INVARIANT MEASURES FOR HOROSPHERICAL ACTIONS AND ANOSOV GROUPS.

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Abstract. Let Γ be an Anosov subgroup of a connected semisimple real linear Lie group $G$. For a maximal horospherical subgroup $N < G$, we show that the space of all non-trivial $NM$-invariant ergodic and $A$-quasi-invariant Radon measures on $\Gamma \backslash G$, up to proportionality, is homeomorphic to $\mathbb{R}^{\text{rank } G - 1}$, where $A$ is a maximal real split torus and $M$ is a maximal compact subgroup which normalizes $N$.

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1. Introduction

Let $G$ be a connected real semisimple linear Lie group and $\Gamma < G$ a Zariski dense discrete subgroup. A subgroup $N$ of $G$ is called horospherical if there exists a diagonalizable element $a \in G$ such that

$$N = \{ g \in G : a^{k} ga^{-k} \to \infty \quad \text{as} \quad k \to +\infty \},$$

or equivalently, $N$ is the unipotent radical of a parabolic subgroup of $G$.

We are interested in the measure rigidity property of horospherical subgroup actions on the homogeneous space $\Gamma \backslash G$. When $\Gamma$ is a lattice, i.e., when $\Gamma \backslash G$ has finite volume, the well-known theorem of Dani [10] gives a complete classification of Radon measures (=locally finite Borel measures) invariant

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under a horospherical subgroup of $G$. This rigidity phenomenon extends to any unipotent subgroup action by the celebrated theorem of Ratner in \[34\].

When $G$ has rank one and $\Gamma$ is geometrically finite, the horospherical subgroup action on $\Gamma \backslash G$ is known to be essentially uniquely ergodic; there exists a unique non-trivial Radon measure on $\Gamma \backslash G$ which is ergodic for a horospherical subgroup of $G$, called the Burger-Roblin measure \([8, 35, 46]\). When $\Gamma$ is geometrically infinite, there may be a continuous family of horospherically invariant ergodic measures as first discovered by Babillot and Ledrappier \([2, 3]\). For a certain family of geometrically infinite groups, a complete classification of horospherically invariant ergodic measures has been obtained; see \([35, 39, 24, 27, 22, 23]\), etc. We refer to a recent article by Landesberg and Lindenstrauss \([22]\) for a more precise description on the rank one case.

When $G$ has rank at least 2 and $\Gamma$ has infinite co-volume in $G$, very little, if any, is known about invariant measures. The work of Quint \([30]\) on a higher rank version of Patterson-Sullivan theory supplies a continuous family of maximal horospherically invariant Burger-Roblin measures, as was introduced in \([11]\).

In this paper, we focus on a special class of discrete subgroups, called Anosov subgroups. In the rank one case, this class consists of all convex cocompact subgroups, and hence the class of Anosov subgroups can be considered as a generalization of convex cocompact subgroups of rank one Lie groups to higher rank.

When $\Gamma < G$ is Anosov, we show that all of these Burger-Roblin measures are ergodic for maximal horospherical foliations and classify all ergodic non-trivial Radon measures for maximal horospherical foliations, which are also quasi-invariant under Weyl chamber flow. In particular, we establish a homeomorphism between the space of these measures and the interior of the projective limit cone of $\Gamma$, which is again homeomorphic to $\mathbb{R}^{\text{rank } G - 1}$.

In order to formulate our main result precisely, we begin with the definition of an Anosov subgroup of $G$. Let $P$ be a minimal parabolic subgroup of $G$ and $\mathcal{F} := G/P$ the Furstenberg boundary. We denote by $\mathcal{F}^{(2)}$ the unique open $G$-orbit in $\mathcal{F} \times \mathcal{F}$. A Zariski dense discrete subgroup $\Gamma < G$ is called an Anosov subgroup if it is a finitely generated word hyperbolic group which admits a $\Gamma$-equivariant embedding $\zeta$ of the Gromov boundary $\partial \Gamma$ into $\mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y$ in $\partial \Gamma$.

First introduced by Labourie \([21]\) as the images of Hitchin representations of surface groups \([17, 13]\), this definition is due to Guichard and Wienhard \([16]\), who showed that Anosov subgroups (more precisely, Anosov representations) form an open subset in the representation variety $\text{Hom}(\Gamma, G)$. The class of Anosov groups include Schottky subgroups and hence any Zariski dense discrete subgroup of $G$ contains an Anosov subgroup. We also refer to the work of Kapovich, Leeb and Porti \([19]\) for other equivalent characterizations of Anosov groups, as well as to excellent survey articles by Kassel and Wienhard \([20, 45]\) on higher Teichmüller theory.
We let $P = NMA$ be the Langlands decomposition of $P$, so that $N$ is the unipotent radical of $P$, $A$ is a maximal real split torus of $G$, and $M$ is a compact subgroup which commutes with $A$.

The limit set $\Lambda$ of $\Gamma$ is the minimal $\Gamma$-invariant closed subset of $F$. It follows that the set

$$\mathcal{E} := \{[g] \in \Gamma \backslash G : gP \in \Lambda\}$$

is the unique minimal $P$-invariant closed subset of $\Gamma \backslash G$. We call a $P$-quasi-invariant measure on $\Gamma \backslash G$ non-trivial if it is supported on $\mathcal{E}$.

**Theorem 1.1.** For any Anosov subgroup $\Gamma \lhd G$, the space $Q_\Gamma$ of all non-trivial $NM$-invariant ergodic and $A$-quasi-invariant Radon measures on $\Gamma \backslash G$, up to constant multiples, is homeomorphic to $\mathbb{R}^{\text{rank}G - 1}$.

In order to describe the explicit homeomorphism, we need to define Burger-Roblin measures on $\mathcal{E}$. Denote by $\mathfrak{a}$ the Lie algebra of $A$ and fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ so that $\log N$ is the sum of positive root subspaces. Fix a maximal compact subgroup $K$ of $G$ so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Let $\mu : G \to \mathfrak{a}^+$ denote the Cartan projection map. We denote by $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ the limit cone of $\Gamma$, which is the smallest closed cone containing $\mu(\Gamma)$. This is known to be a convex cone with non-empty interior [4, Thm. 1.2].

Let $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R}$ denote the growth indicator function of $\Gamma$ (see Definition 2.16). For Anosov subgroups, the following spaces are homeomorphic to each other:

\[
\int(\mathbb{P}\mathcal{L}_\Gamma) \simeq D_\Gamma^\psi := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma, \psi(v) = \psi_\Gamma(v) \text{ at some } v \in \text{int} \mathcal{L}_\Gamma\}
\]

where $\text{int}(\mathbb{P}\mathcal{L}_\Gamma)$ denotes the interior of $\mathbb{P}\mathcal{L}_\Gamma$ ([51], [37], [6], cf. Proposition 4.3). Since $\text{int}(\mathcal{L}_\Gamma)$ is an open convex cone of $\mathfrak{a}^+$, $\text{int}(\mathbb{P}\mathcal{L}_\Gamma) \simeq \mathbb{P}\text{int}(\mathcal{L}_\Gamma)$ is homeomorphic to $\mathbb{R}^{\text{rank}G - 1}$.

For each $\psi \in D_\Gamma^\psi$, there exists a $(\Gamma, \psi)$-Patterson-Sullivan probability measure on the limit set $\Lambda$, constructed by Quint [30]. For $\Gamma$ Anosov, such a measure is $\Gamma$-ergodic and exists uniquely, which we denote by $\nu_\psi$. We also denote by $\nu_\psi$ the $M$-invariant lift of $\nu_\psi$ on $F \simeq K/M$ to $K$ by abuse of notation. The Burger-Roblin measure $m_\psi^\text{BR}$ on $\Gamma \backslash G$ is induced from the following $\Gamma$-invariant measure $\tilde{m}_\psi^\text{BR}$ on $G$ given in terms of the Iwasawa coordinates $G = KAN$: for $g = k(\exp b)n \in KAN$,

\[
dm_\psi^\text{BR}(g) = e^{\psi(b)}dndbdnu_\psi(k)
\]

where $dn$ and $db$ are Lebesgue measures on $N$ and $a$ respectively. The support of $m_\psi^\text{BR}$ is given by

$$\mathcal{E} := \{[g] \in \Gamma \backslash G : gP \in \Lambda\};$$

this is the unique minimal $P$-invariant closed subset of $\Gamma \backslash G$.

In the rest of the introduction, we let $\Gamma \lhd G$ be an Anosov subgroup. The following is a more elaborate version of Theorem 1.1.
Theorem 1.4 (Classification). The map $\psi \mapsto [m^{\text{BR}}_\psi]$ defines a homeomorphism between $D^*_\Gamma$ and $Q_\Gamma$. In particular, $Q_\Gamma$ is homeomorphic to $\text{int}(\mathbb{P}L_\Gamma) \simeq \mathbb{R}^{\text{rank}G-1}$.

While the $P$-ergodicity of $m^{\text{BR}}_\psi$ follows from the $\Gamma$-ergodicity of $\nu_\psi$, the well-definedness of the above map is a most significant part of Theorem 1.4:

Theorem 1.5 (Ergodicity). For each $\psi \in D^*_\Gamma$, $m^{\text{BR}}_\psi$ is $\text{NM}$-ergodic.

A Radon measure $\mu$ on $\Gamma \backslash G$ is called $P$-semi-invariant if there exists a character $\chi : P \to \mathbb{R}^*$ such that $p_*\mu = \chi(p)\mu$ for all $p \in P$. Note that any $P$-semi-invariant Radon measure is necessarily $\text{NM}$-invariant. We show that any $P$-semi-invariant Radon measure on $\mathcal{E}$ is of the form $m^{\text{BR}}_\psi$ for some $\psi \in D^*_\Gamma$ (Proposition 10.23). Hence Theorem 1.5 implies:

Corollary 1.6. Any $P$-semi-invariant Radon measure on $\mathcal{E}$ is $\text{NM}$-ergodic. In particular, $Q_\Gamma$ coincides with the space of all $P$-semi-invariant Radon measures on $\mathcal{E}$, up to constant multiples.

Discussion on the proofs. Defining a $\Gamma$-invariant Radon measure $\hat{\nu}_\psi$ on $\mathcal{H} := G/NM \simeq F \times a$ by

$$d\hat{\nu}_\psi(gNM, b) = e^{\psi(b)}d\nu_\psi(gP)db,$$

the standard duality theorem implies that the $\text{NM}$-ergodicity of $m^{\text{BR}}_\psi$ is equivalent to the $\Gamma$-ergodicity of $\hat{\nu}_\psi$.

Generalizing the observation of Schmidt 40 (also see 35) to a higher rank situation, the $\Gamma$-ergodicity of $\hat{\nu}_\psi$ follows if the closed subgroup, say $E_{\nu_\psi} < a$ (Proposition 9.2):

Definition 1.7. An element $v \in a$ is called a $\nu_\psi$-essential value, if for any Borel set $B \subset F$ with $\nu_\psi(B) > 0$ and any $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$B \cap \gamma B \cap \{\xi \in F : \|\beta_\xi(o, \gamma o) - v\| < \varepsilon\}$$

has a positive $\nu_\psi$-measure, where $o = [K] \in G/K$ and $\beta : F \times G/K \times G/K \to a$ denotes the $a$-valued Busemann function (see Def. 3.1).

Recalling that the Jordan projection $\lambda(\Gamma)$ of $\Gamma$ generates a dense subgroup of $a$ 34, the following is the main ingredient of our proof of Theorem 1.5:

Lemma 1.8. For each $\psi \in D^*_\Gamma$, there exists a finite subset $F_\psi \subset \Gamma$ such that $\lambda(\Gamma - F_\psi) \subset E_{\nu_\psi}$.

In particular, $E_{\nu_\psi} = a$.

Among other things, the following three key properties of Anosov groups play important roles in the proof of Lemma 1.8:

(1) (Antipodality) $\Lambda \times \Lambda - \{(\xi, \xi)\} \subset \mathcal{F}^{(2)}$;
(2) (Regularity) If $\gamma_i \to \infty$ in $\Gamma$, then $\alpha(\mu(\gamma_i)) \to \infty$ for each simple root $\alpha$ of $\text{Lie}(G)$ with respect to $a^+$;
Lemma 1.11. There exists a constant $D > 0$ such that any discrete geodesic ray $[e,x]$ in $\Gamma$ tending to $x \in \partial \Gamma$ is contained in the $D$-neighborhood of some $gA^+$ in $G$ where $g \in G$ satisfies $gP = \zeta(x)$.

(1) is a part of the definition of an Anosov subgroup. (2) follows from the fact that the limit cone of $\Gamma$ is contained in the interior of $a^+$ (see Lemma 7.2), (3) is proved in [19] (see Proposition 5.10).

We give an outline of the proof of Lemma 1.8. We mention that many aspects of our proof can be simplified for a special class of $\psi \in D^*_\Gamma$ with certain strong positivity property (cf. Lemma 5.1); however as our eventual goal is the classification theorem as stated in Theorem 1.4, we need to address all $\psi \in D^*_\Gamma$ which makes the proof much more intricate and requires the full force of the Anosov property of $\Gamma$.

Let $\psi \in D^*_\Gamma$. Fix $\gamma_0 \in \Gamma$ and let $\xi_0 \in F$ denote the attracting fixed point of $\gamma_0$. For any fixed $\varepsilon > 0$, we aim to show that for any Borel set $B \subset F$ with $\nu_p(B) > 0$, there exists $\gamma \in \Gamma$ such that

$$\sigma_p(B \cap \gamma_0^{-1}B \cap \{\xi \in F : \|\beta\gamma\xi_0^{-1}\xi_0 - \lambda(\gamma_0)\| < \varepsilon\}) > 0;$$

this implies that $\lambda(\gamma_0)$ is an essential value of $\nu_p$.

For $p \in G/K$, we define

$$d'_{\psi,p}(\xi_1, \xi_2) = e^{-[\xi_1, \xi_2]_{\psi,p}}$$

for any $\xi_1 \neq \xi_2$ in $\Lambda$, where $[\cdot, \cdot]_{\psi,p}$ denotes the $\psi$-Gromov product based at $p$ (Def. 3.1). Its well-definedness is due to the antipodality (1). In the rank one case, this is simply the restriction of the classical visual metric to the limit set $\Lambda$. In general, it is not even symmetric but we show that any sufficiently small power of $d'_{\psi,p}$ is comparable to some genuine metric on $\Lambda:

**Theorem 1.10.** For all sufficiently small $s > 0$, there exist $C_s > 0$, and a metric $d_s$ on $\Lambda$ such that for all $\xi_1 \neq \xi_2$ in $\Lambda$,

$$C_s^{-1}d'_{\psi,p}(\xi_1, \xi_2)^s \leq d_s(\xi_1, \xi_2) \leq C_s d'_{\psi,p}(\xi_1, \xi_2)^s.$$

As a consequence, $d'_{\psi,p}$ can be used to define virtual balls with respect to which Vitali type covering lemma can be applied. Consider the family $D(\gamma\xi_0, r) := B_p(\gamma\xi_0, \frac{1}{r}e^{-\psi(a(\xi_0^{-1}p,p)+\frac{1}{2}\xi(\gamma_0^{-1}p,p))}r)$ where $a(q,p)$ denotes the $a$-valued distance from $q$ to $p$ (Def. 2.4). We then show that for all sufficiently small $r > 0$, there are infinitely many $D(\gamma\xi_0, r)$ which satisfies (1.9) (Lemma 10.12). The key ingredient in this step is the following:

**Lemma 1.11.** There exists $C = C(\Gamma, \psi, p) > 0$ such that for all $\gamma \in \Gamma$ and $\xi \in \Lambda$,

$$-\psi(a(p, \gamma p)) - C \leq \psi(\beta\xi(\gamma p, p)) \leq \psi(a(\gamma p, p)) + C.$$

In the rank one case, a stronger statement $-d(p, q) \leq \beta\xi(q, p) \leq d(p, q)$ holds for all $q, p \in G/K$ and $\xi \in F$. For a special type of $\psi$ which we call strongly positive, there is a direct generalization of this fact (see Lemma 5.1). For a general $\psi \in D^*_\Gamma$, our proof of Lemma 1.11 is based on the property that the orbit map $\gamma \mapsto \gamma(0) \in G/K$ sends a shadow in the word hyperbolic
group $\Gamma$ to a shadow in the symmetric space $G/K$ (Proposition 5.12). We also need the following lemma, which is of independent interest: we denote $|\cdot|$ the word length on $\Gamma$ for a fixed finite symmetric generating set.

**Lemma 1.12.** There exists $R > 0$ such that for any $\gamma_1, \gamma_2 \in \Gamma$ with $|\gamma_1 \gamma_2| = |\gamma_1| + |\gamma_2|$, we have

$$\|\mu(\gamma_1 \gamma_2) - \mu(\gamma_1) - \mu(\gamma_2)\| < R.$$

We emphasize that this lemma does not follow from the property of Anosov groups that $(\Gamma, |\cdot|) \to G$ is a quasi-isometric embedding [16, Thm. 1.7], due to the multiplicative constant.

To establish (1.9) for a general Borel subset $B \subset F$ with positive $\nu_\psi$-measure, we would like to approximate $B$ by some $D(\gamma_0, r)$ satisfying (1.9). In this step, we prove the following, extending the corresponding result in the rank one case ([25], [1], [26]):

**Theorem 1.13.** For any $\Gamma$-Patterson-Sullivan measure $\nu$ on $\Lambda$, the set of Myrberg limit points (Def. 8.1) has full $\nu$-measure.

It follows that for the $AM$-invariant Bowen-Margulis-Sullivan measure $\nu_{BMS}^\psi$ on $\Gamma \backslash G$, almost all points have dense $A^+M$ orbits (Corollary 8.11). Using the property that virtual balls $B_p(\gamma_0, r)$ satisfy a covering lemma (Lemma 6.10) which is a consequence of Theorem 1.10 we show that $\nu_\psi$-almost all Myrberg limit points satisfy the Lebesgue density type statement for the family $\{D(\gamma_0, r) : \gamma \in \Gamma, 0 < r < r_0\}$ (Proposition 10.17). This gives a desired approximation of $B$ by some $D(\gamma_0, r)$ satisfying (1.9).

In section 2, we go over basic definitions and properties of Zariski dense discrete subgroups of $G$. In section 3, we discuss the notion of $\alpha$-valued Gromov product and define the generalized BMS measures for a pair of $(\Gamma, \psi)$-conformal densities on $F$. From section 4, we assume that $\Gamma$ is Anosov. In section 4, we observe that the BMS measures $\nu_{BMS}^\psi$ is $AM$-ergodic for each $\psi \in D_\Gamma^\psi$. Sections 5 and 6 are devoted to proving Lemma 1.11 and Theorem 1.10 respectively. In section 7, we prove that the space of PS-measures on $\Lambda$ is parametrized by $D_\Gamma^\psi$. In section 8, we show that the set of Myrberg limit points of $\Gamma$ has full measure for any PS-measure on $\Lambda$. In section 9, we discuss the relation between the set of essential values of $\nu_\psi$ and the $N\Gamma$-ergodicity of $\nu_{BMS}^\psi$. In the final section 10, we prove Theorems 1.5 and 1.4.

2. LIMIT SET AND LIMIT CONE.

Let $G$ be a connected, semisimple real Lie group with finite center, and $\Gamma < G$ be a Zariski dense discrete subgroup. We fix, once and for all, a Cartan involution $\theta$ of the Lie algebra $\mathfrak{g}$ of $G$, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of $\theta$, respectively. We denote by $K$ the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$, and by $X = G/K$ the associated symmetric space. We also choose a
maximal abelian subalgebra \( a \) of \( \mathfrak{p} \). Choosing a closed positive Weyl chamber \( a^+ \) of \( a \), let \( A := \exp a \) and \( A^+ = \exp a^+ \). The centralizer of \( A \) in \( K \) is denoted by \( M \), and we set \( N \) to be the contracting horospherical subgroup for \( A \) for \( a \in \text{int} A^+ \), \( N = \{ g \in G : a^{-n}ga^n \to e \text{ as } n \to +\infty \} \). Note that \( \log N \) is the sum of all positive root subspaces for our choice of \( A^+ \). Similarly, we will also need to consider the expanding horospherical subgroup \( N^+ := \{ g \in G : a^n ga^{-n} \to e \text{ as } n \to +\infty \} \). We set
\[
P^+ = MAN^+, \quad P = P^- = MAN^-;
\]
they are minimal parabolic subgroups of \( G \) which are opposite to each other.

The quotient \( F = G/P \) is known as the Furstenberg boundary of \( G \), and is isomorphic to \( K/M \).

Let \( N_K(a) \) be the normalizer of \( a \) in \( K \). Let \( W := N_K(a)/M \) denote the Weyl group. Fixing a left \( G \)-invariant and right \( K \)-invariant Riemannian metric on \( G \) induces a \( W \)-invariant inner product on \( a \), which we denote by \( \langle \cdot, \cdot \rangle \). The identity coset \([e] \) in \( G/K \) is denoted by \( o \).

Denote by \( w_0 \in W \) the unique element in \( W \) such that \( \text{Ad}_{w_0} a^+ = -a^+ \). Note that \( w_0Pw_0^{-1} = P^+ \).

**Definition 2.1** (Visual map). For each \( g \in G \), we define
\[
g^+ := gP \in G/P \quad \text{and} \quad g^- := gw_0P \in G/P.
\]
For all \( g \in G \), and \( m \in M \), observe that \( (gm)^\pm = g^\pm \) and that \( g^\pm = g(e^\pm) \). Let \( F^{(2)} \) denote the unique open \( G \)-orbit in \( F \times F \):
\[
F^{(2)} = G(e^+, e^-) = \{(gP, gw_0P) \in G/P \times G/P : g \in G \}.
\]
Note that the stabilizer of \((e^+, e^-)\) is the intersection \( P^- \cap P^+ = MA \).

