EIGENVALUES OF CONGRUENCE COVERS OF GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

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ABSTRACT. Let $G = \mathrm{SO}(n, 1)^\circ$ for $n \geq 2$ and Γ a geometrically finite Zariski dense subgroup of G which is contained in an arithmetic subgroup of G. Denoting by $\Gamma(q)$ the principal congruence subgroup of Γ of level q, and fixing a positive number λ_0 strictly smaller than $(n-1)^2/4$, we show that, as $q \to \infty$ along primes, the number of Laplacian eigenvalues of the congruence cover $\Gamma(q)\backslash\mathbb{H}^n$ smaller than λ_0 is at most of order $[\Gamma : \Gamma(q)]^c$ for some $c = c(\lambda_0) > 0$.

1. INTRODUCTION

Let G denote identity component of the special orthogonal group $\mathrm{SO}(n, 1)$ for $n \geq 2$. As well-known, G is the group of orientation preserving isometries of the real hyperbolic space $\mathbb{H}^n$. Denote by Δ the negative of the Laplace-Beltrami operator of $\mathbb{H}^n$. On any complete hyperbolic manifold M of dimension n, Δ acts on the space of smooth functions on M with compact support and admits a unique extension to an unbounded self-adjoint positive operator on $L^2(M)$. We denote by $\sigma(M)$ its spectrum of M. For instance, $\sigma(\mathbb{H}^n)$ is known to be $[\frac{(n-1)^2}{4}, \infty)$.

There exists a torsion-free discrete subgroup Γ of G such that $M = \Gamma\backslash\mathbb{H}^n$. The limit set $\Lambda(\Gamma)$ of Γ is the smallest non-empty closed Γ-invariant subset of the geometric boundary of $\mathbb{H}^n$. The convex core $\mathcal{C}(M)$ of M is the quotient by Γ of the smallest convex subset of $\mathbb{H}^n$ containing all geodesics connecting points in $\Lambda(\Gamma)$. We say that M (or Γ) is *geometrically finite* if the unit neighborhood of $\mathcal{C}(M)$ has finite volume. Geometrically finite manifolds are natural generalizations of manifolds with finite volume.

For a geometrically finite hyperbolic manifold M of infinite volume, Lax and Phillips [12] showed that $\sigma(M)$ is the disjoint union of the discrete spectrum consisting of finitely many eigenvalues contained in

$[0, \frac{(n-1)^2}{4})$ with finite multiplicities and the essential spectrum which is a closed sub-interval of $[\frac{(n-1)^2}{4}, \infty)$.

In this note, we are interested in the growth of the number of discrete eigenvalues of congruence covers of a fixed hyperbolic manifold, which itself is a covering (with infinite degree) of an arithmetic hyperbolic manifold of finite volume.

Let G be defined over $\mathbb{Q}$ via a $\mathbb{Q}$-embedding $G \to \mathrm{GL}_N$ for some positive integer N. We set $G(\mathbb{Z}) = G \cap \mathrm{GL}_N(\mathbb{Z})$. Fixing a subgroup Γ of $G(\mathbb{Z})$, and a positive integer q, we consider the q-th principal congruence subgroup $\Gamma(q)$ of Γ given by

$$\Gamma(q) := \{\gamma \in \Gamma : \gamma \equiv e \mod q\}.$$

We denote by $\delta = \delta(\Gamma)$ the critical exponent of Γ, i.e., the abscissa of convergence of the Poincaré series $\mathcal{P}(t) = \sum_{\gamma \in \Gamma} e^{-td(o,\gamma(o))}$ for $o \in \mathbb{H}^n$. In the rest of the introduction, we assume that Γ is torsion-free, geometrically finite and Zariski dense in G. For a fixed $0 < \lambda_0 < \frac{(n-1)^2}{4}$, denote by

$$\mathcal{N}(\lambda_0, \Gamma(q))$$

the number of eigenvalues of $\Gamma(q)\backslash\mathbb{H}^n$ contained in the interval $[0, \lambda_0]$, counted with multiplicities. Here is our main theorem:

Theorem 1.1. *There exists $\eta > 0$ such that for any $0 < \lambda_0 < \frac{(n-1)^2}{4}$,*

$$\mathcal{N}(\lambda_0, \Gamma(q)) \ll [\Gamma : \Gamma(q)]^{\frac{\delta - s_0}{\eta}} \quad \text{for any } q \text{ prime}, \tag{1.2}$$

where $\frac{n-1}{2} < s_0 \le n-1$ such that $\lambda_0 = s_0(n-1-s_0)$ and the implied constant depends only on λ_0.

