RIGIDITY OF KLEINIAN GROUPS VIA SELF-JOININGS: MEASURE THEORETIC CRITERION

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Abstract. Let \( n, m \geq 2 \). Let \( \Gamma < \text{SO}^0(n + 1, 1) \) be a Zariski dense convex cocompact subgroup and \( \Lambda \subset \mathbb{S}^n \) be its limit set. Let \( \rho : \Gamma \to \text{SO}^0(m + 1, 1) \) be a Zariski dense convex cocompact faithful representation and \( f : \Lambda \to \mathbb{S}^m \) the \( \rho \)-boundary map. Let

\[
\Lambda_f := \bigcup \left\{ C \cap \Lambda : C \subset \mathbb{S}^n \text{ is a circle such that } f(C \cap \Lambda) \text{ is contained in a proper sphere in } \mathbb{S}^m \right\}.
\]

When there exists at least one \( \Lambda \)-doubly stable circle in \( \mathbb{S}^n \) (e.g., \( \Omega = \mathbb{S}^n - \Lambda \) is disconnected), we prove the following dichotomy:

- either \( \Lambda_f = \Lambda \) or \( \mathcal{H}^\delta(\Lambda_f) = 0 \),

where \( \mathcal{H}^\delta \) is the Hausdorff measure of dimension \( \delta = \dim_H \Lambda \). Moreover, in the former case, we have \( n = m \) and \( \rho \) is a conjugation by a Möbius transformation on \( \mathbb{S}^n \). Our proof uses ergodic theory for directional diagonal flows and conformal measure theory of discrete subgroups of higher rank semisimple Lie groups, applied to the self-joining subgroup \( \Gamma_\rho = (\text{id} \times \rho)(\Gamma) < \text{SO}^0(n + 1, 1) \times \text{SO}^0(m + 1, 1) \).

1. Introduction

Let \( \mathbb{H}^{n+1} \) denote the \((n + 1)\)-dimensional real hyperbolic space for \( n \geq 2 \). The group of its orientation-preserving isometries is given by the identity component \( \text{SO}^0(n + 1, 1) \) of the special orthogonal group. A discrete subgroup \( \Gamma < \text{SO}^0(n + 1, 1) \) is called \textit{convex cocompact} if the convex core of the associated hyperbolic manifold \( \Gamma \backslash \mathbb{H}^{n+1} \) is compact. Let \( \Gamma < \text{SO}^0(n + 1, 1) \) be a Zariski dense convex cocompact subgroup for \( n \geq 2 \), and

\[
\rho : \Gamma \to \text{SO}^0(m + 1, 1)
\]

be a Zariski dense convex cocompact faithful representation where \( m \geq 2 \). For simplicity, we will call a discrete faithful representation \( \rho : \Gamma \to \text{SO}^0(m + 1, 1) \) a deformation of \( \Gamma \) into \( \text{SO}^0(m + 1, 1) \). If \( \Gamma < \text{SO}^0(n + 1, 1) \) is cocompact and \( n = m \), Mostow strong rigidity theorem \([17]\) says that \( \rho \) is always algebraic, more precisely, it is given by a conjugation by a Möbius transformation on \( \mathbb{S}^n \). However in other cases, Marden’s isomorphism theorem and the Teichmüller theory imply that there exists a continuous family of
convex cocompact deformations, modulo the conjugations by Möbius transformation on $S^m$ (cf. [15, section 5]).

Let $\Lambda \subset S^n$ denote the limit set of $\Gamma$, which is the set of all accumulation points of $\Gamma(o)$ in $S^n$, $o \in \mathbb{H}^{n+1}$. Let $\mathcal{H}^\delta$ be the $\delta$-dimensional Hausdorff measure on $S^n$, where $\delta$ is the Hausdorff dimension of $\Lambda$ with respect to the spherical metric on $S^n$. Sullivan [18, Theorem 7] showed that for $\Gamma$ convex cocompact, we have

$$0 < \mathcal{H}^\delta(\Lambda) < \infty.$$

The main aim of this paper is to present a criterion on when $\rho$ is algebraic, in terms of the Hausdorff measure of the union of all circular slices of $\Lambda$ that are mapped into circles, or more generally into some proper spheres in $S^m$ by the $\rho$-boundary map. More precisely, by Tukia [21], there is a unique $\rho$-equivariant continuous embedding

$$f : \Lambda \to S^m,$$

called the $\rho$-boundary map. We consider all circular slices of $\Lambda$ which are mapped into some proper spheres in $S^m$ by $f$:

$$\Lambda_f := \bigcup \left\{ C \cap \Lambda : C \subset S^n \text{ is a circle such that } f(C \cap \Lambda) \text{ is contained in a proper sphere in } S^m \right\}.$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$f(C \cap \Lambda)$ is contained in a circle}
\end{figure}

We emphasize that the boundary map $f$ is defined only on $\Lambda$ and therefore our definition of $\Lambda_f$ involves the image of the intersection $C \cap \Lambda$ under $f$, but not the whole circle $C$ (see Figure 1). If $n = m$ and $f$ is a Möbius transformation of $S^n$, then $f$ clearly maps all circles to circles and hence $\Lambda_f = \Lambda$. The following main theorem of this paper says that in all other cases, $\Lambda_f$ has zero $\mathcal{H}^\delta$-measure. In other words, if $\mathcal{H}^\delta(\Lambda_f) > 0$, then $f$ is the restriction of a Möbius transformation of $S^n$ and $\rho$ is algebraic.

**Theorem 1.1.** Let $n, m \geq 2$. Let $\Gamma < SO^\circ(n + 1, 1)$ be a Zariski dense convex cocompact subgroup such that the ordinary set $\Omega = S^n - \Lambda$ has at least two components. Let $\rho : \Gamma \to SO^\circ(m + 1, 1)$ be a Zariski dense convex cocompact deformation and $f : \Lambda \to S^m$ the $\rho$-boundary map. Then

either $\Lambda_f = \Lambda$ or $\mathcal{H}^\delta(\Lambda_f) = 0$.

In the former case, we have $n = m$, $f$ extends to some $g \in \text{Möb}(S^n)$ and $\rho$ is a conjugation by $g$. 
When \( n = m = 2 \), the topological version of the above theorem was obtained in our earlier paper [11] for all finitely generated Kleinian groups. Theorem 1.1 provides its measure theoretic version. See Theorem 5.2 for the topological version for general \( n, m \geq 2 \).

