ZARISKI DENSE NON-TEMPERED SUBGROUPS IN HIGHER RANK OF NEARLY OPTIMAL GROWTH

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Abstract. We provide the first example of a Zariski dense discrete non-lattice subgroup Γ_0 of a higher rank simple Lie group G, which is non-tempered in the sense that the associated quasi-regular representation $L^2(\Gamma_0\backslash G)$ is non-tempered.

In fact, let Γ be the fundamental group of a closed hyperbolic n-manifold with properly embedded totally geodesic hyperplane for $n \geq 3$. We prove that there exists a non-empty open subset $\mathcal{O}$ of $\mathrm{Hom}(\Gamma, \mathrm{SO}(n,2))$ such that for any $\sigma \in \mathcal{O}$, the image $\sigma(\Gamma)$ is a Zariski dense and non-tempered Anosov subgroup of $\mathrm{SO}(n,2)$. Moreover the growth indicator of $\sigma(\Gamma)$ is nearly optimal, that is, it almost realizes the supremum of growth indicators of all non-lattice discrete subgroups given by property (T) of $\mathrm{SO}(n,2)$.

Contents

1. Introduction

Let G be a connected semisimple real algebraic group. Let $\Gamma < G$ be a discrete subgroup of G. Denote by dx a G-invariant measure on the homogeneous space $\Gamma\backslash G$. Consider the Hilbert space $L^2(\Gamma\backslash G) = L^2(\Gamma\backslash G, dx)$. The right translation action of G on $\Gamma\backslash G$ induces a unitary representation of G on $L^2(\Gamma\backslash G)$, called the quasi-regular representation.

A unitary representation $(\pi, \mathcal{H})$ of G is called *tempered* if it is weakly contained in the regular representation $L^2(G)$, i.e., any diagonal matrix coefficients of $(\pi, \mathcal{H})$ can be approximated by a convex linear combination of diagonal matrix coefficients of $L^2(G)$, uniformly on compact subsets of G. This notion was introduced and studied by Harish-Chandra.

Definition 1.1. We say that Γ is a *tempered* subgroup of G if the quasi-regular representation $L^2(\Gamma\backslash G)$ is tempered.

Tempereness of Γ is equivalent to the condition that all matrix coefficients of $L^2(\Gamma\backslash G)$ are $L^{2+\varepsilon}$-integrable for any $\varepsilon > 0$ [9]. If G has Kazhdan's property (T), that is, all simple factors of G have rank at least 2 or are isogeneous to $\mathrm{Sp}(n,1)$ or $F_4^{(-20)}$, then it is a consequence of a quantitative version of property (T) that there exists $p = p_G > 0$ such that for any non-lattice discrete subgroup $\Gamma < G$, all matrix coefficients of the quasi-regular representation $L^2(\Gamma\backslash G)$ are L^p-integrable ([8], [28], [22]).

In rank one groups, it is known that any lattice admits a non-elementary infinite index normal subgroup [10]. There are also convex cocompact subgroups of $\mathrm{SO}(n,1)$, $n \geq 2$, whose critical exponents are arbitrarily close to the volume entropy of the hyperbolic n-space $\mathbb{H}^n$, that is, $n-1$ [26, Sec. 6]. Such groups are then examples of Zariski dense subgroups that are non-tempered, by [27, Thm 1.4] and [7, Thm 4.2] respectively. Although there were known examples of non-tempered discrete subgroups ([6, Example B], [3]) of higher rank Lie groups, they were all lattices of a proper algebraic subgroup of G. It remained an open question whether there exists a *Zariski dense* non-lattice discrete subgroup of a higher rank simple group G that is non-tempered. The main goal of this article is to answer this question in the affirmative:

Theorem 1.2. *For any $n \geq 3$, there exists a Zariski dense non-lattice discrete subgroup of $\mathrm{SO}(n,2)$ that is non-tempered.*

Temperedness of Γ can be determined in terms of the growth indicator ψ_Γ. Fix a Cartan decomposition $G = K\exp(\mathfrak{a}^+)K$ where K is a maximal compact subgroup and $\mathfrak{a}^+$ is a positive Weyl chamber of a Cartan subalgebra $\mathfrak{a}$. For $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K\exp\mu(g)K$, called the Cartan projection of g.

For a discrete subgroup Γ of G, denote by $\mathcal{L}_\Gamma \subset \mathfrak{a}^+$ the limit cone of Γ, which is defined as the asymptotic cone of $\mu(\Gamma)$. The growth indicator $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R}\cup\{-\infty\}$, introduced by Quint [29], is a higher rank version of the critical exponent. It is $-\infty$ outside the limit cone $\mathcal{L}_\Gamma$. For each $v \in \mathcal{L}_\Gamma$, the value $\psi_\Gamma(v)$ represents the exponential growth rate of Γ in the direction v:

$$\psi_\Gamma(v) = \|v\| \cdot \inf_{v\in\mathcal{C}} \limsup_{T\to\infty} \frac{\log \#\{\gamma\in\Gamma : \mu(\gamma)\in\mathcal{C}, \|\mu(\gamma)\| \leq T\}}{T} \tag{1.1}$$

where the infimum is taken over all open cones $\mathcal{C} \subset \mathfrak{a}^+$ containing v. This definition is independent of the choice of a norm $\|\cdot\|$ on $\mathfrak{a}$.

Denote by $\rho = \rho_G$ the half sum of all positive roots of $(\mathrm{Lie}\, G, \mathfrak{a})$ counted with multiplicity. The linear form $2\rho \in \mathfrak{a}^*$ precisely represents the exponential volume growth rate of G: for any $v \in \mathfrak{a}^+$,

$$2\rho(v) = \|v\| \cdot \inf_{v\in\mathcal{C}} \limsup_{T\to\infty} \frac{\log \mathrm{Vol}\{g\in G : \mu(g)\in\mathcal{C}, \|\mu(\gamma)\| \leq T\}}{T}$$

where the infimum is taken over all open cones $\mathcal{C} \subset \mathfrak{a}^+$ containing v. We have

$$\psi_\Gamma \leq 2\rho \quad \text{on } \mathfrak{a}^+$$

for any discrete subgroup $\Gamma < G$ and $\psi_\Gamma = 2\rho$ for Γ lattices [30]. If G has Kazhdan's property (T), there exists a constant $\eta_G > 0$ such that for any non-lattice discrete subgroup Γ of G, we have

$$\psi_\Gamma \leq (2 - \eta_G)\rho \quad \text{on } \mathfrak{a}^+$$

([8, Theorem 4.4], [31, Theorem 5.1], see also [24, Theorem 7.1]).

Definition 1.3. We say that a discrete subgroup $\Gamma < G$ has the slow growth if

$$\psi_\Gamma \leq \rho \quad \text{on } \mathfrak{a}^+.$$

So the slow growth of Γ means that Γ grows at most half as fast as the volume growth of G. It turns out that the slow growth property of Γ determines the temperedness:

$$\psi_\Gamma \leq \rho \text{ on } \mathfrak{a}^+ \quad \text{if and only if} \quad \Gamma \text{ is tempered.}$$

This was shown in [13] for Borel-Anosov subgroups, and in [25] for general discrete subgroups.

Theorem 1.4, which is a more elaborate version of theorem 1.2, presents the first example of a Zariski dense discrete subgroup in a higher rank simple Lie group G *without* slow growth. Moreover, these examples have nearly optimal growth. For $n \geq 3$, the identity component of the special orthogonal group $\mathrm{SO}^\circ(n, 2)$ is a simple Lie group of rank two. As discussed in section 4, we can identify its positive Weyl chamber $\mathfrak{a}^+$ with

$$\mathfrak{a}^+ = \{v = (v_1, v_2, 0, \cdots, 0, -v_2, -v_1) \in \mathbb{R}^{n+2} : v_1 \geq v_2 \geq 0\}.$$

The set of simple roots of $\mathrm{SO}^\circ(n, 2)$ is given by $\alpha_1(v) = v_1 - v_2$ and $\alpha_2(v) = v_2$, and ρ is the following:

$$\rho(v) = \frac{1}{2}\left(nv_1 + (n-2)v_2\right)$$

for any $v = (v_1, v_2, 0, \cdots, 0, -v_2, -v_1) \in \mathfrak{a}^+$.