In [8], Hamenstädt showed that the number of eigenvalues of $\Gamma\backslash\mathbb{H}^n$ smaller than λ_0 is bounded from above by $b^{\mathrm{Vol}(\mathcal{C}(\Gamma\backslash\mathbb{H}^n))}$ for some $b > 0$ depending only on the dimension n. In case when Γ is a lattice, a stronger upper bound of $c \cdot \mathrm{Vol}(\Gamma\backslash\mathbb{H}^n)$ (c independent of Γ) was previously known by Buser-Colbois-Dodziuk [1].

Since $\Gamma(q)$ is a normal subgroup of Γ of finite index, the limit set of $\Gamma(q)$ is equal to the limit set of Γ, and hence

$$\frac{\mathrm{Vol}(\mathcal{C}(\Gamma(q)\backslash\mathbb{H}^n))}{\mathrm{Vol}(\mathcal{C}(\Gamma\backslash\mathbb{H}^n))} = [\Gamma : \Gamma(q)].$$

Therefore Theorem 1.1 implies the following much stronger upper bound for congruence coverings of an arithmetic manifold $\Gamma\backslash\mathbb{H}^n$ of infinite volume: as $q \to \infty$ along primes,

$$\mathcal{N}(\lambda_0, \Gamma(q)) \ll \mathrm{Vol}(\mathcal{C}(\Gamma(q)\backslash\mathbb{H}^n))^{\frac{\delta - s_0}{\eta}}. \tag{1.3}$$

Remark 1.4. (1) We can relax the restriction on q so that q is square-free with no divisors from a fixed finite set of primes. The necessity of avoiding a finite set of primes is basically because the strong approximation property applied to the lift $\tilde{\Gamma}$ of Γ in the Spin cover $\tilde{G}$ says $\tilde{\Gamma}(p)\backslash\tilde{\Gamma} = \tilde{G}(\mathbb{F}_p)$ for all but finitely many primes p.

(2) As $\Gamma(q)$ is geometrically finite, the discrete spectrum of $\Gamma(q)\backslash\mathbb{H}^n$ is non-empty only when $\delta > \frac{n-1}{2}$, in which case, $\delta(n-1-\delta)$ is the smallest discrete eigenvalue and has multiplicity one [22].

(3) In our proof of Theorem 1.1, η can be taken to be any number smaller than the uniform spectral gap

$$\liminf_{q:\text{primes}}(\delta - s_{1q})$$

where $s_{1q} < \delta$ is such that $s_{1q}(n-1-s_{1q})$ is the second smallest eigenvalue of $\Gamma(q)\backslash\mathbb{H}^n$. The existence of a *positive* uniform spectral gap follows from the works of Bourgain-Gamburd [2] and of Bourgain-Gamburd-Sarnak [4] for $n = 2$. Their result has been generalized by Salehi-Golsefidy-Varju [17] for a general connected semisimple algebraic group. These methods do not provide an explicit estimate on η.

For the following discussion, we find it convenient to put $\rho = \frac{n-1}{2}$ and to use the parametrization $\lambda_s = s(n-1-s)$ so that

$$[0, \tfrac{(n-1)^2}{4}) = \{\lambda_s : s \in (\rho, n-1]\}.$$

For each $s \in (\rho, n-1]$, we denote by

$$m(\lambda_s, \Gamma(q))$$

the multiplicity of λ_s occurring as a discrete eigenvalue of Δ in $\Gamma(q)\backslash\mathbb{H}^n$. Note that

$$\mathcal{N}(s_0(n-1-s_0), \Gamma(q)) = \sum_{s_0 \le s \le n-1} m(\lambda_s, \Gamma(q)).$$