**Remark 1.2.** If \( \Gamma < \text{SO}^0(3,1) \) is convex cocompact with \( \Lambda \) connected, then \( \Omega \) is disconnected [14, Chapter IX]; hence Theorem 1.1 applies.

We say that a circle \( C \subset S^n \) is \( \Lambda \)-doubly stable if for any sequence of circles \( C_k \) converging to \( C \),

\[
\# \limsup (C_k \cap \Lambda) \geq 2.
\]

If \( \Omega \) is disconnected, there exists a \( \Lambda \)-doubly stable circle (Lemma 4.2). Indeed, we prove Theorem 1.1 under a weaker condition there exists a \( \Lambda \)-doubly stable circle (Theorem 5.1).

In terms of the quasiconformal deformation indicated in Figure 2, our theorem implies that the union of circular slices of the left limit set which are mapped into circles has zero \( H^\delta \)-measure.

![Figure 2. Non-trivial quasiconformal deformation](image)

Note that \( (n + 2) \)-distinct points on \( S^n \) form the set of vertices of a unique ideal hyperbolic \( (n + 1) \)-simplex of \( \mathbb{H}^{n+1} \). Gromov-Thurston’s proof of Mostow rigidity theorem ([6], [20]) uses the fact that a homeomorphism of \( S^n \) mapping vertices of every maximal volume \( (n + 1) \)-simplex of \( \mathbb{H}^{n+1} \) to vertices of a maximal volume \( (n + 1) \)-simplex is a Möbius transformation.

Any \( (n + 2) \)-distinct points on \( S^n \) form vertices of a zero-volume \( (n + 1) \)-simplex of \( \mathbb{H}^{n+1} \) if and only if they lie in some codimension one sphere in \( S^n \), and every circle in \( S^n \) is contained in a codimension one sphere in \( S^n \). We also prove the following higher dimensional version of [11, Theorem 1.3], which answered McMullen’s question for \( n = 2 \):

**Theorem 1.3.** Let \( \Gamma < \text{SO}^0(n+1,1) \) be a Zariski dense discrete subgroup. Suppose that there exists a \( \Lambda \)-doubly stable circle in \( S^n \). If the \( \rho \)-boundary map \( f : \Lambda \to \overline{S^n} \) maps vertices of every \( (n + 1) \)-simplex of zero-volume to vertices of an \( (n + 1) \)-simplex of zero-volume, then \( f \) extends to a Möbius transformation of \( S^n \).

\[\text{Image credit: Curtis McMullen and Yongquan Zhang} \]
We obtain a stronger statement that unless $f$ extends to a Möbius transformation, the union of all vertices of $(n+1)$-simplexes of zero-volume whose images under $f$ form vertices of zero-volume $(n + 1)$-simplexes has empty interior in $\Lambda$.

**On the proof of Theorem 1.1.** We use the theory of Anosov representations. Consider the following self-joining subgroup of $G = \text{SO}^\circ(n + 1, 1) \times \text{SO}^\circ(m + 1, 1)$:

$$\Gamma_\rho := (\text{id} \times \rho)(\Gamma) = \{(\gamma, \rho(\gamma)) : \gamma \in \Gamma\}.$$  

The crucial point is that, under the assumption that both $\Gamma$ and $\rho(\Gamma)$ are Zariski dense and convex cocompact and not conjugate to each other, we have that

$\Gamma_\rho$ is a Zariski dense Anosov subgroup of $G$

with respect to a minimal parabolic subgroup

(see the discussion around (2.2)). Hence the recent classification theorem on higher rank conformal measures by Lee-Oh [13] (Theorem 2.3) and the ergodicity theorem of Burger-Landesberg-Lee-Oh [3] (Theorem 2.4) apply to our setting, yielding that for any $\Gamma_\rho$-conformal measure on the limit set of $\Gamma_\rho$, the associated Bowen-Margulis-Sullivan measure on $\Gamma_\rho \setminus G$ is ergodic and conservative for a unique one-parameter diagonal flow $A_u = \{\exp tu : t \in \mathbb{R}\}$ where $u$ is a vector in the interior of the positive Weyl chamber.

A general higher rank conformal measure seems mysterious. However, the graph structure of our self-joining group $\Gamma_\rho$ allows us to pin down a very explicit $\Gamma_\rho$-conformal measure, which we call the graph-conformal measure [10]. Indeed, under the convex cocompactness hypothesis on $\Gamma$, the graph-conformal measure is given by the pushforward measure $(\text{id} \times f)_* (\mathcal{H}^\delta|_{\Lambda})$, and this is the reason why we can relate the Hausdorff measure $\mathcal{H}^\delta|_{\Lambda}$ with dynamics on the Anosov homogeneous space $\Gamma_\rho \setminus G$ in the proof of Theorem 1.1.

The conclusion of Theorem 1.1 follows if we show that $\Gamma_\rho$ cannot be Zariski dense in $G$ (Lemma 2.2). We give a proof by contradiction. Suppose that $\Gamma_\rho$ is Zariski dense. Considering the action of $\Gamma_\rho$ on the space $\Upsilon_{\rho}$ of all ordered pairs $Y = (C, S)$ of a circle $C \subset S^n$ and a codimension one sphere $S \subset S^m$ intersecting the limit set $\Lambda_\rho \subset S^n \times S^m$ of $\Gamma_\rho$, we are then able to prove, together with the work of Guivarch-Raugi [8] and the aforementioned ergodicity and conservativity result for the directional diagonal flows, that for $\mathcal{H}^\delta|_{\Lambda}$-almost all $\xi \in \Lambda$, the $\Gamma_\rho$-orbit of $Y \in \Upsilon_{\rho}$ containing $(\xi, f(\xi))$ is dense in the space $\Upsilon_{\rho}$.

On the other hand, we show that the existence of a $\Lambda$-doubly stable circle in $S^n$ implies that for any $Y_0 = (C_0, S_0) \in \Upsilon_{\rho}$ with $f(C_0 \cap \Lambda) \subset S_0$, the orbit $\Gamma_\rho Y_0$ cannot be dense in $\Upsilon_{\rho}$ (Theorem 4.1). This shows that $\Gamma_\rho$ cannot be Zariski dense when $\mathcal{H}^\delta(\Lambda_f) > 0$. We also show that when $\Omega$ is disconnected, a $\Lambda$-doubly stable circle exists (Lemma 4.2).
Analogous question for rational maps. We close the introduction by the following question which seems natural in view of Sullivan’s dictionary between Kleinian groups and rational maps ([19], [16]).