We say that \( \xi, \eta \in F \) are in general position if \((\xi, \eta) \in F^{(2)} \). The Bruhat decomposition says that \( G \) is the disjoint union \( \cup_{w \in W} N^-wP^+ \), and \( N^-P^+ \) is Zariski open and dense in \( G \). Hence \((\xi, \eta) \notin F^{(2)} \) if and only if \((\xi, \eta) \in G(e^+, we^-) \) for some \( w \in W - \{e\} \).

**Cartan projection and \( a \)-valued distance.**

**Definition 2.2** (Cartan projection). For each \( g \in G \), there exists a unique element \( \mu(g) \in a^+ \), called the Cartan projection of \( g \), such that
\[
g \in K \exp(\mu(g))K.
\]
When \( \mu(g) \in \text{int } a^+ \) and \( g = k_1\mu(g)k_2 \), \( k_1, k_2 \) are determined uniquely up to mod \( M \), more precisely, if \( g = k_1\mu(g)k_2 \), then for some \( m \in M \), \( k_1 = k_1'm \) and \( k_2 = m^{-1}k_2' \). We write \( \kappa_1(g) := [k_1] \in K/M \) and \( \kappa_2(g) := k_2 \in M \backslash K \).

**Lemma 2.3.** [4 Lem. 4.6] For any compact subset \( L \subset G \), there exists a compact subset \( Q = Q(L) \subset a \) such that for all \( g \in G \),
\[
\mu(Lg) \subset \mu(g) + Q.
\]
Definition 2.4 (a-valued distance). We define \( a : X \times X \to a \) by
\[
\sigma(p, q) = \mu(g^{-1}h)
\]
where \( p = g(o) \) and \( q = h(o) \).

**Accumulation of points of \( X \) on \( \mathcal{F} \).** Let \( \Pi \) denote the set of all simple roots of \( g \) with respect to \( a^+ \).

**Definition 2.5.** We say that
\begin{enumerate}
  \item \( v_i \to \infty \) regularly in \( a^+ \) if \( \alpha(v_i) \to \infty \) as \( i \to \infty \) for all \( \alpha \in \Pi \).
  \item \( a_i \to \infty \) regularly in \( A^+ \) if \( \log a_i \to \infty \) regularly in \( a^+ \).
  \item \( g_i \to \infty \) regularly in \( G \) if \( \mu(g_i) \to \infty \) regularly in \( a^+ \).
\end{enumerate}

If \( a_i \to \infty \) regularly in \( A^+ \), then for all \( n \in N^+ \), \( \lim_{i \to \infty} a_i h_i^{-1} = e \), uniformly on compact subsets of \( N \).

**Lemma 2.6.** If \( \xi_i \to \xi_0 \) in \( \mathcal{F} \) and all of \( \xi_i \) and \( \xi_0 \) are in general position with \( e^- \), then \( a_i \xi_i \to e^+ \) for any sequence \( a_i \to \infty \) regularly in \( A^+ \).

**Proof.** The hypothesis implies that \( \xi_i = n_i e^+ \) for a bounded sequence \( n_i \in N^+ \). Hence \( a_i \xi_i = a_i n_i e^+ = (a_i n_i a_i^{-1}) e^+ \to e^+ \) as \( a_i \to \infty \) regularly in \( A^+ \).

**Definition 2.7.** A sequence \( g_i \in G \) is said to converge to \( \xi \in \mathcal{F} \), if \( g_i \to \infty \) regularly and \( \lim_{i \to \infty} k_i(g_i)^+ = \xi \). In such a case, we write \( \lim_{i \to \infty} g_i = \xi \).

**Lemma 2.8.** Consider a sequence \( g_i = k_i a_i h_i^{-1} \) where \( k_i \in K, a_i \in A^+, h_i \in G \) satisfy that \( k_i^+ \to k_0^+ \) in \( K \), \( h_i \to h_0 \) in \( G \), and \( a_i \to \infty \) regularly in \( A^+ \). Then for any \( \xi \in \mathcal{F} \) in general position with \( h_0^- \), we have
\[
\lim_{i \to \infty} g_i \xi = k_0^+.
\]

**Proof.** As \( (\xi, h_0^-) \in \mathcal{F}(2) \), we have \( (h_0^- \xi, e^-) \in \mathcal{F}(2) \). Since \( \mathcal{F}(2) \) is open and \( h_0^- \xi \to h_0^- \xi \), \( h_0^- \xi \) is in general position with \( e^- \) for all large \( i \). By Lemma 2.6, \( a_i h_0^{-1} \xi \to e^+ \). Therefore \( \lim g_i \xi = \lim k_i e^+ = k_0^+ \).

**Lemma 2.9.** If \( g_i \in G \) is a sequence such that \( g_i o \to \xi \), then \( g_i z \to \xi \) for any \( z \in X \).

**Proof.** Write \( g_i = k_i a_i l_i^{-1} \in KA^+ K \). The hypothesis implies that \( a_i \to \infty \) regularly in \( A^+ \), and \( k_i^+ \to \xi \). Let \( k_0 \in K \) be such that \( k_0^+ = \xi \), and \( g \in G \) be such that \( g(o) = z \). Write \( g_i g = k_i^+ a_i l_i^{-1} \in KA^+ K \). We need to show that \( k_i^+ \to k_0^+ \). As \( k_i^+ \to k_0^+ \), it suffices to show that any limit of the sequence \( k_i^+ k_i^+ \) belongs to \( M = \text{Stab}_K e^+ \).

Set \( q_i := k_i^{-1} k_i^+ \). Let \( q \) be a limit of \( q_i \). By passing to a subsequence, we may suppose \( q_i \to q \in K \). Since \( d(o, z) = d(g_i o, g_i z) = d(a_i o, q_i a_i o) \), the sequence \( h_i^{-1} := a_i^{-1} q_i a_i^+ \) is bounded. Passing to a subsequence, let us assume that \( h_i \to h_0 \) in \( G \) as \( i \to \infty \). Choose \( \eta \in \mathcal{F} \) that is in general position with both \( h_0^- \) and \( e^- \). Then \( \lim a_i h_i^{-1} \eta = e^+ \) and \( \lim q_i a_i^+ \eta = q^+ \) by
Lemma 2.8 Since \(a_i h_i^{-1} \eta = q_i a_i' \eta\), we get \(e^+ = g^+ = q(e^+)\). This implies \(q \in \text{Stab}_K e^+ = M\). \(\square\)

**Lemma 2.10.** If \(a_i \to \infty\) regularly in \(A^+\) and \(g_i \to g\) in \(G\), then \(g_i a_i(o) \to g^+\) and \(g_i a_i^{-1}(o) \to g^-\).

Proof. Write \(g_i a_i = k_i b_i \ell_i^{-1} \in K A^+ K\). As the sequence \(g_i\) is bounded, it follows from Lemma 2.3 that \(\ell_i \to \infty\) regularly in \(A^+\). In order to show that \(g_i a_i(o) \to g^+\), it suffices to show that if \(k_i \to k_0\), then \(k_0^+ = g^+\). By passing to a subsequence, we may assume that \(\ell_i \to \ell_0\) in \(K\). Choose \(\xi \in F\) which is in general position with both \(\ell_0^+\) and \(e^−\). Then \(g_i a_i \xi \to k_0^+\) by Lemma 2.8. On the other hand, as \((\xi, e^-) \in F^{(2)}\), \(g_i a_i \xi \to g^+\) by Lemma 2.6 Hence \(g^+ = k_0^+\), proving the first claim. Now the second claim follows since \(g_i a_i^{-1} = g_i w_0 b_i w_0^{-1}\) for some \(b_i \in A^+\), and \(g_i w_0 b_i w_0^{-1}(o) = g_i w_0 b_i(o) \to (gw_0)^+ = g^−\). \(\square\)

**Limit set and Limit cone.** Denote by \(m_o\) the \(K\)-invariant probability measure on \(F \simeq K/M\).

**Definition 2.11** (Limit set). The limit set \(\Lambda\) of \(\Gamma\) is defined as the set of all points \(\xi \in F\) such that the Dirac measure \(\delta_\xi\) is a limit point in the space of Borel probability measures on \(F\) of \(\{\gamma, m_o : \gamma \in \Gamma\}\).

Benoist showed that \(\Lambda\) is the minimal \(\Gamma\)-invariant closed subset of \(F\). Moreover, \(\Lambda\) is Zariski dense in \(F\) [4, Section 3.6].

**Lemma 2.12.** We have
\[
\Lambda := \left\{ \lim_{i \to \infty} \gamma_i g \in F : \gamma_i \in \Gamma, g \in G \right\}.
\]

Proof. This follows from the fact that for all \(a_i \to \infty\) regularly in \(A^+\), \((a_i)_* m_o\) converges to the Dirac measure at \(e^+\) and Lemma 2.9 \(\square\)

Any element \(g \in G\) can be written as the commuting product \(g_h g_e g_u\), where \(g_h\), \(g_e\) and \(g_u\) are unique elements which are conjugate to elements of \(A^+, K\) and \(N\), respectively. When \(g_h\) is conjugate to an element of \(\text{int } A^+\), \(g\) is called loxodromic; in such a case, \(g_u = e\). If a loxodromic element \(g \in G\) satisfies \(\varphi^{-1} g_h \varphi \in \text{int } A^+\) for \(\varphi \in G\), then
\[
y^+_\gamma := \varphi^+ \quad \text{and} \quad y^-_\gamma := \varphi^-,
\]
are called the attracting and repelling fixed points of \(g\) respectively.

**Lemma 2.14.** [4, Lem. 3.6] The set
\[
\{(y^+_\gamma, y^-_\gamma) \in \Lambda \times \Lambda : \gamma \text{ is a loxodromic element of } \Gamma\}
\]
is dense in \(\Lambda \times \Lambda\).

The Jordan projection of \(g\) is defined as \(\lambda(g) \in a^+\), where \(\exp \lambda(g)\) is the element of \(A^+\) conjugate to \(g_h\).

**Definition 2.15** (Limit cone). The limit cone \(\mathcal{L}_\Gamma \subset a^+\) of \(\Gamma\) is defined as the smallest closed cone containing the Jordan projection \(\lambda(\Gamma)\).
The limit cone $L_\Gamma$ is a convex subset of $a^+$ with non-empty interior [4 Thm. 1.2]. It is also the smallest closed cone containing $\mu(\Gamma)$ [11 Lem. 2.18].

**Definition 2.16 (Growth indicator function).** The growth indicator function $\psi_\Gamma : a^+ \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(tu) = t\psi_\Gamma(u)$, such that for any unit vector $u \in a^+$,

$$\psi_\Gamma(u) := \inf_{\text{open cones } C \subset a^+} \limsup_{t \to \infty} \frac{1}{t} \log \# \{ \gamma \in \Gamma : \mu(\gamma) \in C, \|\mu(\gamma)\| \leq t \}.$$

We may consider $\psi_\Gamma$ as a function on $a$ by setting $\psi_\Gamma = -\infty$ outside of $a^+$. Quint showed the following:

**Theorem 2.17.** [29 Thm. IV.2.2] The growth indicator function $\psi_\Gamma$ is concave, upper-semicontinuous, and satisfies $L_\Gamma = \{ u \in a^+ : \psi_\Gamma(u) > -\infty \}$.

Moreover, $\psi_\Gamma$ is non-negative on $L_\Gamma$ and positive on $\text{int} \, L_\Gamma$.

3. $a$-valued Gromov product and generalized BMS-measures

**$a$-valued Busemann function.** The Iwasawa cocycle $\sigma : G \times F \rightarrow a$ is defined as follows: for $(g, \xi) \in G \times F$, $\sigma(g, \xi) \in a$ is determined by the condition

$$gk \in K \exp(\sigma(g, \xi))N$$

where $k \in K$ is such that $\xi = [k]$.

**Definition 3.1.** The $a$-valued Busemann function $\beta : F \times G/K \times G/K \rightarrow a$ is defined as follows: for $\xi \in F$ and $[g], [h] \in G/K$,

$$\beta_\xi([g], [h]) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi).$$

Observe that the Busemann function is continuous in all three variables. To ease the notation, we will write $\beta_\xi(g, h) = \beta_\xi([g], [h])$. We can check that

$$\beta_\xi(g, h) + \beta_\xi(h, q) = \beta_\xi(g, q),$$

$$\beta_\xi(gh, gq) = \beta_\xi(h, q), \text{ and}$$

$$\beta_\xi(e, g) = -\sigma(g^{-1}, \xi).$$

Geometrically, if $\xi = [k] \in K/M$, then for any unit vector $u \in a^+$,

$$\langle \beta_\xi(g, h), u \rangle = \lim_{t \to +\infty} d([g], \xi_t) - d([h], \xi_t),$$

where $\xi_t = k \exp(tu)o \in G/K$.

**Lemma 3.3.** For any loxodromic element $g \in G$ and $p \in X$,

$$\beta_{y_g^+}(p, gp) = \lambda(g) \text{ and } \beta_{y_g^-}(p, gp) = -\lambda(g^{-1}).$$
Proof. We have \( g = \varphi am \varphi^{-1} \) for some \( \varphi \in G, a \in A^+ \), and \( m \in M \). Hence \( y_g^+ = \varphi^+, y_g^- = \varphi^-, \) and \( \lambda(g) = \log a \). Let \( h \in G \) be so that \( p = h.o. \) Then,

\[
\beta_{y_g^+}(p, gp) = \beta_{\varphi^+}(ho, gho) = \sigma(h^{-1}, \varphi^+) - \sigma(h^{-1}g^{-1}, \varphi^+) = -\sigma(g^{-1}, \varphi^+).
\]

Writing \( \varphi = k b \) with \( k \in K \) and \( b \in P \), we have

\[
g^{-1}k = \varphi(am)^{-1} \varphi^{-1}k = kb(am)^{-1}b^{-1} \in Ka^{-1}N.
\]

This gives \( \sigma(g^{-1}, \varphi^+) = -\lambda(g) \), and hence the first identity. The second identity follows from the first, by replacing \( g \) with \( g^{-1} \), since \( y_{g^{-1}}^+ = y_g^- \). \( \square \)

\textbf{Definition 3.4} (Opposition involution). The involution \( i : a \to a \) defined by

\[
i(u) = -\text{Ad}_{w_0}(u)
\]

is called the opposition involution; it preserves \( a^+ \). Note that for all \( g \in G \),

\[
\lambda(g^{-1}) = i(\lambda(g)) \quad \text{and} \quad \mu(g^{-1}) = i(\mu(g)).
\]

\textbf{Definition 3.5}. For \( (\xi, \eta) \in F^{(2)} \), we can define the \( a \)-valued Gromov product by

\[
\mathcal{G}(\xi, \eta) := \sigma(g, e^+) + i\sigma(g, e^-) = \beta_{g^+}(e, g) + i\beta_{g^-}(e, g).
\]

where \( g \in G \) satisfies \( g^+ = \xi \) and \( g^- = \eta \).

The definition does not depend on the choice of the representative \( [g] \in G/AM \), and for all \( h \in G \) and \((x, y) \in F^{(2)} \), we have the following identity:

\[
(3.6) \quad \mathcal{G}(hx, hy) - \mathcal{G}(x, y) = \sigma(h, x) + i\sigma(h, y).
\]

Observe that \( \mathcal{G}(y, x) = i\mathcal{G}(x, y) \) and hence the Gromov product is not symmetric in general.

\textbf{Lemma 3.7.} [3] There exists a family of irreducible representations \( (\rho_\alpha, V_\alpha) \), \( \alpha \in \Pi \), of \( G \) so that (1) the highest weight \( \chi_\alpha \) of \( \rho_\alpha \) is a positive integral multiple of the fundamental weight \( \varpi_\alpha \) corresponding to \( \alpha \), (2) the highest weight space of \( V_\alpha \) is one dimensional, and (3) the weights of \( \rho_\alpha \) are \( \chi_\alpha, \chi_\alpha - \alpha \) and weights of the form \( \chi_\alpha - \alpha - \sum_{\beta \in \Pi} n_\beta \beta \) with \( n_\beta \in \mathbb{N} \).

For \( \alpha \in \Pi \), denote by \( V_\alpha^+ \) the highest weight space, and \( V_\alpha^- \) its unique complementary \( A \)-invariant subspace in \( V_\alpha \). For each \( \alpha \in \Pi \), \( \rho_\alpha(P)V_\alpha^+ = V_\alpha^+ \), and the map \( g \mapsto (\rho_\alpha(g)V_\alpha^+)^{\alpha \in \Pi} \) factors through a proper immersion

\[
\mathcal{F} = G/P \to \prod_{\alpha \in \Pi} \mathbb{P}(V_\alpha).
\]

Let \( \langle \cdot, \cdot \rangle_\alpha \) be a \( K \)-invariant inner product on \( V_\alpha \) with respect to which \( A \) is symmetric; then \( V_\alpha^+ \) and \( V_\alpha^- \) are orthogonal to each other. We denote by \( \| \cdot \|_\alpha \) the norm on \( V_\alpha \) induced by \( \langle \cdot, \cdot \rangle_\alpha \). For \( \varphi \in V_\alpha^* \), \( \| \varphi \|_\alpha \) means the operator norm of \( \varphi \).
Lemma 3.8. For all $\alpha \in \Pi$ and $g \in G$,
\[ \chi_{\alpha}(G(g^+, g^-)) = -\log \frac{|\varphi(v)|}{\|\varphi\|_\alpha \|v\|_\alpha} \]
where $\varphi \in V^*_\alpha$, $v \in V_\alpha$ are any elements such that $v \in gV^+_\alpha$ and $\ker \varphi = gV^<_\alpha$.

Proof. If we define $G'(g^+, g^-)$ to be the unique element of $a$ satisfying (3.9), it is shown in [36, Lem 4.12] that $G'$ satisfies (3.6). Hence for all $h \in G$,
\[ G'(h^+, h^-) - G'(e^+, e^-) = G(h^+, h^-) - G(e^+, e^-). \]
Observe that $G'(e^+, e^-) = 0$; take $\varphi$ to be the projection $V \to V^+_\alpha$ parallel to $V^<_\alpha$. Since $V^+_\alpha$ and $V^<_\alpha$ are orthogonal, it follows that $\|\varphi\|_\alpha = 1$. Now for $v \in V^+_\alpha$, we have
\[ \frac{|\varphi(v)|}{\|\varphi\|_\alpha \|v\|_\alpha} = \frac{\|v\|}{\|v\|_\alpha} = 1. \]
Since $G(e^+, e^-) = 0$, we conclude $G = G'$ on $\mathcal{F}^{(2)}$. \hfill \Box

Remark 3.10. In view of this lemma, our definition of Gromov product differs by $-i$ from the one given in [36].

Patterson-Sullivan measures on $\Lambda$.

Definition 3.11 (Conformal measures). Given $\psi \in a^*$ and a closed subgroup $\Gamma < G$, a Borel probability measure $\nu_\psi$ on $\mathcal{F}$ is called a $(\Gamma, \psi)$-conformal measure if, for any $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,
\[ \frac{d\gamma_* \nu_\psi}{d\nu_\psi} (\xi) = e^{\psi(\beta_\xi(e, \gamma))} = e^{-\psi(\sigma(\gamma^{-1}, \xi))} \]
where $\gamma_* \nu_\psi(Q) = \nu_\psi(\gamma^{-1} Q)$ for any Borel subset $Q \subset \mathcal{F}$.

If $2\rho$ denotes the sum of all positive roots of $G$ with respect to $a^+$, then a $(G, 2\rho)$-conformal measure is precisely the $K$-invariant probability measure $m_\alpha$ on $\mathcal{F}$.

Definition 3.13 (Patterson-Sullivan measures). For $\psi \in a^*$, a $(\Gamma, \psi)$-conformal measure supported on $\Lambda$ will be called a $(\Gamma, \psi)$-PS measure. By a $\Gamma$-PS measure on $\Lambda$, we mean a $(\Gamma, \psi)$-PS measure for some $\psi \in a^*$.

Set
\[ D_\Gamma := \{ \psi \in a^* : \psi \geq \psi_\Gamma \}, \]
which is a non-empty set [31, Section 4.1]. The following collection of linear forms is of particular importance:
\[ D^*_\Gamma := \{ \psi \in D_\Gamma : \psi(u) = \psi_\Gamma(u) \text{ for some } u \in L_\Gamma \cap \text{int } a^+ \}. \]
When $\psi(u) = \psi_\Gamma(u)$, we say $\psi$ is tangent to $\psi_\Gamma$ at $u$.

Generalizing the work of Patterson and Sullivan ([28], [41]), Quint [30] constructed a $(\Gamma, \psi)$-PS measure for every $\psi \in D^*_\Gamma$. 
Generalized BMS-measure $m_{\nu_1,\nu_2}$. Given a pair of $\Gamma$-conformal measures on $\mathcal{F}$, we now define an $MA$-quasi invariant measure on $\Gamma \backslash G$, which we call a generalized BMS-measure.

**Definition 3.14** (Hopf parametrization). The map

$$gM \to (g^+, g^-, b = \beta_g(e, g))$$

gives a homeomorphism between $G/M$ and $\mathcal{F}(2) \times a$, which is called the Hopf parametrization of $G/M$.