Sarnak and Xue formulated a conjecture on the upper bound of $m(\lambda_s, \Gamma(q))$ when Γ is an arithmetic group of a connected semisimple algebraic group G defined over $\mathbb{Q}$ [19]. In our setting of $G = \mathrm{SO}(n,1)^\circ$, their conjecture can be stated as follows:

Conjecture 1.5 (Sarnak-Xue)**.** *If $[G(\mathbb{Z}) : \Gamma] < \infty$, then for any fixed $s \in (\rho, n-1]$, as $q \to \infty$,*

$$m(\lambda_s, \Gamma(q)) \ll_\epsilon [\Gamma : \Gamma(q)]^{\frac{(n-1)-s}{\rho}+\epsilon} \quad \textit{for any } \epsilon > 0.$$

Fixing $o \in \mathbb{H}^n$ and denoting by d the hyperbolic distance in $\mathbb{H}^n$, consider the ball $B_T := \{g \in G : d(g(o), o) \le T\}$; so that

$$\#\Gamma(q) \cap B_T := \#\{\gamma \in \Gamma(q) : d(\gamma(o), o) \le T\}.$$

Conjecture 1.6 (Sarnak-Xue)**.** *Let* $[G(\mathbb{Z}) : \Gamma] < \infty$*. For any* $T \gg 1$ *and* $q \gg 1$*,*

$$\#\Gamma(q) \cap B_T \ll_\epsilon \frac{e^{(n-1+\epsilon)T}}{[\Gamma : \Gamma(q)]} + e^{\rho T} \quad \textit{for any } \epsilon > 0$$

where $\rho = (n-1)/2$ *and the implied constant is independent of* T *and* q*.*

Sarnak and Xue proved that if Γ is a uniform lattice in G, then Conjecture 1.6 implies Conjecture 1.5; in fact, their methods prove a stronger statement that the same upper bound works for $\mathcal{N}(s(n-1-s), \Gamma(q))$ as well. This implication has been extended to all lattices in G by Huntley and Katznelson [9]. Sarnak and Xue [19] proved Conjecture 1.6 when Γ is a cocompact arithmetic subgroup of $\mathrm{SO}(n,1)$ for $n = 2, 3$ and hence settled Conjecture 1.5 in this case.

In view of the conjectures (1.5) and (1.6) and the above discussion, we pose the following two conjectures: let $\Gamma < G(\mathbb{Z})$ be geometrically finite and Zariski dense in G with its critical exponent $\delta > \rho := (n-1)/2$.

Conjecture 1.7. *For any* $s \in (\rho, \delta]$*, as* $q \to \infty$*,*

$$\mathcal{N}(s(n-1-s), \Gamma(q)) \ll_\epsilon [\Gamma : \Gamma(q)]^{\frac{\delta - s}{\delta - \rho} + \epsilon} \quad \textit{for any } \epsilon > 0.$$

Remark 1.8. It is proved [10] that for any $\eta > 0$,

$$m(\lambda_s, \Gamma(q)) \gg [\Gamma : \Gamma(q)]^{2/n(n-1) - \eta}.$$

Therefore Conjecture 1.7 implies the following spectral gap:

$$\limsup_q s_{1q} \le \delta - \frac{\delta - \rho}{n(n-1)}$$

where $s_{1q}(n-1-s_{1q})$ is the second smallest eigenvalue of $\Gamma(q)\backslash\mathbb{H}^n$. We thank Dubi Kelmer for this observation.

