{n=-\infty}^{\infty}\int_{\mathrm{supp}(\Psi_2)}\Psi_1\circ\zeta^n(x,\xi,m,s+t)\cdot\Psi_2(x,\xi,m,s)\,d\tilde{\mathsf{M}}\\ =&\sum_{n=0}^{\infty}\int_{\tilde{\Omega}}\Psi_1\circ\zeta^n(x,\xi,m,s+t)\cdot\Psi_2(x,\xi,m,s)\,d\tilde{\mathsf{M}}\\ +&\sum_{n=1}^{\infty}\int_{\tilde{\Omega}}\Psi_1\circ\zeta^{-n}(x,\xi,m,s+t)\cdot\Psi_2(x,\xi,m,s)\,d\tilde{\mathsf{M}}.\end{aligned}$$

The first term of the last equation is $I_t(\Psi_1,\Psi_2)$. For the second term, note that for any $(x,\xi,m,s)\in\mathrm{supp}(\Psi_2)$,

$$\Psi_1\circ\zeta^{-n}(x,\xi,m,s+t)=\Psi_1(\sigma^{-n}(x),\xi-f_n(x),\theta_n^{-1}(x)m,s+t+\tau_n(x))$$

which is 0 if t is large enough, as $\tau_n(x)>0$.

Therefore the second term is 0 for t large enough, proving the claim (4.8). Therefore if $\Psi_1,\Psi_2\in C_c(\tilde{\Omega}^+)$, Theorem 4.7 follows from Proposition 4.3.

Let $\Psi_1,\Psi_2\in C_c(\tilde{\Omega})$. Then for any $\epsilon>0$, we can find a sufficiently large $k\geq 1$, F_1,F_2,ω_1, and ω_2 in $C_c(\tilde{\Omega}^+)$ such that for all $(x,\xi,m,s)\in\tilde{\Omega}$,

$$\begin{aligned}&|\Psi_i\circ\zeta^k(x,\xi,m,s)-F_i(x,\xi+f_k(x),\theta_k^{-1}(x)m,s-\tau_k(x))|\\ <&\epsilon\cdot\omega_i(x,\xi-f_k(x),\theta_k^{-1}(x)m,s-\tau_k(x)).\end{aligned}$$

We then deduce by applying the previous case to F_i and ω_i that

$$\begin{aligned}
&\lim t^{d/2} \int \psi_1(ga_t)\psi_2(g) d\mathsf{m}^{\mathrm{BMS}}(g) \\
=&\lim t^{d/2} I_t(\Psi_1, \Psi_2) \\
=&\lim t^{d/2} I_t(\Psi_1 \circ \zeta^k, \Psi_2 \circ \zeta^k) \\
=&\lim \left(t^{d/2} I_t(F_1, F_2) + O(\epsilon \cdot t^{d/2}(I_t(F_1, \omega_2) + I_t(F_2, \omega_1) + I_t(\omega_1, \omega_2))) \right) \\
=&\tfrac{1}{(2\pi\sigma)^{d/2}} \tilde{\mathsf{M}}(\Psi_1)\tilde{\mathsf{M}}(\Psi_2) + O(\epsilon).
\end{aligned}$$

As $\tilde{\mathsf{M}}(\Psi_i) = \mathsf{m}^{\mathrm{BMS}}(\psi_i)$ and $\epsilon > 0$ is arbitrary, this finishes the proof. □

Remark 4.9. We remark that the methods of our proof of Theorem 4.7 can be extended to other Gibbs measures on $\Gamma\backslash G$.

4.2. **Correlation functions for** $(\Gamma\backslash G, a_t, \mathsf{m}^{\mathrm{Haar}})$**.** We can deduce the asymptotic of the correlation functions for the Haar measure Theorem 1.6 from that for the BMS measure Theorem 4.7 via the following theorem:

Theorem 4.10. *Suppose that there exists a function* $J : (0, \infty) \to (0, \infty)$ *such that for any* $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$,

$$\lim_{t\to+\infty} J(t) \int \psi_1(ga_t)\psi_2(g) d\mathsf{m}^{\mathrm{BMS}}(g) = \mathsf{m}^{\mathrm{BMS}}(\psi_1)\mathsf{m}^{\mathrm{BMS}}(\psi_2). \tag{4.11}$$

Then for any $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$,

$$\lim_{t\to+\infty} J(t)e^{(D-\delta)t} \int \psi_1(ga_t)\psi_2(g) d\mathsf{m}^{\mathrm{Haar}}(g) = \mathsf{m}^{\mathrm{BR}_+}(\psi_1)\mathsf{m}^{\mathrm{BR}_-}(\psi_2). \tag{4.12}$$

The main idea of this theorem appeared first in Roblin's thesis [57] and was further developed and used in ([44], [40], [45]). The key ingredients of the arguments are the product structures of the measures m^{BMS} and m^{Haar} and the study of the transversal intersections for the translates of horospherical pieces by the flow a_t. The verbatim repetition of the proof of [45, Theorem 5.8] while replacing H by N^-AM proves Theorem 4.10.

Using theorem 4.10, we deduce the following from Theorem 4.7:

Theorem 4.13. *Let* $\psi_1, \psi_2 \in C_c(\Gamma\backslash G)$. *Then*

$$\lim_{t\to+\infty} t^{d/2} e^{(D-\delta)t} \int_{\Gamma\backslash G} \psi_1(ga_t)\psi_2(g)\, d\mathsf{m}^{\mathrm{Haar}}(g) = \frac{\mathsf{m}^{\mathrm{BR}_+}(\psi_1)\mathsf{m}^{\mathrm{BR}_-}(\psi_2)}{(2\pi\sigma)^{d/2}\mathsf{m}^{\mathrm{BMS}}(\Gamma_0\backslash G)}.$$

Theorems 1.5 and 1.7 are consequences of this theorem: if $\mu \in \mathcal{P}_{\mathsf{acc}}(\Gamma\backslash G)$, then $d\mu = \psi_2\, d\mathsf{m}^{\mathrm{Haar}}$ for some $\psi_2 \in C_c(\Gamma\backslash G)$ with $\int \psi_2\, d\mathsf{m}^{\mathrm{Haar}} = 1$. Hence

$$\int \psi_1 d\mu_t = \int \psi_1(ga_t)\psi_2(g)\, d\mathsf{m}^{\mathrm{Haar}}(g).$$

Hence Theorem 1.5 follows if we put $\alpha(t) = t^{d/2}e^{(D-\delta)t}$ and $c_\mu := \frac{\mathsf{m}^{\mathrm{BR}_-}(\psi_2)}{(2\pi\sigma)^{d/2}\mathsf{m}^{\mathrm{BMS}}(\Gamma_0\backslash G)}$.