Fixing a pair of $\Gamma$-conformal measures $\nu_1, \nu_2$ on $\mathcal{F}$ for a pair of linear forms $\psi_1, \psi_2 \in a^*$, define the following Radon measure $\tilde{m}_{\nu_1,\nu_2}$ on $G/M$ as follows: for $g = (g^+, g^-, b) \in \mathcal{F}(2) \times a$,

$$d\tilde{m}_{\nu_1,\nu_2}(g) = e^{\psi_1(\beta_g(e, g)) + \psi_2(\beta_g^- (e, g))} d\nu_1(g^+) d\nu_2(g^-) db,$$

where $db = \ell(b)$ is the Lebesgue measure on $a$. This measure is left $\Gamma$-invariant, and hence induces a measure on $\Gamma \backslash G/M$. We denote by $m_{\nu_1,\nu_2}$ its $M$-invariant lift to $\Gamma \backslash G$. It is $A$-semi-invariant as

$$a_* m_{\nu_1,\nu_2} = e^{(\psi_2 - \psi_1)(\log a)} m_{\nu_1,\nu_2}$$

for all $a \in A$ (see [11, Lem. 3.6]).

**BMS-measures: $m_{\psi}^{BMS}$**. Let $\psi \in D_\Gamma^*$ and let $\nu_\psi$ be a $(\Gamma, \psi)$-PS measure. We set

$$m_{\psi}^{BMS} : = m_{\nu_\psi, \mu_\psi}$$

and call it the Bowen-Margulis-Sullivan measure associated to $\nu_\psi$. It is right $MA$-invariant and its support is given by

$$\Omega : = \{ x \in \Gamma \backslash G : x^\pm \in \Lambda \};$$

since $\Lambda$ is $\Gamma$-invariant, the condition $x^\pm \in \Lambda$ is well-defined. When the rank of $G$ is at least 2, $m_{\nu_\psi}^{BMS}$ is expected to be an infinite measure unless $\Gamma$ is a lattice. Note that for $[g] \in G/M$,

$$dm_{\nu_\psi}^{BMS}[g] = e^{\psi(g^+ g^-)} d\nu_\psi(g^+) d\nu_\psi(g^-) db.$$

**N-invariant BR-measures: $m_{\psi}^{BR}$**. We set

$$m_{\psi}^{BR} : = m_{\nu_\psi, m_\psi}$$

and call it the $N$-invariant Burger-Roblin measure associated to $\nu_\psi$. See [11, Section 3] for the equivalence of this definition with the one given in [1.3]. The support of $m_{\psi}^{BR}$ is given by

$$\mathcal{E} : = \{ x \in \Gamma \backslash G : x^+ \in \Lambda \}.$$
4. Anosov groups and AM-ergodicity of BMS measures

Let $\Gamma$ be a Zariski dense discrete subgroup of $G$, and set $\Lambda^{(2)} := (\Lambda \times \Lambda) \cap F^{(2)}$.

**Definition 4.1.** We say that $\Gamma < G$ is Anosov, if it is a finitely generated word hyperbolic group and there exists a $\Gamma$-equivariant homeomorphism $\zeta : \partial \Gamma \to \Lambda$ such that $(\zeta(x),\zeta(y)) \in \Lambda^{(2)}$ for all $x \neq y \in \partial \Gamma$, where $\partial \Gamma$ denotes the Gromov boundary of $\Gamma$.

Such $\zeta$ is Hölder continuous and exists uniquely ([21, Prop. 3.2] and [6, Lem. 2.5]). We call it the limit map of $\Gamma$. We note that the antipodal property follows directly:

$\Lambda \times \Lambda - \{(\xi,\xi)\} \subset F^{(2)}$.

In the literature, this definition is referred to as $P$-Anosov for a minimal parabolic subgroup $P$ of $G$. See [16], [15] and [19] for equivalent characterizations of Anosov subgroups.

In the rest of this section, let $\Gamma$ be an Anosov subgroup of $G$. The following theorem was proved by Quint [31, Prop. 3.2 and Thm. 4.7] for Schottky groups and by Sambarino [37, Coro. 3.12, 3.13 and 4.9] for general Anosov subgroups in view of the results in [6]:

**Theorem 4.2.**
1. $L_\Gamma \subset \text{int} a^+ \cup \{0\}$ and $\Gamma$ consists of loxodromic elements.
2. $\psi_\Gamma$ is strictly concave and analytic on $\text{int} L_\Gamma$.
3. $D^*_u = \{\psi \in D_T : \psi(u) = \psi_\Gamma(u) \text{ for some } u \in \text{int } L_\Gamma\}$.
4. For any $\psi \in D^*_u$, $\psi > 0$ on $L_\Gamma - \{0\}$.
5. For any $\psi \in D^*_u$, there exists a unique $(\Gamma,\psi)$-PS measure $\nu_\psi$ on $F$.

(1) and (3) imply that if $\psi \in D_\Gamma$ is tangent to $\psi_\Gamma$ at some $u \in L_\Gamma - \{0\}$, then $u \in \text{int } L_\Gamma$. The uniqueness of $(\Gamma,\psi)$-PS-measure $\nu_\psi$ given in (4) implies that $(\Lambda,\nu_\psi)$ is $\Gamma$-ergodic. For $u \in \text{int } L_\Gamma$, we denote by $D_u \psi_\Gamma$ the directional derivative of $\psi_\Gamma$ at $u$.

**Proposition 4.3.** For $u \in \text{int } L_\Gamma$, $D_u \psi_\Gamma \in D^*_u$ and $D_u \psi_\Gamma(u) = \psi_\Gamma(u)$. Moreover, the map $u \mapsto D_u \psi_\Gamma$ induces a homeomorphism between the set of unit vectors in $\text{int } L_\Gamma$ ($\simeq \text{int } P L_\Gamma$) and $D^*_u$.

**Proof.** See ([37, Thm. A], [11, Lem. 2.23]) for the first claim. The well-definedness and surjectivity of the map in the second claim follows from it, and the injectivity follows from the strict concavity of $\psi_\Gamma$. \hfill $\square$

**AM-ergodicity of $m^{BMS}_\psi$.** We fix $\psi \in D^*_\Gamma$ and set

$$\nu := \nu_\psi \quad \text{and} \quad m^{BMS}_\psi := m^{BMS}_{\nu_\psi}.$$  

The composition $c := \psi \circ \sigma : \Gamma \times \Lambda \to \mathbb{R}$ is a Hölder cocycle satisfying $c(\gamma, y_\gamma^+) = \psi(\lambda(\gamma)) > 0$ for all $\gamma \in \Gamma$. 
Consider the action of \( \Gamma \) on \( \Lambda^{(2)} \times \mathbb{R} \) given as follows: for \( \gamma \in \Gamma \) and \((\xi, \eta, t) \in \Lambda^{(2)} \times \mathbb{R}, \)

\[
\gamma.(\xi, \eta, t) = (\gamma \xi, \gamma \eta, t + c(\zeta, \xi)).
\]

The \( \mathbb{R} \)-action defined by

\[
\tau_s(\xi, \eta, t) = (\xi, \eta, t + s)
\]

will be called translation flow. The following is proved in [37, Thm. 3.2] when \( \Sigma = \pi_1(M) \) for a closed negatively curved manifold \( M \), and can be extended for general Anosov groups, using ingredients from [6]. The sketch of the proof can be found in [9, Appendix A].

**Theorem 4.5.** The action of \( \Gamma \) on \( \Lambda^{(2)} \times \mathbb{R} \) is proper and cocompact, and the measure \( e^{\psi \theta} \nu_\psi \otimes \nu_\psi \otimes ds \) induces the measure of maximal entropy, say \( m_\psi \), for \( \{ \tau_t \} \) on \( \Gamma \backslash \Lambda^{(2)} \times \mathbb{R} \). In particular, \( m_\psi \) is \( \{ \tau_t \} \)-ergodic.

**Corollary 4.6.** For any \( \psi \in D^*_\Gamma \), the \( \text{AM} \)-action on \( (\Gamma \backslash G, m_\psi^{\text{BMS}}) \) is ergodic and if \( \text{rank} G \geq 2 \), \( |m_\psi^{\text{BMS}}| = \infty \).

**Proof.** The \( \mathbb{R} \)-ergodicity of \( m_\psi \) is equivalent to ergodicity of \( (\Lambda^{(2)}, \Gamma, \nu_\psi \otimes \nu_\psi |_{\Lambda^{(2)}}) \), which is again equivalent to the \( \text{AM} \)-ergodicity of \( m_\psi^{\text{BMS}} \). Consider the projection map \( \pi : \Gamma \backslash \Lambda^{(2)} \times a \rightarrow \Gamma \backslash \Lambda^{(2)} \times \mathbb{R} \) induced by the \( \Gamma \)-equivariant map \( \Lambda^{(2)} \times a \rightarrow \Lambda^{(2)} \times \mathbb{R} \) given by \((\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))\).

Note that \( m_\psi^{\text{BMS}} \) disintegrates over the measure \( m_\psi^\# = e^{\psi \theta} \nu_\psi \otimes \nu_\psi \otimes ds \) with conditional measure being the Lebesgue measure on \( \ker \psi \) so that \( m_\psi^{\text{BMS}} \simeq m_\psi^\# \otimes \text{Leb}_{\ker \psi} \) (cf. [36, Prop. 3.5]). This gives the infinitude of \( |m_\psi^{\text{BMS}}| \) when \( G \) has rank at least 2. \( \square \)

5. **Comparing \( a \)-valued Busemann functions and distances via \( \psi \)**

When \( G \) has rank one, the maximum and minimum of Busemann function \( \beta_\xi(p, q), \xi \in \mathcal{F} \) are always achieved as \( \pm d(p, q) \).

A higher rank generalization of this fact can be stated as in the following lemma. Let \((a^+)\wedge \) denote the following dual cone of \( a^+ \):

\[
(a^+)\wedge := \{ w \in a : \langle w, v \rangle \geq 0 \text{ for all } v \in a^+ \}.
\]

**Lemma 5.1.** Let \( \psi \in a^* \) be strongly positive, in the sense that \( \psi \geq 0 \) on \((a^+)\wedge \). Then for any \( p, q \in X \) and \( \xi \in \mathcal{F} \), we have

\[
-\psi(\underline{a}(q, p)) \leq \psi(\beta_\xi(p, q)) \leq \psi(\underline{a}(p, q)).
\]
Proof. We use notations introduced in Lemma 3.7. As \( \psi \) is strongly positive, we have \( \psi \) is a linear combination of \( \varpi_\alpha, \alpha \in \Pi \) with non-negative coefficients. Therefore it suffices to prove the claim when \( \psi = \varpi_\alpha \) for \( \alpha \in \Pi \). Note that \( \varpi_\alpha = c_\alpha \chi_\alpha \) for some \( c_\alpha > 0 \). Write \( q = gq \) and \( p = hq \) for some \( g, h \in G \). Note that \( \varpi_\alpha(a(p, q)) = \varpi_\alpha(\mu(g^{-1}h^{-1}g)) = c_\alpha \log \| \rho_\alpha(g^{-1}h^{-1}g) \| \). Write \( g^{-1}h = k^+ \) for some \( k \in K \) and \( g^{-1}h^{-1}g = k'^+ \in KAN \). Then for a unit vector \( v \in V_\alpha \),

\[
\varpi_\alpha(\beta_\xi(p, q)) = \varpi_\alpha(\log a) = c_\alpha \log \| \rho_\alpha(g^{-1}h^{-1}g)\rho_\alpha(k)v \| .
\]

Since

\[
\| \rho_\alpha(g^{-1}) \|^\gamma - 1 \leq \| \rho_\alpha(g)v \| \leq \| \rho_\alpha(g) \|
\]

for any \( g \in G \) and a unit vector \( v \in V_\alpha \), we get

\[
- \varpi_\alpha(a(q, p)) \leq - \varpi_\alpha(\beta_\xi(p, q)) \leq \varpi_\alpha(a(p, q)).
\]

The inequality (5.2) does not hold for a general \( \psi \in D_\Gamma^* \). We establish the following modification for Anosov groups, which is the main goal of this section:

**Theorem 5.3.** Let \( \Gamma < G \) be Anosov and \( \psi \in D_\Gamma^* \). For any \( p \in X \), there exists \( C = C(\Gamma, \psi, p) > 0 \) such that for all \( \gamma \in \Gamma \) and \( \xi \in \Lambda \),

\[
-\psi(a(p, \gamma p)) - C \leq \psi(\beta_\xi(\gamma p, p)) \leq \psi(a(\gamma p, p)) + C.
\]

When \( p = q \), this theorem implies that for all \( \gamma \in \Gamma \) and \( \xi \in \Lambda \),

\[
-\psi(\mu(\gamma)) - C \leq \psi(\sigma(\gamma^{-1}, \xi)) \leq \psi(\mu(\gamma^{-1})) + C.
\]

We begin by noting that \( \psi(a(\gamma p, p)) \) is always positive possibly except for finitely many \( \gamma \)'s:

**Lemma 5.4.** Let \( p \in X \) and \( \psi \in D_\Gamma^* \). For any sequence \( \gamma_i \to \infty \) in \( \Gamma \),

\( \psi(a(\gamma_i p, p)) \to \infty \).

**Proof.** It suffices to check that \( \psi(\mu(\gamma_i)) \to +\infty \) as \( i \to \infty \). Setting \( t_i := \| \mu(\gamma_i) \|^{-1} \), passing to a subsequence, we may assume that \( t_i \mu(\gamma_i) \) converges to some unit vector \( u \in a \). As \( u \in L_\Gamma \), we have \( \psi(u) > 0 \) by Lemma 4.2. Since \( \psi(t_i \mu(\gamma_i)) \to \psi(u) \) and \( \psi(\mu(\gamma_i)) = t_i^{-1} \psi(t_i \mu(\gamma_i)) \), we have \( \psi(\mu(\gamma_i)) \to +\infty \).

The following is the main ingredient of the proof of Theorem 5.3:

**Proposition 5.5.** For \( p \in X \), there exists \( C = C(\Gamma, p) > 0 \) such that for each \( (\gamma, \xi) \in \Gamma \times \Lambda \), we can find \( \gamma_1 = \gamma_1(\xi), \gamma_2 = \gamma_2(\xi) \in \Gamma \) satisfying

1. \( \gamma = \gamma_1 \gamma_2 \) and \( |\gamma| = |\gamma_1| + |\gamma_2| \);
2. \( \| \beta_\xi(\gamma p, p) + \mu(\gamma_1) - \mu(\gamma_2^{-1}) \| \leq C \);
3. \( \| a(\gamma p, p) - \mu(\gamma_1^{-1}) - \mu(\gamma_2^{-1}) \| \leq C \).
Proof of Theorem 5.3 using Proposition 5.5: For $\gamma \in \Gamma$ and $\xi \in \Lambda$, choose $\gamma_1, \gamma_2 \in \Gamma$ as in Proposition 5.5. Then
\[
\psi(\beta_\xi(\gamma p, p)) \leq \psi(\mu(\gamma_2^{-1}) - \mu(\gamma_1)) + C\|\psi\|
\leq \psi(\mu(\gamma_2^{-1}) + \mu(\gamma_1^{-1})) + C\|\psi\|
\leq \psi(\mu(\gamma p, p)) + 2C\|\psi\|,
\]
where the second inequality is valid because $\psi(\mu(\gamma_1^{\pm 1})) \geq 0$. Similarly, we get
\[
\psi(\beta_\xi(\gamma p, p)) \geq \psi(\mu(\gamma_2^{-1}) - \mu(\gamma_1)) - C\|\psi\| \leq -\psi(\mu(\gamma_2) + \mu(\gamma_1)) - C\|\psi\|.
\]

The rest of this section is devoted to a proof of Proposition 5.5 in which shadows of $\mathcal{F}$ and $\partial \Gamma$ as well as their relationship play important roles.

Shadows in $\mathcal{F}$. Let $q \in X$ and $r > 0$. the shadows of the ball $B(q, r)$ viewed from $p \in X$ and $\xi \in \mathcal{F}$ are respectively defined as
\[
O_r(p, q) := \{ gk^+ \in \mathcal{F} : \text{int} A^+o \cap B(q, r) \neq \emptyset \}
\]
where $g \in G$ satisfies $p = g(o)$, and
\[
O_r(\xi, q) := \{ h^+ \in \mathcal{F} : h^- = \xi, ho \in B(q, r) \}.
\]
The latter can be approximated by the former, as we have:

**Lemma 5.6.** [42, Prop. 8.64] If a sequence $q_i \in X$ converges to $\xi \in \mathcal{F}$, then for any $r > 0$, $q \in X$ and $\varepsilon > 0$, we have
\[
O_r(\xi, q) \subset O_{r+\varepsilon}(q_i, q)
\]
for all sufficiently large $i$.

We also have the following analogue of Sullivan’s shadow lemma:

**Lemma 5.7.** [42, Prop. 8.66] There exists $\kappa > 0$ such that for any $p, q \in X$ and $r > 0$, we have
\[
\sup_{\xi \in O_r(p, q)} \|\beta_\xi(p, q) - a(p, q)\| \leq \kappa r.
\]

This implies Theorem 5.3 for those $\xi \in O_r(\gamma p, p)$. In order to control the value of $\beta_\xi(\gamma p, p)$ when $\xi \notin O_r(\gamma p, p)$, we use the Anosov property of $\Gamma$. Let us recall some basic terminologies for hyperbolic groups for which we refer to [7] and [18].
**Discrete geodesics.** Let $\Gamma < G$ be a finitely generated word hyperbolic group. We fix a finite symmetric set of generators $S = \{s_1^\pm, \ldots, s_k^\pm\}$ of $\Gamma$ once and for all. Let $|\cdot| : \Gamma \to \mathbb{N} \cup \{0\}$ denote the word length associated to $S$. We denote by $d_w$ the associated left-invariant word metric, that is, $d_w(\gamma_1, \gamma_2) := |\gamma_1^{-1}\gamma_2|$ for $\gamma_1, \gamma_2 \in \Gamma$.

A finite sequence $(\gamma_0, \cdots, \gamma_n)$ of elements of $\Gamma$ will be called a finite path if $\gamma_i^{-1}\gamma_{i+1} \in S$ for all $i$. Such a path will be called a geodesic segment if $|\gamma_0^{-1}\gamma_n| = n$. Infinite and bi-infinite paths can be defined analogously. They will be called geodesic rays and geodesic lines, respectively, if all of their finite subpaths are geodesic segments.

Let $\partial \Gamma$ denote the Gromov boundary of $\Gamma$, that is, $\partial \Gamma$ is the set of equivalence classes of geodesic rays, where two rays are equivalent to each other if and only if their Hausdorff distance is finite. For a geodesic ray $(\gamma_0, \gamma_1, \cdots)$, we will refer to the unique element of $\partial \Gamma$ represented by the ray as $[\gamma_0, \gamma_1, \cdots]$.

For any $x, y \in \Gamma \cup \partial \Gamma$, there exists a discrete geodesic starting from $x$ and ending at $y$, which may not be unique. By $[x, y]$, we mean one of those geodesics and by $[x, y]$ we mean $[x, y] - \{y\}$.

A geodesic triangle is a union of three geodesics, pairwise sharing a common endpoint in $\Gamma \cup \partial \Gamma$. Since $\Gamma$ is hyperbolic, there exists $\delta = \delta(\Gamma, S) > 0$ such that for any geodesic triangle $\Delta$, we can find a point on each edge of $\Delta$ so that the set of these triples has diameter less than $\delta$.

**Shadows in $\partial \Gamma$.** For $R > 0$ and $\gamma_1, \gamma_2 \in \Gamma$, the shadow of the ball $B_R(\gamma_2)$ viewed from $\gamma_1$ is given by

$$O_R(\gamma_1, \gamma_2) = \{x \in \partial \Gamma : [\gamma_1, x] \cap B_R(\gamma_2) \neq \emptyset \text{ for some geodesic ray } [\gamma_1, x]\}.$$

Shadows satisfy the equivariance property: for any $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $R > 0$,

$$\gamma O_R(\gamma_1, \gamma_2) = O_R(\gamma \gamma_1, \gamma \gamma_2).$$

**Lemma 5.9.** There exist $R_0 > 1$ and $N_0 > 0$ such that the following holds: if $\gamma_1, \gamma_2 \in \Gamma$ with $|\gamma_1|, |\gamma_2| \geq N_0$ satisfies $|\gamma_1 \gamma_2| = |\gamma_1| + |\gamma_2|$, then for all $R \geq R_0$,

$$O_R(\gamma_1 \gamma_2, e) \cap O_R(\gamma_1 \gamma_2, \gamma_1) \cap O_R(\gamma_1, e) \neq \emptyset.$$

**Proof.** Since $|\gamma_1 \gamma_2| = |\gamma_1| + |\gamma_2|$, there exists a geodesic segment $[\gamma_1 \gamma_2, e]$ passing through $\gamma_1$, say $\alpha = (\gamma_1 \gamma_2, \cdots, \gamma_1, \cdots, e)$. Since $\Gamma$ is word hyperbolic, there exists $C = C(\Gamma) > 0$ such that $\alpha$ lies in the $C$-neighborhood of some geodesic line, say $(\cdots, u_{-1}, u_0, u_1, \cdots)$. Set $N_0 := 4C$. Choose $u_m, u_n$, and $u_\ell$ to be elements closest to $\gamma_1 \gamma_2$, $\gamma_1$, and $e$, respectively.

We claim that $|m - \ell| \geq \max(|m - n|, |n - \ell|)$. By the triangle inequality,

$$|n - \ell| = d_w(u_n, u_\ell) \leq d_w(\gamma_1, e) + 2C = |\gamma_1| + 2C;$$

$$|m - n| = d_w(u_m, u_n) \leq d_w(\gamma_1 \gamma_2, \gamma_1) + 2C = |\gamma_2| + 2C.$$
Set $x$, $\ell$. Then the claim implies that 

$$|m - \ell| \geq |\gamma_1| + |\gamma_2| - 2C \geq \max(|\gamma_1|, |\gamma_2|) - 2C + N_0$$

$$= \max(|\gamma_1|, |\gamma_2|) + 2C \geq \max(|n - \ell|, |m - n|).$$

This proves the claim.