Conjecture 1.9. *For* $T \gg 1$*, as* $q \to \infty$*,*

$$\#\Gamma(q) \cap B_T \ll_\epsilon \frac{e^{(\delta+\epsilon)T}}{[\Gamma : \Gamma(q)]} + e^{\rho T} \quad \textit{for any } \epsilon > 0.$$

Proposition 1.10. *Conjecture 1.9 implies Conjecture 1.7.*

Note that Conjecture 1.7 implies the following limit formula, which is also suggested by the Plancherel formula of $L^2(G)$ given by Harish-Chandra:

Conjecture 1.11. *For any* $s \in (\rho, \delta]$,

$$\lim_{q\to\infty} \frac{\mathcal{N}(s(n-1-s), \Gamma(q))}{[\Gamma : \Gamma(q)]} = 0.$$

Conjecture 1.11 is known to be true if Γ is a co-compact lattice by DeGeorge and Wallach [5] or if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ by Sarnak [18]. It does not seem to be known for a general (arithmetic) lattice, although Savin proved it for those s whose corresponding eigenfunctions are cusp-forms [20]. We also refer to [6] where an analogous problem was answered positively for $\Gamma = \mathrm{SL}_n(\mathbb{Z})$.

The proof of Conjecture 1.5 by Sarnak-Xue [19] for co-compact arithmetic lattices of $\mathrm{SO}(2,1)$ and $\mathrm{SO}(3,1)$ uses number theoretic arguments which give a very sharp uniform upper bound (Conjecture 1.6) for the number of lattice points $\Gamma(q)$ in a ball, using explicit realizations of arithmetic groups as units of certain division algebras over number fields. The presence of ρ in the denominator of the exponent in Conjecture 1.5 (Theorem for the cases in discussion) is due to the sharpness of this counting technique.

This approach won't be possible for general Zariski dense subgroups, as such an explicit arithmetic realization is not available for this rather wild class of groups (which makes one wonder that Conjecture 1.9 is perhaps too bold).

Instead, we use a recent work of Mohammadi and the author [15] where uniform counting results for orbits of $\Gamma(q)$'s were obtained with an error term.

Another important ingredient is the so-called *Collar Lemma* on uniform estimates on the size of eigenfunctions away from flares and cusps of $\Gamma(q)\backslash\mathbb{H}^n$; this was first obtained by Gamburd in [7] for $\mathrm{SO}(2,1)$ and generalized by Magee [13] for all $\mathrm{SO}(n,1)$.

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In the rest of this paper, let Γ be geometrically finite and Zariski dense, with critical exponent $\delta > (n-1)/2$. We continue notations from the introduction.

2.1. **Lattice point counts.** We identify $\mathbb{H}^n = G/K$ for $K = \mathrm{SO}(n)$ and denote by $o \in \mathbb{H}^n$ whose stabilizer is K. As before, set $B_T := \{g \in G : d(g(o), o) \le T\}$.

We first recall the following lattice point counting theorem in [15]:

Theorem 2.1. *There exists $\eta > 0$ such that for any prime q,*

$$\#\Gamma(q) \cap B_T = c \cdot \frac{e^{\delta T}}{[\Gamma : \Gamma(q)]} + O(e^{(\delta-\eta)T}) \quad \text{for all } T \gg 1$$

where both $c > 0$ and the implied constant are independent of q.

We remark that the above type of lattice point counting theorem is stated in [15] under a uniform spectral gap hypothesis. However since B_T is a bi-K-invariant subset, one needs only a uniform *spherical* spectral gap, as defined in [15], in proving Theorem 2.1. We recall that a uniform spherical spectral gap means a uniform lower bound for the differences for the smallest two eigenvalues for the Laplace operator in $L^2(\Gamma(q)\backslash\mathbb{H}^n)$. And the uniform spherical spectral gap property for $L^2(\Gamma(q)\backslash\mathbb{H}^n)$ for q primes (or for q square-free with no divisors from a fixed finite set of primes) follows from [17] by the transfer principle from combinatorial spectral gap to an archimedean gap ([4], also see [11]).

2.2. Fix a Haar measure dg on G. This induces an invariant measure on $\Gamma(q)\backslash G$ for which we use the same notation dg by abuse of notation. The right translation action of G on $L^2(\Gamma(q)\backslash G, dg)$ preserves the measure dg, and hence yields a unitary representation of G. Let $\mathcal{C}$ denote the Casimir operator of G. The action of $\mathcal{C}$ on K-invariant smooth functions on G is given by $-\Delta$. For each $s \in (\rho, \delta]$, we denote by π_s the spherical complementary series representation of G on which $\mathcal{C}$ acts by the scalar $-\lambda_s$. Then the multiplicity $m(\lambda_s, \Gamma(q))$ is equal to the multiplicity of π_s occurring as a sub-representation of $L^2(\Gamma(q)\backslash G)$. This follows from the well-known correspondence between positive definite spherical functions of G and the spherical unitary dual of G.