For Theorem 1.7, note that when $\Gamma_0 < G$ is cocompact, all the measures $\mathsf{m}^{\mathrm{BR}_+}$, $\mathsf{m}^{\mathrm{BR}_-}$ and $\mathsf{m}^{\mathrm{BMS}}$ coincide with $\mathsf{m}^{\mathrm{Haar}}$. Hence $c_\mu = \frac{1}{(2\pi\sigma)^{d/2}\mathsf{m}^{\mathrm{Haar}}(\Gamma_0\backslash G)}$ depends only on Γ.

5. The A-ergodicity of generalized BMS measures

In this section, let Γ be a non-elementary discrete subgroup of $G = \mathrm{Isom}_+(\tilde{X})$, and $\Omega \subset \Gamma\backslash G/M$ denote the non-wandering set of the geodesic flow $\{a_t\}$.

5.1. **Generalized BMS-measures.** Let $\tilde{F}$ be a Γ-invariant Hölder continuous function on $\mathrm{T}^1(\tilde{X})$. Let $\chi : \Gamma \to \mathbb{R}$ be an additive character of Γ.

For all $x \neq y \in \tilde{X}$, we define

$$\int_x^y \tilde{F} := \int_0^{d(x,y)} \tilde{F}(va_t)dt$$

where v is the unique unit tangent vector based at x such that va_t is a vector based at y. The Gibbs cocycle for the potential $\tilde{F}$ is a map $C_{\tilde{F}} : \partial_\infty \tilde{X} \times \tilde{X} \times \tilde{X} \to \mathbb{R}$ defined by

$$(\xi, x, y) \mapsto C_{\tilde{F},\xi}(x,y) = \lim_{t\to+\infty} \int_y^{\xi_t} \tilde{F} - \int_x^{\xi_t} \tilde{F}$$

where $t \mapsto \xi_t$ is any geodesic ray toward the point ξ.

Definition 5.1. For $\sigma \in \mathbb{R}$, a twisted conformal density of dimension σ for $(\Gamma, \tilde{F}, \chi)$ is a family of finite measures $\{\nu_x : x \in \tilde{X}\}$ on $\partial(\tilde{X})$ such that for any $\gamma \in \Gamma$, $x, y \in \tilde{X}$ and $\xi \in \partial(\tilde{X})$,

$$\gamma_* \mu_x = e^{-\chi(\gamma)} \mu_{\gamma x} \quad \text{ and } \quad \frac{d\mu_x}{d\mu_y}(\xi) = e^{-C_{\tilde{F}-\sigma,\xi}(x,y)}.$$

The twisted critical exponent $\delta_{\Gamma,\tilde{F},\chi}$ of $(\Gamma, \tilde{F}, \chi)$ is given by

$$\limsup_{n\to+\infty} \frac{1}{n} \log \sum_{\gamma\in\Gamma, n-1<d(o,\gamma(o))\leq n} \exp\left(\chi(\gamma) + \int_o^{\gamma(o)} \tilde{F}\right).$$

When χ is trivial, we simply write it as $\delta_{\Gamma,\tilde{F}}$. It can be seen that $\delta_{\Gamma,\tilde{F}} \leq \delta_{\Gamma,\tilde{F},\chi}$ for any character χ of Γ.

Suppose that

$$\delta_{\Gamma,\tilde{F},\chi} < \infty.$$

Then there exists a twisted conformal density of dimension $\delta_{\Gamma,\tilde{F},\chi}$ for $(\Gamma, \tilde{F}, \chi)$ whose support is precisely the limit set of Γ; we call it a twisted Patterson-Sullivan density, or a twisted PS density for brevity. Denote $\iota : \mathrm{T}^1(\tilde{X}) \to \mathrm{T}^1(\tilde{X})$ to be the flip map, $v \mapsto -v$. It is shown in [48, Proposition 11.8] that $\delta_{\Gamma,\tilde{F},\chi} = \delta_{\Gamma,\tilde{F}\circ\iota,-\chi}$. We define the following generalized BMS measure:

Definition 5.2 (Generalized BMS measures)**.** Let $\{\mu_x : x \in \tilde{X}\}$ and $\{\mu_x^\iota : x \in \tilde{X}\}$ be twisted PS densities for $(\Gamma, \tilde{F}, \chi)$ and $(\Gamma, \tilde{F}\circ\iota, -\chi)$ respectively. Set $\delta_0 = \delta_{\Gamma,\tilde{F},\chi}$. A generalized BMS measure $\tilde{\mathsf{m}} = \tilde{\mathsf{m}}_\Gamma$ on $\mathrm{T}^1(\tilde{X}) = G/M$ associated to the pair $\{\mu_x : x \in \tilde{X}\}$ and $\{\mu_x^\iota : x \in \tilde{X}\}$ is defined by

$$d\tilde{\mathsf{m}}(u) = e^{C_{\tilde{F}-\delta_0,u^+}(o,u)+C_{\tilde{F}\circ\iota-\delta_0,u^-}(o,u)} d\mu_o(u^+) d\mu_o^\iota(u^-) ds \tag{5.3}$$

using the Hopf parametrization of $\mathrm{T}^1(\tilde{X})$.

By abuse of notation, we use the notation $\tilde{\mathsf{m}}$ for the M-lift of $\tilde{\mathsf{m}}$ to G.

As χ and $-\chi$ cancels with each other, we can check that the measure $\tilde{\mathsf{m}}_\Gamma$ is Γ-invariant. It induces

- an A-invariant measure $\mathsf{m}_\Gamma^\dagger$ on $\Gamma\backslash G/M$ supported on Ω and
- an AM-invariant measure m_Γ on $\Gamma\backslash G$.

When there is no ambiguity about Γ, we will drop the subscript Γ for simplicity. When $\tilde{F}=0$ and χ is the trivial character, m is precisely equal to the BMS measure $\mathsf{m}^{\mathrm{BMS}}$ on $\Gamma\backslash G$ defined in section 2.

5.2. **The A-ergodicity of generalized BMS measures.** The generalized Sullivan's dichotomy says that the dynamical system $(\Gamma\backslash G/M, A, \mathsf{m}^\dagger)$ is either conservative and ergodic, or completely dissipative and non-ergodic [48].

We will extend this dichotomy for the A-action on $(\Gamma\backslash G, \mathsf{m})$ using the density of the transitivity group shown by Winter.

