INVARIANT MEASURES FOR HOROSPHERICAL ACTIONS AND ANOSOV GROUPS.

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Abstract. Let Γ be an Anosov subgroup (with respect to a minimal parabolic subgroup) of a connected semisimple real linear Lie group G. For a maximal horospherical subgroup N of G, we show that the space of all non-trivial NM-invariant ergodic and A-quasi-invariant Radon measures on $\Gamma\backslash G$, up to proportionality, is homeomorphic to $\mathbb{R}^{\operatorname{rank} G-1}$, where A is a maximal real split torus and M is a maximal compact subgroup which normalizes N. One of the main ingredients is to establish the NM-ergodicity of all Burger-Roblin measures.

Contents

1. Introduction

Let G be a connected real semisimple linear Lie group and $\Gamma < G$ a Zariski dense discrete subgroup. A subgroup N of G is called *horospherical* if there exists a diagonalizable element $a \in G$ such that

$$N = \{g \in G : a^k g a^{-k} \to \infty \quad \text{as } k \to +\infty\},$$

or equivalently, N is the unipotent radical of a parabolic subgroup of G. We assume that N is a maximal horospherical subgroup, which exists uniquely up to conjugation. We are interested in the measure rigidity property of the N-action on the homogeneous space $\Gamma\backslash G$. When Γ is a lattice, i,e.,

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when $\Gamma\backslash G$ has finite volume, the well-known measure rigidity theorems of Furstenberg [15], Veech [51] and Dani [13] give a complete classification of Radon measures (=locally finite Borel measures) invariant by N. This rigidity phenomenon extends to any unipotent subgroup action by the celebrated theorem of Ratner in [39].

When G has rank one and Γ is geometrically finite, the horospherical subgroup action on $\Gamma\backslash G$ is known to be essentially uniquely ergodic; there exists a unique non-trivial invariant ergodic Radon measure on $\Gamma\backslash G$, called the Burger-Roblin measure ([9], [40], [53]). When Γ is geometrically infinite, there may be a continuous family of horospherically invariant ergodic measures as first discovered by Babillot and Ledrappier ([2], [3]). For a certain class of geometrically infinite groups, a complete classification of horospherically invariant ergodic measures has been obtained; see [44], [45], [29], [32], [26], [27], etc. We refer to a recent article by Landesberg and Lindenstrauss [26] for a more precise description on the rank one case.

When G has rank at least 2 and Γ has infinite co-volume in G, very little is known about invariant measures. The works of Burger [10] and Quint [35] on a higher rank version of the Patterson-Sullivan theory supply a continuous family of maximal horospherically invariant Burger-Roblin measures, as was introduced in [14].

In this paper, we focus on a special class of discrete subgroups, called *Anosov* subgroups. In the rank one case, this class coincides with the class of convex cocompact subgroups, and hence the class of Anosov subgroups can be considered as a generalization of convex cocompact subgroups of rank one Lie groups to higher rank.

When $\Gamma < G$ is Anosov, we show that all of these Burger-Roblin measures are ergodic for maximal horospherical foliations, and classify all ergodic *non-trivial* Radon measures for maximal horospherical foliations, which are also quasi-invariant under Weyl chamber flow. In particular, we establish a homeomorphism between the space of these measures and the interior of the projective limit cone of Γ, which is again homeomorphic to $\mathbb{R}^{\operatorname{rank} G-1}$.

In order to formulate our main result precisely, we begin with the definition of an Anosov subgroup of G. Let P be the normalizer of N, i.e., a minimal parabolic subgroup of G and $\mathcal{F} := G/P$ the Furstenberg boundary. We denote by $\mathcal{F}^{(2)}$ the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$. A Zariski dense discrete subgroup $\Gamma < G$ is called an *Anosov subgroup* (with respect to P) if it is a finitely generated word hyperbolic group which admits a Γ-equivariant embedding ζ of the Gromov boundary $\partial\Gamma$ into $\mathcal{F}$ such that $(\zeta(x), \zeta(y)) \in \mathcal{F}^{(2)}$ for all $x \neq y$ in $\partial\Gamma$.

First introduced by Labourie [25] as the images of Hitchin representations of surface groups ([21], [16]), this definition is due to Guichard and Wienhard [20], who showed that Anosov subgroups (more precisely, Anosov representations) form an open subset in the representation variety $\operatorname{Hom}(\Gamma, G)$. The class of Anosov groups include Schottky subgroups [36] and hence any Zariski dense discrete subgroup of G contains an Anosov subgroup ([5],

[37]). We also refer to the work of Kapovich, Leeb and Porti [23] for other equivalent characterizations of Anosov groups, as well as to excellent survey articles by Kassel [24] and Wienhard [52] on higher Teichmüller theory. Although there is a notion of Anosov subgroups with respect to a general parabolic subgroup, we refer to Remark 1.9 for the reason why we only consider Anosov subgroups with respect to minimal parabolic subgroups in this paper.

We let $P = NMA$ be the Langlands decomposition of P, so that N is the unipotent radical of P, A is a maximal real split torus of G, and M is a compact subgroup which commutes with A. Note that any maximal horospherical subgroup arises in this way, i.e., as the unipotent radical of a minimal parabolic subgroup.

The limit set Λ of Γ is the unique minimal Γ-invariant closed subset of $\mathcal{F}$. Hence the following set

$$\mathcal{E} := \{[g] \in \Gamma\backslash G : gP \in \Lambda\}$$

is the unique minimal P-invariant closed subset of $\Gamma\backslash G$. We call a P-quasi-invariant measure on $\Gamma\backslash G$ *non-trivial* if it is supported on $\mathcal{E}$.

Theorem 1.1. *For any Anosov subgroup $\Gamma < G$, the space $\mathcal{Q}_\Gamma$ of all non-trivial NM-invariant ergodic and A-quasi-invariant Radon measures on $\Gamma\backslash G$, up to constant multiples, is homeomorphic to $\mathbb{R}^{rank\, G-1}$.*

In order to describe the explicit homeomorphism, we need to define Burger-Roblin measures on $\mathcal{E}$. Denote by $\mathfrak{a}$ the Lie algebra of A and fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ so that $\log N$ is the sum of positive root subspaces. Fix a maximal compact subgroup K of G so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Let $\mu : G \to \mathfrak{a}^+$ denote the Cartan projection map (Def. 2.2). We denote by $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ the limit cone of Γ, which is the smallest closed cone containing $\mu(\Gamma)$ (Def. 2.16). Let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of Γ (Def. 2.17).

For Anosov subgroups, the following two spaces are homeomorphic to each other:

$$D_\Gamma^\star := \{\psi \in \mathfrak{a}^* : \psi \ge \psi_\Gamma, \psi(v) = \psi_\Gamma(v) \text{ for some } v \in \operatorname{int} \mathcal{L}_\Gamma\} \simeq \operatorname{int}(\mathbb{P}\mathcal{L}_\Gamma)$$

where $\operatorname{int}(\mathbb{P}\mathcal{L}_\Gamma)$ denotes the interior of the projective limit cone $\mathbb{P}\mathcal{L}_\Gamma$ (Proposition 4.4). Since $\operatorname{int}(\mathcal{L}_\Gamma)$ is a non-empty open convex cone of $\mathfrak{a}^+$ [5, Thm. 1.2], it follows that $D_\Gamma^\star$ is homeomorphic to $\mathbb{R}^{\operatorname{rank} G-1}$.

For a linear form $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on the limit set Λ is called a (Γ, ψ)-Patterson Sullivan measure if for all $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(o,\gamma o))} \tag{1.2}$$

where $o = [K] \in G/K$ and $\beta : \mathcal{F} \times G/K \times G/K \to \mathfrak{a}$ denotes the $\mathfrak{a}$-valued Busemann function (Def. 3.3). Quint constructed a (Γ, ψ)-Patterson-Sullivan measure for each $\psi \in D_\Gamma^\star$ [35]; for Γ Anosov, this measure exists uniquely (hence Γ-ergodic), which we denote by ν_ψ.

In the rest of the introduction, we let $\Gamma < G$ be an Anosov subgroup. By a Patterson-Sullivan measure on Λ, we mean a (Γ, ψ)-Patterson-Sullivan measure on Λ for some $\psi \in \mathfrak{a}^*$. We show:

Theorem 1.3. *The map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D_\Gamma^\star$ and the space of all Patterson-Sullivan measures on Λ. Moreover, Patterson-Sullivan measures are pairwise mutually singular.*

We also denote by ν_ψ the M-invariant lift of ν_ψ on $\mathcal{F} \simeq K/M$ to K by abuse of notation. The Burger-Roblin measure m_ψ^{BR} on $\Gamma\backslash G$ is induced from the following Γ-invariant measure $\tilde{m}_\psi^{\mathrm{BR}}$ on G: for $g = k(\exp b)n \in KAN$,

$$d\tilde{m}_\psi^{\mathrm{BR}}(g) = e^{\psi(b)} dn\, db\, d\nu_\psi(k) \tag{1.4}$$

where dn and db are Lebesgue measures on N and $\mathfrak{a}$ respectively.

The following is a more elaborate version of Theorem 1.1:

Theorem 1.5 (Classification)**.** *The map $\psi \mapsto [m_\psi^{\mathrm{BR}}]$ defines a homeomorphism between $D_\Gamma^\star$ and $\mathcal{Q}_\Gamma$.*

While the P-ergodicity of m_ψ^{BR} follows from the Γ-ergodicity of ν_ψ, the well-definedness of the above map is the most significant part of Theorem 1.5:

Theorem 1.6 (Ergodicity)**.** *For each $\psi \in D_\Gamma^\star$, m_ψ^{BR} is NM-ergodic.*

Since $\mathcal{E}$ is a second countable topological space, Theorem 1.6 implies:

Corollary 1.7. *For m_ψ^{BR} almost all $x \in \mathcal{E}$,*

$$\overline{xNM} = \mathcal{E}.$$

A Radon measure m on $\Gamma\backslash G$ is called P-semi-invariant if there exists a character $\chi : P \to \mathbb{R}^*$ such that $p_* m = \chi(p) m$ for all $p \in P$. Note that any P-semi-invariant Radon measure is necessarily NM-invariant. We show that any P-semi-invariant Radon measure on $\mathcal{E}$ is of the form m_ψ^{BR} for some $\psi \in D_\Gamma^\star$ (Proposition 10.25). Hence Theorem 1.6 implies:

Corollary 1.8. *The space of all P-semi-invariant Radon measures on $\mathcal{E}$ coincides with $\mathcal{Q}_\Gamma$, up to constant multiples.*

Remark 1.9. We remark that although there are Anosov subgroups with respect to a more general parabolic subgroup, we do not know how to define Burger-Roblin measures for those groups. For Anosov subgroups with respect to a minimal parabolic subgroup, which are considered here, the Hopf-parametrization of G/M as $\mathcal{F}^{(2)} \times A$ were used to define Burger-Roblin measures on $\Gamma\backslash G/M$. However if P is not minimal, then in the Langlands decomposition of $P = NMA$, M is not compact, and a Zariski dense subgroup Γ does not act properly discontinously on G/M [4].

Discussion on the proof of Theorem 1.6. Fix $\psi \in D_\Gamma^\star$. Defining a Γ-invariant Radon measure $\widehat{\nu}_\psi$ on $\mathcal{H} := G/NM \simeq \mathcal{F} \times \mathfrak{a}$ by

$$d\widehat{\nu}_\psi(gP, b) = e^{\psi(b)} d\nu_\psi(gP)\, db,$$

the standard duality theorem implies that the NM-ergodicity of m_ψ^{BR} is equivalent to the Γ-ergodicity of $\widehat{\nu}_\psi$.

Generalizing the observation of Schmidt [46] (also see [40]) to a higher rank situation, the Γ-ergodicity of $\widehat{\nu}_\psi$ follows if the closed subgroup, say $\mathsf{E}_{\nu_\psi} = \mathsf{E}_{\nu_\psi}(\Gamma)$, consisting of all ν_ψ-essential values is equal to $\mathfrak{a}$ (Proposition 9.2):

Definition 1.10. An element $v \in \mathfrak{a}$ is called a (Γ, ν_ψ)-*essential value*, if for any $\varepsilon > 0$ and Borel set $B \subset \mathcal{F}$ with $\nu_\psi(B) > 0$, there exists $\gamma \in \Gamma$ such that

$$B \cap \gamma B \cap \{\xi \in \mathcal{F} : \|\beta_\xi(o, \gamma o) - v\| < \varepsilon\}$$

has a positive ν_ψ-measure.

Recalling that the Jordan projection $\lambda(\Gamma)$ of Γ generates a dense subgroup of $\mathfrak{a}$ [5], the following is the main ingredient of our proof of Theorem 1.6:

Proposition 1.11. *For each $\psi \in D_\Gamma^\star$, there exists a finite subset $F_\psi \subset \lambda(\Gamma)$ such that*

$$\lambda(\Gamma) - F_\psi \subset \mathsf{E}_{\nu_\psi}(\Gamma).$$

In particular, $\mathsf{E}_{\nu_\psi}(\Gamma) = \mathfrak{a}$.

See Proposition 10.2 for a more general version stated for any Zariski dense normal subgroup of Γ.

Among other things, the following three key properties of Anosov groups play important roles in the proof of Proposition 1.11:

1. (Antipodality) $\Lambda \times \Lambda - \{(\xi, \xi)\} \subset \mathcal{F}^{(2)}$;
2. (Regularity) If $\gamma_i \to \infty$ in Γ, then $\alpha(\mu(\gamma_i)) \to \infty$ for each simple root α of $\mathrm{Lie}(G)$ with respect to $\mathfrak{a}^+$;
3. (Morse property) There exists a constant $D > 0$ such that any discrete geodesic ray $[e, x)$ in Γ tending to $x \in \partial\Gamma$ is contained in the D-neighborhood of some gA^+ in G where $g \in G$ satisfies $gP = \zeta(x)$.

(1) is a part of the definition of an Anosov subgroup. (2) follows from the fact that $\mathcal{L}_\Gamma \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$ ([36], [42], [7], see Lemma 7.2). (3) is proved in [23] (Proposition 5.11).

Many aspects of our proof of Proposition 1.11 can be simplified for a special class of $\psi \in D_\Gamma^\star$ with certain strong positivity property (cf. Lemma 5.1); however as our eventual goal is the classification theorem as stated in Theorem 1.1, we need to address all $\psi \in D_\Gamma^\star$ which makes the proof much more intricate and requires the full force of the Anosov property of Γ.

Fix $\gamma_0 \in \Gamma$ and let $\xi_0 \in \mathcal{F}$ denote its attracting fixed point. For any $\varepsilon > 0$, we aim to show that for any Borel subset $B \subset \mathcal{F}$ with $\nu_\psi(B) > 0$, there exists $\gamma \in \Gamma$ such that

$$\nu_\psi(B \cap \gamma\gamma_0\gamma^{-1}B \cap \{\xi \in \mathcal{F} : \|\beta_\xi(o, \gamma\gamma_0\gamma^{-1}o) - \lambda(\gamma_0)\| < \varepsilon\}) > 0; \tag{1.12}$$

this implies that $\lambda(\gamma_0) \in \mathsf{E}_{\nu_\psi}(\Gamma)$.

For $p \in G/K$, we define

$$d_{\psi,p}(\xi_1, \xi_2) = e^{-[\xi_1,\xi_2]_{\psi,p}}$$

for any $\xi_1 \neq \xi_2$ in Λ, where $[\cdot,\cdot]_{\psi,p}$ denotes the ψ-Gromov product based at p (Def. 6.1). Its well-definedness is due to the antipodality (1). In the rank one case, this is simply the restriction of the classical visual metric to the limit set Λ. In general, it is not even symmetric but we show that any sufficiently small power of $d_{\psi,p}$ is comparable to some genuine metric on Λ:

Theorem 1.13. *For all sufficiently small $s > 0$, there exist a metric d_s on Λ and $C_s > 0$ such that for all $\xi_1 \neq \xi_2$ in Λ,*

$$C_s^{-1} d_s(\xi_1, \xi_2) \leq d_{\psi,p}(\xi_1, \xi_2)^s \leq C_s d_s(\xi_1, \xi_2).$$

Remark 1.14. In the process of proving this theorem, we also show that the Gromov product on $\partial\Gamma$ and the ψ-Gromov product $[\cdot,\cdot]_{\psi,p}$ are equivalent to each other (see Theorem 6.13).

As a consequence of Theorem 1.13, $d_{\psi,p}$ can be used to define virtual balls with respect to which Vitali type covering lemma can be applied (Lemma 6.12). Consider the family

$$D(\gamma\xi_0, r) := \mathbb{B}_p(\gamma\xi_0, \frac{1}{3}e^{-\psi(\underline{a}(\gamma^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma^{-1}p,p))}r),\ \gamma \in \Gamma,\ r > 0$$

where $\underline{a}(q, p)$ denotes the $\mathfrak{a}^+$-valued distance from q to p (Def. 2.4). We then show that for all sufficiently small $r > 0$, there are infinitely many $D(\gamma_i\xi_0, r)$ satisfying (1.12) (Lemma 10.12). The key ingredient in this step is the following:

Lemma 1.15. *There exists $C = C(\psi, p) > 0$ such that for all $\gamma \in \Gamma$ and $\xi \in \Lambda$,*

$$-\psi(\underline{a}(p, \gamma p)) - C \leq \psi(\beta_\xi(\gamma p, p)) \leq \psi(\underline{a}(\gamma p, p)) + C.$$

In the rank one case, a stronger statement $-d(p, q) \leq \beta_\xi(q, p) \leq d(p, q)$ holds for all $q, p \in G/K$ and $\xi \in \mathcal{F}$, which generalizes to strongly positive linear forms (Lemma 5.1). For a general $\psi \in D_\Gamma^\star$, our proof of Lemma 1.15 is based on the property that the orbit map $\gamma \mapsto \gamma(o)$ sends a shadow in the word hyperbolic group Γ to a shadow in the symmetric space G/K (Proposition 5.13), as well as the following lemma, which is of independent interest: we denote by $|\cdot|$ the word length on Γ with respect to a fixed finite symmetric generating subset.

Lemma 1.16. *There exists $R > 0$ such that for any $\gamma_1, \gamma_2 \in \Gamma$ with $|\gamma_1\gamma_2| = |\gamma_1| + |\gamma_2|$, we have*

$$\|\mu(\gamma_1\gamma_2) - \mu(\gamma_1) - \mu(\gamma_2)\| < R.$$

We emphasize that this lemma does not follow from the property of Anosov groups that $(\Gamma, |\cdot|) \to G$ is a quasi-isometric embedding [20, Thm. 1.7], due to the non-trivial multiplicative constant.

To establish (1.12), we approximate a general Borel subset $B \subset \mathcal{F}$ by some $D(\gamma\xi_0, r)$ satisfying (1.12). In this step, we prove the following higher rank generalization of Tukia's theorem [50, Thm. 4A] (see also [30], [1], [31]):

Theorem 1.17. *For any Patterson-Sullivan measure ν on Λ, the set of Myrberg limit points (Def. 8.1) has full ν-measure.*

It follows that for the AM-invariant Bowen-Margulis-Sullivan measure m_ψ^{BMS} on $\Gamma\backslash G$, almost all points have dense A^+M orbits (Corollary 8.11). Using the property that virtual balls $\mathbb{B}_p(\gamma\xi_0, r)$ satisfy a covering lemma (Lemma 6.12) which is a consequence of Theorem 1.13, we show that ν_ψ-almost all Myrberg limit points satisfy the Lebesgue density type statement for the family $\{D(\gamma\xi_0, r) : \gamma \in \Gamma, r > 0\}$ (Proposition 10.17). By Theorem 1.17, this gives a desired approximation of B by some $D(\gamma\xi_0, r)$ satisfying (1.12).

We finally remark that in our subsequent work [28], we present refined versions of Theorems 1.1 and 1.6 buliding on the main results of this paper.

Organization: In section 2, we go over basic definitions and properties of Zariski dense discrete subgroups of G. In section 3, we discuss the notion of $\mathfrak{a}$-valued Gromov product and define the generalized BMS measures for a pair of (Γ, ψ)-conformal densities on $\mathcal{F}$. From section 4, we assume that Γ is Anosov. In section 4, we observe that the BMS measures m_ψ^{BMS} is AM-ergodic for each $\psi \in D_\Gamma^\star$. Sections 5 and 6 are devoted to proving Lemma 1.15 and Theorem 1.13 respectively. In section 7, we prove that the space of PS-measures on Λ is homeomorphic to $D_\Gamma^\star$, which is the first part of Theorem 1.3. In section 8, we show that the set of Myrberg limit points of Γ has full measure for any PS-measure on Λ. In section 9, we discuss the relation between the set of essential values of ν_ψ and the NM-ergodicity of m_ψ^{BR}. In the final section 10, we prove Theorems 1.6, 1.5 and the second part of Theorem 1.3.

2. Limit set and Limit cone.

Let G be a connected, semisimple real Lie group with finite center, and $\Gamma < G$ be a Zariski dense discrete subgroup. We fix, once and for all, a Cartan involution θ of the Lie algebra $\mathfrak{g}$ of G, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We denote by K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and by $X = G/K$ the associated symmetric space. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$, let $A := \exp \mathfrak{a}$ and $A^+ = \exp \mathfrak{a}^+$. The centralizer of A in K is denoted by M, and we set N to be the contracting horospherical subgroup: for $a \in \operatorname{int} A^+$, $N = \{g \in G : a^{-n} g a^n \to e \text{ as } n \to +\infty\}$. Note that $\log N$ is the sum of all positive root subspaces for our choice of A^+. Similarly, we also consider the expanding horospherical subgroup N^+: for $a \in \operatorname{int} A^+$, $N^+ := \{g \in G : a^n g a^{-n} \to e \text{ as } n \to +\infty\}$. We set

$$P^+ = MAN^+, \quad \text{and} \quad P = P^- = MAN^-;$$

they are minimal parabolic subgroups of G which are opposite to each other. The quotient $\mathcal{F} = G/P$ is known as the Furstenberg boundary of G, and is isomorphic to K/M.

Let $\mathrm{N}_K(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K. Let $\mathcal{W} := \mathrm{N}_K(\mathfrak{a})/M$ denote the Weyl group. Fixing a left G-invariant and right K-invariant Riemannian metric on G induces a $\mathcal{W}$-invariant inner product on $\mathfrak{a}$, which we denote by $\langle \cdot, \cdot \rangle$. The identity coset $[e]$ in G/K is denoted by o.

Denote by $w_0 \in \mathcal{W}$ the unique element in $\mathcal{W}$ such that $\mathrm{Ad}_{w_0} \mathfrak{a}^+ = -\mathfrak{a}^+$; it is the longest Weyl element. Note that $w_0 P w_0^{-1} = P^+$.

Definition 2.1 (Visual map)**.** For each $g \in G$, we define

$$g^+ := gP \in G/P \quad \text{and} \quad g^- := g w_0 P \in G/P.$$

For all $g \in G$ and $m \in M$, observe that $g^{\pm} = (gm)^{\pm} = g(e^{\pm})$. Let $\mathcal{F}^{(2)}$ denote the unique open G-orbit in $\mathcal{F} \times \mathcal{F}$:

$$\mathcal{F}^{(2)} = G(e^+, e^-) = \{(g^+, g^-) \in \mathcal{F} \times \mathcal{F} : g \in G\}.$$

Note that the stabilizer of (e^+, e^-) is the intersection $P^- \cap P^+ = MA$.

We say that $\xi, \eta \in \mathcal{F}$ are in general position if $(\xi, \eta) \in \mathcal{F}^{(2)}$. The Bruhat decomposition says that G is the disjoint union $\cup_{w \in \mathcal{W}} N^- w P^+$, and $N^- P^+$ is Zariski open and dense in G. Hence $(\xi, \eta) \notin \mathcal{F}^{(2)}$ if and only if $(\xi, \eta) \in G(e^+, we^-)$ for some $w \in \mathcal{W} - \{e\}$.

Cartan projection and $\mathfrak{a}^+$-valued distance.

Definition 2.2 (Cartan projection)**.** For each $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$, called the Cartan projection of g, such that

$$g \in K \exp(\mu(g)) K.$$

When $\mu(g) \in \operatorname{int} \mathfrak{a}^+$ and $g = k_1 \mu(g) k_2$, k_1, k_2 are determined uniquely up to mod M, more precisely, if $g = k_1' \mu(g) k_2'$, then for some $m \in M$, $k_1 = k_1' m$ and $k_2 = m^{-1} k_2'$. We write

$$\kappa_1(g) := [k_1] \in K/M \quad \text{and} \quad \kappa_2(g) := [k_2] \in M \backslash K.$$

Lemma 2.3. [5, Lem. 4.6] *For any compact subset $L \subset G$, there exists a compact subset $Q = Q(L) \subset \mathfrak{a}$ such that for all $g \in G$,*

$$\mu(LgL) \subset \mu(g) + Q.$$

Definition 2.4 ($\mathfrak{a}^+$-valued distance)**.** We define $\underline{a} : X \times X \to \mathfrak{a}^+$ by

$$\underline{a}(p, q) := \mu(g^{-1}h)$$

where $p = g(o)$ and $q = h(o)$.