Define the bi-K-invariant function of G:

$$\psi_s(g) := \langle \pi_s(g) v_s, v_s \rangle$$

where v_s is the unique K-fixed unit vector in π_s, up to a scalar. Then $\psi_s(g)$ is a positive function such that for all $g \in G$,

$$\psi_s(g) \asymp e^{(s-2\rho)d(o,g(o))} \tag{2.2}$$

where the implied constants are independent of $g \in G$ (cf. [15]).

2.3. Proof of Theorem 1.1. We need to show that there exists $\eta > 0$ such that for any $\rho < s_0 \le n-1$, we have, as $q \to \infty$ along primes,

$$\sum_{s \in [s_0, n-1]} m(\lambda_s, \Gamma(q)) \ll [\Gamma : \Gamma(q)]^{\frac{\delta - s_0}{\eta}},$$

where the implied constant is independent of q.

We follow a general strategy of [19] (also see [7]).

Consider the following bi-K-invariant functions of G:

$$f_o(g) := \chi_{B_T}(g)\psi_{s_0}(g) \quad \text{and}$$
$$F_o(g) := f_o * \check{f}_o(g) = \int_{h \in G} f_o(gh^{-1})\check{f}_o(h)dh$$

where χ_{B_T} is the characteristic function of B_T and $\check{f}_o(g) := \overline{f_o(g^{-1})}$.

By [19], we have

$$F_o(g) \ll \begin{cases} e^{2(s_0-\rho)T} e^{-\rho d(g(o),o)} & \text{if } d(o, g(o)) \le 2T \\ 0 & \text{if } d(o, g(o)) > 2T. \end{cases} \tag{2.3}$$

The spherical transform $\hat{f}$ of a bi-K-invariant function f is given by

$$\hat{f}(\lambda_s) = \int_G f(g)\psi_s(g)dg.$$

Hence

$$\hat{f}_o(\lambda_s) = \int_G \chi_{B_T}(g)\psi_{s_0}(g)\psi_s(g)dg$$

and the associated spherical transform $\hat{F}_o(\lambda_s)$ is given by $|\hat{f}_o(\lambda_s)|^2$.

By (2.2), it follows that for all $s \ge s_0$,

$$\hat{F}_o(\lambda_s) \gg e^{(2s_0+2s-4\rho)T} \ge e^{(4s_0-4\rho)T} \tag{2.4}$$

where the implied constant depends only on s_0.

Define the automorphic kernel K_q on $\Gamma(q)\backslash G \times \Gamma(q)\backslash G$ as follows:

$$K_q(g_1, g_2) := \sum_{\gamma \in \Gamma(q)} F_o(g_1^{-1}\gamma g_2).$$

Note $K_q(g_1k_1, g_2k_2) = K_q(g_1, g_2)$ for any $k_1, k_2 \in K$.

Let $\{\lambda_{j,q}\}$ be the multi-set of discrete eigenvalues of $\Gamma(q)\backslash\mathbb{H}^n$ which is finite by Lax-Phillips and let $\{\phi_{j,q}\}$ be corresponding real-valued eigenfunctions with L^2-norm one in $L^2(\Gamma(q)\backslash\mathbb{H}^n)$. We may understand $\phi_{j,q}$ as a function on $\Gamma(q)\backslash G$ which is right K-invariant. Let $s_{j,q} \in (\rho, \delta]$ be such that $\lambda_{j,q} = s_{j,q}(n-1-s_{j,q})$.

