ORBIT CLOSURES OF UNIPOTENT FLOWS ON HYPERBOLIC MANIFOLDS OF INFINITE VOLUME

MINJU LEE AND HEE OH

Abstract. Let $G = \text{SO}^0(d, 1)$, $\Gamma < G$ a torsion-free convex cocompact discrete subgroup, and $M = \Gamma \backslash \mathbb{H}^d$ be the associated hyperbolic manifold. Let $U < G$ be any unipotent subgroup, or more generally, any connected subgroup generated by unipotent elements in it. When the core of $M$ has totally geodesic boundary, we classify all possible closures of $U$-orbits in $\Gamma \backslash G$. We also prove topological equidistribution theorems for any infinite sequence of maximal closed orbits of reductive subgroups containing unipotent elements. Several geometric applications are also described. In particular, we obtain that for any $k \geq 1$,

1. the closure of a $k$-horocycle intersecting the core of $M$ is a properly immersed submanifold, parallel to a geodesic submanifold;

2. the closure of a geodesic $k + 1$-plane intersecting the core of $M$ is a properly immersed geodesic submanifold.

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1. Introduction

Let $G$ be a connected simple linear Lie group and $\Gamma < G$ be a discrete subgroup. An element $g \in G$ is called unipotent if all of its eigenvalues are one, and a subgroup of $G$ is called unipotent if all of its elements are unipotent. Let $U$ be a connected unipotent subgroup of $G$, or more generally, any connected closed subgroup of $G$ generated by unipotent elements in it. We are interested in the action of $U$ on the homogeneous space $\Gamma \backslash G$ by right translations.

If the volume of the homogeneous space $\Gamma \backslash G$ is finite, i.e., if $\Gamma$ is a lattice in $G$, then Moore’s ergodicity theorem [33] says that for almost all $x \in \Gamma \backslash G$, $xU$ is dense in $\Gamma \backslash G$. While this theorem does not provide any information for a given point $x$, the celebrated Ratner’s orbit closure theorem [38] states that

\[(1.1) \quad \text{the closure of every } U\text{-orbit is homogeneous,}\]

that is, for any $x \in \Gamma \backslash G$, $\overline{xU} = xL$ for some connected closed subgroup $L < G$ containing $U$. Ratner’s proof is based on her classification of all $U$-invariant ergodic measures [37] and the work of Dani and Margulis [11] on the non-divergence of unipotent flow. Prior to her work, some important special cases of (1.1) were established by Margulis [23], Dani-Margulis ([9], [10]) and Shah ([43], [42]) by topological methods. This theorem is a fundamental result with numerous applications. In this theorem, the hypothesis on the finiteness of the volume of the homogeneous space $\Gamma \backslash G$ is very crucial; the statement is false in general without such hypothesis.

McMullen, Mohammadi and Oh proved an analogue of (1.1) for certain homogeneous spaces of $SO(3,1)$ of infinite volume, arising as the frame bundles of rigid acylindrical hyperbolic 3-manifolds ([28], [29]). Our goal in this paper is to show that the same type of orbit closure result holds in the homogeneous spaces of $SO(d,1)$ for any $d \geq 3$, which are frame bundles of rigid hyperbolic $d$-manifolds of infinite volume.

We present a complete hyperbolic $d$-manifold $M = \Gamma \backslash \mathbb{H}^d$ as the quotient of the hyperbolic space by the action of a discrete subgroup

\[\Gamma < G = SO^\circ(d,1) \simeq \text{Isom}^+(\mathbb{H}^d)\]

where $SO^\circ(d,1)$ denotes the identity component of $SO(d,1)$. The geometric boundary of $\mathbb{H}^d$ can be identified with the sphere $S^{d-1}$. The limit set $\Lambda \subset S^{d-1}$ of $\Gamma$ is the set of all accumulation points of an orbit $\Gamma(x)$ in the compactification $\mathbb{H}^d \cup S^{d-1}$ for $x \in \mathbb{H}^d$. In the entire paper, we assume $M$ is non-elementary, that is, $\Lambda$ has at least 3 points. This is equivalent to the condition that $\Gamma$ is not virtually abelian.

The convex core of $M$ is a submanifold of $M$ given by the quotient

\[\text{core } M = \Gamma \backslash \text{hull}(\Lambda)\]

where $\text{hull}(\Lambda) \subset \mathbb{H}^d$ is the smallest convex subset containing all geodesics in $\mathbb{H}^d$ connecting points in $\Lambda$. We note that core $M$ has non-empty interior if
and only if $M$ is non-Fuchsian, equivalently, $\Gamma$ is Zariski dense in $G$. When core $M$ is compact, $M$ is called convex cocompact. We note that for a convex cocompact non-Fuchsian manifold $M$, $\text{vol}(M) = \infty$ if and only if $\Lambda \neq S^{d-1}$ if and only if core $M$ has non-empty boundary.

**Definition 1.1.** We call a hyperbolic $d$-manifold $M$ rigid if the convex core of $M$ is a compact submanifold with non-empty interior and with totally geodesic boundary.

Rigid hyperbolic manifolds of finite volume, that is, those whose convex core has no boundary, are simply closed hyperbolic manifolds. Rigid hyperbolic manifolds of infinite volume can also be characterized as convex cocompact hyperbolic manifolds whose limit set satisfies

$$S^{d-1} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

where $B_i$’s are round balls with mutually disjoint closures (see Figure 1). The double of the convex core of a rigid hyperbolic $d$-manifold of infinite volume is a closed hyperbolic $d$-manifold.

**Figure 1.** Limit set of a rigid hyperbolic 4-manifold of infinite volume

Rigid hyperbolic manifolds of infinite volume are constructed in the following way. Begin with a closed hyperbolic $d$-manifold $N_0$ with a fixed collection of finitely many properly embedded totally geodesic hypersurfaces. Cut $N_0$ along those hypersurfaces to obtain a compact hyperbolic manifold $W$ with totally geodesic boundary hypersurfaces. There is a canonical procedure of extending each boundary hypersurface to a Fuchsian end, which results in a rigid hyperbolic manifold $M$ which is diffeomorphic to the interior of $W$.

By Mostow rigidity theorem, there are only countably infinitely many rigid hyperbolic manifolds of dimension at least 3. On the other hand,
for a fixed closed hyperbolic $d$-manifold $N_0$ with infinitely many properly immersed geodesic hypersurfaces,\(^{1}\) one can produce infinitely many non-isometric rigid hyperbolic $d$-manifolds of infinite volume; for each properly immersed geodesic hypersurface $f_i : \mathbb{H}^{d-1} \to N_0$, there is a finite covering $N_i$ of $N_0$ such that $f_i$ lifts to $\mathbb{H}^{d-1} \to N_i$ and $S_i := f_i(\mathbb{H}^{d-1})$ is properly imbedded in $N_i$ [19]. Cutting and pasting $N_i$ along $S_i$ as described above produces a rigid hyperbolic manifold $M_i$ of infinite volume. When the volume of $S_i$ are distinct, $M_i$’s are not isometric to each other.

**Orbit closures.** In the rest of the introduction, we assume that for $d \geq 2,$ $M$ is a rigid hyperbolic $d$-manifold.

The homogeneous space $\Gamma \backslash G$ can be regarded as the bundle $FM$ of oriented frames over $M$. Let $A = \{a_t : t \in \mathbb{R}\} < G$ denote the one parameter subgroup of diagonalizable elements whose right translation action on $\Gamma \backslash G$ corresponds to the frame flow. Let $N \simeq \mathbb{R}^{d-1}$ denote the contracting horospherical subgroup:

\[
N = \{g \in G : a_{-t} ga_t \to e \text{ as } t \to +\infty\}.
\]

We denote by $RFM$ the renormalized frame bundle of $M$ given by

\[
RFM := \{x \in \Gamma \backslash G : xA \text{ is bounded}\},
\]

and also set

\[
RF^+ M := \{x \in \Gamma \backslash G : xA^+ \text{ is bounded}\}
\]

where $A^+ = \{a_t : t \geq 0\}$. When $\text{Vol}(M) < \infty$, we have $\Lambda = S^{d-1}$ and hence

\[
RFM = RF^+ M = \Gamma \backslash G.
\]

In general, $RFM$ projects into core $M$ (but not surjective in general) and $RF^+ M$ projects onto $M$ under the basepoint projection $\Gamma \backslash G \to M$. The sets $RFM$ and $RF^+ M$ are precisely non-wandering sets for the actions of $A$ and $N$ respectively [48].

Let $U$ be a non-trivial connected unipotent subgroup of $N$. If $x \not\in RF^+ M$, then the map $U \to xU \subset \Gamma \backslash G$ given by $u \mapsto xu$ is a proper isometric immersion, and hence every orbit $xU$ is closed in $\Gamma \backslash G$. On the other hand, $xU$ is dense in $RF^+ M$ for almost all $x \in RF^+ M$, with respect to a unique $N$-invariant locally finite measure, called the Burger-Roblin measure by the work of Mohammadi-Oh [32] for $d = 3$ and Maucourant and Schapira for $d \geq 3$ general [31] (see section 10).

\(^{1}\)Any closed arithmetic hyperbolic manifold has infinitely many properly immersed geodesic hypersurfaces provided it has at least one. This is due to the presence of Hecke operators.
**Orbit closures are homogeneous.** As any connected unipotent subgroup is contained in $N$, up to a conjugation, the following theorem gives a complete classification of the closure of an orbit of a connected unipotent subgroup of $G$:

**Theorem 1.2.** Let $U < G$ be a connected unipotent subgroup contained in $N$. For any $x \in \text{RF}_+ M$, the closure of $xU$ is homogeneous in $\text{RF}_+ M$, that is,

\[(1.2) \quad \overline{xU} = xL \cap \text{RF}_+ M\]

where $xL$ is a closed orbit of a connected closed subgroup $L$ containing $U$.

More precisely, $L$ is a reductive subgroup of $G$ with compact center.

If the orbit $xU$ is bounded, then $xL$ is a compact subset contained in $\text{RF}_+ M$, and hence in (1.2), we have

\[\overline{xU} = xL.\]

When $xU$ is unbounded, which is a typical case, taking the intersection with $\text{RF}_+ M$ is necessary in (1.2), as $xL$ is not contained in $\text{RF}_+ M$ unless it is compact.

For any non-trivial connected subgroup $U < \tilde{N} \simeq \mathbb{R}^{d-1}$, we denote by $H(U)$ the smallest connected simple Lie subgroup of $G$ which contains both $U$ and $A$. If $U \simeq \mathbb{R}^{k-1}$, then $H(U) \simeq \text{SO}^0(k, 1)$.

The set $\text{RF}_+ M \cdot H(U)$ turns out to be a minimal closed $H(U)$-invariant subset containing $\text{RF}_+ M$. We have the following classification of $H(U)$-orbit closures of points in $\text{RF}_+ M$:

**Theorem 1.3.** For any $x \in \text{RF}_+ M$, and any non-trivial connected subgroup $U < \tilde{N}$,

\[\overline{xH(U)} = xL \cap \text{RF}_+ M \cdot H(U)\]

where $xL$ is a closed orbit of a connected closed subgroup $L$ containing $U$. More precisely, $L = H(\tilde{U})C$ where $\tilde{U} < N$ is a connected subgroup containing $U$ and $C$ is a connected closed subgroup of the centralizer of $H(\tilde{U})$.

Any connected subgroup of $G$ generated by unipotent one-parameter subgroups is conjugate to either $U < \tilde{N}$ or $H(U)$. Therefore Theorems 1.2 and 1.3 give a classification of orbit closures for all such subgroups of $G$, which can also be presented as follows in a unified manner:

**Corollary 1.4.** Let $H < G$ be a connected closed subgroup generated by unipotent elements in it. Assume that $H$ is normalized by $A$. For any $x \in \text{RF}_+ M$, the closure of $xH$ is homogeneous in $\text{RF}_+ M$, that is,

\[(1.3) \quad \overline{xH} \cap \text{RF}_+ M = xL \cap \text{RF}_+ M\]

where $xL$ is a closed orbit of a connected closed subgroup $L$ containing $U$. Moreover, $L$ is a reductive subgroup of $G$ with compact center.
Remark 1.5. If $\Gamma$ is contained in $G(\mathbb{Q})$ for some $\mathbb{Q}$-structure of $G$, and $[g]L$ is a closed orbit appearing in Corollary 1.4, then $L$ is defined by the condition that $gLg^{-1}$ is the smallest connected $\mathbb{Q}$-subgroup of $G$ containing $gHg^{-1}$.

We also prove the following topological analogue of Mozes-Shah equidistribution theorem [34]:

**Theorem 1.6.** If $x_i L_i \subset \Gamma \setminus G$ is an infinite sequence of maximal closed orbits where $x_i \in \text{RF} \ M$ and $L_i \leq G$ is a connected reductive subgroup normalized by $\Lambda$ with non-compact semisimple part, then

$$\lim x_i L_i \cap \text{RF} \ M = \text{RF} \ M$$

where the limit is taken in the Hausdorff topology on the space of all closed subsets in $\Gamma \setminus G$.

**Horospheres, geodesic planes and spheres.** We now state the geometric implications of our main theorems on the closures of horospheres and geodesic planes of the manifold $M$, as well as on the $\Gamma$-orbit closure of a sphere in $S^{d-1}$. A $k$-horosphere in $\mathbb{H}^d$ is a Euclidean sphere of dimension $k$ which is tangent to a point in $\partial \mathbb{H}^d$. A $k$-horosphere in $M$ is simply the image of a $k$-horosphere in $\mathbb{H}^d$ under the covering map $\mathbb{H}^d \rightarrow M = \Gamma \setminus \mathbb{H}^d$. A geodesic $k$-plane $P$ in $M$ is the image of a totally geodesic immersion $f : \mathbb{H}^k \rightarrow M$, or equivalently, the image of a geodesic $k$-subspace of $\mathbb{H}^d$ under the covering map $\mathbb{H}^d \rightarrow M$. If $f$ factors through the covering map $\mathbb{H}^k \rightarrow S_0 := \Gamma_0 \setminus \mathbb{H}^k$ for a rigid hyperbolic $k$-manifold $S_0$, we call $P = f(\mathbb{H}^k)$ a rigid hyperbolic $k$-plane.

**Corollary 1.7.** Let $M = \Gamma \setminus \mathbb{H}^d$ be a rigid hyperbolic manifold, and let $1 \leq k \leq d - 1$.

1. The closure of any $k$-horosphere of $M$ intersecting core $M$ is a properly immersed $m$-dimensional submanifold, parallel to a rigid geodesic $m$-plane of $M$ for some $m \geq k$.
2. The closure of any geodesic $(k + 1)$-plane of $M$ intersecting core $M$ is a properly immersed rigid geodesic $m$-plane of $M$ for some $m \geq k$.
3. Any infinite sequence of maximal properly immersed geodesic $(k+1)$-planes $P_i$ of $M$ intersecting core $M$ becomes dense in $M$, i.e.,

$$\lim P_i = M$$

where the limit is taken in the Hausdorff topology on the space of all closed subsets in $M$.
4. If $\text{vol}(M) = \infty$, there are only finitely many maximal bounded properly immersed geodesic $(k+1)$-planes in $M$.

By abuse of notation, let $\pi$ denote both base point projection maps $G \rightarrow \mathbb{H}^d$ and $\Gamma \setminus G \rightarrow M$ where we consider an element $g \in G$ as an oriented frame over $\mathbb{H}^d$. Let $H' = \text{SO}^0(k,1) \text{SO}(d-k)$, $2 \leq k \leq d - 1$. The quotient space $G/H'$ parametrizes all oriented $(k-1)$-spheres in $S^{d-1}$, which we denote by $C^{k-1}$. 

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By abuse of notation, let $\pi$ denote both base point projection maps $G \rightarrow \mathbb{H}^d$ and $\Gamma \setminus G \rightarrow M$ where we consider an element $g \in G$ as an oriented frame over $\mathbb{H}^d$. Let $H' = \text{SO}^0(k,1) \text{SO}(d-k)$, $2 \leq k \leq d - 1$. The quotient space $G/H'$ parametrizes all oriented $(k-1)$-spheres in $S^{d-1}$, which we denote by $C^{k-1}$. 

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We also prove the following topological analogue of Mozes-Shah equidistribution theorem [34]:

**Theorem 1.6.** If $x_i L_i \subset \Gamma \setminus G$ is an infinite sequence of maximal closed orbits where $x_i \in \text{RF} \ M$ and $L_i \leq G$ is a connected reductive subgroup normalized by $\Lambda$ with non-compact semisimple part, then

$$\lim x_i L_i \cap \text{RF} \ M = \text{RF} \ M$$

where the limit is taken in the Hausdorff topology on the space of all closed subsets in $\Gamma \setminus G$.
To an $H'$-orbit $gH' \subset G$, the image $\pi(gH') \subset \mathbb{H}^d$ is an oriented geodesic $k$-plane and the boundary $\partial(\pi(gH')) \subset S^{d-1}$ is an oriented $(k-1)$-sphere, and passing to the quotient space $\Gamma \backslash G$, this gives bijections among:

1. the space of all closed $H'$-orbits $xH' \subset \Gamma \backslash G$ for $x \in RF_M$;
2. the space of all oriented properly immersed geodesic $k$-planes $P$ in $M$ intersecting core $M$;
3. the space of all closed $\Gamma$-orbits of oriented $(k-1)$-spheres $C \in \mathcal{C}^{k-1}$ with $\#C \cap \Lambda \geq 2$

If $U := H' \cap N$, then any $(k-1)$-horosphere in $M$ intersecting core $M$ is given by $\pi(xU)$ for some $x \in RF_M$.

In view of these correspondences, Corollary 1.7 follows from Theorems 1.2, 1.3 and 1.6, and we also obtain the following description on $\Gamma$-orbits of a sphere of any positive dimension.

**Corollary 1.8.** Let $1 \leq k \leq d-2$.

1. Let $C \in \mathcal{C}^k$ with $\#C \cap \Lambda \geq 2$. Then there exists an $m$-sphere $S \in \mathcal{C}^m$ such that $\Gamma S$ is closed in $\mathcal{C}^m$ and

\[ \overline{\Gamma C} = \{ D \in \mathcal{C}^k : D \cap \Lambda \neq \emptyset, D \subset \Gamma S \} \]

2. Let $C_i \in \mathcal{C}^k$ be an infinite sequence of $k$-spheres with $\#C_i \cap \Lambda \geq 2$ such that $\Gamma C_i$ is closed in $\mathcal{C}^k$. Assume that $\Gamma C_i$ is maximal in the sense that, that there is no proper sphere $S \subset S^{d-1}$ which properly contains $C_i$ and that $\Gamma S$ is closed. Then as $i \to \infty$,

\[ \lim_{i \to \infty} \Gamma C_i = \{ D \in \mathcal{C}^k : D \cap \Lambda \neq \emptyset \} \]

where the limit is taken in the Hausdorff topology on the space of all closed subsets in $\mathcal{C}^k$.

3. If $\Lambda \neq S^{d-1}$, there are only finitely many maximal closed $\Gamma$-orbits of spheres of positive dimension contained in the limit set $\Lambda$.

**Remark 1.9.** (1) As mentioned before, the main theorems of this paper for $d = 3$ were proved by McMullen, Mohammadi and Oh ([28], [29]), which was an important starting point of this paper. We also mention that analogous theorems were obtained in [4] for geometrically finite acylindrical hyperbolic 3-manifolds.

(2) The rigid hyperbolic 3-manifold of infinite volume has a huge deformation space parametrized by the product of the Teichmüller spaces of the boundary components of core $M$ (cf. [20]), and any convex cocompact acylindrical hyperbolic 3-manifold is a quasi-conformal conjugation of a rigid hyperbolic 3-manifold [27]. In [30], an analog of Theorem 1.3 was obtained for all convex cocompact acylindrical hyperbolic 3-manifolds. We refer to [28] for counterexamples of the orbit closure theorem 1.3 in the presence of an essential cylinder. For dimension higher than 3, Kerckhoff and Storm showed that a rigid hyperbolic $d$-manifold $M = \Gamma \backslash \mathbb{H}^d$ of infinite volume does not allow
any non-trivial deformation, in the sense that the representation of $\Gamma$ into $G$ is infinitesimally rigid [16].

**Remark 1.10.** We discuss an implication of Theorem 1.2 on the $U$-invariant measure classification question on $RF_+ M$. There exists a canonical geometric $U$-invariant measure on each closed orbit $xL$ in Theorem 1.2: we may write $L = L_{nc} \cdot C$ where $L_{nc}$ is the unique connected simple non-compact normal subgroup of $L$, and $C$ is a compact subgroup centralizing $L_{nc}$. In fact, $L_{nc} \cong SO^0(m, 1)$ for some $m \geq 2$, and $L_{nc} \cap N = L \cap N$ is a horospherical subgroup of $L_{nc}$. Denoting by $p : L \to L_{nc}$ the canonical projection, the subgroup $p(Stab_L(x))$ is a convex cocompact Zariski dense subgroup of $L_{nc}$, and hence there exists a unique $L \cap N$-invariant locally finite measure on $p(Stab_L(x)) \backslash L_{nc}$, called the Burger-Roblin measure ([6], [40], [35], [48]). Now its $C$-invariant lift to $(L \cap Stab_L(x)) \backslash L$ defines a unique $(L \cap N)C$-invariant locally finite measure, say $m_{xL}^{BR}$, whose support is equal to $xL \cap RF_+ M$. Moreover $m_{xL}^{BR}$ is $U$-ergodic (cf. section 10). A natural question is the following:

*is every $U$-invariant ergodic locally finite Borel measure in $RF_+ M$ proportional to some $m_{xL}^{BR}$?*

An affirmative answer would provide an analogue of Ratner’s measure classification [37] in this setup. Theorem 1.2 implies that the answer is yes, at least in terms of the support of the measure.

2. **Outline of the proof**

Our proof is almost entirely topological in the sprit of the works ([23], [9], [10], [43], and [42]); the only place which is not topological is in the part where we use the ergodicity of the Burger-Roblin measure to guarantee that there are many generic points for any unipotent one-parameter subgroup of $N$.

We will explain the strategy of our proof with an emphasis on the difference between finite and infinite volume case and the difference between dimension 3 and higher case.

**Thick recurrence of unipotent flows.** Let $U_0 = \{u_t : t \in \mathbb{R}\}$ be a one-parameter unipotent subgroup of $N$. The main obstacle of carrying out unipotent dynamics in a homogeneous space of infinite volume is the scarcity of recurrence of unipotent flow. In a compact homogeneous space, every $U_0$-orbit stays in a compact set for the obvious reason. Already in a noncompact homogeneous space of finite volume, understanding the recurrence of $U_0$-orbit is an important issue. Margulis showed that any $U_0$-orbit is recurrent to a compact subset [21], and Dani-Margulis [11] showed that for any $x \in \Gamma \backslash G$, and for any $\varepsilon > 0$, there exists a compact subset $\Omega \subset \Gamma \backslash G$ such that

$$\ell\{t \in [0, T] : xu_t \in \Omega\} \geq (1 - \varepsilon)T$$
for all large $T \gg 1$, where $\ell$ denotes the Lebesgue measure on $\mathbb{R}$. This non-
divergence of unipotent flows is an important ingredient of Ratner’s orbit
closure theorem [38].

In contrast, when $\Gamma \backslash G$ has infinite volume, for any compact subset $\Omega \subset\Gamma \backslash G$, and for almost all $x$ (with respect to any Borel measure $\mu$ on $\mathbb{R}$),

$$\mu\{t \in [0, T] : xu_t \in \Omega\} = o(T)$$

for all $T \gg 1$ [1]. Nonetheless, the pivotal reason that we can work with rigid hyperbolic
manifolds of infinite volume is the following \textit{thick} recurrence property that
they possess: there exists $k > 1$ such that for any $y \in RF M$, the return time

$$T(y) = \{t \in \mathbb{R} : yu_t \in RF M\}$$

is $k$-thick, in the sense that for any $\lambda > 0$,

$$(2.1) \quad T(y) \cap ([−k\lambda, \lambda] \cup [\lambda, k\lambda]) \neq \emptyset.$$ 

This recurrence property was first observed by McMullen, Mohammadi and Oh [28] in the setting of a rigid hyperbolic $3$-manifold, in order to get an
additional invariance of a relative $U_0$-minimal subset with respect to $RF M$
by studying the polynomial divergence property of $U_0$-orbits of two nearby
$RF M$-points.

**Beyond $d = 3$.** In a higher dimensional case, the presence of intermediate
closed orbits introduces a variety of serious hurdles. We call the collection of
all intermediate closed orbits as the singular set and its complement a generic
set. One of the important new ingredients of this paper is the avoidance of the singular set along the $k$-thick recurrence of $U_0$-orbits to $RF M$ for a sequence of $RF M$-points limiting at a generic point, whose analog in the
finite volume case was proved by Dani-Margulis [12] and also independently
by Shah [41] based on the linearization methods.

To explain it further, let us fix some notations first. Let $U$ be a connected
unipotent subgroup contained in $N$. Recall that $H(U)$ denotes the connected simple Lie subgroup of $G$ containing both $A$ and $U$. We denote by $C(H(U))$
the centralizer of $H(U)$ and by $N(H(U))$ the identity component of the
normalizer of $H(U)$.

Let $\mathcal{L}_U$ denote the collection of all closed subgroups of $G$ which are of the form

$$L = H(\tilde{U})C$$

such that $[g]L$ is closed for some $[g] \in RF M$ and $L \cap g^{-1} \Gamma g$ is Zariski dense
in $L$ where $U < \tilde{U} < N$ and $C$ is a closed subgroup of $C(H(\tilde{U}))$. We also
define the collection

$$Q_U := \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N(U)\}$$

which turns out to be equal to

$$Q_U = \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N\}.$$
Road map for induction. The main theorems 1.2, 1.4 and 1.6 are deduced from the following theorem, which is proved by the induction argument on the co-dimension of $U$ inside the horospherical subgroup of an intermediate closed orbit.