**Question 1.4.** Let \( h_1, h_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be rational maps of degree at least 2 whose Julia sets are not contained in circles. Suppose that \( h_2 = F \circ h_1 \circ F^{-1} \) for some quasiconformal homeomorphism \( F : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \). Suppose that for the Julia set \( J = J_{h_1} \) of \( h_1 \), there exists a \( J \)-doubly stable circle in \( \hat{\mathbb{C}} \). Let

\[
J_F := \bigcup \left\{ C \cap J : \begin{array}{l}
C \subset \hat{\mathbb{C}} \text{ is a circle such that} \\
F(C \cap J) \text{ is contained in a circle}
\end{array} \right\}.
\]

(1) If \( J_F = J \), is \( F \in \text{M"ob}(\hat{\mathbb{C}}) \)?
(2) Suppose that \( h_1, h_2 \) are hyperbolic. Let \( \delta = \dim_H J \). Is it true that either \( J_F = J \) or \( \mathcal{H}^\delta(J_F) = 0 \)?

**Organization.** The main goal of section 2 is to prove Theorem 2.6, which we deduce from the classification of conformal measures in [13] and the ergodicity and conservativity of directional diagonal flows in [3] with respect to the Bowen-Margulis-Sullivan measure associated to the \( \Gamma_\rho \)-conformal measure constructed from the \( \delta \)-dimensional Hausdorff measure on \( \Lambda \). The main theorem of section 3 is Theorem 3.3 which we deduce from Theorem 2.6 and a theorem of Guivarc'h-Raugi (Theorem 3.2). In section 4 we discuss an obstruction to dense \( \Gamma_\rho \)-orbits in the space \( \Upsilon_\rho \) when a \( \Lambda \)-doubly stable circle exists. In section 5 we give a proof of Theorem 1.1. We also discuss a topological version of Theorem 1.1 without convex cocompactness assumption (Theorem 5.2).

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2. Ergodicity and graph-conformal measure

Let \((X_1, d_1)\) and \((X_2, d_2)\) be rank one Riemannian symmetric spaces. Let \(G\) be the product \(G_1 \times G_2\) where \(G_1 = \text{Isom}^\circ(X_1)\) and \(G_2 = \text{Isom}^\circ(X_2)\) are connected simple real algebraic groups of rank one. Then \(G = \text{Isom}^\circ X\) where \(X = X_1 \times X_2\) is the Riemannian product. We fix a Cartan involution \(\theta\) of the Lie algebra \(\mathfrak{g}\) of \(G\), and decompose \(\mathfrak{g}\) as \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\), where \(\mathfrak{k}\) and \(\mathfrak{p}\) are the \(+1\) and \(−1\) eigenspaces of \(\theta\), respectively. We denote by \(K\) the maximal compact subgroup of \(G\) and choose a maximal abelian subalgebra \(\mathfrak{a}\) of \(\mathfrak{p}\). Choosing a closed positive Weyl chamber \(\mathfrak{a}^+\) of \(\mathfrak{a}\), let \(A := \exp \mathfrak{a}\) and \(A^+ = \exp \mathfrak{a}^+\). The centralizer of \(A\) in \(K\) is denoted by \(M\), and we let \(N^+\) and \(N = N^-\) be the horospherical subgroups so that \(\log N^+\) and \(\log N^-\) are the sum of all negative and positive root subspaces for our choice of \(A^+\) respectively. We set

\[
P^+ = MAN^+, \quad P = P^- = MAN;
\]
they are minimal parabolic subgroups of $G$ that are opposite to each other. The quotient $F = G/P$ is known as the Furstenberg boundary of $G$, and is isomorphic to $K/M$. Let $N_K(a)$ be the normalizer of $a$ in $K$ and let $W := N_K(a)/M$ denote the Weyl group. Let $w_0 \in N_K(a)$ be the unique element in $W$ such that $w_0Pw_0^{-1} = P^+$. For each $g \in G$, we define

$$g^+ := gP \in F \quad \text{and} \quad g^- := gw_0P \in F.$$ 

An element $g \in G$ is loxodromic if $g = hamh^{-1}$ for some $a \in \text{int} \, A^+$, $m \in M$ and $h \in G$. The Jordan projection of $g$ is defined to be $\lambda(g) := \log a \in \text{int} \, a^+.$

In the rest of the section, let $\Delta$ be a Zariski dense discrete subgroup of $G$. The limit cone $L_{\Delta} \subset a^+$ is defined as the smallest closed cone containing all Jordan projections of loxodromic elements of $\Delta$. It is a convex subset of $a^+$ with non-empty interior [II Section 1.2]. Benoist showed that there exists a unique $\Delta$-minimal subset of $F$, which is called the limit set of $\Delta$. We denote it by $\Lambda_{\Delta}$. Bowen-Margulis-Sullivan measures. Let $F_i$ be the Furstenberg boundary of $G_i$, which is equal to the geometric boundary $\partial X_i$. For each $i = 1, 2$, the Busemann function $\beta_{\xi_i}(x_i, y_i)$ is defined as

$$\beta_{\xi_i}(x_i, y_i) = \lim_{t \to \infty} d_i(\xi_{i,t}, x_i) - d_i(\xi_{i,t}, y_i)$$

where $\xi_{i,t}$ is a geodesic ray toward to $\xi_i$. For $\xi = (\xi_1, \xi_2) \in F = F_1 \times F_2$ and $x = (x_1, x_2), y = (y_1, y_2) \in X$, the $a$-valued Busemann function is defined componentwise:

$$\beta_{\xi}(x, y) = (\beta_{\xi_1}(x_1, y_1), \beta_{\xi_2}(x_2, y_2)) \in a$$

where we have identified $a = a_1 \oplus a_2$ with $\mathbb{R}^2$.

In the following we fix $o = (o_1, o_2) \in X$ so that the stabilizer of $o$ is $K$.

**Definition 2.1.** For a linear form $\psi \in a^*$, a Borel probability measure $\nu$ on $F$ is called a $(\Delta, \psi)$-conformal measure (with respect to $o$) if for any $g \in \Delta$ and $\xi \in F$,

$$\frac{dg_*\nu}{d\nu}(\xi) = e^{\psi(\beta_{\xi}(o, go))}$$

where $g_*\nu(B) = \nu(g^{-1}B)$ for any Borel subset $B \subset F$. By a $\Delta$-conformal measure, we mean a $(\Delta, \psi)$-conformal measure for some $\psi \in a^*$.