Theorem 1.4. *Let $n \geq 3$ and let Γ be the fundamental group of a closed hyperbolic n-manifold with properly embedded totally geodesic hyperplane. For any $\varepsilon > 0$, there exists a non-empty open subset $\mathcal{O} = \mathcal{O}(\varepsilon)$ of* $\mathrm{Hom}(\Gamma, \mathrm{SO}^\circ(n, 2))$ *such that for any $\sigma \in \mathcal{O}$, we have the following:*

1. *$\sigma(\Gamma)$ is a Zariski dense, $\{\alpha_1\}$-Anosov*[1]*, and non-tempered subgroup of* $\mathrm{SO}^\circ(n, 2)$ *without slow growth;*
2. *for all $v \in \mathfrak{a}^+$, we have*

$$\psi_{\sigma(\Gamma)}(v) \leq \left(\frac{2(n-1)}{n} + \varepsilon\right)\rho(v);$$

[1] see Def. 6.1 for the definition of Anosov subgroups

(3) *for some unit vector* $v_\sigma \in \mathfrak{a}^+$, *we have*

$$\psi_{\sigma(\Gamma)}(v_\sigma) \geq \left(\frac{2(n-1)}{n} - \varepsilon\right)\rho(v_\sigma). \tag{1.2}$$

Moreover, $\sigma(\Gamma)$ *has nearly optimal growth in the sense that*

$$\psi_{\sigma(\Gamma)}(v_\sigma) \geq \sup_{\Lambda} \psi_\Lambda(v_\sigma) - \varepsilon \tag{1.3}$$

where Λ *ranges over* all *non-lattice discrete subgroups of* $\mathrm{SO}^\circ(n,2)$.

We have an upper bound on the growth of arbitrary non-lattice discrete subgroups coming from the effective property (T) of G ([28], see Proposition 4.1). It follows from (1.2) and (1.3) that this bound is nearly realized by our construction. At least in $\mathrm{SO}(n,2)$, this means that there is no hope of improving growth gap theorems (e.g. [24]) by additionally assuming Zariski density. It remains an interesting question whether such an improvement is possible in other higher rank groups, for example in $\mathrm{SL}_n(\mathbb{R}), n \geq 3$.

Remark 1.5. We note that there are many examples of Zariski dense discrete subgroups in higher rank which are tempered, e.g., the image of any Hitchin representation of a surface group into a real split simple algebraic group of higher rank is tempered ([13], [11]).

Our construction of a non-tempered Zariski dense subgroup of $\mathrm{SO}(n,2)$ goes as follows. We start with a uniform lattice Γ in $\mathrm{SO}(n,1)$ which is an amalgamated product of two subgroups over a uniform lattice in $\mathrm{SO}(n-1,1)$. For $n \geq 3$, any lattice of the group $\mathrm{SO}(n,1)$ is non-tempered in $\mathrm{SO}(n,2)$ (Corollary 4.5). The inclusion $\mathrm{id}_\Gamma : \Gamma \hookrightarrow \mathrm{SO}(n,2)$ can be deformed using the bending construction ([18], [19]), yielding a discrete Zariski dense subgroup Γ_1 of $\mathrm{SO}(n,2)$. The heart of the paper is to show that Γ_1 is non-tempered. We present two proofs. In the first, we consider the Chabauty topology on the space of closed subgroups of $\mathrm{SO}(n,2)$ and show that the property of being non-tempered is open, by studying the convergence of the matrix coefficients of quasi-regular representations[2]. As a consequence we deduce that all sufficiently small (discrete) deformations of $\mathrm{SO}(n,1)$ remain non-tempered, so Γ_1 is a non-tempered Zariski dense subgroup, proving Theorem 1.2. For the second proof, we study how the growth indicator of Γ evolves under the deformation, using the property that Γ is an Anosov subgroup. We use that the limit cone of the deformation is known to vary continuously in this setting [19] and that certain critical exponent of Γ_1 varies continuously [5]. This allows us to show that for small deformations, the growth indicator of Γ_1 can be controlled by the growth indicator of Γ and hence it is not smaller than the half-sum of positive roots ρ, proving Theorem 1.4.

[2]It was pointed out to us that this part of the argument could be replaced by a general result of Fell [15, Theorem 4.2] on the continuity of induction. We have decided to keep our writeup since it is a very explicit construction that gives a slightly stronger statement on the convergence of K-finite matrix coefficients for semisimple real Lie groups.

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2. Convergence of matrix coefficients and Chabauty topology

Let G be a locally compact second countable group. Let $\mathfrak{C} = \mathfrak{C}_G$ denote the space of all closed subgroups of G equipped with the Chabauty topology, that is, a sequence of closed subgroups H_n converges to H as $n \to \infty$ if for any element $h \in H$, there exists a sequence $h_n \in H_n$ with $h_n \to h$ and the limit of any convergence sequence $h_{n_k} \in H_{n_k}$ belongs to H. The space $\mathfrak{C}$ is a compact space. When a sequence H_i converges to a closed subgroup H, we say that H is the Chabauty limit of H_i. Note that the Chabauty limit of a sequence of discrete subgroups is not necessarily a discrete subgroup.

For a unimodular closed subgroup H of G, denote by ν_H a Haar measure on H. For $s \in C_c(G)$ and any locally finite measure ν on H we write

$$\nu(s) := \int s(h) d\nu(h).$$

Note that for a non-negative function $s \in C_c(G)$ with $\nu_H(s) \neq 0$, the normalized measure $\nu_H(s)^{-1}\nu_H$ is independent of the choice of a Haar measure ν_H. Let $\mathcal{M}(G)$ be the space of all locally finite Borel measures on G, equipped with the weak*-topology. Throughout the paper, e denotes the identity element of a relevant group.

Proposition 2.1. *Let Γ_n be a sequence of discrete subgroups of G converging to a closed subgroup H in the Chabauty topology. Then H is unimodular, and for any non-negative function $s \in C_c(G)$ with $s(e) > 0$, we have*

$$\lim_{n\to\infty} \nu_{\Gamma_n}(s)^{-1}\nu_{\Gamma_n} = \nu_H(s)^{-1}\nu_H \quad \text{in } \mathcal{M}(G). \tag{2.1}$$

Proof. Consider a non-negative function $s \in C_c(G)$ with $s(e) > 0$. For simplicity, set $\nu_n = \nu_{\Gamma_n}$ and $\nu_n' := \nu_n(s)^{-1}\nu_n$. Then $\nu_n'(s) = 1$.

First we show that the sequence ν_n' is relatively compact in $\mathcal{M}(G)$. Since $s(e) > 0$, it follows from the continuity of s that there exists a symmetric neighborhood U of e such that

$$\kappa := \inf_{g\in U^2} s(g) > 0.$$

Fix any compact subset C of G. Let

$$m_C := \max\{\#F \mid F \subset C, g_1U \cap g_2U = \emptyset \text{ for all } g_1 \neq g_2 \in F\}.$$

Note that

$$m_C \leq \frac{\nu_G(CU)}{\nu_G(U)}.$$

For any $n \in \mathbb{N}$, choose a maximal subset

$$F_n \subset \Gamma_n \cap C$$

such that $g_1 U \cap g_2 U = \emptyset$ for all $g_1 \neq g_2 \in F_n$. Then $\Gamma_n \cap C \subset F_n U^2$, so

$$\#(\Gamma_n \cap C) \leq \# F_n \cdot \#(\Gamma_n \cap U^2) \leq \frac{m_C}{\kappa} \int s(g) d\nu_n(g).$$

Therefore for all $n \in \mathbb{N}$, we have

$$\nu_n'(C) \leq \frac{m_C}{\kappa}.$$

Since C is an arbitrary compact subset of G, it follows that the sequence ν_n', $n \in \mathbb{N}$, forms a relatively compact subset of $\mathcal{M}(G)$.

Let $\nu \in \mathcal{M}(G)$ be a weak-* limit of the sequence ν_n'. By construction, ν is a locally finite measure supported on H and $\nu(s) = 1$. It remains to show that ν is a Haar measure on H. Let $\varphi \in C_c(G)$ and $h \in H$. Let $\gamma_n \in \Gamma_n$ be a sequence with $\lim_{n\to\infty} \gamma_n = h$. Then, since ν_n' is a Haar measure of Γ_n, we get

$$\begin{aligned} \left| \int \varphi(g) - \varphi(hg) d\nu(g) \right| \leq & \left| \int \varphi(g) d\nu(g) - \int \varphi(g) d\nu_n'(g) \right| \\ & + \left| \int \varphi(\gamma_n g) d\nu_n'(g) - \int \varphi(hg) d\nu_n'(g) \right|. \end{aligned}$$

The right hand side converges to 0 as $n \to \infty$, so ν is indeed left H-invariant. Similarly, we can show that ν is also a right H-invariant. This proves that H is unimodular. Since $\nu(s) = 1$, we have $\nu = \nu_H(s)^{-1} \nu_H$ and thus the desired convergence (2.1) follows from $\nu_n' \to \nu$. □

Remark 2.2. The normalization of measures by the integral of s is necessary in the above proposition. For example, if $G = \mathrm{SL}_2(\mathbb{F}_p((t)))$ and

$$\Gamma_n := \left\{ \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \mid f(t) = a_n t^n + a_{n+1} t^{n+1} + \cdots + a_{2n} t^{2n} \in \mathbb{F}_p[t] \right\},$$

then, as $n \to \infty$, Γ_n converges to the trivial subgroup $\{e\}$ in the Chabauty topology, but the sequence ν_{Γ_n} of counting measures on Γ_n fails to converge on the account of mass near identity blowing up to infinity.