Now possibly after flipping the geodesic, we may assume that $m \leq \ell$. Then the claim implies that $\ell - m = |m - n| + |n - \ell|$ and hence $m \leq n \leq \ell$.

Set $x := [u_0, u_1, u_2, \cdots] \in \partial \Gamma$. Choose geodesic rays $[\gamma_1 \gamma_2, x)$ and $[\gamma_1, x)$. Since the Hausdorff distance between $[\gamma_1 \gamma_2, x)$ and the ray $(u_m, u_{m+1}, \cdots)$ is less than $d_w(\gamma_1 \gamma_2, u_m) + \delta \leq C + \delta$, it follows that there exist $v_1, v_2 \in \Gamma$ lying on $[\gamma_1 \gamma_2, x]$, such that $d_w(u_m, v_1) < C + \delta$ and $d_w(u_m, v_2) < C + \delta$.

Since the Hausdorff distance between $[\gamma_1, x]$ and the ray $(u_n, u_{n+1}, \cdots)$ is less than $d_w(\gamma_1, u_n) + \delta < C + \delta$, there exists $v_3 \in \Gamma$ lying on $[\gamma_1, x]$ such that $d_w(u_n, v_3) < C + \delta$. These altogether imply that

$$x \in O_{2C+\delta}(\gamma_1 \gamma_2, e) \cap O_{2C+\delta}(\gamma_1 \gamma_2, \gamma_1) \cap O_{2C+\delta}(\gamma_1, e).$$

$$\square$$

In the rest of this section, we assume that $\Gamma$ is an Anosov subgroup of $G$. The following Morse property of Kapovich-Leeb-Porti [19, Prop. 5.16] says that a discrete geodesic line (resp. ray) of $\Gamma$ is contained in a uniform neighborhood of some $A$-orbit (resp. $A^+$-orbit) in $X$.

**Proposition 5.10** (Morse property). For any Anosov subgroup $\Gamma < G$, there exists $R_1 = R_1(\Gamma) > 0$ such that

1. If $(\cdots, \gamma_{-1}, \gamma_0, \gamma_1, \cdots)$ is a geodesic line in $(\Gamma, d_w)$, then

$$\sup_{k \in \mathbb{Z}} d(\gamma_{k} o, g A o) \leq R_1$$

for any $g \in G$ such that $g^+ = \zeta([\gamma_0, \gamma_1, \cdots])$, $g^- = \zeta([\gamma_0, \gamma_{-1}, \cdots])$.

2. If $(\gamma_0, \gamma_1, \cdots)$ is a geodesic ray in $(\Gamma, d_w)$, then

$$\sup_{k \in \mathbb{N}} d(\gamma_{k} o, g A^+ o) \leq R_1$$

where $g \in \gamma_0 K$ is the unique element satisfying $g^+ = \zeta([\gamma_0, \gamma_1, \cdots])$.

Using this proposition, we will show that shadows in the Gromov boundary $\partial \Gamma$ are mapped to shadows in the Furstenberg boundary $F$ by the limit map $\zeta: \partial \Gamma \to \Lambda$ (Proposition 5.12). We will need the following lemma:

**Lemma 5.11.** There exists $C > 0$ such that for all $\gamma \in \Gamma$, $||\mu(\gamma)|| \leq C|\gamma|$, i.e., $d(o, \gamma o) \leq C d_w(e, \gamma)$.

**Proof.** We use notations from Lemma 3.7. Since $\chi_{\alpha}, \alpha \in \Pi$ form a dual basis of $a^*$, $||\cdot||_* := \sum_{\alpha \in \Pi} |\chi_\alpha(\cdot)|$ defines a norm on $a$. Hence we may replace $||\cdot||$ by $||\cdot||_*$. Let $\gamma \in \Gamma$ be arbitrary, and write $\gamma = s_1 \cdots s_{\ell}$ with $s_i \in S$ and
ℓ = |γ|. Since \( e^{\chi_\alpha(g)} = \|\rho_\alpha(g)\|_\alpha \) for all \( g \in G \) and \( \|\rho_\alpha(s_1 \cdots s_t)\|_\alpha \leq \|\rho_\alpha(s_1)\|_\alpha \cdots \|\rho_\alpha(s_t)\|_\alpha \), it follows that
\[
\chi_\alpha(\mu(s_1 \cdots s_t)) \leq \chi_\alpha(\mu(s_1)) + \cdots + \chi_\alpha(\mu(s_t))
\]
for all \( \alpha \in \Pi \). Noting that \( \chi_\alpha \) is positive on \( a^+ \), we have
\[
\|\mu(\gamma)\|_s = \sum_{\alpha \in \Pi} (\chi_\alpha(\mu(\gamma)) = \sum_{\alpha \in \Pi} \chi_\alpha(\mu(\gamma)) \leq \sum_{\alpha \in \Pi} (\chi_\alpha(\mu(s_1)) + \cdots + \chi_\alpha(\mu(s_t))) \leq C|\gamma|
\]
where \( C := \max \{ \sum_{\alpha \in \Pi} \chi_\alpha(\mu(s)) : s \in S \} \).

The Anosov property of \( \Gamma \) was not used to establish the inequality in Lemma 5.1. We remark that for Anosov groups, the lower bound \( C^{-1}|\gamma| - C \leq \|\mu(\gamma)\| \) holds as well, but we will not need this fact.

**Proposition 5.12** (Shadows go to shadows). There exists \( c > 0 \) such that for all \( R > 1 \) and \( \gamma, \gamma' \in \Gamma \),
\[
\zeta(O_R(\gamma', \gamma)) \subset O_c R(\gamma', \gamma, o).
\]

**Proof.** By (5.8), it suffices to consider the case \( \gamma' = e \). Let \( x \in O_R(e, \gamma) \).
By definition of \( O_R(e, \gamma) \), there exists a geodesic ray \( (\gamma'_0 = e, \gamma'_1, \gamma'_2, \cdots) \) representing \( x \) such that \( d_w(\gamma'_m, \gamma) < R \) for some \( m \in \mathbb{N} \). Let \( R_1 > 0 \) be the constant from Proposition 5.10 and \( k \in K \) be an element such that \( k^+ = \zeta((e, \gamma'_1, \gamma'_2, \cdots)) \). Then by Proposition 5.10(2), there exists \( a \in A^+ \) such that \( d(\gamma'_m o, k a o) \leq R_1 \). By Lemma 5.11 we have
\[
d(\gamma o, \gamma'_m o) = \|\mu(\gamma^{-1}\gamma'_m)\| < C d_w(\gamma, \gamma'_m) < CR.
\]
Therefore
\[
d(\gamma o, k a o) \leq d(\gamma o, \gamma'_m o) + d(\gamma'_m o, k a o) \leq CR + R_1.
\]
This implies that \( \zeta(x) \in O_{CR + R_1}(o, \gamma, o) \). Since \( R > 1 \), the conclusion follows by setting \( c := C + R_1 \). \( \square \)

**Corollary 5.13.** There exists \( R_2 > 0 \) such that for all \( \gamma_1, \gamma_2 \in \Gamma \) with \( |\gamma_1 \gamma_2| = |\gamma_1| + |\gamma_2| \), we have
\[
\|\mu(\gamma_1 \gamma_2) - \mu(\gamma_1) - \mu(\gamma_2)\| \leq R_2.
\]

**Proof.** Let \( N_0 \) and \( R_0 \) be given by Lemma 5.9. If one of \( |\gamma_1|, |\gamma_2| \) is less than \( N_0 \), then the claim holds by Lemma 2.3. Now assume that \( |\gamma_1|, |\gamma_2| \geq N_0 \). Then by Lemma 5.9 and Proposition 5.12 we can choose
\[
\xi \in O_{c R_0}(\gamma_1 \gamma_2 o, o) \cap O_{c R_0}(\gamma_1 o, \gamma_1 o) \cap O_{c R_0}(\gamma_1 o, o)
\]
where \( c \) is as in Proposition 5.12. By Lemma 5.7 and the cocycle identity
\[
\beta_\xi(\gamma_1 \gamma_2 o, o) = \beta_\xi(\gamma_1 \gamma_2 o, \gamma_1 o) + \beta_\xi(\gamma_1 o, o),
\]
we have
\[ \|a(\gamma_1 o, o) - a(\gamma_1 o, o) - a(\gamma_2 o, o)\| \leq 3\kappa c R_0. \]
Since \( a(g o, o) = i \mu(g) \) for all \( g \in G \) and \( i \) preserves \( \|\| \),
\[ \|\mu(\gamma_1 g) - \mu(\gamma_1) - \mu(\gamma_2)\| \leq 3\kappa c R_0. \]

\[\square\]

**Proof of Proposition 5.5:** We may assume that \( p = o \) by Lemma 2.3.

Let \( \gamma \in \Gamma \) and \( \xi \in \Lambda \) be arbitrary. If \( \gamma = \gamma_1 \gamma_2 \), we have
\[ \beta_\xi(\gamma o, o) = \beta_\xi(\gamma_1 \gamma_2 o, \gamma_1 o) + \beta_\xi(\gamma_1 o, o) \]
\[ = \beta_{\gamma_1^{-1}} \xi(\gamma_2 o, o) - \beta_{\gamma_1^{-1}} \xi(\gamma_1^{-1} o, o). \]
We claim that we can find \( \gamma_1, \gamma_2 \in \Gamma \) so that
\[ \gamma = \gamma_1 \gamma_2, \ |\gamma| = |\gamma_1| + |\gamma_2|, \text{ and } \]
(5.14) \[ \gamma_1^{-1} \xi \in O_{c(\delta+1)}(\gamma_2 o, o) \cap O_{c(\delta+1)}(\gamma_1^{-1} o, o) \]
where \( c \) is as in Proposition 5.12.

If \( \xi \in O_{c(\delta+1)}(o, \gamma o) \), then we may simply set \( \gamma_1 = \gamma \) and \( \gamma_2 = e \). In general, we find \( \gamma_1 \) as follows. Consider a geodesic triangle \( \Delta \) whose vertices are \( e, \gamma \in \Gamma \), and \( \zeta^{-1}(\xi) \in \partial \Gamma \). Since \( \Gamma \) is hyperbolic, we can find three points on \( \Delta \), one on each edge, whose diameter is less than \( \delta \) (See Figure 1). Let \( \gamma_1 \in \Gamma \) be the point on the geodesic segment joining \( e \) and \( \gamma \), and set \( \gamma_2 := \gamma_1^{-1} \gamma \). We then have \( |\gamma| = |\gamma_1| + |\gamma_2| \), and \( \zeta^{-1}(\xi) \in O_{\delta}(\gamma, \gamma_1) \cap O_{\delta}(e, \gamma_1) \).

![Figure 1. Geodesic triangle](image)

By Proposition 5.12, \( \xi \in O_{c(\delta+1)}(\gamma o, \gamma_1 o) \cap O_{c(\delta+1)}(o, \gamma_1 o) \), and hence the claim follows by applying \( \gamma_1^{-1} \) on both sides.

By (5.14) and Lemma 5.7, we have
\[ \|\beta_{\gamma_1^{-1}} \xi(\gamma_2 o, o) - \mu(\gamma_2)\| \leq \kappa c(\delta+1), \text{ and } \|\beta_{\gamma_1^{-1}} \xi(\gamma_1^{-1} o, o) - \mu(\gamma_1^{-1})\| \leq \kappa c(\delta+1). \]
Hence (2) holds with a choice of \( C \geq 2\kappa c(\delta+1) \). Since \( |\gamma| = |\gamma_1| + |\gamma_2| \) and \( S \) is symmetric, \( |\gamma^{-1}| = |\gamma_1^{-1}| + |\gamma_2^{-1}| \). As \( a(\gamma o, o) = \mu(\gamma^{-1}) \), we have
\[ \|a(\gamma o, o) - \mu(\gamma_1^{-1}) - \mu(\gamma_2^{-1})\| \leq R_2 \]
by Corollary 5.13. Hence (3) holds with any \( C \geq R_2 \). \( \Box \)

6. Virtual visual metrics via \( \psi \)-Gromov product

In this section, we let \( \Gamma < G \) be an Anosov subgroup, and fix \( \psi \in D_1^\Gamma \). The main aim here is to show that the following \( \psi \)-Gromov product defines a virtual visual metric on \( \Lambda \) up to a small multiplicative constant.

**Definition 6.1.** For \((\xi_1, \xi_2) \in \mathcal{F}(2)\), its \( \psi \)-Gromov product based at \( o \) is defined by

\[
[\xi_1, \xi_2]_{\psi, o} := \psi(\mathcal{G}(\xi_1, \xi_2))
\]

where \( \mathcal{G} \) is the a-valued Gromov product defined in Definition 3.5. For \( p = g(o) \in X \), we set

\[
[\xi_1, \xi_2]_{\psi, p} := [g^{-1}\xi_1, g^{-1}\xi_2]_{\psi, o}.
\]

Define \( d_{\psi, p} = d_p : \mathcal{F}(2) \to \mathbb{R} \) by

\[
d_p(\xi_1, \xi_2) = e^{-[\xi_1, \xi_2]_{\psi, p}}.
\]

We set \( d_p(\xi, \xi) = 0 \). It follows from (3.6) that for all \( g \in G \) and \( p \in X \), we have

\[
d_{gp}(\xi_1, \xi_2) = e^{-\psi(\beta_{g_1}((gp)p) + i\beta_{g_2}(gp,p))}d_p(\xi_1, \xi_2) = d_p(g^{-1}\xi_1, g^{-1}\xi_2).
\]

The following is the main theorem of this section:

**Theorem 6.4.** Fix \( p \in X \). For all sufficiently small \( \varepsilon > 0 \), there exist a constant \( C_\varepsilon = C_\varepsilon(p) > 0 \) and a metric \( d_\varepsilon = d_\varepsilon(p) \) on \( \Lambda \) such that for all \( \xi_1 \neq \xi_2 \in \Lambda \),

\[
C_\varepsilon^{-1}d_{\psi, p}(\xi_1, \xi_2)^\varepsilon \leq d_\varepsilon(\xi_1, \xi_2) \leq C_\varepsilon d_{\psi, p}(\xi_1, \xi_2)^\varepsilon.
\]

**Weak ultrametric inequality.** A well-known construction [14, Section 7.3] shows the existence of a metric in Theorem 6.4 provided there exists \( C > 0 \) such that for all \( \xi_1, \xi_2, \xi_3 \in \Lambda \), we have

1. (weak symmetry) \( d_p(\xi_1, \xi_2) \leq e^C d_p(\xi_2, \xi_1) \);
2. (weak ultrametric inequality) \( d_p(\xi_1, \xi_3) \leq e^C \max(d_p(\xi_1, \xi_2), d_p(\xi_2, \xi_3)) \).

Hence Theorem 6.4 follows from the following proposition:

**Proposition 6.5.** There exists \( C = C(p) > 0 \) such that for all \( \xi_1, \xi_2, \xi_3 \in \Lambda \), we have

\[
[\xi_1, \xi_2]_p \geq [\xi_2, \xi_1]_p - C;
\]

\[
[\xi_1, \xi_3]_p \geq \min([\xi_1, \xi_2]_p, [\xi_2, \xi_3]_p) - C.
\]

In the case of \( X = \mathbb{H}^2 \), the classical Gromov product satisfies that there exists a uniform constant \( C > 0 \) such that for any \( x, y \in \partial \mathbb{H}^2 \),

\[
|\mathcal{G}(x, y) - 2d(o, z)| \leq C
\]

where \( z \) is the unique projection of \( o \) to the geodesic connecting \( x \) and \( y \). In the following lemma 6.6, we establish the analogous property for a-valued Gromov products on \( \Lambda(2) \) using the Morse property of Anosov groups.
Lemma 6.6. There exists $C_1 = C_1(\Gamma) > 0$ such that for any $x \neq y$ in $\partial \Gamma$, and $\gamma \in \pi_{[x,y]}(e)$, we have
$$\|G(\zeta(x), \zeta(y)) - \mu(\gamma) - i \mu(\gamma)\| \leq C_1.$$ In particular, $G$ is almost symmetric on $\Lambda$: for any $\xi_1 \neq \xi_2 \in \Lambda$,
$$\|G(\xi_1, \xi_2) - G(\xi_2, \xi_1)\| \leq 2C_1.$$ Proof. Let $\alpha := (u_0 = e, u_1, u_2, \cdots)$ and $\alpha' := (v_0 = e, v_1, v_2, \cdots)$ be geodesic representatives of $x$ and $y$, respectively. Let $\gamma \in \pi_{[x,y]}(e)$ be arbitrary, and $f, g, h \in \mathcal{G}$ be elements satisfying the following:
- $f(o) = o$, and $f^+ = \zeta(x)$;
- $g(o) = o$, and $g^+ = \zeta(y)$;
- $h^+ = \zeta(x)$, and $h^- = \zeta(y)$.
Since $\Gamma$ is hyperbolic, the diameter of the set $\pi_{\alpha'}(x) \cup \pi_{\alpha}(y) \cup \pi_{[x,y]}(e)$ is at most $C\delta$ for some uniform constant $C > 1$. In particular, we can find $k, \ell \in \mathbb{N}$ such that the set $\{u_k, v_\ell, \gamma\}$ has diameter less than $C\delta$.

Applying Proposition 5.10(1) to the geodesic line $[x, y]$, we have $d(ho, \gamma o) < R_1$ after replacing $h$ with some element of $hA$. Hence by Lemma 2.3 there exists $C' = C'(R_1) > 0$ such that
$$\|\mu(h) - \mu(\gamma)\| \leq C'.$$
Similarly, applying Proposition 5.10(2) to the geodesic ray $\alpha$, we find $a_1 \in A^+$ such that $d(fa_1 o, u_k o) < R_1$. Since $d_w(u_k, \gamma) < \delta$, we have
$$d(u_k o, \gamma o) = d(o, u_k^{-1} \gamma o) < \sup\{\|\mu(\gamma')\| : |\gamma'| \leq C\delta\}.$$ Now that
$$d(fa_1 o, ho) \leq d(fa_1 o, u_k o) + d(u_k o, \gamma o) + d(\gamma o, ho),$$
we get $h^+ = \zeta(x) \in O_R(o, ho)$ with $R = 2R_1 + \sup\{\|\mu(\gamma')\| : |\gamma'| \leq C\delta\}$. Similar argument shows that $h^- = \zeta(y) \in O_R(o, ho)$. Hence
$$\|\beta_{h^+}(o, ho) - \mu(h)\| \leq \kappa R \quad \text{and} \quad \|\beta_{h^-}(o, ho) - \mu(h)\| \leq \kappa R$$
by Lemma 5.7. Since
$$G(\zeta(x), \zeta(y)) = G(h^+, h^-) = \beta_{h^+}(o, ho) + i \beta_{h^-}(o, ho),$$
we have
$$\|G(\zeta(x), \zeta(y)) - \mu(h) - i \mu(h)\| \leq 2\kappa R.$$ Hence the conclusion follows from (6.7) by setting $C_1 := 2(\kappa R + C')$. \qed

Lemma 6.8. For any compact subset $C \subset X$, the set $\{\beta_x(p, o) : x \in \mathcal{F}, p \in C\}$ is bounded.