A key technical ingredient we need is the following result of Gamburd for $n = 2$ [7] and of Magee for n general [13]:

Theorem 2.5. *Fix a closed interval $I \subset (\rho, \delta]$. There exists a compact subset $\Omega \subset \Gamma\backslash G$ and $C > 0$ such that for any integer $q \geq 1$ and any $s_{j,q} \in I$,*

$$\int_{\Omega_q} |\phi_{j,q}(g)|^2 dg \geq C$$

where $\Omega_q := \pi_q^{-1}(\Omega)$ for the canonical projection $\pi_q : \Gamma(q)\backslash G \to \Gamma\backslash G$.

By applying the pretrace formula to K_q, we deduce

$$K_q(g, g) = \sum_{\lambda_{j,q}} \hat{F}_o(\lambda_{j,q})|\phi_{j,q}(g)|^2 + \mathcal{E}$$

where the term $\mathcal{E}$ is the contribution from the continuous spectrum. The positivity of $\hat{F}_o$ yields that $\mathcal{E} \geq 0$, and hence

$$K_q(g, g) \geq \sum_{\lambda_{j,q}} \hat{F}_o(\lambda_{j,q})|\phi_{j,q}(g)|^2.$$

By integrating $K_q(g, g)$ over the compact subset Ω_q of Theorem 2.5, we obtain

$$\int_{\Omega_q} K_q(g, g)dg \geq C \cdot \sum_{s_{j,q} \in I} \hat{F}_o(\lambda_{j,q}).$$

Therefore setting $I := [s_0, \delta]$, we deduce from (2.4) the following:

$$\int_{\Omega_q} K_q(g, g)dg \gg e^{4(s_0-\rho)T} \left(\sum_{s \in I} m(\lambda_s, \Gamma(q)) \right).$$

On the other hand, using (2.3),

$$\begin{aligned} \int_{\Omega_q} K_q(g, g)dg &= \sum_{\gamma \in \Gamma(q)} \int_{\Omega_q} F_o(g^{-1}\gamma g)dg \\ &= \sum_{\gamma \in \Gamma(q)} \sum_{\gamma_0 \in \Gamma(q)\backslash\Gamma} \int_{\Omega} F_o(g^{-1}\gamma_0^{-1}\gamma\gamma_0 g)dg \\ &= [\Gamma : \Gamma(q)] \cdot \sum_{\gamma \in \Gamma(q)} \int_{\Omega} F_o(g^{-1}\gamma g)dg \end{aligned}$$

Using the compactness of Ω, and Theorem 2.1, there exists $R > 1$ such that

$$\begin{aligned}\sum_{\gamma\in\Gamma(q)} \int_\Omega F_o(g^{-1}\gamma g)dg &\ll e^{2(s_0-\rho)T} \sum_{\gamma\in\Gamma(q), d(o,\gamma o)<2T+R} e^{-\rho d(o,\gamma o)} \\ &\ll e^{2(s_0-\rho)T} \int_0^{2T+R} e^{-\rho t}(\#\Gamma(q)\cap B_t)dt \\ &\ll e^{2(s_0-\rho)T}\left(\tfrac{1}{[\Gamma:\Gamma(q)]}e^{(2\delta-2\rho)T} + e^{(2\delta-2\rho-2\eta)T}\right).\end{aligned}$$

Putting these together, we have

$$\left(\sum_{s\geq s_0} m(\lambda_s, \Gamma(q))\right) \ll e^{(-2s_0+2\delta)T} + [\Gamma:\Gamma(q)]\cdot e^{(-2s_0+2\delta-2\eta)T}.$$

Hence by setting T so that $e^{2T} = [\Gamma:\Gamma(q)]^{1/\eta}$, we deduce

$$\left(\sum_{s\geq s_0} m(\lambda_s, \Gamma(q))\right) \ll [\Gamma:\Gamma(q)]^{\frac{\delta-s_0}{\eta}},$$

as desired, proving Theorem 1.1.

Note that replacing the use of Theorem 2.1 by Conjecture 1.9 for an upper bound of $\#\Gamma(q)\cap B_T$ yields

$$\mathcal{N}(s_0(n-1-s_0), \Gamma(q)) \ll_\epsilon [\Gamma:\Gamma(q)]^{\frac{\delta-s_0}{\delta-\rho}+\epsilon},$$

which is Conjecture 1.7. This verifies Proposition 1.10.

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