Definition 5.4 (Transitivity group)**.** Fix $g\in\Omega$. We define the transitivity subgroup $\mathcal{H}_\Gamma(g) < AM$ as follows: $ma\in\mathcal{H}_\Gamma(g)$ if and only if there is a sequence $h_i\in N^-\cup N^+$, $i=1,\dots,k$ and $\gamma\in\Gamma$ such that

$$\gamma g h_1 h_2 \dots h_r \in \Omega \text{ for all } 0\le r\le k, \text{ and}$$
$$\gamma g h_1 h_2 \dots h_k = gam.$$

Lemma 5.5. [65, Theorem 3.14] *Let Γ be Zariski dense. The transitivity group $\mathcal{H}_\Gamma(g)$ is dense in AM for any $g\in\Omega$.*

Theorem 5.6. *Suppose that Γ is Zariski dense. Let m be a generalized BMS-measure on $\Gamma\backslash G$ associated to $(\Gamma,\tilde{F},\chi)$. If $(\Gamma\backslash G/M, A, \mathsf{m}^\dagger)$ is conservative, then $(\Gamma\backslash G, A, \mathsf{m})$ is conservative and ergodic.*

In particular, if Γ is of divergence type, then $(\Gamma\backslash G, A, \mathsf{m}^{\mathrm{BMS}})$ is conservative and ergodic.

Proof. Since the A-action on $(\Gamma\backslash G/M, \mathsf{m}^\dagger)$ is conservative and $\Gamma\backslash G$ is a principal M-bundle over $\Gamma\backslash G/M$, it follows that the A-action on $(\Gamma\backslash G, \mathsf{m})$ is conservative as well: we can decompose $\Gamma\backslash G$ as $\Omega_C\cup\Omega_D$ where Ω_C and Ω_D are respectively the conservative and the dissipative parts of the A-action, that is, $x\in\Omega_C$ iff xa_{t_i} comes back to a compact subset for some $t_i\to\infty$. Note that $\Omega_C M$ is the conservative part for the geodesic flow, and must have the full $\mathsf{m}^{\mathrm{BMS}}$-measure by the assumption. Since a_t and M commutes, $\Omega_C=\Omega_C M$, hence the claim follows.

We will now prove the A-ergodicity using the conservativity of the A-action, following Sullivan's argument which is based on Hopf's ratio ergodic theorem (see also [57], [12]).

Fix a positive Lipschitz map $\rho:\Gamma\backslash G\to\mathbb{R}$ with $\int\rho d\,\mathsf{m}=1$; fix $o\in\Omega$ and fix a positive continuous non-increasing function r on $\mathbb{R}_{>0}$ which is affine on each $[n,n+1]$ and $r(n)=1/(2^n\mathsf{m}(B(o,n+1)))$. Then $g\mapsto r(d(o,g(o)))$ is Lipschitz and belongs to $L^1(\mathsf{m})$; so by normalizing it, we get a function ρ with desired properties (see [48, Ch 5] for more details).

By the conservativity,

$$\int_0^{\pm\infty}\rho(xa_t)dt=\pm\infty$$

for m-almost all $x\in\Gamma\backslash G$.

Now by the conservativity of the A-action, we can apply the Hopf ratio theorem which says for any $\psi\in C_c(\Gamma\backslash G)$,

$$\lim_{T\to\pm\infty}\frac{\int_0^T\psi(xa_t)dt}{\int_0^T\rho(xa_t)dt}$$

converges almost everywhere to an L^1-function $\tilde{\psi}^{\pm}$, and $\tilde{\psi}^{+} = \tilde{\psi}^{-}$ almost everywhere. Moreover the A-action is ergodic if and only if $\tilde{\psi}^{\pm}$ is constant almost everywhere.

Using the uniform continuity of ψ and ρ, we can first show that the limit functions $\tilde{\psi}^{\pm}$ coincide a.e. with an N^+ and N^- invariant measurable function $\tilde{\psi}$. Denote by ψ^* the lift to $\tilde{\psi}$ to G. Consider the Borel sigma algebra $\mathcal{B}(G)$ on G, and define subalgebras $\Sigma_{\pm} := \{B \in \mathcal{B}(G) : B = \Gamma B N^{\pm}\}$ and $\tilde{\Sigma} := \Sigma_- \wedge \Sigma_+$. That is, $B \in \tilde{\Sigma}$ if and only if there exist $B_{\pm} \in \Sigma_{\pm}$ such that $\tilde{\mathsf{m}}(B \Delta B_{\pm}) = 0$. It is shown in [65, Thm.4.3] that the density of the transitivity group and the ergodicity of AM-action on $(\Gamma\backslash G, \tilde{\mathsf{m}})$ implies that $\tilde{\Sigma}$ is trivial (we note that the proof of [65, Thm.4.3] does not require the finiteness of $\tilde{\mathsf{m}}$). Both conditions are satisfied under our hypothesis by Lemma 5.5 and the ergodicity of the system $(\Gamma\backslash G/M, A, \mathsf{m}^{\dagger})$ as remarked before. It now follows that ψ^* coincides with a constant function almost everywhere. This proves the A-ergodicity on $(\Gamma\backslash G, \mathsf{m})$.

The last part follows since $(\Gamma\backslash G/M, A, \mathsf{m}^{\mathrm{BMS}})$ is conservative and ergodic when Γ is of divergence type [64]. $\square$

Corollary 5.7. *Let Γ_0 be a Zariski dense convex cocompact subgroup of G. Let $\tilde{F}$ be a Γ_0-invariant Hölder continuous function on $\tilde{X}$ and $\chi : \Gamma_0 \to \mathbb{R}$ be a character. Suppose $\delta_{\Gamma_0, \tilde{F}, \chi} < \infty$.*

(1) If m_{Γ_0} is a generalized BMS-measure on $\Gamma_0\backslash G$ associated to $(\Gamma_0, \tilde{F}, \chi)$, then $(\Gamma_0\backslash G, A, \mathsf{m}_{\Gamma_0})$ is ergodic and conservative.

(2) Let $\Gamma < \Gamma_0$ be a normal subgroup with $\Gamma\backslash\Gamma_0 = \mathbb{Z}^d$, and $\chi = 0$ be the trivial character. Let m_{Γ} be the measure on $\Gamma\backslash G$ induced by the generalized BMS-measure $\tilde{\mathsf{m}}_{\Gamma_0}$ on G associated to $(\Gamma_0, \tilde{F}, 0)$. Then $(\Gamma\backslash G, A, \mathsf{m}_{\Gamma})$ is ergodic and conservative if and only if $d \leq 2$.

Proof. Since Γ_0 is convex cocompact, $\mathsf{m}^{\dagger}_{\Gamma_0}$ on $\Gamma_0\backslash G/M$ is compactly supported and hence conservative. Therefore Theorem 5.6 shows the claim (1).