**Theorem 2.1.** Let $x \in RF M$.

1. $(H(U)$-orbit closure) We have
   \[ \overline{xH(U)} = xL \cap RF_+ M \cdot H(U) \]
   where $xL$ is closed for some $L \in \mathcal{L}_U$.

2. $(U$-orbit closure) If $\overline{xU}$ is contained in a closed orbit $x\hat{L}$ for some $\hat{L} \in \mathcal{Q}_U$, then
   \[ \overline{xU} = xL \cap RF_+ M \]
   where $xL$ is closed for some $L \in \mathcal{Q}_U$ contained in $\hat{L}$.

3. (Equidistribution) If $x\hat{L}$ is closed for $\hat{L} \in \mathcal{L}_U$ and $y_i L_i v_i \subset x\hat{L}$ is a sequence of closed orbits where $y_i \in RF M$, $L_i \in \mathcal{L}_U$, and $v_i \in \hat{L} \cap N$ satisfy either of the following:
   - $v_i \to \infty$ modulo $L_i$ or
   - $v_i$ is bounded modulo $L_i$ and $y_i L_i$ are all distinct,
   then
   \[ \limsup_{i \to \infty} (y_i L_i v_i \cap RF_+ M) = x\hat{L} \cap RF_+ M. \]

Let us say $(1)_m$ holds, if (1) is true for all $U$ satisfying $\text{co-dim}_N(U) \leq m$. We will say $(2)_m$ holds, if (2) is true for all $U$ and $\hat{L}$ satisfying $\text{co-dim}_{L \cap N}(U) \leq m$ and similarly for $(3)_m$.

The base case of $m = 0$ is trivial except for $(2)_0$. The claim $(2)_0$ requires the extension of the minimality of a horospherical subgroup action in the presence of compact factors in the ambient group $\hat{L}$ (see section 11).

We show that the validity of $(1)_m$, $(2)_m$, and $(3)_m$ implies that of $(1)_{m+1}$ and the validity of $(1)_{m+1}$, $(2)_m$, and $(3)_m$ implies that of $(2)_{m+1}$ and $(3)_{m+1}$.

In order to give an outline of the proof of $(1)_{m+1}$, we suppose that $\text{co-dim}_N(U) \leq m + 1$. Let $F := RF_+ M \cdot H(U)$, $F^*$ denote the interior of $F$, and $\partial F := F - F^*$ the boundary of $F$. For any $x \in \partial F \cap RF M$, $xH(U)$ lies in the compact homogeneous space a connected subgroup isomorphic to $\text{SO}^0(d - 1, 1)$, and hence Theorem 2.1 follows from the finite volume case ([38], [44]).

Therefore, we let $x \in F^* \cap RF M$, and consider
\[ X := \overline{xH(U)} \subset F. \]

The strategy in proving $(1)_{m+1}$ for $X$ consists of two steps:

1. (Find) Find a closed $L$-orbit $x_0 L$ with $x_0 \in F^* \cap RF M$ such that $x_0 L \cap F$ contained in $X$ for some $L \in \mathcal{L}_U$;
(2) (Enlarge) If $X \not\subset x_0 L C(H(U))$, then enlarge $x_0 L$ to a bigger closed orbit $x_1 \bar{L} \cap F \subset X$ where $x_1 \in F^* \cap \RF M$ and $\bar{L} \in L_{\bar{U}}$ for some $\bar{U} < N$ of dimension strictly larger than $\dim U$.

The enlargement process must end after finitely many steps because of the dimension reason. Finding a closed orbit as in (1) is based on the study of the relative $U$-minimal sets and the unipotent blow up argument using the polynomial divergence of $U$-orbits of nearby $RF M$-points. To explain some ideas behind the enlargement step, suppose that we are given an intermediate closed $L$-orbit with $x_0 L \cap F \subset X$, and a one-parameter subgroup $U_0 = \{u_t\}$ of $U$ such that $x_0 U_0$ is dense in $x_0 L \cap \RF M$. Such $L$ turns out be reductive always, and hence the Lie algebra of $G$ can be decomposed into the $\Ad(L)$-invariant subspaces $l \oplus l^\perp$ where $l$ denotes the Lie algebra of $L$. Suppose that we could arrange a sequence $x_0 \ell_i \to x_0$ in $X$ for some $g_i \to e$ such that writing $g_i = \ell_i \exp q_i$ where $\ell_i \in L$ and $q_i \in l^\perp$, the following conditions are satisfied:

- $\exp q_i \notin N(U) \cup N(U_0)$;
- $x_0 \ell_i \in \RF M$.

Then for any $\varepsilon > 0$, the $k$-thick return property of $x_0 \ell_i \in \RF M$ along $U_0$ yields a sequence $u_i \in U_0$ such that

$$x_0 \ell_i u_i \to x_1 \in \RF M \cap x_0 L \quad \text{and} \quad u_i^{-1} \exp(q_i) u_i \to v$$

for some unipotent element $v \in N - L$ of size between $\varepsilon$ and $k\varepsilon$. This gives us a point

$$x_1 v \in X.$$ 

(2.2) If we could guarantee that $x_1$ is a generic point for $U$ in $x_0 L$ in the sense that its $U$-orbit $x_1 U$ is not contained in any proper closed orbit inside $x_0 L$, then $\overline{x_1 U}$ must be equal to $x_0 L \cap \RF M$ by the induction hypothesis (2)$_m$ since the codimension of $U$ inside $L \cap N$ is at most $m$. Then

$$\overline{x_1 v U} = \overline{x_1 U v} = (x_0 L \cap \RF M) v \subset X.$$ 

Now the double coset $A v A$ contains a one-parameter unipotent semigroup $V^+$, and this gives us $(x_0 L \cap \RF M) V^+ \subset X$ using the $A$-invariance of $X$.

(2.3) Assuming that $x_0 \in F^*$,

we can promote $V^+$ to a one-parameter group $V$, and find an orbit of $\hat{U} := UV$ of a bigger unipotent subgroup contained in $X$. This enables us to use the induction hypothesis (2)$_m$ again to complete the enlargement step. Note that if $x_1$ is not generic for $U$ in $x_0 L$, the closure of $x_1 U$ may be stuck in a smaller closed orbit inside $x_0 L$, in which case $\overline{x_1 U v}$ may not be not bigger than $x_0 L$ in terms of the dimension, resulting in no progress.

We now explain how we establish (2.2).
Avoidance of the singular set along the thick return time. Let $U_0 = \{u_t\}$ be a one parameter unipotent subgroup of $U$. We denote by $\mathcal{S}(U_0)$, called the singular set for $U_0$, the collection of all closed $U_0$-invariant subsets of the form $xL$ where $L \in \mathcal{L}_{U_0}$ is a proper connected closed subgroup of $G$. Its complement in $\Gamma \setminus G$ is denoted by $\mathcal{G}(U_0)$, and called the set of generic elements of $U_0$. We have

$$\mathcal{S}(U_0) = \bigcup_{H \in \mathcal{H}} \Gamma \setminus \Gamma X(H,U_0)$$

where $\mathcal{H}$ is the countable collection of all proper closed connected subgroups $H$ of $G$ such that $\Gamma \setminus \Gamma H$ is closed and $H \cap \Gamma$ is Zariski dense in $H$, and $X(H,U_0) := \{g \in G : gU_0g^{-1} \subset H\}$ (Proposition 5.9). We define $\mathcal{E}$ to be the collections of subsets of $\mathcal{S}(U_0) \cap \text{RF}M$ of the form

$$\bigcup \Gamma \setminus \Gamma H_i D_i \cap \text{RF}M$$

where $H_i \in \mathcal{H}$ is a finite collection, and $D_i$ is a compact subset of $X(H_i,U_0)$. The following avoidance theorem is a main ingredient of this paper: let $k$ be given by (2.1) for $M = \Gamma \setminus \mathbb{H}^n$:

**Theorem 2.2 (Avoidance theorem).** There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$ with

$$\mathcal{S}(U_0) \cap \text{RF}M = \bigcup_{j=1}^{\infty} E_j$$

satisfying the following: for each $j \in \mathbb{N}$ and for any compact subset $F \subset \text{RF}M - E_{j+1}$, there exists an open neighborhood $O_j = O_j(F)$ of $E_j$ such that for any $x \in F$,

$$(2.4) \quad \{t \in \mathbb{R} : xu_t \in \text{RF}M - O_j\}$$

is $2k$-thick.

It is crucial that the thickness size of the set (2.4), which is given by $2k$ here, can be controlled independently of the compact subsets $E_j$ for applications in the orbit closure theorem. If $E_j$ does not contain any closed orbit of a connected subgroup of $G$, then obtaining $E_{j+1}$ and $O_j$ is much simpler. In general, $E_j$ may contain infinitely many intermediate closed orbits, and our proof is based on a careful analysis on the graded intersections of those closed orbits and a combinatorial argument, which we call *inductive search argument*. That is, we find an RF $M$-point in any given window $[-2k\lambda, -\lambda] \cup [\lambda, 2k\lambda]$, avoiding the sequence of neighborhoods of all graded intersections in a finite number of steps depending only on the dimension of $G$ and the quality $k$ of the thick recurrence time to RF $M$ of $U_0$-orbits. This process is quite delicate, compared to the finite volume case treated in ([12], [41]) in which case the set $\{t : xu_t \in \text{RF}M\}$, being equal to $\mathbb{R}$, possesses the Lebesgue measure which can be used to measure the time outside of a neighborhood of $E_j$'s. Thanks to this property, together with the countability of closed orbits in $E_j$, up to translations by $N(U_0)$, it was sufficient in
loc. cit. to use the induction hypothesis only on the self-intersections from each closed orbit in $E_j$ at a time, rather than from all closed orbits in $E_j$ simultaneously.

We deduce from Theorem 2.2 the following:

**Theorem 2.3** (Accumulation on a generic point). Suppose that (2)$_m$ and (3)$_m$ hold in Theorem 2.1. Then the following holds for any closed connected subgroup $U < N$ with co-dim$_U(U) = m + 1$. Let $U_0 = \{u_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of $U$. Let $x_i \in RF$ be a sequence converging to $x_0 \in \mathcal{G}(U_0)$ as $i \to \infty$. Then for any given sequence $T_i \to \infty$,

$$\limsup_{i \to \infty} \{x_i u_t \in RF : T_i \leq |t| \leq 2kT_i\}$$

contains a sequence $\{y_N : N = 1, 2, \ldots\}$ such that $\limsup_{N \to \infty} y_N U$ contains a point in $\mathcal{G}(U_0) \cap RF$.\(^2\)

Again, it is important that $2k$ is independent of $x_i$ in the above theorem. We prove two independent but related versions of Theorem 2.3 in section 14 depending on the relation of $x_i$ with the set $RF$; we use Theorem 14.2 for the proof of (1)$_{m+1}$ and Theorem 14.3 for the proof of (2)$_{m+1}$.

**Comparison with the finite volume case.** We remark that if $\Gamma \backslash G$ is compact, the methods of Dani-Margulis [12] show that if $x_i$ converges to $x \in \mathcal{G}(U_0)$, and $\varepsilon > 0$, then we can find a sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$, and neighborhoods $\mathcal{O}_i$ of $E_i$ such that $\mathcal{G}(U_0) = \bigcup E_j$, $x_i \not\in \bigcup_{j \leq i} \mathcal{O}_j$ and for all $j \leq i$,

$$\ell\{t \in [0, T] : x_i u_t \in \mathcal{O}_j\} \leq \frac{\varepsilon}{2^i} T$$

and hence

$$\ell\{t \in [0, T] : x_i u_t \in \bigcup_{j \leq i} \mathcal{O}_j\} \leq \varepsilon T$$

for any $T > 0$, where $\ell$ denotes the Lebesgue measure on $\mathbb{R}$. This implies that the limsup set as in (2.5) always contains an element of $\mathcal{G}(U_0)$, without using the induction hypothesis. This is the reason why (3)$_m$ is not needed in obtaining (1)$_{m+1}$ and (2)$_{m+1}$ in Theorem 17.3 for the finite volume case.\(^3\)

In comparison, we are not able to get a generic point at once in Theorem 2.3; only with the help of the induction hypothesis (2)$_m$ and (3)$_m$, we can get a generic point for $U_0$ after taking the limsup of the $U$-orbits of all accumulating points from the $2k$-thick sets obtained in Theorem 2.2.

\(^2\)Here we allow a constant sequence $y_N = y$ in which case $\limsup_{N \to \infty} y_N U$ is understood as $\mathcal{G}(U)$ and hence $y \in \mathcal{G}(U_0)$.

\(^3\)We give a summary of our proof for the case when $\Gamma \backslash G$ is compact and has at least one $SO(d-1, 1)$ closed orbit in the appendix to help readers understand the whole scheme of the proof.
Generic points in $F^*$ as limits of RF $M$-points. In the inductive argument, it is important to find a closed orbit $x_0L$ based at a point $x_0 \in F^*$ in order to guarantee (2.3). Another reason why this is critical is the following: Implementing Theorem 2.3 and its versions Theorems 14.2 and 14.3 requires having a sequence of RF $M$-points of $X$ accumulating on a generic point of $x_0L$ with respect to any given one-parameter subgroup $U_0$ of $U$; that is, how do we find a sequence $x_i$ accumulating on a generic point $x$ inside $RF M \cap X$? The advantage of having a closed orbit $xL$ with $x \in F^*$ is that any generic $U_0$-point in $xL \cap RF M$ can be approximated by a sequence of RF $M$-point in $F^* \cap X$ (Lemma 12.2).

We also point out that we use the ergodicity theorem obtained in [32] and [31] to guarantee there are many $U_0$-generic points in any closed orbit $x_0L$ as above.

Existence of proper closed orbits in $\Gamma \backslash G$. In our setting, $\Gamma \backslash G$ always contains a closed orbit $xL$ for some $x \in RF M$ and a proper subgroup $L \in \mathcal{L}_U$; namely those closed orbits of SO$^\circ (d-1,1)$ over the boundary of core $M$. This fact was crucially used in deducing $(2)_{m+1}$ from $(1)_{m+1}, (2)_m, (3)_m$.

Organization of the paper.

- In section 3, we set up notations for certain Lie subgroups of $G$, review some basic facts and gather preliminaries about them and geodesic planes of $M$.
- In section 4, for each unipotent subgroup $U$ of $G$, we define the minimal $H(U)$-invariant closed subset $F_{H(U)} \subset \Gamma \backslash G$ containing RF$_+ M$ and study some properties of this set and its interior for a rigid hyperbolic manifold case.
- In section 5, we define the singular set $\mathcal{S}(U)$, more generally $\mathcal{S}(U, x_0L)$ for a closed orbit $x_0L \subset \Gamma \backslash G$, and prove a structure theorem and a countability theorem for a general convex cocompact manifold.
- In sections 6, we prove a combinatorial lemma 6.4, called Inductive search lemma and Proposition 6.3 on the property of intersection of a global thick set and a uniformly bounded number of families of triples of intervals equipped with certain properties on the relative sizes, which will be used in the proof of Theorem 7.11.
- In section 7, we construct families of triples of intervals which satisfies the hypothesis of Proposition 6.3 based on a careful analysis of the graded intersections of the singular set and the linearization, and prove Theorem 2.2 (Theorem 7.11).
- In sections 8, we study the unipotent blowup lemmas using quasi-regular maps and properties of thick subsets.
- In section 9, we study the translates of relative $U$-minimal sets $Y$ into the orbit closure of an RF $M$ point.
• In section 10, we review the ergodicity theorem of [32] and [31] and deduce the density of almost all orbits of a connected unipotent subgroup in RF+ M.
• In section 11, the minimality of a horospherical subgroup action is obtained in the presence of compact factors.
• In section 12, we prove several geometric lemmas which are needed to modify a sequence limiting on a generic point to a sequence of RF M-points which still converges to a generic point.
• In section 13, we begin to prove Theorem 2.1 with the base case m = 0 proved and the orbit closure of a singular U-orbit is classified under the induction hypothesis.
• In section 14 we prove two propositions on how to get additional invariance, which is a main tool in the enlargement step of the proof of 2.1.
• We prove (1)m+1, (2)m+1 and (3)m+1 respectively in sections 15, 16 and 17.
• In the appendix, we give an outline of our proof in the case when Γ\G is compact with at least one SO0(d − 1, 1)-closed orbit.

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3. Lie subgroups and geodesic planes

In this section, we fix notation and recall some background about certain Lie subgroups of SO(d, 1) as well as some basic notions regarding geodesic planes of a hyperbolic manifold. These notation will be used throughout the paper.

Let G denote the connected simple Lie group SO0(d, 1) for d ≥ 2. The group G acts continuously on \( \mathbb{H}^d \cup \mathbb{S}^{d-1} \), preserving the hyperbolic metric on \( \mathbb{H}^d \). As a Lie group, we have \( G \cong \text{Isom}^+(\mathbb{H}^d) \). In order to present a family of subgroups of G explicitly, we fix a quadratic form \( Q(x_1, \ldots, x_{d+1}) = 2x_1x_{d+1} + x_1^2 + x_2^2 + \cdots + x_d^2 \), and identify \( G = \text{SO}^0(Q) \). The Lie algebra of G is then given as:

\[
\mathfrak{so}(d, 1) = \{ X \in \mathfrak{sl}_{d+1}(\mathbb{R}) : X^t Q + QX = 0 \}
\]

where

\[
Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \text{Id}_{d-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

\(^4\)Our appendix is different from his proof in that we prove that (1)m implies (2)m using the existence of a closed SO(d − 1, 1)-orbit, while he shows that (2)m implies (1)m.
A subset $S \subset G$ is Zariski closed if $S$ is defined to be the zero set \( \{(x_{ij}) \in G : p_1(x_{ij}) = \cdots = p_\ell(x_{ij}) = 0\} \) for a finite collection of polynomials with real coefficients in variables \((x_{ij}) \in M_{d+1}(\mathbb{R})\). The Zariski closure of a subset $S \subset G$ means the smallest Zariski closed subset of $G$ containing $S$.

**Subgroups of $G$.** Inside $G$, we have the following subgroups:

\[
K = \text{SO}(d + 1) \cap G \simeq \text{SO}(d),
\]

\[
A = \left\{ a_t = \begin{pmatrix} e^t & 0 & 0 \\ 0 & \text{Id}_{d-1} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}
\]

\[
M = \text{the centralizer of } A \text{ in } K \simeq \text{SO}(d - 1),
\]

\[
N^+ = \{ \exp u^+(x) : x \in \mathbb{R}^{d-1} \},
\]

\[
N^- = \{ \exp u^-(x) : x \in \mathbb{R}^{d-1} \}
\]

where

\[
u^-(x) = \begin{pmatrix} 0 & x^t & 0 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix}
\]

and

\[
u^+(x) = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^t & 0 \end{pmatrix}.
\]

The Lie algebra of $M$ consists of matrices of the form

\[
m(C) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

where $C \in M_{d-1}(\mathbb{R})$ is a skew-symmetric matrix, i.e., $X = -X$.

The hyperbolic space $\mathbb{H}^d$, the unit tangent bundle $T^1 \mathbb{H}^d$ and the oriented frame bundle $F \mathbb{H}^d$ can be identified with $G/K$, $G/M$ and $G$ respectively. The action of $G$ on the hyperbolic space $\mathbb{H}^d = G/K$ extends continuously to a conformal action of $G$ on the compactification $S^{d-1} \cup \mathbb{H}^d$.

If $g \in G$ corresponds to a frame $(v_1, \cdots, v_d) \in F \mathbb{H}^d$, we define $g^+, g^- \in S^{d-1}$ to be the forward and backward end points of the directed geodesic tangent to $v_1$ respectively. The right translation action of $A$ on $G = F \mathbb{H}^d$ defines the frame flow and we have

\[
g^\pm = \lim_{t \to \pm \infty} \pi(ga_t)
\]

where $\pi : G = F \mathbb{H}^d \to \mathbb{H}^d$ is the basepoint projection.

For a subset $S \subset G$, we denote by $N_G(S)$ and $C_G(S)$ the normalizer of $S$ and the centralizer of $S$ respectively. We denote by $N(S) = N_G(S)^\circ$ and $C(S) = C_G(S)^\circ$ the identity components of $N_G(S)$ and $C_G(S)$ respectively.

The subgroups $N^\pm$ are respectively the expanding and the contracting horospherical subgroups of $G$ for the action of $A$. As we will be using the subgroup $N^-$ frequently, we simply write $N = N^-$. We often identify the subgroup $N^\pm$ with $\mathbb{R}^{d-1}$ via the map $u^\pm(x) \to x$. For a subgroup $U \subset N^\pm$, we use the notation $U^\perp$ for the orthogonal complement of $U$ in
\(N^\pm\) as a vector subgroup of \(N^\pm\). We use the notation \(B_U(r)\) to denote the ball of radius \(r\) centered at 0 in \(U\) for a Euclidean metric on \(N = \mathbb{R}^{d-1}\).

For each non-trivial connected subgroup \(U < N\), we denote by
\[
H(U)
\]
the connected closed subgroup of \(G\) generated by \(U\) and the transpose of \(U\). It is the smallest connected simple Lie subgroup of \(G\) containing \(A\) and \(U\). If \(U\) has dimension \(k\), then \(H(U)\) is isomorphic to \(\text{SO}^0(k+1,1)\).

We set
\[
H'(U) := N(H(U)) = H(U) C(H(U)),
\]
which is a connected reductive Lie subgroup of \(G\) with compact center. We have
\[
C(U) = C(H(U)) N, \quad \text{and} \quad N(U) = N A C_1(U) C_2(U)
\]
where
\[
(3.1) \quad C_1(U) = C(H(U)) = M \cap C(U), \quad \text{and} \quad C_2(U) = H(U) \cap M \cap C(U^\perp).
\]

**Example.** Fix \(e_1, \ldots, e_{d-1}\) of \(\mathbb{R}^{d-1}\) and define \(U^\pm_k\) to be the subgroup of \(N^\pm\) spanned by \(e_1, \ldots, e_k\).

Then
\[
H(U_k) = \langle U^+_k, U^-_k \rangle = \text{SO}^0(k+1,1)
\]
\[
C(H(U_k)) = \text{SO}(d-k-1)
\]
\[
H'(U_k) = \text{SO}^0(k+1,1) \text{SO}(d-k-1) = N(H(U_k)).
\]

Any connected closed subgroup \(U < N\) and \(H(U)\) are respectively conjugate to \(U_k\) and to \(H(U_k)\) by an element of \(M\).

**The complementary subspaces** \(\mathfrak{h}_U^\perp\) and \(\mathfrak{h}^\perp\). Denote by \(\mathfrak{g}\) the Lie algebra of \(G\), and by \(\mathfrak{h}_U \subset \mathfrak{g}\) the Lie algebra of \(H(U)\) for \(U = U_k\). Since \(H(U)\) is reductive, the restriction of the adjoint representation of \(G\) to \(H(U)\) is completely reducible, and hence there exists an \(\text{Ad}(H(U))\)-invariant complementary subspace \(\mathfrak{h}_U^\perp\) to \(\mathfrak{h}_U\) so that
\[
\mathfrak{g} = \mathfrak{h}_U \oplus \mathfrak{h}_U^\perp.
\]

Denote by \(\mathfrak{u}^\perp\) the subspace \(\text{Lie}(U^\perp)\), and by \((\mathfrak{u}^\perp)^t\) its transpose. Then \(\mathfrak{h}_U^\perp\) can be given explicitly as follows:
\[
(3.2) \quad \mathfrak{h}_U^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0
\]
where \(\mathfrak{m}_0\) is given by
\[
\left\{ m(C) : C^t = -C, C = \begin{pmatrix} 0 & Y \\ -Y^t & Z \end{pmatrix}, Z \in \text{M}_{d-1-k}(\mathbb{R}), Y \in \text{M}_{k \times (d-1-k)}(\mathbb{R}) \right\};
\]
to see this, it is enough to check that \(\dim(\mathfrak{g}) = \dim(\mathfrak{h}_U) + \dim(\mathfrak{h}_U^\perp)\) and that \(\mathfrak{h}_U^\perp\) is \(\text{Ad}(H(U))\)-invariant, which can be done by direct computation.

Similarly, setting \(\mathfrak{h} := \text{Lie}(H'(U))\), the \(\text{Ad}(H'(U))\)-invariant complementary subspace to \(\mathfrak{h}\) in \(\mathfrak{g}\) is given by
\[ \mathfrak{h}^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus m'_0 \]

where
\[ m'_0 := \left\{ m(C) : C = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}, Z^t = -Z \right\}. \]

It follows from the inverse function theorem that the product maps \( H_U \times \mathfrak{h}_U^\perp \to G \) and \( H'_U \times \mathfrak{h}_U^\perp \to G \) given by \( (h, X) \mapsto h \exp X \) are local diffeomorphisms onto an open neighborhood of \( e \) in \( G \).

**Definition 3.1.** For a connected reductive subgroup \( L < G \), we denote by \( L_{nc} \) the maximal connected normal semisimple subgroup of \( L \) with no compact factors.

A connected reductive subgroup \( L \) of \( G \) is an almost direct product of
\[ L = L_{nc}CT \]
where \( C \) is a connected semisimple compact normal subgroup of \( L \) and \( T \) is the central torus of \( L \). If \( L \) contains a unipotent element, then \( L_{nc} \) is non-trivial, and simple, containing a conjugate of \( A \), and the center of \( L \) is compact.