Proof. This follows from Lemma 5.1 by setting $\psi = \sum_{\alpha \in \Pi} \varpi_{\alpha}$. \qed
**Proof of Proposition 6.5.** We first claim that it suffices to consider the case $p = o$. Observe that the identity \( (6.3) \) gives that for any $\xi_1 \neq \xi_2 \in \Lambda$,

\[
[\xi_1, \xi_2]_p - [\xi_1, \xi_2]_o = \psi(\beta_{\xi_1}(p, o) + i \beta_{\xi_2}(p, o)).
\]

Now Lemma \(6.8\) shows the existence of $C = C(p, \psi) > 0$ such that $|[\xi_1, \xi_2]_p - [\xi_1, \xi_2]_o| \leq C$, and hence the claim. Now let $p = o$. It is immediate from Lemma \(6.6\) that the first inequality holds with $C > 2C_1||\psi||$. Set $x_i := \zeta^{-1}(\xi_i) \in \partial \Gamma$ for $i = 1, 2, 3$. For each $i$, we fix a geodesic line $[x_i, x_{i+1}]$ joining $x_i$ and $x_{i+1}$, and choose $\gamma_{i+2} \in \pi_{[x_i, x_{i+1}]}(e)$, where all the indices are to be interpreted mod 3. By the hyperbolicity of $\Gamma$, at least one of the following holds:

1. $d_w(\gamma_1, \gamma_2) < C\delta$;
2. $d_w(\gamma_2, \gamma_3) < C\delta$;
3. There exists $\gamma' \in \Gamma$ and a geodesic segment $(e, \cdots, \gamma', \cdots, \gamma_2)$ such that the diameter of $\{\gamma', \gamma_1, \gamma_3\}$ is at most $C\delta$.

where $C > 0$ is a uniform constant. Let $C_1 > 0$ be a constant from Lemma \(6.6\) so that we have

\[
(6.9) \quad [\xi_1, \xi_3]_o \geq \psi(\mu(\gamma_2) + i \mu(\gamma_2)) - C_1||\psi||.
\]

We first consider the case (1). Since

\[
d(\gamma_1o, \gamma_2o) \leq d_w(\gamma_1, \gamma_2) \max_{s \in S} d(o, s) < C\delta \max_{s \in S} d(o, s)
\]

it follows from Lemma \(2.3\) that

\[
||\mu(\gamma_1) - \mu(\gamma_2)|| \leq C_2
\]

for some uniform $C_2 > 0$. Hence

\[
[\xi_1, \xi_3]_o \geq \psi(\mu(\gamma_1) + i \mu(\gamma_1)) - C_1||\psi|| - 2C_2
\]

\[
\geq [\xi_2, \xi_3]_o - 2C_1||\psi|| - 2C_2 \text{ by Lemma } 6.6
\]

\[
\geq \min([\xi_1, \xi_2]_o, [\xi_2, \xi_3]_o) - 2C_1||\psi|| - 2C_2.
\]

The case (2) can be handled similarly by interchanging the roles of $\gamma_2$ and $\gamma_3$. Finally in case (3), let $R_2$ be as in Corollary \(5.13\). Since $(e, \cdots, \gamma', \cdots, \gamma_2)$ is a geodesic, we have by Corollary \(5.13\) that

\[
||\mu(\gamma_2) - \mu(\gamma') - \mu(\gamma'^{-1}\gamma_2)|| \leq R_2.
\]

By \(6.9\) and the fact $\psi(\mu((\gamma'^{-1}\gamma_2)^{\pm 1})) \geq 0$, we deduce

\[
[\xi_1, \xi_3]_o \geq \psi(\mu(\gamma') + i \mu(\gamma')) - C_1||\psi|| - 2R_2||\psi||
\]

\[
\geq \psi(\mu(\gamma_1) + i \mu(\gamma_1)) - (C_1 + 2C_2 + 2R_2)||\psi||,
\]

as the diameter of $\{\gamma', \gamma_1, \gamma_3\}$ is less than $\delta$. The rest of the proof is similar to case (1). □
Covering lemma. For $\xi \in \Lambda$ and $r > 0$, set
\[ \mathbb{B}_p(\xi, r) := \{ \eta \in \Lambda : d_{\psi, p}(\xi, \eta) < r \}. \]
Using Theorem 6.4, we prove the following:

Lemma 6.10 (Covering lemma). There exists $N_0 = N_0(\psi, p) > 0$ such that for any $\xi_1, \xi_2 \in \Lambda$ and $r_1 \geq r_2 > 0$, if $\mathbb{B}_p(\xi_1, r_1) \cap \mathbb{B}_p(\xi_2, r_2) \neq \emptyset$, then $\mathbb{B}_p(\xi_1, 3N_0r_1) \supset \mathbb{B}_p(\xi_2, r_2)$. Moreover, $N_0(\psi, p)$ can be taken uniformly for $p$ in a fixed compact subset.

Proof. Let $N_0$ be as given by Lemma 6.11 below. Choose $\xi_3 \in \mathbb{B}_p(\xi_1, r_1) \cap \mathbb{B}_p(\xi_2, r_2)$. Then we have $d_p(\xi_1, \xi_3) < r_1$ and $d_p(\xi_2, \xi_3) < r_2$. Now by Lemma 6.11, for an arbitrary $\xi' \in \mathbb{B}_p(\xi_2, r_2)$,
\[ d_p(\xi_1, \xi') \leq N_0(d_p(\xi_1, \xi_3) + d_p(\xi_2, \xi_3) + d_p(\xi_2, \xi')) < N_0(r_1 + 2r_2) \leq 3N_0r_1. \]
This proves the lemma. \hfill \Box

Lemma 6.11. There exists $N_0 = N_0(\psi, p) \in \mathbb{N}$ such that for any $\xi_1, \xi_2, \xi_3, \xi_4 \in \Lambda$,
\[ d_p(\xi_1, \xi_4) \leq N_0(\max(d_p(\xi_1, \xi_2), d_p(\xi_2, \xi_1)) + \max(d_p(\xi_2, \xi_3), d_p(\xi_3, \xi_2)) + \max(d_p(\xi_3, \xi_4), d_p(\xi_4, \xi_3))). \]
Moreover, $N_0(\psi, p)$ can be taken uniformly for $p$ in a fixed compact subset.

Proof. Choose $\varepsilon > 0$ sufficiently small so that Theorem 6.4 holds, and set $d = d_{\varepsilon}, C = C_{\varepsilon}$. We then have
\[ d_p(\xi_1, \xi_4)^{\varepsilon} \leq C d(\xi_1, \xi_4) \]
\[ \leq C(d(\xi_1, \xi_2) + d(\xi_2, \xi_3) + d(\xi_3, \xi_4)) \]
\[ \leq C^2(\max(d_p(\xi_1, \xi_2), d_p(\xi_2, \xi_1))))^{\varepsilon} \]
\[ \leq \max(d_p(\xi_3, \xi_4), d_p(\xi_4, \xi_3)))^{\varepsilon}. \]
Since $(a^\varepsilon + b^\varepsilon + c^\varepsilon)^{1/\varepsilon} \leq \alpha(a + b + c)$ for all $a, b, c \geq 0$ for some uniform constant $\alpha = \alpha(\varepsilon) > 0$, it suffices to take the $1/\varepsilon$ power in each side of the above. Now the second part follows from (6.3) and Lemma 6.8. \hfill \Box

7. Conical points, divergence type and classification of PS measures

In this section, we show that for Anosov groups, the space of all PS-measures on $\Lambda$ is parametrized by $D^+_\Gamma$.

Conical limit points. For a discrete subgroup $\Gamma < G$ and $x \in \Gamma \setminus G$, we mean by $\limsup x A^+ M$ the set of all limit points $\lim_{i \to \infty} x a_i m_i$ where $a_i \to \infty$ in $A^+$ and $m_i \in M$. 
**Definition 7.1** (Conical limit points). We call $\xi \in \mathcal{F}$ a conical limit point of $\Gamma$ if $\limsup gA^+ \mathcal{F} \neq \emptyset$ for some $g \in G$ with $g^+ = \xi$. Equivalently, $\xi \in \mathcal{F}$ is conical if there exists $R > 0$ such that $\xi \in O_R(o, \gamma_i o)$ for some sequence $\gamma_i \to \infty$ in $\Gamma$. We denote by $\Lambda_c$ the set of all conical limit points of $\Gamma$.

**Lemma 7.2** (Regularity property). Let $\Gamma$ be Anosov. If $\gamma_i g_i a_i$ is a bounded sequence where $g_i \in G$ is bounded, $\gamma_i \in \Gamma$ and $a_i \to \infty$ in $A^+$, then $a_i \to \infty$ regularly in $A^+$.

In particular, for any $x \in \Gamma \setminus G$,

$$
\limsup_{i \to \infty} xA^+ M = \{ \lim_{i \to \infty} x a_i m_i : m_i \in M, a_i \to \infty \text{ regularly in } A^+ \}.
$$

**Proof.** We prove this lemma only using the property that $L_\Gamma - \{ 0 \} \subset \text{int} \, a^+$, which holds for Anosov groups by Theorem 4.2. As $g_i$ and $\gamma_i g_i a_i$ are bounded sequences, the sequence $\mu(\gamma_i \gamma_i^{-1}) - \log a_i$ is also bounded by Lemma 2.3 Hence it suffices to show that $\mu(\gamma_i \gamma_i^{-1}) \to \infty$ regularly. This follows easily from the property $L_\Gamma - \{ 0 \} \subset \text{int} \, a^+$ by considering the sequence of unit vectors $\| \mu(\gamma_i \gamma_i^{-1}) \|^{-1} \mu(\gamma_i \gamma_i^{-1})$.

We deduce from Proposition 5.10

**Proposition 7.3.** For $\Gamma$ Anosov, there exists $R_0 > 0$ such that for any $g \in G$ with $g^+ \in \Lambda$, there exist $a_i \to \infty$ regularly in $A^+$ and $\gamma_i \in \Gamma$ such that $d(o, \gamma_i g_i a_i o) < R_0$. In particular,

$$
\Lambda = \Lambda_c.
$$

**Proof.** We first check that $\Lambda_c \subset \Lambda$. Let $g^+ \in \Lambda_c$ for some $g \in G$. Then there exists $\gamma_i \in \Gamma$ and $a_i m_i \to \infty$ in $A^+ M$ such that $\gamma_i g_i a_i m_i$ is bounded. By Lemma 7.2, it follows that $a_i \to \infty$ regularly in $A^+$. Hence by Lemma 2.10 $g_i m_i \to g^+$ as $i \to \infty$. Since $d(g_i m_i, \gamma_i^{-1} o)$ is bounded, $\gamma_i^{-1} o \to g^+$ as $i \to \infty$. By Lemma 2.12 $g^+ \in \Lambda$.

Let $g^+ = \xi \in \Lambda$ and $x \in \partial \Gamma$ be such that $\xi = \zeta(x)$. Choose a geodesic ray $r = (\gamma_0 = e, \gamma_1, \gamma_2, \cdots)$ representing $x$. Note that if $g^+ = h^+$, then for any sequence $a_i \to \infty$ in $A^+$, there exists $h_i \in \Gamma$ such that $d(g_i a_i, h_i o) < 1$ for all sufficiently large $i$. Hence we may assume that $g \in K$. By Proposition 5.10 $\gamma_i o$ is contained in the $R_0$-neighborhood of $g A^+ o$, with $R_0$ given therein. This proves the claim.

**Classification of PS measures on $\Lambda$.**

**Lemma 7.4.** Let $\psi_i \in a^*$ and $\nu_{\psi_i}$ be a $(\Gamma, \psi_i)$-PS measure for $i = 1, 2$. If $\nu_{\psi_1} = \nu_{\psi_2}$, then $\psi_1 = \psi_2$.

**Proof.** Suppose that $\nu_{\psi_1} = \nu_{\psi_2}$. Then for all $\gamma \in \Gamma$ and $\xi \in \Lambda$, we have

$$
\psi_1(\beta_\xi(e, \gamma)) = \psi_2(\beta_\xi(e, \gamma)).
$$

By setting $\xi = y^+_\xi$, we obtain $\lambda(\gamma) \in \ker(\psi_1 - \psi_2)$ for all $\gamma \in \Gamma$, by Lemma 3.3 Hence $L_\Gamma \subset \ker(\psi_1 - \psi_2)$. Since $L_\Gamma$ has nonempty interior [1 Thm. 1.2], this implies that $\psi_1 = \psi_2$. 

Remark 7.5. When $\Gamma$ is an Anosov subgroup, $\nu_{\psi_1}$ and $\nu_{\psi_2}$ are even mutually singular to each other whenever $\psi_1 \neq \psi_2$ (See Theorem 10.19 below).

We denote by $\mathcal{S}_\Gamma$ the space of all PS measures on $\Lambda$. Recall that for $\psi \in D^*_1$, Quint constructed a $(\Gamma, \psi)$-PS measure on $\Lambda$ [30]. In the Anosov case, such a measure is unique, which we denote by $\nu_{\psi}$. By Lemma 7.4 the map $\psi \mapsto \nu_{\psi}$ from $D^*_1$ to $\mathcal{S}_\Gamma$ is injective.

**Theorem 7.6.** For $\Gamma \prec G$ Anosov, the map $\psi \mapsto \nu_{\psi}$ is a bijection between $D^*_1$ and $\mathcal{S}_\Gamma$.

We need to prove that the map $\psi \mapsto \nu_{\psi}$ is surjective.

**Lemma 7.7** (Size of shadow). Let $\Gamma \prec G$ be Anosov and $\psi \in \mathfrak{a}^*$. For a $(\Gamma, \psi)$-conformal measure $\nu_{\psi}$ on $\mathcal{F}$, there exists $R = R(\nu_{\psi}) > 0$ with the following property: for all $r \geq R$, there exists $C = C(r) > 0$ such that for all $\gamma \in \Gamma$,

$$C^{-1}e^{-\psi(\mu(\gamma))} \leq \nu_{\psi}(O_r(o, \gamma o)) \leq Ce^{-\psi(\mu(\gamma))}.$$

**Proof.** We claim that there exists $R > 0$ such that

$$c := \inf_{\gamma \in \Gamma} \nu_{\psi}(O_R(\gamma^{-1}o, o)) > 0.$$

Suppose not. Then there exist $R_i \to \infty$ and $\gamma_i \in \Gamma$ with $\nu(O_{R_i}(\gamma_i^{-1}o, o)) < 1/i$. Let $\gamma_i = k_ia_i\ell_i \in KA^+ K$ be the Cartan decomposition of $\gamma_i$. Passing to a subsequence, we may assume that $\ell_i \to \ell_0$ as $i \to \infty$. Note that $a_i \to \infty$ regularly in $A^+$ as $\Gamma$ is Anosov. And hence $\lim_{i \to \infty} O_{R_i}(a_i^{-1}o, o) = N^+ e^+$. Since $O_{R_i}(\gamma_i^{-1}o, o) = \ell_i^{-1}O_{R_i}(a_i^{-1}o, o)$, we obtain $\nu(\ell_i^{-1}N^+ e^+) = 0$. Since $N^+ e^+$ is Zariski open, this contradicts the fact that $\Lambda \subset \text{ supp } \nu_{\psi}$ is Zariski dense in $\mathcal{F}$. This proves the claim.

Now let $\gamma \in \Gamma$ and $r > R$ be arbitrary. By Lemma 5.7, for all $\xi \in O_r(\gamma^{-1}o, o)$, we have

$$\|\beta_\xi(\gamma^{-1}o, o) - \mu(\gamma)\| \leq \kappa r.$$

Since

$$\nu_{\psi}(O_r(o, \gamma o)) = \int_{O_r(\gamma^{-1}o, o)} e^{-\psi(\beta_\xi(\gamma^{-1}o, o))} d\nu_{\psi}(\xi),$$

it remains to set $C = \max(c^{-1}, 1) e^{\|\psi\|\kappa r}$. \qed

**Lemma 7.8.** [29, Lem. III.1.3] Let $\theta : \mathfrak{a} \to \mathbb{R}$ be a continuous function satisfying $\theta(tu) = t\theta(u)$ for all $t \geq 0$ and $u \in \mathfrak{a}$. If $\theta(u) > \psi_\Gamma(u)$ for all $u \in \mathfrak{a} - \{0\}$, then

$$\sum_{\gamma \in \Gamma} e^{-\theta(\mu(\gamma))} < \infty.$$

**Lemma 7.9.** Let $\Gamma \prec G$ be Anosov and $\psi \in \mathfrak{a}^*$. If there exists a $(\Gamma, \psi)$-PS measure on $\Lambda$, then

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty.$$
In particular, for any $\psi \in D^*_\Gamma$, we have $\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty$.

Proof. By Proposition 7.3, $\Lambda = \Lambda_c$. Hence $\Lambda$ is an increasing union $\bigcup_{N=1}^{\infty} \Lambda_N$, where

$$\Lambda_N := \{\xi \in \Lambda : \text{there exists } \gamma_i \to \infty \text{ in } \Gamma \text{ such that } \xi \in O_N(o, \gamma_i o)\}.$$ 

Hence $\nu(\Lambda_N) > 0$ for some $N_0 \geq 1$. Suppose that there exists a $(\Gamma, \psi)$-conformal measure, say $\nu$. Fix $N \geq \max\{R(\nu), N_0\}$, and set $C := C(N)$ where $R(\nu)$ and $C(N)$ are as in Lemma 7.7. Observe that

$$\Lambda_N \subset \bigcup\{O_N(o, \gamma o) : \gamma \in \Gamma, d(o, \gamma o) > m\}.$$ 

for any $m \geq 1$. Hence

$$0 < \nu(\Lambda_N) \leq \sum_{d(o, \gamma o) > m} \nu(O_N(o, \gamma o)) \leq C \sum_{d(o, \gamma o) > m} e^{-\psi(\mu(\gamma))}.$$ 

Since $m > 1$ is arbitrary, the conclusion follows. \hfill \Box

If $u_\Gamma \in a^+$ is the unit vector in the unique direction of maximal growth given by $\psi_T(u_\Gamma) = \max_{||u||=1} \psi_T(u)$, then $D_{u_\Gamma} \psi_T(\cdot) = \delta_\Gamma(\cdot)$ where $\delta_\Gamma = \psi_T(u_\Gamma)$ (cf. [11, Lem. 2.23]).

We have the following corollary of Lemma 7.9 in view of Proposition 4.3

Corollary 7.10 (Divergence property). Let $\Gamma < G$ be Anosov. For any unit vector $u \in \text{int} \, L_\Gamma$, $\sum_{\gamma \in \Gamma} e^{-D_u \psi_T(\mu(\gamma))} = \infty$. In particular,

$$\sum_{\gamma \in \Gamma} e^{-\delta_\Gamma(u_\Gamma, \mu(\gamma))} = \infty.$$ 

For convex cocompact groups in rank one Lie groups, there exists a unique Patterson-Sullivan measure on $\Lambda$ and its dimension is given by the critical exponent of $\Gamma$. The following proposition, which implies Theorem 7.6, generalizes such phenomenon:

Proposition 7.11. Let $\Gamma < G$ be an Anosov subgroup and $\psi \in a^*$. If there exists a $(\Gamma, \psi)$-PS measure, then $\psi \in D^*_\Gamma$.

Proof. Suppose that there exists a $(\Gamma, \psi)$-PS measure, say $\nu_\psi$, for $\psi \in a^*$. We note that $\psi \geq \psi_T$ by [29, Thm. 8.1]. We need to show $\psi(u) = \psi_T(u)$ for some $u \in \text{int} \, L_\Gamma$.

By Theorem 4.2, it suffices to show that $\psi(u) = \psi_T(u)$ for some $u \in L_\Gamma - \{0\}$. Suppose not. Then $\psi(u) > \psi_T(u)$ for all $u \in a - \{0\}$. By Lemma 7.8, we have

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty.$$ 

This is a contradiction by Lemma 7.9. \hfill \Box

Remark 7.12. The shadow lemma and the property that $\Lambda = \Lambda_c$ implies that any $(\Gamma, \psi)$-conformal measure on $\mathcal{F}$ has no atom in $\Lambda$. 
Let $\Lambda^\dagger \subset \mathcal{F}$ be the set of all accumulation points of $\{\gamma \xi : \gamma \in \Gamma, \xi \in \mathcal{F}\}$. Clearly, $\Lambda^\dagger$ is a $\Gamma$-invariant closed subset containing $\Lambda$, and $\Gamma$ acts properly discontinuously on $\mathcal{F} - \Lambda^\dagger$. If $G$ has rank one, $\Lambda^\dagger = \Lambda$. In general, $\Lambda^\dagger$ is a proper subset of $\mathcal{F}$ [16, Thm. 1.9].

Lemma 7.13. Any atom-free $\Gamma$-ergodic conformal measure $\nu$ on $\mathcal{F}$ is supported on $\Lambda^\dagger$.

Proof. Suppose that $\nu(\Lambda^\dagger) \neq |\nu|$. Then $\nu(\Lambda^\dagger) = 0$ by the ergodicity assumption. Since $\nu$ is atom free, we can find $\xi_1, \xi_2 \in \text{supp} \nu - \Lambda^\dagger$ such that $\Gamma \xi_1 \neq \Gamma \xi_2$. Since $\Gamma$ acts properly discontinuously on $\mathcal{F} - \Lambda^\dagger$, there exist open neighborhoods $U_i$ of $\xi_i$ in $\mathcal{F} - \Lambda^\dagger$ such that $\Gamma U_1 \cap \Gamma U_2 = \emptyset$. Since $\xi_i \in \text{supp} \nu$, we have $\nu(\Gamma U_i) > 0$, contradicting the $\Gamma$-ergodicity of $\nu$. \hfill $\square$

It will be interesting to know whether there exists a $\Gamma$-conformal measure on $\Lambda^\dagger$ which is not supported on $\Lambda$ for Anosov groups.

8. Myrberg Limit Points of Anosov Groups

In this section, we discuss the notion of Myrberg limit points. We show that for Anosov groups, the set of Myrberg limit points has a co-measure zero in $\Lambda$ for any PS measure on $\Lambda$. In the rank one case, this was proved in ([25], [1], [26]). Let $\Gamma < G$ be a Zariski dense discrete subgroup.

Definition 8.1 (Myrberg points). Let $p \in X$. We call a point $\xi_0 \in \Lambda$ a Myrberg limit point for $\Gamma$ if for any $\xi \neq \eta$ in $\Lambda$, there exists a sequence $\gamma_i \in \Gamma$ such that $\gamma_i p \to \xi$ and $\gamma_i \xi_0 \to \eta$ as $i \to \infty$.

Note that this definition is independent of the choice of $p \in X$ by Lemma 2.9. We denote by $\Lambda_M \subset \Lambda$ the set of all Myrberg limit points for $\Gamma$.

When $G$ is of rank one, a Myrberg limit point $\xi \in \Lambda$ is characterized by the property that any geodesic ray toward $\xi$ is dense in the space of all geodesics connecting limit points. The following proposition generalizes this to a general Anosov subgroup.

Proposition 8.2. Let $\Gamma$ be Anosov. We have $\xi_0 \in \Lambda_M$ if and only if for any $g \in G$ with $g^+ = \xi_0$, 
\[ \limsup \Gamma \setminus \Gamma g A^+ M = \Omega. \]

Let $\Gamma < G$ be an Anosov subgroup for the rest of this section.