For (2), denote by $\mathsf{m}^{\dagger}_{\Gamma}$ the measure on $\Gamma\backslash G/M$ induced by $\tilde{\mathsf{m}}_{\Gamma_0}$. The A-action on $(\Gamma\backslash G/M, \mathsf{m}^{\dagger}_{\Gamma})$ is ergodic and conservative if and only if $d = 1, 2$ ([56],[66]). Hence (2) again follows from Theorem 5.6. $\square$

Theorem 1.10 is a special case of Corolloary 5.7(2).

6. Babillot-Ledrappier measures and measure classification

Let G be a connected simple linear Lie group of rank one. Let Γ_0 be a discrete subgroup of G. For a subset S of G and $\epsilon > 0$, let S_ϵ denote the intersection of S and the ϵ-ball of e in G. We also use the notation $S_{O(\epsilon)}$ to denote the set $S_{C\epsilon}$ for some constant $C > 0$ depending only on G and Γ_0.

6.1. Closing lemma.

Let

$$\mathcal{B}(\epsilon) := (N^+_\epsilon N^- \cap N^-_\epsilon N^+ AM) M_\epsilon A_\epsilon.$$

Lemma 6.1 (Closing lemma)**.** [38, Lemma 3.1] *There exist $T_0 > 1$, and $\epsilon_0 > 0$ depending only on G for which the following holds: if*

$$g_0 \mathcal{B}(\epsilon) a_T m \cap \gamma g_0 \mathcal{B}(\epsilon) \neq \emptyset$$

for some $T > T_0$, $0 < \epsilon < \epsilon_0$, $m \in M$, $g_0 \in G$ *and* $\gamma \in G$, *then there exists* $g \in g_0\mathcal{B}(2\epsilon)$ *such that*

$$\gamma = ga_0m_0g^{-1}$$

where $a_0 \in A$ *and* $m_0 \in M$ *satisfy* $a_0 \in a_TA_{O(\epsilon)}$ *and* $m_0 \in mM_{O(\epsilon)}$.

6.2. **Generalized length spectrum.** In the rest of this section, let Γ be a normal subgroup of a Zariski dense convex cocompact subgroup Γ_0 of G. For $\gamma \in \Gamma_0$, we have $\gamma = gamg^{-1}$ for some $g \in G$, $am \in AM$. The element g is determined in G/M, and m is determined only up to conjugation in M. Fix a section $\mathcal{L} = N^+AN^-$ for G/M. Let

$$\Gamma_0^\star = \{\gamma \in \Gamma_0 : \gamma = gamg^{-1} \text{ for some } g \in \mathcal{L}\}.$$

For $\gamma \in \Gamma_0^\star$, there are unique $g \in \mathcal{L}, l(\gamma) \in \mathbb{R}_{>0}$ and $m_\gamma \in M$ such that

$$\gamma = ga_{l(\gamma)}m_\gamma g^{-1}.$$

For $\gamma \in \Gamma_0$, we write $f(\gamma) \in \Gamma\backslash\Gamma_0$ for its image under the projection map $\Gamma_0 \to \Gamma\backslash\Gamma_0$. Hence we can write

$$\Gamma f(\gamma)g = \Gamma g a_{l(\gamma)}m_\gamma.$$

Definition 6.2. The generalized length spectrum $\mathcal{GL}(\Gamma_0, \Gamma)$ of Γ_0 relative to Γ is defined as

$$\mathcal{GL}(\Gamma_0, \Gamma) := \{(f(\gamma), a_{l(\gamma)}, m_\gamma) \in \Gamma\backslash\Gamma_0 \times A \times M : \gamma \in \Gamma_0^\star\}.$$

Proposition 6.3. *Suppose that* Γ *is a normal subgroup of* Γ_0 *and that* $(\Gamma\backslash G, a_t)$ *satisfies the topological mixing property: for any two open subsets* $U, V \subset \Gamma\backslash G$,

$$Ua_t \cap V \neq \emptyset$$

for all sufficiently large $t > 1$. *Then the subgroup generated by* $\mathcal{GL}(\Gamma_0, \Gamma)$ *is dense in* $\Gamma\backslash\Gamma_0 \times A \times M$.

Proof. Consider $(\xi, a_T, m) \in \Gamma\backslash\Gamma_0 \times A \times M$. The assumption on the topological mixing property implies that for any small $\epsilon > 0$, there is $T_\epsilon > 0$ such that

$$\Gamma\mathcal{B}(\epsilon)ma_T \cap \Gamma\xi\mathcal{B}(\epsilon) \neq \emptyset$$

for all $T > T_\epsilon$. That is, for all $T > T_\epsilon$, there exist $g_1, g_2 \in \mathcal{B}(\epsilon)$ such that $g_1ma_T = \xi\gamma g_2$ for some $\gamma \in \Gamma$. Set $\gamma_0 := \xi\gamma \in \Gamma_0$; so $f(\gamma_0) = \xi$. The closing lemma 6.1 implies that there exists $g \in \mathcal{B}(2\epsilon)$ such that $\gamma_0 = ga_{T_0}m_{\gamma_0}g^{-1}$ with $a_{T_0-T} \in A_{O(\epsilon)}$ and $m_{\gamma_0} = mM_{O(\epsilon)}$.

Hence for any $\epsilon > 0$ and for any $(\xi, a_T, m) \in \Gamma\backslash\Gamma_0 \times A \times M$ for T sufficiently large, we can find $\gamma \in \Gamma_0^\star$ such that $(f(\gamma), a_{l(\gamma)}, m_\gamma)$ is within an $O(\epsilon)$-neighborhood of (ξ, a_T, m). This proves the claim. $\square$

We deduce the following from Proposition 6.3 and Theorem 1.6:

Corollary 6.4. *Suppose that* Γ *is a co-abelian subgroup of* Γ_0. *Then the group generated by* $\mathcal{GL}(\Gamma_0, \Gamma)$ *is dense in* $\Gamma\backslash\Gamma_0 \times A \times M$.

6.3. **The N^--ergodicity of generalized BMS measures.** Let $\tilde{F}$ be a Γ_0-invariant Hölder continuous function on $\mathrm{T}^1(\tilde{X})$ and $\chi : \Gamma_0 \to \mathbb{R}$ be a character. Suppose $\delta_{\Gamma_0,\tilde{F},\chi} < \infty$. Let $\tilde{\mathsf{m}}_{\Gamma_0}$ be a generalized BMS-measure on G associated to $(\Gamma, \tilde{F}, \chi)$. By Corollary 5.7, the induced measure m_{Γ_0} on $\Gamma_0\backslash G$ is A-ergodic. We denote by

$$\mathsf{m} = \mathsf{m}_\Gamma$$

the AM-invariant measure on $\Gamma\backslash G$ induced by $\tilde{\mathsf{m}}_{\Gamma_0}$.