**Proposition 3.2.** If \( L < G \) is a connected reductive subgroup normalized by \( A \) and containing a unipotent element, then
\[ L = H(U)C \]
where \( U < N \) is a non-trivial connected subgroup and \( C \) is a closed subgroup of \( C(H(U)) \). In particular, \( L_{nc} \) and \( N(L_{nc}) \) are equal to \( H(U) \) and \( H'(U) \) respectively.

**Proof.** In the decomposition 3.4, \( L_{nc} \) is now normalized by \( A \). Therefore it suffices to prove that a connected closed non-compact simple subgroup \( H < G \) normalized by \( A \) is of the form \( H = H(U) \) where \( U < N \) is a non-trivial connected subgroup.

Consider first the case when \( A < H \). Let \( \mathfrak{h} \) be the Lie algebra of \( H \), and \( \mathfrak{a} \) be the Lie algebra of \( A \). Since \( \mathfrak{h} \) is simple, its root space decomposition for the adjoint action of \( \mathfrak{a} \) is of the form \( \mathfrak{h} = \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^- \) where \( \mathfrak{u}^\pm \) are the sum of all positive and negative root subspaces respectively. Since the sum of all negative root subspaces for the adjoint action of \( \mathfrak{a} \) on \( \mathfrak{g} \) is \( \text{Lie}(N^-) \), it follows that \( U := \exp(\mathfrak{u}^-) < N^- \) and \( H = H(U) \).

Now for the general case, \( H \) contains a conjugate \( gAg^{-1} \) for some \( g \in G \). Hence \( g^{-1}Hg = H(U) \). Since \( H(U) \) contains both \( A \) and \( g^{-1}Ag \), they must be conjugate within \( H(U) \), so \( A = h^{-1}g^{-1}Agh \) for some \( h \in H(U) \). Hence \( gh \in N_G(A) = AM \). Therefore \( H = gH(U)g^{-1} \) is equal to \( mH(U)m^{-1} \) for some \( m \in M \). Since \( m \) normalizes \( N \), \( H = mH(U)m^{-1} \) is of the form \( H(U') \) for \( U' < N \), as desired. \( \square \)
Corollary 3.3. Any connected closed subgroup \( S \) of \( G \) generated by unipotent elements is conjugate to either \( U \) or \( H(U) \) for some non-trivial connected subgroup \( U < N \).

Proof. By the Levi decomposition, \( S = LV \) where \( L \) is reductive and \( V \) is the unipotent radical of \( S \). If \( L \) is trivial, the claim follows since any connected unipotent subgroup can be conjugate into \( N \). Suppose that \( L \) is not trivial. As \( S \) is generated by unipots, \( L \) does not have compact factors, and hence up to conjugation, \( L = H(U) \) for some non-trivial \( U < N \) by Proposition 3.2. Unless \( V \) is trivial, the normalizer of \( V \) is contained in a conjugate of \( N \). Hence \( V = \{ e \} \). \( \square \)

Totally geodesic immersed planes. Let \( \Gamma \) be a discrete torsion free subgroup of \( G \), and consider the associated hyperbolic manifold \( M = \Gamma \backslash \mathbb{H}^d = \Gamma \backslash G/K \).

The limit set of \( \Gamma \) is defined by
\[
\Lambda = \mathbb{S}^{d-1} \cap \overline{\Gamma x}
\]
where \( x \) is any point in \( \mathbb{H}^d \), and the closure is taken in the compactification \( \mathbb{S}^{d-1} \cup \mathbb{H}^d \). The convex core of \( M \) is given by
\[
\text{core } M = \Gamma \backslash \text{hull}(\Lambda)
\]
where hull(\( \Lambda \)) is the smallest convex subset containing all geodesics with both end points in \( \Lambda \). Note that core \( M \) contains all bounded geodesics in \( M \).

The hyperbolic manifold \( M \) is called convex cocompact, if core \( M \) is compact.

We denote by \( F M \simeq \Gamma \backslash G \) the bundle of all oriented orthonormal frames over \( M \). We denote by
\[
\pi : \Gamma \backslash G \to M = \Gamma \backslash G/K
\]
the base-point projection. By abuse of notation, we also denote by
\[
\pi : G \to \mathbb{H}^d = G/K
\]
the base-point projection.

The images of the orbits of \( H'(U_{k-1}) = \text{SO}^o(k, 1) \cdot \text{SO}(d-k) \) under \( \pi \) give rise to all oriented totally geodesic immersed \( k \)-planes in \( M \). As \( H'(U) = m^{-1}H'(U_{k-1})m \) for some \( m \in M \) and \( H'(U) \subset H(U) \cdot M \), the same is true for \( H'(U) \) and \( H(U) \)-orbits for any connected subgroup \( U < N \).

Fix \( k \geq 2 \) and let
\[
H = \text{SO}^o(k, 1) \quad \text{and} \quad H' = \text{SO}^o(k, 1) \cdot \text{SO}(d-k).
\]
Let \( C_0 \) denote the unique oriented \((k-1)\)-sphere in \( \mathbb{S}^{d-1} \) stabilized by \( H' \). Then \( \tilde{S}_0 := \text{hull}(C_0) \) is the unique oriented totally geodesic subspace of \( \mathbb{H}^d \) stabilized by \( H' \), and \( \partial \tilde{S}_0 = C_0 \).
The group $G$ acts transitively on the space of all oriented $(k - 1)$ spheres in $S^{d-1}$ giving rise to the isomorphisms of $G/H'$ with

$$G^{k-1} = \text{the space of all oriented } (k - 1)\text{-spheres in } S^{d-1}$$

and with

the space of all oriented totally geodesic $k$-planes of $\mathbb{H}^d$.

We discuss the fundamental group of an immersed geodesic $k$-plane $S \subset M$. Choose a totally geodesic subspace $\tilde{S}$ of $\mathbb{H}^d$ which covers $S$. Then $\tilde{S} = g\tilde{S}_0$ for some $g \in G$, and the stabilizer of $\tilde{S}$ in $G$ is equal to $gH'g^{-1}$. We have

$$\Gamma_{\tilde{S}} = \{ \gamma \in \Gamma : \gamma(\tilde{S}) = \tilde{S} \} = \Gamma \cap gH'g^{-1}$$

and get an immersion $\tilde{f} : \Gamma_{\tilde{S}} \backslash \tilde{S} \to M$ with image $S$. Consider the projection map

$$p : gH'g^{-1} \to gHg^{-1}.$$ 

Then $p$ is injective on $\Gamma_{\tilde{S}}$ and

$$\Gamma_{\tilde{S}} \backslash \tilde{S} \cong p(\Gamma_{\tilde{S}}) \backslash \tilde{S}$$

is an isomorphism, since $gC(H)g^{-1}$ acts trivially on $\tilde{S}$. Hence $\tilde{f}$ gives an immersion

(3.8) $$f : p(\Gamma_{\tilde{S}}) \backslash \tilde{S} \to M$$

with image $S$. The immersion $f$ is generically injective, as $\tilde{S}$ is oriented.

We refer to

$$\pi_1(S) \cong p(\Gamma_{\tilde{S}}) \cdot gHg^{-1}$$

as the fundamental group of $S$ (with orientation), and say $S$ properly immersed if $f$ is a proper map.

**Proposition 3.4.** Let $x \in \Gamma \backslash G$, and set $S = \pi(xH') \subset M$. Then

1. $xH'$ is closed in $\Gamma \backslash G$ if and only if $S$ is properly immersed in $M$.
2. If $M$ is convex cocompact and $S$ is properly immersed, then $S$ is convex cocompact and

$$\partial \tilde{S} \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}}))$$

for any geodesic subspace $\tilde{S} \subset \mathbb{H}^d$ which covers $S$.

**Proof.** Choose a representative $g \in G$ of $x$ and consider the totally geodesic subspace $\tilde{S} := g\tilde{S}_0$. Then $S$ is given as the image of the map $f : p(\Gamma_{\tilde{S}}) \backslash \tilde{S} \to M = \Gamma \backslash \mathbb{H}^n$.

Now the closedness of $xH'$ in $\Gamma \backslash G$ is equivalent to the properness of the map $(H' \cap g^{-1}\Gamma g) \backslash H' \to \Gamma \backslash G$ induced from map $h \mapsto xh$. This in turn is equivalent to the properness of the induced map $(H' \cap g^{-1}\Gamma g) \backslash (H' \cap K) \to \Gamma \backslash G / K$. 

If $\Delta$ is the image of $H' \cap g^{-1} \Gamma g$ under the projection map $H' \to H$, then
the natural injective map
$$\Delta \backslash H / H \cap K \to (H' \cap g^{-1} \Gamma g) \backslash H' / H' \cap K$$
is an isomorphism.

Since
$$p(\Gamma \tilde{S}) \backslash \tilde{S} = p(\Gamma \tilde{S}) \backslash gH / (H \cap K) \simeq \Delta \backslash H / (H \cap K),$$
the first claim follows. The second claim follows from [35].

4. RIGID HYPERBOLIC MANIFOLDS

In this section, we introduce the definition of a rigid hyperbolic manifold, and the associated subsets $\text{RF}_M$ and $\text{RF}_+ M$ of $\Gamma \backslash G$, which are non-wandering subsets for $A$ and $N$ respectively. For each connected subgroup $U < N$, we define a closed $H(U)$-invariant subset $F_{H(U)}$ of $\Gamma \backslash G$. We discuss equivalent formulations of $F_{H(U)}$ and study some basic properties of frames in its interior and boundary. At the end of the section, we address the global thickness of the return time of any unipotent one-parameter subgroup of $N$ to $\text{RF}_M$.

**Definition 4.1.** A convex cocompact hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d$ is called a rigid hyperbolic manifold of infinite volume if one of the following equivalent conditions holds:

1. its convex core has non-empty interior and non-empty totally geodesic boundary.

2. the domain of discontinuity of $\Gamma$

$$\Omega := S^{d-1} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

is a dense union of infinitely many round balls with mutually disjoint closures.

By a rigid hyperbolic manifold $M$, we mean either that $M$ is rigid with infinite volume in the above sense, or that $M$ is compact. We note that if $M = \Gamma \backslash \mathbb{H}^d$ is rigid, then $\Gamma$ is Zariski dense in $G$. As $M$ is convex cocompact, there are only finitely many $\Gamma$-orbits of $B_i$’s, say $\Gamma B_1, \cdots, \Gamma B_k$, and the boundary of core $M$ is given by $\bigcup_{i=1}^{k} \Gamma \text{hull}(B_i) \subset \Gamma \backslash \mathbb{H}^d$.

In the whole section, let $M$ be a rigid hyperbolic manifold of infinite volume.

**Renormalized frame bundle.** The renormalized frame bundle $\text{RF}_M \subset F M$ is defined as the following $A M$-invariant subset

$$\text{RF}_M = \{ [g] \in \Gamma \backslash G : g^\pm \in \Lambda \} = \{ x \in \Gamma \backslash G : xA \text{ is bounded} \}$$
i.e., the closed set consisting of all oriented frames $(v_1, \cdots, v_d)$ such that the complete geodesic through $v_1$ is contained in core $M$. 
Similarly, we define
\[ RF_+ M = \{ [g] \in \Gamma\backslash G : g^+ \in A \} = \{ x \in \Gamma\backslash G : xA^+ \text{ is bounded} \} \]
which is a closed $NAM$-invariant subset (here $A^+ = \{ a_t : t \geq 0 \}$).

**Lemma 4.2.** For $x \in RF_+ M$, $\overline{xA^+}$ meets $RF$.

*Proof.* Take any sequence $a_i \to \infty$ in $A^+$. Since $xA^+$ is bounded, $xa_i$ converges to some $x_0 \in \overline{xA^+}$ by passing to a subsequence. On the other hand, $x_0A \subset \limsup(xa_i)(a_i^{-1}A^+) \subset \overline{xA^+}$. Hence $x_0A$ is bounded, implying $x_0 \in RF$ as desired. \(\square\)

$H(U)$-invariant subsets: $F_{H(U)}, F^*_H(U), \partial F_{H(U)}$. Fix a non-trivial connected subgroup $U < N$, and consider the associated subgroups $H(U)$ and $H'(U)$ as defined in section 3.

We define
\[ (4.1) \quad F_{H(U)} := RF_+ M \cdot H(U) \]
which captures all of the non-trivial dynamics of the $H(U)$-action. The closedness of $F_{H(U)}$ is an easy consequence of compactness of the limit set $\Lambda$. It is also $C(H(U))$-invariant, since $RF_+ M$ is $M$-invariant and $C(H(U)) \subset M$, and hence $H'(U)$-invariant.

**Lemma 4.3.** We have
\[ F_{H(U)} = \{ x \in \Gamma\backslash G : \pi(xH(U)) \cap \text{core } M \neq \emptyset \}. \]

*Proof.* Denote by $Q$ the subset on the righthand side of (1). To show $F_{H(U)} \subset Q$, let $x \in F_{H(U)}$. Using $H(U)$, we may assume that $x \in RF_+ M$. By Lemma 4.2, $\overline{xA^+}$ contains $x_0 \in RF$. Since $x_0A$ is bounded,
\[ \pi(x_0A) \subset \pi(xH(U)) \cap \text{core } M \]
because $\text{core } M$ contains all bounded geodesics. Therefore $x \in Q$. To show the other inclusion $Q \subset F_{H(U)}$, we use the hypothesis on $M$ being rigid. Suppose $x = [g] \not\in F_{H(U)}$. Then the boundary of $\pi(gH(U))$ is disjoint from the limit set $\Lambda$, and hence it must be contained in a connected component, say $B_i$, of $\Omega$. Hence $\pi(gH(U))$ is contained in the interior of $\text{hull}(B_i)$, which is disjoint from $\text{hull}(\Lambda)$, by the convexity of $B_i$. Therefore the orbit $\Gamma \pi(gH(U))$ is a closed subset of $\mathbb{H}^d$, disjoint from $\text{hull}(\Lambda)$. Hence $x \not\in Q$, proving the claim. \(\square\)

Note also that
\[ (4.2) \quad RF M \cdot H(U) = \{ x \in \Gamma\backslash G : \pi(xH(U)) \cap \text{core } M \neq \emptyset \}. \]

**Lemma 4.4.** We have
\[ (4.3) \quad RF M \cdot H(U) = RF_+ M \cdot H(U). \]
Proof. We will prove the following equivalent statement:

\[(4.4) \quad RF M \cdot H'(U) = RF_+ M \cdot H'(U).\]

Let \(k = \dim(U).\) Under the correspondence between \(H'(U)\)-orbits in \(\Gamma\backslash G\) and \(\Gamma\)-orbits in the space \(C_k\) of all oriented \(k\)-spheres, \(RF M \cdot H'(U)\) and \(RF_+ M \cdot H'(U)\) correspond to the collection of \(C \in C_k\) such that \(#C \cap \Lambda \geq 2,\) and \(#C \cap \Lambda \geq 1\) respectively.

Let \(C \in C_k\) be such that \(C \cap \Lambda = \{\xi\}.\) Since \(C - \xi\) is connected, \(C - \xi\) is contained in one component of \(\Omega,\) say \(B,\) such that \(C \cap \partial B = \{\xi\}.\) Let \(C_i \in C_k\) be the sphere whose center is same as that of \(C,\) and whose radius is the sum of \(1/i\) and the radius of \(C.\) Clearly \(#C_i \cap \partial B \geq 2,\) and \(C_i \to C.\) This proves (4.4).

By Lemma 4.3, the interior of \(F_{H(U)}\) is then given by

\[F_{H(U)}^* = \{x \in \Gamma \backslash G : \pi(xH(U)) \cap M^* \neq \emptyset\}\]

where \(M^*\) denotes the interior of the core of \(M.\)

We denote by \(\partial F_{H(U)}\) the boundary of \(F_{H(U)},\) that is,

\[\partial F_{H(U)} = F_{H(U)} - F_{H(U)}^*.\]

When there is no room for confusion, we will omit the subscript \(H(U)\) and simply write \(F, F^*\) and \(\partial F,\)

We call an oriented frame \(g = (v_1, \ldots, v_d) \in FM = G\) a boundary frame if the first \((d-1)\) vectors \(v_1, \ldots, v_{d-1}\) are tangent to the boundary of core \(M.\)

We denote by \(BF M\) the set of all boundary frames of \(M;\) it is a union of compact \(H := SO^\circ(d-1,1)\)-orbits:

\[BF M = \bigcup_{i=1}^k z_i \tilde{H}\]

such that \(\pi(z_i \tilde{H}) = \Gamma \backslash \Gamma \text{hull}(B_i).\)

Consider the one-dimensional subgroup \(\tilde{V} = \mathbb{R}e_{d-1}\) of \(N = \mathbb{R}^{d-1}.\) If \(U\) is contained in \(\tilde{H} \cap N = \mathbb{R}^{d-2},\) then the boundary of \(F_{H(U)}\) is given by:

\[\partial F_{H(U)} = BF M \cdot \tilde{V}^+ H'(U)\]

for some one-parameter semigroup \(\tilde{V}^+\) of \(\tilde{V}\) and

\[\partial F_{H(U)} \cap RF M = BF M \cdot C(H(U)).\]

For a general proper connected subgroup \(U < N, mUm^{-1} \subset \tilde{H} \cap N\) for some \(m \in M,\) and

\[\partial F_{H(U)} \cap RF M = BF Mm C(H(U))\]

where \(BF Mm\) is now a union of finitely many \(m^{-1}\tilde{H}m\)-compact orbits.

Lemma 4.5. Let \(U < \tilde{H} \cap N.\)

1. For \(z \in BF M\) and \(v \in \tilde{V},\) if \(zv \in RF M,\) then \(zv \in F_{H(U)}^*.\)
(2) For \( y \in RF_+ M \cap F^*_H(U) \), we have
\[ yU \cap RF M \neq \emptyset. \]

**Proof.** Let \( z = [g] \), and \( B \) be a component of \( \Omega \) such that \( \partial B \) corresponds to the sphere \( \partial (gH) \). Then \( g^+ = (gv)^+ \in \partial B \) but \( (gv)^- \in \Lambda - \overline{B} \). This means that \( \pi(gvH(U)) \) meets at least two components of \( \Omega \) and hence \( zv \in F^*_H(U) \).

Now, suppose \( y = [g] \in RF_+ M \), and \( yU \cap RF M = \emptyset \). Let \( C \) denote the sphere which is the boundary of \( \pi(gH(U)) \). Then there exists a component \( B \) of \( \Omega \) such that \( C \subset \overline{B} \), and \( \#(C \cap \partial B) = 1 \). Hence there exists a \((d-2)\)-sphere \( S \) such that \( C \subset S \subset \overline{B} \).

Since \( C \subset S \), there exists \( c \in C(H(U)) \) such that \( gc \) is a frame tangent to \( \text{hull}(S) \), and \( (gc)^+ \in \partial B \). This implies \( gc \in BF M \cdot \overline{V} \). Hence \( y \notin F^*_H(U) \), proving the claim. \( \square \)

**Properly immersed geodesic planes in the rigid case.**

**Lemma 4.6.** For any sphere \( C \) in \( S^{d-1} \) with \( \#C \cap \Lambda \geq 2 \), the intersection \( C \cap \Lambda \) is Zariski dense in \( C \).

**Proof.** Write
\[ C - (C \cap \Lambda) = \bigcup_{i \in I}(C \cap B_i) \]
where \( I \) is the collection of all \( i \) such that \( C \cap B_i \neq \emptyset \).

If \( I \) is a finite set, it means that the intersection \( C \cap \Lambda \) contains an open subset \( C - \bigcup_{i \in I}(C \cap B_i) \). If \( C - \bigcup_{i \in I}(C \cap B_i) = \emptyset \), then \( C \subset \overline{B_i} \) for some \( i \), as \( B_i \)'s are disjoint. Since \( \#C \cap \Lambda \geq 2 \), it follows that \( C \subset \partial B_i \subset \Lambda \). Hence the claim follows. If \( C - \bigcup_{i \in I}(C \cap B_i) \) is non-empty, \( C \cap \Lambda \) contains a non-empty open subset of \( C \); this implies the Zariski density of \( C \cap \Lambda \).

If \( I \) is infinite, then \( C \cap \Lambda \) contains infinitely many \( C \cap \partial B_i \)'s, each of which is an irreducible codimension-1 real subvariety of \( C \). It follows that the Zariski-closure of \( C \cap \Lambda \) has dimension strictly greater than \( \dim(C) - 1 \), hence is equal to \( C \). \( \square \)

In (3.8), if \( p(\Gamma_S) \backslash \tilde{S} \) is a rigid hyperbolic \( k \)-manifold and \( f \) is proper, then the image \( S = \text{Im}(f) \) is referred to as a properly immersed rigid hyperbolic \( k \)-submanifold.

**Proposition 4.7.** Suppose that \( xH'(U) \) is closed for \( x = [g] \in RF M \). Then
(1) the identity component of the Zariski closure of \( g^{-1}\Gamma g \cap H'(U) \) is of the form \( H(U)C \) for some closed subgroup \( C < C(H(U)) \);
(2) \( S = \pi(xH'(U)) \) is a properly immersed rigid hyperbolic submanifold.

**Proof.** Suppose that \( xH' \) is closed for \( x = [g] \in RF M \). Consider the totally geodesic subspace \( \tilde{S} := g\tilde{S}_0 \), and set \( C := \partial(\tilde{S}) \). Set \( S := \pi(xH') \). By Propositions 3.4, \( S \) is properly immersed, convex cocompact and \( C \cap \Lambda \) is equal to the limit set of \( p(\Gamma_S) \). It is well-known that any proper algebraic
subgroup in $G$ stabilizes either a point, or a proper sphere in $S^{d-1}$. Therefore by Lemma 4.6, $p(\Gamma_\tilde{S})$ is Zariski dense in $gHg^{-1}$, proving (1).

Write $C - (C \cap \Lambda) = \bigcup_{i \in I} (C \cap B_i)$ as the union of non-empty intersections $C \cap B_i$. If $I$ is a finite set, $C \cap \Lambda$ contains an open subset of $C$. This implies that $p(\Gamma_\tilde{S})$ is a convex compact subgroup whose limit set has Hausdorff dimension $k - 1$, and hence $p(\Gamma_\tilde{S})$ is a uniform lattice; so $S$ is a compact hyperbolic $k$-submanifold. If $I$ is an infinite set, then $S$ is a rigid hyperbolic $k$-submanifold of infinite volume. This proves (2). □

**Thick return time to** $RF M$. The following lemma shows that $RF M$ has a thick return property under the action of any one-dimensional subgroup $U$ of $N$.

We begin by recalling the various notions of thick subsets of $\mathbb{R}$, following [28] and [30].

**Definition 4.8.** Fix $k > 1$.

- A subset $T \subset \mathbb{R}$ is locally $k$-thick at $t$ if for any $\lambda > 0$,
  $$T \cap (t \pm [\lambda, k\lambda]) \neq \emptyset.$$
- A subset $T \subset \mathbb{R}$ is $k$-thick if $T$ is locally $k$-thick at 0.
- A subset $T \subset \mathbb{R}$ is $k$-thick at $\infty$ if
  $$T \cap (t \pm [\lambda, k\lambda]) \neq \emptyset$$
  for all sufficiently large $\lambda \gg 1$.
- A subset $T \subset \mathbb{R}$ is globally $k$-thick if $T \neq \emptyset$ and $T$ is locally $k$-thick at every $t \in T$.

We will frequently use the fact that if $T_i$ is a sequence of $k$-thick subsets, then $\limsup T_i$ is also $k$-thick, and that if $T$ is $k$-thick, so is $-T$.

**Proposition 4.9.** There exists a constant $k > 1$ depending only on $M$ such that for any one-parameter unipotent subgroup $U = \{u_t : t \in \mathbb{R}\}$ of $N^\pm$, and any $y \in RF M$,

$$T(y) := \{t \in \mathbb{R} : yu_t \in RF M\}$$

is globally $k$-thick.

**Proof.** We first prove the case when $U < N$. Let $s \in T(y)$ be arbitrary. To show that $T(y)$ is locally $k$-thick at $s$, by replacing $y$ with $yu_s \in RF M$, we may assume $s = 0$. Let $\mathbb{H}^d$ be an upper half space model so that $g^+ = g(\infty)$ and $g^- = g(0)$ for $g \in G$. We write

$$\Omega = S^{d-1} - \Lambda = \bigcup_{i=1}^\infty B_i$$

where $B_i$'s are connected components of $\Omega$. Note that

$$\eta := \inf \{d(\text{hull}(B_i), \text{hull}(B_j)) : i \neq j\}$$
is strictly positive because $2\eta$ is bounded below by the shortest length of a closed geodesic in the hyperbolic double of core $M$, which is a closed hyperbolic manifold. We may assume that $y = [g]$ where $g^+ = \infty$ and $g^- = 0$. Note that

$$T(y) = \{ t \in \mathbb{R} : gu_t(0) \in \Lambda \}.$$  

Choose $k > 1$ so that

$$d(\text{hull}([-k, -1]), \text{hull}([1, k])) = 2\eta$$

where $d$ is the hyperbolic distance in the upper half plane $\mathbb{H}^2$ with $\partial\mathbb{H}^2 = \mathbb{R} \cup \{ \infty \}$. For any $w \in \mathbb{R}^{d-1}$, denoting by $d_w$ the hyperbolic distance of the plane above the line $\mathbb{R}w$, we have that

$$d_w(\text{hull}([-kt, -t] \cdot w), \text{hull}([t, kt] \cdot w))$$

is independent of $w \in \mathbb{R}^{d-1}$ and $t > 0$, because both the dilation centered at 0 and the $(d - 2)$-dimensional rotation with respect to the vertical axis above 0 are hyperbolic isometries.