On the other hand, we can skip the normalization if the group G has no small subgroup property. We say that a locally compact group G has no small subgroup if there exists a neighborhood of e in G which does not contain any non-trivial subgroup of G; this notion was first introduced in [23]. It is a well-known fact that a real Lie group G has no small subgroup; this can be easily seen, using the fact that the exponential map is a diffeomorphism of a neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of the e in G.

Proposition 2.3. *Suppose that G has no small subgroup property (e.g., real Lie group). Let Γ_n be a sequence of discrete subgroups of G which converges to a discrete subgroup Γ in the Chabauty topology. Then as $n \to \infty$*

$$\lim_{n\to\infty} \sum_{\gamma\in\Gamma_n} \delta_\gamma = \sum_{\gamma\in\Gamma} \delta_\gamma \quad \text{in } \mathcal{M}(G)$$

where δ_γ denotes the Dirac measure at $\{\gamma\}$.

Proof. Let $\nu_n := \sum_{\gamma\in\Gamma_n} \delta_\gamma$ and $\nu := \sum_{\gamma\in\Gamma} \delta_\gamma$. Let $\varphi \in C_c(G)$. We need to show that

$$\lim_{n\to\infty} \int \varphi d\nu_n = \int \varphi d\nu.$$

Let $\varepsilon > 0$ be arbitrary. Fix a compact subset $C \subset G$ and $\varphi \in C_c(G)$ supported on C. Enlarging C if needed, we may assume that $\Gamma \cap \partial C = \emptyset$. By the hypothesis that G has no small subgroup property, there is an open neighborhood $U \subset G$ of the identity e which contains no non-trivial subgroup of G. We choose an open symmetric neighborhood $U_1 \subset G$ of e such that

1. $U_1^2 \subset U$;
2. $\gamma U_1^5 \subset C$ for all $\gamma \in \Gamma \cap C$;
3. the collection $\gamma U_1^5, \gamma \in \Gamma \cap C$, are pairwise disjoint;
4. for all $\gamma \in C \cap \Gamma$ and $u \in U_1$,

$$|\varphi(\gamma) - \varphi(\gamma u)| \leq \frac{\varepsilon}{\#(\Gamma \cap C)}.$$

Consider the following compact subset

$$C_1 := C \setminus \bigcup_{\gamma\in\Gamma\cap C} \gamma U_1.$$

Note that $\Gamma \cap C_1 = \emptyset$. Since the sequence Γ_n converges to Γ in the Chabauty topology, we have $\Gamma_n \cap C_1 = \emptyset$ and for each fixed $\gamma \in \Gamma \cap C$, there exists $n_0 = n_0(\gamma) \geq 1$ such that

$$\Gamma_n \cap \gamma U_1 \neq \emptyset \quad \text{ for all } n \geq n_0.$$

Since $\Gamma \cap C$ is finite, we have $n_0 := \max\{n_0(\gamma) : \gamma \in \Gamma \cap C\} < \infty$.

On the other hand, we claim that for any $\gamma \in C \cap \Gamma$ and $n \geq 1$,

$$\#(\Gamma_n \cap \gamma U_1) \leq 1.$$

Indeed, suppose there exists some element $\gamma_n \in \Gamma_n \cap \gamma U_1$. Then

$$\gamma_n^{-1}(\Gamma_n \cap \gamma U_1) = \Gamma_n \cap (\gamma_n^{-1}\gamma U_1) \subset \Gamma_n \cap U_1^2.$$

By the no-small-subgroups property of G, we have either $\Gamma_n \cap U_1^2 = \{e\}$ or there is some element $\gamma_n' \in \Gamma_n \cap (U_1^4 \setminus U_1^2)$; otherwise $\Gamma_n \cap U_1^2$ would be a non-trivial subgroup. In the second case, we would have

$$\gamma_n \gamma_n' \in \gamma_n(U_1^4 \setminus U_1^2) \subset \gamma U_1^5 \setminus \gamma U_1 \subset C \setminus \gamma U_1.$$

Using property (3), we get $\gamma_n\gamma_n' \in C_1$, contradicting the fact that $\Gamma_n \cap C_1 = \emptyset$. Therefore we must have $\Gamma_n \cap U_1^2 = \{e\}$. This implies that $\Gamma_n \cap \gamma U_1 = \{\gamma_n\}$, proving the claim.

Therefore for all $\gamma \in \Gamma \cap C$ and $n \geq n_0$, we have a unique element $\gamma_n \in \Gamma_n$ such that $\Gamma_n \cap \gamma U_1 = \{\gamma_n\}$, and $\gamma_n \to \gamma$ as $n \to \infty$. Since

$$\int \varphi d\nu_n = \sum_{\gamma \in \Gamma \cap C} \varphi(\gamma_n) \quad \text{for all } n \geq n_0,$$

we get from (4) that for all $n \geq n_0$,

$$\left| \int \varphi d\nu - \int \varphi d\nu_n \right| \leq \sum_{\gamma \in \Gamma \cap C} |\varphi(\gamma) - \varphi(\gamma_n)| \leq \varepsilon.$$

This finishes the proof. □

Let G be unimodular and dg a Haar measure on G. For a closed unimodular subgroup H of G, there exists a unique G-invariant measure $d_{H\backslash G}$ on $H\backslash G$ such that for all $\psi \in C_c(G)$,

$$\int_G \psi dg = \int_{H\backslash G} \int_H \psi(hg) d\nu_H(h) d_{H\backslash G}(Hg).$$

We then have a unitary representation of G on the Hilbert space

$$L^2(H\backslash G) = \{f : H\backslash G \to \mathbb{R} : \int_{H\backslash G} |f|^2 d_{H\backslash G} < \infty\}$$

by right translations: $g \mapsto g.f$ for $g \in G$ and $f \in L^2(H\backslash G)$.

Proposition 2.4. *Let Γ_n be a sequence of discrete subgroups of G which converges to a closed unimodular subgroup H in the Chabauty topology. Let $K < G$ be a compact subgroup of G.*

For any vectors $v, w \in L^2(H\backslash G)$, there exist sequences $v_n, w_n \in L^2(\Gamma_n\backslash G)$, $n \in \mathbb{N}$ such that

(1) *for all $g \in G$,*

$$\lim_{n\to\infty} \langle v_n, g.w_n\rangle_{L^2(\Gamma_n\backslash G)} = \langle v, g.w\rangle_{L^2(H\backslash G)},$$

and the convergence is uniform on compact subsets of G;

(2) *we have*

$$\lim_{n\to\infty} \|v_n\|_{L^2(\Gamma_n\backslash G)} = \|v\|_{L^2(H\backslash G)} \ \ \& \ \ \lim_{n\to\infty} \|w_n\|_{L^2(\Gamma_n\backslash G)} = \|w\|_{L^2(H\backslash G)};$$

(3) *we have that for all $n \in \mathbb{N}$,*

$$\dim\langle K.v_n\rangle \leq \dim\langle K.v\rangle \ \ \& \ \ \dim\langle K.w_n\rangle \leq \dim\langle K.w\rangle.$$

Proof. Since $C_c(H\backslash G)$ is dense in $L^2(H\backslash G)$, the matrix coefficient $g \mapsto \langle v, g.w\rangle_{L^2(H\backslash G)}$ can be approximated by the matrix coefficients for continuous compactly supported functions, uniformly on compact subsets of G. This approximation can be done without increasing the dimensions of the spaces spanned by the K-orbits of v and w. In fact, let u_m be a sequence of compactly supported right K-invariant functions on $H\backslash G$ converging to the constant function 1 uniformly on compact subsets of $H\backslash G$. Since the multiplication by u_m is K-equivariant, we have $\dim\langle K.(u_m v)\rangle \leq \dim\langle K.v\rangle$, similarly for w. The matrix coefficients $g \mapsto \langle u_m v, g.u_m w\rangle_{L^2(H\backslash G)}$ converge to $g \mapsto \langle v, g.w\rangle_{L^2(H\backslash G)}$ uniformly on compact sets. Thus we have shown that v, w can be replaced by compactly supported functions, spanning K-invariant subspaces of equal or smaller dimension. We need one more step to replace them by continuous functions.