Note that any essentially N^--invariant measurable function ψ in $(\Gamma\backslash G, \mathsf{m})$ is almost everywhere equal to a measurable N^--invariant function [67].

Define $H = H(\mathsf{m})$ to be the set of $(\xi, a, m) \in \Gamma\backslash\Gamma_0 \times A \times M$ such that for all N^--invariant measurable function ψ in $L^\infty(\Gamma\backslash G, \mathsf{m})$,

$$\psi(x) = \psi(\xi^{-1}xam)$$

for m-almost all $x \in \Gamma\backslash G$.

It is easy to check that H is a closed subgroup.

Proposition 6.5. *If $\Gamma < \Gamma_0$ is co-abelian, then*

$$H(\mathsf{m}) = \Gamma\backslash\Gamma_0 \times A \times M.$$

Proof. By Corollary 6.4, it suffices to show that

$$\mathcal{GL}(\Gamma_0, \Gamma) \subset H.$$

Fix $(f(\gamma), a_{l(\gamma)}, m_\gamma) \in \mathcal{GL}(\Gamma_0, \Gamma)$, that is, for some unique $g \in \mathcal{L}$,

$$f(\gamma)\Gamma g = \Gamma g a_{l(\gamma)} m_\gamma.$$

Let $\pi : \Gamma\backslash G \to \Gamma_0\backslash G$ be the canonical projection. Let $[g] = \Gamma g$ and let $B(\pi[g], \epsilon)$ denote the ϵ-ball around $\pi[g]$. Let $p > 0$ be such that for any $x \in B(\pi[g], \epsilon/p)$ and any $0 \le t < 2\ell(\gamma)$, we have

$$d(xa_t, \pi[g]a_t) < \epsilon/2.$$

Fix $\epsilon > 0$, and set $\mathfrak{S}_k = \mathfrak{S}_k(\epsilon) \subset \Gamma\backslash G$ be the set of x's such that $\pi(x)a_t \in B(\pi[g], \epsilon/p)$ for some $t \in [k\ell(\gamma), (k+1)\ell(\gamma))$.

Since m_0 is A-ergodic,

$$\mathsf{m}(\Gamma\backslash G - \cup_k \mathfrak{S}_k) = 0.$$

Let $x \in \mathfrak{S}_k$. By replacing g by ga_t for some $0 \le t < \ell(\gamma)$, we have

$$d(xa_{k\ell(\gamma)}, \xi[g]) \le \epsilon/2 \text{ and } d(xa_{k\ell(\gamma)+\ell(\gamma)}, \xi[g]a_{\ell(\gamma)}) \le \epsilon/2.$$

Since $\xi[g]a_{\ell(\gamma)} = f(\gamma)\xi[g]m_\gamma^{-1}$ and the metric d is right M-invariant and left G-invariant,

$$d(xa_{k\ell(\gamma)+\ell(\gamma)}, \xi[g]a_{\ell(\gamma)}) = d(f(\gamma)^{-1}xa_{k\ell(\gamma)+\ell(\gamma)}m_\gamma, \xi[g]).$$

Hence

$$d(f(\gamma)^{-1}xa_{(k+1)\ell(\gamma)}m_\gamma, xa_{k\ell(\gamma)}) \le \epsilon.$$

Since the product map $N^- \times A \times M \times N^+ \to G$ is a diffeomorphism onto an open neighborhood of e in G, there exist an element $n^-amn^+ \in N^-_{O(\epsilon)}A_{O(\epsilon)}M_{O(\epsilon)}N^+_{O(\epsilon)}$ such that

$$xa_{k\ell(\gamma)} = xf(\gamma)^{-1}a_{(k+1)\ell(\gamma)}m_\gamma n^- amn^+.$$

Set

$$x^* := x(a_{(k+1)\ell(\gamma)}m_\gamma)n^-(a_{(k+1)\ell(\gamma)}m_\gamma)^{-1} \in xN^-$$

and

$$T_k(x) := f(\gamma)^{-1}x^* a_{\ell(\gamma)} m_\gamma am.$$

Note

$$x = T_k(x) a_{k\ell(\gamma)} n^+ a_{k\ell(\gamma)}^{-1} \in T_k(x) N^+_{O(e^{-k}\epsilon)}.$$

Now suppose that $(f(\gamma), a_\gamma, m_\gamma) \notin H$, that is, there exists an N^--invariant measurable function $\psi \in L^\infty(\mathsf{m})$, a compact set W with $\mathsf{m}(W) > 0$, and $\epsilon_0 > 0$ such that $\psi(xh) = \psi(x)$ for all $x \in W$ and $h \in N^-$ and

$$\psi(x) > \psi(f(\gamma)^{-1} x a_\gamma m_\gamma) + \epsilon_0$$

for all $x \in W$. Since $x^* \in xN^-$, we have for all $x \in W$,

$$\psi(x) > \psi(f(\gamma)^{-1} x^* a_\gamma m_\gamma) + \epsilon_0.$$

If we consider a sequence of non-negative continuous functions η_δ with integral one and which is supported in $(AM)_\delta$, then as $\delta \to 0$, the convolution $\psi * \eta_\delta$ converges to ψ almost everywhere. By replacing ψ with $\psi * \eta_\delta$ for small δ, we may assume that ψ is continuous for the AM-action. Moreover, using Luzin's theorem, we may assume that ψ is uniformly continuous on W by replacing W by a smaller subset if necessary.

Since

$$T_k(x) = f(\gamma)^{-1} x^* a_{\ell(\gamma)} m_\gamma am \to x$$

as $k \to \infty$ and $am \in (AM)_{O(\epsilon)}$. we will get a contradiction if we can find $x \in W$ with $T_k(x) \in W$ for an arbitrarily large k.

Hence it remains to prove the following claim:

(1) $\lim_k \mathsf{m}(\{x \in W \cap \mathfrak{S}_k : T_k(x) \notin W\}) = 0$;
(2) $\limsup_k \mathsf{m}(W \cap \mathfrak{S}_k) > 0$.