Suppose that $T(y)$ is not locally $k$-thick at 0. Then there exist $w \in U$ and $t > 0$ such that

$$([-kt, -t] \cdot w \cup [t, kt] \cdot w) \cap \Lambda = \emptyset.$$  

Since each component of $\Omega$ is convex and $0 \notin \Omega$, it follows that $[-kt, -t] \cdot w$ and $[t, kt] \cdot w$ lie in distinct components of $\Omega$, say $B_i$ and $B_j$, $(i \neq j)$. But this yields

$$\eta/2 = d_w(\text{hull}([-kt, -t] \cdot w), \text{hull}([t, kt] \cdot w))$$

$$\geq d(\text{hull}(B_i), \text{hull}(B_j)) \geq \eta$$

which is a contradiction. The case of $U < N^-$ is proved similarly, just replacing the role of $g^+$ and $g^-$ in the above arguments. $\square$

5. Structure of singular sets

Let $G = \text{SO}^0(d, 1)$, and let $\Gamma < G$ be a convex cocompact torsion-free Zariski-dense subgroup. Let $U < G$ be a connected closed subgroup of $G$ generated by unipotent elements in it.

In this section, we define the singular set $\mathcal{S}(U)$ associated to $U$ and study the structural property of $\mathcal{S}(U)$. The singular set $\mathcal{S}(U)$ is defined so that it contains all closed orbits of intermediate closed subgroups between $U$ and $G$.

**Definition 5.1** (Singular set). We define the singular set $\mathcal{S}(U)$ for $U$ as follows:

$$\mathcal{S}(U) = \left\{ x \in \Gamma \setminus G : \begin{array}{l}
\text{there exists a proper connected} \\
\text{closed subgroup } W \supseteq U \text{ such that } xW \\
\text{is closed and } \text{Stab}_W(x) \text{ is Zariski dense in } W.
\end{array} \right\}.$$
Definition 5.2 (Definition of \( \mathcal{H} \)). We denote by \( \mathcal{H} \) the collection of all proper closed connected subgroups \( H < G \) containing a unipotent element such that

- \( \Gamma \backslash \Gamma H \) is closed, and
- \( H \cap \Gamma \) is Zariski dense in \( H \).

Lemma 5.3. If \( H \in \mathcal{H} \), then \( H \) is a reductive subgroup of \( G \), and hence is of the form \( gH(U)Cg^{-1} \) for some connected subgroup \( U < N \), a closed subgroup \( C < C(H(U)) \) and \( g \in G \) such that \( [g] \in RF \ M \).

Proof. If \( R \) is the unipotent radical of \( H \), then since \( H \cap \Gamma \) is Zariski dense in \( H \), then \( R \cap \Gamma \) is also Zariski dense in \( R \) [36]. Since \( \Gamma \) has no unipotent elements, \( R = \{ e \} \). So \( H \) is reductive. By Proposition 3.2, \( H \) is of the form \( gH(U)Cg^{-1} \) as describe above; \( [g]H(U) \cap RF \ M \neq \emptyset \), and hence we can take \( g \in RF \ M \) by modifying it with an element of \( H(U) \) if necessary. \( \square \)

Therefore, the non-compact semisimple part \( H_{nc} \) of \( H \) is uniquely defined for each \( H \in \mathcal{H} \) by (3.4), and we define:

Definition 5.4 (Definition of \( \mathcal{H}^* \)).

\[
\mathcal{H}^* := \{ N(H_{nc}) : H \in \mathcal{H} \}.
\]

Proposition 5.5. If \( H \in \mathcal{H} \), then

- \( H \cap \Gamma \) is finitely generated;
- \( [N(H_{nc}) \cap \Gamma ; H \cap \Gamma] < \infty \).

Proof. Let \( p \) denote the projection map \( N(H_{nc}) \to H_{nc} \). Note that \( p \) is an injective map on \( N(H_{nc}) \cap \Gamma \), as \( \Gamma \) is torsion free and the kernel of \( p \) is a compact subgroup.

It follows from Lemma 5.3 that \( N(H_{nc}) \) is a direct product \( H_{nc} C(H_{nc}) \). Since \( H \in \mathcal{H} \), \( [e]H \) is closed, and hence \( [e]N(H_{nc}) \) is closed, as \( H \) is co-compact in \( N(H_{nc}) \). It follows that both \( p(H \cap \Gamma) \) and \( p(N(H_{nc}) \cap \Gamma) \) are convex cocompact Zariski-dense subgroups of \( H_{nc} \) by Proposition 3.4. As any convex cocompact subgroup is finitely generated [3], \( p(H \cap \Gamma) \) is finitely generated. Hence \( H \cap \Gamma \) is finitely generated by the injectivity of \( p|_{H \cap \Gamma} \).

Since \( p(H \cap \Gamma) \) is a normal subgroup of \( p(N(H_{nc}) \cap \Gamma) \), it follows that \( p(H \cap \Gamma) \) has finite index in \( p(N(H_{nc}) \cap \Gamma) \) by Lemma 5.6 below. Since \( p|_{N(H_{nc}) \cap \Gamma} \) is injective, it follows that \( H \cap \Gamma \) has finite index in \( N(H_{nc}) \cap \Gamma \). \( \square \)

Lemma 5.6. Let \( \Gamma_1, \Gamma_2 \) be non-elementary convex cocompact subgroups of \( G \). If \( \Gamma_2 \) is a normal subgroup of \( \Gamma_1 \), then \( [\Gamma_1 : \Gamma_2] < \infty \).

Proof. Let \( \Lambda_i \) be the limit set of \( \Gamma_i \) for \( i = 1, 2 \). We have \( \Lambda_1 \subset \Lambda_2 \). As \( \Gamma_2 \) is normalized by \( \Gamma_1 \), \( \Lambda_2 \) is \( \Gamma_1 \)-invariant. Since \( \Gamma_1 \) is non-elementary, \( \Lambda_1 \) is a minimal \( \Gamma_1 \)-invariant closed subset. Hence \( \Lambda_1 = \Lambda_2 \). Now for \( M_i := \Gamma_i \bmod \), core \( M_1 = \Gamma_1 \backslash \text{hull}(\Lambda_2) \) is covered by core \( M_2 = \Gamma_2 \backslash \text{hull}(\Lambda_2) \). Since core \( M_2 \) is compact, it follows that \( [\Gamma_1 : \Gamma_2] < \infty \). \( \square \)
Corollary 5.7 (Countability). The collection $\mathcal{H}$ is countable, and the map $H \to N(H_{nc})$ defines a bijection between $\mathcal{H}$ and $\mathcal{H}^\ast$.

Proof. Since there are only countably many finitely generated subgroups of $\Gamma$, the first claim follows. Since $H \cap \Gamma$ has finite index in $N(H_{nc}) \cap \Gamma$ by Proposition 5.5, $H$ is determined as the identity component of the Zariski closure of $N(H_{nc}) \cap \Gamma$. This proves the second claim. $\square$

In the rigid case, there is a one to one correspondence between $\mathcal{H}$ and the collection of all closed $H'(\tilde{U})$-orbits of points in $RF M$ for $\tilde{U} < N$: if $H \in \mathcal{H}$, then $H = gH(\tilde{U})Cg^{-1}$ for some $U < \tilde{U} < N$ and $g \in G$ with $[g] \in RF M$ and $[g]H'(\tilde{U})$ is closed. Conversely, if $[g]H'(\tilde{U})$ is closed for some $[g] \in RF M$, then the identity component of the Zariski closure of $\Gamma \cap gH'(\tilde{U})g^{-1}$ is given by $gH(\tilde{U})Cg^{-1}$ for some closed subgroup $C < C(H(\tilde{U}))$ by Proposition 4.7, and hence $gH(\tilde{U})Cg^{-1} \in \mathcal{H}$. Therefore Corollary 5.7 implies the following corollary by Propositions 3.4 and 4.7.

Corollary 5.8. When $M$ is rigid and $k \geq 2$, there are only countably many properly immersed geodesic $k$-planes intersecting core $M$.

For a subgroup $H < G$, define

$$X(H,U) = \{g \in G : gUg^{-1} \subset H\}. \tag{5.2}$$

Note that $X(H,U)$ is left-$N_G(H)$ and right-$N_G(U)$-invariant, and for any $g \in G$,

$$X(gHg^{-1},U) = gX(H,U). \tag{5.3}$$

For $H \in \mathcal{H}$ and any unipotent subgroup $U < G$, observe that

$$X(H,U) = X(H_{nc},U) = X(N(H_{nc}),U); \tag{5.4}$$

this follows since any unipotent element of $N(H_{nc})$ is contained in $H_{nc}$.

Proposition 5.9. We have

$$\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}^\ast} \Gamma \backslash \Gamma X(H,U).$$

Proof. If $x = [g] \in \mathcal{S}(U)$, then there exists a proper connected closed subgroup $W$ of $G$ containing $U$ such that $[g]W$ is closed and $\text{Stab}_W(x)$ is Zariski dense in $W$. This means $H := gWg^{-1} \in \mathcal{H}$ and $g \in X(H,U)$. Since $X(H,U) = X(N(H_{nc}),U)$, and $N(H_{nc}) \in \mathcal{H}^\ast$, this proves the inclusion $\subset$.

Conversely, let $g \in X(N(H_{nc}),U)$ for some $H \in \mathcal{H}$. Set $W := g^{-1}Hg$. Then $U \subset W$. $[g]W = \Gamma Hg$ is closed and $\text{Stab}_W([g]) = g^{-1}(\Gamma \cap H)g$ is Zariski dense in $W$. Hence $[g] \in \mathcal{S}(U)$. $\square$
Singular subset of a closed orbit. Let \( L < G \) be a connected reductive subgroup of \( G \) containing unipotent elements. For a closed orbit \( x_0 L \) of \( x_0 \in RFM \), and a connected subgroup \( U_0 < L \cap N \), we define the singular set \( \mathcal{S}(U_0, x_0 L) \) by

\[
\mathcal{S}(U_0, x_0 L) = \left\{ x \in x_0 L : \begin{array}{l}
\text{there exists a connected closed subgroup } W < L, \\
\text{containing } U \text{ such that } \dim(W_{nc}) < \dim(L_{nc}), \\
xW \text{ is closed and } \text{Stab}_W(x) \text{ is Zariski dense in } W
\end{array}\right\}.
\]

It follows from the proof of Proposition 5.9 and Lemma 5.3 that the subgroup \( W \) in the definition 5.1 is conjugate to \( H(U) \). Hence \( W \) being a proper subgroup of \( G \) is same as requiring \( \dim(W_{nc}) < \dim(G) \). Therefore \( \mathcal{S}(U_0) = \mathcal{S}(U_0, \Gamma \setminus G) \), and we have

\[
\mathcal{S}(U_0, x_0 L) = \bigcup_{H \in \mathcal{H}_{x_0 L}} \Gamma \setminus \Gamma X(H, U_0)
\]

where \( \mathcal{H}_{x_0 L} \) consists of all subgroups \( H \in \mathcal{H}^* \) such that \( g_0^{-1} H g_0 \) is a subgroup of \( L \) with \( \dim(H_{nc}) < \dim(L_{nc}) \) and \( x_0 = [g_0] \).

Then the generic set \( \mathcal{G}(U_0, x_0 L) \) is defined by

\[
\mathcal{G}(U_0, x_0 L) := x_0 L - \mathcal{S}(U_0, x_0 L).
\]

Definition of \( \mathcal{L}_U \) and \( Q_U \). For a connected subgroup \( U < N \), we define the collection \( \mathcal{L}_U \) of all subgroups of the form \( H(U)C \) where \( U < \hat{U} < N \) and \( C \) is a closed subgroup of \( C(H(\hat{U})) \) satisfying the following:

\[
\mathcal{L}_U := \left\{ L = H(\hat{U})C : \begin{array}{l}
\text{for some } [g] \in RFM, [g]L \text{ is closed in } \Gamma \setminus G \\
\text{and } L \cap g^{-1} \Gamma g \text{ is Zariski dense in } L
\end{array}\right\}.
\]

Observe that for \( L = H(\hat{U})C \neq G \), the condition \( L \in \mathcal{L}_U \) with \([g]L \) closed is equivalent to the condition that

\[
gLg^{-1} \in \mathcal{H}.
\]

Lemma 5.10. Let \( L_1, L_2 \in \mathcal{L}_U \) with \( xL_1 \) and \( xL_2 \) closed. If \( (L_1)_{nc} = (L_2)_{nc} \), then \( L_1 = L_2 \).

Proof. If \( L_1 \) or \( L_2 \) is equal to \( G \), then the claim is trivial. Suppose that both \( L_1 \) and \( L_2 \) are proper subgroups of \( G \). If \( x = [g] \), then both subgroups \( H_1 := gL_1g^{-1} \) and \( H_2 := gL_2g^{-1} \) belong to \( \mathcal{H} \). Since \( (H_1)_{nc} = (H_2)_{nc} \), we have \( H_1 = H_2 \) by Corollary 5.7. Hence \( L_1 = L_2 \). \( \square \)

We also define

\[
Q_U := \{ vL^{-1} : L \in \mathcal{L}_U, v \in N(U) \}.
\]

Lemma 5.11. We have

\[
Q_U = \{ vL^{-1} : L \in \mathcal{L}_U, v \in U^- \}.
\]
Proof. Since \( N(U) = ANC_1(U)C_2(U) \), and the collection \( \mathcal{L}_U \) is invariant under a conjugation by an element of \( AU C_1(U)C_2(U) \), the claim follows. \( \square \)

Proposition 5.12. Consider a closed orbit \( x_0L \) for \( L \in \mathcal{Q}_U \) and \( x_0 \in RF M \). Let \( U_0 \) be a connected subgroup of \( U \). If \( x \in \mathcal{S}(U_0, x_0L) \), then there exists a subgroup \( Q \in \mathcal{Q}_{U_0} \) such that

\[
\begin{align*}
&\bullet \dim Q_{nc} < \dim L_{nc}; \\
&\bullet xQ \text{ is closed}; \\
&\bullet \overline{xU_0} \subset xQ.
\end{align*}
\]

Proof. Write \( x_0 = [g_0] \). If \( x = [g] \in \mathcal{S}(U_0, x_0L) \), then \( g \in X(H, U_0) \) for some \( H \in \mathcal{H} \) such that \( H < g_0Lg_0^{-1} \) and \( \dim(H_{nc}) < \dim(L_{nc}) \). By Lemma 5.3, \( H = qH(\hat{U})Cq^{-1} \) for some \( U_0 < \hat{U} < L \cap N \) and some \( [q] \in RF M \). Hence \( [q]H(\hat{U})C \) is closed. Since \( g \in X(H, U_0) \), we have \( gU_0g^{-1} \subset qH(\hat{U})q^{-1} \), that is, \( q^{-1}g \in X(H(\hat{U}), U_0) \).

By Lemma 5.13 below, we have

\[
q^{-1}g \in N_G(H(\hat{U}))N_G(U_0).
\]

Write \( q^{-1}g = sv \) for \( s \in N_G(H(\hat{U})) \) and \( v \in N_G(U_0) \). Then

\[
xU_0 \subset [q]H(\hat{U})Cq^{-1}g = [q]sH(\hat{U})Cv.
\]

Hence setting \( Q := v^{-1}H(\hat{U})Cv \), we have \( Q \in \mathcal{Q}_{U_0} \) and

\[
\overline{xU_0} \subset xQ
\]

Note that \( xQ \) is closed, as it is equal to \( [q]sH(\hat{U})Cv \), and that \( \dim H_{nc} = \dim Q_{nc} \) is strictly smaller than \( \dim L_{nc} \). \( \square \)

We now prove the following lemma used in the proof of Proposition 5.12.

Lemma 5.13. For \( 0 < U < N \), we have

\[
X(H(U), U_0) = N_G(H(U))N_G(U_0).
\]

Proof. Without loss of generality, we may assume \( U = U_m \) and \( U_0 = U_\ell \) with \( \ell \leq m \leq d - 1 \). Set \( H = H(U_m) \). If \( m = d - 1 \), then \( H = G \), and the statement is trivial. Assume \( m \leq d - 2 \) below. We will prove the inclusion \( X(H, U_\ell) \subset N_G(H)N_G(U_0) \), as the other one is clear. Let \( g \in X(H, U_\ell) \) be arbitrary. By multiplying \( g \) by an element of \( N_G(H) \) on the left as well as by an element of \( N_G(U_m) \) on the right, we will reduce \( g \) to an element of \( N_G(U_m) \), which implies the claim.

Using the upper-half space model, fix \( o = (0, \cdots, 0, 1) \in \mathbb{H}^d \), so that the maximal compact subgroup \( K \) is the stabilizer of \( o \), and let \( e_0, \cdots, e_{d-1} \) be the standard basis for \( T_o \mathbb{H}^d \cong \mathbb{R}^d \). The map

\[
g \mapsto (g.e_0, \cdots, g.e_{d-1})
\]
gives the identification $G \simeq F\mathbb{H}^d$. Under the identification $\partial\mathbb{H}^d = \mathbb{R}^{d-1} \cup \{\infty\}$, the group $H' := HC(H)$ consists of all frames tangent to a geodesic plane $\mathbb{H}^{m+1} = \mathbb{R}^m \times \{0\}^{d-1-m} \times \mathbb{R}^+ \subset \mathbb{H}^d$.

In view of the Iwasawa decomposition $G = KAN$, since $AN < N_G(U_0)$, we may assume that $g = k \in K$. As $k \in X(H, U_\ell)$, we have

\[
\begin{align*}
\text{(5.9)} \quad kU_\ell & \subset Hk.
\end{align*}
\]

Let $\pi : G \rightarrow \mathbb{H}^d$ denote the basepoint projection. It follows from (5.9) that $k(\infty) = kU_\ell(\infty) \subset \partial\mathbb{H}^{m+1}$, since $\pi(H'k) = \mathbb{H}^{m+1}$, and that $\langle k.e_1, \ldots, k.e_\ell \rangle \subset \langle e_0, e_1, \ldots, e_m \rangle$ as subspaces of $T_o\mathbb{H}^d$. Since the compact subgroup $K_m := K \cap H$ acts transitively on $\partial\mathbb{H}^{m+1}$, by multiplying an element of $K_m$ to $k$ on the left, we may assume that $k(\infty) = \infty$. Now that $k$ fixes the vertical geodesic joining $o \in \mathbb{H}^d$ and $\infty$, it follows $k \in M$ and hence

\[
\langle k.e_1, \ldots, k.e_\ell \rangle \subset \langle e_1, \ldots, e_m \rangle.
\]

Now that $K_m \cap M$ acts transitively on space of $\ell$-tuples of orthonormal vectors in the subspace $\langle e_1, \ldots, e_m \rangle$, we may assume $k.e_1 = e_1, \ldots, k.e_\ell = e_\ell$ after multiplying an element of $K_m \cap M$ to $k$ on the left. This implies that $k \in C_1(U_0)$ and hence the proof is complete, as $C_1(U_0) \subset N_G(U_0)$.

6. Inductive search lemma

In this section, we prove a combinatorial lemma 6.4, which we call an inductive search lemma, and use it to prove Proposition 6.3 the thickness of a certain subset of $\mathbb{R}$, constructed by the intersection of a global thick subset $T$ and finite families of triples of subsets of $\mathbb{R}$ with controlled regularity, degree and the multiplicity with respect to $T$. This proposition is a key ingredient in the proof of the avoidance theorem 7.11 in section 7.

**Definition 6.1.** Let $J^* \subset I$ be open subsets of $\mathbb{R}$.

- For $\beta > 0$, the pair $(I, J^*)$ is said to be $\beta$-regular if for every component $J^0$ of $J^*$ and the component $I^0$ of $I$ containing $J^0$, we have

\[
J^0 + \pm \beta|J^0| \subset I^0.
\]

- The degree of $(I, J^*)$ is the minimal integer $\delta \in \mathbb{N}$ such that for each connected component $I^0$ of $I$, the number of connected components of $J^*$ contained in $I^0$ is bounded by $\delta$.

**Definition 6.2.** Let $\mathcal{X}$ be a family of countably many triples $(I, J^*, J')$ of subsets of $\mathbb{R}$ with $I \supset J^* \supset J'$.

- Given $\beta > 0$ and $\delta \in \mathbb{N}$, we say that $\mathcal{X}$ is $\beta$-regular of degree $\delta$ if for every triple $(I, J^*, J') \in \mathcal{X}$, the pair $(I, J^*)$ is $\beta$-regular with degree at most $\delta$.

- Given a subset $T \subset \mathbb{R}$, we say that $\mathcal{X}$ is of $T$-multiplicity free if for any two distinct triples $(I_1, J^*_1, J'_1)$ and $(I_2, J^*_2, J'_2)$ of $\mathcal{X}$, we have

\[
I_1 \cap J^*_2 \cap T = \emptyset.
\]
For a family $\mathcal{X} = \{(I_\lambda, J^*_{\lambda}, J'_\lambda) : \lambda \in \Lambda\}$, we will use the notation

$$I(\mathcal{X}) := \bigcup_{\lambda \in \Lambda} I_\lambda, \quad J^*(\mathcal{X}) := \bigcup_{\lambda \in \Lambda} J^*_{\lambda} \quad \text{and} \quad J'(\mathcal{X}) := \bigcup_{\lambda \in \Lambda} J'_{\lambda}.$$

If $\mathcal{X}$ is a union $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_n$, we write

$$I(\mathcal{X}) = \bigcup_{i=1}^n I(\mathcal{X}_i), \quad J^*(\mathcal{X}) = \bigcup_{i=1}^n J^*(\mathcal{X}_i), \quad \text{and} \quad J'(\mathcal{X}) = \bigcup_{i=1}^n J'(\mathcal{X}_i).$$

The goal of this section is to prove:

**Proposition 6.3 (Thickness of $T - J'(\mathcal{X})$).** Given $k, m, \delta \in \mathbb{N}$, there exists $\beta_0 = \beta_0(m, k, \delta)$ for which the following holds: if $T \subset \mathbb{R}$ is a globally $k$-thick set, $\mathcal{X}$ is the union $\bigcup_{i=1}^\ell X_i$, where $\ell \leq m$ and $X_1, \ldots, X_\ell$ are $\beta_0$-regular families of triples of subsets of $\mathbb{R}$ with degree $\delta$ and of $T$-multiplicity free, and $0 \in T - I(\mathcal{X})$, then

$$T - J'(\mathcal{X})$$

is a $2k$-thick set.

We prove Proposition 6.3 using the inductive search lemma 6.4. The case of $m = 1$ and $\delta = 1$ is easy. As the statement of the lemma is rather complicated in a general case, we first explain a simpler case where $m = \ell = 2$ and $\delta = 1$ in order to motivate the statement.

For simplicity, let us show $T - (J'(X_1) \cup J'(X_2))$ is $4k$-thick instead of $2k$-thick, given that $X_1$ and $X_2$ are $8k^2$-regular families of degree 1, and of $T$-multiplicity free. Assuming $x = 0$, for any $r > 0$, we need to find

$$t \in \pm(r, 4kr) \cap \left(T - (J'(X_1) \cup J'(X_2))\right).$$

First, we know that there exists $t_1 \in \pm(2r, 2kr) \cap T$ as $T$ is locally $k$-thick at 0. If $t_1 \notin J'(X_1) \cup J'(X_2)$, then we are done. So we assume that $t_1 \in J'(X_1)$. Our strategy is then to search for a sequence in $T$ of length at most 4 starting with $t_1$, say $(t_1, t_2, t_3, t_4)$ such that

$$\frac{|t_{i-1}|}{\sqrt{2}} \leq |t_i| \leq \sqrt{2}|t_{i-1}| \quad \text{for each } i = 2, 3, 4,$$

and the last element $t_4$ does not belong to the union $J'(X_1) \cup J'(X_2)$. This will imply $|t_1|/2 \leq |t_4| \leq 2|t_1|$ and hence

$$t := t_4 \in \pm(r, 4kr) \cap \left(T - J'(\mathcal{X})\right)$$

as desired, because $2r \leq |t_1| \leq 2kr$.

We next sketch how we find $t_2$ from $t_1$ and so on. Let $t_1 \in J'_1$ where $(I_1, J^*_{1}, J'_1) \in X_1$. Since $T$ is locally $k$-thick at $t_1$, there exists

(6.1) $t_2 \in t_1 \pm (|J^*_{1}|, k|J'_1|) \cap T$.

We will refer to $t_1$ as a pivot for searching $t_2$ in (6.1), as $t_2$ was found in a symmetric interval around $t_1$. Note $t_2 \in I_1$ as $(I_1, J^*_{1})$ is $k$-regular. This implies that $t_2 \notin J'(X_1)$ as the family $X_1$ is of $T$-multiplicity free.
Now we will assume \( t_2 \in J_2^* \) for some triple \((I_2, J_2^*, J_2') \in \mathcal{X}_2\), since otherwise, \( t_2 \notin J'(X_2) \) and we are done (cf. Figure 2).

Theorem 3.1: \( J^* \subset I \) drawn on one of the two copies of \( \mathbb{R} \), according to where \((I, J^*, J')\) belongs.

To search for the next point \( t_3 \in T \), we choose between two candidates \( t_1 \) and \( t_2 \) for our pivot - we will choose \( t_1 \) if \(|J_1^*| \geq |J_2^*|\), and \( t_2 \) otherwise. Without loss of generality, we will assume \(|J_1^*| \geq |J_2^*|\). Then we can find
\[
 t_3 \in t_1 \pm 2k(|J_1^*|, k|J_1^*|) \cap T,
\]
because \( T \) is locally \( k \)-thick at \( t_3 \). Note \( t_3 \in I_1 \) as the pair \((I_1, J_1^*)\) is \( 2k^2 \)-regular. This implies \( t_3 \notin J'(X_1) \) as \( X_1 \) is of \( T \)-multiplicity free. Now we can assume \( t_3 \in J_3' \) for some \((I_3, J_3^*, J_3') \in \mathcal{X}_2\), otherwise we are done. One can check that \( J_3^* \) cannot coincide with \( J_2^* \). We claim that
\[
|J_1^*| \geq |J_3^*|
\]
(cf. Figure 3).