Two points $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ are in general position if $\xi_i \neq \eta_i$ for each $i = 1, 2$. Let $F^{(2)}$ be the set of all pairs $(\xi, \eta) \in F \times F$ which are in general position. The map $G \to F^{(2)} \times a$, $g \mapsto (g^+, g^-, \beta_{g^+}(o, go))$ induces a $G$-equivariant homeomorphism $G/M \simeq F^{(2)} \times a$, called the Hopf-parametrization.
For a \((\Delta, \psi)\)-conformal measure \(\nu\) supported on the limit set \(\Lambda_\Delta\) for some \(\psi \in a^*\), we can define the following Borel measure on \(G/M\) using the Hopf-parametrization:

\[
d\tilde{m}_{BMS}^\nu(gM) = e^{\psi(\beta_+(o,go) + \beta_-(o,go))} d\nu(g^+)d\nu(g^-) db
\]

where \(db\) is the Haar measure on \(a\). By integrating over the fiber of \(G \to G/M\) with respect to the Haar measure of \(M\), we will consider \(\tilde{m}_{BMS}^\nu\) as a Radon measure on \(G\), which is then a left \(\Delta\)-invariant and right \(AM\)-invariant measure. We denote by \(m_{BMS}^\nu\) the Radon measure on \(\Delta \backslash G\) induced by \(\tilde{m}_{BMS}^\nu\). This measure is called the Bowen-Margulis-Sullivan measure associated to \(\nu\). Its support is

\[
\Omega_\Delta = \{[g] \in \Delta \backslash G : g^\pm \in \Lambda_\Delta\}.
\]

We refer to [5] for a detailed discussion on the construction of this measure.

**Self-joinings of convex cocompact groups.** In the rest of the section, we will consider the following special type of discrete subgroups of \(G\). Let \(\Gamma < G_1\) be a Zariski dense convex cocompact subgroup and \(\rho : \Gamma \to G_2\) be a Zariski dense convex cocompact faithful representation. Define the self-joining of \(\Gamma\) via \(\rho\):

\[
\Gamma_\rho := (id \times \rho)(\Gamma) = \{ (\gamma, \rho(\gamma)) : \gamma \in \Gamma\}
\]

which is a discrete subgroup of \(G\).

It follows from the convex cocompactness assumption for \(\Gamma\) and \(\rho(\Gamma)\) that if we fix a word metric \(\cdot\) on \(\Gamma\) for some finite generating set and fix \(o_1 \in X_1\) and \(o_2 \in X_2\), then there exist constants \(C, C' > 0\) such that for all \(\gamma \in \Gamma\),

\[
\min\{d_1(\gamma o_1, o_1), d_2(\rho(\gamma) o_2, o_2)\} \geq C|\gamma| - C'.
\]

In other words, \(\Gamma_\rho\) is an Anosov subgroup of \(G\) with respect to a minimal parabolic subgroup ([12], [7], [9]). This enables us to use the general theory developed for Anosov subgroups. We remark that ergodic theory for self-joining groups of convex cocompact groups was first studied in [2].

Since both \(G_1\) and \(G_2\) are simple, we have the following equivalence between Zariski density of the self-joining and the rigidity of \(\rho\).

**Lemma 2.2 ([11, Lemma 4.1]).** The subgroup \(\Gamma_\rho\) is Zariski dense in \(G\) if and only if \(\rho\) does not extend to a Lie group isomorphism \(G_1 \to G_2\).

Since \(\Gamma\) and \(\rho(\Gamma)\) are convex cocompact, there exists a unique \(\rho\)-equivariant continuous embedding \(f : \Lambda \to F_2\) [21]. Hence the limit set \(\Lambda_\rho \subset F\) of \(\Gamma_\rho\) is written as

\[
\Lambda_\rho = (id \times f)(\Lambda)
\]

where \(id \times f : \Lambda \to \Lambda_\rho\) is the diagonal embedding. We denote by \(L_\rho \subset a^+\) the limit cone of \(\Gamma_\rho\):

\[
L_\rho = L_{\Gamma_\rho}.
\]

Since \(\Gamma_\rho\) is Anosov, the following Theorems [2.3 and 2.4] are special cases of theorems proved in those respective papers.
Theorem 2.3 (Classification of conformal measures, [13]). Suppose that \( \Gamma_\rho \) is Zariski dense in \( G \). The space of unit vectors in \( \text{int} \mathcal{L}_\rho \) is in bijection with the space of all \( \Gamma_\rho \)-conformal measures on \( \Lambda_\rho \). Moreover, each \( \Gamma_\rho \)-conformal measure on \( \Lambda_\rho \) is a \( (\Gamma_\rho, \psi) \)-conformal measure for a unique linear form \( \psi \in \mathfrak{a}^* \).

We will denote this bijection by

\[
u_u \mapsto \nu_u^*.
\] (2.3)

For each unit vector \( u \in \text{int} \mathcal{L}_\rho \), we also denote by \( \psi_u \in \mathfrak{a}^* \) the (unique) linear form associated to \( \nu_u \), that is, \( \nu_u \) is \( (\Gamma_\rho, \psi_u) \)-conformal.

Ergodicity. For simplicity, we set

\[
\tilde{m}_{\text{BMS}} u := \tilde{m}_{\text{BMS}} \nu_u \quad \text{and} \quad m_{\text{BMS}} u := m_{\text{BMS}} \nu_u.
\]

For any non-zero vector \( u \in \mathfrak{a} \), we consider the following one-parameter semigroup/subgroup:

\[
A^+_u := \{ a_{tu} : t \geq 0 \} \quad \text{and} \quad A_u := \{ a_{tu} : t \in \mathbb{R} \},
\]

where \( a_{tu} = \exp tu \). The following ergodicity result due to Burger-Landesberg-Lee-Oh [3] is the main ingredient of our proof of Theorem 1.1:

Theorem 2.4 (Ergodicity of directional flows, [3]). Suppose that \( \Gamma_\rho \) is Zariski dense in \( G \). For any unit vector \( u \in \text{int} \mathcal{L}_\rho \), \( (m_{\text{BMS}} u, \Gamma_\rho \setminus G) \) is ergodic and conservative for the \( A_u \)-action. In particular, for \( m_{\text{BMS}} u \)-almost all \( x \), \( xA^+_u \) is dense in \( \Omega_{\Gamma_\rho} \).

Graph-conformal measure. Let \( \nu_{\Gamma} \) be the \( \Gamma \)-conformal measure supported on the limit set \( \Lambda \) of \( \Gamma \); since \( \Gamma \) is convex cocompact, it exists uniquely [18]. It turns out that the measure \( (\text{id} \times f)_* \nu_{\Gamma} \) is a \( \Gamma_\rho \)-conformal measure, where \( \text{id} \times f : \Lambda \to \Lambda_\rho \) is the diagonal embedding. We called this measure the graph-conformal measure in [10]. More precisely, we have the following lemma, thanks to which we were able to apply Theorem 2.4 in the proof of Theorem 1.1: we denote by \( \delta_{\Gamma} \) the critical exponent of \( \Gamma_\rho \).