Lemma 8.3. Let $b_i \in A$ be a sequence tending to $\infty$ such that $w^{-1} b_i^{-1} w \in A^+$ for some $w \in \mathcal{W}$. If $\gamma_i g \beta_i \to h$ for some $h, g \in G$ and $\gamma_i \in \Gamma$, then $\gamma_i g \beta_i \to h w^+ \in \mathcal{F}$. In particular, if $b_i \in A^+$, then $\gamma_i g \beta_i \to h^-$. \hfill $\square$
Let \((\cdot|\cdot)\) denote the Gromov product in the hyperbolic space \(\Gamma\) based at \(e \in \Gamma\). The space \(\Gamma \cup \partial \Gamma\) is a compact space with the topology given as follows: a sequence \(\gamma_i \in \Gamma\) converges to \(x \in \partial \Gamma\) if and only if \(\lim_{i \to \infty} (\gamma_i|v_i) = \infty\) for any geodesic ray \((e, v_1, v_2, \cdots)\) representing \(x\).

The following is proved in [19] Coro. 5.8:

**Theorem 8.4** (The limit map as a continuous extension of the orbit map). For any \(p \in X\), the map \(\Gamma \cup \partial \Gamma \to X \cup \mathcal{F}\) given by \(\gamma \mapsto \gamma p\) for \(\gamma \in \Gamma\) and \(x \mapsto \zeta(x)\) for \(x \in \partial \Gamma\) is continuous.

We need the following basic fact about word hyperbolic groups.

**Lemma 8.5.** Let \(x \neq y\) in \(\partial \Gamma\). If \(\gamma_i \in \Gamma\) is a sequence such that \((\gamma_i, \gamma_i y) \to (x', y')\) in \(\partial \Gamma \times \partial \Gamma\), then \(\gamma_i\) converges to either \(x'\) or \(y'\).

**Proof.** Choose a geodesic line \([x, y]\), and its representative \((\cdots, u_2, u_1, u_0 = v_0, v_1, v_2, \cdots)\). Note that \(x = [u_0, u_1, u_2, \cdots]\) and \(y = [v_0, v_1, v_2, \cdots]\). It suffices to show that \(\gamma_i u_0\) converges to either \(x'\) or \(y'\). Suppose not. Then by passing to a subsequence we have \(\gamma_i u_0 \to z'\) where \(z' \notin \{x', y'\}\). Since \((z'|x'), (z'|y') < \infty\), there exists a subsequence \(n_k\) such that \(\sup_k (\gamma_k u_0|\gamma_k u_{n_k}) + (\gamma_k u_0|\gamma_k v_{n_k}) < \infty\). Let \(L_k^- := [\gamma_k u_0, \gamma_k u_{n_k}]\) and \(L_k^+ := [\gamma_k u_0, \gamma_k v_{n_k}]\), so that \(\sup_k d_{\omega}(e, L_k^\pm) < \infty\). The thin triangle property of the hyperbolic group \(\Gamma\) implies that if the projection of \(e\) to the geodesic segment \(L_k^- \cup L_k^+\) lies in \(L_k^\pm\), then \(d_{\omega}(e, \gamma_k u_0)\) is equal to \(d_{\omega}(e, L_k^\pm)\) up to a uniform additive constant. And hence \(d_{\omega}(e, \gamma_k u_0)\) is uniformly bounded, which is a contradiction as \(\gamma_k \to \infty\) as \(k \to \infty\). \(\square\)

The following is immediate from Theorem 8.4 and Lemma 8.5.

**Corollary 8.6.** Let \(\gamma_i \in \Gamma\) be an infinite sequence such that \((\gamma_i, \gamma_i^g) \to (\xi', \eta')\) in \(\Lambda^{(2)}\) as \(i \to \infty\). Then for any \(p \in X\), \(\gamma_i p\) converges to either \(\xi'\) or \(\eta'\).

**Lemma 8.7.** Let \(g \in G\) be such that \(g^\pm \in \Lambda\). If \(\gamma_i g^\pm \to \xi\) as \(i \to \infty\) for some \(\gamma_i \in \Gamma\), then \(\gamma_i g_0 \to \xi\) as \(i \to \infty\).

**Proof.** Set \(x^\pm := \zeta^{-1}(g^\pm)\) and \(y = \zeta^{-1}(\xi)\). Since \(\zeta : \partial \Gamma \to \Lambda\) is a homeomorphism, we have \(\gamma_i x^\pm \to y\) as \(i \to \infty\). By Lemma 8.5, we have \(\gamma_i \to y\) as \(i \to \infty\). By Lemma 8.4 we get \(\gamma_i g_0 \to \xi\) as \(i \to \infty\). By Lemma 2.9, \(\lim_{i \to \infty} \gamma_i g_0 = \xi\) as desired. \(\square\)

Since the fibers of the visual map \(g \mapsto g^+\) are \(P\)-orbits, the following lemma is an easy consequence of the regularity lemma 7.2.

**Lemma 8.8.** If \(g^+ = h^+\), then \(\limsup \Gamma g A^+ M = \limsup \Gamma h A^+ M\).

**Proof of Proposition 8.2** Set \(\hat{\Omega} := \{g \in G : g^\pm \in \Lambda\}\). Suppose \(\xi_0 \in \Lambda_M\) and \(g^+ = \xi_0\). We claim that \(\Gamma g A^+ M = \hat{\Omega}\). By Lemma 8.8 we may assume that \(g^- \in \Lambda\). Let \(h \in \hat{\Omega}\). As \(\xi_0 \in \Lambda_M\), there exists \(\gamma_i \in \Gamma\) such that \(\gamma_i g^+ \to h^+\) and \(\gamma_i g_0 \to h^-\). By Lemma 8.5 by passing to a subsequence,
\( \gamma_i g^- \) converges to \( h^- \). Therefore \( \gamma_i gA^M \to hA^M \) in \( G/A^M \); there exists \( b_i m_i \subset A^M \) such that \( \gamma_i gb_i m_i \to h \). We claim that \( b_i \in A^+ \) for all large \( i \). If not, by passing to a subsequence, we have \( m_i^{-1} \) converges to some \( m_0 \in M \) and there exists \( w \in h^{-1} h^+ = A^+ \). Then \( \gamma_i g a_i \to h m_0 w \). By Lemma 8.3 \( \gamma_i g a_i \to h m_0 w \), and hence \( h m_0 w \to h^- \). It follows that \( w = e \), yielding a contradiction. Therefore \( h \in \text{lim sup} \Gamma gA^+ M \), proving the claim.

Now suppose that \( \text{lim sup} \Gamma gA^+ M = \hat{\Omega} \). We claim that \( g^+ \in \Lambda_M \). Let \( \xi \neq \xi' \) be an \( \xi \in \Lambda \), and let \( h \in G \) be such that \( h^+ = \xi \) and \( h^+ = \xi' \). By the hypothesis and Lemma 7.2, there exist \( \gamma_i \in \Gamma \), \( m_i \subset M \) and \( a_i \to \infty \) regularly in \( A^+ \) such that \( \gamma_i g a_i m_i \to h \) in \( G \). Then \( \gamma_i g^+ \to h^+ = \xi \). By Lemma 8.3 \( \gamma_i g a_i m_i \) converges to \( h \). Hence \( g^+ \in \Lambda_M \).

**Theorem 8.9.** For any \( \Gamma \)-\( PS \)-measure \( \nu \) on \( \Lambda \), \( \Lambda_M \) has a full \( \nu \)-measure.

**Proof.** By Theorem 7.6 \( \nu = \nu_\psi \) for some \( \psi \in D^*_\Gamma \). Let \( m_\psi \) be the measure in Theorem 4.5. Let \( \tilde{Z}_\psi \subset \Gamma \Lambda(2) \times \mathbb{R} \) denote the set of elements which has dense \{ \( \tilde{\tau} : t \geq 0 \) \} orbit, and \( \tilde{Z}_\psi \) be its lift in \( \Lambda(2) \times \mathbb{R} \). Denote by \( \pi : \Lambda(2) \times \mathbb{R} \to \Lambda \) the projection map \( \pi(\xi, \eta, t) = \xi \). Since \( m_\psi \) is a \{ \( \tilde{\tau} \} \)-ergodic finite measure on \( \Gamma \Lambda(2) \times \mathbb{R} \), \( Z_\psi \) has full \( m_\psi \)-measure. Hence it suffices to prove that \( \pi(\tilde{Z}_\psi) \subset \Lambda_M \).

Let \( \xi \in \pi(\tilde{Z}_\psi) \) and \( (\eta_1, \eta_2) \in \Lambda(2) \) be arbitrary. We need to show that there exists \( \gamma_i \in \Gamma \) such that \( \gamma_i \xi \to \eta_1 \) and \( \gamma_i \xi \to \eta_2 \) as \( i \to \infty \). Choose \( (\xi, \xi', 0) \in Z_\psi \). By definition, we can find \( \gamma_i \in \Gamma \) and \( t_i \to +\infty \) such that

\[
\gamma_i(x, \xi', t_i) = (\gamma_i(x), \gamma_i(x'), t_i + \psi(\gamma_i(x'), \gamma_i(\xi'))) \equiv (\gamma_i(\xi), \gamma_i(\xi'), t_i + \psi(\gamma_i(\xi), \gamma_i(\xi'))) \equiv (\gamma_i(\xi), \gamma_i(\xi'), t_i + \psi(\gamma_i(\xi), \gamma_i(\xi'))) \text{ converges. Write } x = \xi^{-1}(\xi), \; x' = \xi'^{-1}(\xi'), \; y_1 = \xi^{-1}(\eta_1), \; y_2 = \xi'^{-1}(\eta_2), \text{ and choose } u \in [x, x'].
\]

Since the triangle \( [\gamma_i(x), \gamma_i(x')] \cup [\gamma_i(u), \gamma_i(x')] \cup [\gamma_i(u), \gamma_i(x')] \) is \( \delta \)-thin, it follows that for all \( i \), either \( \gamma_i(x) \in O_\delta(u, \gamma_i(u)) \) or \( \gamma_i(x') \in O_\delta(u, \gamma_i(u)) \). We claim the latter holds for all large \( i \).

Suppose not. Then by passing to a subsequence, we may assume that \( \gamma_i x \in O_\delta(u, \gamma_i u) \) for all \( i \). Then for some uniform constant \( c > 0 \), \( \gamma_i \xi \in O_{c(\delta + 1)}(u, \gamma_i u) \) by Proposition 5.12 and hence

\[
|\psi(\beta_{\gamma_i}(u, \gamma_i u)) - \psi(\mu(\xi))| < ||\psi||\kappa(c(\delta + 1))
\]

by Lemma 5.7. Since \( \psi(\mu(\xi)) \to +\infty \) as \( i \to \infty \) by Lemma 5.4 and \( \psi(\beta_{\gamma_i}(u, \gamma_i u)) \) and \( \psi(\beta_{\gamma_i}(o, \gamma_i o)) \) are uniformly close to each other, \( \psi(\beta_{\gamma_i}(o, \gamma_i o)) \to +\infty \). This contradicts the hypothesis that the sequence \( t_i + \psi(\beta_{\gamma_i}(o, \gamma_i o)) \) converges to a finite number as \( i \to \infty \). It follows that for all sufficiently large \( i \),

\[
(8.10) \quad \gamma_i x' \in O_\delta(u, \gamma_i u).
\]

On the other hand, \( \gamma_i u \to y_\ell \) for some \( \ell \in \{1, 2\} \) by Lemma 8.5. Since \( \gamma_i x' \to y_2 \), and \( O_\delta(u, \gamma_i u) \) converges to \( y_\ell \), (8.10) implies that \( \gamma_i u \to y_2 \).

Therefore \( \gamma_i o \to \eta_2 \) by Lemma 8.4. \( \Box \)
In the rank one case, $A = \{a_t\}$ is the union of $A^+ = \{a_t : t \geq 0\}$ and $A^- = \{a_t : t \leq 0\}$, and the $AM$-ergodicity of the BMS measure implies that for almost all $x \in \Gamma \setminus G$, $xA^+M$ is dense in $\Omega = \{x \in \Gamma \setminus G : x^\pm \in \Lambda\}$. In general, $A = \cup_{w \in W} wA^+w^{-1}$, and we have the following corollary of Theorem 8.9.

**Corollary 8.11.** Let $\psi \in D^*_\Gamma$. For $m^{\text{BMS}}_\psi$-almost all $x \in \Omega$, 

$$xwA^+w^{-1}M \quad \begin{cases} \text{dense in } \Omega & \text{if } w = e, w_0; \\ \text{proper} & \text{otherwise.} \end{cases}$$

*Proof.* Note that if $(g^+, gw^+) \in \mathcal{F}^{(2)}$ for $w \in W$, then $w = w_0$. Choose $g \in G$ so that $\Gamma g = x$. Since $g^+ \in \Lambda$ and $\Lambda \times \Lambda - \{(\xi, \xi)\} \subset \mathcal{F}^{(2)}$, $gw^+ \notin \Lambda$ for all $w \neq e, w_0$. Hence it follows from Proposition 7.3 that for each $w \in W - \{e, w_0\}$, $\lim sup xwA^+M = \emptyset$, or equivalently, the map $A^+M \to xwA^+M$ is proper. On the other hand, Theorem 8.9 implies that for $m^{\text{BMS}}_\psi$-almost all $x \in \Omega$, each $xA^+M$ and $xw_0A^+w_0^{-1}M$ is dense in $\Omega$. \hfill $\Box$

9. **Criterion for ergodicity via essential values**

In this section, let $\Gamma < G$ be a Zariski dense discrete subgroup, and let $\nu_\psi$ be a $(\Gamma, \psi)$-conformal measure on $\mathcal{F}$ for $\psi \in \mathfrak{a}^*$. Consider the action of $G$ on $\mathcal{F} \times \mathfrak{a}$ by $g(\xi, v) = (g\xi, v + \beta_\xi(g^{-1}, v))$. Then the map $g \mapsto (g^+, b := \beta_{g^+}(e, g))$ induces a $G$-equivariant homeomorphism $G/NM \simeq \mathcal{F} \times \mathfrak{a}$.

Using this homeomorphism, we define a $\Gamma$-invariant Radon measure $\tilde{\nu}_\psi$ on $G/NM \simeq \mathcal{F} \times \mathfrak{a}$ by 

$$d\tilde{\nu}_\psi([g]) = d\nu_\psi(g^+) e^{\psi(b)} db.$$ 

Since $dm^{\text{BR}}_\psi = d\tilde{\nu}_\psi dm dn$, the $NM$-ergodicity of $m^{\text{BR}}_\psi$ is equivalent to the $\Gamma$-ergodicity of $\tilde{\nu}_\psi$. For simplicity, we set $\nu := \nu_\psi$ and $\tilde{\nu} = \tilde{\nu}_\psi$ for the rest of the section.

Schmidt gave a characterization of $\Gamma$-ergodicity of $\tilde{\nu}$ using the notion of $\nu$-essential values in the rank one case ([10], see also [35, Prop. 2.1]).

**Definition 9.1.** An element $v \in \mathfrak{a}$ is called a $\nu$-essential value, if for any Borel set $B \subset \mathcal{F}$ with $\nu(B) > 0$, and any $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that 

$$B \cap \gamma B \cap \{\xi \in \mathcal{F} : \|\beta_\xi(o, \gamma o) - v\| < \varepsilon\}$$

has a positive $\nu$-measure.

Let $\mathcal{E}_\nu$ denote the set of all $\nu$-essential values in $\mathfrak{a}$. It is easy to see that $\mathcal{E}_\nu$ is a closed subgroup of $\mathfrak{a}$. The main goal of this section is to prove the following criterion of $\Gamma$-ergodicity of $\tilde{\nu}$, which can be considered as a higher rank version of [35, Prop. 2.1].

**Proposition 9.2.** $(G/NM, \Gamma, \tilde{\nu})$ is ergodic if and only if $(G/P, \Gamma, \nu)$ is ergodic and $\mathcal{E}_\nu = \mathfrak{a}$. 

Suppose that \( \nu \) is a uniform positive lower bound for the volume of \( O \).

We will then find a subset \( A \) of \( \mathbb{R}^d \) such that \( \| h \|_{\nu} \leq C \) for some \( C > 0 \) independent of \( \xi \).

By replacing \( h \) with \( -h \) if necessary, we may assume that \( \nu \{ x \in G/NM : h(x) \neq h(xa) \} > 0 \) for some \( \log a \in E \).

We will then find a subset \( A^* = A^*(a) \subset G/NM \) with \( \nu(A^*) > 0 \) and \( \gamma \in \Gamma \) such that \( h(\gamma^{-1}x) \neq h(x) \) for all \( x \in A^* \); this contradicts the \( \Gamma \)-invariance of \( h \).

Similarly, \( \nu \{ x \in G/NM : h(x) \neq h(xa) \} > 0 \). Hence there exist \( r, \varepsilon > 0 \) such that

\[
Q_a := \{ x \in G/NM : h(x) < r - C\varepsilon < r + C\varepsilon < h(xa) \}
\]

has a positive \( \nu \)-measure. Now we can choose a ball \( O = B_a(v_0, \varepsilon/2) \subset a \) such that

\[
\nu((F \times O) \cap Q_a) > 0.
\]

Set \( F_a := \{ \xi \in F : \{ \xi \} \times O \cap Q_a \neq \emptyset \} \).

We claim that

\[ (9.4) \quad \text{if } (\xi, w) \in F_a \times O, \text{ then } h(\xi, w + \log a) > r > h(\xi, w). \]

Note that there exists \( v \in a \) with \( \| v \| < \varepsilon \) such that \( (\xi, w + v) \in Q_a \) and hence

\[
|h(\xi, w)| \leq |h(\xi, w) - h(\xi, w + v)| + |h(\xi, w + v)| < C\| v \| + (r - C\varepsilon) \leq r.
\]

Similarly,

\[
|h(\xi, w + \log a)| \geq |h(\xi, w + v + \log a)| - |h(\xi, w + \log a) - h(\xi, w + v + \log a)| > (r + C\varepsilon) - C\tau \| v \| > r,
\]

which verifies the claim \( (9.4) \).

Since \( \log a \in E \) and \( \nu(F_a) > 0 \), there exists \( \gamma \in \Gamma \) such that

\[
A := F_a \cap \gamma F_a \cap \{ \xi \in G/P : \| \beta(\xi, o, \gamma o) - \log a \| < \varepsilon \}
\]

has a positive \( \nu \)-measure. For \( \xi \in A \), set

\[
O_\xi := \{ w \in O : (\beta(\xi, o, \gamma o) - \log a) + w \in O \}.
\]

Since \( \| \beta(\xi, o, \gamma o) - \log a \| < \varepsilon \), and \( O \) is a Euclidean ball of diameter \( \varepsilon \), there is a uniform positive lower bound for the volume of \( O_\xi \). It follows that

\[
A^* := \bigcup_{\xi \in A} \{ \xi \} \times O_\xi
\]

has positive \( \nu \)-measure. We now claim that \( h \circ \gamma^{-1} > h \) on \( A^* \).

Let \( (\xi, w) \in A^* \). Since \( (\xi, w) \in F_a \times O \), \( (9.4) \) implies that \( h(\xi, w) < r \).
Write $\gamma^{-1}(\xi, w) = (\gamma^{-1} \xi, w + (\beta \xi (o, \gamma o) - \log a) + \log a)$. Since $\gamma^{-1} \xi, w + (\beta \xi (o, \gamma o) - \log a) \in F_a \times O$, (9.4) says that
\[ h(\gamma^{-1}(\xi, w)) > r; \]
this proves the claim. 

\textbf{Proof of Proposition 9.2.} Assume that $(G/NM, \Gamma, \tilde{\nu})$ is ergodic. Let $\pi : G/NM \to G/P$ denote the projection map. Since $\pi_* \tilde{\nu}$ is absolutely continuous with respect to $\nu$, it follows that $(G/P, \Gamma, \nu)$ is ergodic.

To show $E = a$, fix an arbitrary Borel set $B \subset G/P$ of positive $\nu$-measure. For any $w \in a$ and $\varepsilon > 0$, we define
\[ B_{w, \varepsilon} := \{ (\xi, v) \in G/P \times a : \xi \in B, \|v - w\| < \varepsilon \} \subset G/NM. \]
Observe that
\[ \tilde{\nu}(B_{0, \varepsilon}) = \int_{G/P} \int_a 1_{B_{0, \varepsilon}}(\xi, b)e^{\psi(b)} db d\nu(\xi) \geq \text{Vol}(B_0(0, \varepsilon)) e^{-\|\psi\|\varepsilon} \nu(B) > 0. \]
Hence it follows from the ergodicity of $(G/NM, \Gamma, \tilde{\nu})$ that $\tilde{\nu}(G/NM - \Gamma B_{0, \varepsilon}) = 0$. In particular, there exists $\gamma \in \Gamma$ such that $\tilde{\nu}(B_{w, \varepsilon} \cap \gamma B_{0, \varepsilon}) > 0$. Finally, note that if $(\xi, v) \in B_{w, \varepsilon} \cap \gamma B_{0, \varepsilon}$, then $\xi \in B \cap \gamma B$, and
\[ \|\beta \xi (e, \gamma) - w\| \leq \|\beta \xi (e, \gamma) - v\| + \|v - w\| \leq \varepsilon + \varepsilon = 2\varepsilon. \]
This, together with the fact $\pi_* \tilde{\nu} \ll \nu$, implies that
\[ \nu(B \cap \gamma B \cap \{ \xi \in G/P : \|\beta \xi (e, \gamma) - w\| \leq 2\varepsilon \}) > 0, \]
which finishes the proof of $(\Rightarrow)$.