Assume to the contrary that \(|J_3^*| > |J_1^*|\). Then we would have \(|t_2 - t_1| < k|J_3^*| \) and \(|t_1 - t_3| < 2k^2|J_3^*|\), which implies that \( t_2 \in I_3 \), as the pair \((I_3, J_3^*)\) is \((2k^2 + k)\)-regular. This is a contradiction as \( J_2' \cap I_3 \cap T = \emptyset \) from the beginning.

Finally, we will choose \( t_3 \) as a pivot and search for \( t_4 \). By the local \( k \)-thickness of \( T \) at \( t_3 \in T \), we can find
\[
 t_4 \in t_3 \pm (k|J_3^*|) \cap T.
\]
Since the pair \((I_3, J_3^*)\) is \( k \)-regular, we have \( t_4 \in I_3 \). From the fact that the pair \((I_1, J_1^*)\) is \((2k^2 + k)\)-regular, one can check that \( t_4 \notin J'(X_1) \) and hence \( t_4 \notin J'(X_1) \cup J'(X_2) \) (cf. Figure 4).
Figure 3. \(|J_1^*| \geq |J_3^*|\), otherwise \(t_2\) would also be in \(I_3\).

It remains to check that \(|t_{i-1}|/\sqrt{2} \leq |t_i| \leq \sqrt{2}|t_{i-1}|\) for each \(i = 2, 3, 4\). This does not necessarily hold for the current sequence, but will hold after passing to a subsequence where \(t_{i-1}\) becomes a pivot for searching \(t_i\) for all \(i\). In the previous case, \((t_1, t_3, t_4)\) will be such a subsequence, as \(t_2\) was not a pivot for searching \(t_3\).
Now note that \(|t_{i-1}| - \beta|J^*_i| > 0\) from the \(\beta\)-regularity of \((I_{i-1}, J^*_{i-1})\) as \(t_{i-1} \in J^*_i\) and \(0 \notin I_{i-1}\). On the other hand, observe that
\[
t_i \in t_{i-1} \pm C_i(|J^*_{i-1}|, k|J^*_{i-1}|) \cap T
\]
for some \(C_i \leq 2 \beta^2\). This gives us the desired upper bound for \(|t_i/t_{i-1}|\), as
\[
|t_i| < |t_{i-1}| + C|J^*_{i-1}| \leq (1 + \beta |t_{i-1}|
\]
and \(1 + C_i/\beta \leq \sqrt{2}\). The lower bounds is obtained similarly, completing the proof for \(m = 2\) and \(\delta = 1\).

The general case reduces to the case of \(\delta = 1\), by replacing \(m\) by \(m\delta\), and the previous argument suggests that we look for a sequence that is almost geometric in a sense that the ratio \(|t_i|/|t_{i-1}|\) is coarsely a constant, which finally lands on \(T - J'(\mathcal{X})\) in a time controlled by \(n\).

The following lemma gives an inductive argument for the search of \(t' \in T - J'(\mathcal{X})\) of size comparable to \(|t|\) for any given \(t \in T \cap J'(\mathcal{X})\). That is to say, for given \(n \in \mathbb{N}\), if we choose \(\beta = \beta(n)\) large enough, then for any \(\beta\)-regular family \(\mathcal{X}_1, \ldots, \mathcal{X}_n\) starting from an arbitrary \(t_1 \in T\), we can always find such a sequence in \(T\) of length at most \(2^n\).

**Lemma 6.4** (Inductive search lemma). Let \(k > 1\), \(n \in \mathbb{N}\) and \(\varepsilon > 0\) be fixed. There exists \(\beta = \beta(n, k, \varepsilon)\) for which the following holds: Let \(T \subset \mathbb{R}\) is a globally \(k\)-thick set, and \(\mathcal{X}_1, \ldots, \mathcal{X}_n\) be \(\beta\)-regular families of countably many triples \((I_\lambda, J^*_\lambda, J'_\lambda)\) with degree 1, and of \(T\)-multiplicity free. Assume \(0 \notin I(\mathcal{X})\). For any \(t \in T \cap J(\mathcal{X})\) and any \(1 \leq r \leq n\), we can find a sequence
\[
t_1(= t), t_2, \ldots, t_m \in T
\]
with \(2 \leq m \leq 2^r\) and \(t_1, \ldots, t_{m-1}\) belonging to distinct members \(J^*_1, \ldots, J^*_{m-1}\) of \(J^*(\mathcal{X})\), which satisfies the following conditions.

1. Either \(t_m \notin J'(\mathcal{X})\), or \(t_m \in J^*_m\) for some \(J^*_m \in J^*(\mathcal{X})\) distinct from \(J^*_1, \ldots, J^*_{m-1}\), and \(t_1, \ldots, t_m\) intersect at least \(r + 1\) number of \(J^*(\mathcal{X})\)’s.
2. For any \(2 \leq j \leq m\), there exists an increasing sequence \(1 = j_0 < j_1 < \cdots < j_a = j\) such that for each \(1 \leq b \leq a\),
\[
|t_{j_b} - t_{j_b-1}| < ((4k)^{r+1} - 1)k|J^*_{j_b-1}|.
\]
3. For each \(1 \leq i \leq m\),
\[
(1 - \varepsilon)^{m-1}|t_1| \leq |t_i| \leq (1 + \varepsilon)^{m-1}|t_1|.
\]

In particular, for any \(t \in T \cap J'(\mathcal{X})\), there exists \(t' \in T - J'(\mathcal{X})\) such that
\[
(1 - \varepsilon)^{2^n-1}|t| \leq |t'| \leq (1 + \varepsilon)^{2^n-1}|t|.
\]

**Proof.** We set \(\beta = \beta(n, k, \varepsilon) = 2^{3n+5}k^{n+3}\varepsilon^{-1}\). Write \(J^*(\mathcal{X}) = \bigcup_{i \in \Lambda} J^*_i\). Consider the increasing sequence \(Q(r) = (4k)^{r+1} - 1\) for \(r \in \mathbb{N}\). Note that
\[
Q(1) \geq 2 \text{ and } Q(r + 1) \geq 4Q(r)k + 1.
\]
Moreover we can check that
\[ \beta > \max((Q(n) + 4Q(n-1))k + 1, (Q(n+1) + (2^n + 2)Q(n))k, Q(n)e^{-1}) \]

We proceed by induction on \( r \).

First consider the case when \( r = 1 \). Let \( t_1 = t \) and \( t_1 \in J_1^{*} \) for some \((I_1, J_1^{*}, J_1') \in \mathcal{X} \). As \( T \) is globally \( k \)-thick, we can choose
\[ t_2 \in (t_1 \pm Q(1)(|J_1'|, k|J_1'|)) \cap T. \tag{6.2} \]
In case when \( t_2 \in J'(\mathcal{X}) \), we will take \( J_2^{*} \in J^{*}(\mathcal{X}) \) to be any interval containing \( t_2 \). We claim that \( t_1, t_2 \) is our desired sequence with \( m = 2 \).

(1): If \( t_2 \in J'(\mathcal{X}) \), then \( t_2 \in J_2^{*} - J_1^{*} \) implies that \( J_1^{*} \) and \( J_2^{*} \) are distinct. Since \( \beta > Q(1)k \), by \( \beta \)-regularity of \((I_1, J_1^{*})\), we have \( t_2 \in I_1 \). Since \( t_2 \in J_2^{*} \), it follows \( t_1 \) and \( t_2 \) intersect two number of \( J^{*}(\mathcal{X}_1) '\)’s.

(2): \( |t_1 - t_2| < Q(1)k|J_1'| = ((4k)^2 - 1)k|J_1'| \).

(3): Note that \( 0 \not\in I_1 \), since \( 0 \not\in I(\mathcal{X}) \). By the \( \beta \)-regularity of \((I_1, J_1^{*})\), we have \( t_1 - \beta |J_1'| < I_1 \). Since \( 0 \not\in I_1 \) and \( \beta > e^{-1}Q(1)k \), we have \( |t_1| - e^{-1}Q(1)k|J_1'| > 0 \).

On the other hand, by (6.2),
\[ |t_2 - t_1| \leq Q(1)k|J_1'\| \leq \varepsilon|t_1|. \]

In particular,
\[ |t_2| < |t_1| + |t_2 - t_1| < |t_1| + Q(1)k|J_1'| < (1 + \varepsilon)|t_1| \text{ and } \]
\[ |t_2| > |t_1| - |t_2 - t_1| > |t_1| - Q(1)k|J_1'| > (1 - \varepsilon)|t_1|. \]

This proves the base case.

Next, assume the induction hypothesis for \( r \). Hence we have a sequence
\[ t_1(= t) \in J_1^{*}, t_2 \in J_2^{*}, \cdots, t_{m-1} \in J_{m-1}^{*}, \text{ and } t_m \]
in \( T \) with \( m \leq 2^r \) satisfying the three conditions listed in the lemma.

If \( t_m \not\in J'(\mathcal{X}) \), the same sequence would satisfy the hypothesis for \( r+1 \) and we are done. Now we assume that \( t_m \in J'(\mathcal{X}) \), and that \( t_1, \cdots, t_m \) intersect at least \((r+1)\) numbers of \( J^{*}(\mathcal{X}_i) '\)’s. We may assume that they intersect exactly \((r+1)\)-number of \( J^{*}(\mathcal{X}_i) '\)’s, which we may label as \( J^{*}(\mathcal{X}_1), \cdots, J^{*}(\mathcal{X}_{r+1}) \), since if they intersect more than \((r+1)\) of them, we are already done.

Let \( J_r^{*} \) be the largest interval among \( J_1^{*}, \cdots, J_m^{*} \). Again using the global \( k \)-thickness of \( T \), we choose
\[ s_1 \in (t_{\ell} \pm Q(r+1)(|J_{\ell}^{*}|, k|J_{\ell}^{*}|)) \cap T. \tag{6.3} \]

First, consider the case when \( s_1 \not\in J'(\mathcal{X}) \). We will show that the points \( t_1, \cdots, t_m, s_1 \) satisfy the conditions in the lemma. Indeed, the conditions (1) and (2) are immediate. To show (3), since \( \beta > e^{-1}Q(r+1)k \) and \( 0 \not\in I_\ell \)(as \( 0 \not\in I(\mathcal{X}) \)), by applying the \( \beta \)-regularity to the pair \((I_\ell, J_{\ell}^{*})\), we have
\[ |t_\ell| - e^{-1}Q(r+1)k|J_{\ell}^{*}| > 0. \]
It follows that
\[|s_1| \leq |t_\ell| + |s_1 - t_\ell| < |t_\ell| + Q(r + 1)k|J^*_\ell| < (1 + \varepsilon)|t_\ell| \leq (1 + \varepsilon)^m|t_1|;\]
\[|s_1| \geq |t_\ell| - |s_1 - t_\ell| > |t_\ell| + Q(r + 1)k|J^*_\ell| > (1 - \varepsilon)|t_\ell| \geq (1 - \varepsilon)^m|t_1|.
\]
This proves (3).

For the rest of the proof, we now assume that \(s_1 \in J'(X)\). Apply the induction hypothesis for \(r \to 0 \notin I(X)\) and \(s_1 \in T \cap J'(X)\) to obtain
\[s_1 \in \tilde{J}_1^*, \ s_2 \in \tilde{J}_2^*, \ldots, \ s_{m' - 1} \in \tilde{J}_{m' - 1}^*, \ \text{and} \ s_{m'}
\]
with \(m' \leq 2^r\) satisfying the corresponding condition. We claim that the sequence
\[(6.4) \quad t_1, \ldots, t_m, s_1, \ldots, s_{m'}
\]
of length \(m + m' \leq 2^{r+1}\) satisfies the conditions of the lemma for \(r + 1\).

We first claim that \(J^*_p\) and \(J^*_j\) are distinct for all \(1 \leq i \leq m\) and \(1 \leq j \leq m' - 1\). To see that, choose \(J^*_p\) to be the largest interval among \(J^*_1\)'s, and \(\tilde{J}_q^*\) to be the largest one among \(J^*_j\)'s, and let \(J^*_q\) be the larger one between \(J^*_p\) and \(\tilde{J}_q^*\).

By the induction hypothesis, we have
\[
\max_{1 \leq i,j \leq m} |t_i - t_j| < 2Q(r)k|J^*_p| \quad \text{and} \quad \max_{1 \leq i,j \leq m'} |s_i - s_j| < 2Q(r)k|\tilde{J}_q^*|.
\]

Now for \(t_i \in J^*_1(1 \leq i \leq m)\) and \(s_j \in \tilde{J}_j^*(1 \leq j < m')\), we estimate:
\[(6.5) \quad |s_j - t_i| \geq |s_1 - t_\ell| - |t_1 - t_\ell| - |s_1 - s_j| > Q(r + 1)|J^*_\ell| - 2Q(r)k|J^*_p| - 2Q(r)k|\tilde{J}_q^*| \geq (Q(r + 1) - 4Q(r)k)|J^*_q| \geq |J^*_q|.
\]
This in particular means that \(s_j \notin J^*_p\) and \(t_i \notin \tilde{J}_j^*\). Hence
\[J^*_1 \neq \tilde{J}_j^*.
\]

We now begin checking the conditions (1), (2) and (3).

(1): If \(s_{m'} \notin J'(X)\), there is nothing to check. Now assume \(s_{m'} \in J'(X)\), and by induction hypothesis for \(r\) on the sequence \((s_1, \ldots, s_{m'})\), we can choose \(\tilde{J}_{m'}^*\) containing \(s_{m'}\) distinct from other \(J_j^*\)'s. In particular, \(\tilde{J}_1^*, \ldots, \tilde{J}_{m'}^*\) belong to at least \((r + 1)\) number of \(J_j^*(X_i)'s\). Observe that in the estimate (6.5), there is no harm in allowing \(j = m'\) in addition to \(j < m'\). This shows that \(\tilde{J}_{m'}^*\) is also distinct from all \(J_j^*\)'s.

Therefore, unless the following inclusion
\[(6.6) \quad \{\tilde{J}_1^*, \ldots, \tilde{J}_{m'}^*\} \subset J^*(X_1) \cup \cdots \cup J^*(X_{r+1}),
\]
holds, we are done. Suppose on the contrary that (6.6) holds. Without loss of generality, we assume that

\[ J^*_\ell \in J^*(\mathcal{X}_{r+1}). \]

**Claim:** We have

\[ |J^*_\ell| = \max_{1 \leq i \leq m, 1 \leq j \leq m'} (|J^*_i|, |\tilde{J}^*_j|). \]

Recall that \(|J^*_\ell|\) was chosen to be maximal among \(|J^*_1|, \ldots, |J^*_m|\). Hence, if the claim does not hold, then we can take \(j\) to be the least number such that \(|\tilde{J}^*_j| > |J^*_\ell|\). Then by induction hypothesis for (2), there exists an increasing sequence \(1 = j_0 < j_1 < \cdots < j_a = j\) such that

\[ |s_{j_a} - s_{j_{a-1}}| < Q(r)k|\tilde{J}^*_{j_a-1}| \]

\[ \cdots \]

\[ |s_{j_2} - s_{j_1}| < Q(r)k|\tilde{J}^*_{j_1}|. \]

In particular all of the right hand sides in the above inequalities are less than or equal to \(Q(r)k|J^*_\ell|\), by the choice of \(j\). Therefore, we get

\[ |t_\ell - s_j| \leq |t_\ell - s_{j_0}| + |s_{j_0} - s_{j_1}| + \cdots + |s_{j_{a-1}} - s_{j_a}| \]

\[ \leq Q(r + 1)k|J^*_\ell| + Q(r)k|J^*_\ell| + \cdots + Q(r)k|J^*_\ell| \]

\[ \leq (Q(r + 1) + 2^r Q(r))k|J^*_\ell|. \]

Now as \(J^*_1, \ldots, J^*_m\) intersect \((r + 1)\) families \(J^*(\mathcal{X}_1), \ldots, J^*(\mathcal{X}_{r+1})\) and \(\tilde{J}^*_j\) lies in one of these families, there exists \(J^*_i\) that belongs to the same family as \(\tilde{J}^*_j\). Recall that the induction hypothesis for \(t_1, \ldots, t_m\) gives us

\[ |t_\ell - t_i| \leq 2Q(r)k|J^*_\ell|. \]

Since \(\beta > (Q(r + 1) + (2^r + 2)Q(r))k\), we have

\[ |t_i - s_j| \leq |t_i - t_\ell| + |t_\ell - s_j| \]

\[ \leq (Q(r + 1) + (2^r + 2)Q(r))k|J^*_\ell| \]

\[ \leq \beta|\tilde{J}^*_j|. \]

Applying the \(\beta\)-regularity to the pair \((\tilde{I}_j, \tilde{J}^*_j)\), we conclude that

\[ t_i \in \tilde{I}_j \cap J^*_i \cap T. \]

Since \(J^*_i\) and \(\tilde{I}_j\) belong to the same family, this is a contradiction. This proves the claim (6.7).

We now claim that the following inclusion holds:

\[ \{\tilde{J}^*_1, \ldots, \tilde{J}^*_m'\} \subset J^*(\mathcal{X}_1) \cup \cdots \cup J^*(\mathcal{X}_r). \]

Note that this finishes the proof since \(\tilde{J}^*_1, \ldots, \tilde{J}^*_m'\) must intersect at least \((r + 1)\) number of \(J^*(\mathcal{X}_i)\)'s by induction hypothesis. In order to prove the
Lastly, also by the induction hypothesis on the sequence $t_1, \ldots, s_{m'}$ in the second line, to estimate the term $|s_1 - s_j|$.

Next, applying the $\beta$-regularity to the pair $(I_\ell, J^*_j)$, we conclude that $s_j \in I_\ell$. Since $s_j \in \tilde{J}^*_j$, it follows $I_\ell \cap \tilde{J}^*_j \cap T \neq \emptyset$. This contradicts that $\mathcal{X}_{r+1}$ is of $T$-multiplicity free, as both $\tilde{J}^*_j$ and $I_\ell$ belong to the same family $\mathcal{X}_{r+1}$. This completes the proof of condition (1).

(2) : Let $1 \leq j \leq m'$ be arbitrary. First, recall from (6.3) that we chose $|s_1 - t_\ell| \leq Q(r + 1)k|J^*_\ell|$ in the choice of the sequence $t_1, \ldots, t_m, s_1, \ldots, s_{m'}$ in (6.4).

Secondly, by the induction hypothesis on the sequence $s_1, \ldots, s_{m'}$, there exists a sequence $1 = j_0 < j_1 < \cdots < j_a = j$ such that for each $1 \leq b \leq a$,

$$|s_{j_b} - s_{j_b-1}| < Q(r)k|J^*_{j_b-1}|.$$ 

Lastly, also by the induction hypothesis on the sequence $t_1, \ldots, t_m$, there exists $1 = j'_0 < j'_1 < \cdots < j'_b = \ell$ such that for each $1 \leq b' \leq a'$,

$$|s_{j'_b} - s_{j'_b-1}| < Q(r)k|J^*_{j'_b-1}|.$$ 

These three observations prove (2) for $r + 1$, by concatenating the two index sets $1 = j'_0 < j'_1 < \cdots < j'_a = \ell$ for $t$ and $1 = j_0 < j_1 < \cdots < j_a = j$ for $s$.

(3) : We already have observed that the inequality $\beta > \varepsilon^{-1}Q(r + 1)k$ implies that

$$(1 - \varepsilon)^m|t_1| \leq |s_1| \leq (1 + \varepsilon)^m|t_1|.$$ 

Combining this with the induction hypothesis, we deduce that

$$(1 - \varepsilon)^{m'-1}|s_1| \leq |s_i| \leq (1 + \varepsilon)^{m'-1}|s_1|$$

for all $1 \leq i \leq m'$ would give us the conclusion.

Finally, the last statement of the lemma is obtained from the case $r = n$, since there are only $n$-number of $J^*(\mathcal{X}_i)'s$; hence the second possibility of (1) cannot arise for $r = n$. \hfill $\square$

Proof of Proposition 6.3. We may assume that $\mathcal{X}_i$’s are all of degree 1, by replacing each $\mathcal{X}_i$’s with $\delta$-many families associated to it. Hence we have $m\delta$-number of $\mathcal{X}_i$’s given. Set $n = m\delta$.

We set

$$\beta_0(m, k, \delta) = 2^{3m\delta + 5k^{m\delta} + 3\varepsilon^{-1}},$$

where $\varepsilon = \varepsilon_{m\delta}$ satisfies $\left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{2m\delta - 1} \leq 2$. 

Note that $\beta_0(m, k, \delta)$ is equal to the number $\beta(n, k, \varepsilon_n)$ from Lemma 6.4. We may assume $x = 0$ without loss of generality. Let $\lambda > 0$. We need to find a point
\begin{equation}
(6.9) \quad t' \in \left( [-2k\lambda, -\lambda] \cup [\lambda, 2k\lambda] \right) \cap \left( T - \bigcup_{i \in \Lambda} J'(X_i) \right).
\end{equation}

Choose $s > 0$ such that
\begin{equation}
(6.10) \quad (1 - \varepsilon)^{-2^{n-1}} \lambda \leq s \leq 2(1 + \varepsilon)^{-2^{n-1}} \lambda.
\end{equation}

Since $T$ is globally $k$-thick, there exists $t \in \left( [-ks, -s] \cup [s, ks] \right) \cap T$.

If $t \not\in \bigcup_{i=1}^n J'(X_i)$, then by choosing $t' = t$, we are done. Now suppose $t \in \bigcup_{i=1}^n J'(X_i)$.

Since $0 \not\in \bigcup_{i=1}^n I(X_i)$, by applying Lemma 6.4 to $t \in T \cap \bigcup_{i=1}^n J'(X_i)$, we obtain $t' \in T - \bigcup_{i=1}^n J'(X_i)$ such that
\begin{equation}
(1 - \varepsilon)^2 \lambda |t| \leq |t'| \leq (1 + \varepsilon)2^{n-1}|t|.
\end{equation}

Note that
\begin{equation}
|t'| \leq (1 + \varepsilon)2^{n-1}|t| \leq (1 + \varepsilon)^2 \lambda \leq 2k\lambda.
\end{equation}

Similarly, we have
\begin{equation}
|t'| \geq (1 - \varepsilon)^2 \lambda |t| \geq (1 - \varepsilon)^2 \lambda s \geq \lambda.
\end{equation}

This completes the proof since $t'$ satisfies (6.9).

7. Avoidance of the singular set

Let $G = \text{SO}^\circ(d, 1)$, $\Gamma < G$ be a convex cocompact non-elementary subgroup and let
\[ U = \{ u_t \} < N \]
be a one-parameter unipotent subgroup. Let $\mathcal{J}(U)$, $\mathcal{G}(U) X(H, U)$, and $\mathcal{H}^*$ be as defined in section 5. In particular, $\mathcal{J}(U)$ is a countable union:
\[ \mathcal{J}(U) = \bigcup_{H \in \mathcal{H}^*} \Gamma \backslash \Gamma X(H, U). \]

The main goal of this section is to prove the avoidance Theorem 7.11 for any rigid hyperbolic manifold. For this, we extend the linearization method developed by Dani and Margulis [12] to our setting and via a careful analysis of the graded intersections of the union $\bigcup \Gamma \backslash \Gamma H_i D_i \cap \text{RF} M$ for finitely many groups $H_i \in \mathcal{H}^*$ and compact subsets $D_i \subset X(H_i, U)$, we construct families of triples of subsets of $\mathbb{R}$ satisfying the conditions of Proposition 6.3 relative to the global $k$-thick subset of the return time to $\text{RF} M$ under $U$ given in Proposition 4.9.
**Linearization.** Let $H \in \mathcal{H}^\ast$. Then $H$ is reductive and has index at most 2 in its Zariski closure. There exists an $\mathbb{R}$-regular representation $\rho_H : G \to \text{GL}(V_H)$ with a point $p_H \in V_H$, such that $H$ has index at most 2 in the isotropy group $\text{Stab}_G(p_H)$ and the orbit $p_H G$ is closed [2, 7.7]. This implies that $p_H G$ is also Zariski closed. Since $\Gamma \setminus \Gamma H$ is closed, it follows that $p_H \Gamma$ is a closed (and hence discrete) subset of $V_H$.

Let $\eta_H : G \to V_H$ denote the orbit map defined by $\eta_H(g) = p_H g$ for all $g \in G$.

Note that $X(H,U)$ is Zariski closed in $G$, and is left $N_G(H)$-invariant. Since $p_H G$ is Zariski closed in $V_H$, it follows that $A_H := p_H X(H,U)$ is Zariski closed in $V_H$ and $X(H,U) = \eta_H^{-1}(A_H)$.

Since $[N_G(H) : H] \leq 2$, we have

$$(7.1) \quad p_H N_G(H) \subset \{ \pm p_H \}$$

Following [18], for given $C > 0$ and $\alpha > 0$, a function $f : \mathbb{R} \to \mathbb{R}$ is called $(C,\alpha)$-good if for any interval $I \subset \mathbb{R}$ and $\varepsilon > 0$, we have

$$\ell\{ t \in I : |f(t)| \leq \varepsilon \} \leq C \cdot \left( \frac{\varepsilon}{\sup_{t \in I} |f(t)|} \right)^\alpha \cdot \ell(I)$$

where $\ell$ is a Lebesgue measure on $\mathbb{R}$.