Accumulation of points of X on $\mathcal{F}$. Let Π denote the set of all simple roots of $\mathfrak{g}$ with respect to $\mathfrak{a}^+$.

Definition 2.5. We write that

(1) $v_i \to \infty$ regularly in $\mathfrak{a}^+$ if $\alpha(v_i) \to \infty$ as $i \to \infty$ for all $\alpha \in \Pi$;
(2) $a_i \to \infty$ regularly in A^+ if $\log a_i \to \infty$ regularly in $\mathfrak{a}^+$;
(3) $g_i \to \infty$ regularly in G if $\mu(g_i) \to \infty$ regularly in $\mathfrak{a}^+$.

If $a_i \to \infty$ regularly in A^+, then for all $n \in N^+$,

$$\lim_{i \to \infty} a_i n a_i^{-1} = e$$

uniformly on compact subsets of N.

Definition 2.6. We call Γ *regular* if for any sequence $\gamma_i \in \Gamma$ going to ∞ in G, $\gamma_i \to \infty$ regularly in G.

Lemma 2.7. *If the closure of $\{(\xi_i, e^-) : i = 1, 2, \cdots\}$ is contained in $\mathcal{F}^{(2)}$, then $a_i \xi_i \to e^+$ for any sequence $a_i \to \infty$ regularly in A^+.*

Proof. The hypothesis implies that $\xi_i = n_i e^+$ for a bounded sequence $n_i \in N^+$. Hence $a_i \xi_i = a_i n_i e^+ = (a_i n_i a_i^{-1}) e^+ \to e^+$ as $a_i \to \infty$ regularly in A^+. □

Definition 2.8. (1) A sequence $g_i \in G$ is said to converge to $\xi \in \mathcal{F}$, if $g_i \to \infty$ regularly in G and $\lim_{i \to \infty} \kappa_1(g_i)^+ = \xi$.
(2) A sequence $p_i = g_i(o) \in X$ is said to converge to $\xi \in \mathcal{F}$ if g_i does.

Lemma 2.9. *Consider a sequence $g_i = k_i a_i h_i^{-1}$ where $k_i \in K, a_i \in A^+, h_i \in G$ satisfy that $k_i^+ \to k_0^+$ in K, $h_i \to h_0$ in G, and $a_i \to \infty$ regularly in A^+. Then for any $\xi \in \mathcal{F}$ in general position with h_0^-, we have*

$$\lim_{i \to \infty} g_i \xi = k_0^+.$$

Proof. As $(\xi, h_0^-) \in \mathcal{F}^{(2)}$, we have $(h_0^{-1}\xi, e^-) \in \mathcal{F}^{(2)}$. Since $\mathcal{F}^{(2)}$ is open and $h_i^{-1}\xi \to h_0^{-1}\xi$, we have $(h_i^{-1}\xi, e^-) \in \mathcal{F}^{(2)}$ for all large i. By Lemma 2.7, $a_i h_i^{-1} \xi \to e^+$ as $i \to \infty$. Therefore $\lim_{i \to \infty} g_i \xi = \lim_{i \to \infty} k_i(a_i h_i^{-1} \xi) = k_0^+$. □

Lemma 2.10. *If $g_i \in G$ converges to $\xi \in \mathcal{F}$, then $\lim_{i \to \infty} g_i p = \xi$ for any $p \in X$.*

Proof. Write $g_i = k_i a_i \ell_i^{-1} \in KA^+K$. The hypothesis implies that $a_i \to \infty$ regularly in A^+ and $k_i^+ \to \xi$ as $i \to \infty$. Let $k_0 \in K$ be such that $k_0^+ = \xi$, and $g \in G$ be such that $g(o) = p$. Write $g_i g = k_i' a_i' (\ell_i')^{-1} \in KA^+K$. We need to show that $\lim_{i\to\infty} k_i' = k_0^+$. As $k_i^+ \to k_0^+$, it suffices to show that any limit of the sequence $k_i^{-1}k_i'$ belongs to $M = \operatorname{Stab}_K e^+$.

Set $q_i := k_i^{-1}k_i'$. Let q be a limit of the sequence q_i. By passing to a subsequence, we may suppose $q_i \to q \in K$. Since $d(o,p) = d(g_i o, g_i p) = d(a_i o, q_i a_i' o)$, the sequence $h_i^{-1} := a_i^{-1} q_i a_i'$ is bounded. Passing to a subsequence, assume that h_i converges to some $h_0 \in G$ as $i \to \infty$. Choose $\eta \in \mathcal{F}$ that is in general position with both h_0^- and e^-. Then $\lim_{i\to\infty} a_i h_i^{-1}\eta = e^+$ and $\lim_{i\to\infty} q_i a_i'\eta = q^+$ by Lemma 2.9. Since $a_i h_i^{-1}\eta = q_i a_i'\eta$, we get $e^+ = q^+ = q(e^+)$. This implies $q \in \operatorname{Stab}_K e^+ = M$. □

Lemma 2.11. *If $g_i \to g$ in G and $a_i \to \infty$ regularly in A^+, then for any $p \in X$, $\lim_{i\to\infty} g_i a_i(p) = g^+$ and $\lim_{i\to\infty} g_i a_i^{-1}(p) = g^-$.*

Proof. By Lemma 2.10, it suffices to consider the case when $p = o$. Write $g_i a_i = k_i b_i \ell_i^{-1} \in KA^+K$. As the sequence g_i is bounded, it follows from Lemma 2.3 that $b_i \to \infty$ regularly in A^+. In order to show that $g_i a_i(o) \to g^+$, it suffices to show that if $k_i \to k_0$, then $k_0^+ = g^+$. By passing to a subsequence, we may assume that $\ell_i \to \ell_0$ in K. Choose $\xi \in \mathcal{F}$ which is in general position with both ℓ_0^- and e^-. Then $g_i a_i \xi \to k_0^+$ by Lemma 2.9. On the other hand, as $(\xi, e^-) \in \mathcal{F}^{(2)}$, $g_i a_i \xi \to g^+$ by Lemma 2.7. Hence $g^+ = k_0^+$, proving the first claim. Now the second claim follows since $g_i a_i^{-1} = g_i w_0 b_i w_0^{-1}$ for some $b_i \in A^+$, and $g_i w_0 b_i w_0^{-1}(o) = g_i w_0 b_i(o) \to (g w_0)^+ = g^-$. □

Limit set and Limit cone. Denote by m_o the K-invariant probability measure on $\mathcal{F} \simeq K/M$.

Definition 2.12 (Limit set)**.** The limit set Λ of Γ is defined as the set of all points $\xi \in \mathcal{F}$ such that the Dirac measure δ_ξ is a limit point of $\{\gamma_* m_o : \gamma \in \Gamma\}$ in the space of Borel probability measures on $\mathcal{F}$.

Benoist showed that Λ is the unique minimal Γ-invariant closed subset of $\mathcal{F}$. Moreover, Λ is Zariski dense in $\mathcal{F}$ ([5, Section 3.6], see also [14, Lem. 2.10] for a stronger statement).

Lemma 2.13. *We have*

$$\Lambda = \left\{ \lim_{i\to\infty} \gamma_i p \in \mathcal{F} : \gamma_i \in \Gamma, p \in X \right\}.$$

Proof. Let $(\gamma_i)_* m_o \to \delta_\xi$, and write $\gamma_i = k_i a_i \ell_i^{-1} \in KA^+K$. Suppose $k_i \to k$. Then $(a_i)_* m_o \to \delta_{k^{-1}\xi}$. It follows that $a_i \to \infty$ regularly in A^+ and $k^{-1}\xi = e^+$, i.e., $\xi = k^+$. Hence $\gamma_i \to \xi$. This proves the inclusion $\subset$. If $\gamma_i p \to \xi$ and $\gamma_i = k_i a_i \ell_i^{-1} \in KA^+K$, then $a_i \to \infty$ regularly and $k_i^+ \to \xi$. Since $(a_i)_* m_o$ converges to δ_{e^+}, we have $(\gamma_i)_* m_o \to \delta_\xi$. This proves the other inclusion. □

Any element $g \in G$ can be written as the commuting product $g_h g_e g_u$, where g_h, g_e and g_u are unique elements which are conjugate to elements of A^+, K and N, respectively. When g_h is conjugate to an element of $\operatorname{int} A^+$,

g is called *loxodromic*; in such a case, $g_u = e$. If a loxodromic element $g \in G$ satisfies $\varphi^{-1} g_h \varphi \in \operatorname{int} A^+$ for $\varphi \in G$, then

$$y_g^+ := \varphi^+ \qquad \text{and} \qquad y_g^- := \varphi^- \tag{2.14}$$

are called the attracting and repelling fixed points of g respectively.

Lemma 2.15. [5, Lem. 3.6] *The set*

$$\{(y_\gamma^+, y_\gamma^-) \in \Lambda \times \Lambda : \gamma \text{ is a loxodromic element of } \Gamma\}$$

is dense in $\Lambda \times \Lambda$.

The Jordan projection of g is defined as $\lambda(g) \in \mathfrak{a}^+$, where $\exp \lambda(g)$ is the element of A^+ conjugate to g_h.

Definition 2.16 (Limit cone)**.** The *limit cone* $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ of Γ is defined as the smallest closed cone containing the *Jordan projection* $\lambda(\Gamma)$. Alternatively, it can be defined as the smallest closed cone containing $\mu(\Gamma)$ [14, Lem. 2.18].

The limit cone $\mathcal{L}_\Gamma$ is a convex subset of $\mathfrak{a}^+$ with non-empty interior [5, Thm. 1.2].

Definition 2.17 (Growth indicator function)**.** The growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined as a homogeneous function, i.e., $\psi_\Gamma(tu) = t\psi_\Gamma(u)$, such that for any unit vector $u \in \mathfrak{a}^+$,

$$\psi_\Gamma(u) := \inf_{\substack{\text{open cones } \mathcal{C} \subset \mathfrak{a}^+ \\ u \in \mathcal{C}}} \limsup_{t \to \infty} \frac{1}{t} \log \#\{\gamma \in \Gamma : \mu(\gamma) \in \mathcal{C}, \|\mu(\gamma)\| \le t\}.$$

We may consider ψ_Γ as a function on $\mathfrak{a}$ by setting $\psi_\Gamma = -\infty$ outside of $\mathfrak{a}^+$. Quint showed the following:

Theorem 2.18. [34, Thm. IV.2.2] *The growth indicator function* ψ_Γ *is concave, upper-semicontinuous, and satisfies*

$$\mathcal{L}_\Gamma = \{u \in \mathfrak{a}^+ : \psi_\Gamma(u) > -\infty\}.$$

Moreover, ψ_Γ *is non-negative on* $\mathcal{L}_\Gamma$ *and positive on* $\operatorname{int} \mathcal{L}_\Gamma$.

3. $\mathfrak{a}$-valued Gromov product and generalized BMS-measures

Iwasawa cocycle and $\mathfrak{a}$-valued Busemann function. The Iwasawa decomposition says that the product map $K \times A \times N \to G$ is a diffeomorphism.

Definition 3.1. The Iwasawa cocycle $\sigma : G \times \mathcal{F} \to \mathfrak{a}$ is defined as follows: for $(g, \xi) \in G \times \mathcal{F}$, $\sigma(g, \xi) \in \mathfrak{a}$ is the unique element satisfying

$$gk \in K \exp(\sigma(g, \xi)) N \tag{3.2}$$

where $k \in K$ is such that $\xi = k^+$.

It satisfies the cocycle relation

$$\sigma(g_1 g_2, \xi) = \sigma(g_1, g_2 \xi) + \sigma(g_2, \xi)$$

for all $g_1, g_2, g_3 \in G$ and $\xi \in \mathcal{F}$.

Definition 3.3. The $\mathfrak{a}$-valued Busemann function $\beta : \mathcal{F} \times X \times X \to \mathfrak{a}$ is defined as follows: for $\xi \in \mathcal{F}$ and $g(o), h(o) \in X$,

$$\beta_\xi(g(o), h(o)) := \sigma(g^{-1}, \xi) - \sigma(h^{-1}, \xi).$$

Observe that the Busemann function is continuous in all three variables. To ease the notation, we will write $\beta_\xi(g, h) = \beta_\xi(g(o), h(o))$. We can check that for all $g, h, q \in G$ and $\xi \in \mathcal{F}$,

$$\begin{aligned} \beta_\xi(g, h) + \beta_\xi(h, q) &= \beta_\xi(g, q), \\ \beta_{g\xi}(gh, gq) &= \beta_\xi(h, q), \text{ and} \\ \beta_\xi(g, e) &= \sigma(g^{-1}, \xi). \end{aligned} \tag{3.4}$$

Geometrically, if $\xi = k^+ \in \mathcal{F}$ for $k \in K$, then for any unit vector $u \in \mathfrak{a}^+$,

$$\langle \beta_\xi(g, h), u \rangle = \lim_{t \to +\infty} d(g(o), \xi_t) - d(h(o), \xi_t)$$

where $\xi_t = k \exp(tu)o \in X$.

Lemma 3.5. *For any loxodromic element $g \in G$ and $p \in X$,*

$$\beta_{y_g^+}(p, gp) = \lambda(g) \quad \textit{and} \quad \beta_{y_g^-}(p, gp) = -\lambda(g^{-1}).$$

Proof. Let $\varphi \in G$ be so that $g = \varphi a m \varphi^{-1}$ for some $a \in A^+$ and $m \in M$. If $p = h(o)$ for $h \in G$, then, since g^{-1} fixes φ^+,

$$\beta_{y_g^+}(p, gp) = \beta_{\varphi^+}(ho, gho) = \sigma(h^{-1}, \varphi^+) - \sigma(h^{-1}g^{-1}, \varphi^+) = -\sigma(g^{-1}, \varphi^+).$$

Writing $\varphi = kb$ with $k \in K$ and $b \in P$, we have

$$g^{-1}k = \varphi(am)^{-1}\varphi^{-1}k = kb(am)^{-1}b^{-1} \in Ka^{-1}N.$$

This gives $\sigma(g^{-1}, \varphi^+) = \log a^{-1} = -\lambda(g)$, and hence the first identity. The second identity follows from the first, by replacing g with g^{-1}, since $y_{g^{-1}}^+ = y_g^-$. ∎

$\mathfrak{a}$-valued Gromov product.

Definition 3.6 (Opposition involution)**.** The involution $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ defined by

$$\mathrm{i}(u) = -\operatorname{Ad}_{w_0}(u)$$

is called the opposition involution; it preserves $\mathfrak{a}^+$. Note that for all $g \in G$,

$$\lambda(g^{-1}) = \mathrm{i}(\lambda(g)) \quad \text{and} \quad \mu(g^{-1}) = \mathrm{i}(\mu(g)).$$

It follows that

$$\mathrm{i}(\mathcal{L}_\Gamma) = \mathcal{L}_\Gamma \quad \text{and} \quad \psi_\Gamma \circ \mathrm{i} = \psi_\Gamma. \tag{3.7}$$

Definition 3.8. *We define the $\mathfrak{a}$-valued Gromov product on $\mathcal{F}^{(2)}$ as follows: for $(\xi, \eta) \in \mathcal{F}^{(2)}$,*

$$\mathcal{G}(\xi, \eta) := \beta_{g^+}(e, g) + \mathrm{i}\, \beta_{g^-}(e, g)$$

where $g \in G$ satisfies $g^+ = \xi$ and $g^- = \eta$.

The definition does not depend on the choice of a representative of $[g] \in G/AM$. For all $h \in G$ and $(x,y) \in \mathcal{F}^{(2)}$, we have the following identity:

$$\mathcal{G}(hx,hy) - \mathcal{G}(x,y) = \sigma(h,x) + \mathrm{i}\,\sigma(h,y). \tag{3.9}$$

As $\mathcal{G}(y,x) = \mathrm{i}\,\mathcal{G}(x,y)$, the Gromov product is not symmetric in general.

Lemma 3.10. [49] *There exists a family of irreducible representations (ρ_α, V_α), $\alpha \in \Pi$, of G so that*

1. *the highest weight χ_α of ρ_α is a positive integral multiple of the fundamental weight ϖ_α corresponding to α;*
2. *the highest weight space of ρ_α is one dimensional.*

For $\alpha \in \Pi$, denote by V_α^+ the highest weight space of ρ_α, and by $V_\alpha^<$ its unique complementary A-invariant subspace in V_α. We have $\rho_\alpha(P)V_\alpha^+ = V_\alpha^+$, and hence the map $g \mapsto (\rho_\alpha(g)V_\alpha^+)_{\alpha\in\Pi}$ factors through a proper immersion

$$\mathcal{F} = G/P \to \prod_{\alpha\in\Pi} \mathbb{P}(V_\alpha).$$

Let $\langle\cdot,\cdot\rangle_\alpha$ be a K-invariant inner product on V_α with respect to which A is symmetric; then V_α^+ and $V_\alpha^<$ are orthogonal to each other. We denote by $\|\cdot\|_\alpha$ the norm on V_α induced by $\langle\cdot,\cdot\rangle_\alpha$. For $\varphi \in V_\alpha^*$, $\|\varphi\|_\alpha$ means the operator norm of φ. We also use the notation $\|\cdot\|_\alpha$ for a bi-$\rho_\alpha(K)$-invariant norm on $\mathrm{GL}(V_\alpha)$.

Lemma 3.11. *For all $\alpha \in \Pi$ and $g \in G$,*

$$\chi_\alpha(\mathcal{G}(g^+,g^-)) = -\log \frac{|\varphi(v)|}{\|\varphi\|_\alpha \|v\|_\alpha} \tag{3.12}$$

where $v \in gV_\alpha^+$ and $\varphi \in V_\alpha^*$ is such that $\ker\varphi = gV_\alpha^<$.

Proof. If we define $\mathcal{G}'(g^+,g^-)$ to be the unique element of $\mathfrak{a}$ satisfying (3.12), it is shown in [41, Lem 4.12] that $\mathcal{G}'$ satisfies (3.9). Hence for all $h \in G$,

$$\mathcal{G}'(h^+,h^-) - \mathcal{G}'(e^+,e^-) = \mathcal{G}(h^+,h^-) - \mathcal{G}(e^+,e^-).$$

Observe that $\mathcal{G}'(e^+,e^-) = 0$; take φ to be the projection $V \to V_\alpha^+$ parallel to $V_\alpha^<$. Since V_α^+ and $V_\alpha^<$ are orthogonal, it follows that $\|\varphi\|_\alpha = 1$. Now for $v \in V_\alpha^+$, we have

$$\frac{|\varphi(v)|}{\|\varphi\|_\alpha \|v\|_\alpha} = \frac{\|v\|_\alpha}{\|v\|_\alpha} = 1.$$

Since $\mathcal{G}(e^+,e^-) = 0$, we conclude $\mathcal{G} = \mathcal{G}'$ on $\mathcal{F}^{(2)}$. ◻

Remark 3.13. In view of this lemma, our definition of Gromov product differs by $-\mathrm{i}$ from the one given in [41].

Patterson-Sullivan measures on Λ.

Definition 3.14 (Conformal measures)**.** Given a closed subgroup $\Gamma < G$ and $\psi \in \mathfrak{a}^*$, a Borel probability measure ν on $\mathcal{F}$ is called a (Γ, ψ)-conformal measure if, for any $\gamma \in \Gamma$ and $\xi \in \mathcal{F}$,

$$\frac{d\gamma_* \nu}{d\nu}(\xi) = e^{\psi(\beta_\xi(e,\gamma))} \tag{3.15}$$

where $\gamma_* \nu(Q) = \nu(\gamma^{-1} Q)$ for any Borel subset $Q \subset \mathcal{F}$.

If 2ρ denotes the sum of all positive roots of G with respect to $\mathfrak{a}^+$, then a $(G, 2\rho)$-conformal measure is precisely the K-invariant probability measure m_o on $\mathcal{F}$.

Fix a Zariski dense discrete subgroup $\Gamma < G$ in the rest of this section.

Definition 3.16 (Patterson-Sullivan measures)**.** For $\psi \in \mathfrak{a}^*$, a (Γ, ψ)-conformal measure supported on the limit set Λ will be called a (Γ, ψ)-PS measure. By a PS measure on Λ, we mean a (Γ, ψ)-PS measure for some $\psi \in \mathfrak{a}^*$.

Set

$$D_\Gamma := \{\psi \in \mathfrak{a}^* : \psi \geq \psi_\Gamma\}.$$

The following collection of linear forms is of particular importance:

$$D_\Gamma^\star := \{\psi \in D_\Gamma : \psi(u) = \psi_\Gamma(u) \text{ for some } u \in \mathcal{L}_\Gamma \cap \operatorname{int} \mathfrak{a}^+\}. \tag{3.17}$$

By (3.7), $\psi \circ \mathrm{i} \in D_\Gamma^\star$ for all $\psi \in D_\Gamma^\star$. The concavity of ψ_Γ and the non-emptiness of $\operatorname{int} \mathcal{L}_\Gamma$ imply that $D_\Gamma^\star$ is non-empty by the Hahn-Banach theorem. When $\psi(u) = \psi_\Gamma(u)$, we say ψ is tangent to ψ_Γ at u.

Generalizing the work of Patterson and Sullivan ([33], [47]), Quint [35] constructed a (Γ, ψ)-PS measure for every $\psi \in D_\Gamma^\star$.

Generalized BMS-measure m_{ν_1,ν_2}. Given a pair of Γ-conformal measures on $\mathcal{F}$, we now define an MA-semi invariant measure on $\Gamma \backslash G$, which we call a generalized BMS-measure.

Definition 3.18 (Hopf parametrization)**.** The map

$$gM \to (g^+, g^-, b = \beta_{g^+}(e, g))$$

gives a homeomorphism between G/M and $\mathcal{F}^{(2)} \times \mathfrak{a}$, which is called the Hopf parametrization of G/M.

Fixing a pair of Γ-conformal measures $\nu_{\psi_1}, \nu_{\psi_2}$ on $\mathcal{F}$ for a pair of linear forms $\psi_1, \psi_2 \in \mathfrak{a}^*$, we define a Radon measure $\tilde{m}_{\nu_{\psi_1},\nu_{\psi_2}}$ on G/M as follows: for $g = (g^+, g^-, b) \in \mathcal{F}^{(2)} \times \mathfrak{a}$,

$$d\tilde{m}_{\nu_{\psi_1},\nu_{\psi_2}}(g) = e^{\psi_1(\beta_{g^+}(e,g)) + \psi_2(\mathrm{i}\beta_{g^-}(e,g))}\, d\nu_{\psi_1}(g^+) d\nu_{\psi_2 \circ \mathrm{i}}(g^-) db, \tag{3.19}$$

where $db = d\ell(b)$ is the Lebesgue measure on $\mathfrak{a}$. This measure is left Γ-invariant, and hence induces a measure on $\Gamma \backslash G/M$. We denote by $m_{\nu_{\psi_1},\nu_{\psi_2}}$ its M-invariant lift to $\Gamma \backslash G$. It is A-semi-invariant as

$$a_* m_{\nu_{\psi_1},\nu_{\psi_2}} = e^{(\psi_2 - \psi_1)(\log a)} m_{\nu_{\psi_1},\nu_{\psi_2}} \tag{3.20}$$

for all $a \in A$ [14, Lem. 3.6].
BMS-measures: $m^{\mathrm{BMS}}_{\nu_\psi}$. Let $\psi \in \mathfrak{a}^*$ and let ν_ψ be a (Γ, ψ)-PS measure. We set

$$m^{\mathrm{BMS}}_{\nu_\psi} := m_{\nu_\psi, \nu_\psi} \tag{3.21}$$

and call it the Bowen-Margulis-Sullivan measure associated to ν_ψ. It is right MA-invariant and its support is given by

$$\Omega := \{x \in \Gamma\backslash G : x^\pm \in \Lambda\};$$

since Λ is Γ-invariant, the condition $x^\pm \in \Lambda$ is well-defined. When the rank of G is at least 2, $m^{\mathrm{BMS}}_{\nu_\psi}$ is expected to be an infinite measure unless Γ is a lattice. Note that for $[g] \in G/M$,

$$dm^{\mathrm{BMS}}_{\nu_\psi}[g] = e^{\psi(\mathcal{G}(g^+, g^-))} d\nu_\psi(g^+) d\nu_{\psi \circ \mathrm{i}}(g^-) db. \tag{3.22}$$

***N*-invariant BR-measures: $m^{\mathrm{BR}}_{\nu_\psi}$.** We set

$$m^{\mathrm{BR}}_{\nu_\psi} := m_{\nu_\psi, m_o} \tag{3.23}$$

and call it the N-invariant Burger-Roblin measure associated to ν_ψ. See [14, Section 3] for the equivalence of this definition with the one given in (1.4). The support of $m^{\mathrm{BR}}_{\nu_\psi}$ is given by

$$\mathcal{E} := \{x \in \Gamma\backslash G : x^+ \in \Lambda\}.$$

4. Anosov groups and AM-ergodicity of BMS measures

Let Γ be a Zariski dense discrete subgroup of G, and set $\Lambda^{(2)} := (\Lambda \times \Lambda) \cap \mathcal{F}^{(2)}$.