Lemma 2.5. [10, Proposition 4.9] The measure

\[
(id \times f)_* \nu_{\Gamma}
\]

is a \( (\Gamma_\rho, \sigma_1) \)-conformal measure supported on \( \Lambda_\rho \), where \( \sigma_1 \in \mathfrak{a}^* \) is the linear form given by \( \sigma_1(t_1, t_2) = \delta_{\Gamma}t_1 \) for \( (t_1, t_2) \in \mathfrak{a} \).

We now deduce Theorem 2.6 from Theorems 2.3 and 2.4: first, there exists a unique unit vector

\[
u_{\rho} \in \text{int} \mathcal{L}_\rho \quad \text{such that} \quad (id \times f)_* \nu_\Gamma = \nu_{\rho}.
\] (2.4)

Hence if we write \( \Omega_{\rho} := \Omega_{\Gamma_\rho} \), we get the following main theorem of this section:

\footnote{The critical exponent \( \delta_{\Gamma_\rho} \) is the abscissa of convergence of the Poincaré series \( \sum_{\gamma \in \Gamma} e^{-sd_1(o_1, \gamma o_1)} \) for \( o_1 \in X_1 \).}
Theorem 2.6. Suppose that $\Gamma_\rho$ is Zariski dense. Then there exists an $(\text{id} \times f)_*\nu_\Gamma$-conull subset
\[
\Lambda'_\rho \subset \Lambda_\rho
\]
such that for any $g \in G$ with $g^+ \in \Lambda'_\rho$, the closure $[g]A_u^+\rho$ contains $\Omega_\rho$.

Proof. Since $\tilde{m}_{u,\text{BMS}}$ is equivalent to the product measure $d\nu_u \times d\nu_u \times da \times dm$ where $da$ and $dm$ denote Haar measures on $A$ and $M$ respectively, it follows from Theorem 2.4 that there exists a $\nu_u$-conull subset $\Lambda'_\rho \subset \Lambda_\rho$ such that for all $\xi \in \Lambda'_\rho$, there exists $g_0 \in G$ with $g_0^+ = \xi$ and $g_0^- \in \Lambda_\rho$ such that $[g_0]A_u^+\rho$ is dense in $\Omega_\rho$. Hence the claim follows by the following Lemma 2.7. □

Lemma 2.7 (see [11, Corollary 2.3]). Let $u \in \text{int} a^+$ and $\Delta < G$ be a Zariski dense discrete subgroup. If $[g_0]A_u^+$ is dense in $\Omega_\Delta$, then for any $g \in G$ with $g^+ = g_0^+$, the closure $[g]A_u^+$ contains $\Omega_\Delta$.

3. Orbits in the space of circle-sphere pairs

Let $G_1 = \text{SO}^o(n + 1, 1)$, $n \geq 2$ and $G_2 = \text{SO}^o(m + 1, 1)$, $m \geq 2$. We set
\[
\Upsilon = \{Y = (C, S) : C \subset S^n \text{ a circle}, S \subset S^m \text{ a codimension one sphere}\}.
\]

Let $G = G_1 \times G_2$. The group $G$ acts on $\Upsilon$ componentwise:
\[
(g_1, g_2)(C, S) = (g_1C, g_2S)
\]
for $(g_1, g_2) \in G_1 \times G_2$ and $(C, S) \in \Upsilon$. Let $\Delta < G$ be a Zariski dense discrete subgroup. Then $\Delta$ acts on the space
\[
\Upsilon_\Delta = \{Y \in \Upsilon : Y \cap \Lambda_\Delta \neq \emptyset\},
\]
which is a closed subset of $\Upsilon$.

Denseness of $\Upsilon_\Delta^\ast$. Let
\[
\Upsilon^\ast_\Delta := \{Y \in \Upsilon_\Delta : \#Y \cap \Lambda_\Delta \geq 2\}.
\]

Theorem 3.1. The subset $\Upsilon^\ast_\Delta$ is dense in $\Upsilon_\Delta$.

Recalling that $P = MAN$ and $F = G/P \simeq K/M$, we have $G/AN \simeq K$. Consider the projection $\pi : G/AN = K \to G/P = K/M$, and set
\[
\tilde{\Lambda}_\Delta = \pi^{-1}(\Lambda_\Delta) \subset G/AN = K.
\]

Since $M \simeq \text{SO}(n) \times \text{SO}(m)$ is connected, the following is a special case of a theorem of Guivarch and Raugi [8]:

Theorem 3.2 ([8 Theorem 2]). The action of $\Delta$ on $\tilde{\Lambda}_\Delta$ is minimal.
Indeed, this theorem is a key ingredient of the proof of Theorem 3.1, which we now begin.

**Proof of Theorem 3.1.** For simplicity, we write Λ for ΛΔ in this proof. Write \( K = K_1 \times K_2 \) where \( K_1 = K \cap (G_1 \times \{e\}) = \text{SO}(n+1) \) and \( K_2 = K \cap (\{e\} \times G_2) = \text{SO}(m+1) \), and similarly, we write \( M = M_1 \times M_2 = \text{SO}(n) \times \text{SO}(m) \). Via the projection \( K_i \to K_i/M_i = \mathcal{F}_i \), we can think of a point of \( K_i \) as an orthonormal frame \( f_\xi \) based at \( \xi \in \mathcal{F}_i \). Hence an element of \( K \) is a pair of orthonormal frames \((f_{\xi_1}, f_{\xi_2}) \in K_1 \times K_2 \). For an infinite sequence \((\xi_{1,j}, \xi_{2,j}) \in \mathcal{F}_1 \times \mathcal{F}_2 \) converging to \((\xi_1, \xi_2)\), we say that the convergence is \((1,1)\)-tangential to the frame \((f_{\xi_1}, f_{\xi_2})\) if, for each \( i = 1, 2 \), the sequence of unit vectors \( \frac{f_{\xi_{i,j}}}{\|f_{\xi_{i,j}}\|} \) at \( \xi_i \) converges to the first vector of the frame \( f_{\xi_i} \) as \( j \to \infty \).