Note that T_k maps x to $f(\gamma)^{-1} x a_{(k+1)\ell(\gamma)} m_\gamma a_{-k\ell(\gamma)} am$. Since m is AM-invariant, $\Gamma\backslash\Gamma_0$-invariant, and $n^- \in G_\epsilon$, there exists $\kappa = \kappa(\epsilon) > 1$ such that $\mathsf{m}(T_k^{-1}(Q)) \leq \kappa \mathsf{m}(Q)$ for any Borel set $Q \subset \Gamma\backslash G$ and any $k \geq 1$

Since for $x \in \mathfrak{S}_k$, $d(x, T_k(x)) \to 0$, we have

$$\mathsf{m}(\{x \in W \cap \mathfrak{S}_k : T_k(x) \notin W\}) \leq \mathsf{m}(T_k^{-1}(WG_\eta - W)) \leq \kappa \cdot \mathsf{m}(WG_\eta - W)$$

for small small $\eta > 0$ which goes to 0 as $k \to \infty$.

Now as W is a compact subset, we have $\mathsf{m}(WG_\eta - W) \to 0$ as $\eta \to 0$, and hence the first claim follows.

For the second claim, suppose the claim fails. Note that for any $t > 0$, there exists $k \in \mathbb{N}$ such that $\pi^{-1}(B(\pi[g], \epsilon)) a_{-t} \subset \mathfrak{S}_k$. Hence we have

$$\limsup_t \mathsf{m}(W \cap \pi^{-1}(B(\pi[g], \epsilon) a_{-t}) = 0.$$

This would mean that

$$\limsup_s \frac{1}{s} \int_0^s \int_W 1_{B(\pi[g],\epsilon)}(\pi(x) a_t) d\mathsf{m} dt = 0.$$

However, by the Fubini theorem, the Birkhoff ergodic theorem (as m_0 is A-ergodic) and the Lebesgue dominated covergence theorem, we deduce

$$\begin{aligned}\frac{1}{s} \int_0^s \int_W 1_{B(\pi[g],\epsilon)}(\pi(x) a_t) d\mathsf{m} dt =& \int_W \frac{1}{s} \int_0^s 1_{B(\pi[g],\epsilon)}(\pi(x) a_t) dt d\mathsf{m} \\ \to& \mathsf{m}_0(B(\pi[g], \epsilon)) \mathsf{m}(W).\end{aligned}$$

Therefore this proves the claim. ☐

Theorem 6.6. *If* $\Gamma\backslash\Gamma_0$ *is abelian, then the* N^-*-action on* $(\Gamma\backslash G, \mathsf{m})$ *is ergodic, i.e. any* N^-*-invariant measurable function on* $(\Gamma\backslash G, \mathsf{m})$ *is a constant.*

Proof. Since $H = H(\mathsf{m}) = \Gamma\backslash\Gamma_0 \times A \times M$ by Proposition 6.5, any N^--invariant measurable function ψ on $(\Gamma\backslash G, \mathsf{m})$ gives rise an AMN^--invariant function $\tilde{\psi}$ on $(\Gamma_0\backslash G, \mathsf{m}_{\Gamma_0})$.

Since m_{Γ_0} is A-ergodic, $\tilde{\psi}$ is constant m_{Γ_0} almost everywhere. It follows that ψ is constant m-almost everywhere. Therefore the claim is proved. □

6.4. **Babillot-Ledrappier measures.** Since Γ_0 is convex cocompact, we have

$$\delta_{\Gamma_0,\chi} < \infty$$

for any character χ of Γ_0 (see [48, Proof of Prop. 11.8]).

Suppose that Γ is co-abelian in Γ_0, and consider the space $(\Gamma\backslash\Gamma_0)^*$ of characters of Γ_0 which vanish on Γ. Fix a character $\chi \in (\Gamma\backslash\Gamma_0)^*$. There exists a unique twisted PS density $\{\mu_{\chi,x} : x \in \tilde{X}\}$ for $(\Gamma_0, 0, \chi)$ supported on $\Lambda(\Gamma_0)$ [48, Corollary 11.13].

Definition 6.7. Define the following measure on $\mathrm{T}^1(\tilde{X})$ using the Hopf parametrization:

$$d\tilde{\mathsf{m}}_\chi(u) = e^{\delta_{\Gamma_0,\chi}\beta_{u^+}(o,u)+D\beta_{u^-}(o,u)} d\mu_{\chi,o}(u^+) d\mathsf{m}_o(u^-) ds,$$

where $d\mathsf{m}_o$ is the Lebesgue density on $\partial(\tilde{X})$.

One can check that the measure $\tilde{\mathsf{m}}_\chi$ satisfies the following properties:

(1) identifying $\mathrm{T}^1(\tilde{X})$ with G/M, $\tilde{\mathsf{m}}_\chi$ is N^--invariant (i.e., Lebesgue measures on each N^- leaf) and quasi A-invariant;
(2) as χ vanishes on Γ, $\tilde{\mathsf{m}}_\chi$ is Γ-invariant;
(3) for any $\gamma \in (\Gamma\backslash\Gamma_0)^*$, $\gamma_*\tilde{\mathsf{m}}_\chi = e^{-\chi(\gamma)}\tilde{\mathsf{m}}_\chi$.

Denote by m_χ the MN^--invariant measure on $\Gamma\backslash G$ induced by $\tilde{\mathsf{m}}_\chi$; we call it a Babillot-Ledrappier measure. When $\chi = 0$ is the trivial character, m_0 coincides with the Burger-Roblin mesure m^{BR_-} up to a constant multiple.

Consider the generalized BMS measure m_{Γ_0} on $\Gamma\backslash G$ associated to the pair $\{\mu_{\chi,x}\}$ and $\{\mu_{-\chi,x}\}$. Then m_{Γ_0} and m_χ have the same transverse measure

$$e^{\delta_{\Gamma_0,\chi}\beta_{u^+}(o,u)} d\mu_{\chi,o}(u^+) ds dm.$$

Hence the N^--ergodicity of m_χ is equivalent to the N^--ergodicity of m_Γ.

Using Theorem 6.6, we obtain:

Theorem 6.8. *For each* $\chi \in (\Gamma\backslash\Gamma_0)^*$*, the measure* m_χ *is* N^-*-ergodic.*

Moreover when Γ_0 is cocompact, Sarig [59] and Ledrappier [33, Corollary 1.4] showed that any N^-M invariant ergodic measure on $\Gamma\backslash G$ is one of m_χ's, by showing that such a measure should be A-quasi-invariant and then using the classification of Babillot on N^-M-invariant and A-quasi-invariant measures [4].

Theorem 6.9. *Suppose that* $\Gamma_0 < G$ *is cocompact and that* $\Gamma < \Gamma_0$ *is co-abelian. Then any* N^-*-invariant ergodic measure is proportional to* m_χ *for some* $\chi \in (\Gamma\backslash\Gamma_0)^*$.