**Lemma 7.1.** For given $C > 0$ and $\alpha > 0$, consider functions $p_1, p_2, \ldots, p_k : \mathbb{R} \to \mathbb{R}$ satisfying the $(C,\alpha)$-good property. For $0 < 1 < \delta$, set

$$I = \{ t \in \mathbb{R} : \max_i |p_i(t)| < 1 \} \quad \text{and} \quad J(\delta) = \{ t \in \mathbb{R} : \max_i |p_i(t)| < \delta \}.$$

For any $\beta > 1$, there exists $\delta = \delta(C,\alpha,\beta) > 0$ such that the pair $(I, J(\delta))$ is $\beta$-regular.

**Proof.** We prove that the conclusion holds for

$$\delta := \left( \frac{1}{(1 + \beta) C} \right)^\frac{1}{\alpha}.$$

First, note that the function $q(t) := \max_i |p_i(t)|$ also has the $(C,\alpha)$-property. Let $J' = (a,b)$ be a component of $J(\delta)$, and $I'$ be the component of $I$ containing $J'$. Note that $I'$ is an open interval and $(a, \infty) \cap I' = (a, c)$ for some $b \leq c \leq \infty$. We claim

$$J' + \beta |J'| \subset (a, \infty) \cap I' \subset I'.$$

We may assume $c < \infty$; otherwise the inclusion is trivial. We claim that $q(c) = 1$. Since $\{ t \in \mathbb{R} : q(t) < 1 \}$ is open and $c$ is the boundary point of $I'$, $q(c) \geq 1$. If $q(c)$ were strictly bigger than 1, since $\{ t \in \mathbb{R} : q(t) > 1 \}$ is open,
$I'$ would be disjoint from an open interval around $c$, which is impossible. Hence $q(c) = 1$.

Now that $\sup \{ q(t) : t \in (a, \infty) \cap I' \} = q(c) = 1$, by applying the $(C, \alpha)$-good property of $q$ on the interval $(a, \infty) \cap I'$, we get

$$\ell(J') \leq \ell\{ t \in (a, \infty) \cap I' : |q(t)| \leq \delta \} \leq C\delta^\alpha \cdot \ell((a, \infty) \cap I').$$

Now as $J' = (a, b)$ and $(a, \infty) \cap I'$ are nested intervals with one common endpoint, it follows from the inequality $C\delta^\alpha < 1/(1 + \beta)$ that

$$J' + \beta |J'| \subset (a, \infty) \cap I' \subset I'.$$

Similarly, applying the $(C, \alpha)$-good property of $q$ on $(-\infty, b) \cap I'$, we deduce that

$$J' - \beta |J'| \subset I'.$$

This proves that $(I, J(\delta))$ is $\beta$-regular. \qed

**Proposition 7.2.** Let $V$ be a finite dimensional real vector space, $\theta \in \mathbb{R}[V]$ be a polynomial and $A = \{ v \in V : \theta(v) = 0 \}$. Then for any compact subset $D \subset A$ and any $\beta > 0$, there exists a compact neighborhood $D' \subset A$ of $D$ which has a $\beta$-regular size with respect to $D$ in the following sense: for any neighborhood $\Phi$ of $D'$, there exists a neighborhood $\Psi$ of $D$ such that for any $q \in V - \Phi$ and for any one-parameter unipotent subgroup $\{ u_t \} \subset \text{GL}(V)$, the pair $(I(q), J(q))$ is $\beta$-regular where

$$I(q) = \{ t \in \mathbb{R} : qu_t \in \Phi \} \quad \text{and} \quad J(q) = \{ t \in \mathbb{R} : qu_t \in \Psi \}.$$

Furthermore, the degree of $(I(q), J(q))$ is at most $(\deg \theta + 2) \cdot \dim(V)$.

**Proof.** Choose a norm on $V$ so that $\| \cdot \|^2$ is a polynomial function on $V$. Since $D$ is compact, we can find $R > 0$ such that

$$D \subset \{ v \in V : \| v \| < R \}.$$

Then we set

$$D' = \{ v \in V : \theta(v) = 0, \| v \| < R/\sqrt{\delta} \},$$

where $0 < \delta < 1$ is to be specified later. Note that if $\Phi$ is a neighborhood of $D'$, there exists $\eta > 0$ such that

$$\{ v \in V : \theta(v) < \eta, \| v \| < (R + \eta)/\sqrt{\delta} \} \subset \Phi.$$

We will take $\Psi$ to be

$$\Psi = \{ v \in V : \theta(v) < \eta\delta, \| v \| < (R + \eta) \}.$$

Set

$$\tilde{I}(q) = \{ t \in \mathbb{R} : \theta(qu_t) < \eta, \| qu_t \| < (R + \eta)/\sqrt{\delta} \}.$$

Since $\tilde{I}(q) \subset I(q)$ for $0 < \delta < 1$, it suffices to find $\delta$ (and hence $D'$ and $\Psi$) so that the pair $(\tilde{I}(q), J(q))$ is $\beta$-regular.
If we set
\[ \psi_1(t) := \frac{\theta(qu)}{\eta} \quad \text{and} \quad \psi_2(t) := \left( \frac{\|qu\|\sqrt{\delta}}{(R + \eta)} \right)^2, \]
then
\[ \tilde{I}(q) = \{ t \in \mathbb{R} : \max(\psi_1(t), \psi_2(t)) < 1 \}; \]
\[ J(q) = \{ t \in \mathbb{R} : \max(\psi_1(t), \psi_2(t)) < \delta \}. \]

As \( \psi_1 \) and \( \psi_2 \) are polynomials, they have the \((C, \alpha)\)-property for an appropriate choice of \( C \) and \( \alpha \). Therefore by applying Lemma 7.1, by choosing \( \delta \) small enough, we can make the pair \((\tilde{I}(q), J(q))\) \( \beta \)-regular for any \( \beta > 0 \).

Note that the degrees of \( \psi_1 \) and \( \psi_2 \) are bounded by \( \deg \theta \cdot \dim(V) \) and \( 2 \dim(V) \) respectively. Therefore \( J(q) \) cannot have more than \((\deg \theta + 2) \cdot \dim(V)\) number of components. Hence the proof is complete. \( \square \)

**Collection \( \mathcal{E} \).** Recalling \( \mathcal{S}(U) = \bigcup_{H \in \mathcal{H}^*} \Gamma \backslash \Gamma X(H, U) \), we define \( \mathcal{E} = \mathcal{E}_U \) to be the collection of all subsets of \( \mathcal{S}(U) \cap RF M \) which can be written as
\[
(7.2) \quad E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i^{-1}(D_i) \cap RF M
\]
where \( \{H_i : i \in \Lambda\} \) is a finite collection and \( D_i \subset A_{H_i} \) is a compact subset. In this expression, we always use the minimal index set \( \Lambda \) for \( E \).

**Lemma 7.3.** In the expression \((7.2)\) for \( E \in \mathcal{E} \), the collection \( \{H_i : i \in \Lambda\} \) is not redundant, in the sense that
- no \( \gamma H_j \gamma^{-1} \) is contained in \( H_i \) for all triples \((i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma \) except for the trivial cases of \( i = j \) and \( \gamma \in N_G(H_i) \).

**Proof.** Observe that if \( \gamma H_j \gamma^{-1} \subset H_i \) for some \( \gamma \in \Gamma \), then \( \Gamma H_{H_i}^{-1}(D_{j}) \subset \Gamma H_{H_i}^{-1}(\gamma D_j) \), and hence by replacing \( D_i \) by a new compact subset \( D_i \cup \gamma D_j \subset X(H_i, U) \), we may remove \( j \) from the index set \( \Lambda \) of \( E \). This contradicts the minimality of \( \Lambda \). \( \square \)

Observe that for any subgroups \( H_1, H_2 \) of \( G \), and \( g \in G \),
\[
X(H_1 \cap gH_2 g^{-1}, U) = X(H_1, U) \cap X(gH_2g^{-1}, U) = X(H_1, U) \cap gX(H_2, U).
\]

Note that for \( D_i \subset A_{H_i} \), and \( \gamma \in \Gamma \), the intersection \( \eta^{-1}_{H_i}(D_1) \cap \gamma \eta^{-1}_{H_2}(D_2) \) only depends on the \((\Gamma_{H_1}, \Gamma_{H_2})\)-double coset of \( \gamma \) where \( \Gamma_{H_i} = \text{Stab}_G(p_{H_i}) \).

**Proposition 7.4.** Let \( H_1, H_2 \in \mathcal{H}^* \). Then for any compact subset \( D_i \subset A_{H_i} \) for \( i = 1, 2 \) and a compact subset \( K \subset \Gamma \backslash G \), there exists a finite set \( \Delta \subset (H_1 \cap \Gamma) \backslash (H_2 \cap \Gamma) \) such that
\[
\{ K \cap (\Gamma \backslash \Gamma H_i^{-1}(D_1) \cap \gamma \eta_{H_2}^{-1}(D_2)) \}_{\gamma \in \Delta} = \{ K \cap (\Gamma \backslash \Gamma H_i^{-1}(D_1) \cap \gamma \eta_{H_2}^{-1}(D_2)) \}_{\gamma \in \Delta}
\]
where the latter set consists of distinct elements.
Moreover for each \( \gamma \in \Delta \), there exists a compact subset \( C_0 \subset \eta_{H_1}^{-1}(D_1) \cap \gamma \eta_{H_2}^{-1}(D_2) \subset X(H_1 \cap \gamma H_2 \gamma^{-1} \cap U) \) such that
\[
K \cap \Gamma \setminus (\eta_{H_1}^{-1}(D_1) \cap \gamma \eta_{H_2}^{-1}(D_2)) = \Gamma \setminus \Gamma C_0.
\]

Proof. For simplicity, write \( \eta_{H_i} = \eta_i \) and \( p_i = p_{H_i} \). Let \( K_0 \subset G \) be a compact set such that \( K = \Gamma \setminus \Gamma K_0 \). We fix \( \gamma \in \Gamma \), and define for any \( \gamma' \in \Gamma \),
\[
K_{\gamma'} = \{ g \in K_0 : \gamma' g \in \eta_1^{-1}(D_1) \cap \gamma \eta_2^{-1}(D_2) \}.
\]
By definition, one can check
\[
K \cap \Gamma \setminus (\eta_1^{-1}(D_1) \cap \gamma \eta_2^{-1}(D_2)) = \Gamma \setminus (\cup_{\gamma' \in \Gamma} K_{\gamma'}).
\]
If this set is nonempty, then \( K_{\gamma'} \neq \emptyset \) for some \( \gamma' \in \Gamma \) and
\[
p_1 \gamma' g \in p_1 D_1, \quad p_2 \gamma^{-1} \gamma' g \in p_2 D_2
\]
for some \( g \in K_0 \). In particular,
\[
(7.3) \quad p_1 \gamma' \in p_1 D K_0^{-1}, \quad p_2 \gamma^{-1} \in p_2 D K_0^{-1} \gamma'^{-1}.
\]

As \( p_1 \Gamma \) is discrete, \( [\text{Stab}_G(p_1) : H_1] \leq 2 \), and \( p_1 D_1 K_0^{-1} \) is compact, the first condition of (7.3) implies that there exists a finite set \( \Delta_0 \subset G \) such that \( \gamma' \in (H_1 \cap \Gamma) \Delta_0 \). Writing \( \gamma' = h \delta_0 \) where \( h \in H_1 \cap \Gamma \), and \( \delta_0 \in \Delta_0 \), the second condition of (7.3) implies
\[
p_2 \gamma^{-1} h \in p_2 D_2 K_0^{-1} \delta_0^{-1}.
\]
As \( p_2 D_2 K_0^{-1} \Delta_0^{-1} \) is compact and \( p_2 \Gamma \) is discrete, there exists a finite set \( \Delta \subset G \) such that \( \gamma^{-1} h \in (H_2 \cap \Gamma) \Delta \). Hence, if \( K \cap \Gamma \setminus (\eta_1^{-1}(D_1) \cap \gamma \eta_2^{-1}(D_2)) \neq \emptyset \), then \( \gamma \in (H_1 \cap \Gamma) \Delta (H_2 \cap \Gamma) \). This completes the proof of the first claim.

For the second claim, it suffices to set \( C_0 := \bigcup_{\gamma' \in \Delta} K_{\gamma'} \). □

**Proposition 7.5.** Let \( H_1, H_2 \in \mathcal{H}^* \) such that \( H_1 \cap H_2 \) contains a unipotent element. Let \( H_0 \) be the smallest connected closed subgroup of \( H_1 \cap H_2 \) containing all unipotent elements of \( H_1 \cap H_2 \) such that \( \Gamma \setminus \Gamma H_0 \) is closed. Then \( H_0 \in \mathcal{H}^* \).

Proof. We need to show that \( \Gamma \cap H_0 \) is Zariski dense in \( H_0 \). Let \( L \) be the subgroup of \( H_0 \) generated by all unipotent elements in \( H_0 \). Note that \( L \) is a normal subgroup of \( H_0 \) and hence \( (H_0 \cap \Gamma)L \) is a subgroup of \( H_0 \). If \( F \) is the identity component of the closure of \( (H_0 \cap \Gamma)L \), then \( \Gamma \setminus \Gamma F \) is closed. By the minimality assumption on \( H_0 \), we have \( F = H_0 \). Hence \( (H_0 \cap \Gamma)L = H_0 \); so \( [e]L = [e]H_0 \). We can then apply [41, Corollary 2.12] and deduce that \( H_0 \cap \Gamma \) is Zariski dense in \( H_0 \). □

**Corollary 7.6.** Let \( H_1, H_2 \in \mathcal{H}^* \) and \( \gamma \in \Gamma \) be satisfying that \( X(H_1 \cap \gamma H_2 \gamma^{-1} \cap U) \neq \emptyset \). Then there exists a subgroup \( H \in \mathcal{H}^* \) contained in \( H_1 \cap \gamma H_2 \gamma^{-1} \) such that for any compact subsets \( D_i \subset A_{H_i}, i = 1, 2 \), there exists a compact subset \( D_0 \subset A_{H_0} \) such that
\[
K \cap \Gamma \setminus (\eta^{-1}_{H_1}(D_1) \cap \gamma \eta_{H_2}^{-1}(D_2)) = K \cap \Gamma \setminus \Gamma \eta_{H_1}^{-1}(D_0).
\]
**Definition 7.7** (Self-intersection operator on $E_U$). We define an operator $s : E_U \cup \{\emptyset\} \rightarrow E_U \cup \{\emptyset\}$ as follows: we set $s(\emptyset) = \emptyset$. For a non-empty subset

$$E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma \eta_{H_i}^{-1}(D_i) \cap RF M \in E,$$

we define

$$s(E) := \bigcup_{i,j \in \Lambda} \bigcup_{\gamma_{ij} \in \Gamma} \Gamma \backslash \Gamma \eta_{H_i}^{-1}(D_i) \cap \gamma \eta_{H_j}^{-1}(D_j) \cap RF M$$

where $\gamma_{ij} \in \Gamma$ ranges over all $\gamma \in \Gamma$ such that

$$\dim(H_i \cap H_j \gamma^{-1})_{nc} < \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}.$$ 

By Proposition 7.4 and Corollary 7.6, we have:

**Corollary 7.8.**

1. For $E \in E_U$, we have $s(E) \in E_U$.
2. For $E_1, E_2 \in E_U$, we have $E_1 \cap E_2 \in E_U$.

Hence for $E \in E_U$ as in (7.5), $s(E)$ is of the form

$$s(E) = \bigcup_{i \in s(\Lambda)} \Gamma \backslash \Gamma \eta_{H_i}^{-1}(D_i) \cap RF M$$

where $s(\Lambda)$ is a (minimal) finite index set, $H_i \in \mathcal{H}$ with $X(H_i, U) \neq \emptyset$ and

$$\max\{\dim(H_i)_{nc} : i \in s(\Lambda)\} < \max\{\dim(H_i)_{nc} : i \in \Lambda\}.$$ 

Hence, $s$ maps $E$ to $E \cup \{\emptyset\}$ and

$$s^{dimG}(E) = \emptyset.$$ 

**Definition 7.9.** For a compact subset $K \subset \Gamma \backslash G$ and $E \in \mathcal{E}$, we say that $K$ does not have any self-intersection point of $E$, or simply $K$ is $E$-self intersection-free, if

$$K \cap s(E) = \emptyset.$$
 Proposition 7.10. Let $E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma \eta_{H_i}^{-1}(D_i) \cap RF$ $M \in \mathcal{E}$ where $D_i \subset \Lambda_{H_i}$ is a compact subset and $\Lambda$ is a finite subset. Let $K \subset RF$ $M$ be a compact subset which is $E$-self intersection free. Then there exists a collection of open subsets $\Omega_i$ of $D_i$, $i \in \Lambda$, such that for $O := \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma \eta_{H_i}^{-1}(\Omega_i)$, the compact subset $K$ is $O$-self intersection free, in the sense that, if $K \cap \Gamma \backslash \Gamma \eta_{H_i}^{-1}(\Omega_i) \cap \gamma \eta_{H_j}^{-1}(\Omega_j) \neq \emptyset$ for some $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$, then $i = j$ and $\gamma \in N_G(H_i) \cap \Gamma$.

Proof. For each $k \in \mathbb{N}$ and $i \in \Lambda$, let $\Omega_i(k)$ be the $1/k$-neighborhood of the compact subset $D_i$. Since $\Lambda$ is finite, if the proposition does not hold, by passing to a subsequence, there exist $i, j \in \Lambda$ and a sequence $\gamma_k \in \Gamma$ such that $K \cap \Gamma \backslash \Gamma \eta_{H_i}^{-1}(\Omega_i(k)) \cap \gamma_k \eta_{H_j}^{-1}(\Omega_j(k)) \neq \emptyset$ and

\[(i, j, \gamma_k) \notin \{(i, i, \gamma) : i \in \Lambda, \gamma \in N_G(H_i) \cap \Gamma\}.

Hence there exist $g_k = h_kw_k \in \Lambda_{H_i}(k)$ and $g_k' = h'_kw'_k \in \Lambda_{H_j}(k)$ such that $g_k = \gamma_k g_k$ where $[g_k] \in K$.

Now $w_k \to w \in D_i$ and $w_k' \to w' \in D_j$. There exists $\delta_k \in \Gamma$ such that $\delta_k g_k \in \hat{K}$ where $\hat{K}$ is a compact subset such that $K = \Gamma \backslash \hat{K}$, so $\delta_k g_k \to g_0$. Since $\Gamma H_i$ and $\Gamma H_j$ are closed, we have $\delta_k h_k \to \delta_0 h_i$ and $\delta_k \gamma_k h_k' \to \delta_0 \gamma_j h_j$ where $\delta_0, \delta_0' \in \Gamma$ and $h_i \in H_i$ and $h_j \in H_j$. As $\Gamma[H_i]$ and $\Gamma[H_j]$ are discrete in $G/H_i$ and $G/H_j$ respectively, we have

\[(\delta_0^{-1} \delta_k \in H_i \quad \text{and} \quad (\delta_0')^{-1} \delta_k \gamma_k \in H_j)

for all sufficiently large $k$.

Therefore $g_0 = \delta_0 h_i w = \delta_0 h_j w' \in \delta_0(\eta_{H_i}^{-1}(D_i)) \cap \delta_0^{-1} \delta_0' \eta_{H_j}^{-1}(D_j))$ and $[g_0] \in K$.

Hence $K \cap \Gamma \backslash \Gamma \eta_{H_i}^{-1}(D_i) \cap \delta_0^{-1} (\delta_0' \eta_{H_j}^{-1}(D_j)) \neq \emptyset$ where $\delta := \delta_0^{-1} \delta_0' \in \Gamma$.

Since $K \cap s(E) \neq \emptyset$, this implies that $\Gamma(\eta_{H_i}^{-1}(D_i) \cap \delta \eta_{H_j}^{-1}(D_j)) \subset s(E)$. By the definition of $s(E)$,

$$\dim(H_i \cap \delta H_j \delta^{-1})_{nc} = \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}.$$ 

Since $H_i = N((H_i)_{nc})$, and similarly for $H_j$, we have $H_i \cap \delta H_j \delta^{-1}$ is either $H_i$ or $\delta H_j \delta^{-1}$, that is, $\delta H_j \delta^{-1} \subset H_i$ or $H_i \subset \delta H_j \delta^{-1}$. By Lemma 7.3, this implies $i = j$ and $\delta \in N_G(H_i) \cap \Gamma$. It follows from (7.7) that $\gamma_k \in N_G(H_i) \cap \Gamma$ for all large $k$. This is a contradiction to (7.6), completing the proof. $\square$

In the rest of this section, we assume that $M = \Gamma \backslash \mathbb{H}^d$ is a rigid hyperbolic manifold, and let $k$ be as given by Proposition 4.9.
Theorem 7.11 (Avoidance theorem I). Let $U = \{u_t\} < N$ be a one-parameter subgroup. For any $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}$ such that the following holds: If $F \subset RF M$ is a compact set disjoint from $E'$, then there exists a neighborhood $\mathcal{O}^\circ$ of $E \cap RF M$ such that for all $x \in F$,

$$\{t \in \mathbb{R} : xu_t \in RF M - \mathcal{O}^\circ\}$$

is $2k$-thick.

Proof. \(\spadesuit\) 1. The constant $\beta_0$: We write $\mathcal{H}^* = \{H_i\}$. For simplicity, set $V_i = V_{H_i}$ and $p_i = p_{H_i}$. Let $\theta_i$ be the defining polynomial of the algebraic variety $A_{H_i}$.

Set

$$m := 2 \cdot \dim(G); \quad \text{and} \quad \delta := \max_{H_i \in \mathcal{H}} (\deg \theta_i + 2) \dim(V_i).$$

Note that if $H_i$ is conjugate to $H_j$, then $\theta_i$ and $\theta_j$ have same degree and $\dim(V_i) = \dim(V_j)$. Since there are only finitely many conjugacy classes in $\mathcal{H}^*$, the constant $\delta$ is finite. Now let

$$\beta_0 := \beta_0(m, k, \delta) = 2^{3m\delta+5}k^{m\delta+3}\epsilon^{-1}$$

be given as in Proposition 6.3 where $\epsilon = \epsilon_{m\delta}$ satisfies $\left(\frac{1+\epsilon}{1-\epsilon}\right)^{2m\delta-1} \leq 2$.

\(\spadesuit\) 2. Definition of $E_n$ and $E'_n$: We write

$$E = \bigcup_{i \in \Lambda_0} \Gamma \backslash \Gamma_\eta^{-1}_{H_i}(D_i) \cap RF M$$

for some finite minimal set $\Lambda_0$. Set

$$\ell := \max_{i \in \Lambda_0} \dim(H_i)_{nc}.$$

We define $E_n, E'_n \in \mathcal{E}_U$ for all $1 \leq n \leq \ell$ inductively as follows: Set

$$E_\ell := E \quad \text{and} \quad \Lambda_\ell := \Lambda_0.$$

For each $i \in \Lambda_\ell$, let $D_i'$ be a compact subset of $A_{H_i}$ containing $D_i$ such that $D_i'$ has a $\beta_0$-regular relative size with respect to $D_i$ as in Proposition 7.2. Set

$$E_i' := \bigcup_{i \in \Lambda_\ell} \Gamma \backslash \Gamma_\eta^{-1}_{H_i}(D_i') \cap RF M.$$

Suppose that $E_{n+1}, E'_{n+1} \in \mathcal{E}_U$ are given for $n \geq 1$. Then define

$$E_n := E \cap s(E'_{n+1}).$$

Then by Corollary 7.8, $E_n$ belongs to $\mathcal{E}_U$ and hence can be written as

$$E_n = \bigcup_{i \in \Lambda_\ell} \Gamma \backslash \Gamma_\eta^{-1}_{H_i}(D_i) \cap RF M$$

where $D_i$ is a compact subset of $X(H_i, U)$, so that $\Lambda_{\ell}$ is a minimal index set.
For each $i \in \Lambda_n$, let $D'_i$ be a compact subset of $A_{H_i}$ containing $D_i$ such that $D'_i$ has a $\beta_0$-regular relative size with respect to $D_i$ as in Proposition 7.2. Set
\[
E'_n := \bigcup_{i \in \Lambda_n} \Gamma \backslash \Gamma H_i^{-1}(D'_i) \cap RF M.
\]
Hence we get a sequence of compact (possibly empty) subsets of $E$:
\[
E_1, E_2, \cdots, E_{\ell-1}, E_\ell = E,
\]
and a sequence of compact sets
\[
E'_1, E'_2, \cdots, E'_{\ell-1}, E'_\ell = E'.
\]
Note that $s(E_1) = s(E'_1) = \emptyset$ by the dimension reason.\(^5\)

\[\blacktriangleleft\textbf{Outline of the plan:}\]

Let $F \subset RF M$ be a compact set disjoint from $E'$. For $x \in F$, we set
\[
T(x) := \{t \in \mathbb{R} : xu_t \in RF M\}
\]
which is a globally $k$-thick set by Proposition 4.9. We will construct
\begin{itemize}
\item a neighborhood $\mathcal{O}'$ of $E'$ disjoint from $F$, and
\item a neighborhood $\mathcal{O}^\circ$ of $E \cap RF M$
\end{itemize}
such that for $x \notin \mathcal{O}'$, we have
\[
\{t \in \mathbb{R} : xu_t \in RF M - \mathcal{O}^\circ\} = T(x) - J'(\mathcal{X})
\]
where $\mathcal{X} = \mathcal{X}(x)$ is the union of $\ell \leq m$ number of $\beta_0$-regular families $\mathcal{X}_i$ of triples $(I(q), J^*(q), J'(q))$ of subsets of $\mathbb{R}$ with degree $\delta$ and of $T(x)$-multiplicity free. Once we do that, the theorem is a consequence of Proposition 6.3. Construction of such $\mathcal{O}'$ and $\mathcal{O}^\circ$ requires an inductive process on $E_n$'s.

\[\blacktriangleleft\textbf{Inductive construction of $K_n$, $\mathcal{O}'_{n+1}$, $\mathcal{O}_{n+1}$, and $\mathcal{O}^*_{n+1}$:}\]

Let
\[
K_0 := RF M.
\]
For each $i \in \Lambda_1$, there exists a neighborhood $\Omega'_i$ of $D'_i$ in $V_i$ such that for
\[
\mathcal{O}'_i := \bigcup_{i \in \Lambda_1} \Gamma \backslash \Gamma H_i^{-1}(\Omega'_i),
\]
the compact subset $K_0$ is $\mathcal{O}'_1$-self intersection free by Lemma 7.10, since $s(E'_1) = \emptyset$.