Let $\tilde{\phi}_m \in C_c(G)$ be a sequence of non-negative continuous functions with $\int \tilde{\phi}_m(g)dg = 1$ and support contained in some neighborhood $\tilde{U}_m$ of e such that $\tilde{U}_m \to \{e\}$ as $m \to \infty$. Define $\phi_m \in C_c(G)$ by

$$\phi_m(g) = \int_K \tilde{\phi}_m(k^{-1}gk)dk \quad \text{for } g \in G,$$

where dk is the probability Haar measure on K. Clearly, ϕ_m is non-negative, continuous and $\int \phi_m(g)dg = 1$. The support of ϕ_m is contained in $U_m := \{k\tilde{U}_m k^{-1} : k \in K\}$. Note that $U_m \to \{e\}$ as $m \to \infty$; otherwise, we have, by passing to a subsequence, $k_m g_m k_m^{-1} \to g$ for some $k_m \in K$ converging to $k_0 \in K$, $g_m \in \tilde{U}_m$ and $g \neq e$. Since $g_m \to e$ as $m \to \infty$, this is a contradiction.

Consider the convolution $v * \phi_m$:

$$v * \phi_m(Hg) = \int_G v(Hgx)\phi_m(x^{-1})dx \quad \text{for } Hg \in H\backslash G$$

and similarly for $w * \phi_m$. The functions $v * \phi_m$ and $w * \phi_m$ are continuous compactly supported functions on $H\backslash G$.

Since the sequence ϕ_m is an approximate identity, the matrix coefficient $g \mapsto \langle v * \phi_m, g.w * \phi_m\rangle_{L^2(H\backslash G)}$ converges to $g \mapsto \langle v, g.w\rangle_{L^2(H\backslash G)}$, uniformly on compact sets. Furthermore, because ϕ_m is K-conjugation invariant, the map $v \mapsto v * \phi_m$ commutes with the action of K: $k.(v * \phi_m) = (k.v) * \phi_m$ for all $k \in K$. It follows that

$$\dim\langle K.(v * \phi_m)\rangle \leq \dim\langle K.v\rangle,$$

and similarly for w. Therefore, we may assume without loss of generality that $v, w \in C_c(H\backslash G)$.

First, let $\tilde{v}_0 \in C(G)$ be the lift of v to G, i.e., for all $g \in G$, $\tilde{v}_0(g) := v(Hg)$. We note that

$$\dim\langle K.v\rangle = \dim\langle K.\tilde{v}_0\rangle$$

Now, we choose a right K-invariant non-negative function $\varphi \in C_c(G)$ such that $\int_H \varphi(hg)d\nu_H(h) = 1$ for every $g \in H \operatorname{supp} v \cup H \operatorname{supp} w$.

Define $\tilde{v} \in C_c(G)$ by $\tilde{v}(g) := \varphi(g)\tilde{v}_0(g)$ for all $g \in G$. Then for each $g \in G$, we have

$$\int_H \tilde{v}(hg)d\nu_H(h) = v(g).$$

Moreover

$$\dim\langle K.\tilde{v}\rangle \leq \dim\langle K.\tilde{v}_0\rangle = \dim\langle K.v\rangle.$$

Choose a non-negative function $s \in C_c(G)$ such that $s(e) > 0$ and

$$\int_H s(h) d\nu_H(h) = 1.$$

Set $\alpha_n := \sum_{\gamma \in \Gamma_n} s(\gamma)$, and define $v_n \in C_c^\infty(\Gamma_n \backslash G)$ as follows: for all $g \in G$,

$$v_n(g) := \alpha_n^{-1/2} \sum_{\gamma \in \Gamma_n} \tilde{v}(\gamma g).$$

Then

$$\dim \langle K.v_n \rangle \leq \dim \langle K.\tilde{v} \rangle \leq \dim \langle K.v \rangle.$$

Let $\tilde{w} \in C_c(G)$ and $w_n \in C_c^\infty(\Gamma_n \backslash G)$ be functions constructed in the same way for the vector w.

We claim that for all $g \in G$,

$$\langle v_n, g.w_n \rangle_{L^2(\Gamma_n \backslash G)} \to \langle v, g.w \rangle_{L^2(H \backslash G)},$$

uniformly on compact subsets of G. Indeed,

$$\begin{aligned} \langle v_n, g.w_n \rangle_{L^2(\Gamma_n \backslash G)} &= \alpha_n^{-1} \int_{\Gamma_n \backslash G} \left(\sum_{\gamma \in \Gamma_n} \tilde{v}(\gamma x) \sum_{\gamma' \in \Gamma_n} \tilde{w}(\gamma' x g) \right) dx \\ &= \int_G \tilde{v}(x) \left(\alpha_n^{-1} \sum_{\gamma' \in \Gamma_n} \tilde{w}(\gamma' x g) \right) dx. \end{aligned} \tag{2.2}$$

Proposition 2.1 yields the weak-* convergence of measures

$$\alpha_n^{-1} \sum_{\gamma' \in \Gamma_n} \delta_{\gamma'} \to d\nu_H.$$

It follows that

$$\lim_{n \to \infty} \alpha_n^{-1} \sum_{\gamma' \in \Gamma_n} \tilde{w}(\gamma' x g) = \int_H \tilde{w}(hxg) d\nu_H(h)$$

and the convergence is uniform for all g and x in a given compact subset of G. Since $\tilde{v}$ is compactly supported, we get

$$\lim_{n \to \infty} \langle v_n, g.w_n \rangle_{L^2(\Gamma_n \backslash G)} = \int_G \tilde{v}(x) \int_\Gamma \tilde{w}(hxg) d\nu_H(h) dx,$$

and the convergence is uniform for all g in a given compact subset of G. Since

$$\begin{aligned} &\int_G \tilde{v}(x) \int_\Gamma \tilde{w}(hxg) d\nu_H(h) dg = \int_G \tilde{v}(x) w(Hxg) dx \\ &= \int_{H \backslash G} v(Hx) w(Hxg) d_{H \backslash G}(Hx) = \langle v, g.w \rangle_{L^2(H \backslash G)}, \end{aligned}$$

this finishes the proof of (1) and (3). The claim (2) follows since the above argument applies when $v = w$ and $g = e$ and hence gives $\langle v_n, v_n \rangle_{L^2(\Gamma_n \backslash G)} \to \langle v, v \rangle_{L^2(H \backslash G)}$ and similarly for w_n and w. $\square$

Remark 2.5. This proposition implies that if Γ_n converges to H in the Chabauty topology, then $L^2(H\backslash G)$ is weakly contained in $\bigoplus_{n=n_0}^{\infty} L^2(\Gamma_n\backslash G)$ for all $n_0 \geq 1$.[3]

3. Temperedness is a closed condition in $\operatorname{Hom}(\Gamma, G)$

Let G be a connected semisimple real algebraic group. Let P be a minimal parabolic subgroup of G with a fixed Langlands decomposition $P = MAN$ where A is a maximal real split torus of G, M is the maximal compact subgroup of P, which commutes with A, and N is the unipotent radical of P. We denote by $\mathfrak{g}$ and $\mathfrak{a}$ the Lie algebras of G and A respectively. We fix a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ so that $\log N$ consists of positive root subspaces. Let $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a})$ denote the set of all positive roots for $(\mathfrak{g}, \mathfrak{a}^+)$. For each $\alpha \in \Sigma^+$, let $m(\alpha)$ be its multiplicity. We also write $\Pi \subset \Sigma^+$ for the set of all simple roots. We denote by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m(\alpha)\alpha \tag{3.1}$$

the half sum of the positive roots for $(\mathfrak{g}, \mathfrak{a}^+)$, counted with multiplicity.

We fix a maximal compact subgroup K of G so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds, that is, for any $g \in G$, there exists a unique element $\mu(g) \in \mathfrak{a}^+$ such that $g \in K \exp \mu(g) K$.