Let
\[
\mathcal{E} = \left\{ (f_{\xi_1}, f_{\xi_2}) \in \bar{\Lambda} : \text{there exists a sequence } (\xi_{1,j}, \xi_{2,j}) \in \Lambda \text{ converging to } (f_{\xi_1}, f_{\xi_2}) \text{ (1,1)-tangentially) } \right\}.
\]

We first note that \( \mathcal{E} \) is non-empty. Since \( \Delta \) is Zariski dense in \( G \), \( \Delta \) contains a loxodromic element, say, \( g \in \Delta \). Denote by \( y_g \in \mathcal{F} \) the attracting fixed point of \( g \). Choose \( \zeta \in \Lambda \) which is in general position with \( y_g \). Then the sequence \( g^\ell \zeta \) converges to \( y_g \) as \( \ell \to +\infty \). The claim follows from the compactness of the unit sphere in the tangent space of \( \mathcal{F} \) at \( y_g \).

On the other hand, since the action of \( G \) on \( \mathcal{F} \) is conformal and \( \Lambda \) is \( \Delta \)-invariant, \( \mathcal{E} \) is a \( \Delta \)-invariant subset of \( \bar{\Lambda} \). Hence by Theorem 3.2,
\[
\mathcal{E} = \bar{\Lambda}.
\]

Let \( Y = (C, S) \in \Upsilon_\Delta \). We will construct a sequence \( Y_k = (C_k, S_k) \in \Upsilon_\Delta \) converging to \( Y \) as \( k \to \infty \). Choose \( \xi = (\xi_1, \xi_2) \in Y \cap \Lambda \). Choose a unit vector \( v_1 \) at \( \xi_1 \) tangent to \( C \) and a unit vector \( v_2 \) at \( \xi_2 \) tangent to \( S \). For each \( i = 1, 2 \), choose an orthonormal frame \( f_{\xi_i} \) in \( \mathcal{F}_i \) based at \( \xi_i \) whose first vector is \( v_i \). Since \( (f_{\xi_1}, f_{\xi_2}) \in \bar{\Lambda} \) and \( \mathcal{E} \) is dense in \( \bar{\Lambda} \), we can find a sequence \( (f_{\eta_{1,k}}, f_{\eta_{2,k}}) \in \mathcal{E} \) converging to \( (f_{\xi_1}, f_{\xi_2}) \) as \( k \to \infty \). Hence, for each \( k \), there exists a sequence \( \{(\eta_{1,j}^{(k)}, \eta_{2,j}^{(k)}) \in \Lambda : j = 1, 2, \cdots \} \) converging \((1,1)\)-tangentially to \((f_{\eta_{1,k}}, f_{\eta_{2,k}})\) as \( j \to \infty \). Since \( (f_{\eta_{1,k}}, f_{\eta_{2,k}}) \to (f_{\xi_1}, f_{\xi_2}) \) as \( k \to \infty \), we can choose large enough \( j_k \) for each \( k \) so that the following holds for each \( i = 1, 2 \):

1. \( \eta_{i,j_k}^{(k)} \to \xi_i \) as \( k \to \infty \); and
2. the unit tangent vector \( \frac{\eta_{i,j_k}^{(k)}}{||\eta_{i,j_k}^{(k)}||} \) at \( \eta_{i,k} \) converges to \( v_i \) as \( k \to \infty \).

Now we are ready to construct a sequence \( Y_k = (C_k, S_k) \in \Upsilon_\Delta \):

1. Fix \( z_1 \in C - \{\xi_1\} \) and let \( C_k \) be the circle passing through \( z_1, \eta_{1,k} \) and \( \eta_{1,j_k}^{(k)} \).
2. Fix \( z_2 \in S - \{\xi_2\} \). The tangent space \( T_{\xi_2}S \) of \( S \) at \( \xi_2 \) is a codimension one subspace of the tangent space \( T_{\xi_2} \mathcal{F}_2 \). Noting that \( v_2 \in T_{\xi_2}S \),
we can choose unit tangent vectors \( w_1, \ldots, w_{m-2} \in T_{\xi_2}S \) so that \( v_2, w_1, \ldots, w_{m-2} \) form a basis of \( T_{\xi_2}S \). For each \( \ell = 1, \ldots, m-2 \), we choose a sequence \( \zeta_{\ell,k} \in F_2 \) converging to \( \xi_2 \) such that the unit vectors \( \frac{n_{2,k}\zeta_{\ell,k}}{\|n_{2,k}\zeta_{\ell,k}\|} \) converges to \( w_{\ell} \) as \( k \to \infty \). Then for each \( k \geq 1 \) large enough, the set
\[
\{ z_2, \eta_{2,k}, \eta_{2,jk}, \zeta_{1,k}, \ldots, \zeta_{m-2,k} \}
\]
has cardinality \((m+1)\) and hence uniquely determines an \((m-1)\)-dimensional sphere in \( F_2 = S^m \), which we set to be \( S_k \).

Since \((C_k, S_k) \cap \Lambda\) contains two distinct points \((\eta_{1,k}, \eta_{2,k})\) and \((\eta_{1,jk}, \eta_{2,jk})\), we have
\[
(C_k, S_k) \in \Upsilon^* \Delta.
\]
Moreover, as \( k \to \infty \), \( C_k \) converges to the unique circle passing through \( z_1 \) and tangent to \( v_1 \), which must be \( C \), and \( S_k \) converges to the unique sphere passing through \( z_2 \) and whose tangent space at \( \xi_2 \) is same as \( T_{\xi_2}S \), which must be \( S \). Therefore \((C_k, S_k) \in \Upsilon^* \Delta\) converges to \( Y = (C, S) \). This finishes the proof of Theorem 3.1.

**Dense orbits.** Let \( \Gamma < SO^\circ(n+1,1) \) be a convex cocompact subgroup where \( n \geq 2 \). Then \( \nu_\Gamma \) is equal to \( \delta_\Gamma \)-dimensional Hausdorff measure \( \mathcal{H}^\delta_\Gamma|_\Lambda \) and \( \delta := \delta_\Gamma \) is equal to the Hausdorff dimension of \( \Lambda \) by [18]. Let \( \rho : \Gamma \to SO^\circ(m+1,1) \) be a Zariski dense convex cocompact faithful representation. Let \( \Gamma_\rho := (id \times \rho)(\Gamma) < G \) and
\[
\Upsilon_\rho := \Upsilon_{\Gamma_\rho} = \{ Y = (C, S) \in \Upsilon : Y \cap \Lambda_\rho \neq \emptyset \}.
\]

**Theorem 3.3.** Suppose that \( \Gamma_\rho \) is Zariski dense. Then there exists a \( \mathcal{H}^\delta|_\Lambda \)-conull \( \Lambda' \subset \Lambda \) such that for any \( Y \in \Upsilon_\rho \) intersecting \((id \times f)(\Lambda')\) non-trivially,
\[
\Gamma_\rho Y = \Upsilon_\rho.
\]

**Proof.** Since \( G \) acts transitively on \( \Upsilon \) as homeomorphisms, we have the homeomorphism
\[
\Upsilon \simeq G/H
\]
where \( H = \text{Stab}(Y_0) \) is the stabilizer of some \( Y_0 = (C_0, S_0) \in \Upsilon \). Noting that \( H^\circ \) is a semisimple real algebraic subgroup conjugate to \((SO^\circ(2,1) \times SO(n-1)) \times SO^\circ(m,1) \), we may choose \( Y_0 \) so that \( H \supset A \) and that \( H \cap P \) is a minimal parabolic subgroup of \( H \).