Definition 4.1. We say that $\Gamma < G$ is Anosov, if it is a finitely generated word hyperbolic group admitting a Γ-equivariant homeomorphism $\zeta : \partial\Gamma \to \Lambda$ such that $(\zeta(x), \zeta(y)) \in \Lambda^{(2)}$ for all $x \neq y \in \partial\Gamma$, where $\partial\Gamma$ denotes the Gromov boundary of Γ.

Such ζ is Hölder continuous and exists uniquely ([25, Prop. 3.2] and [7, Lem. 2.5]). We call it the limit map of Γ. We note that the antipodal property of Λ follows directly:

$$\Lambda \times \Lambda - \{(\xi, \xi)\} = \Lambda^{(2)}. \tag{4.2}$$

In the literature, this definition is referred to as P-Anosov for a minimal parabolic subgroup P of G. See [20], [19] and [23] for equivalent characterizations of Anosov subgroups.

In the rest of this section, let Γ be an Anosov subgroup of G. The following theorem was proved by Quint [36, Prop. 3.2 and Thm. 4.7] for Schottky groups and by Sambarino [42, Coro. 3.12, 3.13 and 4.9] and by [14, Thm. 7.9] for general Anosov subgroups in view of the results in [12]:

Theorem 4.3. (1) *$\mathcal{L}_\Gamma \subset \operatorname{int}\mathfrak{a}^+ \cup \{0\}$ and every non-trivial element of Γ is loxodromic.*
(2) *ψ_Γ is strictly concave and analytic on $\operatorname{int}\mathcal{L}_\Gamma$.*
(3) *$D_\Gamma^\star = \{\psi \in D_\Gamma : \psi(u) = \psi_\Gamma(u)$ for some $u \in \operatorname{int}\mathcal{L}_\Gamma\}$.*
(4) *For any $\psi \in D_\Gamma^\star$, $\psi > 0$ on $\mathcal{L}_\Gamma - \{0\}$.*
(5) *For any $\psi \in D_\Gamma^\star$, there exists a unique (Γ, ψ)-PS measure, say ν_ψ, on $\mathcal{F}$. In particular, ν_ψ is Γ-ergodic.*

(1) and (3) imply that if $\psi \in D_\Gamma$ is tangent to ψ_Γ at some $u \in \mathcal{L}_\Gamma - \{0\}$, then $u \in \operatorname{int}\mathcal{L}_\Gamma$. Note also that (1) implies that any Anosov subgroup is regular as defined in Def. 2.6. For $u \in \mathcal{L}_\Gamma$, we denote by $D_u\psi_\Gamma$ the directional derivative of ψ_Γ at u, whenever it exists.

Proposition 4.4. *For each $u \in \operatorname{int}\mathcal{L}_\Gamma$, $\psi_u := D_u\psi_\Gamma \in D_\Gamma^\star$ and $D_u\psi_\Gamma(u) = \psi_\Gamma(u)$. Moreover, the map $u \mapsto \psi_u$ induces a homeomorphism between the set of unit vectors of $\operatorname{int}\mathcal{L}_\Gamma$ ($\simeq \operatorname{int}\mathbb{P}\mathcal{L}_\Gamma$) and $D_\Gamma^\star$. Hence $D_\Gamma^\star \simeq \mathbb{R}^{\mathrm{rank}\, G-1}$.*

Proof. See ([42, Thm. A], [14, Lem. 2.23]) for the first claim. The well-definedness and surjectivity of the map $u \mapsto D_u\psi_\Gamma$ follows from it, and the injectivity follows from the strict concavity of ψ_Γ as in Theorem 4.3(2). Continuity follows from the analyticity of ψ_Γ on $\operatorname{int}\mathcal{L}_\Gamma$. We claim that if $D_{u_i}\psi_\Gamma \to D_u\psi_\Gamma$ for some unit vectors $u_i, u \in \operatorname{int}\mathcal{L}_\Gamma$, then $u_i \to u$. Let $v \in \mathcal{L}_\Gamma$ be a limit of the sequence u_i. By passing to a subsequence, assume $u_i \to v$. By the upper-semi continuity of ψ_Γ (Theorem 2.18), we have

$$\psi_\Gamma(v) \geq \limsup_{i\to\infty} \psi_\Gamma(u_i).$$

Since $\psi_\Gamma(u_i) = D_{u_i}\psi_\Gamma(u_i)$ and $D_{u_i}\psi_\Gamma \to D_u\psi_\Gamma$, we get $\psi_\Gamma(v) \geq D_u\psi_\Gamma(v)$. Since $D_u\psi_\Gamma \in D_\Gamma$, we have $\psi_\Gamma(v) = D_u\psi_\Gamma(v)$. It follows from Theorem 4.3(1) and (3) that $v \in \operatorname{int}\mathcal{L}_\Gamma$. Since $\psi_\Gamma(u) = D_u\psi_\Gamma(u)$, the strict concavity of ψ_Γ on $\operatorname{int}\mathcal{L}_\Gamma$ implies that $u = v$, establishing the homeomorphism. Since $\operatorname{int}(\mathcal{L}_\Gamma)$ is a non-empty open convex cone of $\mathfrak{a}^+$, $\operatorname{int}(\mathbb{P}\mathcal{L}_\Gamma) \simeq \mathbb{P}\operatorname{int}(\mathcal{L}_\Gamma)$ is homeomorphic to $\mathbb{R}^{\operatorname{rank} G-1}$. □

We denote by $\nabla\psi_\Gamma$ the gradient of ψ_Γ so that $D_u\psi_\Gamma(v) = \langle \nabla\psi_\Gamma(u), v\rangle$ for $u \in \operatorname{int}\mathcal{L}_\Gamma$ and $v \in \mathfrak{a}$. Set

$$\mathsf{O}_\Gamma := \{\nabla\psi_\Gamma(u) \in \mathfrak{a} : u \in \operatorname{int}\mathcal{L}_\Gamma\}, \tag{4.5}$$

which is an open convex cone of $\mathfrak{a} - \{0\}$. By Proposition 4.4, the map $w \mapsto \langle w, \cdot\rangle$ gives a homeomorphism between $\{\nabla\psi_\Gamma(u) \in \mathfrak{a} : u \in \operatorname{int}\mathcal{L}_\Gamma, \|u\| = 1\}$ and $D_\Gamma^\star$, and hence a homeomorphism

$$\mathsf{O}_\Gamma \simeq \mathbb{R}_+ D_\Gamma^\star.$$

Quint showed that there exists a unique unit vector, say $u_\Gamma \in \operatorname{int}\mathfrak{a}^+$, such that $\psi_\Gamma(u_\Gamma) = \max_{\|u\|=1} \psi_\Gamma(u)$. The vector u_Γ is called the direction of maximal growth of Γ. If we set $\delta_\Gamma := \psi_\Gamma(u_\Gamma)$, then $\nabla\psi_\Gamma(u_\Gamma) = \delta_\Gamma u_\Gamma$ and

$$\delta_\Gamma = \limsup_{T\to\infty} \frac{\log \#\{\gamma \in \Gamma : \|\mu(\gamma)\| < T\}}{T}.$$

Consider the following dual cone to $\mathcal{L}_\Gamma$:

$$\mathcal{L}_\Gamma^* := \{w \in \mathfrak{a} : \langle w, v\rangle \geq 0 \text{ for all } v \in \mathcal{L}_\Gamma\}.$$

Note that $\operatorname{int}\mathcal{L}_\Gamma^* = \{w \in \mathfrak{a} : \langle w, v\rangle > 0 \text{ for all non-zero } v \in \mathcal{L}_\Gamma\}$.

Lemma 4.6. *We have* $\mathsf{O}_\Gamma = \operatorname{int}\mathcal{L}_\Gamma^*$. *In particular,*

$$\mathfrak{a}^+ - \{0\} \subset \mathsf{O}_\Gamma \subset \mathfrak{a} - \{0\}.$$

Proof. If $w \in \mathsf{O}_\Gamma$, then $\langle w, \cdot\rangle \in \mathbb{R}_+ D_\Gamma^\star$, and hence by Theorem 4.3 (3), $\langle w, v\rangle > 0$ for all $v \in \mathcal{L}_\Gamma - \{0\}$. Hence $\mathsf{O}_\Gamma \subset \operatorname{int}\mathcal{L}_\Gamma^*$. Now suppose $w \in \operatorname{int}\mathcal{L}_\Gamma^*$. Setting $\psi(v) := \langle w, v\rangle$, we claim that $\psi \in \mathbb{R}_+ D_\Gamma^\star$; this implies $\operatorname{int}\mathcal{L}_\Gamma^* \subset \mathsf{O}_\Gamma$. Since $w \in \operatorname{int}\mathcal{L}_\Gamma^*$, $\psi > 0$ on $\mathcal{L}_\Gamma - \{0\}$ and hence $c := \max_{\|v\|=1, v\in\mathcal{L}_\Gamma} \psi(v) > 0$. Since $\delta_\Gamma c^{-1}\psi \geq \psi_\Gamma$ on $\mathcal{L}_\Gamma$, and hence on $\mathfrak{a}$, it follows that for some $\varepsilon > 0$, $\varepsilon\psi \in D_\Gamma^\star$, i.e., $\psi \in \mathbb{R}_+ D_\Gamma^\star$.

Since $\mathcal{L}_\Gamma - \{0\} \subset \operatorname{int}\mathfrak{a}^+$ and the angle between any two walls of $\mathfrak{a}^+$ is at most $\pi/2$, the second claim follows. $\square$

***AM*-ergodicity of m_ψ^{BMS}.** We fix $\psi \in D_\Gamma^\star$ and set

$$\nu := \nu_\psi \quad \text{and} \quad m_\psi^{\mathrm{BMS}} := m_{\nu_\psi}^{\mathrm{BMS}}. \tag{4.7}$$

The composition $c := \psi \circ \sigma : \Gamma \times \Lambda \to \mathbb{R}$ is a Hölder cocycle satisfying $c(\gamma, y_\gamma^+) = \psi(\lambda(\gamma)) > 0$ for all non-trivial $\gamma \in \Gamma$.

Consider the action of Γ on $\Lambda^{(2)} \times \mathbb{R}$ given as follows: for $\gamma \in \Gamma$ and $(\xi, \eta, t) \in \Lambda^{(2)} \times \mathbb{R}$,

$$\gamma.(\xi, \eta, t) = (\gamma\xi, \gamma\eta, t + c(\gamma, \xi)).$$

The $\mathbb{R}$-action on $\Lambda^{(2)} \times \mathbb{R}$ defined by

$$\tau_s(\xi, \eta, t) = (\xi, \eta, t + s)$$

will be called translation flow.

The following is proved in [43, Thm. 3.2] when Γ is the fundamental group of a closed negatively curved manifold, and can be extended for general Anosov groups, using ingredients from [7]. The sketch of the proof can be found in [11, Appendix A].

Theorem 4.8. *The action of* Γ *on* $\Lambda^{(2)} \times \mathbb{R}$ *is proper and cocompact, and the measure* $d\tilde{\mathsf{m}}_\psi(\xi, \eta, t) = e^{\psi(\mathcal{G}(\xi,\eta))} d\nu_\psi(\xi) \otimes d\nu_{\psi\circ\mathrm{i}}(\eta) \otimes dt$ *induces the measure of maximal entropy, say* m_ψ, *for* $\{\tau_s : s \in \mathbb{R}\}$ *on* $\Gamma\backslash\Lambda^{(2)} \times \mathbb{R}$. *In particular,* m_ψ *is* $\{\tau_s : s \in \mathbb{R}\}$*-ergodic.*

In terms of the Hopf parametrization, Γ acts on $\Lambda^{(2)} \times \mathfrak{a} = \operatorname{supp}\tilde{m}_\psi^{\mathrm{BMS}}$ as follows: for $\gamma \in \Gamma$ and $(\xi, \eta, v) \in \Lambda^{(2)} \times \mathfrak{a}$,

$$\gamma.(\xi, \eta, v) = (\gamma\xi, \gamma\eta, v + \sigma(\gamma, \xi)).$$

Corollary 4.9. *For any* $\psi \in D_\Gamma^\star$, *the* AM*-action on* $(\Gamma\backslash G, m_\psi^{\mathrm{BMS}})$ *is ergodic and if* $\operatorname{rank} G \geq 2$, $|m_\psi^{\mathrm{BMS}}| = \infty$.

Proof. The $\{\tau_s : s \in \mathbb{R}\}$-ergodicity of m_ψ is equivalent to ergodicity of $(\Lambda^{(2)}, \Gamma, \nu_\psi \otimes \nu_{\psi\circ\mathrm{i}}|_{\Lambda^{(2)}})$, which is again equivalent to the AM-ergodicity of m_ψ^{BMS}. Consider the projection map $\pi : \Gamma \setminus \Lambda^{(2)} \times \mathfrak{a} \to \Gamma \setminus \Lambda^{(2)} \times \mathbb{R}$ induced by the Γ-equivariant map $\Lambda^{(2)} \times \mathfrak{a} \to \Lambda^{(2)} \times \mathbb{R}$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi(v))$. Then π is a principal $\ker\psi$-bundle, which is trivial as $\ker\psi$ is a vector group. It follows that there exists a $\ker\psi$-equivariant homeomorphism between $\Gamma \setminus \Lambda^{(2)} \times \mathfrak{a}$ and $\left(\Gamma \setminus \Lambda^{(2)} \times \mathbb{R}\right) \times \ker\psi$. Therefore m_ψ^{BMS} disintegrates over the measure m_ψ with conditional measure being the Lebesque measure on $\ker\psi \simeq \mathbb{R}^{\mathrm{rank}G-1}$ so that $m_\psi^{\mathrm{BMS}} \simeq \mathsf{m}_\psi \otimes \mathrm{Leb}_{\ker\psi}$ (cf. [41, Prop. 3.5]). This gives the infinitude of $|m_\psi^{\mathrm{BMS}}|$ when G has rank at least 2. $\square$

5. Comparing $\mathfrak{a}$-valued Busemann functions and distances via ψ

When G has rank one, for any $p, q \in X$, the maximum and minimum of Busemann function $\beta_\xi(p, q), \xi \in \mathcal{F}$ are always achieved as $\pm d(p, q)$. A higher rank generalization of this fact can be stated as follows.

Lemma 5.1. *Let $\psi \in \mathfrak{a}^*$ be strongly positive, in the sense that ψ is a non-negative linear combination of fundamental weights ϖ_α, $\alpha \in \Pi$. Then for any $p, q \in X$ and $\xi \in \mathcal{F}$, we have*

$$-\psi(\underline{a}(q, p)) \leq \psi(\beta_\xi(p, q)) \leq \psi(\underline{a}(p, q)). \tag{5.2}$$

Proof. We use notations introduced in Lemma 3.10. Since ϖ_α is a positive multiple of χ_α, it suffices to prove the claim when $\psi = \chi_\alpha$ for $\alpha \in \Pi$.

Write $q = go$, and $p = hq$ for some $g, h \in G$. Note that

$$\chi_\alpha(\underline{a}(p, q)) = \chi_\alpha(\mu(g^{-1}h^{-1}g)) = \log\|\rho_\alpha(g^{-1}h^{-1}g)\|_\alpha.$$

Write $g^{-1}\xi = k^+$ for some $k \in K$ and $g^{-1}h^{-1}gk = k'an \in KAN$. Then

$$\beta_\xi(p, q) = \sigma(g^{-1}h^{-1}g, k^+) = \log a.$$

Hence for a unit vector $v \in V_\alpha$,

$$\chi_\alpha(\beta_\xi(p, q)) = \log\|\rho_\alpha(g^{-1}h^{-1}g)\rho_\alpha(k)v\| \leq \log\|\rho_\alpha(g^{-1}h^{-1}g)\|_\alpha = \chi_\alpha(\underline{a}(p, q)).$$

Since $\|\rho_\alpha(g^{-1})\|^{-1} \leq \|\rho_\alpha(g)v\|$ and $\chi_\alpha(\underline{a}(q, p)) = \log\|\rho_\alpha(g^{-1}hg)\|_\alpha$, we also get

$$\chi_\alpha(\beta_\xi(p, q)) \geq \log\|\rho_\alpha(g^{-1}hg)\|_\alpha^{-1} = -\chi_\alpha(\underline{a}(q, p)).$$

$\square$

There are $\psi \in D_\Gamma^\star$'s which are not strongly positive (see Lemma 4.6). We establish the following modification for Anosov groups, which is the main goal of this section:

Theorem 5.3. *Let $\Gamma < G$ be Anosov. For any $\psi \in D_\Gamma^\star$ and $p \in X$, there exists $C = C(\psi, p) > 0$ such that for all $\gamma \in \Gamma$ and $\xi \in \Lambda$,*

$$-\psi(\underline{a}(p, \gamma p)) - C \leq \psi(\beta_\xi(\gamma p, p)) \leq \psi(\underline{a}(\gamma p, p)) + C.$$

We begin by noting that $\psi(\underline{a}(\gamma p, p))$ is always positive possibly except for finitely many γ's:

Lemma 5.4. *Let $\psi \in D_\Gamma^\star$ and $p \in X$. For any sequence $\gamma_i \to \infty$ in Γ, $\psi(\underline{a}(\gamma_i p, p)) \to +\infty$.*

Proof. By Lemma 2.3, it suffices to check that $\psi(\mu(\gamma_i)) \to +\infty$ as $i \to \infty$. Setting $t_i := \|\mu(\gamma_i)\|^{-1}$, passing to a subsequence, we may assume that $t_i\mu(\gamma_i)$ converges to some unit vector $u \in \mathfrak{a}$. As $u \in \mathcal{L}_\Gamma$, we have $\psi(u) > 0$ by Lemma 4.3. Since $\psi(t_i\mu(\gamma_i)) \to \psi(u)$ and $\psi(\mu(\gamma_i)) = t_i^{-1}\psi(t_i\mu(\gamma_i))$, we have $\psi(\mu(\gamma_i)) \to +\infty$. □

The following is the main ingredient of the proof of Theorem 5.3:

Proposition 5.5. *For $p \in X$, there exists $C = C(p) > 0$ such that for each $(\gamma, \xi) \in \Gamma \times \Lambda$, we can find $\gamma_1 = \gamma_1(\xi), \gamma_2 = \gamma_2(\xi) \in \Gamma$ satisfying*

(1) $\gamma = \gamma_1\gamma_2$ *and* $|\gamma| = |\gamma_1| + |\gamma_2|$;
(2) $\|\beta_\xi(\gamma p, p) + \mu(\gamma_1) - \mu(\gamma_2^{-1})\| \le C$;
(3) $\|\underline{a}(\gamma p, p) - \mu(\gamma_1^{-1}) - \mu(\gamma_2^{-1})\| \le C$.

Proof of Theorem 5.3 using Proposition 5.5: For $\gamma \in \Gamma$ and $\xi \in \Lambda$, choose $\gamma_1, \gamma_2 \in \Gamma$ as in Proposition 5.5. Then

$$\begin{aligned}\psi(\beta_\xi(\gamma p, p)) &\le \psi(\mu(\gamma_2^{-1}) - \mu(\gamma_1)) + C\|\psi\| \\ &\le \psi(\mu(\gamma_2^{-1}) + \mu(\gamma_1^{-1})) + C\|\psi\| \\ &\le \psi(\underline{a}(\gamma p, p)) + 2C\|\psi\|,\end{aligned}$$

where the second inequality is valid because $\psi(\mu(\gamma_1^{\pm 1})) \ge 0$. Similarly, we get

$$\begin{aligned}\psi(\beta_\xi(\gamma p, p)) &\ge \psi(\mu(\gamma_2^{-1}) - \mu(\gamma_1)) - C\|\psi\| \\ &\ge -\psi(\mu(\gamma_2) + \mu(\gamma_1)) - C\|\psi\|.\end{aligned}$$

Since $\mathrm{i}\,\mu(g^{-1}) = \mu(g)$, $\mathrm{i}\,\underline{a}(p, q) = \underline{a}(q, p)$ and the norm is i-invariant, we get $\psi(\beta_\xi(\gamma p, p)) \ge \underline{\psi}(a(p, \gamma p)) - 2C\|\psi\|$. □

The rest of this section is devoted to a proof of Proposition 5.5 in which shadows of $\mathcal{F}$ and $\partial\Gamma$ as well as their relationship play important roles.

Shadows in $\mathcal{F}$. Let $q \in X$ and $r > 0$. The shadows of the ball $B(q, r)$ viewed from $p \in X$ and $\xi \in \mathcal{F}$ are respectively defined as

$$O_r(p, q) := \{gk^+ \in \mathcal{F} : k \in K,\ gk\,\mathrm{int}\,A^+o \cap B(q, r) \ne \emptyset\}$$

where $g \in G$ satisfies $p = g(o)$, and

$$O_r(\xi, q) := \{h^+ \in \mathcal{F} : h^- = \xi, ho \in B(q, r)\}.$$

The following two lemmas 5.6 and 5.7 are proved for $G = \mathrm{SL}_n(\mathbb{R})$ in [48]. For $r > 0$, we set $A_r = \{a \in A : \|\log a\| \le r\}$, and $A_r^+ = A_r \cap A^+$.

Lemma 5.6. *If a sequence $q_i \in X$ converges to $\xi \in \mathcal{F}$, then for any $r > 0$, $q \in X$ and $\varepsilon > 0$, we have, for all sufficiently large i,*

$$O_r(\xi, q) \subset O_{r+\varepsilon}(q_i, q).$$

Proof. Since $O_r(\xi, g(o)) = gO_r(g^{-1}\xi, o) = kgO_r(e^+, o)$ for $k \in K$ with $k^+ = g^{-1}\xi$, it suffices to consider the case when $q = o$ and $\xi = e^+$. Since $q_i \to e^+$, we have $q_i = k_i a_i o$ for some $k_i \to e$ in K and $a_i \to \infty$ regularly in A^+. Fixing $\varepsilon > 0$, we need to show $O_r(e^+, o) \subset O_{r+\varepsilon}(k_i a_i o, o)$ for all large $i \gg 1$. As $k_i \to e$, $k_i^{-1} O_r(e^+, o) \subset O_{r+\varepsilon/2}(e^+, o)$ for all sufficiently large i. Therefore it suffices to show that $O_r(e^+, o) \subset O_{r+\varepsilon}(a_i o, o)$ for all large i.

Let $\xi \in O_r(e^+, o)$, that is, $\xi = h^+$ for some $h \in Pw_0 \cap KA_r^+K$. Note that the sequence $a_i^{-1} h w_0^{-1} a_i$ is bounded as $a_i \in A^+$ and $hw_0^{-1} \in P$. If we write $a_i^{-1} h = \tilde{k}_i \tilde{a}_i \tilde{n}_i \in KAN$, then the following gives the KAN^+-decomposition of $a_i^{-1} h w_0^{-1} a_i$:

$$a_i^{-1} h w_0^{-1} a_i = \tilde{k}_i \tilde{a}_i \tilde{n}_i w_0^{-1} a_i = (\tilde{k}_i w_0^{-1})(w_0 \tilde{a}_i w_0^{-1} a_i)(a_i^{-1} w_0 \tilde{n}_i w_0^{-1} a_i) \in KAN^+.$$

As the product map $K \times A \times N^+ \to G$ is a diffeomorphism, there exists $R > 1$ such that $\{w_0 \tilde{a}_i w_0^{-1} a_i \in A : i \in \mathbb{N}\} \subset A_R$. Moreover as the sequence $a_i^{-1} w_0 \tilde{n}_i w_0^{-1} a_i$ must be bounded while $a_i \to \infty$ regularly in A^+, it follows that $\tilde{n}_i \to e$ as $i \to \infty$. We now claim that for all sufficiently large $i \gg 1$,

$$a_i \tilde{k}_i \operatorname{int} A^+ \cap KA_{r+\varepsilon}^+ K \neq \emptyset$$

and hence $\xi = h^+ = a_i \tilde{k}_i^+ \in O_{r+\varepsilon}(a_i o, o)$. Set $b_i = w_0 a_i^{-1} w_0^{-1}$. Then

$$a_i \tilde{k}_i b_i = a_i \tilde{k}_i (w_0 a_i^{-1} w_0^{-1}) = h \tilde{n}_i^{-1} (\tilde{a}_i^{-1} w_0 a_i^{-1} w_0^{-1}).$$

Since $\tilde{n}_i \to e$ as $i \to \infty$ abd $\tilde{a}_i^{-1} w_0 a_i^{-1} w_0^{-1} \in A_R$, we can find $\tilde{b}_i \in b_i A_R$ such that $a_i \tilde{k}_i \tilde{b}_i \to h$ as $i \to \infty$. On the other hand, there exists a neighborhood $\mathcal{O}$ of e in G such that $KA_r^+ K\mathcal{O} \subset KA_{r+\varepsilon}K$ [18, Thm. 2.1]. Since $h \in KA_r^+K$ and $\tilde{b}_i \in \operatorname{int} A^+$ for all large $i \gg 1$, we obtain that $a_i \tilde{k}_i \tilde{b}_i \in KA_{r+\varepsilon}K$ for all sufficiently large i. This finishes the proof. ☐

The following is an analogue of Sullivan's shadow lemma:

Lemma 5.7. *There exists $\kappa > 0$ such that for any $p, q \in X$ and $r > 0$, we have*

$$\sup_{\xi \in O_r(p,q)} \|\beta_\xi(p, q) - \underline{a}(p, q)\| \leq \kappa r.$$

We will prove this lemma using the following:

Lemma 5.8. (1) *There exists $c_1 > 1$ such that for all $r \geq 0$,*

$$KA_r^+ K \subset KA_{c_1 r} N.$$

(2) *There exists $c_2 > 1$ such that for all $g \in G$ and $r \geq 0$, we have*

$$\mu(gKA_r^+ K) \subset \mu(g) + \log A_{c_2 r}.$$

Proof. We use notations introduced in Lemma 3.10. Since $\chi_\alpha, \alpha \in \Pi$, form a dual basis of $\mathfrak{a}^*$, $\|\cdot\|_* := \sum_{\alpha\in\Pi} |\chi_\alpha(\cdot)|$ defines a norm on $\mathfrak{a}$. Let $k \in K$ and $a \in A_r^+$ be arbitrary. Write $ak = k'bn \in KAN$. Let $\alpha \in \Pi$. For $v_\alpha \in V_\alpha^+ \backslash \{0\}$, we have

$$\|\rho_\alpha(ak)v_\alpha\|_\alpha = \|\rho_\alpha(k'bn)v_\alpha\|_\alpha = \|\rho_\alpha(b)v_\alpha\|_\alpha = e^{\chi_\alpha(\log b)}\|v_\alpha\|_\alpha.$$

On the other hand, we have

$$e^{-\chi_\alpha(\log a)}\|v_\alpha\|_\alpha \le \|\rho_\alpha(ak)v_\alpha\|_\alpha \le e^{\chi_\alpha(\log a)}\|v_\alpha\|_\alpha.$$

Hence $|\chi_\alpha(\log b)| \le |\chi_\alpha(\log a)|$ for all $\alpha \in \Pi$; so $\|\log b\|_* \le \|\log a\|_*$. Since $\|\cdot\|_*$ and $\|\cdot\|$ are comparable, the first claim follows.