Let dg be a Haar measure on G. The right translation action of G on itself induces the regular representation $L^2(G) = L^2(G, dg)$:

$$g.f(x) = f(xg) \quad \text{for all } x, g \in G \text{ and } f \in L^2(G).$$

Following Harish-Chandra, we call a unitary representation $(\pi, \mathcal{H})$ of G *tempered* if π is weakly contained in the regular representation $L^2(G)$. For $v, w \in \mathcal{H}$, the function $g \mapsto \langle g.v, w\rangle$ is called the matrix coefficient of π with respect to v, w. For any $p > 0$, a unitary representation $(\pi, \mathcal{H})$ of G is said to be almost L^p-integrable if all of its matrix coefficients are $L^{p+\varepsilon}$-integrable for any $\varepsilon > 0$.

Denote by $\Xi = \Xi_G$ the Harish-Chandra function of G. It is a bi-K-invariant function satisfying that for any $\varepsilon > 0$, there exist $c, c_\varepsilon > 0$ such that

$$ce^{-\rho(v)} \leq \Xi(\exp v) \leq c_\varepsilon e^{-(1-\varepsilon)\rho(v)} \quad \text{for all } v \in \mathfrak{a}^+.$$

We will use the following characterization of a tempered representation of G given by Cowling, Haggerup and Howe:

Theorem 3.1. [9] *For a unitary representation $(\pi, \mathcal{H})$ of G, the following are equivalent:*

(1) *π is tempered;*
(2) *π is almost L^2-integrable;*

---

[3] Since submitting this paper, we have learned that this conclusion already follows from [15, Theorem 4.2].

(3) *for any K-finite unit vectors $v_1, v_2 \in \mathcal{H}$ and any $g \in G$,*

$$|\langle \pi(g)v_1, v_2\rangle| \le (\dim\langle \pi(K)v_1\rangle \cdot \dim\langle \pi(K)v_2\rangle)^{1/2}\, \Xi_G(g).$$

Definition 3.2. We say that a unimodular subgroup H is a *tempered* subgroup of G (or G-tempered) if the quasi-regular representation $L^2(H\backslash G)$ is a tempered representation of G.

Lemma 3.3. [3, Lem 3.2] *Let H be a unimodular closed subgroup of G. If H is G-tempered, then any unimodular closed subgroup $H' < H$ is also G-tempered.*

We show that temperedness is a closed condition both for the Chabauty topology and the algebraic topology (Theorems 3.4 and 3.6).

Theorem 3.4. *The Chabauty limit of a sequence of tempered discrete subgroups of G is unimodular and tempered.*

Proof. Suppose that Γ_n is a sequence of tempered discrete subgroups converging to a closed subgroup H in the Chabauty topology. We have H unimodular by Proposition 2.1. We claim that $L^2(H\backslash G)$ is tempered. Suppose not. By Theorem 3.1, there exist K-finite unit vectors $v, w \in L^2(H\backslash G)$ and $g \in G$ such that

$$\langle v, g.w\rangle_{L^2(H\backslash G)} > \Xi(g)\dim\langle K.v\rangle^{1/2}\dim\langle K.w\rangle^{1/2}. \tag{3.2}$$

By Proposition 2.4, there exist $v_n, w_n \in L^2(\Gamma_n\backslash G)$ such that $\|v_n\| \to \|v\|, \|w_n\| \to \|w\|$ as $n \to \infty$, $\dim\langle K.v_n\rangle \le \dim\langle K.v\rangle, \dim\langle K.w_n\rangle \le \dim\langle K.w\rangle$, and $\langle v, g.w\rangle_{L^2(H\backslash G)} = \lim_{n\to\infty}\langle v_n, g.w_n\rangle_{L^2(\Gamma_n\backslash G)}$. We deduce that for all n large enough,

$$\langle v_n, g.w_n\rangle_{L^2(\Gamma_n\backslash G)} > \Xi(g)\dim\langle K.v_n\rangle^{1/2}\dim\langle K.w_n\rangle^{1/2}.$$

This is a contradiction since $L^2(\Gamma_n\backslash G)$ is tempered.

Alternatively, one can use [15, Theorem 4.2] that $L^2(H\backslash G)$ is weakly contained in the direct sum $\bigoplus_{n=1}^{\infty} L^2(\Gamma_n\backslash G)$. If Γ_n were all tempered, we would deduce that $L^2(H\backslash G)$ is weakly contained in $\bigoplus_{n=1}^{\infty} L^2(G)$, hence in $L^2(G)$, which then implies that H is tempered. $\square$

Definition 3.5. We say that a sequence of discrete subgroups Γ_i of G converges to a discrete subgroup Γ algebraically if there exists a sequence of isomorphisms

$$\chi_i : \Gamma \to \Gamma_i$$

such that for all $\gamma \in \Gamma$, $\chi_i(\gamma)$ converges to γ as $i \to \infty$. In other words, χ_i converges to the natural inclusion id_Γ in $\mathrm{Hom}(\Gamma, G)$. In this case, Γ is called the algebraic limit of Γ_i

Theorem 3.6. *The algebraic limit of a sequence of tempered discrete subgroups of G is tempered.*

Proof. Let Γ_i be a sequence of tempered discrete subgroups of G which converges to a discrete subgroup Γ algebraically. By passing to a subsequence if necessary, we may assume that Γ_i converges to a closed subgroup H in the Chabauty topology. Since Γ is the algebraic limit of Γ_i, we have

$$\Gamma < H.$$

By Theorem 3.4, H is unimodular and tempered. Since any closed unimodular subgroup of a tempered subgroup is tempered by Lemma 3.3, Γ is tempered as desired. □

The following is an equivalent formulation of Theorem 3.6:

Theorem 3.7. *If a discrete subgroup Γ is a non-tempered subgroup of G, there exists an open neighborhood $\mathcal{O}$ of id_Γ in $\mathrm{Hom}(\Gamma, G)$ such that for any $\sigma \in \mathcal{O}$, $\sigma(\Gamma)$ is non-tempered.*

4. Growth indicator of a lattice of SO$(n,1)$ as a subgroup of SO$(n,2)$

Let $G = \mathrm{SO}^\circ(n,2)$ for $n \geq 2$. Consider the quadratic form

$$Q(x_1, \cdots, x_{n+2}) = x_1 x_{n+2} + x_2 x_{n+1} + \sum_{i=3}^{n} x_i^2.$$

We realize G as the identity component of the following special orthogonal group

$$\mathrm{SO}(Q) = \{g \in \mathrm{SL}_{n+2}(\mathbb{R}) : Q(gX) = Q(X) \text{ for all } X \in \mathbb{R}^{n+2}\}.$$

Consider the diagonal subgroup

$$A = \{\mathrm{diag}(e^{t_1}, e^{t_2}, 1 \cdots, 1, e^{-t_2}, e^{-t_1}) : t_1, t_2 \in \mathbb{R}\},$$

which is a maximal real split torus of G. We denote by $\mathfrak{g}$ the Lie algebra of G and set

$$\mathfrak{a} = \{v = \mathrm{diag}(v_1, v_2, 0, \cdots, 0, -v_2, -v_1) : v_1, v_2 \in \mathbb{R}\} = \log A.$$

For simplicity, we write $v = (v_1, v_2, 0, \cdots, 0, -v_2, -v_1)$ for an element of $\mathfrak{a}$. Choose a positive Weyl chamber

$$\mathfrak{a}^+ = \{v = (v_1, v_2, 0, \cdots, 0, -v_2, -v_1) : v_1 \geq v_2 \geq 0\}. \tag{4.1}$$

Since G is invariant under the Cartan involution $g \mapsto g^{-T}$,

$$K = \{g \in G : gg^T = e\} = G \cap \mathrm{SO}(n+2)$$

is a maximal compact subgroup of G and we have the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$. We denote by $\mu : G \to \mathfrak{a}^+$ the Cartan projection of G.