By Proposition 7.2, there exists a neighborhood $\Omega_i$ of $D_i$ such that the pair $(I(q), J(q))$ is $\beta_0$-regular for all $q \in V_i - \Omega'_i$ where
\[
I(q) = \{t \in \mathbb{R} : qu_t \in \Omega'_i\} \quad \text{and} \quad J(q) = \{t \in \mathbb{R} : qu_t \in \Omega_i\}.
\]
Set
\[
\mathcal{O}_1 := \bigcup_{i \in \Lambda_1} \Gamma \backslash \Gamma H_i^{-1}(\Omega_i).
\]
Since $E_1 = \bigcup_{i \in \Lambda_1} \Gamma \backslash \Gamma H_i^{-1}(D_i) \cap RF M$, $\mathcal{O}_1$ is a neighborhood of $E_1 = s(E'_1) \cap E$. Now the compact subset $s(E'_2) - \mathcal{O}_1$ is contained in $s(E'_2) - E$, which
\(^5\)In fact $E_{\ell-1} = \emptyset$ for all $i \geq d - 1$, but we won’t use this information.
is relatively open in \( s(E'_2) \). Therefore we can take a neighborhood \( O'_1 \) of \( s(E'_2) - O_1 \) so that

\[ \overline{O'_1} \cap E = \emptyset. \]

We will now define the following quadruple \( K_n, O'_{n+1}, O_{n+1} \) and \( s^*_{n+1} \) for each \( 1 \leq n \leq \ell - 1 \) inductively:

- a compact subset \( K_n = K_{n-1} - (O_n \cup O^*_n) \subset RF M \),
- a neighborhood \( O'_{n+1} \) of \( E'_{n+1} \),
- a neighborhood \( O_{n+1} \) of \( E_{n+1} \) and
- a neighborhood \( s^*_{n+1} \) of \( s(E'_{n+2}) - O_{n+1} \) so that

\[ s(E'_{n+2}) \subset O_{n+1} \cup O^*_{n+1} \text{ and } \overline{O^*_{n+1}} \cap E = \emptyset. \]

Assume that the sets \( K_{n-1}, O'_n, O_n \) and \( s^*_n \) are defined. We define

\[ K_n := K_{n-1} - (O_n \cup O^*_n). \]

For each \( i \in \Lambda_{n+1} \), let \( \Omega'_i \) be a neighborhood of \( D'_i \) in \( V_i \) such that for \( O'_{n+1} := \bigcup_{i \in \Lambda_{n+1}} \eta_i^{-1}(\Omega'_i) \), \( K_n \) is \( O'_{n+1} \)-self intersection free. Since \( O_n \cup O^*_n \) is a neighborhood of \( s(E'_{n+1}) \), which is the set of all intersection points of \( E'_{n+1} \), such collection of \( \Omega'_i, i \in \Lambda_{n+1} \) exists by Lemma 7.10.

Since \( F \subset RF M \) is compact and disjoint from \( E' \), we can also assume \( \Gamma \setminus \Gamma \eta_i^{-1}(\Omega'_i) \) is disjoint from \( F \), by shrinking \( \Omega'_i \) if necessary. More precisely, writing \( F = \Gamma \setminus \Gamma \hat{F} \) for some compact subset \( \hat{F} \subset G \), this can be achieved by choosing a neighborhood \( \Omega'_i \) of \( D'_i \) disjoint from \( p_i \Gamma \hat{F} \); and this is possible since \( p_i \Gamma \hat{F} \) is a closed set disjoint from a compact subset \( D'_i \).

After choosing \( \Omega'_i \) for each \( i \in \Lambda_{n+1} \), define the following neighborhood of \( E'_{n+1} \):

\[ O'_{n+1} := \bigcup_{i \in \Lambda_{n+1}} \Gamma \setminus \Gamma \eta_i^{-1}(\Omega'_i). \]

We will now define \( O_{n+1} \). By Lemma 7.2, there exists a neighborhood \( \Omega_i \) of \( D_i \) such that the pair \( (I(q), J(q)) \) is \( \beta_0 \)-regular for all \( q \in V_i - \Omega'_i \) where

\[ I(q) = \{ t \in \mathbb{R} : qt \in \Omega'_i \} \text{ and } J(q) = \{ t \in \mathbb{R} : qt \in \Omega_i \}. \]

We then define the following neighborhood of \( E_{n+1} = s(E'_{n+2}) \cap E \):

\[ O_{n+1} := \bigcup_{i \in \Lambda_n} \Gamma \setminus \Gamma \eta_i^{-1}(\Omega_i). \]

Now since the compact subset \( s(E'_{n+2}) - O_{n+1} \) is contained in the set \( s(E'_{n+2}) - E \), which is relatively open inside \( s(E'_{n+2}) \), we can take a neighborhood \( s^*_{n+1} \) of \( s(E'_{n+2}) - O_{n+1} \) so that

\[ \overline{s^*_{n+1}} \cap E = \emptyset. \]

Noting that \( O_{n+1} \cup O^*_{n+1} \) is a neighborhood of \( s(E'_{n+2}) \), this finishes the inductive construction.
5. Definition of $O'$ and $O^*$: We define:

$$O' := \bigcup_{n=1}^{\ell} O'_n, \quad O := \bigcup_{n=1}^{\ell} O_n, \quad O^* := \bigcup_{n=1}^{\ell} O_n^*.$$ 

Note that $O'$ and $O$ are neighborhoods of $E'$ and $E$ respectively. Since $E \cap O^* = \emptyset$, the following defines a neighborhood of $E$:

$$(7.9) \quad O^\circ := O - O^*.$$ 

6. Construction of $\beta_0$-regular families of $T(x)$-multiplicity free:

Fix $x \in F \subset RF M - O'$. Choose a representative $g \in G$ of $x$.

For each $q \in \bigcup_{i \in \Lambda_n} p_i \Gamma g$, we define the following subsets of $\mathbb{R}$:

- $I(q) = \{ t : xu_t \in O'_n \}$ and
- $J(q) = \{ t : xu_t \in O_n \}.$

In general, $I(q)$’s have high multiplicity among $q$’s in $\bigcup_{i \in \Lambda_n} p_i \Gamma g$, but the following subset $I'(q)$’s will be multiplicity-free, and this is is why we defined $K_{n-1}$ as carefully as above.

For each $q \in \bigcup_{i \in \Lambda_n} p_i \Gamma g$, we define the following subsets of $I(q)$:

- $I'(q) := \{ t : \text{for some } a \geq 0, \ [t, t + a] \subset I(q) \text{ and } xu_{t+a} \in K_{n-1} \};$
- $J^*(q) := I'(q) \cap J(q);$
- $J'(q) := \{ t \in J(q) : xu_t \in K_{n-1} \}.$

Observe that $I'(q)$ and $J^*(q)$ are unions of finitely many intervals, $J'(q) \subset T(x)$ and that

$$J'(q) \subset J^*(q) \subset I'(q).$$

Now, for each $n = 1, \ldots, \ell$, define the family

$$X_n = \{(I(q), J^*(q), J'(q)) : q \in \bigcup_{i \in \Lambda_n} p_i \Gamma g \}.$$ 

By splitting each $X_n$ into two families, we may assume that $q$ and $-q$ does not simultaneously appear as indices for $X_n$. That will increase the number of $X_n$’s at most twice. Hence we have at most $m$- number of $X_n$’s.

We claim that each $X_n$ is a $\beta_0$-regular family with degree at most $\delta$ and $T(x)$-multiplicity free.

Note for each $q \in p_i \Gamma g$, the number of connected components of $J^*(q)$ is less than or equal to that of $J(q)$. Now that $J^*(q) \subset J(q)$ and all the pairs $(I(q), J(q))$ are $\beta_0$-regular pairs of degree at most $\delta$, it follows that $X_n$’s are $\beta_0$-regular families with degree at most $\delta$.

We now claim that $X_n$ has $T(x)$-multiplicity free, that is, for any distinct indices $q_1, q_2 \in \bigcup_{i \in \Lambda_n} p_i \Gamma g$ of $X_n$,

$$I(q_1) \cap J'(q_2) = \emptyset.$$ 

We first show that

$$I'(q_1) \cap I'(q_2) = \emptyset.$$
Assume to the contrary that there exists $t \in I'(q_1) \cap I'(q_2)$ for $q_1 = p_i \gamma_1 g$ and $q_2 = p_j \gamma_2 g$. Then for some $a \geq 0$, we have $[t, t + a] \subset I(q_1) \cap I(q_2)$ together with $x_{u_{t+a}} \in K_{n-1}$. In particular,

$$x_{u_{t+a}} \in \Gamma \setminus \Gamma(\gamma_1^{-1} \eta_1^{-1} (\Omega_i) \cap \gamma_2^{-1} \eta_2^{-1} (\Omega_j)) \cap K_{n-1}.$$ 

Since $K_{n-1}$ is $O_n'$-self intersection free, we deduce from Proposition 7.10 that this happens only when $i = j$, and $\gamma_1 \gamma_2^{-1} \in N_G(H_i) \cap \Gamma$. Hence

$$q_1 \in \{\pm q_2\}.$$ 

By our assumption on $X_n$, we must have $q_1 = q_2$, and hence the disjointness of $I'(q)$'s is established. Now suppose that there exists an element $t \in I(q_1) \cap J'(q_2)$. Then by the disjointness of $I'(q_1)$ and $I'(q_2)$, it follows that

$$t \in (I(q_1) - I'(q_1)) \cap J'(q_2).$$ 

By the definition of $I'(q_1)$, we have $x_{u_t} \notin K_{n-1}$. This contradicts the assumption that $t \in J'(q_2)$.

\[\Diamond \text{7. Completing the proof:}\] We need to check that the condition $t \notin J'(\mathcal{X})$ implies that $x_{u_t} \notin O^\circ$ where $O^\circ$ is given in (7.9).

Write the neighborhood $O^\circ$ as the disjoint union

$$O^\circ = \bigcup_{n=1}^\ell (O_n - (\bigcup_{i \leq n-1} O_i \cup O^*)).$$ 

Suppose $x_{u_t} \in O^\circ$. Let $n \leq \ell$ be such that

$$x_{u_t} \in O_{n+1} - (\bigcup_{i \leq n} O_i \cup O^*).$$ 

If $x_{u_t} \notin J'(\mathcal{X})$, then $x_{u_t} \notin K_{n-1}$, that is,

$$x_{u_t} \in \bigcup_{i=1}^n (O_i \cup O_i^*).$$ 

This is a contradiction.

\[\square\]

As $\mathcal{H}^*$ is countable and $X(H_i, U)$ is $\sigma$-compact, the intersection $\mathcal{J}(U) \cap RF M$ can be exhausted by the union of increasing sequence of $E_j \in \mathcal{E}_U$'s. Therefore, we deduce:

**Corollary 7.12.** There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ such that $\mathcal{J}(U) \cap RF M = \bigcup_{j=1}^\infty E_j$ satisfying the following: for each $j \in \mathbb{N}$, there exists an open neighborhood $O_j$ of $E_j$ such that for all $x \in RF M - E_{j+1}$,

$$\{t \in \mathbb{R} : x_{u_t} \in RF M - O_j\}$$

is a $2k$-thick set for all $j \in \mathbb{N}$.
Corollary 7.13. Assume that \( x_i \to x \) in \( RFM \) with \( x \in \mathcal{G}(U) \). Then for each \( j \in \mathbb{N} \), there exists a neighborhood \( \mathcal{O}_j \) of \( E_j \) such that
\[
\{ t \in \mathbb{R} : x_i u_t \in RFM - \mathcal{O}_j \}
\]
is 2\( k \)-thick for sufficiently large \( i \)'s.

Proof. We fix \( j \in \mathbb{N} \). Then there exists \( i_0 \in \mathbb{N} \) such that \( x_i \not\in E_{j+1} \) for all \( i \geq i_0 \). Applying Proposition 7.11 to a compact subset \( F = \{ x_i : i \geq i_0 \} \) of \( RFM \), we obtain a neighborhood \( \mathcal{O}_j \) of \( E_j \) such that
\[
\{ t \in \mathbb{R} : x_i u_t \in RFM - \mathcal{O}_j \}
\]
is 2\( k \)-thick for all \( i \geq i_0 \). \( \square \)

Indeed we will apply this corollary for the sequence \( \{ x_i \} \) contained in a closed orbit \( x_0L \) of a proper connected closed subgroup \( L < G \), which can be proved in the same way:

Theorem 7.14 (Avoidance Theorem II). Suppose that \( x_0L \) is closed for some \( x_0 \in RFM \) and a connected reductive subgroup \( L \) containing \( U \). Then there exists a sequence of compact subsets \( E_1 \subset E_2 \subset \cdots \) in \( \mathcal{E}_U \) such that \( \mathcal{G}(U, x_0L) \cap RFM = \bigcup_{j=1}^\infty E_j \), satisfying the following: if \( x_i \to x \) in \( RFM \cap x_0L \) with \( x \in \mathcal{G}(U, x_0L) \), then for each \( j \in \mathbb{N} \), there exist \( i_j \geq 1 \) and an open neighborhood \( \mathcal{O}_j \subset x_0L \) of \( E_j \) such that
\[
\{ t \in \mathbb{R} : x_i u_t \in RFM - \mathcal{O}_j \}
\]
is a 2\( k \)-thick set for all \( i \geq i_j \).

8. LIMITS OF UNIPOTENT BLOWUPS

We assume that \( M \) is a rigid hyperbolic manifold and fix \( k > 1 \) as given by Proposition 4.9.

In the whole section, we fix a connected subgroup \( U < N \). For a given sequence \( g_i \to e \), and a sequence of \( k \)-thick subsets \( T_i \) of a one-parameter unipotent subgroup \( U_0 < U \), we study the following set
\[
\limsup T_i g_i U
\]
under certain conditions on \( g_i \). The basic tool used is the so-called the quasi-regular map associated to the sequence \( g_i \) introduced in the work of Margulis-Tomanov [26] to study the object \( \limsup U_0 g_i U \) in the finite volume case. For our application, we need a somewhat precise information on the shape of the set \( \limsup T_i g_i U \) for which we also carry out some explicit matrix computations.

Let \( U^\perp \) denote the orthogonal complement of \( U \) in \( N \approx \mathbb{R}^{d-1} \) as defined in section 3. Recall from (3.1) that
\[
N(U) = AN C_1(U) C_2(U)
\]
where \( C_1(U) = C(H(U)) \) and \( C_2(U) = H(U) \cap M \cap C(U^\perp) \). Since \( N(U) \) is the identity component of \( N_G(U) \), for a sequence \( g_i \to e \), the condition
\( g_i \in N_G(U) \) means \( g_i \in N(U) \) for all sufficiently large \( i \gg 1 \). Note that \( AU^\perp C_2(U) \) is a subgroup of \( G \).

**Lemma 8.1.** For a given sequence \( g_i \to e \) in \( G - N(U) \), there exists a one-parameter subgroup \( U_0 < U \) such that the following holds: for any given sequence of \( k \)-thick subsets \( T_i \subset U_0 \), there exist \( t_i \to \infty \) in \( T_i \), and \( u_i \in U \) such that as \( i \to \infty \),

\[
u_{u_i}g_iu_{t_i} \to \alpha\]

for some nontrivial element \( \alpha \in AU^\perp C_2(U) - C_2(U) \). Moreover, \( \alpha \) can be made arbitrarily close to \( e \).

**Proof.** Set \( L := AU^\perp MN^+ \). Note that

\[
N(U) \cap L = AU^\perp C_1(U) C_2(U)
\]

and that the product map from \( U \times L \) to \( G \) is a diffeomorphism onto a Zariski open neighborhood of \( e \) in \( G \).

Following [26], we will construct a quasi-regular map

\[
\psi : U \to N(U) \cap L
\]

associated to the sequence \( g_i \).

Except for a Zariski closed subset of \( U \), the product \( g_iu \) can be written as an element of \( UL \) in a unique way. We denote by \( \psi_i : U \to L \) its \( L \)-component so that

\[
 g_iu \in U\psi_i(u).
\]

By Chevalley’s theorem, there exists an \( \mathbb{R} \)-regular representation \( G \to GL(W) \) with a distinguished point \( p \in W \) such that \( U = Stab(p) \). Then \( pG \) is locally closed, and

\[
N_G(U) = \{ g \in G : pgu = pg \text{ for all } u \in U \}.
\]

As \( U \) is a connected unipotent subgroup of \( G \), isomorphic to \( \mathbb{R}^k \) for some \( 1 \leq k \leq d - 1 \), the map \( \tilde{\phi}_i : U \to W \) defined by

\[
\tilde{\phi}_i(u) = pg_iu
\]

is a polynomial map in \( k \)-variables of degree uniformly bounded for all \( i \), and \( \tilde{\phi}_i(e) \to p \).

As \( g_i \notin N_G(U) \), \( \tilde{\phi}_i \) is non-constant. Denote by \( B(p, r) \) the ball of radius \( r \) centered at \( p \), fixing a norm \( \| \cdot \| \) on \( W \).

Since \( pG \) is open in its closure, we can find \( \lambda_0 > 0 \) such that

\[
B(p, \lambda_0) \cap \overline{pG} \subset pG.
\]

Without loss of generality, we may assume that \( \lambda_0 = 2 \) by renormalizing the norm. Now define

\[
\lambda_i := \sup\{ \lambda \geq 0 : \tilde{\phi}_i(B_U(\lambda)) \subset B(p, 2) \}
\]
where \( B_U(\lambda) \) denotes the ball of radius \( \lambda \) in \( U \simeq \mathbb{R}^k \) centered at 0. Note that \( \lambda_i < \infty \) as \( \phi_i \) is nonconstant, and \( \lambda_i \to \infty \) as \( i \to \infty \), as \( g_i \to e \). We define
\[
\phi_i : U \to W \text{ by } \phi_i(u) := \tilde{\phi}_i(\lambda_i u).
\]
This forms a sequence of equi-continuous polynomials on \( U = \mathbb{R}^k \).

Therefore, after passing to a subsequence, \( \phi_i \) converges to a non-constant polynomial \( \phi \) uniformly on every compact subset of \( U \). Moreover \( \sup \{ ||\phi(u) - p|| : u \in B_U(1) \} = 1 \), \( \phi(B_U(1)) \subset pL \), and \( \phi(0) = p \).

Now the following defines a non-constant rational map defined on a Zariski open dense neighborhood of \( \mathcal{U} \) of \( e \) in \( U \):
\[
\psi := \rho_L^{-1} \circ \phi
\]
where \( \rho_L \) is the restriction to \( L \) of the orbit map \( g \mapsto p.g \).

We have \( \psi(e) = e \) and
\[
\psi(u) = \lim_{i \to \infty} \phi_i(\lambda_i u)
\]
where the convergence is uniform on compact subsets of \( \mathcal{U} \) and
\[
\psi(u) \in L \cap NC(U) = AU \perp C_1(U) C_2(U).
\]

Since \( \psi \) is non-constant, there exists a one-parameter subgroup \( U_0 < U \) such that \( \psi|_{U_0} \) is non-constant. Now let \( T_i \) be a sequence of \( k \)-thick sets in \( U_0 \simeq \mathbb{R} \). Then \( T_i/\lambda_i \) is also a \( k \)-thick set, and so is
\[
T_\infty := \lim_{i \to \infty} \sup \{ T_i/\lambda_i \} \subset U_0.
\]

Finally, for all \( t \in T_\infty \), there exists a sequence \( t_i \in T_i \) such that \( t_i/\lambda_i \to t \) as \( i \to \infty \) (by passing to a subsequence). Since \( \psi \circ \lambda_i \to \psi \) uniform on compact subsets,
\[
\psi(t) = \lim_{i \to \infty} \phi_i(\lambda_i) (t_i/\lambda_i) = \lim_{i \to \infty} \phi_i(t_i).
\]

By the definition of \( \psi_i \), this means that there exists \( u_i \in U \) such that
\[
\psi(t) = \lim_{i \to \infty} u_i g_i u_{t_i}.
\]

Since \( \psi|_{U_0} \) is non-constant, \( \psi \) is continuous, and an uncountable set \( T_\infty \) accumulates on \( 0 \), the image \( \psi(T_\infty) \) contains a non-trivial element \( \alpha \) of \( AU \perp C_1(U) C_2(U) \) which can be taken arbitrarily close to \( e \).

We now claim that if \( \alpha \) is sufficiently close to \( e \), then it belongs to \( AU \perp C_2(U) \). Consider \( H'(U) := H(U) C_1(U) \), and let \( \mathfrak{h} \) denote its Lie algebra.

Now for all \( i \) large enough, using the decomposition \( g = \mathfrak{h} \oplus \mathfrak{h}^\perp \) in (3.3), we can write \( g_i = c_i d_i r_i \) where \( c_i \in C_1(U) \), \( d_i \in H(U) \) and \( r_i \in \exp(\mathfrak{h}^\perp) \).

Now since \( c_i \) commutes with \( U \), we can write
\[
u_i g_i u_{t_i} = (u_i u_{t_i}) c_i (u_i^{-1} d_i u_{t_i}) (u_i^{-1} r_i u_{t_i}).
On the other hand, we have
\[ \lim_i pu_i g_i u_{t_i} = \lim_i pc_i (u_{t_i}^{-1} d_i u_{t_i}^{-1}) (u_{t_i}^{-1} r_i u_{t_i}) = p\alpha. \]

Since \( c_i \to e, u_{t_i} d_i u_{t_i}^{-1} \in H(U), \) and \( u_{t_i} r_i u_{t_i}^{-1} \in \exp(h^\perp), \) it follows that both sequences \( u_{t_i} d_i u_{t_i}^{-1} \) and \( u_{t_i} r_i u_{t_i}^{-1} \) must converge, say to \( h \in H(U) \) and \( q \in \exp(h^\perp), \) respectively.

Hence \( \alpha = hq \) by replacing \( h \) by \( uh \) for some \( u \in U. \) On the other hand, we can write \( \alpha = avc_1 c_2 \in AU^\perp C_1(U) C_2(U). \)

So \( hq = avc_1 c_2. \) Note that \( c := c_1 c_2 \in C(H(U)) H(U) = H'(U). \) We get

\[ (a^{-1} h c^{-1})(c q c^{-1}) = v. \]

Now, when \( \alpha \) is sufficiently close to \( e, \) all elements appearing in (8.2) are also close to \( e. \)

Note that \( H'(U) \times h^\perp \to G \) given by \( (h', X) \to h' \exp X \) is a local diffeomorphism onto a neighborhood of \( e. \) Since \( (a^{-1} h c^{-1}) \in H'(U), \) and \( c c^{-1}, v \in \exp h^\perp, \) we have \( a^{-1} h c^{-1} = e \) and \( c c^{-1} = v \) for \( \alpha \) sufficiently small. In particular,

\[ a^{-1} h c_2^{-1} = c_1^{-1} \in H(U) \cap C(H(U)) = \{ e \}. \]

Hence \( c_1 = e. \)

It follows that \( \alpha \in AU^\perp C_2(U), \) as desired.

We further claim that we can choose \( \alpha \) outside of \( C_2(U). \) As \( C_2(U) \) is a compact subgroup, we can choose a \( C_2(U) \)-invariant Euclidean norm \( \| \cdot \| \) on \( W. \) If \( \alpha = \psi(t) \in C_2(U) \) for some \( t \in T_\infty \subset U_0, \) then \( t \) is one of the at most two solutions of the quadratic equation \( \| \psi(t) \|^2 = \| p \|^2. \) Therefore, except for finitely many \( t \in T_\infty, \) \( \alpha \in AU^\perp C_2(U) - C_2(U). \) This finishes the proof. \( \square \)

The following lemma is similar to Lemma 8.1, but here we consider the case when \( U \) is the whole horospherical subgroup \( N. \) In this restrictive case, the limiting element can be taken inside \( A. \)

**Lemma 8.2.** Let \( T_i \subset N \) be a sequence of \( k \)-thick subsets in the sense that for any one-parameter unipotent subgroup \( U_0 < N, T_i \cap U_0 \) is a \( k \)-thick subset of \( U_0 \simeq \mathbb{R}. \) For any sequence \( g_i \to e \) in \( G/N_G(N), \) there exists \( t_i \to \infty \) in \( T_i \) and \( u_i \in N \) such that

\[ u_i g_i u_{t_i} \to a \]

for some nontrivial element \( a \in A. \) Moreover, \( a \) can be chosen to be arbitrarily close to \( e. \)

**Proof.** We first consider the case when \( g_i \) belongs to the opposite horospherical subgroup \( N^+. \) We will use the notations \( u^+ \) and \( u^- \) defined in Section 3. Write \( g_i = \exp(u^+(w_i)) \) for some \( w_i \in \mathbb{R}^{d-1}. \) For \( x \in \mathbb{R}^{d-1}, \) set \( u_x := \exp(u^-(x)) \in N. \)
Let $\varepsilon > 0$ be arbitrary. Since $T_i$ is a $k$-thick subset of $N$, there exists $\alpha_i \in \mathbb{R}$ such that

$$\varepsilon < |\alpha_i| : \|w_i\|^2 < k\varepsilon.$$ 

Setting $u_{x_i} := u_{\alpha_i} w_i \in T_i$ and $y_i := -\alpha_i w_i \left(1 - \frac{\|w_i\|^2 \alpha_i}{2}\right)^{-1}$, we compute:

$$u_{y_i} g_i u_{x_i} = \begin{pmatrix} \left(1 - \frac{\alpha_i \|w_i\|^2}{2}\right)^{-2} & 0 & 0 \\ w_i \left(1 - \frac{\alpha_i \|w_i\|^2}{2}\right)^{-1} & I_{d-1} & 0 \\ -\frac{\|w_i\|^2}{2} & -w_i \left(1 - \frac{\alpha_i \|w_i\|^2}{2}\right) & \left(1 - \frac{\alpha_i \|w_i\|^2}{2}\right)^2 \end{pmatrix}$$

The condition for the size of $\alpha_i$ guarantees that, by passing to a subsequence, the sequence $u_{x_i} g_i u_{y_i}$ converges to an element

$$a = \text{diag}(\alpha, I_{d-1}, \alpha^{-1}) \in A,$$

for $\alpha \in \left[\frac{1}{(1-k\varepsilon)^2}, \frac{1}{(1-k\varepsilon)^2}\right] \cup \left[\frac{1}{(1+k\varepsilon)^2}, \frac{1}{(1+k\varepsilon)^2}\right]$ as $i \to \infty$. This proves the claim when $g_i \in N^+$.