Note that $e^{\chi_\alpha(\mu(g))} = \|\rho_\alpha(g)\|_\alpha$ for $g \in G$. For all $k \in K$ and $a \in A^+$,

$$\|\rho_\alpha(gka)\|_\alpha \le \|\rho_\alpha(g)\|_\alpha \|\rho_\alpha(k)\|_\alpha \|\rho_\alpha(a)\|_\alpha = e^{\chi_\alpha(\log a)}\|\rho_\alpha(g)\|_\alpha,$$

and similarly,

$$\|\rho_\alpha(g)\|_\alpha \le \|\rho_\alpha(gka)\|_\alpha \|\rho_\alpha(k^{-1})\|_\alpha \|\rho_\alpha(a^{-1})\|_\alpha = e^{\chi_\alpha(\log a)}\|\rho_\alpha(gka)\|_\alpha.$$

It follows that $-\chi_\alpha(\log a) \le \chi_\alpha(\mu(gka) - \mu(g)) \le \chi_\alpha(\log a)$. This means that $\|\mu(gka) - \mu(g)\|_* \le \|\log a\|_*$, finishing the proof. $\square$

Proof of Lemma 5.7. It suffices to prove the claim for $p = o$ and $q = a^{-1}o$ for $a \in A^+$. Let $\xi = k^+ \in O_r(o, a^{-1}o)$ for $k \in K$. Then there exists $b \in \operatorname{int} A^+$ such that $d(kbo, a^{-1}o) < r$. Hence $akb \in KA_r^+K$. Now note that $ak \in Ke^{-\beta_\xi(o,a^{-1}o)}N$ by the definition of $\beta_\xi(o, a^{-1}o)$ and hence

$$akb \in Kbe^{-\beta_\xi(o,a^{-1}o)}N \cap KA_r^+K.$$

By Lemma 5.8(1), $be^{-\beta_\xi(o,a^{-1}o)} \in A_{c_1 r}$. On the other hand, $b \in Ka^{-1}KA_r^+K$, and hence $\log b \in \mu(a^{-1}KA_r^+K)$. By Lemma 5.8(2), $b \in a^{-1}A_{c_2 r}$. Since $\underline{a}(o, a^{-1}o) = \log(a^{-1})$, the lemma is now proved.

Lemma 5.7 implies Theorem 5.3 for those $\xi \in O_r(\gamma p, p)$. In order to control the value of $\beta_\xi(\gamma p, p)$ when $\xi \notin O_r(\gamma p, p)$, we use the Anosov property of Γ. Let us recall some basic terminologies for hyperbolic groups for which we refer to [8] and [22].

Discrete geodesics. Let Γ be a finitely generated word hyperbolic group. We fix a finite symmetric generating subset S of Γ once and for all. Let $|\cdot| : \Gamma \to \mathbb{N} \cup \{0\}$ denote the word length associated to S. We denote by d_{w} the associated left-invariant word metric, that is, $d_{\mathsf{w}}(\gamma_1, \gamma_2) := |\gamma_1^{-1}\gamma_2|$ for $\gamma_1, \gamma_2 \in \Gamma$.

A finite sequence $(\gamma_0, \cdots, \gamma_n)$ of elements of Γ will be called a finite path if $\gamma_i^{-1}\gamma_{i+1} \in S$ for all i. Such a path will be called a geodesic segment if $|\gamma_0^{-1}\gamma_n| = n$. Infinite and bi-infinite paths can be defined analogously. They will be called geodesic rays and geodesic lines, respectively, if all of their finite subpaths are geodesic segments.

Let $\partial\Gamma$ denote the Gromov boundary of Γ, that is, $\partial\Gamma$ is the set of equivalence classes of geodesic rays, where two rays are equivalent to each other if and only if their Hausdorff distance is finite. For a geodesic ray $(\gamma_0, \gamma_1, \cdots)$, we use the notation $[\gamma_0, \gamma_1, \cdots]$ for its equivalence class in $\partial\Gamma$.

Let $(\cdot|\cdot)$ denote the Gromov product in the hyperbolic space Γ based at $e \in \Gamma$: $(\gamma_1|\gamma_2) := \frac{1}{2}\left(d_{\mathsf{w}}(\gamma_1, e) + d_{\mathsf{w}}(\gamma_2, e) - d_{\mathsf{w}}(\gamma_1, \gamma_2)\right)$. This extends to $\partial\Gamma$: for $x, y \in \partial\Gamma$, $(x|y) := \sup \liminf_{i,j\to\infty}(\gamma_i|\gamma'_j)$ where the supremum is taken over all sequences γ_i and γ'_j such that $x = \lim \gamma_i$ and $y = \lim \gamma'_j$. The union $\Gamma \cup \partial\Gamma$ is a compact space with the topology given as follows: a sequence $\gamma_i \in \Gamma$ converges to $x \in \partial\Gamma$ if and only if $\lim_{i\to\infty}(\gamma_i|v_i) = \infty$ for any geodesic ray $(e, v_1, v_2, \cdots)$ representing x.

For any $x, y \in \Gamma \cup \partial\Gamma$, there exists a discrete geodesic starting from x and ending at y, which may not be unique. By $[x, y]$, we mean one of those geodesics and by $[x, y)$ we mean $[x, y] - \{y\}$.

A geodesic triangle is a union of three geodesics, pairwise sharing a common endpoint in $\Gamma \cup \partial\Gamma$. Since Γ is hyperbolic, there exists $\delta = \delta(\Gamma, S) > 0$ such that for any geodesic triangle Δ, we can find a point on each edge of Δ so that the set of these triples has diameter less than δ.

Shadows in $\partial\Gamma$. For $R > 0$ and $\gamma_1, \gamma_2 \in \Gamma$, the shadow of the ball $B_R(\gamma_2)$ viewed from γ_1 is given by

$$O_R(\gamma_1, \gamma_2) = \{x \in \partial\Gamma : \ [\gamma_1, x] \cap B_R(\gamma_2) \neq \emptyset \text{ for some geodesic ray } [\gamma_1, x]\}.$$

Shadows satisfy the equivariance property: for any $\gamma, \gamma_1, \gamma_2 \in \Gamma$ and $R > 0$,

$$\gamma O_R(\gamma_1, \gamma_2) = O_R(\gamma\gamma_1, \gamma\gamma_2). \tag{5.9}$$

Lemma 5.10. *There exist $R_0 > 1$ and $N_0 > 0$ such that the following holds: if $\gamma_1, \gamma_2 \in \Gamma$ with $|\gamma_1|, |\gamma_2| \geq N_0$ satisfies $|\gamma_1\gamma_2| = |\gamma_1| + |\gamma_2|$, then for all $R \geq R_0$,*

$$O_R(\gamma_1\gamma_2, e) \cap O_R(\gamma_1\gamma_2, \gamma_1) \cap O_R(\gamma_1, e) \neq \emptyset.$$

Proof. Since $|\gamma_1\gamma_2| = |\gamma_1| + |\gamma_2|$, there exists a geodesic segment $[\gamma_1\gamma_2, e]$ passing through γ_1, say $\alpha = (\gamma_1\gamma_2, \cdots, \gamma_1, \cdots, e)$. Since Γ is word hyperbolic, there exists $C > 0$ such that α lies in the C-neighborhood of some geodesic line, say $(\cdots, u_{-1}, u_0, u_1, \cdots)$. Set $N_0 := 4C$. Choose u_m, u_n, and u_ℓ to be elements closest to $\gamma_1\gamma_2$, γ_1, and e, respectively.

We claim that $|m - \ell| \geq \max(|m - n|, |n - \ell|)$. By the triangle inequality,

$$|n - \ell| = d_{\mathsf{w}}(u_n, u_\ell) \leq d_{\mathsf{w}}(\gamma_1, e) + 2C = |\gamma_1| + 2C;$$
$$|m - n| = d_{\mathsf{w}}(u_m, u_n) \leq d_{\mathsf{w}}(\gamma_1\gamma_2, \gamma_1) + 2C = |\gamma_2| + 2C.$$

Since $|\gamma_1\gamma_2| = |\gamma_1| + |\gamma_2|$ and $|\gamma_1\gamma_2| \leq d_{\mathsf{w}}(u_m, u_\ell) + 2C = |m - \ell| + 2C$, it follows that

$$\begin{aligned} |\gamma_2| - 2C &\geq \max(|\gamma_1|, |\gamma_2|) - 2C + N_0 \\ &= \max(|\gamma_1|, |\gamma_2|) + 2C \\ &\geq \max(|n - \ell|, |m - n|). \end{aligned}$$

This proves the claim.

Now possibly after flipping the geodesic, we may assume that $m \le \ell$. Then the claim implies that $\ell - m = |m-n| + |n-\ell|$ and hence $m \le n \le \ell$. Set $x := [u_0, u_1, u_2, \cdots] \in \partial\Gamma$. Choose geodesic rays $[\gamma_1\gamma_2, x)$ and $[\gamma_1, x)$. Since the Hausdorff distance between $[\gamma_1\gamma_2, x)$ and the ray $(u_m, u_{m+1}, \cdots)$ is at most $d_{\mathsf{w}}(\gamma_1\gamma_2, u_m) + \delta \le C + \delta$, it follows that there exist $v_1, v_2 \in \Gamma$ lying on $[\gamma_1\gamma_2, x]$ such that $d_{\mathsf{w}}(u_n, v_1) < C + \delta$ and $d_{\mathsf{w}}(u_\ell, v_2) < C + \delta$. Since the Hausdorff distance between $[\gamma_1, x)$ and the ray $(u_n, u_{n+1}, \cdots)$ is at most $d_{\mathsf{w}}(\gamma_1, u_n) + \delta < C + \delta$, there exists $v_3 \in \Gamma$ lying on $[\gamma_1, x)$ such that $d_{\mathsf{w}}(u_\ell, v_3) < C + \delta$. These altogether imply that

$$x \in O_{2C+\delta}(\gamma_1\gamma_2, e) \cap O_{2C+\delta}(\gamma_1\gamma_2, \gamma_1) \cap O_{2C+\delta}(\gamma_1, e).$$

$\square$

In the rest of this section, we assume that Γ is an Anosov subgroup of G. The following Morse property of Kapovich-Leeb-Porti [23, Prop. 5.16] says that a discrete geodesic line (resp. ray) of Γ is contained in a uniform neighborhood of some A-orbit (resp. A^+-orbit) in X.

Proposition 5.11 (Morse property)**.** *For any Anosov subgroup $\Gamma < G$, there exists $R_1 > 0$ such that*

(1) *If $(\cdots, \gamma_{-1}, \gamma_0, \gamma_1, \cdots)$ is a geodesic line in (Γ, d_{w}), then*
$$\sup_{k\in\mathbb{Z}} d(\gamma_k o, gAo) \le R_1$$
for any $g \in G$ such that $g^+ = \zeta([\gamma_0, \gamma_1, \cdots])$, $g^- = \zeta([\gamma_0, \gamma_{-1}, \cdots])$.

(2) *If $(\gamma_0, \gamma_1, \cdots)$ is a geodesic ray in (Γ, d_{w}), then*
$$\sup_{k\in\mathbb{N}} d(\gamma_k o, gA^+o) \le R_1$$
where $g \in \gamma_0 K$ is the unique element satisfying $g^+ = \zeta([\gamma_0, \gamma_1, \cdots])$.

Using this proposition, we will show that shadows in the Gromov boundary $\partial\Gamma$ are mapped to shadows in the Furstenberg boundary $\mathcal{F}$ by the limit map $\zeta : \partial\Gamma \to \Lambda$ (Proposition 5.13). We will need the following lemma:

Lemma 5.12. *There exists $C > 0$ such that for all $\gamma \in \Gamma$, $\|\mu(\gamma)\| \le C|\gamma|$. In particular, $d(o, \gamma o) \le C d_{\mathsf{w}}(e, \gamma)$.*

Proof. We use notations from Lemma 3.10. Consider the norm $\|\cdot\|_* := \sum_{\alpha\in\Pi} |\chi_\alpha(\cdot)|$ on $\mathfrak{a}$. Let $\gamma \in \Gamma$ be arbitrary, and write $\gamma = s_1 \cdots s_\ell$ with $s_i \in S$ and $\ell = |\gamma|$. Since $\chi_\alpha(\mu(g)) = \log\|\rho_\alpha(g)\|_\alpha$ for all $g \in G$ and $\|\rho_\alpha(s_1\cdots s_\ell)\|_\alpha \le \|\rho_\alpha(s_1)\|_\alpha \cdots \|\rho_\alpha(s_\ell)\|_\alpha$, it follows that for each $\alpha \in \Pi$,

$$\chi_\alpha(\mu(s_1\cdots s_\ell)) \le \chi_\alpha(\mu(s_1)) + \cdots + \chi_\alpha(\mu(s_\ell)).$$

Noting that χ_α is positive on $\mathfrak{a}^+$, we have

$$\begin{aligned}\|\mu(\gamma)\|_* &= \sum_{\alpha\in\Pi} |\chi_\alpha(\mu(\gamma))| = \sum_{\alpha\in\Pi} \chi_\alpha(\mu(\gamma)) \\ &\le \sum_{\alpha\in\Pi} \big(\chi_\alpha(\mu(s_1)) + \cdots + \chi_\alpha(\mu(s_\ell))\big) \le C|\gamma|\end{aligned}$$

where $C := \max\left\{\sum_{\alpha\in\Pi}\chi_\alpha(\mu(s)) : s \in S\right\}$. Since $\|\cdot\|$ and $\|\cdot\|_*$ are comparable, this finishes the proof. $\square$

Proposition 5.13 (Shadows go to shadows)**.** *There exists $c > 0$ such that for all $R > 1$ and $\gamma, \gamma' \in \Gamma$,*

$$\zeta(O_R(\gamma', \gamma)) \subset O_{cR}(\gamma' o, \gamma o).$$

Proof. By (5.9), it suffices to consider the case $\gamma' = e$. Let $x \in O_R(e, \gamma)$. By the definition of $O_R(e, \gamma)$, there exists a geodesic ray $(\gamma_0' = e, \gamma_1', \gamma_2', \cdots)$ representing x such that $d_{\mathsf{w}}(\gamma_m', \gamma) < R$ for some $m \in \mathbb{N}$. Let $R_1 > 0$ be the constant from Proposition 5.11, and $k \in K$ be an element such that $k^+ = \zeta([e, \gamma_1', \gamma_2', \cdots])$. Then by Proposition 5.11(2), there exists $a \in A^+$ such that $d(\gamma_m' o, kao) \le R_1$. By Lemma 5.12, we have

$$d(\gamma o, \gamma_m' o) = \|\mu(\gamma^{-1}\gamma_m')\| < C d_{\mathsf{w}}(\gamma, \gamma_m') < CR.$$

Therefore

$$d(\gamma o, kao) \le d(\gamma o, \gamma_m' o) + d(\gamma_m' o, kao) \le CR + R_1.$$

This implies that $\zeta(x) \in O_{CR+R_1}(o, \gamma o)$. Since $R > 1$, the conclusion follows by setting $c := C + R_1$. $\square$

Corollary 5.14. *There exists $R_2 > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma$ with $|\gamma_1\gamma_2| = |\gamma_1| + |\gamma_2|$, we have*

$$\|\mu(\gamma_1\gamma_2) - \mu(\gamma_1) - \mu(\gamma_2)\| \le R_2.$$

Proof. Let N_0 and R_0 be given by Lemma 5.10. If one of $|\gamma_1|, |\gamma_2|$ is less than N_0, then the claim holds by Lemma 2.3. Now assume that $|\gamma_1|, |\gamma_2| \ge N_0$. Then by Lemma 5.10 and Proposition 5.13, we can choose

$$\xi \in O_{cR_0}(\gamma_1\gamma_2 o, o) \cap O_{cR_0}(\gamma_1\gamma_2 o, \gamma_1 o) \cap O_{cR_0}(\gamma_1 o, o)$$

where c is as in Proposition 5.13. By Lemma 5.7 and the cocycle identity

$$\beta_\xi(\gamma_1\gamma_2 o, o) = \beta_\xi(\gamma_1\gamma_2 o, \gamma_1 o) + \beta_\xi(\gamma_1 o, o),$$

we have

$$\|\underline{a}(\gamma_1\gamma_2 o, o) - \underline{a}(\gamma_1 o, o) - \underline{a}(\gamma_2 o, o)\| \le 3\kappa c R_0.$$

Since $\underline{a}(go, o) = \mathrm{i}\,\mu(g)$ for all $g \in G$ and i preserves $\|\cdot\|$,

$$\|\mu(\gamma_1\gamma_2) - \mu(\gamma_1) - \mu(\gamma_2)\| \le 3\kappa c R_0.$$

$\square$

Proof of Proposition 5.5: We may assume that $p = o$ by Lemma 2.3. Let $\gamma \in \Gamma$ and $\xi \in \Lambda$ be arbitrary. If $\gamma = \gamma_1\gamma_2$, we have

$$\beta_\xi(\gamma o, o) = \beta_\xi(\gamma o, \gamma_1 o) - \beta_\xi(o, \gamma_1 o).$$

We claim that we can find $\gamma_1, \gamma_2 \in \Gamma$ so that $\gamma = \gamma_1\gamma_2$, $|\gamma| = |\gamma_1| + |\gamma_2|$, and

$$\xi \in O_{c(\delta+1)}(\gamma o, \gamma_1 o) \cap O_{c(\delta+1)}(o, \gamma_1 o) \tag{5.15}$$

where $c > 0$ is as in Proposition 5.13.

If $\xi \in O_{c(\delta+1)}(o, \gamma o)$, then we may simply set $\gamma_1 = \gamma$ and $\gamma_2 = e$. In general, we find γ_1 as follows. Consider a geodesic triangle Δ whose vertices are $e, \gamma \in \Gamma$, and $\zeta^{-1}(\xi) \in \partial\Gamma$. Since Γ is hyperbolic, we can find three points on Δ, one on each edge, whose diameter is less than δ (See Figure **??**). Let $\gamma_1 \in \Gamma$ be the point on the geodesic segment joining e and γ, and set $\gamma_2 := \gamma_1^{-1}\gamma$. We then have $|\gamma| = |\gamma_1| + |\gamma_2|$, and $\zeta^{-1}(\xi) \in O_\delta(\gamma, \gamma_1) \cap O_\delta(e, \gamma_1)$. Now the claim follows from Proposition 5.13.

Therefore, by Lemma 5.7,

$$\max(\|\beta_\xi(\gamma o, \gamma_1 o) - \mu(\gamma_2^{-1})\|, \|\beta_\xi(o, \gamma_1 o) - \mu(\gamma_1)\|) \leq \kappa c(\delta + 1)$$

and hence

$$\|\beta_\xi(\gamma o, o) + \mu(\gamma_1) - \mu(\gamma_2^{-1})\| \leq 2\kappa c(\delta + 1).$$

Since $|\gamma| = |\gamma_1| + |\gamma_2|$ and S is symmetric, we have $|\gamma^{-1}| = |\gamma_1^{-1}| + |\gamma_2^{-1}|$. As $\underline{a}(\gamma o, o) = \mu(\gamma^{-1})$, we have, by Corollary 5.14,

$$\|\underline{a}(\gamma o, o) - \mu(\gamma_1^{-1}) - \mu(\gamma_2^{-1})\| \leq R_2.$$

Hence it suffices to set $C := \max(2\kappa c(\delta + 1), R_2)$. $\square$

6. Virtual visual metrics via ψ-Gromov product

In this section, we let $\Gamma < G$ be an Anosov subgroup, and fix $\psi \in D_\Gamma^\star$. The main aim here is to show that exponentiating the following ψ-Gromov product defines a virtual visual metric on Λ up to a small power.

Definition 6.1. The ψ-Gromov product based at o is a function $\mathcal{F}^{(2)} \to \mathbb{R}$ defined as follows: for any $(\xi_1, \xi_2) \in \mathcal{F}^{(2)}$,

$$[\xi_1, \xi_2]_{\psi,o} := \psi(\mathcal{G}(\xi_1, \xi_2))$$

where $\mathcal{G}$ is the $\mathfrak{a}$-valued Gromov product defined in Definition 3.8. For $p = g(o) \in X$, we set

$$[\xi_1, \xi_2]_{\psi,p} := [g^{-1}\xi_1, g^{-1}\xi_2]_{\psi,o}.$$

For simplicity, we set $[\xi_1, \xi_2]_p := [\xi_1, \xi_2]_{\psi,p}$.

Define $d_p = d_{\psi,p} : \mathcal{F}^{(2)} \to \mathbb{R}_{\geq 0}$ by

$$d_p(\xi_1, \xi_2) = e^{-[\xi_1, \xi_2]_p}. \tag{6.2}$$

It follows from (3.9) that for all $g \in G$, $p \in X$, and $(\xi_1, \xi_2) \in \mathcal{F}^{(2)}$, we have

$$d_{gp}(\xi_1, \xi_2) = e^{-\psi(\beta_{\xi_1}(gp,p) + \mathrm{i}\,\beta_{\xi_2}(gp,p))} d_p(\xi_1, \xi_2) = d_p(g^{-1}\xi_1, g^{-1}\xi_2). \tag{6.3}$$

We set $[\xi, \xi]_p = +\infty$ and $d_p(\xi, \xi) = 0$ for all $\xi \in \mathcal{F}$. By the antipodal property (4.2), $[\cdot, \cdot]_p$ and d_p are defined on all of $\Lambda \times \Lambda$. The following is the main theorem of this section:

Theorem 6.4. *Fix $p \in X$. For all sufficiently small $\varepsilon > 0$, there exist a metric $d_\varepsilon = d_\varepsilon(p)$ on Λ and a constant $C_\varepsilon = C_\varepsilon(p) > 0$ such that for all $\xi_1, \xi_2 \in \Lambda$,*

$${C_\varepsilon}^{-1} d_{\psi,p}(\xi_1, \xi_2)^\varepsilon \leq d_\varepsilon(\xi_1, \xi_2) \leq C_\varepsilon d_{\psi,p}(\xi_1, \xi_2)^\varepsilon.$$

This is an analogue of [8, Part III, Prop. 3.21] for Gromov hyperbolic spaces.

Weak ultrametric inequality. A well-known construction [17, Section 7.3] shows the existence of a metric in Theorem 6.4, provided there exists $C > 0$ such that for all $\xi_1, \xi_2, \xi_3 \in \Lambda$, we have

(1) (weak symmetry) $d_p(\xi_1, \xi_2) \leq e^C d_p(\xi_2, \xi_1)$;

(2) (weak ultrametric inequality) $d_p(\xi_1, \xi_3) \leq e^C \max(d_p(\xi_1, \xi_2), d_p(\xi_2, \xi_3))$.

Hence Theorem 6.4 follows from the following proposition:

Proposition 6.5. *There exists $C = C(p) > 0$ such that for all $\xi_1, \xi_2, \xi_3 \in \Lambda$, we have*

$$[\xi_1, \xi_2]_p \geq [\xi_2, \xi_1]_p - C;$$
$$[\xi_1, \xi_3]_p \geq \min([\xi_1, \xi_2]_p, [\xi_2, \xi_3]_p) - C.$$

In the case of $X = \mathbb{H}^2$, the classical Gromov product satisfies that there exists a uniform constant $C > 0$ such that for any $x, y \in \partial\mathbb{H}^2$,

$$|\mathcal{G}(x, y) - 2d(o, z)| \leq C$$

where z is the unique projection of o to the geodesic connecting x and y. In the following lemma 6.6, we establish the analogous property for $\mathfrak{a}$-valued Gromov products on $\Lambda \times \Lambda$ using the Morse property of Anosov groups.