We then have two simple (restricted) roots α_1 and α_2 for $(\mathfrak{g}, \mathfrak{a})$ given by

$$\alpha_1(v) = v_1 - v_2 \quad \text{and} \quad \alpha_2(v) = v_2 \quad \text{for all } v \in \mathfrak{a}.$$

By explicit computation of $\mathfrak{g}$, we can see that the set of all positive roots of $\mathfrak{g}$ is given by

$$\Sigma^+(\mathfrak{g},\mathfrak{a}) = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}.$$

The sum of root subspaces is given by

$$\left\{ \begin{pmatrix} 0 & x & Y_1 & z & 0 \\ 0 & 0 & Y_2 & 0 & -z \\ & & & -Y_1^t & -Y_2^t \\ & & & 0 & -x \\ & & & 0 & 0 \end{pmatrix} : x, z \in \mathbb{R}, Y_1, Y_2 \in \mathbb{R}^{n-2} \right\}.$$

where the subspaces corresponding to $x \in \mathbb{R}$, $Y_1 \in \mathbb{R}^{n-2}$, $Y_2 \in \mathbb{R}^{n-2}$, and $z \in \mathbb{R}$ are root subspaces for α_1, $\alpha_1+\alpha_2$, α_2 and $\alpha_1+2\alpha_2$ respectively. Hence the multiplicities are given by

$$m(\alpha_1) = m(\alpha_1+2\alpha_2) = 1$$

and

$$m(\alpha_1+\alpha_2) = m(\alpha_2) = n-2.$$

Since $(\alpha_1+\alpha_2)(v) = v_1$ and $(\alpha_1+2\alpha_2)(v) = v_1+v_2$, the half sum of all positive roots counted with multiplicity is

$$\rho(v) = \sum_{\alpha\in\Sigma^+} m(\alpha)\alpha(v) = \frac{1}{2}(nv_1 + (n-2)v_2) \quad \text{for } v \in \mathfrak{a}^+. \tag{4.2}$$

Bound on growth indicator for general non-lattice subgroups. Recall the definition of the growth indicator of a discrete subgroup of G from (1.1). For any discrete subgroup Γ of G, the growth indicator ψ_Γ is concave and upper-semicontinuous. Since $\dim \mathfrak{a}^+ = 2$, it follows that ψ_Γ is continuous on the limit cone $\mathcal{L}_\Gamma$.

The quantitative Kazhdan's property (T) of the group G obtained in [28] yields the following explicit upper bound:

Proposition 4.1. *For any non-lattice discrete subgroup Γ of G, we have*

$$\psi_\Gamma(v) \le (n-1)v_1 + (n-2)v_2 \quad \textit{for all } v \in \mathfrak{a}^+.$$

Proof. By [24, Theorem 7.1], we have

$$\psi_\Gamma(v) \le (2\rho - \Theta)(v) \text{ for all } v \in \mathfrak{a}^+$$

where Θ is the half sum of all roots in a maximal strongly orthogonal system of $\Sigma^+(\mathfrak{g},\mathfrak{a})$. Since $\{\alpha_1, \alpha_1+2\alpha_2\}$ is a maximal strongly orthogonal system, we have

$$\Theta(v) = v_1 \quad \text{for all } v \in \mathfrak{a}^+.$$

Therefore

$$(2\rho - \Theta)(v) = (n-1)v_1 + (n-2)v_2,$$

proving the claim. ∎

Growth indicator for discrete subgroups of G that are lattices of H. Let $H = \mathrm{SO}^\circ(n,1)$. The restriction of the quadratic form Q to the hyperplane $V := \{x_1 = x_{n+2}\}$ yields a quadratic form $Q_0 = Q|_V$ in $(n+1)$ variables. We identify

$$H = \mathrm{SO}^\circ(n,1) = \{g \in G : g(V) = V\} = \mathrm{SO}^\circ(Q_0).$$

Since H is invariant under the Cartan involution $g \mapsto g^{-T}$, the intersection $K \cap H$ is a maximal compact subgroup of H. Denoting by $\mathfrak{h}$ the Lie algebra of H, we have

$$\mathfrak{h} \cap \mathfrak{a} = \{\mathrm{diag}(0, v_2, 0, \cdots, 0, -v_2, 0) : v_2 \in \mathbb{R}\}.$$

Note that the Cartan projection $\mu(H)$ is equal to $\mathfrak{a}^+ \cap \ker \alpha_2$:

$$\mu(H) = \{v = (v_1, 0, \cdots, 0, -v_1) : v_1 \geq 0\}.$$

To see that, apply the Weyl element switching the first two rows (and hence the last two rows) to $\mathfrak{h} \cap \mathfrak{a}$, resulting in $\{(v_2, 0, \ldots, 0, -v_2) : v_2 \in \mathbb{R}\} = \ker \alpha_2$.

Proposition 4.2. *Let $\Gamma < G$ be a discrete subgroup such that Γ is a lattice of H. Then*

$$\psi_\Gamma(v) = \begin{cases} (n-1)v_1 & \text{for } v = (v_1, 0, \cdots, 0 - v_1),\ v_1 \geq 0 \\ -\infty & \text{for } v \notin \mu(H) \end{cases} \tag{4.3}$$

In other words,

$$\psi_\Gamma \leq \frac{2(n-1)}{n} \rho \quad \text{on } \mathfrak{a}^+. \tag{4.4}$$

with the equality on $\mu(H)$.

Proof. Since Γ is a lattice of H, the limit cone of Γ satisfies

$$\mathcal{L}_\Gamma = \mu(H) = \mathfrak{a}^+ \cap \ker \alpha_2.$$

Hence for $v \notin \mu(H)$, $\psi_\Gamma(v) = -\infty$. Let $\|\cdot\|$ denote the norm on $\mathfrak{a}$ induced from the Riemannian metric on G/K. Since $H/H \cap K \subset G/K$ is an isometric embedding, we have that for all $h \in H$, $\|\mu(h)\|$ is equal to the Riemannian distance $d_{H/H\cap K}(ho, o)$ in $H/(H \cap K)$. Since ψ_Γ is independent of the choice of a norm, we may assume that for all $h \in H$, $\|\mu(h)\|$ is equal to the hyperbolic distance $d_{\mathbb{H}^n}(ho, o)$ by identifying $H/(H \cap K) \simeq \mathbb{H}^n$, which is equivalent to $\|(v_1, 0, \cdots, 0, -v_1)\| = v_1$. Since $\Gamma < H$ is a lattice, we have

$$\#\{\gamma \in \Gamma : d_{\mathbb{H}^n}(\gamma o, o) < T\} \sim Ce^{(n-1)T} \quad \text{as } T \to \infty$$

(cf. [12], [14]). Hence for $v = (v_1, 0, \cdots, 0, -v_1)$ with $v_1 \geq 0$,

$$\psi_\Gamma(v) = \|v\| \limsup_{T \to \infty} \frac{\log \#\{\gamma \in \Gamma : \|\mu(\gamma)\| \leq T\}}{T} = (n-1)v_1.$$

Since $\rho(v_1, 0, \cdots, 0, -v_1) = \frac{n}{2} v_1$ by (6.4), the claim follows. $\square$

Remark 4.3. Note that the upper bound (4.4) already follows from Proposition 4.1. The above proposition shows that that upper bound is optimal for the case at hand.

We recall the following criterion on the temperedness of $L^2(\Gamma\backslash G)$.

Theorem 4.4. *([13], [25, Theorem 5.1]) For any discrete subgroup Γ of a connected semisimple real algebraic group G, we have*

$$\psi_\Gamma \le \rho \text{ if and only if } \Gamma \text{ is a tempered subgroup of } G.$$

Moreover, if $\psi_\Gamma \le (1+\eta)\rho$, then $L^2(\Gamma\backslash G)$ is almost L^p for $p \le \frac{2}{1-\eta}$.

That $L^2(\Gamma\backslash G)$ is almost L^p means that every matrix coefficient of the quasi-regular representation $L^2(\Gamma\backslash G)$ is $L^{p+\varepsilon}$-integrable for any $\varepsilon > 0$. By Theorem 3.1, a discrete subgroup Γ is G-tempered if and only if $L^2(\Gamma\backslash G)$ is almost L^2.

Since $\psi_\Gamma = \frac{2(n-1)}{n}\rho$ on $\mu(H)$ by Proposition 4.2, we obtain the following examples of non-tempered subgroups of G:

Corollary 4.5. *Let Γ be a lattice of $H = \mathrm{SO}^\circ(n,1)$, considered as a subgroup of $G = \mathrm{SO}^\circ(n,2)$. Then*

$$\Gamma \text{ is } G\text{-tempered if and only if } n = 2.$$

Moreover, for each $n \ge 2$,

$$L^2(\Gamma\backslash G) \text{ is almost } L^n.$$

5. Deformations and non-tempered Zariski dense examples

Let $G = \mathrm{SO}^\circ(n,2)$ and $H = \mathrm{SO}^\circ(n,1) = \mathrm{Isom}^+(\mathbb{H}^n)$. Let Γ be a torsion-free uniform lattice of H such that $M = \Gamma\backslash\mathbb{H}^n$ is a closed hyperbolic n-manifold with properly embedded totally geodesic hyperplane S.