In particular, if \( g \in G \) is such that the closure of \([g]A_u^+ \) contains \( \Omega_\rho \) for some \( u \in \text{int } \mathfrak{a}^+ \), then the closure of \( \Gamma_\rho gY_0 \) contains \( \Upsilon^*_\rho \), and hence by Theorem 3.1
\[
\Gamma_\rho gY_0 = \Upsilon_\rho.
\]
Since \( \Gamma < \text{SO}^0(n+1,1) \) is convex cocompact, we have that \( \mathcal{H}^\delta \vert _\Lambda \) is the unique \( \Gamma \)-conformal measure on \( \Lambda \), up to a constant multiple [18]. Therefore Theorem 3.3 follows from Theorem 2.6 and Lemma 2.5.

4. Doubly Stable Condition

In this section, let \( \Gamma < \text{SO}^0(n+1,1) \) be a discrete group, \( n \geq 2 \), which is not necessarily convex cocompact. Let \( \Lambda \subset \mathbb{S}^n \) denote its limit set. The \( \limsup \) of a sequence of subsets \( S_k \) is defined as \( \limsup S_k = \bigcap_{k \in \mathbb{N}} \bigcup_{j \geq k} S_j \).

We say that a circle \( C \subset \mathbb{S}^n \) is \( \Lambda \)-doubly stable if for any sequence of circles \( C_k \) converging to \( C \),

\[
\# \limsup (C_k \cap \Lambda) \geq 2.
\]

If \( \Omega \) is disconnected, there exists a \( \Lambda \)-doubly stable circle (Lemma 4.2).

**Theorem 4.1.** Let \( \Gamma < \text{SO}^0(n+1,1) \) be a discrete subgroup and \( \rho: \Gamma \rightarrow \text{SO}^0(m+1,1), m \geq 2 \), be a discrete faithful representation with a boundary map \( f: \Lambda \rightarrow \mathbb{S}^m \). Assume that there exists at least one \( \Lambda \)-doubly stable circle. If \( (C_0, S_0) \in \Upsilon_\rho \) such that \( f(C_0) \cap \Lambda \subset S_0 \), then

\[
\overline{\rho(C_0, S_0)} \neq \Upsilon_\rho.
\]

**Proof.** Let \( C \subset \mathbb{S}^n \) be a \( \Lambda \)-doubly stable circle. Then for any sequence of circles \( C_k \subset \mathbb{S}^n \) converging to \( C \) as \( k \rightarrow \infty \), we have

\[
\# \limsup (C_k \cap \Lambda) \geq 2. \tag{4.1}
\]

It follows that \( \#C \cap \Lambda \geq 2 \). We first claim that there exists a codimension one sphere \( S \subset \mathbb{S}^m \) such that

\[
\#S \cap f(C \cap \Lambda) = 1. \tag{4.2}
\]

Since \( C \cap \Lambda \) is not homemorphic to \( \mathbb{S}^m \), \( m \geq 2 \), the image \( f(C \cap \Lambda) \) is a proper compact subset of \( \mathbb{S}^m \). Therefore we can find a minimal closed \( m \)-ball \( B \subset \mathbb{S}^m \) containing \( f(C \cap \Lambda) \). By the minimality of \( B \), there exists \( \xi_0 \in C \cap \Lambda \) such that \( f(\xi_0) \) lies in the boundary of \( B \). Now any codimension one sphere \( S \) in \( \mathbb{S}^m \) such that \( C \cap B = \{ f(\xi_0) \} \) satisfies (4.2).

Set \( Y = (C, S) \). Since \( (\xi_0, f(\xi_0)) \in (C, S) \), we have \( Y \in \Upsilon_\rho \). We claim that for any \( (C_0, S_0) \in \Upsilon_\rho \) such that \( f(C_0) \cap \Lambda \subset S_0 \), we have \( Y \notin \overline{\rho(C_0, S_0)} \); this implies the theorem. Suppose not. Then there exists a sequence \( \gamma_k \in \Gamma \) such that \( \gamma_k C_0 \rightarrow C \) and \( \rho(\gamma_k) S_0 \rightarrow S \) as \( k \rightarrow \infty \). By (4.1), we have

\[
\# \limsup (\gamma_k C_0 \cap \Lambda) \geq 2. \tag{4.3}
\]

By the \( \rho \)-equivariance of \( f \), we have

\[
f(\gamma_k C_0 \cap \Lambda) = f(\gamma_k (C_0 \cap \Lambda)) = \rho(\gamma_k) f(C_0 \cap \Lambda) \subset \rho(\gamma_k) S_0.
\]

Hence

\[
\limsup f(\gamma_k C_0 \cap \Lambda) \subset \limsup \rho(\gamma_k) S_0 = S.
\]

Since \( \limsup f(\gamma_k C_0 \cap \Lambda) \subset f(C \cap \Lambda) \) and \( f \) is injective, it follows from (4.3) that \( \#S \cap f(C \cap \Lambda) \geq 2 \). This contradicts (4.2), proving the claim.\( \square \)
Lemma 4.2. Let $\Gamma < \text{SO}^o(n+1,1)$ be a discrete subgroup. If $\Omega$ is disconnected, then $\Lambda$ is doubly stable.

Proof. Let $\Omega_1, \Omega_2$ be distinct connected components of $\Omega$ and fix any $\xi \in \Lambda$. Let $C$ be a circle containing $\xi$ and intersecting $\Omega_1$ and $\Omega_2$.