For $\gamma \in \Gamma$ and any geodesic segment α in Γ, we define the set of projections of γ to α by

$$\pi_\alpha(\gamma) := \{\gamma' \in \alpha : d_{\mathsf{w}}(\gamma, \gamma') = d_{\mathsf{w}}(\gamma, \alpha)\}.$$

Since Γ is hyperbolic, the diameter of $\pi_\alpha(\gamma)$ is less than 4δ.

Lemma 6.6. *There exists $C_1 > 0$ such that for any $x \neq y$ in $\partial\Gamma$ and $\gamma \in \pi_{[x,y]}(e)$, we have*

$$\|\mathcal{G}(\zeta(x), \zeta(y)) - (\mu(\gamma) + \mathrm{i}\,\mu(\gamma))\| \leq C_1.$$

In particular, $\mathcal{G}$ is almost symmetric on Λ: for any $\xi_1 \neq \xi_2 \in \Lambda$,

$$\|\mathcal{G}(\xi_1, \xi_2) - \mathcal{G}(\xi_2, \xi_1)\| \leq 2C_1.$$

Proof. Let $\alpha := (u_0 = e, u_1, u_2, \cdots)$ and $\alpha' := (v_0 = e, v_1, v_2, \cdots)$ be geodesic representatives of x and y, respectively. Let $\gamma \in \pi_{[x,y]}(e)$ be arbitrary, and $f, g, h \in G$ be elements satisfying the following:

- $f(o) = o$ and $f^+ = \zeta(x)$;
- $g(o) = o$ and $g^+ = \zeta(y)$;
- $h^+ = \zeta(x)$ and $h^- = \zeta(y)$.

Applying Proposition 5.11(1) to the geodesic line $[x, y]$, we have

$$d(ho, \gamma o) < R_1$$

after replacing h with some element of hA. Hence by Lemma 2.3, there exists $C' = C'(R_1) > 0$ such that

$$\|\mu(h) - \mu(\gamma)\| \leq C'. \tag{6.7}$$

Noting

$$\mathcal{G}(\zeta(x),\zeta(y))=\mathcal{G}(h^+,h^-)=\beta_{h^+}(o,ho)+\mathrm{i}\,\beta_{h^-}(o,ho),$$

it is now sufficient to show that for some uniform constant $C_1>0$,

$$\|\beta_{h^+}(o,ho)-\mu(h)\|\le C_1 \quad \text{and} \quad \|\beta_{h^-}(o,ho)-\mu(h)\|\le C_1.$$

By Lemma 5.7, this claim follows if we show

$$h^+,h^-\in O_R(o,ho) \tag{6.8}$$

for some uniform constant $R>0$.

Since Γ is hyperbolic, the diameter of the set $\pi_{\alpha'}(x)\cup\pi_\alpha(y)\cup\pi_{[x,y]}(e)$ is at most $C\delta$ for some uniform constant $C>1$. In particular, we can find $k,\ell\in\mathbb{N}$ such that the set $\{u_k,v_\ell,\gamma\}$ has diameter less than $C\delta$. Applying Proposition 5.11(2) to the geodesic ray α, we find $a_1\in A^+$ such that

$$d(fa_1o,u_ko)<R_1.$$

Since $d_{\mathsf{w}}(u_k,\gamma)=|u_k^{-1}\gamma|\le C\delta$, we have

$$d(u_ko,\gamma o)=\|\mu(u_k^{-1}\gamma)\|\le\sup\{\|\mu(\gamma')\|:|\gamma'|\le C\delta\}.$$

Therefore

$$\begin{aligned}d(fa_1o,ho)&\le d(fa_1o,u_ko)+d(u_ko,\gamma o)+d(\gamma o,ho)\\&\le 2R_1+\sup\{\|\mu(\gamma')\|:|\gamma'|\le C\delta\}.\end{aligned}$$

Setting $R:=2R_1+\sup\{\|\mu(\gamma')\|:|\gamma'|\le C\delta\}$, it follows that $h^+=f^+\in O_R(o,ho)$. Similar argument shows that $h^-=g^+\in O_R(o,ho)$. This proves (6.8). ∎

Lemma 6.9. *For any compact subset $C\subset X$, the set $\{\beta_\xi(p,o):\xi\in\mathcal{F},p\in C\}$ is bounded.*

Proof. This follows from Lemma 5.1 by setting $\psi=\sum_{\alpha\in\Pi}\varpi_\alpha$. ∎

Proof of Proposition 6.5. Observe that the identity (6.3) gives that for any $\xi_1\ne\xi_2\in\Lambda$,

$$[\xi_1,\xi_2]_p-[\xi_1,\xi_2]_o=\psi(\beta_{\xi_1}(p,o)+\mathrm{i}\,\beta_{\xi_2}(p,o)).$$

Now Lemma 6.9 shows the existence of $C=C(p,\psi)>0$ such that $|[\xi_1,\xi_2]_p-[\xi_1,\xi_2]_o|\le C$. Therefore it suffices to show the claim for $p=o$. The first inequality is an immediate consequence of Lemma 6.6 with $C>2C_1\|\psi\|$.

To show the second inequality, let $C_1>0$ be a constant from Lemma 6.6 so that we have

$$[\xi_1,\xi_3]_o\ge\psi(\mu(\gamma_2)+\mathrm{i}\,\mu(\gamma_2))-C_1\|\psi\|. \tag{6.10}$$

Set $x_i:=\zeta^{-1}(\xi_i)\in\partial\Gamma$ for $i=1,2,3$. For each i, we fix a geodesic line $[x_i,x_{i+1}]$ joining x_i and x_{i+1}, and choose $\gamma_{i+2}\in\pi_{[x_i,x_{i+1}]}(e)$, where all the indices are to be interpreted mod 3. By the hyperbolicity of Γ, for some uniform constant $C>0$, there exists $1\le i\le 3$ such that $d_{\mathsf{w}}(\gamma_i,\gamma_{i+1})<C\delta$ and for some $\gamma'\in[e,\gamma_{i+2}]$, the diameter of $\{\gamma',\gamma_i,\gamma_{i+1}\}$ is at most $C\delta$.

We first consider the case when $i = 1$. Since

$$d(\gamma_1 o, \gamma_2 o) \le d_{\mathsf{w}}(\gamma_1, \gamma_2) \max_{s\in S} d(o, s) < C\delta \max_{s\in S} d(o, s),$$

it follows from Lemma 2.3 that for some uniform $C_2 > 0$,

$$\|\mu(\gamma_1) - \mu(\gamma_2)\| \le C_2.$$

In view of (6.10), we now obtain

$$\begin{aligned} [\xi_1, \xi_3]_o &\ge \psi(\mu(\gamma_1) + \mathrm{i}\,\mu(\gamma_1)) - C_1\|\psi\| - 2C_2 \\ &\ge [\xi_2, \xi_3]_o - 2C_1\|\psi\| - 2C_2 \text{ by Lemma 6.6} \\ &\ge \min([\xi_1, \xi_2]_o, [\xi_2, \xi_3]_o) - 2C_1\|\psi\| - 2C_2. \end{aligned}$$

The case $i = 2$ can be handled similarly by interchanging the roles of γ_2 and γ_3. Finally in the case when $i = 3$, let R_2 be as in Corollary 5.14. Since $(e, \cdots, \gamma', \cdots, \gamma_2)$ is a geodesic, we have by Corollary 5.14 that

$$\|\mu(\gamma_2) - \mu(\gamma') - \mu(\gamma'^{-1}\gamma_2)\| \le R_2.$$

By (6.10) and the fact $\psi(\mu((\gamma'^{-1}\gamma_2)^{\pm 1})) \ge 0$, we deduce

$$\begin{aligned} [\xi_1, \xi_3]_o &\ge \psi(\mu(\gamma') + \mathrm{i}\,\mu(\gamma')) - C_1\|\psi\| - 2R_2\|\psi\| \\ &\ge \psi(\mu(\gamma_1) + \mathrm{i}\,\mu(\gamma_1)) - (C_1 + 2C_2 + 2R_2)\|\psi\|, \end{aligned}$$

as the diameter of $\{\gamma', \gamma_1, \gamma_3\}$ is less than δ. The rest of the proof is similar to the case $i = 1$. $\square$

Covering lemma. Using Theorem 6.4, we obtain:

Lemma 6.11 (Triangle inequality)**.** *There exists $N = N(\psi, p) \ge 1$ such that for any $\xi_1, \xi_2, \xi_3 \in \Lambda$,*

$$d_p(\xi_1, \xi_3) \le N\big(d_p(\xi_1, \xi_2) + d_p(\xi_2, \xi_3)\big).$$

In particular, $d_p(\xi_1, \xi_2) \le N d_p(\xi_2, \xi_1)$. Moreover, $N(\psi, p)$ can be taken uniformly for all p in a fixed compact subset of X.

Proof. Choose $\varepsilon > 0$ sufficiently small so that Theorem 6.4 holds, and set $d := d_\varepsilon$, $C := C_\varepsilon$. We then have

$$d_p(\xi_1, \xi_3)^\varepsilon \le C d(\xi_1, \xi_3) \le C(d(\xi_1, \xi_2) + d(\xi_2, \xi_3)) \le C^2(d_p(\xi_1, \xi_2)^\varepsilon + d_p(\xi_2, \xi_3)^\varepsilon).$$

Since $(a^\varepsilon + b^\varepsilon)^{1/\varepsilon} \le \alpha(a + b)$ for all $a, b \ge 0$ for some uniform constant $\alpha = \alpha(\varepsilon) > 0$, it suffices to take the $1/\varepsilon$ power in each side of the above. Now the second part follows from (6.3) and Lemma 6.9. $\square$

For $\xi \in \Lambda$ and $r > 0$, set

$$\mathbb{B}_p(\xi, r) := \{\eta \in \Lambda : d_{\psi,p}(\xi, \eta) < r\}.$$

Lemma 6.12 (Covering lemma)**.** *There exists $N_0(\psi, p) \ge 1$ satisfying the following: for any finite collection $\mathbb{B}_p(\xi_1, r_1), \cdots, \mathbb{B}_p(\xi_n, r_n)$ with $\xi_i \in \Lambda$ and $r_i > 0$, there exists a disjoint subcollection $\mathbb{B}_p(\xi_{i_1}, r_{i_1}), \cdots, \mathbb{B}_p(\xi_{i_\ell}, r_{i_\ell})$ such that*

$$\mathbb{B}_p(\xi_1, r_1) \cup \cdots \cup \mathbb{B}_p(\xi_n, r_n) \subset \mathbb{B}_p(\xi_{i_1}, 3N_0 r_{i_1}) \cup \cdots \cup \mathbb{B}_p(\xi_{i_\ell}, 3N_0 r_{i_\ell}).$$

Moreover, $N_0(\psi, p)$ can be taken uniformly for all p in a fixed compact subset of X.

Proof. Let $N = N(\psi, p)$ be as given by Lemma 6.11. For simplicity, set $B_i := \mathbb{B}_p(\xi_i, r_i)$. We may asume $r_1 \geq \cdots \geq r_n$ without loss of generality and define inductively

$$i_1 = 1,\, i_{j+1} = \min\{i > i_j : B_i \cap (B_{i_1} \cup \cdots \cup B_{i_j}) = \emptyset\},$$

as long as possible, to obtain a maximal disjoint subcollection $\{B_{i_1}, \cdots, B_{i_\ell}\}$. Let $\xi \in B_j$ for some $1 \leq j \leq n$. Then there exists $1 \leq k \leq \ell$ such that $B_j \cap B_{i_k} \neq \emptyset$ and $r_{i_k} \geq r_j$. Choose $\eta \in B_j \cap B_{i_k}$. Then by Lemma 6.11, we have $d_p(\eta, \xi) \leq N(d_p(\eta, \xi_j) + d_p(\xi_j, \xi)) < 2N^2 r_j \leq 2N^2 r_{i_k}$ and $d_p(\xi_{i_k}, \eta) < r_{i_k}$. Hence

$$d_p(\xi_{i_k}, \xi) \leq N(d_p(\xi_{i_k}, \eta) + d_p(\eta, \xi)) < 3N^3 r_{i_k}.$$

Hence it suffices to set $N_0 := N^3$. □

Comparing Gromov products. Although we will not be using it in the rest of the paper, we record the following theorem which is of independent interest:

Theorem 6.13. *For any $\psi \in D_\Gamma^\star$, there exist $c_1 = c_1(\psi) \geq 1, c_2 = c_2(\psi) > 0$ such that for all $x \neq y \in \partial\Gamma$,*

$$c_1^{-1}(x|y) - c_2 \leq \psi(\mathcal{G}(\zeta(x), \zeta(y))) \leq c_1(x|y) + c_2.$$

Note that if $\gamma \in \pi_{[x,y]}(e)$ for $x \neq y$ in $\partial\Gamma$, then $|(x|y) - |\gamma|| \leq C$ for some uniform constant $C > 0$ (cf. [8]). Given this fact, Theorem 6.13 follows immediately from Lemma 6.6 and the following lemma:

Lemma 6.14. *For any $\psi \in D_\Gamma^\star$, there exist constants $C_\psi, c_\psi > 0$ such that for all $\gamma \in \Gamma$,*

$$C_\psi^{-1}|\gamma| - c_\psi \leq \psi(\mu(\gamma)) \leq C_\psi|\gamma|.$$

Proof. Since $\psi > 0$ on $\mathcal{L}_\Gamma$, we have

$$0 < d := \min_{\|u\|=1, u\in\mathcal{L}_\Gamma} \psi(u) \leq D := \max_{\|u\|=1, u\in\mathcal{L}_\Gamma} \psi(u) < \infty.$$

Hence $d\|\mu(\gamma)\| \leq \psi(\mu(\gamma)) \leq D\|\mu(\gamma)\|$ for all $\gamma \in \Gamma$. So the upper bound follows from Lemma 5.12, and the lower bound follows from the well-known property of Anosov groups that for some uniform $C > 0$, $C^{-1}|\gamma| - C \leq \|\mu(\gamma)\|$ for all $\gamma \in \Gamma$ [20]. □

7. Conical points, divergence type and classification of PS measures

In this section, we show that for Anosov groups, the space of all PS-measures on Λ is homeomorphic to $D_\Gamma^\star$.

Conical limit points. For a discrete subgroup $\Gamma < G$ and $x \in \Gamma\backslash G$, we mean by $\limsup xA^+M$ the set of all limit points $\lim\limits_{i\to\infty} xa_im_i$ where $a_i \to \infty$ in A^+ and $m_i \in M$.

Definition 7.1 (Conical limit points)**.** We call $\xi \in \mathcal{F}$ a conical limit point of Γ if $\limsup \Gamma g A^+M \neq \emptyset$ for some $g \in G$ with $g^+ = \xi$. Equivalently, $\xi \in \mathcal{F}$ is conical if there exists $R > 0$ such that $\xi \in O_R(o, \gamma_i o)$ for some sequence $\gamma_i \to \infty$ in Γ. We denote by Λ_c the set of all conical limit points of Γ.

Lemma 7.2. *Let $\mathfrak{c} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$ be a closed convex cone whose interior contains $\mathcal{L}_\Gamma - \{0\}$. If $\gamma_i g_i a_i$ is a bounded sequence where $g_i \in G$ is bounded, $\gamma_i \in \Gamma$ and $a_i \to \infty$ in A^+, then*

$$\log a_i \in \mathfrak{c} \quad \textit{for all sufficiently large } i. \tag{7.3}$$

In particular, for any $x \in \Gamma\backslash G$, $\limsup xA^+M$ coincides with the set

$$\{\lim_{i\to\infty} xa_im_i : m_i \in M,\ \log a_i \to \infty \textit{ in } \mathfrak{c}\}.$$

Proof. As g_i and $\gamma_i g_i a_i$ are bounded sequences, the sequence $\mu(\gamma_i^{-1}) - \log a_i$ is also bounded by Lemma 2.3. Since $\mathcal{L}_\Gamma - \{0\} \subset \operatorname{int} \mathfrak{c}$ and $\mu(\gamma_i^{-1}) \in \mathcal{L}_\Gamma$, it follows that $\log a_i \in \mathfrak{c}$ for all large i. □

We note that for $\mathfrak{c} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$ as above, there exists a constant $s = s(\mathfrak{c}) > 0$ such that for any sequence $v_i \to \infty$ in $\mathfrak{c}$,

$$\min_{\alpha\in\Pi} \liminf_{i\to\infty} \alpha\left(\tfrac{v_i}{\|v_i\|}\right) \geq s.$$

We deduce from Proposition 5.11: recall $\mathcal{E} = \{[g] \in \Gamma\backslash G : g^+ \in \Lambda\}$.

Proposition 7.4. *For Γ Anosov, there exist a compact subset $\mathcal{Q}$ of $\mathcal{E}$ and a closed convex cone $\mathfrak{c} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$ such that for any $x \in \mathcal{E}$, there exists $\log a_i \to \infty$ in $\mathfrak{c}$ such that*

$$xa_i \in \mathcal{Q} \quad \textit{for all } i \geq 1.$$

In particular, $\Lambda = \Lambda_c$.

Proof. For Γ Anosov, we have $\mathcal{L}_\Gamma - \{0\} \subset \operatorname{int} \mathfrak{a}^+$ by Theorem 4.3. Hence we can find a closed convex cone $\mathfrak{c} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}$ such that $\mathcal{L}_\Gamma \subset \operatorname{int} \mathfrak{c} \cup \{0\}$. We first check that $\Lambda_c \subset \Lambda$. Let $g^+ \in \Lambda_c$ for some $g \in G$. Then there exists $\gamma_i \in \Gamma$ and $a_im_i \to \infty$ in A^+M such that $\gamma_i g a_i m_i$ is bounded. By Lemma 7.2, it follows that $\log a_i \to \infty$ in $\mathfrak{c}$. In particular, $a_i \to \infty$ regularly in A^+. Hence by Lemma 2.11, $ga_io \to g^+$ as $i \to \infty$. Since $d(ga_io, \gamma_i^{-1}o)$ is bounded, $\gamma_i^{-1}o \to g^+$ as $i \to \infty$. By Lemma 2.13, $g^+ \in \Lambda$.

Let $g^+ = \xi \in \Lambda$ and $z \in \partial\Gamma$ be such that $\xi = \zeta(z)$. Choose a geodesic ray $r = (\gamma_0 = e, \gamma_1, \gamma_2, \cdots)$ representing z. Note that if $g^+ = h^+$, then for any sequence $a_i \to \infty$ in A^+, there exists $b_i \in A^+$ such that $d(ga_io, hb_io) \leq 1$ for all sufficiently large i. Hence we may assume that $g \in K$ by replacing g by an element of gP. By Proposition 5.11, $\gamma_i o$ is contained in the R_1-neighborhood of gA^+o, with R_1 given therein. Hence for some $a_i \to \infty$

in A^+, $\Gamma\backslash\Gamma g a_i \in \mathcal{Q}$ where $\mathcal{Q} = \Gamma\backslash\Gamma\{h \in G : d(o, ho) \le R_1\} \cap \mathcal{E}$. Hence $g^+ \in \Lambda_c$. Moreover, by Lemma 7.2, $\log a_i \in \mathfrak{c}$ for all sufficiently large i. This finishes the proof. $\square$

Classification of PS **measures on** Λ.

Lemma 7.5. *Let $\psi_i \in \mathfrak{a}^*$ and ν_{ψ_i} be a (Γ, ψ_i)-*PS *measure for $i = 1, 2$. If $\nu_{\psi_1} = \nu_{\psi_2}$, then $\psi_1 = \psi_2$.*

Proof. Suppose that $\nu_{\psi_1} = \nu_{\psi_2}$. Then for all $\gamma \in \Gamma$ and $\xi \in \Lambda$, we have

$$\psi_1(\beta_\xi(e, \gamma)) = \psi_2(\beta_\xi(e, \gamma)).$$

By setting $\xi = y_\gamma^+$, we obtain $\lambda(\gamma) \in \ker(\psi_1 - \psi_2)$ for all $\gamma \in \Gamma$, by Lemma 3.5. Hence $\mathcal{L}_\Gamma \subset \ker(\psi_1 - \psi_2)$. Since $\mathcal{L}_\Gamma$ has nonempty interior [5, Thm. 1.2], this implies that $\psi_1 = \psi_2$. $\square$

Remark 7.6. When Γ is an Anosov subgroup, ν_{ψ_1} and ν_{ψ_2} are even mutually singular to each other whenever $\psi_1 \neq \psi_2$ (See Theorem 10.20 below).

We denote by $\mathcal{S}_\Gamma$ the space of all PS measures on Λ. Recall that for $\psi \in D_\Gamma^\star$, Quint constructed a (Γ, ψ)-PS measure on Λ [35]. In the Anosov case, such a measure is unique, which we denote by ν_ψ. By Lemma 7.5, the map $\psi \mapsto \nu_\psi$ from $D_\Gamma^\star$ to $\mathcal{S}_\Gamma$ is injective.

Theorem 7.7. *For $\Gamma < G$ Anosov, the map $\psi \mapsto \nu_\psi$ is a homeomorphism between $D_\Gamma^\star$ and $\mathcal{S}_\Gamma$.*

In the rank one case, there exists a unique Patterson-Sullivan measure on Λ and its dimension is given by the critical exponent of Γ. The above theorem generalizes such phenomenon.

To prove that the map $\psi \mapsto \nu_\psi$ is surjective, we need the following shadow lemma (cf. [35, Lem. 8.2]):

Lemma 7.8 (Size of shadow)**.** *Let $\Gamma < G$ be a regular Zariski dense discrete subgroup and $\psi \in \mathfrak{a}^*$. For a (Γ, ψ)-conformal measure ν_ψ on $\mathcal{F}$, there exist $R = R(\nu_\psi), C = C(R) > 0$ such that for all $r \ge R$ and for all $\gamma \in \Gamma$,*

$$C^{-1} e^{-\|\psi\|\kappa r} e^{-\psi(\mu(\gamma))} \le \nu_\psi(O_r(o, \gamma o)) \le C e^{\|\psi\|\kappa r} e^{-\psi(\mu(\gamma))}$$

where $\kappa > 0$ is a constant given in Lemma 5.7. In particular, if $\Lambda = \Lambda_c$ in addition, then ν_ψ is atom-free on Λ.

Proof. We claim that there exists $R > 0$ such that

$$c := \inf_{\gamma \in \Gamma} \nu_\psi(O_R(\gamma^{-1} o, o)) > 0.$$

Suppose not. Then there exist $R_i \to \infty$ and $\gamma_i \in \Gamma$ with $\nu_\psi(O_{R_i}(\gamma_i^{-1} o, o)) < 1/i$. Let $\gamma_i = k_i a_i \ell_i \in K A^+ K$ be the Cartan decomposition of γ_i. Passing to a subsequence, we may assume that $\ell_i \to \ell_0$ as $i \to \infty$. Note that $a_i \to \infty$ regularly in A^+ as Γ is regular. And hence $\lim\limits_{i\to\infty} O_{R_i}(a_i^{-1} o, o) = N^+ e^+$. Since $O_{R_i}(\gamma_i^{-1} o, o) = \ell_i^{-1} O_{R_i}(a_i^{-1} o, o)$, we obtain $\nu_\psi(\ell_0^{-1} N^+ e^+) = 0$. Since $N^+ e^+$

is Zariski open in $\mathcal{F}$, this contradicts the fact that $\Lambda \subset \operatorname{supp} \nu_\psi$ is Zariski dense in $\mathcal{F}$. This proves the claim.

Now let $\gamma \in \Gamma$ and $r > R$ be arbitrary. By Lemma 5.7, for all $\xi \in O_r(\gamma^{-1}o, o)$, we have

$$\|\beta_\xi(\gamma^{-1}o, o) - \mu(\gamma)\| \leq \kappa r.$$

Since

$$\nu_\psi(O_r(o, \gamma o)) = \int_{O_r(\gamma^{-1}o,o)} e^{-\psi(\beta_\xi(\gamma^{-1}o,o))}\, d\nu_\psi(\xi),$$

it remains to set $C = \max(c^{-1}, 1)e^{\|\psi\|\kappa r}$.

Suppose that $\Lambda = \Lambda_c$. Then for any $\xi \in \Lambda$, there exist $r > 0$ and a sequence $\gamma_i \to \infty$ in Γ such that $\xi \in \bigcap_i O_r(o, \gamma_i o)$. Since $\nu_\psi(\xi) \leq \nu_\psi(O_r(o, \gamma_i o)) \leq Ce^{-\psi(\mu(\gamma_i))}$ and $\psi(\mu(\gamma_i)) \to +\infty$ as $i \to \infty$, $\nu_\psi(\xi) = 0$. Hence the second claim follows. $\square$

Lemma 7.9. [34, Lem. III.1.3] *Let $\theta : \mathfrak{a} \to \mathbb{R}$ be a continuous function satisfying $\theta(tu) = t\theta(u)$ for all $t \geq 0$ and $u \in \mathfrak{a}$. If $\theta(u) > \psi_\Gamma(u)$ for all $u \in \mathfrak{a} - \{0\}$, then*

$$\sum_{\gamma \in \Gamma} e^{-\theta(\mu(\gamma))} < \infty.$$

If there exists $u \in \mathfrak{a}$ such that $\theta(u) < \psi_\Gamma(u)$, then $\sum_{\gamma\in\Gamma} e^{-\theta(\mu(\gamma))} = \infty$.