Proof. If ν is an ergodic N^--invariant measure on $\Gamma\backslash G$, then the average $\nu^\clubsuit := \int_M m_*\nu dm$ is an ergodic N^-M invariant measure, and hence is proportional to m_χ for some $\chi \in (\Gamma\backslash\Gamma_0)^*$ by the afore-mentioned results ([59], [33]). As m_χ is N^--ergodic by Theorem 6.8, it follows that for almost all $m \in M$, $m_*\nu$ is proportional to m_χ. Pick one such $m \in M$. Since m_χ is M-invariant, we have ν is proportional to $(m^{-1})_* \mathsf{m}_\chi = \mathsf{m}_\chi$. □

7. Applications to counting and equidistribution problems

As before, let Γ be a normal subgroup of a Zariski dense convex cocompact subgroup Γ_0 of G with $\Gamma\backslash\Gamma_0 \simeq \mathbb{Z}^d$ for some $d \geq 0$. We describe counting and equidistribution results for Γ orbits which can be deduced from Theorems 4.7 and 4.13. The deduction process is well-understood (see [44], [40], and [38]).

7.1. **Equidistribution of translates.** Let H be an expanding horospherical subgroup or a symmetric subgroup of G. Recall the definition the PS measure on $gH/(H\cap M) \subset G/M = \mathrm{T}^1(\tilde{X})$:

$$d\tilde{\mu}^{\mathrm{PS}}_{gH}(gh) = e^{\delta \beta_{(gh)^+}} d\nu_o((gh)^+).$$

We denote by $\tilde{\mu}_{gH}$ the $H\cap M$ invariant lift to the orbit gH.

Theorem 7.1. *Let H be the expanding horospherical subgroup or a symmetric subgroup of G, and let $x = \Gamma g \in \Gamma\backslash G$. Let $U \subset H$ be a small open subset such that $u \mapsto xu$ is injective on U. Suppose that $\tilde{\mu}^{\mathrm{PS}}_H(\partial(gU)) = 0$. Then we have*

$$\lim_{t\to\infty} t^{d/2} \int_U \psi(\Gamma g u a_t) d\tilde{\mu}^{\mathrm{PS}}_H(gu) = \frac{\tilde{\mu}^{\mathrm{PS}}_H(gU)}{(2\pi\sigma)^{d/2}} \mathsf{m}^{\mathrm{BMS}}(\psi)$$

and

$$\lim_{t\to\infty} t^{d/2} e^{(D-\delta)t} \int_U \psi(xua_t) du = \frac{\tilde{\mu}^{\mathrm{PS}}_H(gU)}{(2\pi\sigma)^{d/2}} \mathsf{m}^{\mathrm{BR}_+}(\psi).$$

The assumption on H being either symmetric or horospherical ensures the wave front property of [19] which can be used to establish, as $t \to \infty$,

$$\int_U \psi(\Gamma g u a_t) d\tilde{\mu}^{\mathrm{PS}}_H(gu) \approx \langle a_t\psi, \rho_{U,\epsilon}\rangle_{m^{\mathrm{BMS}}} \text{ and } \int_U \psi(xua_t)du \approx \langle a_t\psi, \rho_{U,\epsilon}\rangle \tag{7.2}$$

where $\rho_{U,\epsilon} \in C_c(\Gamma\backslash G)$ is an ϵ-approximation of U, and $\approx$ means that the ratio of the two terms is of size $1 + O(\epsilon)$. Therefore the estimates on the matrix coefficients in Theorems 4.7 and 1.6 can be used to establish Theorem 7.1. We refer to [44, Sec. 3] and [40] for more details.

Suppose that $\Gamma_0\backslash\Gamma_0 H$ is closed. This implies that $\Gamma\backslash\Gamma H$ is closed too. Then $\tilde{\mu}^{\mathrm{PS}}_H$ induces locally finite Borel measures $\mu^{\mathrm{PS}}_{\Gamma_0,H}$ and $\mu^{\mathrm{PS}}_{\Gamma,H}$ on $\Gamma_0\backslash\Gamma_0 H$ and $\Gamma\backslash\Gamma H$ respectively.

As Γ_0 is convex cocompact, $\mu^{\mathrm{PS}}_{\Gamma_0,H}$ is compactly supported [44, Theorem 6.3]. If $H\cap\Gamma < H\cap\Gamma_0$ is of finite index, the measure $\mu^{\mathrm{PS}}_{\Gamma,H}$ is also compactly supported. In this case, by applying Theorem 7.1 to the support of $\mu^{\mathrm{PS}}_{\Gamma,H}$, we get the following:

Theorem 7.3. *Suppose that $\Gamma_0\backslash\Gamma_0 H$ is closed and that $[H\cap\Gamma_0 : H\cap\Gamma] < \infty$. Then*

$$\lim_{t\to\infty} t^{d/2} e^{(D-\delta)t} \int_H \psi([e]ha_t) dh = \frac{|\mu^{\mathrm{PS}}_{\Gamma,H}|}{(2\pi\sigma)^{d/2}} \mathsf{m}^{\mathrm{BR}_+}(\psi). \tag{7.4}$$

In the case when $H = K$ and Γ_0 is torsion free, we have $K\cap\Gamma = K\cap\Gamma_0 = \{e\}$, and Theorem 7.3 provides the equidistribution of Riemannian spheres S_t with respect to $(\Gamma\backslash G, \mathsf{m}^{\mathrm{BR}_+})$ as $t \to \infty$.

7.2. **Distribution of a discrete Γ-orbit on $H\backslash G$.** Let H be either a symmetric subgroup, a horospherical subgroup or the trivial subgroup of G. Consider the homogeneous space $H\backslash G$ and suppose $[e]\Gamma_0$ is discrete in $H\backslash G$. Recall that $v_o \in \mathrm{T}^1(\tilde{X})$ is a vector whose stabilizer is M.