Remark 5.1. For any $n \ge 2$, such Γ exists, for instance, consider a quadratic form $Q_0(x_1,\cdots,x_{n+1}) = \sum_{i=1}^n x_i^2 - \sqrt{d}x_{n+1}^2$ for a square-free integer d. Let $\Gamma < \mathrm{SO}(Q_0)\cap \mathrm{SL}_{n+1}(\mathbb{Z}\sqrt{d})$ be a torsion -free subgroup of finite index. Then Γ is a uniform lattice of $\mathrm{SO}(Q_0)$ [4]. Considering SL_n as a subgroup of SL_{n+1} embedded as the lower diagonal block subgroup, the intersection $\Delta = \Gamma\cap\mathrm{SL}_n$ is a uniform lattice of $\mathrm{SO}(Q_0)\cap\mathrm{SL}_n \simeq \mathrm{SO}(n-1,1)$. Now $M = \Gamma\backslash\mathbb{H}^n$ is a closed hyperbolic n-manifold with a properly embedded geodesic hyperplane $S = \Delta\backslash\mathbb{H}^{n-1}$.

We may assume that $\Gamma\cap\mathrm{SO}(n-1,1) = \Delta$ is a uniform lattice of $\mathrm{SO}(n-1,1)$ by replacing Γ by a conjugate if necessary.

We briefly recall the bending construction of Johnson-Millson [18]. Their bending was constructed with the ambient group $\mathrm{SL}_{n+2}(\mathbb{R})$. We use a modification by Kassel [19, Sec. 6] where the bending was done inside $G = \mathrm{SO}^\circ(n,2)$. There exists a one-parameter subgroup $a_t \in G$ which centralizes $\mathrm{SO}(n-1,1)$. If S is separating, i.e., $M - S$ is the disjoint union of two connected components M_1 and M_2, then $\Gamma = \Gamma_1 *_\Delta \Gamma_2$. Consider the homomorphism $\sigma_t : \Gamma \to G$ given by

$$\sigma_t(\gamma) = \begin{cases} \gamma & \text{for } \gamma \in \Gamma_1 \\ a_t \gamma a_{-t} & \text{for } \gamma \in \Gamma_2. \end{cases}$$

Since a_t commutes with Δ, σ_t is well-defined. If S does not separate M, then Γ is an HNN extension of Δ, and we have a homomorphism σ_t defined similarly (cf. [19, Sec 6.3]).

The following Zariski density and discreteness results were obtained in [19] and [17] respectively:

Proposition 5.2. *For all sufficiently small $t \neq 0$, $\sigma_t(\Gamma)$ is discrete and Zariski dense in $G = \mathrm{SO}^\circ(n, 2)$.*

We now give a proof of Theorem 1.2:

Theorem 5.3. *Let $n \geq 3$. For all sufficiently small $t \neq 0$, the subgroup $\sigma_t(\Gamma)$ is a non-tempered, Zariski dense and discrete subgroup of $G = \mathrm{SO}^\circ(n, 2)$.*

Proof. The subgroup Γ is a non-tempered subgroup of G for $n \geq 3$ by Corollary 4.5. Hence the claim follows from Theorem 3.7 and Proposition 5.2. □

6. Anosov representations and non-temperedness

In this section, we prove a stronger result than Theorem 1.2 using the theory of Anosov representations. We keep the notations for $G = \mathrm{SO}^\circ(n, 2)$, $H = \mathrm{SO}^\circ(n, 1)$, $\mathfrak{a}$, α_1, α_2 etc from Section 4. Let Γ be a torsion-free uniform lattice of H such that the closed hyperbolic manifold $\Gamma\backslash\mathbb{H}^n$ has a properly embedded totally geodesic hyperplane as in Section 5.

Definition 6.1. For a non-empty subset $\theta \subset \Pi = \{\alpha_1, \alpha_2\}$, a finitely generated subgroup Γ_0 of G is called θ-Anosov if there exists $C > 0$ such that for all $\gamma \in \Gamma_0$ and $\alpha \in \theta$, we have

$$\alpha(\mu(\gamma)) \geq C^{-1}|\gamma| - C$$

where $|\gamma|$ denotes the word length of γ with respect to a fixed finite generating subset of Γ_0. A Π-Anosov subgroup is called Borel-Anosov.

Lemma 6.2. *The subgroup Γ is an $\{\alpha_1\}$-Anosov subgroup of G.*

Proof. Note that $\beta_1 := -\alpha_1$ restricted to $\mathfrak{h} \cap \mathfrak{a}$ is a simple root of $(\mathfrak{h}, \mathfrak{h} \cap \mathfrak{a})$ with respect to the choice of a positive Weyl chamber $(\mathfrak{h} \cap \mathfrak{a})^+ = \{v = (0, v_2, 0, \cdots, 0, -v_2, 0) : v_2 \geq 0\}$. Since Γ is a uniform lattice of H, it is in particular a convex cocompact subgroup of H, and hence a $\{\beta_1\}$-Anosov subgroup of H [16]. Therefore there exists $C \geq 1$ such that for all $\gamma \in \Gamma$,

$$\beta_1(\mu_H(\gamma)) \geq C^{-1}|\gamma| - C$$

where μ_H denotes the Cartan projection map of H. Since

$$\beta_1 \circ \mu_H = \alpha_1 \circ \mu|_H,$$

it follows that $\alpha_1(\mu(\gamma)) \geq C^{-1}|\gamma| - C$ for all $\gamma \in \Gamma$. This proves the claim. □

Theorem 6.3. *Let $n \geq 3$, and $G = \mathrm{SO}^\circ(n, 2)$. There exists a non-empty open subset $\mathcal{O}$ of $\mathrm{Hom}(\Gamma, G)$ such that for any $\sigma \in \mathcal{O}$, we have*

1. *σ is injective and discrete;*
2. *$\sigma(\Gamma)$ is a Zariski dense $\{\alpha_1\}$-Anosov subgroup of G;*
3. *$\sigma(\Gamma)$ is not G-tempered.*

By [1, Proposition 8.2], the set of Zariski dense representations of Γ forms an open subset of $\mathrm{Hom}(\Gamma, G)$, which we know is non-empty by Proposition 5.2. Moreover, all Anosov representations are discrete with finite kernel and the set of all $\{\alpha_1\}$-Anosov representations forms an open subset in $\mathrm{Hom}(\Gamma, G)$ by ([16], [20]). Since Γ is assumed to be torsion-free, Theorem 6.3 follows from Theorem 3.7 and non-temperedness of Γ.

In the rest of this section, we will give a different proof of Theorem 6.3(3) using the continuity of limit cones under a small deformation of Γ and the Anosov property of Γ.

For any discrete subgroup Γ_0 of G and any linear form $\psi \in \mathfrak{a}^*$ such that $\psi > 0$ on $\mathcal{L}_{\Gamma_0} - \{0\}$, denote by

$$\delta_{\psi,\Gamma_0}$$

the abscissa of convergence of the series $s \mapsto \sum_{\gamma\in\Gamma_0} e^{-s\psi(\mu(\gamma))}$. This is well-defined and $0 \leq \delta_{\psi,\Gamma_0} < \infty$. Since $\rho > 0$ on $\mathfrak{a}^+ - \{0\}$, δ_{ρ,Γ_0} is well-defined for any discrete subgroup $\Gamma_0 < G$. Theorem 4.4 can be reformulated as follows:

Proposition 6.4. *For any discrete subgroup Γ_0 of a connected semisimple real algebraic group G_0, we have*

$$\delta_{\rho,\Gamma_0} \leq 1 \textit{ if and only if } \Gamma_0 \textit{ is } G_0\textit{-tempered.}$$

Proof. By [21, Theorem 2.5], we have

$$\psi_{\Gamma_0} \leq \delta_{\rho,\Gamma_0} \cdot \rho$$

and $\psi_{\Gamma_0}(v) = \delta_{\rho,\Gamma_0} \cdot \rho(v)$ for some non-zero $v \in \mathfrak{a}^+$. Therefore the claim follows from Theorem 4.4. □

Set

$$\mathfrak{a}_{\alpha_1} = \ker \alpha_2 \quad \text{and} \quad \mathfrak{a}^+_{\alpha_1} = \mathfrak{a}^+ \cap \ker \alpha_2.$$

Let $p_{\alpha_1} : \mathfrak{a} \to \mathfrak{a}_{\alpha_1}$ denote the unique projection invariant under the Weyl element fixing $\mathfrak{a}_{\alpha_1}$ pointwise, which is simply the reflection about $\mathfrak{a}_{\alpha_1}$. The space of linear forms $\mathfrak{a}^*_{\alpha_1}$ can be identified with the set of all linear forms in $\mathfrak{a}^*$ which are invariant under p_{α_1}.