Lemma 7.10. *Suppose that Γ is regular and that $\Lambda = \Lambda_c$ (e.g., Γ is Anosov). If there exists a (Γ, ψ)-PS measure on Λ for some $\psi \in \mathfrak{a}^*$, then*

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} = \infty.$$

In particular, for any $\psi \in D_\Gamma^\star$, we have $\sum_{\gamma\in\Gamma} e^{-\psi(\mu(\gamma))} = \infty$.

Proof. Since $\Lambda = \Lambda_c$, Λ is an increasing union $\bigcup_{N=1}^\infty \Lambda_N$, where

$$\Lambda_N := \{\xi \in \Lambda : \text{ there exists } \gamma_i \to \infty \text{ in } \Gamma \text{ such that } \xi \in O_N(o, \gamma_i o)\}.$$

Hence $\nu(\Lambda_{N_0}) > 0$ for some $N_0 \geq 1$. Suppose that there exists a (Γ, ψ)-conformal measure, say ν. Fix $N \geq \max\{R(\nu), N_0\}$, and set $C' := C(R)e^{\|\psi\|\kappa N}$ where $R(\nu)$ and $C(R)$ are as in Lemma 7.8. Observe that for any $m \geq 1$,

$$\Lambda_N \subset \bigcup_{\gamma\in\Gamma, d(o,\gamma o)>m} O_N(o, \gamma o).$$

Hence

$$0 < \nu(\Lambda_N) \leq \sum_{d(o,\gamma o)>m} \nu(O_N(o, \gamma o)) \leq C' \sum_{d(o,\gamma o)>m} e^{-\psi(\mu(\gamma))}.$$

Since $m > 1$ is arbitrary, the conclusion follows. $\square$

Proof of Theorem 7.7: In order to prove surjectivity, suppose that there exists a (Γ, ψ)-PS measure, say ν_ψ, for $\psi \in \mathfrak{a}^*$. We note that $\psi \geq \psi_\Gamma$ by [35, Thm. 8.1]. We need to show $\psi(u) = \psi_\Gamma(u)$ for some $u \in \operatorname{int} \mathcal{L}_\Gamma$.

By Theorem 4.3, it suffices to show that $\psi(u) = \psi_\Gamma(u)$ for some $u \in \mathcal{L}_\Gamma - \{0\}$. Suppose not. Then $\psi(u) > \psi_\Gamma(u)$ for all $u \in \mathfrak{a} - \{0\}$. By Lemma 7.9, we have

$$\sum_{\gamma \in \Gamma} e^{-\psi(\mu(\gamma))} < \infty.$$

This is a contradiction by Lemma 7.10, proving surjectivity.

If $\psi_i \to \psi$ in $D_\Gamma^\star$, then any weak-limit of ν_{ψ_i} is a (Γ, ψ)-PS measure. By the uniqueness of (Γ, ψ)-conformal measure, ν_{ψ_i} converges to ν_ψ as $i \to \infty$. Hence the map $\psi \mapsto \nu_\psi$ is continuous. Now suppose $\nu_{\psi_i} \to \nu_\psi$ where $\psi_i, \psi \in D_\Gamma^\star$. Since the closed cone generated by $\mu(\Gamma)$ is equal to $\mathcal{L}_\Gamma$ which has non-empty interior, we can find $\gamma_1, \cdots, \gamma_k \in \Gamma$ such that $\mu(\gamma_i)$'s form a basis of $\mathfrak{a}$. For each γ_ℓ and $r > 0$, we have $\nu_{\psi_i}(O_r(o, \gamma_\ell o)) \to \nu_\psi(O_r(o, \gamma_\ell o))$. Hence $\{(\psi_i - \psi)(\mu(\gamma_\ell)) : i = 1, 2, \cdots\}$ is bounded by Lemma 7.8. It follows that $\{\psi_i : i = 1, 2, \cdots\}$ is a relatively compact subset of $\mathfrak{a}^*$. Suppose that $\phi \in \mathfrak{a}^*$ is a limit of $\{\psi_i\}$. By passing to a subsequence, assume that $\psi_i \to \phi \in \mathfrak{a}^*$. Since $\nu_{\psi_i} \to \nu_\psi$, it follows that ν_ψ is a (Γ, ϕ)-PS measure. Since $\psi \mapsto \nu_\psi$ is a bijection between $D_\Gamma^\star$ and $\mathcal{S}_\Gamma$, we have $\phi \in D_\Gamma^\star$ and $\nu_\phi = \nu_\psi$. By Lemma 7.5, we have $\phi = \psi$. Since every limit of the sequence ψ_i is ψ, it follows that ψ_i converges to ψ as $i \to \infty$. This finishes the proof.

Critical exponents. Recall the definition of O_Γ from (4.5). For each unit vector $w \in \mathsf{O}_\Gamma$, consider the Poincare series

$$\mathcal{P}_w(s, p) := \sum_{\gamma \in \Gamma} e^{-s\langle w, \underline{a}(p, \gamma p)\rangle}$$

Define the critical exponent δ_w to be the abscissa of convergence of $\mathcal{P}_w(s, p)$, which is independent of $p \in G/K$:

$$\delta_w := \inf\{s \in \mathbb{R} : \mathcal{P}_w(s, p) < \infty\}. \tag{7.11}$$

Corollary 7.12. *Let $w \in \mathsf{O}$ be a unit vector.*

(1) *For $w = \frac{\nabla \psi_\Gamma(u)}{\|\nabla \psi_\Gamma(u)\|} \in \mathsf{O}$ for $u \in \operatorname{int} \mathcal{L}_\Gamma$ with $\|u\| = 1$, we have*

$$\delta_w = \|\nabla \psi_\Gamma(u)\|.$$

In particular, $w \mapsto \delta_w$ is analytic on $\{w \in \mathsf{O} : \|w\| = 1\}$.

(2) *For any $p \in G/K$, $\mathcal{P}_w(\delta_w, p) = \infty$.*

Proof. (1) follows from Lemmas 7.9 and 7.10 together with Theorem 4.3. (2) is a direct consequence of Lemma 7.10. ☐

8. Myrberg limit points of Anosov groups

In this section, we discuss the notion of Myrberg limit points. We show that for Anosov groups, the set of Myrberg limit points has full measure for any PS measure on Λ. In the rank one case, this was proved by Tukia [50, Thm. 4A]. Let $\Gamma < G$ be a Zariski dense discrete subgroup.

Definition 8.1 (Myrberg points)**.** Let $p \in X$. We call a point $\xi_0 \in \Lambda$ a Myrberg limit point for Γ if, for any $\xi \neq \eta$ in Λ, there exists a sequence $\gamma_i \in \Gamma$ such that $\gamma_i p \to \xi$ and $\gamma_i \xi_0 \to \eta$ as $i \to \infty$.

Note that this definition is independent of the choice of $p \in X$ by Lemma 2.10. We denote by $\Lambda_M \subset \Lambda$ the set of all Myrberg limit points for Γ.

When G is of rank one, a Myrberg limit point $\xi \in \Lambda$ is characterized by the property that any geodesic ray toward ξ is dense in the space of all geodesics connecting limit points. The following proposition generalizes this to a general Anosov subgroup.

Proposition 8.2. *Let Γ be Anosov. We have $\xi_0 \in \Lambda_M$ if and only if for any $g \in G$ with $g^+ = \xi_0$,*

$$\limsup \Gamma\backslash\Gamma g A^+ M = \Omega.$$

Let $\Gamma < G$ be an Anosov subgroup for the rest of this section.

Lemma 8.3. *Let $b_i \in A$ be a sequence tending to ∞ such that $w^{-1}b_i^{-1}w \in A^+$ for some $w \in \mathcal{W}$. If $\gamma_i g b_i \to h$ for some $h, g \in G$ and $\gamma_i \in \Gamma$, then $\lim_{i\to\infty} \gamma_i g o = hw^+ \in \Lambda$. In particular, if $b_i \in A^+$, then $\lim_{i\to\infty} \gamma_i g o = h^-$.*

Proof. Let $c_i := h^{-1}\gamma_i g b_i$ and $a_i := w^{-1}b_i^{-1}w \in A^+$. Then $gw = \gamma_i^{-1} h c_i w a_i$. Hence by Lemma 7.2, $a_i \to \infty$ regularly in A^+. Lemma 2.11 implies that $hc_i w a_i(o) \to hw^+$. Since $\gamma_i g w = h c_i w a_i$, we have $\gamma_i g w(o) = \gamma_i g o \to hw^+$. This proves the first claim by Lemma 2.13. If $b_i \in A^+$, then $w_0^{-1} b_i^{-1} w_0 \in A^+$. Since $w_0^+ = e^-$, the last claim follows. $\square$

The following is proved in [23, Coro. 5.8]:

Theorem 8.4 (The limit map as a continuous extension of the orbit map)**.** *For any $p \in X$, the map $\Gamma \cup \partial\Gamma \to X \cup \mathcal{F}$ given by $\gamma \mapsto \gamma p$ for $\gamma \in \Gamma$ and $x \mapsto \zeta(x)$ for $x \in \partial\Gamma$ is continuous.*

We need the following basic fact about word hyperbolic groups.

Lemma 8.5. *Let $x \neq y$ in $\partial\Gamma$. If $\gamma_i \in \Gamma$ is an infinite sequence such that $(\gamma_i x, \gamma_i y) \to (x', y') \in \partial\Gamma \times \partial\Gamma$, then γ_i converges to either x' or y'.*

Proof. Choose a geodesic line $[x, y]$, and its representative $(\cdots, u_2, u_1, u_0 = v_0, v_1, v_2, \cdots)$. Note that $x = [u_0, u_1, u_2, \cdots]$ and $y = [v_0, v_1, v_2, \cdots]$. It suffices to show that $\gamma_i u_0$ converges to either x' or y'. Suppose not. Then by passing to a subsequence we have $\gamma_i u_0 \to z'$ where $z' \notin \{x', y'\}$. Since $(z'|x'), (z'|y') < \infty$, there exists a subsequence n_k such that $\sup_k (\gamma_k u_0 | \gamma_k u_{n_k}) + (\gamma_k u_0 | \gamma_k v_{n_k}) < \infty$. Let $\mathcal{L}_k^- := [\gamma_k u_0, \gamma_k u_{n_k}]$ and $\mathcal{L}_k^+ := [\gamma_k u_0, \gamma_k v_{n_k}]$, so that $\sup_k d_{\mathsf{w}}(e, \mathcal{L}_k^\pm) < \infty$. The thin triangle property of the hyperbolic group Γ implies that if the projection of e to the geodesic segment $\mathcal{L}_k^- \cup \mathcal{L}_k^+$ lies in $\mathcal{L}_k^{\pm}$, then $d_{\mathsf{w}}(e, \gamma_k u_0)$ is equal to $d_{\mathsf{w}}(e, \mathcal{L}_k^{\mp})$ up to a uniform additive constant. And hence $d_{\mathsf{w}}(e, \gamma_k u_0)$ is uniformly bounded, which is a contradiction as $\gamma_k \to \infty$ as $k \to \infty$. □

The following is immediate from Theorem 8.4 and Lemma 8.5:

Corollary 8.6. *Let $\gamma_i \in \Gamma$ be an infinite sequence such that $(\gamma_i \xi, \gamma_i \eta) \to (\xi', \eta')$ in $\Lambda^{(2)}$ as $i \to \infty$. Then for any $p \in X$, $\gamma_i p$ converges to either ξ' or η'.*

Lemma 8.7. *Let $g \in G$ be such that $g^{\pm} \in \Lambda$. If $\lim_{i \to \infty} \gamma_i g^{\pm} = \xi$ for some infinite sequence $\gamma_i \in \Gamma$, then $\lim_{i \to \infty} \gamma_i g o = \xi$.*

Proof. Set $x^{\pm} := \zeta^{-1}(g^{\pm})$ and $y = \zeta^{-1}(\xi)$. Since $\zeta : \partial\Gamma \to \Lambda$ is a homeomorphism, we have $\gamma_i x^{\pm} \to y$ as $i \to \infty$. By Lemma 8.5, we have $\gamma_i \to y$ as $i \to \infty$. By Theorem 8.4, we get $\lim_{i \to \infty} \gamma_i o = \xi$. By Lemma 2.10, $\lim_{i \to \infty} \gamma_i g o = \xi$ as desired. □

Since the fibers of the visual map $g \mapsto g^+$ are P-orbits, the following lemma is an easy consequence of the regularity lemma 7.2.

Lemma 8.8. *If $g, h \in G$ satisfy $g^+ = h^+$, then*

$$\limsup \Gamma g A^+ M = \limsup \Gamma h A^+ M.$$

Proof of Proposition 8.2. Set $\tilde{\Omega} := \{g \in G : g^{\pm} \in \Lambda\}$. Suppose $\xi_0 \in \Lambda_M$ and $g^+ = \xi_0$. We claim that $\Gamma g A^+ M = \tilde{\Omega}$. By Lemma 8.8, we may assume that $g^- \in \Lambda$. Let $h \in \tilde{\Omega}$. As $\xi_0 \in \Lambda_M$, there exists $\gamma_i \in \Gamma$ such that $\gamma_i g^+ \to h^+$ and $\gamma_i g o \to h^-$. By Lemma 8.5, by passing to a subsequence, $\gamma_i g^-$ converges to h^-. Therefore $\gamma_i g AM \to hAM$ in G/AM; there exists $b_i m_i \in AM$ such that $\gamma_i g b_i m_i \to h$. We claim that $b_i \in A^+$ for all large i. If not, by passing to a subsequence, we have m_i^{-1} converges to some $m_0 \in M$ and there exists $w \in \mathcal{W} - \{e\}$ such that $a_i := w^{-1} b_i w \in A^+$. Then $\gamma_i g w a_i \to h m_0 w$. By Lemma 8.3, $\gamma_i g o \to h m_0 w^-$, and hence $h m_0 w^- = h^-$. It follows that $w = e$, yielding a contradiction. Therefore $h \in \limsup \Gamma g A^+ M$, proving the claim.

Now suppose that $\limsup \Gamma g A^+ M = \tilde{\Omega}$. We claim that $g^+ \in \Lambda_M$. Let $\xi \neq \xi'$ in Λ, and let $h \in G$ be such that $h^+ = \xi$ and $h^- = \xi'$. By the hypothesis and Lemma 7.2, there exist $\gamma_i \in \Gamma$, $m_i \in M$ and $a_i \to \infty$ regularly in A^+ such that $\gamma_i g a_i m_i \to h$ in G. Then $\gamma_i g^+ \to h^+ = \xi$. By Lemma 8.3, $\gamma_i g o \to h^- = \xi'$. Hence $g^+ \in \Lambda_M$.

Theorem 8.9. *For any* PS*-measure ν on Λ, $\nu(\Lambda_M) = 1$.*

Proof. By Theorem 7.7, $\nu = \nu_\psi$ for some $\psi \in D_\Gamma^\star$. Let m_ψ be the $\mathbb{R} := \{\tau_s : s \in \mathbb{R}\}$-ergodic finite measure on $\Gamma \backslash \Lambda^{(2)} \times \mathbb{R}$ in Theorem 4.8. Let $Z_\psi \subset \Gamma \backslash \Lambda^{(2)} \times \mathbb{R}$ denote the set of elements with dense $\mathbb{R}_+$-orbits, and $\tilde{Z}_\psi$ be its lift in $\Lambda^{(2)} \times \mathbb{R}$. By the Birkhoff ergodic theorem, Z_ψ has full m_ψ-measure, and hence $\nu(\pi(\tilde{Z}_\psi)) = \nu(\Lambda)$ where $\pi : \Lambda^{(2)} \times \mathbb{R} \to \Lambda$ denotes the projection map $\pi(\xi, \eta, t) = \xi$. It is now sufficient to prove that $\pi(\tilde{Z}_\psi) \subset \Lambda_{\mathrm{M}}$.

Let $\xi \in \pi(\tilde{Z}_\psi)$ and $(\eta_1, \eta_2) \in \Lambda^{(2)}$ be arbitrary. We need to show that there exists $\gamma_i \in \Gamma$ such that $\gamma_i \xi \to \eta_1$ and $\gamma_i o \to \eta_2$ as $i \to \infty$. Choose $(\xi, \xi', 0) \in \tilde{Z}_\psi$. By definition, we can find $\gamma_i \in \Gamma$ and $t_i \to +\infty$ such that the sequence

$$\gamma_i(\tau_{t_i}.(\xi, \xi', 0)) = \gamma_i(\xi, \xi', t_i) = (\gamma_i\xi, \gamma_i\xi', t_i + \psi(\beta_{\gamma_i\xi}(o, \gamma_i o)))$$

converges to $(\eta_1, \eta_2, 0)$. Write $x = \zeta^{-1}(\xi)$, $x' = \zeta^{-1}(\xi')$, $y_1 = \zeta^{-1}(\eta_1)$, $y_2 = \zeta^{-1}(\eta_2)$, and choose $u \in [x, x']$. Since the triangle $[\gamma_i x, \gamma_i x'] \cup [\gamma_i u, \gamma_i x] \cup [\gamma_i u, \gamma_i x']$ is δ-thin, it follows that for all i, either $\gamma_i x \in O_\delta(u, \gamma_i u)$ or $\gamma_i x' \in O_\delta(u, \gamma_i u)$. We claim the latter holds for all large i.

Suppose not. Then by passing to a subsequence, we may assume that $\gamma_i x \in O_\delta(u, \gamma_i u)$ for all i. Then by Proposition 5.13 and Lemma 5.7, there exists a uniform constant $c > 0$ such that $\gamma_i \xi \in O_{c(\delta+1)}(uo, \gamma_i uo)$ and

$$|\psi(\beta_{\gamma_i\xi}(uo, \gamma_i uo))) - \psi(\mu(\gamma_i))| < \|\psi\|\kappa c(\delta + 1).$$

Since $\psi(\mu(\gamma_i)) \to +\infty$ as $i \to \infty$ by Lemma 5.4, and $\psi(\beta_{\gamma_i\xi}(uo, \gamma_i uo)))$ and $\psi(\beta_{\gamma_i\xi}(o, \gamma_i o)))$ are uniformly close to each other, $\psi(\beta_{\gamma_i\xi}(o, \gamma_i o))) \to +\infty$. This contradicts the hypothesis that the sequence $t_i + \psi(\beta_{\gamma_i\xi}(o, \gamma_i o))$ converges to a finite number as $i \to \infty$. It follows that for all sufficiently large i,

$$\gamma_i x' \in O_\delta(u, \gamma_i u). \tag{8.10}$$

On the other hand, $\gamma_i u \to y_\ell$ for some $\ell \in \{1, 2\}$ by Lemma 8.5. Since $\gamma_i x' \to y_2$ and $O_\delta(u, \gamma_i u)$ converges to y_ℓ, (8.10) implies that $\gamma_i u \to y_2$. Therefore $\gamma_i o \to \eta_2$ by Lemma 8.4. $\square$

In the rank one case, the BMS measure is finite, and $A = \{a_t\}$ is the union of $A^+ = \{a_t : t \geq 0\}$ and $A^- = \{a_t : t \leq 0\}$. The AM-ergodicity of the BMS measure implies that for almost all $x \in \Gamma\backslash G$, $xA^\pm M$ is dense in $\Omega = \{x \in \Gamma\backslash G : x^\pm \in \Lambda\}$. In general, $A = \cup_{w \in \mathcal{W}} wA^+w^{-1}$, and we have the following corollary of Theorem 8.9:

Corollary 8.11. *Let $\psi \in D_\Gamma^\star$. For m_ψ^{BMS}-almost all $x \in \Omega$, each xA^+M and xw_0A^+M is dense in Ω.*

Proof. Note that for $x = \Gamma g \in \Omega$, xwA^+M is dense in Ω if and only if $gw^+ \in \Lambda_M$ by Proposition 8.2. For $w = e$ (resp. $w = w_0$), the claim follows as $\nu_\psi(\Lambda_M) = 1$ (resp. $\nu_{\psi \circ \mathrm{i}}(\Lambda_M) = 1$) by Theorem 8.9. $\square$

We also observe:

Lemma 8.12. *For any $x \in \mathcal{E}$ and $w \in \mathcal{W} - \{e, w_0\}$, the map $A^+M \to xwA^+M$ is proper.*

Proof. Note that if $(g^+, gw^+) \in \mathcal{F}^{(2)}$ for $g \in G$ and $w \in \mathcal{W}$, then $w = w_0$. Choose $g \in G$ so that $\Gamma g = x \in \mathcal{E}$. Since $g^+ \in \Lambda$ and $\Lambda \times \Lambda - \{(\xi, \xi)\} \subset \mathcal{F}^{(2)}$ by the antipodality, $gw^+ \in \Lambda$ can happen only for $w \in \{e, w_0\}$. Suppose for some $\gamma_i \in \Gamma$ and $a_i \to \infty$ in A^+, $\gamma_i gwa_i$ converges to some $h \in G$ as $i \to \infty$.

This means that $d(gwa_i, \gamma_i^{-1}h) \to 0$ as $i \to \infty$, and hence $gw^+ \in \Lambda$. Hence, for each $w \in \mathcal{W} - \{e, w_0\}$, $\limsup xwA^+M = \emptyset$, proving the claim. $\square$

9. Criterion for ergodicity via essential values

In this section, let $\Gamma < G$ be a Zariski dense discrete subgroup, and let ν_ψ be a (Γ, ψ)-conformal measure on $\mathcal{F}$ for $\psi \in \mathfrak{a}^*$. Consider the action of G on $\mathcal{F} \times \mathfrak{a}$ by

$$g(\xi, v) = (g\xi, v + \beta_\xi(g^{-1}, e)).$$

Then the map $g \mapsto (g^+, b := \beta_{g^+}(e, g))$ induces a G-equivariant homeomorphism $G/NM \simeq \mathcal{F} \times \mathfrak{a}$. Using this homeomorphism, we define a Γ-invariant Radon measure $\widehat{\nu}_\psi$ on $G/NM \simeq \mathcal{F} \times \mathfrak{a}$ by

$$d\widehat{\nu}_\psi(gNM) = d\nu_\psi(g^+) e^{\psi(b)}\, db.$$

Since $dm_\psi^{\mathrm{BR}} = d\widehat{\nu}_\psi\, dm\, dn$, the NM-ergodicity of m_ψ^{BR} is equivalent to the Γ-ergodicity of $\widehat{\nu}_\psi$. For simplicity, we set $\nu := \nu_\psi$ and $\widehat{\nu} := \widehat{\nu}_\psi$ for the rest of the section. Schmidt gave a characterization of Γ-ergodicity of $\widehat{\nu}$ using the notion of ν-essential values in the rank one case ([46], see also [40, Prop. 2.1]).

Definition 9.1. An element $v \in \mathfrak{a}$ is called a (ν, Γ)-*essential value*, if for any Borel set $B \subset \mathcal{F}$ with $\nu(B) > 0$ and any $\varepsilon > 0$, there exists $\gamma \in \Gamma$ such that

$$\nu\left(B \cap \gamma^{-1}B \cap \{\xi \in \mathcal{F} : \|\beta_\xi(\gamma^{-1}o, o) - v\| < \varepsilon\}\right) > 0.$$

Let $\mathsf{E}_\nu = \mathsf{E}_\nu(\Gamma)$ denote the set of all (ν, Γ)-essential values in $\mathfrak{a}$. It is easy to see that E_ν is a closed subgroup of $\mathfrak{a}$. The main goal of this section is to prove the following criterion of Γ-ergodicity of $\widehat{\nu}$, which can be considered as a higher rank version of [40, Prop. 2.1] .

Proposition 9.2. *$(G/NM, \Gamma, \widehat{\nu})$ is ergodic if and only if $(G/P, \Gamma, \nu)$ is ergodic and $\mathsf{E}_\nu(\Gamma) = \mathfrak{a}$.*

Fixing ν, we set $\mathsf{E} := \mathsf{E}_\nu(\Gamma)$ in the rest of this section. Our proof of Proposition 9.2 is an easy adaptation of the proof of [40, Prop. 2.1] to a higher rank case. We begin with the following lemma .

Lemma 9.3. *Let $h : G/NM = \mathcal{F} \times \mathfrak{a} \to [0, 1]$ be a Γ-invariant Borel function such that for each $\xi \in \mathcal{F}$, $h(\xi, \cdot)$ is a C-Lipschitz function on $\mathfrak{a}$ for some $C > 0$ independent of ξ. Then for each $\log a \in \mathsf{E}$, $h(xa) = h(x)$ for $\widehat{\nu}$-a.e. $x \in G/NM$.*

Proof. Suppose that $\widehat{\nu}\{x \in G/NM : h(x) \neq h(xa)\} > 0$ for some $\log a \in \mathsf{E}$. We will then find a subset $\mathcal{A}^* = \mathcal{A}^*(a) \subset G/NM$ with $\widehat{\nu}(\mathcal{A}^*) > 0$ and $\gamma \in \Gamma$ such that $h(\gamma^{-1}x) \neq h(x)$ for all $x \in \mathcal{A}^*$; this contradicts the Γ-invariance of h.