Let $C_k$ be a sequence of circles converging to $C$ as $k \to \infty$. We claim that $\# \limsup(C_k \cap \Lambda) \geq 2$. Suppose that $\# \limsup(C_k \cap \Lambda) \leq 1$. We will show that $C \cap \Omega_1$ is a singleton, which is a contradiction since $C \cap \Omega_1$ is an open subset of $C$.

For each $k$, let $I_k \subset C_k$ be a compact interval containing $C_k \cap \Lambda$ with minimal diameter. Since $C_k - I_k$ is a connected subset of $\Omega$, $C_k - I_k \subset W_k$ for some connected component $W_k$ of $\Omega$. After passing to a subsequence and relabeling $\Omega_1$ and $\Omega_2$ if necessary, we may assume that $\Omega_1 \neq W_k$ and hence $\Omega_1 \cap W_k = \emptyset$ for all $k$.

Let $x, y \in C \cap \Omega_1$. Since the sequence $C_k$ converges to $C$, $x = \lim_{k \to \infty} x_k$ and $y = \lim_{k \to \infty} y_k$ for some $x_k, y_k \in C_k$. Since $\Omega_1$ is open, we may assume that $x_k, y_k \in C_k \cap \Omega_1$ for all $k \geq 1$. Hence $x_k, y_k \notin W_k$; so $x_k, y_k \in I_k$.

Since $\# \limsup(C_k \cap \Lambda) \leq 1$, the diameter of $I_k$ tends to 0 as $k \to \infty$. Therefore the distance between $x_k$ and $y_k$ must go to 0 and hence $x = y$. This proves the claim, finishing the proof. \qed

5. Rigidity via circular slices

Let $n, m \geq 2$. Let $\Gamma < \text{SO}^o(n+1,1)$ be a Zariski dense convex cocompact subgroup. Let $\rho : \Gamma \to \text{SO}^o(m+1,1)$ be a Zariski dense convex cocompact deformation and $f : \Lambda \to \mathbb{S}^m$ be its boundary map. Recall

$$\Lambda_f = \bigcup \{ C \cap \Lambda : C \subset \mathbb{S}^n \text{ is a circle such that } f(C \cap \Lambda) \text{ is contained in a } (m-1)\text{-sphere of } \mathbb{S}^m \}.$$  

Theorem 1.1 is a special case of the following:

Theorem 5.1. Suppose that there exists a $\Lambda$-doubly stable circle (e.g. $\Omega$ has at least two components). Then

either $\Lambda_f = \Lambda$ or $\mathcal{H}^\delta(\Lambda_f) = 0$.

In the former case, we have $n = m$, $f$ extends to some $g \in \text{Möb}(\mathbb{S}^n)$ and $\rho$ is a conjugation by $g$.

Proof. Suppose that $\mathcal{H}^\delta(\Lambda_f) > 0$. We need to show that $\Lambda_f = \Lambda$. We claim that $\Gamma_\rho$ cannot be Zariski dense in $G$. Suppose that $\Gamma_\rho$ is Zariski dense. Let $\Lambda' \subset \Lambda$ be the $\mathcal{H}^\delta(\Lambda_\rho)$-conull subset given by Theorem 3.3. Since $\mathcal{H}^\delta(\Lambda_f) > 0$, there exists $\xi_0 \in \Lambda_f \cap \Lambda'$. By the definition of $\Lambda_f$, we can find $Y_0 = (C_0, S_0) \in \Upsilon_\Lambda$ so that $Y_0 \supseteq (\xi_0, f(\xi_0))$ and $f(C_0 \cap \Lambda) \subset S_0$. By the definition of $\Lambda'$ as in Theorem 3.3, we have

$$\Gamma_\rho Y_0 = \Upsilon_\rho.$$
On the other hand, since there exists a Λ-doubly stable circle, Theorem 4.1 implies that \( \Gamma_0 \) is not \( \Gamma_0 \)-Zariski dense. Hence by Theorem 2.4, \( \rho \) extends to a Lie group isomorphism \( \text{SO}^\circ(n+1,1) \rightarrow \text{SO}^\circ(m+1,1) \) and in particular \( n = m \). Since the Lie group automorphism of \( \text{SO}^\circ(n+1,1) \) is a conjugation by some \( g \in \text{Möb}(\mathbb{S}^n) \), it follows that \( \rho \) is a conjugation by \( g \) and by the uniqueness of the \( \rho \)-boundary map, \( f \) is the restriction of \( g \) to \( \Lambda \). Therefore \( \Lambda_f = \Lambda \).

By Lemma 4.2, when \( \Omega \) is disconnected, there exists a Λ-doubly stable circle. Hence the proof is now complete.

**Topological version without convex cocompactness.** The assumption that \( \Gamma \) and \( \rho(\Gamma) \) are convex cocompact was used to apply the ergodicity as in Theorem 2.4. The approach of our paper proves the following theorem without the convex cocompact hypothesis, which was shown in [11] for \( n = m = 2 \):

**Theorem 5.2.** Let \( \Gamma < \text{SO}^\circ(n+1,1) \) be a Zariski dense discrete subgroup. Suppose that there exists a Λ-doubly stable circle. Let \( \rho : \Gamma \rightarrow \text{SO}^\circ(m+1,1) \) be a Zariski dense deformation with a \( \rho \)-boundary map \( f : \Lambda \rightarrow \mathbb{S}^m \). Then

either \( \Lambda_f = \Lambda \) or \( \Lambda_f \) has empty interior in \( \Lambda \).

In the former case, we have \( n = m \), \( f \) extends to some \( g \in \text{Möb}(\mathbb{S}^n) \) and \( \rho \) is a conjugation by \( g \).

For this, we need to replace the ergodicity theorem (Theorem 2.4) by the following theorem of Chow-Sarkar for \( \Delta = \Gamma_\rho \):

**Theorem 5.3 ([4, Theorem 8.1]).** Let \( \Delta < G \) be a Zariski dense discrete subgroup. For any \( u \in \text{int} \mathcal{L}_\Delta \), there exists a dense \( \mathcal{L}_\Delta^u \)-orbit in

\[
\Omega_\Delta := \{ [g] \in \Delta \backslash G : g^\pm \in \Lambda_\Delta \}.
\]

This theorem provides a dense subset \( \Lambda' \subset \Lambda \) such that for any \( Y \subset \Upsilon_\rho \) intersecting \( (\text{id} \times f)(\Lambda') \) non-trivially, \( \Gamma_\rho Y \) is dense in \( \Upsilon_\rho \), which is a topological version of Theorem 3.3. With this replacement, the rest of the proof can be repeated in verbatim. Theorem 1.3 is a direct consequence of Theorem 5.2 and Lemma 4.2.

**References**


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