We now assume that $(G/P, \Gamma, \nu)$ is ergodic and $E = a$. Let $h : G/NM \to [0, 1]$ be a $\Gamma$-invariant Borel function. We need to show that $h$ is constant $\tilde{\nu}$-a.e. Identifying $a \simeq \mathbb{R}^r$ with $r = \text{rank} G$, for each $\tau = (\tau_1, \cdots, \tau_r) \in a$, we define a $\Gamma$-invariant Borel function $h_\tau : G/NM \to \mathbb{R}$ as follows:
\[ h_\tau(x) = \int_0^{\tau_1} \cdots \int_0^{\tau_r} h(x \exp(t_1, \cdots, t_r)) dt_r \cdots dt_1. \]

Note that $h_\tau$ satisfies the hypothesis of Lemma 9.3. Hence by the hypothesis $E_\nu = a$, for each $a \in A$, $h_\tau(x) = h_\tau(xa)$ for $\tilde{\nu}$-a.e. $x \in G/NM$.

Let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of $A$. Then there exists $\Omega_n$ of full $\tilde{\nu}$-measure such that for all $x \in \Omega_n$, $h_\tau(x) = h_\tau(xa_n)$. Set $\Omega := \cap_{n=1}^\infty \Omega_n$. Then for all $x \in \Omega$, we have $h_\tau(x) = h_\tau(xa)$ for all $a \in A$, as $h_\tau(\xi, \cdot)$ is continuous on $a$. Now $h_\tau$ is a $\Gamma$-invariant function on $G/NM$, which is also $A$-invariant $\tilde{\nu}$-a.e. Since $(G/P, \Gamma, \nu)$ is ergodic, there exists $c(\tau) \in \mathbb{R}$ such that $h_\tau = c(\tau) \tilde{\nu}$-a.e. on $G/NM$.

Next, fix $1 \leq i \leq r$ and $\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \cdots, \tau_r \geq 0$, and define
\[ f(t) := (\tau_1, \cdots, \tau_{i-1}, t, \tau_{i+1}, \cdots, \tau_r) \in a. \]
Then $t \mapsto c(f(t))$ is linear; indeed, by definition, we have
\[ h_{f(t+s)} = h_{f(t)} + h_{f(s)} \circ \exp(t e_i) \]
for all \( t, s \geq 0 \) and hence \( c(f(t + s)) = c(f(t)) + c(f(s)) \). We conclude 
\[
c(\tau) = \kappa \tau_1 \cdots \tau_r,
\]
for some \( \kappa \in \mathbb{R} \).

Hence for each \( \tau \in a \), \( h_\tau = \kappa \tau_1 \cdots \tau_r \tilde{\nu} \)-a.e. Since \( |h_{\tau+\sigma} - h_\tau| \leq 2^r \|\sigma\| \|\tau\|^{r-1} \)
and hence \( \tau \to h_\tau \) is continuous, using a countable dense subset of \( a \), we
conclude there exists a subset \( \Omega \) of full \( \tilde{\nu} \)-measure such that
\[
h_\tau(x) = \kappa \tau_1 \cdots \tau_r \quad \text{for all } x \in \Omega \text{ and } \tau \in a.
\]
By restricting \( h_\tau \) to each fiber of \( \pi : G/NM \to G/P \), and applying
the Lebesque differentiation theorem, we conclude that \( \frac{1}{\tau_1 \cdots \tau_r} h_\tau(x) \to h(x) \) as
\( \tau \to 0 \) for \( \tilde{\nu} \)-a.e. \( x \). Consequently, \( h = \kappa \tilde{\nu} \)-a.e., finishing the proof.

10. Ergodicity of \( m_{\psi}^{BR} \) and classification

Let \( \Gamma < G \) be an Anosov subgroup. Recall the \( NM \)-invariant BR measure \( m_{\psi}^{BR} \) defined in (3.19). The following is the main theorem of this section:

**Theorem 10.1.** For each \( \psi \in D_1^\ast \), \( m_{\psi}^{BR} \) is \( NM \)-ergodic.

Recall the definition of \( \tilde{\nu}_\psi \) and \( \nu_\psi \) from section 9. Since \( (F, \Gamma, \nu_\psi) \) is
ergodic by Theorem 4.2, the following proposition implies that \( (G/NM, \Gamma, \tilde{\nu}_\psi) \),
and hence \( (\Gamma \setminus G, NM, m_{\psi}^{BR}) \), is ergodic by Proposition 9.2

**Proposition 10.2.** For any \( \psi \in D_1^\ast \), \( E_{\nu_\psi} = a \).

Most of the section is devoted to the proof of Proposition 10.2

**Lemma 10.3.** For any finite subset \( S \subset \lambda(\Gamma) \), the subgroup generated by
\( \lambda(\Gamma) - S \) is dense in \( a \).

**Proof.** Let \( F \) denote the closure of the subgroup generated by \( \lambda(\Gamma) - S \).
Suppose that \( F \neq a \). Identifying \( a = \mathbb{R}^r \), since \( F \) is infinite, there exist \( 1 \leq k < r \) and \( 0 \leq m \leq r \) such that \( F = \sum_{i=1}^k \mathbb{R}v_i + \sum_{i=m+1}^r \mathbb{Z}w_i \) where \( v_i, w_i \) are
linearly independent vectors. For each \( s = \lambda(\gamma) \in S \), \( \lambda(\gamma^s) = n\lambda(\gamma) \to \infty \)
as \( \gamma \) is loxodromic. Hence there exists \( n_s \in \mathbb{N} \) so that \( n_s\lambda(\gamma) \in F \). Setting \( N := \prod_{s \in S} n_s \), we have \( S \subset \sum_{i=1}^k \mathbb{R}v_i + N^{-1} \sum_{i=m+1}^r \mathbb{Z}w_i \).

Therefore, the closure of the subgroup generated by \( F \cup S \) is contained in
\( \sum_{i=1}^k \mathbb{R}v_i + N^{-1} \sum_{i=m+1}^r \mathbb{Z}w_i \). Since \( \lambda(\Gamma) \subset \sum_{i=1}^k \mathbb{R}v_i + N^{-1} \sum_{i=m+1}^r \mathbb{Z}w_i \) and
\( \lambda(\Gamma) \) generates a dense subgroup by a theorem of Benoist [5], it follows that
\( k = \dim a \), yielding a contradiction. \( \square \)

**Proposition 10.4.** For any \( \psi \in D_1^\ast \) and \( C > 0 \), the set \( \{ \lambda(\gamma) \in a^+ : \gamma \in \Gamma, \psi(\lambda(\gamma)) \geq C \} \) generates a dense subgroup of \( a \).

**Proof.** The result [37] Thm. 3.2] extends to Anosov representations (see also
[29] Thm. A.2-(2)), and hence the cocycle \( c = \psi \circ \sigma \) has a finite exponential
growth rate. In particular, \( \psi(\lambda(\gamma)) \leq C \).

\[
\# \{ \lambda(\gamma) : \gamma \in \Gamma, \psi(\lambda(\gamma)) < C \} \leq \# \{ [\gamma] \in \Gamma : \psi(\lambda(\gamma)) < C \} < \infty.
\]
Hence the claim follows from Lemma 10.3 \( \square \)
Lemma 10.6. There exists a compact subset $C \subset G$ such that for any $\xi \in \Lambda$, there exists $g \in C$ such that $g^+ = \xi$ and $g^- \in \Lambda$.

Proof. In the Gromov hyperbolic space $\Gamma$, there exists a finite subset $F \subset \Gamma$ such that for any $x \in \partial \Gamma$, there exists $y \in \partial \Gamma$ such that $[x, y] \cap F \neq \emptyset$. It suffices to choose a compact subset $C \subset G$ such that $C(o)$ contains the $R_1$-neighborhood of $F(o)$ with $R_1$ given in Proposition 5.10. □

We set $N_0 := \max_{p \in C} N_0(\psi, p) < \infty$

with $N_0(\psi, p)$ and $C$ given by Lemmas 6.10 and 10.6 respectively.

In view of Proposition 10.4, Proposition 10.2 is an immediate consequence of the following:

Proposition 10.7. For any $\gamma_0 \in \Gamma$ with $\psi(\lambda(\gamma_0)) \geq 1 + \log 3 N_0$,

$$\lambda(\gamma_0) \in E_\nu.$$

Essential values of $\nu_\psi$. The rest of this section is devoted to the proof of this proposition. We fix $\gamma_0 \in \Gamma$ with

$$\psi(\lambda(\gamma_0)) \geq \log 3 N_0 + 1.$$

Since $\psi > 0$ on $\lambda(\Gamma)$, we have

$$\psi(i \lambda(\gamma_0)) + \psi(\lambda(\gamma_0)) > \log 3 N_0 + 1.$$

Definition of $B_R(\gamma_0, \varepsilon)$. Let $0 < \varepsilon < \|\psi\|^{-1}$ be an arbitrary number. We fix $g \in C$ such that $g^+ = y^+_{\gamma_0}$ and $g^- \in \Lambda$, given by Lemma 10.6. Set $p := go \in C(o)$, $\xi_0 := y^+_{\gamma_0}$ and $\eta := g^-$. For each $\xi \in \Lambda$ and $r > 0$, set

$$B_\xi(p, r) := \{\eta \in \Lambda : d(p, \xi, \eta) < r\}$$

where $d_{\psi, p} = d_p$ is the virtual visual metric defined in section 6.

For each $\gamma \in \Gamma$, define $r_p(\gamma) > 0$ to be the supremum $r \geq 0$ such that

$$\max_{\xi, \eta, \xi_0, \eta_0} \|\beta_{\xi}(p, \gamma \xi_0, \eta) \pm \lambda(\gamma_0)\| < \varepsilon.$$

For each $R > 0$, we define the family of virtual-balls as follows:

$$B_R(\gamma_0, \varepsilon) = \{B_\xi(p, r) : \gamma \in \Gamma, 0 < r < \min(R, r_p(\gamma))\}.$$

Let $C = C(\Gamma, \psi, p) > 0$ be as in Theorem 5.3. Since $\xi_0 \in O_{\varepsilon/(8\kappa)}(\eta, p)$ where $\kappa > 0$ is as in Lemma 5.7, we can choose $0 < s = s(\gamma_0) < R$ small enough such that

$$B_\xi(p, \psi(\lambda(\gamma_0) + i \lambda(\gamma_0)) + \frac{1}{2}\|\psi\|s + 2C s) \subset \phi_{\varepsilon/(8\kappa)}(\eta, p);$$

$$\sup_{x \in B_{\varepsilon/2}(\xi_0, e^{2s} x)} \|\beta_{x}(p, \gamma_0 \pm p) \pm \lambda(\gamma_0)\| < \varepsilon/4.$$
For each \( \gamma \in \Gamma \) and \( r > 0 \), set
\[
D(\gamma \xi_0, r) := B_p(\gamma \xi_0, \frac{1}{3N_0} e^{-\psi(g(\gamma^{-1}p,p) + i a(\gamma^{-1}p,p))} r).
\]

**Lemma 10.12.** Fix \( R > 0 \). If \( \xi \in \Lambda \) and \( \gamma_i \in \Gamma \) is a sequence such that \( \gamma_i^{-1} \rightarrow \eta \) and \( \gamma_i^{-1} \xi \rightarrow \xi_0 \) as \( i \rightarrow \infty \), then for any \( 0 < r \leq s(\gamma_0) \), there exists \( i_0 = i_0(r) > 0 \) such that \( D(\gamma_i \xi_0, r) \in B_R(\gamma_0, \varepsilon) \) and \( \xi \in D(\gamma_i \xi_0, r) \) for all \( i \geq i_0 \).

In particular, for any \( R > 0 \),
\[
\Lambda_M \subset \bigcup_{D \in B_R(\gamma_0, \varepsilon)} D.
\]

**Proof.** Set \( \Gamma_p := \{ \gamma \in \Gamma : \psi(g(\gamma^{-1}p,p) + i a(\gamma^{-1}p,p)) > 0 \} \); note that \( \Gamma - \Gamma_p \) is a finite subset by Lemma 5.4. Hence we may assume that for all \( i, \gamma_i \in \Gamma_p \).

Since \( \gamma_i^{-1} \rightarrow \eta \) as \( i \rightarrow \infty \), we may assume by Lemma 5.6 that for all \( i, \gamma_i \in \Gamma_p \).

\[
O_{\varepsilon/(8\kappa)}(\eta, p) \subset O_{\varepsilon/(4\kappa)}(\gamma_i^{-1}p, p).
\]

To prove that \( D(\gamma_i \xi_0, r) \in B_R(\gamma_0, \varepsilon) \), we need to check that
\[
\max_{\xi' \in B_p(\gamma_i \xi_0, 3N_0s_i)} \| \beta_{\xi'}(p, \gamma_i) \| \leq \lambda(\gamma_0) < \varepsilon,
\]
where \( s_i = \frac{1}{3N_0} e^{-\psi(g(\gamma_i^{-1}p,p) + i a(\gamma_i^{-1}p,p))} \). Let \( \xi' \in B_p(\gamma_i \xi_0, 3N_0s_i) \). We only prove that \( \| \beta_{\xi'}(p, \gamma_i) \| \leq \lambda(\gamma_0) \), as the other case can be treated similarly. First, observe that

\[
d_p(\xi_0, \gamma_i^{-1} \xi') = d_p(\gamma_i \xi_0, \xi') e^{\psi(\beta_{\gamma_0}(\gamma_i^{-1}p,p) + i \beta_{\gamma_i^{-1} \xi'}(\gamma_i^{-1}p,p))}
\]
\[
\leq e^{\psi(\beta_{\gamma_i^{-1}p,p} + i \beta_{\gamma_i^{-1} \xi'}(\gamma_i^{-1}p,p)) + \psi(\beta_{\gamma_0}(\gamma_i^{-1}p,p) + i \beta_{\gamma_i^{-1} \xi'}(\gamma_i^{-1}p,p))} r
\]
\[
\leq e^{2C \cdot r} \text{ by Theorem 5.3}
\]

Since \( r \leq s(\gamma_0) \), this implies that
\[
\| \beta_{\gamma_i^{-1} \xi'}(p, \gamma_0) \| \leq \varepsilon/4.
\]

Hence, by (6.3), we have

\[
d_p(\xi_0, \gamma_0^{-1} \gamma_i^{-1} \xi') = e^{-\psi(\beta_{\gamma_0}(\gamma_0p,p) + i \beta_{\gamma_i^{-1} \xi'}(\gamma_0p,p))} d_p(\xi_0, \gamma_i^{-1} \xi')
\]
\[
\leq e^{\psi(\lambda(\gamma_0) + \lambda(\gamma_0)) + \frac{1}{2} \| \psi \| \| \varepsilon \| + 2C \cdot r}.
\]

Since \( r \leq s(\gamma_0) \), it follows from (10.14), (10.15), and (10.10) that both \( \gamma_i^{-1} \xi' \) and \( \gamma_0^{-1} \gamma_i^{-1} \xi' \) belong to \( O_{\varepsilon/(8\kappa)}(\eta, p) \). Since, \( \gamma_i^{-1} \xi', \gamma_0^{-1} \gamma_i^{-1} \xi' \in O_{\varepsilon/(4\kappa)}(\gamma_i^{-1}p, p) \) by (10.13), it follows from Lemma 5.7 that
\[
\| \beta_{\gamma_i^{-1} \xi'}(\gamma_i^{-1}p, p) - \beta_{\gamma_0^{-1} \gamma_i^{-1} \xi'}(\gamma_i^{-1}p, p) \| < 2\kappa(\varepsilon/4\kappa) = \varepsilon/2.
\]
Now we have
\[
\|\beta_{i}(p, \gamma_{i}(\gamma_{i}^{-1}p)) - \lambda(\gamma_{0})\| \\
\leq \|\beta_{i}(\gamma_{i}p, \gamma_{i}(\gamma_{i}^{-1}p)) - \beta_{i}(p, \gamma_{i}^{-1}p, \gamma_{i}^{-1}p)\| + \|\beta_{i}(\gamma_{i}^{-1}p, \gamma_{i}^{-1}p) - \beta_{i}(\gamma_{i}^{-1}p, \gamma_{i}^{-1}p)\| \\
= \|\beta_{i}(\gamma_{i}^{-1}p, \gamma_{i}^{-1}p) - \lambda(\gamma_{0})\| + \|\beta_{i}(\gamma_{i}^{-1}p, \gamma_{i}^{-1}p) - \beta_{i}(\gamma_{i}^{-1}p, \gamma_{i}^{-1}p)\| \\
\leq \varepsilon/4 + \varepsilon/2 < \varepsilon,
\]
which verifies that \(D(\gamma_{i}\xi_{0}, r)\) belongs to the family \(B_{\mathcal{R}}(\gamma_{0}, \varepsilon)\).

We now check that \(\xi \in D(\gamma_{i}\xi_{0}, r)\). Since \(\gamma_{i}^{-1}\xi \to \xi_{0}\), we may assume that for all \(i\),
\[(10.16) \quad d_{p}(\xi_{0}, \gamma_{i}^{-1}\xi) < \frac{1}{3N_{0}}e^{-\|\xi\|\varepsilon}r.\]
Since \(r \leq s(\gamma_{0})\), (10.10), (10.13), and (10.16) imply that \(\gamma_{i}^{-1}\xi \in \mathcal{O}_{\varepsilon/(4\kappa)}(\gamma_{i}^{-1}p, p)\).

Since \(\xi_{0} \in \mathcal{O}_{\varepsilon/(4\kappa)}(\gamma_{i}^{-1}p, p)\) as well, we have
\[
\|\beta_{i}(\gamma_{i}^{-1}p, p) - a(\gamma_{i}^{-1}p, p)\| \leq \varepsilon/4 \quad \text{and} \quad \|\beta_{\xi_{0}}(\gamma_{i}^{-1}p, p) - a(\gamma_{i}^{-1}p, p)\| \leq \varepsilon/4,
\]
by Lemma 5.7. Note that
\[
d_{p}(\gamma_{i}\xi_{0}, \xi) = d_{\gamma_{i}^{-1}p}(\xi_{0}, \gamma_{i}^{-1}\xi) \\
= e^{-\psi(\beta_{\xi_{0}}(\gamma_{i}^{-1}p, p) + i\beta_{\gamma_{i}}^{-1}(\gamma_{i}^{-1}p, p))}d_{p}(\xi_{0}, \gamma_{i}^{-1}\xi) \\
\leq e^{-\psi(a(\gamma_{i}^{-1}p, p) + i\gamma_{i}^{-1}p, p)) + \frac{1}{2}\|\psi\|_{1}}e^{-\psi(\gamma_{i}^{-1}p, p))}r \\
\leq \frac{1}{3N_{0}}e^{-\psi(\gamma_{i}^{-1}p, p))}r \quad \text{by (10.16)}.
\]
This proves that \(\xi \in D(\gamma_{i}\xi_{0}, r)\). \(\square\)

Consider the following measure \(\nu_{p} = \nu_{\psi, p} \) on \(\Lambda\):
\[
d\nu_{p}(\xi) = e^{\psi(\beta_{\psi}(\nu_{p}))}d\nu_{\psi}(\xi).
\]

**Proposition 10.17.** Let \(B \subset F\) be a Borel subset with \(\nu_{p}(B) > 0\). Then for \(\nu_{p}\)-a.e. \(\xi \in B\),
\[
\lim_{R \to 0} \sup_{\xi \in D, D \in B_{\mathcal{R}}(\gamma_{0}, \varepsilon)} \frac{\nu_{p}(B \cap D)}{\nu_{p}(D)} = 1.
\]

**Proof.** For a given Borel function \(h : F \to \mathbb{R}\), we define \(h^{*} : F \to \mathbb{R}\) by
\[
h^{*}(\xi) = \lim_{R \to 0} \sup_{\xi \in D, D \in B_{\mathcal{R}}(\gamma_{0}, \varepsilon)} \frac{1}{\nu_{p}(D)} \int_{D} h d\nu_{p}.
\]
By Lemma 10.12, \(h^{*}\) is well defined on \(\Lambda_{M}\). Since \(\Lambda_{M}\) has a full \(\nu_{p}\) measure by Theorem 8.9, \(h^{*}\) is defined \(\nu_{p}\)-a.e. on \(F\). We will prove that \(h = h^{*}\), \(\nu_{p}\)-a.e.; by taking \(h = 1_{B}\), the conclusion of the lemma will follow. Note that \(h = h^{*}\) when \(h\) is continuous. To deal with the general case, we proceed as follows.
Step 1: For all $\alpha > 0$,

$$\nu_p(\{h^* > \alpha\}) \leq \frac{e^{\psi(\lambda(\gamma_0)) + \|\psi\|\varepsilon}}{\alpha} \int_{\mathcal{F}} |h| \, d\nu_p.$$ 

Letting $K$ be an arbitrary compact subset of $\{\xi : h^*(\xi) > \alpha\}$, it suffices to show that

$$\nu_p(K) \leq \frac{e^{\psi(\lambda(\gamma_0)) + \|\psi\|\varepsilon}}{\alpha} \int_{\mathcal{F}} |h| \, d\nu_p.$$ 

Fix $R > 0$. By definition, for each $x \in K$, there exists $D_x \in \mathcal{B}_R(\gamma_0, \varepsilon)$ containing $x$ such that

$$\frac{1}{\nu_p(D_x)} \int_{D_x} h \, d\nu_p > \alpha.$$ 

Since $K$ is compact, there exists a finite subcover of $\{D_x : x \in K\}$, say $D_i = \mathbb{B}_p(\gamma_i \xi_0, s_i) (i = 1, \ldots, n)$ where $\gamma_i \in \Gamma$ and $s_i = \frac{1}{3N_0} e^{-\psi(\gamma_i^{-1} p, p)} r_i$ for some $0 < r_i < R$. We will rearrange the indices so that $s_1 \geq \cdots \geq s_n$ and define inductively

$$i_1 = 1, \ i_{j+1} = \min\{i > i_j : D_i \cap (D_{i_1} \cup \cdots \cup D_{i_j}) = \emptyset\},$$

as long as possible, to obtain $\mathcal{B}^* := \{D_{i_1}, \cdots, D_{i_\ell}\}$. For brevity, we will write $3N_0 D_i := \mathbb{B}_p(\gamma_i \xi_0, 3N_0 s_i)$. For each $1 \leq k \leq n$, there exists $m$ so that $i_m < k \leq i_{m+1}$. By construction, $D_k$ meets one of $D_{i_1}, \cdots, D_{i_m}$, say $D_{i_j}$. By Lemma 6.10, $D_k \subset 3N_0 D_{i_j}$. Therefore

$$\bigcup_{k=1}^n D_k \subset \bigcup_{j=1}^\ell 3N_0 D_{i_j}.$$ 

Now we claim that $3N_0 D_{i_j} \subset \gamma_{ij} \gamma_0^{-1} \gamma_{ij}^{-1} D_{i_j}$: note that for $\xi \in 3N_0 D_{i_j}$,

$$d_p(\gamma_{ij} \xi_0, \gamma_{ij} \gamma_0 \gamma_{ij}^{-1} \xi) = d_{\gamma_{ij} \gamma_0^{-1} \gamma_{ij}^{-1} p}(\gamma_{ij} \xi_0, \xi)$$

$$= e^{-\psi(\beta_{ij} \xi_0, \gamma_{ij} \gamma_0^{-1} \gamma_{ij}^{-1} p, p) + i \beta_{ij} (\gamma_{ij} \gamma_0^{-1} \gamma_{ij}^{-1} p, p)} d_p(\gamma_{ij} \xi_0, \xi)$$

$$\leq 3N_0 e^{-\psi(\lambda(\gamma_0) + i \lambda(\gamma_0)) + \|\psi\| \varepsilon} s_{i_j} < s_{i_j},$$

by [6.3], [5.2], [10.9] and that $\|\psi\| \varepsilon < 1$.