By replacing h with $-h$ if necessary, we may assume that $\widehat{\nu}\{x \in G/NM : h(x) < h(xa)\} > 0$. Hence there exist $r, \varepsilon > 0$ such that

$$Q_a := \{x \in G/NM : h(x) < r - C\varepsilon < r + C\varepsilon < h(xa)\}$$

has a positive $\widehat{\nu}$-measure. Now we can choose a ball $\mathcal{O} = B_{\mathfrak{a}}(v_0, \varepsilon/2) \subset \mathfrak{a}$ such that

$$\widehat{\nu}((\mathcal{F} \times \mathcal{O}) \cap Q_a) > 0.$$

Set $F_a := \{\xi \in \mathcal{F} : (\{\xi\} \times \mathcal{O}) \cap Q_a \neq \emptyset\}$. We claim that

$$\text{if } (\xi, w) \in F_a \times \mathcal{O}, \text{ then } h(\xi, w + \log a) > r > h(\xi, w). \tag{9.4}$$

Note that there exists $v \in \mathfrak{a}$ with $\|v\| < \varepsilon$ such that $(\xi, w + v) \in Q_a$ and hence

$$|h(\xi, w)| \leq |h(\xi, w) - h(\xi, w + v)| + |h(\xi, w + v)| < C\|v\| + (r - C\varepsilon) \leq r.$$

Similarly,

$$\begin{aligned} |h(\xi, w + \log a)| &\geq |h(\xi, w + v + \log a)| - |h(\xi, w + \log a) - h(\xi, w + v + \log a)| \\ &> (r + C\varepsilon) - C\|v\| > r, \end{aligned}$$

which verifies the claim (9.4).

Since $-\log a \in \mathsf{E}$ and $\nu(F_a) > 0$, there exists $\gamma \in \Gamma$ such that

$$\mathcal{A} := F_a \cap \gamma F_a \cap \{\xi \in G/P : \|\beta_\xi(o, \gamma o) + \log a\| < \varepsilon/2\}$$

has a positive ν-measure. For $\xi \in \mathcal{A}$, set

$$\mathcal{O}_\xi := \{w \in \mathcal{O} : w - (\beta_\xi(o, \gamma o) + \log a) \in \mathcal{O}\}.$$

Since $\|\beta_\xi(o, \gamma o) + \log a\| < \varepsilon/2$, and $\mathcal{O}$ is a Euclidean ball of diameter ε, there is a uniform positive lower bound for the volume of $\mathcal{O}_\xi$. It follows that

$$\mathcal{A}^* := \bigcup_{\xi \in \mathcal{A}} \{\xi\} \times \mathcal{O}_\xi$$

has positive $\widehat{\nu}$-measure. We now claim that $h \circ \gamma^{-1} > h$ on $\mathcal{A}^*$.

Let $(\xi, w) \in \mathcal{A}^*$. Since $(\xi, w) \in F_a \times \mathcal{O}$, (9.4) implies that $h(\xi, w) < r$.

Write $\gamma^{-1}(\xi, w) = (\gamma^{-1}\xi, w - (\beta_\xi(o, \gamma o) + \log a) + \log a)$. Since $(\gamma^{-1}\xi, w - (\beta_\xi(o, \gamma o) + \log a)) \in F_a \times \mathcal{O}$, (9.4) says that

$$h(\gamma^{-1}(\xi, w)) > r;$$

this proves the claim. □

Proof of Proposition 9.2. Assume that $(G/NM, \Gamma, \widehat{\nu})$ is ergodic. Let $\pi : G/NM \to G/P$ denote the projection map. Since $\pi_*\widehat{\nu}$ is absolutely continuous with respect to ν, it follows that $(G/P, \Gamma, \nu)$ is ergodic.

To show $\mathsf{E} = \mathfrak{a}$, fix an arbitrary Borel set $B \subset G/P$ of positive ν-measure. For any $w \in \mathfrak{a}$ and $\varepsilon > 0$, we define

$$\mathcal{B}_{w,\varepsilon} := \{(\xi, v) \in G/P \times \mathfrak{a} : \xi \in B, \|v - w\| < \varepsilon\} \subset G/NM.$$

Observe that

$$\widehat{\nu}(\mathcal{B}_{0,\varepsilon}) = \int_{G/P} \int_{\mathfrak{a}} \mathbf{1}_{\mathcal{B}_{0,\varepsilon}}(\xi, b) e^{\psi(b)} \, db \, d\nu(\xi) \geq \mathrm{Vol}(B_{\mathfrak{a}}(0, \varepsilon)) \, e^{-\|\psi\|\varepsilon} \nu(B) > 0.$$

Hence it follows from the ergodicity of $(G/NM, \Gamma, \widehat{\nu})$ that $\widehat{\nu}(G/NM - \Gamma\mathcal{B}_{0,\varepsilon}) = 0$. In particular, there exists $\gamma \in \Gamma$ such that $\widehat{\nu}(\mathcal{B}_{w,\varepsilon} \cap \gamma\mathcal{B}_{0,\varepsilon}) > 0$. Finally, note that if $(\xi, v) \in \mathcal{B}_{w,\varepsilon} \cap \gamma\mathcal{B}_{0,\varepsilon}$, then $\xi \in B \cap \gamma B$, and

$$\|\beta_\xi(e,\gamma) - w\| \le \|\beta_\xi(e,\gamma) - v\| + \|v - w\| \le \varepsilon + \varepsilon = 2\varepsilon.$$

This, together with the fact $\pi_*\widehat{\nu} \ll \nu$, implies that

$$\nu(B \cap \gamma B \cap \{\xi \in G/P : \|\beta_\xi(e,\gamma) - w\| \le 2\varepsilon\}) > 0,$$

which finishes the proof of $(\Rightarrow)$.

We now assume that $(G/P, \Gamma, \nu)$ is ergodic and $\mathsf{E} = \mathfrak{a}$. Let $h : G/NM \to [0,1]$ be a Γ-invariant Borel function. We need to show that h is constant $\widehat{\nu}$-a.e. Identifying $\mathfrak{a} \simeq \mathbb{R}^r$ with $r = \operatorname{rank} G$, for each $\tau = (\tau_1, \cdots, \tau_r) \in \mathfrak{a}$, we define a Γ-invariant Borel function $h_\tau : G/NM \to \mathbb{R}$ as follows:

$$h_\tau(x) = \int_0^{\tau_1} \cdots \int_0^{\tau_r} h(x \exp(t_1, \cdots, t_r))\, dt_r \cdots dt_1.$$

Note that h_τ satisfies the hypothesis of Lemma 9.3. Hence by the hypothesis $\mathsf{E}_\nu = \mathfrak{a}$, for each $a \in A$, $h_\tau(x) = h_\tau(xa)$ for $\widehat{\nu}$-a.e. $x \in G/NM$.

Let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of A. Then there exists Ω_n of full $\widehat{\nu}$-measure such that for all $x \in \Omega_n$, $h_\tau(x) = h_\tau(xa_n)$. Set $\Omega := \cap_{n=1}^\infty \Omega_n$. Then for all $x \in \Omega$, we have $h_\tau(x) = h_\tau(xa)$ for all $a \in A$, as $h_\tau(\xi, \cdot)$ is continuous on $\mathfrak{a}$. Now h_τ is a Γ-invariant function on G/NM, which is also A-invariant $\widehat{\nu}$-a.e.

Since $(G/P, \Gamma, \nu)$ is ergodic, there exists $c(\tau) \in \mathbb{R}$ such that $h_\tau = c(\tau)$ $\widehat{\nu}$-a.e. on G/NM.

Next, fix $1 \le i \le r$ and $\tau_1, \cdots, \tau_{i-1}, \tau_{i+1}, \cdots, \tau_r \ge 0$, and define

$$f(t) := (\tau_1, \cdots, \tau_{i-1}, t, \tau_{i+1}, \cdots, \tau_r) \in \mathfrak{a}.$$

Then $t \mapsto c(f(t))$ is linear; indeed, by definition, we have

$$h_{f(t+s)} = h_{f(t)} + h_{f(s)} \circ \exp(te_i)$$

for all $t, s \ge 0$ and hence $c(f(t+s)) = c(f(t)) + c(f(s))$. We conclude $c(\tau) = \kappa\tau_1 \cdots \tau_r$, for some $\kappa \in \mathbb{R}$.

Hence for each $\tau \in \mathfrak{a}$, $h_\tau = \kappa\tau_1 \cdots \tau_r$ $\widehat{\nu}$-a.e. Since $|h_{\tau+\sigma} - h_\tau| \le 2^r\|\sigma\|\|\tau\|^{r-1}$ and hence $\tau \to h_\tau$ is continuous, using a countable dense subset of $\mathfrak{a}$, we conclude there exists a subset Ω of full $\widehat{\nu}$-measure such that

$$h_\tau(x) = \kappa\tau_1 \cdots \tau_r \quad \text{for all } x \in \Omega \text{ and } \tau \in \mathfrak{a}.$$

By restricting h_τ to each fiber of $\pi : G/NM \to G/P$, and applying the Lebesque differentiation theorem, we conclude that $\frac{1}{\tau_1 \cdots \tau_r} h_\tau(x) \to h(x)$ as $\tau \to 0$ for $\widehat{\nu}$-a.e. x. Consequently, $h = \kappa$ $\widehat{\nu}$-a.e., finishing the proof.

10. Ergodicity of m_ψ^{BR} and classification

Let $\Gamma < G$ be an Anosov subgroup. Recall the NM-invariant BR measure m_ψ^{BR} defined in (3.23). We prove the following theorem in this section:

Theorem 10.1. *For each $\psi \in D_\Gamma^\star$, m_ψ^{BR} is NM-ergodic.*

Recall the definition of $\widehat{\nu}_\psi$ and ν_ψ from section 9. Since $(\mathcal{F}, \Gamma, \nu_\psi)$ is ergodic by Theorem 4.3, the following proposition implies that $(G/NM, \Gamma, \widehat{\nu}_\psi)$, and hence $(\Gamma\backslash G, NM, m_\psi^{\mathrm{BR}})$, is ergodic by Proposition 9.2.

Proposition 10.2. *Let Γ_0 be a Zariski dense normal subgroup of Γ. For any $\psi \in D_\Gamma^\star$, $\mathsf{E}_{\nu_\psi}(\Gamma_0) = \mathfrak{a}$. In particular, $\mathsf{E}_{\nu_\psi}(\Gamma) = \mathfrak{a}$.*

Most of the section is devoted to the proof of Proposition 10.2. We fix a Zariski dense normal subgroup Γ_0 of Γ.

Lemma 10.3. *For any finite subset $S_0 \subset \lambda(\Gamma_0)$, the subgroup generated by $\lambda(\Gamma_0) - S_0$ is dense in $\mathfrak{a}$.*

Proof. Let F denote the closure of the subgroup generated by $\lambda(\Gamma_0) - S_0$. Suppose that $F \neq \mathfrak{a}$. Identifying $\mathfrak{a} = \mathbb{R}^r$, since F is infinite, there exist $1 \leq k < r$ and $0 \leq m \leq r$ such that $F = \sum_{i=1}^k \mathbb{R}v_i + \sum_{i=1}^m \mathbb{Z}w_i$ where v_i, w_i are linearly independent vectors. For each $s = \lambda(\gamma) \in S_0$, $\lambda(\gamma^n) = n\lambda(\gamma) \to \infty$ as γ is loxodromic. Hence there exists $n_s \in \mathbb{N}$ so that $n_s\lambda(\gamma) \in F$. Setting $N := \prod_{s\in S_0} n_s$, we have $S_0 \subset \sum_{i=1}^k \mathbb{R}v_i + N^{-1}\sum_{i=1}^m \mathbb{Z}w_i$.

Therefore, the closure of the subgroup generated by $F \cup S_0$ is contained in $\sum_{i=1}^k \mathbb{R}v_i + N^{-1}\sum_{i=1}^m \mathbb{Z}w_i$. Since $\lambda(\Gamma_0) \subset \sum_{i=1}^k \mathbb{R}v_i + N^{-1}\sum_{i=1}^m \mathbb{Z}w_i$ and $\lambda(\Gamma_0)$ generates a dense subgroup of $\mathfrak{a}$ [6], it follows that $k = \dim \mathfrak{a}$, yielding a contradiction. $\square$

Proposition 10.4. *For any $\psi \in D_\Gamma^\star$ and $C > 0$, the set $\{\lambda(\gamma) \in \mathfrak{a}^+ : \gamma \in \Gamma_0,\, \psi(\lambda(\gamma)) \geq C\}$ generates a dense subgroup of $\mathfrak{a}$.*

Proof. Theorem 3.2 in [43] extends to general Anosov subgroups (see also [11, Thm. A.2-(2)]), and hence the cocycle $c = \psi \circ \sigma$ has a finite exponential growth rate. In particular,

$$\#\{\lambda(\gamma) : \gamma \in \Gamma,\, \psi(\lambda(\gamma)) < C\} \leq \#\{[\gamma] \in [\Gamma] : \psi(\lambda(\gamma)) < C\} < \infty \tag{10.5}$$

where $[\Gamma]$ denotes the set of conjugacy classes in Γ. Hence $\#\{\lambda(\gamma) : \gamma \in \Gamma_0,\, \psi(\lambda(\gamma)) < C\} < \infty$ and the claim follows from Lemma 10.3. $\square$

Lemma 10.6. *There exists a compact subset $\mathcal{C} \subset G$ such that for any $\xi \in \Lambda$, there exists $g \in \mathcal{C}$ such that $g^+ = \xi$ and $g^- \in \Lambda$.*

Proof. In the Gromov hyperbolic space Γ, there exists a finite subset $F \subset \Gamma$ such that for any $x \in \partial\Gamma$, there exists $y \in \partial\Gamma$ such that $[x, y] \cap F \neq \emptyset$. It suffices to choose a compact subset $\mathcal{C} \subset G$ such that $\mathcal{C}(o)$ contains the R_1-neighborhood of $F(o)$ with R_1 given in Proposition 5.11. $\square$

We set

$$N_0 := \max_{p\in\mathcal{C}(o)} N_0(\psi, p) < \infty$$

with $N_0(\psi, p)$ and $\mathcal{C}$ given by Lemmas 6.12 and 10.6 respectively.

In view of Proposition 10.4, Proposition 10.2 is an immediate consequence of the following:

Proposition 10.7. *For any* $\gamma_0 \in \Gamma_0$ *with* $\psi(\lambda(\gamma_0)) \geq 1 + \log 3N_0$,

$$\lambda(\gamma_0) \in \mathsf{E}_{\nu_\psi}(\Gamma_0).$$

Essential values of ν_ψ**.** Most of this section is devoted to the proof of this proposition. We fix $\gamma_0 \in \Gamma_0$ with

$$\psi(\lambda(\gamma_0)) \geq \log 3N_0 + 1.$$

Since $\psi > 0$ on $\lambda(\Gamma) - \{0\}$ by Theorem 4.3(4), we have

$$\psi(\mathrm{i}\,\lambda(\gamma_0)) + \psi(\lambda(\gamma_0)) > \log 3N_0 + 1. \tag{10.8}$$

Definition of $\mathcal{B}_R(\gamma_0, \varepsilon)$**.** Let $0 < \varepsilon < \|\psi\|^{-1}$ be an arbitrary number. We fix $g \in \mathcal{C}$ such that $g^+ = y_{\gamma_0}^+$ and $g^- \in \Lambda$, given by Lemma 10.6. Set $p := go \in \mathcal{C}(o)$, $\xi_0 := y_{\gamma_0}^+$ and $\eta := g^-$.

For $\xi \in \Lambda$ and $r > 0$, set

$$\mathbb{B}_p(\xi, r) := \{\eta \in \Lambda : d_{\psi,p}(\xi, \eta) < r\}$$

where $d_{\psi,p}$ is the virtual visual metric defined in section 6.

For each $\gamma \in \Gamma$, define $r_p(\gamma) > 0$ to be the supremum $r \geq 0$ such that

$$\max_{\xi \in \mathbb{B}_p(\gamma\xi_0, 3N_0 r)} \|\beta_\xi(p, \gamma\gamma_0^{\pm 1}\gamma^{-1}p) \mp \lambda(\gamma_0)\| < \varepsilon. \tag{10.9}$$

For each $R > 0$, we define the family of virtual-balls as follows:

$$\mathcal{B}_R(\gamma_0, \varepsilon) = \{\mathbb{B}_p(\gamma\xi_0, r) : \gamma \in \Gamma, 0 < r < \min(R, r_p(\gamma))\}.$$

Let $C = C(\psi, p) > 0$ be as in Theorem 5.3. Since $\xi_0 \in O_{\varepsilon/(8\kappa)}(\eta, p)$ where $\kappa > 0$ is as in Lemma 5.7, we can choose $0 < s = s(\gamma_0) < R$ small enough such that

$$\mathbb{B}_p(\xi_0, e^{\psi(\lambda(\gamma_0)+\mathrm{i}\ \lambda(\gamma_0))+\frac{1}{2}\|\psi\|\varepsilon+2C}s) \subset O_{\varepsilon/(8\kappa)}(\eta, p); \tag{10.10}$$

$$\sup_{x \in \mathbb{B}_p(\xi_0, e^{2C}s)} \|\beta_x(p, \gamma_0^{\pm 1}p) \mp \lambda(\gamma_0)\| < \varepsilon/4. \tag{10.11}$$

For each $\gamma \in \Gamma$ and $r > 0$, set

$$D(\gamma\xi_0, r) := \mathbb{B}_p(\gamma\xi_0, \frac{1}{3N_0} e^{-\psi(\underline{a}(\gamma^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma^{-1}p,p))}r).$$

Lemma 10.12. *Fix* $R > 0$*. If* $\xi \in \Lambda$ *and* $\gamma_i \in \Gamma$ *is a sequence such that* $\gamma_i^{-1}p \to \eta$ *and* $\gamma_i^{-1}\xi \to \xi_0$ *as* $i \to \infty$*, then for any* $0 < r \leq s(\gamma_0)$*, there exists* $i_0 = i_0(r) > 0$ *such that for all* $i \geq i_0$*,*

$$D(\gamma_i\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon) \quad \text{and} \quad \xi \in D(\gamma_i\xi_0, r).$$

In particular, for any $R > 0$*,*

$$\Lambda_M \subset \bigcup_{D \in \mathcal{B}_R(\gamma_0, \varepsilon)} D.$$

Proof. Set $\Gamma_p := \{\gamma \in \Gamma : \psi(\underline{a}(\gamma^{-1}p, p) + \mathrm{i}\,\underline{a}(\gamma^{-1}p, p)) > 0\}$; note that $\Gamma - \Gamma_p$ is a finite subset by Lemma 5.4. Hence we may assume that for all i, $\gamma_i \in \Gamma_p$. Since $\gamma_i^{-1}p \to \eta$ as $i \to \infty$, we may assume by Lemma 5.6 that for all i,

$$O_{\varepsilon/(8\kappa)}(\eta, p) \subset O_{\varepsilon/(4\kappa)}(\gamma_i^{-1}p, p). \tag{10.13}$$

To prove that $D(\gamma_i\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$, we need to check that

$$\max_{\xi' \in \mathbb{B}_p(\gamma_i\xi_0, 3N_0 s_i)} \|\beta_{\xi'}(p, \gamma_i\gamma_0^{\pm 1}\gamma_i^{-1}p) \mp \lambda(\gamma_0)\| < \varepsilon,$$

where $s_i = \frac{1}{3N_0} e^{-\psi(\underline{a}(\gamma_i^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma_i^{-1}p,p))} r$. Let $\xi' \in \mathbb{B}_p(\gamma_i\xi_0, 3N_0 s_i)$. We only prove that $\|\beta_{\xi'}(p, \gamma_i\gamma_0\gamma_i^{-1}p) - \lambda(\gamma_0)\| < \varepsilon$, as the other case can be treated similarly. First, observe that

$$\begin{aligned} d_p(\xi_0, \gamma_i^{-1}\xi') &= d_p(\gamma_i\xi_0, \xi') e^{\psi(\beta_{\xi_0}(\gamma_i^{-1}p,p)+\mathrm{i}\,\beta_{\gamma_i^{-1}\xi'}(\gamma_i^{-1}p,p))} \\ &\le e^{-\psi(\underline{a}(\gamma_i^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma_i^{-1}p,p))+\psi(\beta_{\xi_0}(\gamma_i^{-1}p,p)+\mathrm{i}\,\beta_{\gamma_i^{-1}\xi'}(\gamma_i^{-1}p,p))} r \\ &\le e^{2C} r \text{ by Theorem 5.3.} \end{aligned} \tag{10.14}$$

Since $r \le s(\gamma_0)$, this implies that

$$\|\beta_{\gamma_i^{-1}\xi'}(p, \gamma_0 p) - \lambda(\gamma_0)\| < \varepsilon/4.$$

Hence, by (6.3), we have

$$\begin{aligned} d_p(\xi_0, \gamma_0^{-1}\gamma_i^{-1}\xi') &= e^{-\psi(\beta_{\xi_0}(\gamma_0 p,p)+\mathrm{i}\,\beta_{\gamma_i^{-1}\xi'}(\gamma_0 p,p))} d_p(\xi_0, \gamma_i^{-1}\xi') \\ &\le e^{\psi(\lambda(\gamma_0)+\mathrm{i}\,\lambda(\gamma_0))+\frac{1}{2}\|\psi\|\varepsilon+2C} r. \end{aligned} \tag{10.15}$$

Since $r \le s(\gamma_0)$, it follows from (10.14), (10.15), and (10.10) that both $\gamma_i^{-1}\xi'$ and $\gamma_0^{-1}\gamma_i^{-1}\xi'$ belong to $\mathcal{O}_{\varepsilon/(8\kappa)}(\eta, p)$. Since, $\gamma_i^{-1}\xi', \gamma_0^{-1}\gamma_i^{-1}\xi' \in O_{\varepsilon/(4\kappa)}(\gamma_i^{-1}p, p)$ by (10.13), it follows from Lemma 5.7 that

$$\|\beta_{\gamma_i^{-1}\xi'}(\gamma_i^{-1}p, p) - \beta_{\gamma_0^{-1}\gamma_i^{-1}\xi'}(\gamma_i^{-1}p, p)\| < 2\kappa(\varepsilon/4\kappa) = \varepsilon/2.$$

Now we have

$$\begin{aligned} &\|\beta_{\xi'}(p, \gamma_i\gamma_0\gamma_i^{-1}p) - \lambda(\gamma_0)\| \\ &\le \|\beta_{\xi'}(\gamma_i p, \gamma_i\gamma_0 p) - \lambda(\gamma_0)\| + \|\beta_{\xi'}(p, \gamma_i p) - \beta_{\xi'}(\gamma_i\gamma_0\gamma_i^{-1}p, \gamma_i\gamma_0 p)\| \\ &= \|\beta_{\gamma_i^{-1}\xi'}(p, \gamma_0 p) - \lambda(\gamma_0)\| + \|\beta_{\gamma_i^{-1}\xi'}(\gamma_i^{-1}p, p) - \beta_{\gamma_0^{-1}\gamma_i^{-1}\xi'}(\gamma_i^{-1}p, p)\| \\ &\le \varepsilon/4 + \varepsilon/2 < \varepsilon, \end{aligned}$$

which verifies that $D(\gamma_i\xi_0, r)$ belongs to the family $\mathcal{B}_R(\gamma_0, \varepsilon)$.

We now check that $\xi \in D(\gamma_i\xi_0, r)$. Since $\gamma_i^{-1}\xi \to \xi_0$, we may assume that for all i,

$$d_p(\xi_0, \gamma_i^{-1}\xi) < \frac{1}{3N_0} e^{-\|\psi\|\varepsilon} r. \tag{10.16}$$

Since $r \le s(\gamma_0)$, (10.10), (10.13), and (10.16) imply that $\gamma_i^{-1}\xi \in \mathcal{O}_{\varepsilon/(4\kappa)}(\gamma_i^{-1}p, p)$. Since $\xi_0 \in \mathcal{O}_{\varepsilon/(4\kappa)}(\gamma_i^{-1}p, p)$ as well, we have

$\|\beta_{\gamma_i^{-1}\xi}(\gamma_i^{-1}p, p) - \underline{a}(\gamma_i^{-1}p, p)\| \le \varepsilon/4$ and $\|\beta_{\xi_0}(\gamma_i^{-1}p, p) - \underline{a}(\gamma_i^{-1}p, p)\| \le \varepsilon/4$,

by Lemma 5.7. Note that

$$\begin{aligned} d_p(\gamma_i\xi_0, \xi) &= d_{\gamma_i^{-1}p}(\xi_0, \gamma_i^{-1}\xi) \\ &= e^{-\psi(\beta_{\xi_0}(\gamma_i^{-1}p,p)+\mathrm{i}\,\beta_{\gamma_i^{-1}\xi}(\gamma_i^{-1}p,p))} d_p(\xi_0, \gamma_i^{-1}\xi) \\ &\le e^{-\psi(\underline{a}(\gamma_i^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma_i^{-1}p,p))+\frac{1}{2}\|\psi\|\varepsilon} d_p(\xi_0, \gamma_i^{-1}\xi) \\ &\le \frac{1}{3N_0} e^{-\psi(\underline{a}(\gamma_i^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma_i^{-1}p,p))} r \text{ by (10.16).} \end{aligned}$$

This proves that $\xi \in D(\gamma_i\xi_0, r)$. ∎

Consider the following measure $\nu_p = \nu_{\psi,p}$ on Λ:

$$d\nu_p(\xi) = e^{\psi(\beta_\xi(o,p))} d\nu_\psi(\xi).$$

Proposition 10.17. *Let $B \subset \mathcal{F}$ be a Borel subset with $\nu_p(B) > 0$. Then for ν_p-a.e. $\xi \in B$,*

$$\lim_{R\to 0} \sup_{\xi\in D, D\in\mathcal{B}_R(\gamma_0,\varepsilon)} \frac{\nu_p(B\cap D)}{\nu_p(D)} = 1.$$

Proof. For a given Borel function $h : \mathcal{F} \to \mathbb{R}$, we define $h^* : \mathcal{F} \to \mathbb{R}$ by

$$h^*(\xi) = \lim_{R\to 0} \sup_{\xi\in D, D\in\mathcal{B}_R(\gamma_0,\varepsilon)} \frac{1}{\nu_p(D)} \int_D h\, d\nu_p.$$

By Lemma 10.12, h^* is well defined on Λ_M. Since Λ_M has a full ν_p measure by Theorem 8.9, h^* is defined ν_p-a.e. on $\mathcal{F}$. We will prove that $h = h^*$, ν_p-a.e.; by taking $h = \mathbf{1}_B$, the conclusion of the lemma will follow. Note that $h = h^*$ when h is continuous. To deal with the general case, we proceed as follows.