The following was obtained by Bridgeman, Canary, Labourie and Sambarino using thermodynamic formalism:

Theorem 6.5. [5] *For any $\psi \in \mathfrak{a}^*_{\alpha_1}$ which is positive on $\mathfrak{a}^+_{\alpha_1} - \{0\}$, the critical exponent $\delta_{\psi,\sigma(\Gamma)}$ varies analytically on any sufficiently small analytic neighborhood of an $\{\alpha_1\}$-Anosov representation of $\mathrm{Hom}(\Gamma, G)$.*

Since Γ is a convex cocompact subgroup of H, the following is a special case of Kassel's theorem [19, Proposition 5.1]:

Proposition 6.6. *For any $\eta > 0$, we have an open neighborhood $\mathcal{O}$ of id_Γ in $\mathrm{Hom}(\Gamma, G)$ such that for any $\sigma \in \mathcal{O}$, the limit cone of $\sigma(\Gamma)$ is contained in $\mathcal{C}_\eta := \{v \in \mathfrak{a}^+ : \|v - \mathfrak{a}_{\alpha_1}\| < \eta\|v\|\}$.*

Remark 6.7. For the bending deformations σ_t discussed in section 5, we always have a non-trivial element of γ (of infinite order) such that $\sigma_t(\gamma) = \gamma$, and hence $\mu(\sigma_t(\gamma)) \in \mu(H) - \{0\}$. Therefore we have the following property: for all sufficiently small $t \neq 0$, the limit cone of $\sigma_t(\Gamma)$ contains the ray $\mu(H)$. Since $\sigma_t(\Gamma)$ is Zariski dense, its limit cone is convex and has non-empty interior [2]. Therefore Proposition 6.6 implies that that the limit cone of $\sigma_t(\Gamma)$ is the convex cone given

$$\mathcal{L}_{\sigma_t(\Gamma)} = \{v = (v_1, v_2, 0, \cdots, -v_2, -v_1) \in \mathfrak{a}^+ : 0 \le v_2 \le c_{\sigma_t} v_1\} \tag{6.1}$$

where $c_{\sigma_t} > 0$ tends to 0 as $t \to 0$.

Recall from Proposition 4.2. that

$$\delta_{\rho,\Gamma} = \frac{2(n-1)}{n}.$$

The following proposition gives an alternative proof of Theorem 6.3(3):

Proposition 6.8. *For any sufficiently small $\varepsilon > 0$, there exists an open neighborhood $\mathcal{O} = \mathcal{O}(\varepsilon)$ of id_Γ in $\mathrm{Hom}(\Gamma, G)$ such that for any $\sigma \in \mathcal{O}$,*

$$\left|\delta_{\rho,\sigma(\Gamma)} - \frac{2(n-1)}{n}\right| < \varepsilon.$$

In particular, for $n \ge 3$, we have $\psi_\Gamma \not\le \rho$; and hence $\sigma(\Gamma)$ is non-tempered in G for all $\sigma \in \mathcal{O}(\frac{n-2}{n})$

Proof. Let ρ' be the restriction of ρ to $\mathfrak{a}_{\alpha_1}$. We may consider ρ' as a linear form on $\mathfrak{a}$ invariant under p_{α_1}. Note that ρ' is non-negative on $\mathfrak{a}_{\alpha_1}^+$.

Let $\varepsilon > 0$. We can find $\eta > 0$ so that for any $v \in \mathcal{C}_\eta = \{v \in \mathfrak{a}^+ : \|v - \mathfrak{a}_{\alpha_1}\| < \eta\|v\|\}$,

$$-\varepsilon\rho(v) \le (\rho - \rho')(v) \le \varepsilon\rho(v).$$

We can take a small neighborhood $\mathcal{O}$ of id_Γ so that for any $\sigma \in \mathcal{O}$, the limit cone of $\sigma(\Gamma)$ is contained in the cone $\mathcal{C}_\eta$ by Proposition 6.6. In particular, $\mu(\sigma(\gamma)) \in \mathcal{C}_\eta$ for all $\gamma \in \Gamma$ except for some finite subset F_σ. Then for any $\sigma \in \mathcal{O}$, we have that for all $s > 0$,

$$\sum_{\gamma \in \Gamma - F_\sigma} e^{-(1-\varepsilon)s\rho(\mu(\sigma(\gamma)))} \ge \sum_{\gamma \in \Gamma - F_\sigma} e^{-s\rho'(\mu(\sigma(\gamma)))}.$$

It follows that

$$\delta_{(1-\varepsilon)\rho,\sigma(\Gamma)} \ge \delta_{\rho',\sigma(\Gamma)} \quad \text{and hence} \quad \delta_{\rho,\sigma(\Gamma)} \ge (1-\varepsilon)\delta_{\rho',\sigma(\Gamma)}.$$

Similarly, we have

$$\sum_{\gamma\in\Gamma-F_\sigma} e^{-(1+\varepsilon)s\rho(\mu(\sigma(\gamma)))} \le \sum_{\gamma\in\Gamma-F_\sigma} e^{-s\rho'(\mu(\sigma(\gamma)))},$$

$$\delta_{(1+\varepsilon)\rho,\sigma(\Gamma)} \le \delta_{\rho',\sigma(\Gamma)} \quad \text{and hence} \quad \delta_{\rho,\sigma(\Gamma)} \le (1+\varepsilon)\delta_{\rho',\sigma(\Gamma)}.$$

Therefore

$$(1-\varepsilon)\delta_{\rho',\sigma(\Gamma)} \le \delta_{\rho,\sigma(\Gamma)} \le (1+\varepsilon)\delta_{\rho',\sigma(\Gamma)}. \tag{6.2}$$

By replacing $\mathcal{O}$ by a smaller neighborhood of id_Γ if necessary, we may assume that

$$|\delta_{\rho',\sigma(\Gamma)} - \delta_{\rho',\Gamma}| \le \varepsilon \quad \text{for all } \sigma\in\mathcal{O} \tag{6.3}$$

by Theorem 6.5.

Hence using that $1 \le \delta_{\rho,\Gamma} = 2(n-1)/n \le 2$, we deduce from (6.2) and (6.3) that

$$|\delta_{\rho,\sigma(\Gamma)} - \delta_{\rho,\Gamma}| < 5\varepsilon \quad \text{for all } \sigma\in\mathcal{O}\ .$$

Since $\delta_{\rho,\Gamma} = 2(n-1)/n$, the claim follows. □

We can also obtain the following estimates for the growth indicator $\psi_{\sigma(\Gamma)}$:

Corollary 6.9. *For any sufficiently small $\varepsilon > 0$, there exists an open neighborhood $\mathcal{O} = \mathcal{O}(\varepsilon)$ of id_Γ in $\mathrm{Hom}(\Gamma, G)$ such that for any $\sigma\in\mathcal{O}$,*

$$\psi_{\sigma(\Gamma)}(v) \le \left(\frac{2(n-1)}{n} + \varepsilon\right)\rho(v) \quad \textit{for all } v\in\mathfrak{a}^+$$

and

$$\psi_{\sigma(\Gamma)}(v_\sigma) \ge \left(\frac{2(n-1)}{n} - \varepsilon\right)\rho(v_\sigma) \quad \textit{for some unit vector } v_\sigma\in\mathfrak{a}^+. \tag{6.4}$$

Moreover, v_σ converges to a unit vector in $\mathfrak{a}_{\alpha_1}$ as $\sigma\to\mathrm{id}_\Gamma$.

Proof. Recall that $\psi_{\sigma(\Gamma)} \le \delta_{\rho,\sigma(\Gamma)}\rho$ and $\psi_{\sigma(\Gamma)}(v_\sigma) = \delta_{\rho,\sigma(\Gamma)}\rho(v_\sigma)$ for some non-zero vector v_σ on the limit cone $\mathcal{L}_{\sigma(\Gamma)}$ [21, Theorem 2.5]. Hence the inequalities follow from Proposition 6.8. The last claim follows from Proposition 6.6. □

Finally, since v_σ is of the form $(v_{\sigma,1}, c_\sigma v_{\sigma,1}, 0, \cdots, -c_\sigma v_{\sigma,1}, -v_{\sigma,1})$ for some $v_{\sigma,1} > 0$ with $c_\sigma \to 0$, the inequality (6.4) and Proposition 4.1 imply the inequality (1.3) in Theorem 1.4. Hence, together with Theorem 6.3, Proposition 6.8 and Corollary 6.9, this completes the proof of Theorem 1.4.

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