Hence

$$\nu_p(3N_0 D_{i_j}) \leq \nu_p(\gamma_{ij} \gamma_0^{-1} \gamma_{ij}^{-1} D_{i_j})$$

$$= \int_{D_{i_j}} e^{\psi(\beta_{ij}(\varepsilon, \gamma_{ij} \gamma_0 \gamma_{ij}^{-1}))} d\nu_p(\xi)$$

$$\leq e^{\psi(\lambda(\gamma_0)) + \|\psi\| \varepsilon} \nu_p(D_{i_j}),$$
where the last inequality follows from \([10.9]\). Therefore,

\[
\nu_p(K) \leq \sum_{j=1}^\ell \nu_p(3N_0D_{ij}) \leq \sum_{j=1}^\ell e^{\psi(\lambda(\gamma_0)) + \|\psi\|\varepsilon} \nu_p(D_{ij})
\]

\[
\leq \frac{e^{\psi(\lambda(\gamma_0)) + \|\psi\|\varepsilon}}{\alpha} \sum_{j=1}^\ell \int_{D_{ij}} h \, d\nu_p \leq \frac{e^{\psi(\lambda(\gamma_0)) + \|\psi\|\varepsilon}}{\alpha} \int_F |h| \, d\nu_p,
\]

which was to be proved.

**Step 2:** \(h(\xi) = h^*(\xi)\) for \(\nu_p\text{-a.e} \xi\).

We first prove that \(h(\xi) \leq h^*(\xi)\) for \(\nu_p\text{-a.e} \xi\). Let \(\alpha > 0\) be arbitrary. It suffices to show that \(\nu_p(\{\xi : h(\xi) - h^*(\xi) > \alpha\}) = 0\). Let \(h_n\) be a continuous function converging to \(h\) in \(L^1(\nu_p)\). Note that \(h^*_n = h_n\) and

\[
\nu_p(\{\xi : h(\xi) - h^*(\xi) > \alpha\}) \\
\leq \nu_p(\{\xi : h(\xi) - h_n(\xi) > \alpha/2\}) + \nu_p(\{\xi : h^*_n(\xi) - h^*(\xi) > \alpha/2\}) \\
\leq \frac{2}{\alpha} \|h - h_n\|_1 + \frac{2}{\alpha} e^{\psi(\lambda(\gamma_0)) + \|\psi\|\varepsilon} \|h - h_n\|_1.
\]

Taking \(n \to \infty\), we get \(\nu_p(\{\xi : h(\xi) - h^*(\xi) > \alpha\}) = 0\), i.e., \(h \leq h^*, \nu_p\text{-a.e}\). Similar argument shows that \(\nu_p(\{\xi : h^*(\xi) - h(\xi) > \alpha\}) = 0\), which completes the proof. \(\square\)

**Proof of Proposition 10.7.** It is easy to check that \(E_\nu = E_{\nu_p}\). Hence it suffices to show \(\lambda(\gamma_0) \in E_{\nu_p}\). Let \(B \subset F\) be a Borel subset with \(\nu_p(B) > 0\) and \(\varepsilon > 0\). By Proposition \([10.17]\) there exists \(\gamma \in \Gamma\) and \(D = B_p(\gamma_0, r) \in \mathcal{B}(\gamma_0, \varepsilon)\) such that

\[
\nu_p(D \cap B) > (1 + e^{-\psi(\lambda(\gamma_0)) - \|\psi\|\varepsilon})^{-1} \nu_p(D).
\]

Fix \(R > r\), so that \(D \in \mathcal{B}_R(\gamma_0, \varepsilon)\).

Since \(r < r_p(\gamma)\), we have

\[
D \subset \{\xi : \|\beta_\gamma(p, \gamma \gamma_0^{-1} p) - \lambda(\gamma_0)\| \leq \varepsilon\} \\
\subset \{\xi : |\psi(\beta_\gamma(p, \gamma \gamma_0^{-1} p)) - \psi(\lambda(\gamma_0))| \leq \|\psi\|\varepsilon\}.
\]

We note that \(\gamma \gamma_0^{-1} D \subset D\): if \(\xi \in D\), by \([6.3]\),

\[
d_p(\gamma \xi_0, \gamma \gamma_0^{-1} \xi) = d_{\gamma \gamma_0^{-1} \gamma^{-1} p}(\gamma \xi_0, \xi) \\
= e^{\psi(\beta_\gamma(p, \gamma \gamma_0^{-1} p) - i\beta_\gamma(p, \gamma \gamma_0^{-1} p))} d_p(\gamma \xi_0, \xi) \\
\leq e^{-\psi(\lambda(\gamma_0) - i\lambda(\gamma_0)) + \|\psi\|\varepsilon r} < r.
\]

Since

\[
B \cap \gamma \gamma_0^{-1} B \cap \{\xi : \|\beta_\gamma(p, \gamma \gamma_0^{-1} p) - \lambda(\gamma_0)\| < \varepsilon\} \supset (D \cap B) \cap \gamma \gamma_0^{-1} (D \cap B),
\]

\[
\int_F |h - h^*| \, d\nu_p = \int_F |h^* - h^*_n| \, d\nu_p + \int_F |h - h_n| \, d\nu_p \\
\leq \frac{\|h - h^*\|_1}{\alpha} + \frac{\|h - h_n\|_1}{\alpha} \\
\leq \frac{\|h - h_n\|_1}{\alpha} + \frac{\|h^* - h_n\|_1}{\alpha}
\]

Therefore,

\[
\int_F |h - h^*| \, d\nu_p \leq \frac{\|h - h_n\|_1}{\alpha} + \frac{\|h^* - h_n\|_1}{\alpha} \\
\leq \frac{\|h - h_n\|_1}{\alpha} + \frac{\|h^* - h_n\|_1}{\alpha}
\]
it suffices to prove that \((D \cap B) \cap \gamma_0^{-1}(D \cap B)\) has a positive \(\nu_p\)-measure. Note that
\[
\nu_p(\gamma_0^{-1}(D \cap B)) = \int_{D \cap B} e^{\psi(\beta_\xi(p,\gamma_0^{-1} \gamma^{-1}p))} d\nu_p(\xi) \\
\geq e^{-\psi(\lambda(\gamma_0)) - \|\psi\| \varepsilon} \nu_p(D \cap B)
\]
and hence by (10.18),
\[
\nu_p(D \cap B) + \nu_p(\gamma_0^{-1}(D \cap B)) > (1 + e^{-\psi(\lambda(\gamma_0)) - \|\psi\| \varepsilon}) \nu_p(D) > \nu_p(D).
\]
Since both \(D \cap B\) and \(\gamma_0^{-1}(D \cap B)\) are contained in \(D\), this implies their intersection has a positive \(\nu_p\)-measure.

**Patterson Sullivan measures are mutually singular.**

**Theorem 10.19.** Let \(\Gamma < G\) be an Anosov subgroup. Then \(\{\nu_\psi : \psi \in D_\Gamma^*\}\) are pairwise mutually singular.

**Proof.** Since \(\Gamma < G\) is Anosov, the family \(\{\nu_\psi : \psi \in D_\Gamma^*\}\) consists of ergodic measures. Hence any \(\nu_1\) and \(\nu_2\) in the family are either mutually singular or absolutely continuous with respect to each other. Now the claim follows from Lemma [10.20] below and Proposition 10.2. \(\square\)

**Lemma 10.20.** For \(i = 1, 2\), let \(\psi_i \in a^*\) be linear forms, \(\nu_i\) be \((\Gamma, \psi_i)\)-PS measures, and assume that \(E_{\nu_2} = a\). Then \(\nu_1 \ll \nu_2\) if and only if \(\psi_1 = \psi_2\).

**Proof.** Suppose that \(\nu_1 \ll \nu_2\) and that \(\psi_1 \neq \psi_2\). Consider the Radon-Nikodym derivative \(f := \frac{d\nu_1}{d\nu_2} \in L^1(\Lambda, \nu_2)\). Note that there exists a \(\nu_2\)-conull set \(E \subset \Lambda\) such that for all \(\xi \in E\) and for all \(\gamma \in \Gamma\), we have
\[
(10.21) \quad f(\gamma^{-1}\xi) = e^{(\psi_1(\psi_2)(\beta_\xi(\epsilon, \gamma)))} f(\xi).
\]
If \(f\) were continuous, then \(f \neq 0\) and by applying \(\xi = y_\gamma^+\) in the above, we get \(\psi_1(\lambda(\gamma)) = \psi_2(\lambda(\gamma))\) for all \(\gamma \in \Gamma\). Since \(\lambda(\gamma)\) generates a dense subgroup of \(a\), it follows that \(\psi_1 = \psi_2\).

In general, we use the hypothesis \(E_{\nu_2} = a\). Choose \(0 < r_1 < r_2\) such that
\[
B := \{\xi \in \Lambda : r_1 < f(\xi) < r_2\}
\]
has a positive \(\nu_2\)-measure. Since \(\psi_1 \neq \psi_2\), we can choose \(w \in a\) such that
\[
(10.22) \quad e^{(\psi_1(\psi_2)(w))} > \frac{2r_2}{r_1}.
\]
Choose \(\varepsilon > 0\) such that \(e^{||\psi_1 - \psi_2||/\varepsilon} < 2\). Since \(\nu_2(B) > 0\) and \(E_{\nu_2} = a\), there exists \(\gamma \in \Gamma\) such that
\[
B' := B \cap \gamma B \cap \{\xi \in \Lambda : \|\beta_\xi(\epsilon, \gamma) - w\| < \varepsilon\}
\]
has a positive \(\nu_2\)-measure. Now note that
\[
\int_{B'} f(\gamma^{-1}\xi) d\nu_2(\xi) > e^{(\psi_1(\psi_2)(w)) - ||\psi_1 - \psi_2||/\varepsilon} \int_{B'} f(\xi) d\nu_2(\xi)
\]
\[
> \frac{r_2}{r_1} \int_{B'} f(\xi) d\nu_2(\xi)
\]

by \(10.21\), \(10.22\), and the choice of \(\varepsilon\). In particular,

\[
\nu_2 \left\{ \xi \in B' : f(\gamma^{-1}\xi) > \frac{r_2}{r_1}f(\xi) \right\} > 0.
\]

It follows that there exists \(\xi \in B' \cap E\) such that

\[
f(\gamma^{-1}\xi) > \frac{r_2}{r_1}f(\xi).
\]

Note that for \(\xi \in B'\), both \(\xi\) and \(\gamma^{-1}\xi \in B\). Hence, by definition of \(B\), for all \(\xi \in B'\), we have

\[
f(\gamma^{-1}\xi) < \frac{r_2}{r_1}f(\xi).
\]

This is a contradiction. \(\square\)

**P-semi-invariant measures.** In this section, we establish that \(P\)-semi-invariant Radon measures supported in \(E = \{x \in \Gamma \setminus G : x^+ \in \Lambda\}\), up to constant multiples, are parametrized by \(D_{\Gamma}^\ast\).

If \(\mu\) is \(P\)-semi-invariant, then there exists a linear form \(\chi_\mu \in a^\ast\) such that for all \(a \in A\),

\[
a_s \mu = e^{-\chi_\mu(\log a)} \mu.
\]

We set \(\psi_\mu = \chi_\mu + 2\rho \in a^\ast\). The first part of the following proposition is known in the rank one case (see e.g. \([2]\), \([8]\), and \([22]\)) and the proof can be easily adapted to the higher rank case.

**Proposition 10.23.** Any \(P\)-semi-invariant Radon measure \(\mu\) on \(\Gamma \setminus G\) is proportional to \(m_{\nu_\psi} m_{\nu_0}\), where \(\nu_\psi\) is a \((\Gamma, \psi_\mu)\)-conformal measure and \(\psi_\mu \in D_{\Gamma}^\ast\). Moreover, if \(\mu\) is supported on \(E\), then \(\psi_\mu \in D_{\Gamma}^\ast\) and \(\mu\) is proportional to \(m_{\nu_\psi}^{BR}\).

**Proof.** For simplicity, set \(\chi = \chi_\mu\) and \(\psi = \psi_\mu\). Let \(\tilde{\mu}\) be the \(\Gamma\)-invariant lift of \(\mu\) to \(G\) and \(\pi : G \to G/P\) be the projection. Choose a section \(c : G/P \to K\) so that \(\pi \circ c = \text{id}\) and consider the measurable isomorphism

\[
G/P \times M \times A \times N \rightarrow G
\]

\[
(\xi, m, a, n) \rightarrow c(\xi)man.
\]

Let \(dm, dn, da\) be the Haar measures on \(M, N,\) and \(A\). As \(\tilde{\mu}\) is \(P\)-semi-invariant Radon measure, there exists \(\tilde{\chi} \in a^\ast\) and a Radon measure \(\nu\) on \(G/P\) such that

\[
d\tilde{\mu}(c(\xi)man) = e^{\tilde{\chi}(\log a)} dn da dm d\nu(\xi).
\]

Because \(d\tilde{\mu}(\cdot a) = e^{\chi(\log a)} d\tilde{\mu}(\cdot)\), we have

\[
\chi = \tilde{\chi} - 2\rho, \text{ or equivalently, } \tilde{\chi} = \psi.
\]

Note that \(G\) is measurably isomorphic to the product \(G/P \times P\) and the left \(\Gamma\)-action with respect to these coordinates is given by \(\gamma \cdot (\xi, p) = (\gamma \cdot \xi, \Phi(\gamma, \xi)p)\) for some \(P\)-valued cocycle \(\Phi : \Gamma \times G/P \to P\) where \(\gamma \in \Gamma\) and \((\xi, p) \in G/P \times P\). One can check that

\[
\Phi(\gamma, \xi) = m(\gamma, \xi) \exp(\beta_\xi(\gamma^{-1}, e)) n(\gamma, \xi)
\]
for some \( m(\gamma, \xi) \in M \) and \( n(\gamma, \xi) \in N \). Hence, for \( p = \text{man} \), the \( \text{MAN} \)-coordinates for \( \Phi(\gamma, \xi)p \) are given by

\[
\Phi(\gamma, \xi)p = (m(\gamma, \xi)m)(\exp(\beta_\xi(\gamma^{-1}, e))a)((ma)^{-1}n(\gamma, \xi)\text{man}).
\]

Since \( \tilde{\mu} \) is left-\( \Gamma \)-invariant, we have for any \( f \in C_c(G) \) and any \( \gamma \in \Gamma \),

\[
\int_G f(g) \, d\tilde{\mu}(g) = \int_G f(g) \, d(\gamma_\ast \tilde{\mu})(g)
\]

\[
= \int_{G/P} \int_P f((\gamma\xi, \Phi(\gamma, \xi)p)e^{\psi(\log a)} \, dn \, da \, dm \, d\nu(\xi)
\]

\[
= \int_{G/P} \int_P f(\xi, p)e^{\psi(\log a - \beta_{\gamma^{-1}}(\gamma^{-1}, e))} \, dn \, da \, dm \, d(\gamma_\ast \nu)(\xi),
\]

where in the last equality, we have used (10.24) and the change of variables \( a' = a \exp(\beta_\xi(e, \gamma^{-1})) \). On the other hand, we have

\[
\int_G f(g) \, d\tilde{\mu}(g) = \int_{G/P} \int_P f(\xi, p)e^{\psi(\log a)} \, dn \, da \, dm \, d\nu(\xi).
\]

By comparing these two identities, we get that for any \( \gamma \in \Gamma \),

\[
d(\gamma_\ast \nu)(\xi) = e^{\psi(\beta_\xi(e, \gamma))} \, d\nu(\xi),
\]

that is, \( \nu \) is a \( (\Gamma, \psi) \)-conformal measure. By [30, Thm. 8.1], \( \psi \in D_\Gamma \).

Finally, recall that for all \( g \in G \) and \( \phi \in C_c(G) \),

\[
\int_N \phi(gn) \, dn = \int_{G/P} \phi(gn)e^{2\rho(\beta_{gn^-}(e, gn^-))} \, dm \, d\nu(\xi).
\]

For \( g = c(\xi)\text{man} \in \text{KAN} \), we have \( \beta_{g^+}(e, g) = \log a \) and \( g^+ = \xi \). Hence, for any \( f \in C_c(G) \),

\[
\int_G f(g) \, d\tilde{\mu}(g) = \int_{G/P} \int_P f(c(\xi)\text{man})e^{\psi(\log a)} \, dn \, da \, dm \, d\nu(\xi)
\]

\[
= \int_{G/M} \int_M f(g)e^{2\rho(\beta_{g^-}(e, g))}e^{\psi(\beta_{g^+}(e, g))} \, dm \, d\nu(\xi)
\]

\[
= |\nu| \, m_{\nu,m_0}(f).
\]

Therefore, \( \hat{\mu} = |\nu| \, m_{\nu,m_0}(f) \).

Now, if \( \mu \) is supported on \( \mathcal{E} \), then \( \nu \) is supported on \( \Lambda \). Hence \( \nu \) is a \( (\Gamma, \psi) \)-PS measure and \( \psi \in D_\Gamma^* \) by Theorem 7.6. It remains to note that \( m_{\nu,m_0}(f) = m^{\text{BR}}(f) \).

Let \( \mathcal{P}_\Gamma \) be the space of all \( P \)-semi-invariant Radon measures on \( \mathcal{E} \) up to proportionality. Let \( \mathcal{Q}_\Gamma \) be the space of all \( \text{NM} \)-invariant, ergodic and \( A \)-quasi-invariant Radon measures supported on \( \mathcal{E} \) up to proportionality.

**Theorem 10.25.** Let \( \Gamma < G \) be an Anosov subgroup. We have \( \mathcal{P}_\Gamma = \mathcal{Q}_\Gamma \) and the map \( D_\Gamma^* \to \mathcal{Q}_\Gamma \) given by \( \psi \mapsto [m^{\text{BR}}] \) gives a homeomorphism between \( D_\Gamma^* \) and \( \mathcal{Q}_\Gamma \).
Proof. It is easy to see that $\mathcal{Q}_\Gamma \subset \mathcal{P}_\Gamma$. The other direction $\mathcal{P}_\Gamma \subset \mathcal{Q}_\Gamma$ follows from Proposition 10.23 and Theorem 10.1.

Let $\tilde{\mathcal{Q}}_\Gamma$ be the space of all $NM$-ergodic $A$-quasi-invariant Radon measures supported on $\{x \in \Gamma G : x^+ \in \Lambda\}$, so that $\mathcal{Q}_\Gamma = \tilde{\mathcal{Q}}_\Gamma/\sim$. Set $\iota(\psi) = m_{\psi}^{BR}$ for $\psi \in D^*_\Gamma$. Since $m_{\psi}^{BR}$ is $NM$-ergodic by Theorem 10.1, the map $\iota$ is well defined and injective by Lemma 10.20. By Proposition 10.23, $\iota(D^*_\Gamma)$ contains precisely one representative of each class in $\mathcal{Q}_\Gamma$. Hence it suffices to show that the map $\iota$ gives a homeomorphism between $D^*_\Gamma$ and its image $\iota(D^*_\Gamma)$.

If $\psi_i \to \psi$ in $D^*_\Gamma$, then any weak-limit of $\nu_{\psi_i}$ is a $(\Gamma, \psi)$-PS measure. By the uniqueness of $(\Gamma, \psi)$-conformal measure, $\nu_{\psi_i}$ converges to $\nu_\psi$ as $i \to \infty$. Hence $\iota$ is continuous. Now, suppose that $m_{\psi_i}^{BR} \to m_{\psi}^{BR}$ for some sequence $\psi_i, \psi \in D^*_\Gamma$. Then the $A$-semi-invariance of the BR-measures given by (3.16) and the convergence $a_*m_{\psi_i}^{BR} \to a_*m_{\psi}^{BR}$ implies that $e^{(2\rho - \psi_i)(\log a)} \to e^{(2\rho - \psi)(\log a)}$ for all $a \in A$. Hence $\psi_i \to \psi$, completing the proof. □

References


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