Definition 7.5. (1) Define a Borel measure $\mathcal{M} = \mathcal{M}_{\Gamma,G}$ on G as follows: for $\psi \in C_c(G)$,

$$\mathcal{M}(\psi) := \frac{1}{(2\pi\sigma)^{d/2}} \int_{k_1 a_t k_2 \in KA^+K} \psi(k_1 a_t k_2) e^{\delta t} t^{d/2} d\nu_o(k_1 v_o^+) dt d\nu_o(k_2^{-1} v_o^-).$$

(2) For H symmetric or horospherical, we have either $G = HA^+K$ or $G = HA^+K \cup HA^-K$. Define a Borel measure $\mathcal{M} = \mathcal{M}_{\Gamma,H\backslash G}$ on $H\backslash G$ as follows: if $G = HA^+K$ and $\psi \in C_c(H\backslash G)$,

$$\mathcal{M}(\psi) := \frac{|\mu^{\mathrm{PS}}_{\Gamma,H}|}{(2\pi\sigma)^{d/2}} \int_{a_t k \in A^+K} e^{\delta t} \cdot t^{d/2} \psi([e]a_t k) dt d\nu_o(k^{-1} v_o^-).$$

If $G = HA^+K \cup HA^-K$ and $\psi \in C_c(H\backslash G)$,

$$\mathcal{M}(\psi) := \sum \frac{|\mu^{\mathrm{PS}}_{\Gamma,H,\pm}|}{(2\pi\sigma)^{d/2}} \int_{a_{\pm t} k \in A^\pm K} e^{\delta t} \cdot t^{d/2} \psi([e]a_t k) dt d\nu_o(k^{-1} v_o^\mp).$$

For a compact subset $B \subset H\backslash G$, define $B_\epsilon^+ = BU_\epsilon$ and $B_\epsilon^- = \cap_{u \in U_\epsilon} Bu$ if $H \neq \{e\}$. For $H = \{e\}$, set $B_\epsilon^+ = U_\epsilon B U_\epsilon$ and $B_\epsilon^- = \cap_{u_1,u_2 \in U_\epsilon} u_1 B u_2$.

Definition 7.6. Let B_T be a family of compact subsets in $H\backslash G$. We say B_T is well-rounded with respect to $\mathcal{M} = \mathcal{M}_{\Gamma,H\backslash G}$ if

(1) $\mathcal{M}(B_T) \to \infty$ as $T \to \infty$;
(2) $\limsup_{\epsilon\to 0} \limsup_{T\to 0} \frac{\mathcal{M}(B^+_{T,\epsilon} - B^-_{T,\epsilon})}{\mathcal{M}(B_T)} = 0$.

The following theorem can be proved using the equidistribution theorem 7.3 for H symmetric or horospherical, and Theorem 1.6 for H trivial (see [42, Sec. 6]).

Theorem 7.7. *Suppose that $[e]\Gamma_0$ is discrete and that $[H \cap \Gamma_0 : H \cap \Gamma] < \infty$. If $\{B_T\}$ is a sequence of compact subsets in $H\backslash G$ which are well-rounded with respect to $\mathcal{M}_{\Gamma,H\backslash G}$, then as $T \to \infty$,*

$$\#[e]\Gamma \cap B_T \sim \mathcal{M}_{\Gamma,H\backslash G}(B_T).$$

This was shown in [44] and [40] for $d = 0$.

7.3. **Distribution of circles.** We also state the following result on the asymptotic distribution of a Γ-orbit of a circle whose proof can be obtained in the same way as in [43] using Theorem 1.6 (see also [30], [34], [46]).

Theorem 7.8. *Let $G = \mathrm{PSL}_2(\mathbb{C})$. Let C_0 be a circle in the complex plane $\mathbb{C}$ such that $\mathsf{P} := \Gamma_0(C_0)$ is discrete in the space of circles in $\mathbb{C}$. Suppose that $\mathrm{Stab}_\Gamma(C_0) = \mathrm{Stab}_{\Gamma_0}(C_0)$. There exists $c > 0$ such that for any compact subset $E \subset \mathbb{C}$ whose boundary is rectifiable, we have*

$$\#\{C \in \mathsf{P} : C \cap E \neq \emptyset, \mathrm{curv}(C) \leq T\} \sim c \frac{T^\delta}{(\log T)^{d/2}} \mathrm{H}^\delta(E)$$

where H^δ is the δ-dimensional Hausdorff measure of the limit set of Γ (with respect to the Euclidean metric).

7.4. **Joint equidistribution of closed geodesics and holonomies.** Recall $X = \Gamma\backslash\tilde{X}$. A primitive closed geodesic C in $\mathrm{T}^1(X)$ is a compact set of the form

$$\Gamma\backslash\Gamma gAM/M = \Gamma\backslash\Gamma gA(v_o)$$

for some $g \in G$. The length of a closed geodesic $C = \Gamma\backslash\Gamma gAM/M$ is given as the co-volume of $AM \cap g^{-1}\Gamma g$ in AM.

For each closed geodesic $C = \Gamma\backslash\Gamma gAM/M$, we write $\ell(C)$ for the length of C, and denote by h_C the unique M-conjugacy class associated to the holonomy class of C: $\Gamma gma_{\ell(C)} = \Gamma gmh_C$. We also write $\mathcal{L}_C$ for the length measure on C

Let M^{c} denote the space of conjugacy classes of M. For $T > 0$, define

$$\mathcal{G}_\Gamma(T) := \{C : C \text{ is a closed geodesic in } \mathrm{T}^1(X),\ \ \ell(C) \le T\}.$$

For each $T > 0$, we define the measure μ_T on the product space $(\Gamma\backslash G/M) \times M^{\mathrm{c}}$: for $\psi \in C(\Gamma\backslash G/M)$ and any class function $\xi \in C(M)$,

$$\mu_T(\psi \otimes \xi) = \sum_{C \in \mathcal{G}_\Gamma(T)} \mathcal{L}_C(\psi)\xi(h_C).$$

We also define a measure η_T by

$$\eta_T(\psi \otimes \xi) = \sum_{C \in \mathcal{G}_\Gamma(T)} \mathcal{D}_C(\psi)\xi(h_C),$$

where $\mathcal{D}_C(\psi) = \ell(C)^{-1}\mathcal{L}_C(\psi)$.

Given Theorem 1.6, the following can be deduced in the same way as the proof of Theorem 5.1 in [38].

Theorem 7.9. *For any bounded $\psi \in C(\Gamma\backslash G/M)$ and a class function $\xi \in C(M)$, we have, as $T \to \infty$,*

$$\mu_T(\psi \otimes \xi) \sim \frac{e^{\delta T}}{(2\pi\sigma)^{2/d}\delta T^{d/2}} \cdot \mathsf{m}^{\mathrm{BMS}}(\psi) \cdot \int_M \xi dm$$

and

$$\eta_T(\psi \otimes \xi) \sim \frac{e^{\delta T}}{(2\pi\sigma)^{2/d}\delta T^{d/2+1}} \cdot \mathsf{m}^{\mathrm{BMS}}(\psi) \cdot \int_M \xi dm.$$

Theorem 1.12 is an easy consequence of this.

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Mathematics department, Yale university, New Haven, CT 06511 and Korea Institute for Advanced Study, Seoul, Korea
E-mail address: hee.oh@yale.edu

Mathematics department, Yale university, New Haven, CT 06511
E-mail address: wenyu.pan@yale.edu