Step 1: For all $\alpha > 0$,

$$\nu_p(\{h^* > \alpha\}) \le \frac{e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}}{\alpha} \int_{\mathcal{F}} |h|\, d\nu_p.$$

Letting Q be an arbitrary compact subset of $\{\xi : h^*(\xi) > \alpha\}$, it suffices to show that

$$\nu_p(Q) \le \frac{e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}}{\alpha} \int_{\mathcal{F}} |h|\, d\nu_p.$$

Fix $R > 0$. By definition, for each $x \in Q$, there exists $D_x \in \mathcal{B}_R(\gamma_0, \varepsilon)$ containing x such that

$$\frac{1}{\nu_p(D_x)} \int_{D_x} h\, d\nu_p > \alpha.$$

Since K is compact, there exists a finite subcover of $\{D_x : x \in Q\}$, say $D_i = \mathbb{B}_p(\gamma_i\xi_0, s_i)(i = 1, \cdots, n)$ where $\gamma_i \in \Gamma$ and $s_i = \frac{1}{3N_0}e^{-\psi(\underline{a}(\gamma_i^{-1}p,p)+\mathrm{i}\,\underline{a}(\gamma_i^{-1}p,p))}r_i$ for some $0 < r_i < R$.

For brevity, we will write $3N_0D_i := \mathbb{B}_p(\gamma_i\xi_0, 3N_0s_i)$. By Lemma 6.12, there exists a disjoint subcollection $\{D_{i_1}, \cdots, D_{i_\ell}\}$ such that

$$\bigcup_{k=1}^{n} D_k \subset \bigcup_{j=1}^{\ell} 3N_0D_{i_j}.$$

Now we claim that $3N_0D_{i_j} \subset \gamma_{i_j}{\gamma_0}^{-1}\gamma_{i_j}^{-1}D_{i_j}$: note that for $\xi \in 3N_0D_{i_j}$,

$$\begin{aligned} d_p(\gamma_{i_j}\xi_0, \gamma_{i_j}\gamma_0\gamma_{i_j}^{-1}\xi) &= d_{\gamma_{i_j}{\gamma_0}^{-1}\gamma_{i_j}^{-1}p}(\gamma_{i_j}\xi_0, \xi) \\ &= e^{-\psi(\beta_{\gamma_{i_j}\xi_0}(\gamma_{i_j}{\gamma_0}^{-1}\gamma_{i_j}^{-1}p,p)+\mathrm{i}\,\beta_\xi(\gamma_{i_j}{\gamma_0}^{-1}\gamma_{i_j}^{-1}p,p))}d_p(\gamma_{i_j}\xi_0, \xi) \\ &\le 3N_0e^{-\psi(\lambda(\gamma_0)+\mathrm{i}\,\lambda(\gamma_0))+\|\psi\|\varepsilon}s_{i_j} < s_{i_j}, \end{aligned}$$

by (6.3), (5.2), (10.9) and (10.8). Hence

$$\begin{aligned} \nu_p(3N_0D_{i_j}) &\le \nu_p(\gamma_{i_j}{\gamma_0}^{-1}\gamma_{i_j}^{-1}D_{i_j}) \\ &= \int_{D_{i_j}} e^{\psi(\beta_\xi(e,\gamma_{i_j}\gamma_0\gamma_{i_j}^{-1}))}\,d\nu_p(\xi) \\ &\le e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}\nu_p(D_{i_j}), \end{aligned}$$

where the last inequality follows from (10.9). Therefore,

$$\begin{aligned} \nu_p(Q) &\le \sum_{j=1}^{\ell}\nu_p(3N_0D_{i_j}) \le \sum_{j=1}^{\ell} e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}\nu_p(D_{i_j}) \\ &\le \frac{e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}}{\alpha}\sum_{j=1}^{\ell}\int_{D_{i_j}} h\,d\nu_p \le \frac{e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}}{\alpha}\int_{\mathcal{F}}|h|\,d\nu_p, \end{aligned}$$

which was to be proved.

Step 2: $h(\xi) = h^*(\xi)$ for ν_p-a.e ξ.

We first prove that $h(\xi) \le h^*(\xi)$ for ν_p-a.e ξ. Let $\alpha > 0$ be arbitrary. It suffices to show that $\nu_p(\{\xi : h(\xi) - h^*(\xi) > \alpha\}) = 0$. Let h_n be a continuous function converging to h in $L^1(\nu_p)$. Note that $h_n^* = h_n$ and

$$\begin{aligned} &\nu_p(\{\xi : h(\xi) - h^*(\xi) > \alpha\}) \\ &\le \nu_p(\{\xi : h(\xi) - h_n(\xi) > \alpha/2\}) + \nu_p(\{\xi : h_n^*(\xi) - h^*(\xi) > \alpha/2\}) \\ &\le \tfrac{2}{\alpha}\|h - h_n\|_1 + \tfrac{2}{\alpha}e^{\psi(\lambda(\gamma_0))+\|\psi\|\varepsilon}\|h - h_n\|_1. \end{aligned}$$

Taking $n \to \infty$, we get

$$\nu_p(\{\xi : h(\xi) - h^*(\xi) > \alpha\}) = 0.$$

As $\alpha > 0$ is arbitrary, it follows that $h \le h^*$, ν_p-a.e. Similar argument shows that $h^* \le h$, ν_p -a.e. $\square$

Proof of Proposition 10.7: It is easy to check that $\mathsf{E}_\nu(\Gamma_0) = \mathsf{E}_{\nu_p}(\Gamma_0)$. Hence it suffices to show $\lambda(\gamma_0) \in \mathsf{E}_{\nu_p}(\Gamma_0)$. Let $B \subset \mathcal{F}$ be a Borel subset with $\nu_p(B) > 0$ and $\varepsilon > 0$. By Proposition 10.17, there exists $D = \mathbb{B}_p(\gamma\xi_0, r) \in \mathcal{B}_R(\gamma_0, \varepsilon)$ for $\gamma \in \Gamma$ and $r > 0$ such that

$$\nu_p(D \cap B) > (1 + e^{-\psi(\lambda(\gamma_0)) - \|\psi\|\varepsilon})^{-1} \nu_p(D). \tag{10.18}$$

Since $r < r_p(\gamma)$, we have

$$\begin{aligned} D &\subset \{\xi : \|\beta_\xi(p, \gamma{\gamma_0}^{\pm}\gamma^{-1}p) \mp \lambda(\gamma_0)\| \le \varepsilon\} \\ &\subset \{\xi : |\psi(\beta_\xi(p, \gamma{\gamma_0}^{\pm}\gamma^{-1}p)) \mp \psi(\lambda(\gamma_0))| \le \|\psi\|\varepsilon\}. \end{aligned}$$

We note that $\gamma\gamma_0\gamma^{-1}D \subset D$: if $\xi \in D$, by (6.3),

$$\begin{aligned} d_p(\gamma\xi_0, \gamma\gamma_0\gamma^{-1}\xi) &= d_{\gamma{\gamma_0}^{-1}\gamma^{-1}p}(\gamma\xi_0, \xi) \\ &= e^{\psi(\beta_{\gamma\xi_0}(p, \gamma\gamma_0^{-1}\gamma^{-1}p) + \mathrm{i}\,\beta_\xi(p, \gamma\gamma_0^{-1}\gamma^{-1}p))} d_p(\gamma\xi_0, \xi) \\ &\le e^{-\psi(\lambda(\gamma_0) + \mathrm{i}\,\lambda(\gamma_0)) + \|\psi\|\varepsilon} r < r. \end{aligned}$$

Since

$$B \cap \gamma\gamma_0\gamma^{-1}B \cap \{\xi : \|\beta_\xi(p, \gamma\gamma_0\gamma^{-1}p) - \lambda(\gamma_0)\| < \varepsilon\} \supset (D \cap B) \cap \gamma\gamma_0\gamma^{-1}(D \cap B),$$

it suffices to prove that $(D \cap B) \cap \gamma\gamma_0\gamma^{-1}(D \cap B)$ has a positive ν_p-measure. Note that

$$\begin{aligned} \nu_p(\gamma\gamma_0\gamma^{-1}(D \cap B)) &= \int_{D \cap B} e^{\psi(\beta_\xi(p, \gamma{\gamma_0}^{-1}\gamma^{-1}p))} \, d\nu_p(\xi) \\ &\ge e^{-\psi(\lambda(\gamma_0)) - \|\psi\|\varepsilon} \nu_p(D \cap B). \end{aligned}$$

Hence by (10.18),

$$\nu_p(D \cap B) + \nu_p(\gamma\gamma_0\gamma^{-1}(D \cap B)) > (1 + e^{-\psi(\lambda(\gamma_0)) - \|\psi\|\varepsilon}) \nu_p(D \cap B) > \nu_p(D).$$

Since both $D \cap B$ and $\gamma\gamma_0\gamma^{-1}(D \cap B)$ are contained in D, this implies that their intersection has a positive ν_p-measure. Since $\gamma\gamma_0\gamma^{-1} \in \Gamma_0$, it follows that $\lambda(\gamma_0) \in \mathsf{E}_{\nu_p}(\Gamma_0)$.

In view of Proposition 9.2, we obtain the following corollary:

Corollary 10.19. *Let Γ_0 be a Zariski dense normal subgroup of an Anosov subgroup $\Gamma < G$. Let $\psi \in D_\Gamma^\star$. If ν_ψ is Γ_0-ergodic, then m_ψ^{BR}, considered as a measure on $\Gamma_0 \backslash G$, is NM-ergodic.*

Patterson Sullivan measures are mutually singular.

Theorem 10.20. *Let $\Gamma < G$ be an Anosov subgroup. Then $\{\nu_\psi : \psi \in D_\Gamma^\star\}$ are pairwise mutually singular.*

Proof. Since $\Gamma < G$ is Anosov, the family $\{\nu_\psi : \psi \in D_\Gamma^\star\}$ consists of Γ-ergodic measures (see the remark following Theorem 4.3). Hence any ν_{ψ_1} and ν_{ψ_2} in this family are either mutually singular or absolutely continuous to each other. Now the claim follows from Lemma 10.21 below, in view of Proposition 10.2. $\square$

Lemma 10.21. *For $i = 1, 2$, let ν_{ψ_i} be a (Γ, ψ_i)-PS measure for some $\psi_i \in \mathfrak{a}^*$. If $\mathsf{E}_{\nu_{\psi_2}} = \mathfrak{a}$ and $\nu_{\psi_1} \ll \nu_{\psi_2}$, then $\psi_1 = \psi_2$.*

Proof. Suppose that $\nu_{\psi_1} \ll \nu_{\psi_2}$ and that $\psi_1 \neq \psi_2$. Consider the Radon-Nikodym derivative $f := \frac{d\nu_{\psi_1}}{d\nu_{\psi_2}} \in L^1(\Lambda, \nu_{\psi_2})$. Note that there exists a ν_{ψ_2}-conull set $E \subset \Lambda$ such that for all $\xi \in E$ and $\gamma \in \Gamma$, we have

$$f(\gamma^{-1}\xi) = e^{(\psi_1 - \psi_2)(\beta_\xi(e,\gamma))} f(\xi). \tag{10.22}$$

If f were continuous, then $f \neq 0$ and by applying $\xi = y_\gamma^+$ in the above, we get $\psi_1(\lambda(\gamma)) = \psi_2(\lambda(\gamma))$ for all $\gamma \in \Gamma$. Since $\lambda(\Gamma)$ generates a dense subgroup of $\mathfrak{a}$, it follows that $\psi_1 = \psi_2$.

In general, we use the hypothesis $\mathsf{E}_{\nu_{\psi_2}} = \mathfrak{a}$. Choose $0 < r_1 < r_2$ such that

$$B := \{\xi \in \Lambda : r_1 < f(\xi) < r_2\}$$

has a positive ν_{ψ_2}-measure. Since $\psi_1 \neq \psi_2$, we can choose $w \in \mathfrak{a}$ such that

$$e^{(\psi_1 - \psi_2)(w)} > \tfrac{2r_2}{r_1}. \tag{10.23}$$

Choose $\varepsilon > 0$ such that $e^{\|\psi_1 - \psi_2\|\varepsilon} < 2$. Since $\nu_{\psi_2}(B) > 0$ and $\mathsf{E}_{\nu_{\psi_2}} = \mathfrak{a}$, there exists $\gamma \in \Gamma$ such that

$$B' := B \cap \gamma B \cap \{\xi \in \Lambda : \|\beta_\xi(e,\gamma) - w\| < \varepsilon\}$$

has a positive ν_{ψ_2}-measure. Now note that

$$\begin{aligned} \int_{B'} f(\gamma^{-1}\xi)\, d\nu_{\psi_2}(\xi) &> e^{(\psi_1-\psi_2)(w) - \|\psi_1 - \psi_2\|\varepsilon} \int_{B'} f(\xi)\, d\nu_{\psi_2}(\xi) \\ &> \tfrac{r_2}{r_1} \int_{B'} f(\xi)\, d\nu_{\psi_2}(\xi) \end{aligned}$$

by (10.22), (10.23), and the choice of ε. In particular,

$$\nu_{\psi_2}\left\{\xi \in B' : f(\gamma^{-1}\xi) > \tfrac{r_2}{r_1} f(\xi)\right\} > 0.$$

It follows that there exists $\xi \in B' \cap E$ such that

$$f(\gamma^{-1}\xi) > \tfrac{r_2}{r_1} f(\xi). \tag{10.24}$$

On the other hand, for $\xi \in B'$, both ξ and $\gamma^{-1}\xi$ belong to B. Hence, by definition of B, for all $\xi \in B'$, we have

$$f(\gamma^{-1}\xi) < \tfrac{r_2}{r_1} f(\xi).$$

This is a contradiction to (10.24). $\square$

P-semi-invariant measures. In this section, we establish that P-semi-invariant Radon measures supported in $\mathcal{E} = \{x \in \Gamma\backslash G : x^+ \in \Lambda\}$, up to constant multiples, are parametrized by $D_\Gamma^\star$.

If μ is P-semi-invariant, then there exists a linear form $\chi_\mu \in \mathfrak{a}^*$ such that for all $a \in A$,

$$a_*\mu = e^{-\chi_\mu(\log a)}\mu.$$

We set $\psi_\mu := \chi_\mu + 2\rho \in \mathfrak{a}^*$. The first part of the following proposition is known in the rank one case (see e.g. [2], [9], and [26]) and the proof can be easily adapted to the higher rank case.

Proposition 10.25. *For any Zariski dense discrete subgroup $\Gamma < G$, any P-semi-invariant Radon measure μ on $\Gamma\backslash G$ is proportional to $m_{\nu_{\psi_\mu}, m_o}$ where ν_{ψ_μ} is a (Γ, ψ_μ)-conformal measure and $\psi_\mu \in D_\Gamma$. Moreover, if μ is supported on $\mathcal{E}$, then μ is proportional to $m^{\mathrm{BR}}_{\psi_\mu}$. If Γ is Anosov, we also have $\psi_\mu \in D_\Gamma^\star$.*

Proof. For simplicity, set $\chi = \chi_\mu$ and $\psi = \psi_\mu$. Let $\tilde{\mu}$ be the Γ-invariant lift of μ to G and $\pi : G \to G/P$ be the projection. Choose a section $c : G/P \to K$ so that $\pi \circ c = \mathrm{id}$ and consider the measurable isomorphism

$$\begin{array}{ccc} G/P \times M \times A \times N & \to & G \\ (\xi,\ m,\ a,\ n) & \to & c(\xi)man. \end{array}$$

Let dm, dn, da be the Haar measures on M, N, and A. As $\tilde{\mu}$ is a P-semi-invariant Radon measure, there exist $\tilde{\chi} \in \mathfrak{a}^*$ and a Radon measure ν on G/P such that

$$d\tilde{\mu}(c(\xi)man) = e^{\tilde{\chi}(\log a)} dn\, da\, dm\, d\nu(\xi).$$

Without loss of generality, we may assume that $|\nu| = 1$. Because $d\tilde{\mu}(\cdot\, a) = e^{\chi(\log a)} d\tilde{\mu}(\cdot)$, we have

$$\chi = \tilde{\chi} - 2\rho, \text{ or equivalently, } \tilde{\chi} = \psi.$$

Note that G is measurably isomorphic to the product $G/P \times P$ and the left Γ-action with respect to these coordinates is given by $\gamma \cdot (\xi, p) = (\gamma \cdot \xi, \Phi(\gamma, \xi)p)$ for some P-valued cocycle $\Phi : \Gamma \times G/P \to P$ where $\gamma \in \Gamma$ and $(\xi, p) \in G/P \times P$. One can check that

$$\Phi(\gamma, \xi) = m(\gamma, \xi) \exp(\beta_\xi(\gamma^{-1}, e)) n(\gamma, \xi)$$

for some $m(\gamma, \xi) \in M$ and $n(\gamma, \xi) \in N$. Hence, for $p = man$, the MAN-coordinates for $\Phi(\gamma, \xi)p$ are given by

$$\Phi(\gamma, \xi)p = \big(m(\gamma, \xi)m\big)\big(\exp(\beta_\xi(\gamma^{-1}, e))a\big)\big((ma)^{-1}n(\gamma, \xi)man\big). \tag{10.26}$$

Since $\tilde{\mu}$ is left-Γ-invariant, we have for any $f \in C_c(G)$ and any $\gamma \in \Gamma$,

$$\begin{aligned}\int_G f(g)\, d\tilde{\mu}(g) &= \int_G f(g)\, d(\gamma_*\tilde{\mu})(g)\\ &= \int_{G/P}\int_P f((\gamma\xi, \Phi(\gamma,\xi)p)e^{\psi(\log a)}\, dn\, da\, dm\, d\nu(\xi)\\ &= \int_{G/P}\int_P f(\xi,p)e^{\psi\left(\log a - \beta_{\gamma^{-1}\xi}(\gamma^{-1},e)\right)}\, dn\, da\, dm\, d(\gamma_*\nu)(\xi),\end{aligned}$$

where in the last equality, we have used (10.26) and the change of variables $a' = a\exp(\beta_\xi(e,\gamma^{-1}))$. On the other hand, we have

$$\int_G f(g)\, d\tilde{\mu}(g) = \int_{G/P}\int_P f(\xi,p)e^{\psi(\log a)} dn\, da\, dm\, d\nu(\xi).$$

By comparing these two identities, we get that for any $\gamma \in \Gamma$,

$$d(\gamma_*\nu)(\xi) = e^{\psi(\beta_\xi(e,\gamma))} d\nu(\xi),$$

that is, ν is a (Γ,ψ)-conformal measure. By [35, Thm. 8.1], $\psi \in D_\Gamma$.

Finally, recall that for all $g \in G$ and $\phi \in C_c(G)$,

$$\int_N \phi(gn)\, dn = \int_{G/P} \phi(gn)e^{2\rho(\beta_{gn^-}(e,gn))} dm_o(gn^-).$$

For $g = c(\xi)man \in KAN$, we have $\beta_{g^+}(e,g) = \log a$ and $g^+ = \xi$. Hence, for any $f \in C_c(G)$,

$$\begin{aligned}\int_G f(g) d\tilde{\mu}(g) &= \int_{G/P}\int_P f(c(\xi)man)e^{\psi(\log a)} dn\, da\, dm\, d\nu(\xi)\\ &= \int_{G/M}\int_M f(g)e^{2\rho(\beta_{g^-}(e,g))}e^{\psi(\beta_{g^+}(e,g))}\, dm\, da\, dm_o(g^-)\, d\nu(g^+)\\ &= \tilde{m}_{\nu,m_o}(f).\end{aligned}$$

Therefore $\tilde{\mu}(f) = \tilde{m}_{\nu,m_o}(f)$.

Now, if μ is supported on $\mathcal{E}$, then ν is supported on Λ. Hence ν is a (Γ,ψ)-PS measure; so $\mu = m^{\mathrm{BR}}_\psi$. When Γ is Anosov, $\psi \in D^\star_\Gamma$ by Theorem 7.7. $\square$

Let $\mathcal{P}_\Gamma$ be the space of all P-semi-invariant Radon measures on $\mathcal{E}$ up to proportionality. Let $\mathcal{Q}_\Gamma$ be the space of all NM-invariant, ergodic and A-quasi-invariant Radon measures supported on $\mathcal{E}$ up to proportionality.

Theorem 10.27. *Let $\Gamma < G$ be an Anosov subgroup. We have $\mathcal{P}_\Gamma = \mathcal{Q}_\Gamma$ and the map $D^\star_\Gamma \to \mathcal{Q}_\Gamma$ given by $\psi \mapsto [m^{\mathrm{BR}}_\psi]$ is a homeomorphism between $D^\star_\Gamma$ and $\mathcal{Q}_\Gamma$. In particular, $\mathcal{Q}_\Gamma$ is homeomorphic to $\mathbb{R}^{\text{rank}\, G-1}$.*

Proof. For $\mu \in \mathcal{Q}_\Gamma$ and $a \in A$, $a_*\mu$ and μ are equivalent to each other, and by the NM-ergodicity of μ, the Radon-Nikodym derivative $da_*\mu/d\mu$ is constant, say $\chi(a)$. Now the function $a \mapsto \chi(a)$ gives the semi-invariance of μ by A and hence by P. This implies $\mathcal{Q}_\Gamma \subset \mathcal{P}_\Gamma$. The other direction $\mathcal{P}_\Gamma \subset \mathcal{Q}_\Gamma$ follows from Proposition 10.25 and Theorem 10.1.

Let $\mathcal{Q}_\Gamma^\spadesuit$ be the space of all NM-ergodic A-quasi-invariant Radon measures supported on $\{x \in \Gamma\backslash G : x^+ \in \Lambda\}$, so that $\mathcal{Q}_\Gamma = \mathcal{Q}_\Gamma^\spadesuit / \sim$. Set $\iota(\psi) = m_\psi^{\mathrm{BR}}$ for $\psi \in D_\Gamma^\star$. Since m_ψ^{BR} is NM-ergodic by Theorem 10.1, the map $\iota : D_\Gamma^\star \to \mathcal{Q}_\Gamma^\spadesuit$ is well defined and injective by Lemma 10.21. By Proposition 10.25, $\iota(D_\Gamma^\star)$ contains precisely one representative of each class in $\mathcal{Q}_\Gamma$. Hence it suffices to show that the map ι gives a homeomorphism between $D_\Gamma^\star$ and its image $\iota(D_\Gamma^\star)$. Continuity of ι follows from Theorem 7.7. Now, suppose that $m_{\psi_i}^{\mathrm{BR}} \to m_\psi^{\mathrm{BR}}$ for some sequence $\psi_i, \psi \in D_\Gamma^\star$. Then the A-semi-invariance of the BR-measures given by (3.20) and the convergence $a_* m_{\psi_i}^{\mathrm{BR}} \to a_* m_\psi^{\mathrm{BR}}$ implies that $\lim_{i\to\infty} e^{(2\rho-\psi_i)(\log a)} m_{\psi_i}^{\mathrm{BR}}(f) = e^{(2\rho-\psi)(\log a)} m_\psi^{\mathrm{BR}}(f)$ for all $f \in C_c(\Gamma\backslash G)$. Since $\lim_{i\to\infty} m_{\psi_i}^{\mathrm{BR}}(f) = m_\psi^{\mathrm{BR}}(f)$, we get $\lim_{i\to\infty} e^{(2\rho-\psi_i)(\log a)} = e^{(2\rho-\psi)(\log a)}$ for all $a \in A$. Hence $\psi_i \to \psi$. This proves that $D_\Gamma^\star$ and $\mathcal{Q}_\Gamma$ are homeomorphic to each other. The last claim follows from Proposition 4.4. $\square$

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