Since the product map $A \times M \times N^+ \times N^- \to G$ is a diffeomorphism onto a Zariski-open neighborhood of $e$ in $G$, we can write $g_i = a_i m_i u_i^+ u_i^-$ for some $a_i \in A$, $m_i \in M$, $u_i^+ \in N^+$ and $u_i^- \in N$ all of which converge to $e$ as $i \to \infty$.

Set $q_i := (a_i m_i) u_i^+ (a_i m_i)^{-1} \in N^+$. By the previous case, we can find $u_{t_i} \in T_i$ and $u_i \in N$ such that $u_{t_i} g_i u_i$ converges to a non-trivial element $a \in A$ while $u_i^{-1}(a_i m_i) u_i (a_i m_i)^{-1} \to e$; this is possible since the adjoint action of $a_i m_i$ on $N \simeq \mathbb{R}^{d-1}$ is a rotation by dilation, which maps a $k$-thick subset to a $k$-thick subset.

Now as $i \to \infty$,

$$u_{t_i} g_i u_i = (u_{t_i} g_i u_i)(u_i^{-1} a_i m_i u_i^{-1} u_i)$$

$$= (u_{t_i} g_i u_i)(u_i^{-1} a_i m_i u_i (a_i m_i)^{-1}) a_i m_i u_i^{-1} \to a,$$

proving the claim.

\[\square\]

**Lemma 8.3.** Let $L$ be any connected reductive subgroup of $G$ normalized by $A$ and $L \cap N$ non-trivial. Let $U_0 < L \cap N$ be a one-parameter unipotent subgroup of $L$. Set $l = \text{Lie}(L)$, and denote by $l^\perp$ the $\text{Ad}(L)$-invariant complementary subspace of $g$. For a given sequence $r_i \to e$ in $\exp(l^\perp) - N(U_0)$ and for any given sequence of $k$-thick subsets $T_i \subset U_0$, there exists $t_i \to \infty$ in $T_i$ such that

$$u_{t_i}^{-1} r_i u_{t_i} \to v$$

for some non-trivial element $v \in (L \cap N)^\perp$, and $v$ can be chosen arbitrarily close to $e$. Moreover, for $n \gg 1$, there exists $v_n \in (L \cap N)^\perp$ such that

$$n \leq \|v_n\| \leq c_k n$$

where $c_k$ is a constant depending only on $k$ and $G$. 
Proof. Without loss of generality, by Proposition 3.2, we may assume that \( L_{nc} = H(U) \) for \( U = U_k = \mathbb{R}^k \) some \( k \geq 1 \) and \( U_0 := \mathbb{R}e_1 \). We write \( r_i = \exp(q_i) \) where \( q_i \to 0 \) in \( l^1 \).

Using the notations introduced in section 3 and setting \( u^\perp = \text{Lie}(U^\perp) = \mathbb{R}^{d-1-k} \), let us write \( q_i \) is of the form

\[
q_i = u^-(x_i) + u^+(y_i) + m(C_i)
\]

where \( x_i \in u^\perp, y_i \in (u^\perp)^t \), and \( C_i = \begin{pmatrix} 1_k & B_i \\ -B_i^t & A_i \end{pmatrix} \) is a skew symmetric matrix, all of which converge to 0 as \( i \to \infty \). We consider \( U_0 \) as \( \{u_s = se_1 \in \mathbb{R}^{d-1}\} \) and define the map \( \psi_i : \mathbb{R} \to l^1 \) by

\[
\psi_i(s) = u_s^{-1} q_i u_s \quad \text{for all} \quad s \in \mathbb{R}
\]

this is well-defined since \( l^1 \) is \( \text{Ad}(L) \)-invariant. Then a direct computation shows

\[(8.3) \quad \psi_i(s) = u^- (x_i + sB_t e_1 + s^2 y_i/2) + u^+(y_i) + m(C_i)\]

where \( C_i \) is a skew-symmetric matrix of the form

\[
C_i = \begin{pmatrix} 0_k & B_i \\ -B_i^t - s y_i e_1 \end{pmatrix}.
\]

Since \( r_i \notin \mathbb{N}(U_0) \), it follows that either \( y_i \neq 0 \) or \( y_i = 0 \) and \( B_i \neq 0 \). Hence \( \psi_i \) is a non-constant polynomial of degree at most 2, and \( \psi_i(0) \to 0 \).

Let \( \lambda_i \in \mathbb{R} \) be defined by

\[
\lambda_i = \sup \{ \lambda > 0 : \psi_i[-\lambda, \lambda] \subseteq \mathbb{R} \}.
\]

Then \( \lambda_i < \infty \) and \( \lambda_i \to \infty \).

Now the rescaled polynomials \( \phi_i = \psi_i \circ \lambda_i : \mathbb{R} \to l^1 \) form an equicontinuous family of polynomials of degree at most 2 and \( \phi_i(0) \to 0 \). Therefore \( \phi_i \) converges to a non-constant polynomial \( \phi : \mathbb{R} \to l^1 \) uniformly on compact subsets. From (8.3), it can be seen easily that the image \( \text{Im}(\phi) \) is contained \( \text{Lie}(N) \cap l^1 \), by considering the two cases of \( y_i \neq 0 \), and \( y_i = 0 \) and \( B_i \neq 0 \) separately.

Now let \( T_i \) be a sequence of \( k \)-thick subsets of \( U_0 \). Let

\[
T_\infty = \lim \sup_{i \to \infty} (T_i / \lambda_i),
\]

which is also a \( k \)-thick subset of \( U_0 \).

Let \( s \in T_\infty \). By passing to a subsequence, there exists \( t_i \in T_i \) such that \( t_i / \lambda_i \to s \) as \( i \to \infty \). As \( \phi_i \to \phi \) uniformly on compact subsets, it follows that

\[
\phi(s) = \lim_{i \to \infty} \psi_i(\lambda_i \cdot t_i / \lambda_i) = \lim_{i \to \infty} u_i^{-1} q_i u_i.
\]

Since \( T_\infty \) accumulates on 0, so does \( \phi(T_\infty) \). Taking the exponential map to each side of the above, the first part of the lemma follows.

The second part of the lemma holds by applying the following fact for degree-4 polynomial \( p(s) = \|\phi(s)\|^2 \):
If \( p \in \mathbb{R}[s] \) is of degree \( \delta \) with \( p(0) = 0 \), and \( T \subset \mathbb{R} \) is a \( k \)-thick set, then \( p(T) \) is \( 2k\delta \)-thick at \( \infty \); see Definition 4.8 for the definition of the \( k \)-thickness at \( \infty \).

By a one-parameter semigroup, we mean a set of the form \( \{ \exp(t\xi) : t \geq 0 \} \) for some \( \xi \) in the Lie algebra of \( G \). Note that the product \( AU^\perp C_2(U) \) is a connected subgroup of \( G \), since \( C_2(U) \) commutes with \( U^\perp \), and \( A \) normalizes \( U^\perp C_2(U) \).

Lemma 8.4. An unbounded one-parameter semigroup \( L \) of \( AU^\perp C_2(U) \) is one of the following form:

\[
\begin{align*}
\{ \exp(t\xi_A) \exp(t\xi_C) & : t \geq 0 \}; \\
\{ (v \exp(t\xi_A)v^{-1}) \exp(t\xi_C) : t \geq 0 \}; \\
\{ \exp(t\xi_V) \exp(t\xi_C) : t \geq 0 \}
\end{align*}
\]

for some \( \xi_A \in \text{Lie}(A) - \{0\}, \xi_C \in \text{Lie}(C_2(U)), v \in U^\perp - \{e\}, \) and \( \xi_V \in \text{Lie}(U^\perp) - \{0\} \).

Proof. By definition, \( L = \{ \exp(t\xi) : t \geq 0 \} \) for some \( \xi \in \text{Lie}(AU^\perp C_2(U)) \). Write \( \xi = \xi_0 + \xi_C \) where \( \xi_0 \in \text{Lie}(AU^\perp) \) and \( \xi_C \in \text{Lie}(C_2(U)) \). Since \( AU^\perp \) commutes with \( C_2(U) \), \( \exp(t\xi) = \exp(t\xi_0) \exp(t\xi_C) \). Hence we only need to show that either \( \xi_0 \in \text{Lie}(U^\perp) \) or

\[
\{ \exp(t\xi_0) : t \geq 0 \} = \{ v \exp(t\xi_A)v^{-1} : t \geq 0 \}
\]

for some \( v \in U^\perp \) and \( \xi_A \in \text{Lie}(A) \). Now if \( \xi_0 \notin \text{Lie}(U^\perp) \), then writing

\[
\xi_0 = \begin{pmatrix} a & x & 0 \\ 0 & 0 & -x \\ 0 & 0 & -a \end{pmatrix} \in \text{Lie}(AU^\perp)
\]

with \( a \neq 0 \), a direct computation shows that \( \xi_0 = v\xi_Av^{-1} \) for the choice of

\[
v = \begin{pmatrix} 0 & x/a & 0 \\ 0 & 0 & -x/a \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \xi_A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix},
\]

proving (8.4). □

9. Translates of relative \( U \)-minimal sets

In the rest of the whole paper, we let \( M \) be a rigid hyperbolic manifold.

In this section, we fix a compact \( A \)-invariant subset \( R \subset RF M \) such that for every \( x \in R \), and for any one-parameter subgroup \( U_0 = \{u_t\} \) of \( U \),

\[
\{ t \in \mathbb{R} : xu_t \in R \}
\]

is \( k \)-thick. In applying the results of this section later, \( R \) will be either \( RF M \) or of the form \( RF M \cap F^* \cap X \) (when it is compact) for a closed \( H(U) \)-invariant subset \( X \).
The main goal of this section is to study the set \( \text{lim sup} Yg_t U \) for a relative \( U \)-minimal subset \( Y \) with respect to \( R \) and for a sequence \( g_i \to e \) under certain hypothesis, using the analysis of \( \text{lim sup} T_i g_i U \) in section 8. For instance, if \( g_i \) is so that \( y_0g_i \subset Y \).

**Definition 9.1.**

- A subset \( Y \subset \Gamma \setminus G \) is \( U \)-minimal if \( yU \) is dense in \( Y \) for any \( y \in Y \).
- A subset \( Y \subset \Gamma \setminus G \) is \( U \)-minimal with respect to \( R \) if \( Y \cap R \neq \emptyset \) and for any \( y \in Y \cap R \), \( yU \) is dense in \( Y \).

A \( U \)-minimal set may not exist, but a \( U \)-minimal set with respect to a compact subset \( R \) always exists by Zorn’s lemma.

In this section, we study how to find an additional invariance of \( Y \) beyond \( U \) under certain conditions.

**Lemma 9.2.** Let \( Y \) be a \( U \)-minimal set of \( \Gamma \setminus G \) with respect to \( R \), and \( L \) be a closed subgroup of \( N(U) \) containing \( U \). Then there does not exist a locally closed \( L \)-orbit through a point in \( Y \cap R \).

**Proof.** Assume to the contrary that \( y_0L \) is locally closed for some \( y_0 \in Y \cap R \). Since \( Y \) is \( U \)-minimal with respect to \( R \), there exists \( u_n \to \infty \) in \( U \) such that \( y_0u_n \to y_0 \) by [4, Lemma 8.2]. Since \( y_0L \) is locally closed, \( y_0L \) is homeomorphic to \( (L \cap \Gamma) \setminus L \). Assuming \( y_0 = [e] \) without loss of generality, there exists \( \delta_n u_n \to e \) as \( n \to \infty \).

Since \( N(U) = ANC_1(U) C_2(U) \), writing \( \delta_n = a_n r_n \) for \( a_n \in A \) and \( r_n \in NC_1(U) C_2(U) \), this cannot hold unless \( a_n \to e \). On the other hand, note that \( a_n \) is nontrivial as \( \Gamma \) contains no elliptic nor parabolic. This is a contradiction, as there exists a positive lower bound for the translation length of \( \Gamma \). \( \square \)

In the rest of this section, we use the following notation:

\[ H = H(U), \quad H' = H'(U), \quad \text{and} \quad F^* = F^*_{H(U)}. \]

**Corollary 9.3.** For every \( U \)-minimal set \( Y \) with respect to \( RF M \) such that \( Y \cap RF M \cap F^* \neq \emptyset \), and for any \( y_0 \in Y \cap RF M \cap F^* \), there exists \( g_n \to e \) in \( G \setminus N(U) \) such that \( y_0 g_n \in Y \cap RF M \).

**Proof.** Define

\[ S_0 = \{ g \in N(U) : Y g \subset Y \}; \]
\[ S = \{ g \in N(U) : Y g = Y \}. \]

Observe that \( S = \{ g \in S_0 : Y g \cap R \neq \emptyset \} \). Pick \( y_0 \in Y \cap R \cap F^* \). Then \( S_0 = \{ g \in N(U) : y_0 g \in Y \} \) by the relative \( U \)-minimality, and \( S \) is a closed subgroup contained in a semigroup \( S_0 \).

There exists an open neighborhood \( O \) such that for all \( y_0 g \in y_0 O \cap Y \), we have \( y_0 g U \cap RF M \neq \emptyset \) using Lemma 4.5 and the fact that \( F^* \) is open, and \( Y \subset RF_+ M \).
It is now sufficient to show that there is no open neighborhood $O'$ of $e$ such that

$$y_0O' \cap Y \subset y_0N(U).$$

By contradiction, let $O'$ be such an open neighborhood.

By shrinking $O'$, we may assume $O' \subset O$. This implies that $y_0O' \cap Y \subset y_0S$, by the hypothesis on $y_0$ and $O$. Therefore $y_0S$ is open in $Y$.

On the other hand, since $U \subset S$, we get $Y = \overline{y_0S}$. Therefore, $y_0S$ is locally closed. This contradicts Lemma 9.2, as there exists no such locally closed $S$-orbit.

\begin{proposition}[Translate of $Y$ inside of $Y$] Let $Y$ be a $U$-minimal set of $\Gamma \setminus G$ with respect to RF $M$ such that $Y \cap RF M \cap F^* \neq \emptyset$.

Then there exists an unbounded one-parameter subsemigroup $L$ inside the subgroup $AU^\perp C_2(U)$ such that

$$YL \subset Y.$$  

\end{proposition}

\begin{proof}
Choose $y_0 \in Y \cap R \cap F^*$. By Corollary 9.3, there exists $g_i \to e$ in $G - N(U)$ such that $y_0g_i \in Y \cap RF M$. Let $U_0 = \{u_t\}$ be a one-parameter subgroup of $U$ as given by Lemma 8.1, with respect to the sequence $g_i$.

Let

$$T_i := \{u_t \in U_0 : y_0g_iu_t \in Y \cap RF M\}$$

which is a $k$-thick subset of $U_0$.

Hence by Lemma 8.1, there exists sequences $t_i \to \infty$ in $T_i$, and $u_i \in U$ such that

$$\lim_{i \to \infty} u_ig_iu_{t_i} = \alpha$$

for some nontrivial $\alpha \in AU^\perp C_2(U) - C_2(U)$. Moreover $\alpha$ can be made arbitrarily close to $e$ in Lemma 8.1. Note that $y_0g_iu_{t_i}$ converges to some $y_1 \in Y \cap RF M$ by passing to a subsequence. Hence as $i \to \infty$,

$$y_0u_i^{-1} = y_0g_iu_{t_i}(u_ig_iu_{t_i})^{-1} \to y_1\alpha^{-1}.$$  

So $y_1\alpha^{-1} \in Y$, and hence $Y\alpha^{-1} \subset Y$, since $y_1 \in Y \cap RF M$. Hence the claim follows. \hfill \Box

\begin{proposition}[Translate of $Y$ inside of $X$] Let $X$ be a closed $H'$-invariant set such that $X \cap R \neq \emptyset$. Let $Y \subset X$ be a $U$-minimal subset with respect to $R$, and assume there exists $y \in Y \cap R$, $g_n \to e$ in $G - H'$ such that $yg_n \in X$. Then there exists some nontrivial $v_0 \in U^\perp$ such that

$$Yv_0 \subset X.$$  

\end{proposition}

\begin{proof}
Let $h$ denote the Lie algebra of $H'$. We may write $g_n = r_nh_n$ where $h_n \in H'$ and $r_n = \exp g_n$ with $g_n \in h^\perp$. By replacing $g_n$ with $g_nh_n^{-1}$, we may assume $g_n = r_n \neq e$. If $r_n \in U^\perp$ for some $n$, then the claim follows from $y_0r_n \in X$ and hence $Yr_n \subset X$. Hence we assume that $r_n \notin U^\perp$ for all $n$. We have from (3.3)

$$h^\perp \cap \operatorname{Lie}(N(U)) = \operatorname{Lie}U^\perp.$$  


Hence \( r_n \notin N(U) \) for all \( n \). Therefore there exists a one-parameter subgroup \( U_0 = \{ u_t \} < U \) such that \( r_n \notin N(U) \). Let
\[
T = \{ t \in \mathbb{R} : yu_t \in R \}.
\]
Since \( y \in R \), it follows that \( T \) is a \( k \)-thick subset of \( U_0 \) by the assumption on \( R \). Hence, by Lemma 8.3, there exists \( t_n \in T \) such that \( u_{t_n}^{-1} r_n u_{t_n} \to v \) for some \( v \in U^\perp \). Observe
\[
(yu_{t_n})(u_{t_n}^{-1} r_n u_{t_n}) = yr_n u_{t_n} \in X.
\]
Passing to a subsequence, \( yu_{t_n} \to y_0 \) for some \( y_0 \in Y \cap R \), and hence \( y_0 v \in X \). It follows \( Y v \subset X \).

Recall that \( F^* \) is equal to the set of all frames whose \( H \)-orbit passes through the interior of core \( M \) in the projection. For a one-parameter subgroup \( V = \{ v_t : t \in \mathbb{R} \} \) and a subset \( I \subset \mathbb{R} \), the notation \( V_I \) means the subset \( \{ v_t : t \in I \} \).

**Lemma 9.6.** Let \( X \) be a closed \( AU \)-invariant set of \( \Gamma \backslash G \), and \( V \) be a one-parameter subgroup of \( U^\perp \). Assume that \( R := X \cap RF M \cap F^* \) is nonempty and compact. If \( x_0 V_I \subset X \) for some \( x_0 \in R \) and an interval \( I \) containing \( 0 \), then \( X \) contains a \( V \)-orbit of a point in \( R \).

**Proof.** Choose a sequence \( a_n \in A \) such that \( \text{lim sup} \, a_n V_I a_n^{-1} \) contains a semigroup \( V^+ \) of \( V \) as \( n \to \infty \). Then
\[
(x_0 a_n^{-1})(a_n V_I a_n^{-1}) = x_0 V_I a_n^{-1} \subset X.
\]
Since \( R \) is compact, so is \( x_0 \overline{A} \cap F^* \), and \( x_0 a_n^{-1} \to x_1 \) as \( n \to \infty \) for some \( x_1 \in x_0 \overline{A} \cap F^* \). Hence \( x_1 V^+ \subset X \).

Since \( x_1 \) belongs to the open set \( F^* \), it follows \( x_1 v_s \in F^* \) for all sufficiently small \( s \in \mathbb{R} \). By Lemma 4.5, this implies that \( x_1 v_s U \cap RF M \neq \emptyset \) for some \( s > 0 \) with \( v_s \in V^+ \). Note that
\[
(x_1 v_s U)(v_s^{-1} V^+) = x_1 V^+ \subset X.
\]
Let \( x_2 \in x_1 v_s U \cap RF M \). Then \( x_2 \in X \), since \( x_1 v_s U \subset X \). Moreover, observe that \( x_2 \in X \cap RF M \cap F^* \) and \( x_2 (v_s^{-1} V^+) \subset X \). Similarly as before, let \( a_n \in A \) be a sequence such that \( a_n (v_s^{-1} V^+) a_n^{-1} \to V \) as \( n \to \infty \). From
\[
(x_2 a_n^{-1})(a_n v_s^{-1} V^+ a_n^{-1}) = x_2 v_s^{-1} V^+ a_n^{-1} \subset X,
\]
we conclude that
\[
x_3 V \subset X
\]
where \( x_3 \) is a limit point of \( x_2 a_n^{-1} \in R \). This finishes the proof. \( \square \)

**Proposition 9.7.** Let \( X \) be a closed \( H' \)-invariant set. Assume \( R = X \cap F^* \cap RF M \) is a nonempty compact set, and let \( Y \subset X \) be a \( U \)-minimal set with respect to \( R \). Assume that there exist \( y \in Y \cap R \) and a sequence \( g_n \to e \) in \( G \) such that \( y g_n \in X \). Then
\[
z V \subset X
\]
for some \( z \in R \) and for some nontrivial connected subgroup \( V < U^\perp \).
Proof. By Lemma 9.6, it suffices to find \( x_0 \in R \), a one-parameter subgroup \( V < U^\perp \) and \( V_I \subset U^\perp \) for some interval \( I < V \) containing 0 such that \( x_0 V_I \subset X \). It follows from Proposition 9.5 that there exists \( v_0 \in U^\perp \) such that \( Y v_0 \subset X \). By Proposition 9.4, there exists an unbounded one-parameter subsemigroup \( L \) inside the subgroup \( A U^\perp C_2(U) \) such that \( Y L \subset Y \). By Lemma 8.4, \( L \) is either of the form

\[
(1) \quad L = \{ (v \exp(t\xi_A)v^{-1}) \exp(t\xi_C) : t \in \mathbb{R} \},
\]

\[
(2) \quad L = \{ \exp(t\xi_V) \exp(t\xi_C) : t \in \mathbb{R} \}
\]

for some \( \xi_A \in \text{Lie}(A) - \{0\}, \xi_C \in \text{Lie}(C_2(U)), \xi_V \in \text{Lie}(V) - \{0\}, \) and \( v \in U^\perp \).

Since \( X \) is \( H'(U) \)-invariant, we may assume \( Y L \subset X \) with \( \xi_C = 0 \).

By Lemma 9.6, it suffices to show that there exists a one-parameter subgroup \( V < U^\perp \) such that \( Y V_I \subset X \) for some interval \( I \) containing 0. In the case (1), the claim is immediate. Hence we assume we are in the case (2). First suppose \( v = e \). Then we have \( Y A v_0 = Y v_0 (v_0^{-1} Av_0) = Y v_0 \subset X \). Then by the \( A \)-invariance of \( X \), we have \( Y (Av_0 A) \subset X \). Since \( Av_0 A \) contains some \( V^+, Y V_I \subset X \) for some interval \( I \) containing 0 as desired.

Next suppose \( v \neq e \). Then since \( YA \subset X \),

\[
Y (v^{-1}A^+ v) A \subset YA \subset X.
\]

We will take \( V = \langle v \rangle \). Now note that \( (v^{-1}A^+ v) A \) contains \( V_I \) for some interval \( I \) containing 0 for any subsemigroup \( A^+ \) of \( A \). Hence \( Y V_I \subset X \) also holds for this case. \( \square \)

10. Density of almost all \( U \)-orbits

Let \( \Gamma < G = \text{SO}^o(d, 1) \) be a Zariski dense convex cocompact subgroup. The action of \( N \) on \( \text{RF}_+ M \) is minimal, and hence any \( N \)-orbit is dense in \( \text{RF}_+ M \) [48]. Given a non-trivial connected subgroup \( U \) of \( N \), there always exist a dense \( U \)-orbit in \( \text{RF}_+ M \) [31]. In this section, we deduce from [32] and [31] that almost all \( U \)-orbit is dense in \( \text{RF}_+ M \) with respect to the Burger-Roblin measure in the rigid hyperbolic manifold case (Corollary 10.4).

The critical exponent \( \delta = \delta_{\Gamma} \) of \( \Gamma \) is defined to be the infimum \( s \) such that the Poincare series \( \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))} \) converges for any \( o \in \mathbb{H}^d \). It is known that \( \delta \) is equal to the Hausdorff dimension of the limit set \( \Lambda \) and \( \delta = d - 1 \) if and only if \( \Gamma \) is a lattice in \( G \) [45].

Denote by \( m^{BR} \) the \( N \)-invariant Burger-Roblin measure supported on \( \text{RF}_+ M \); it is characterized as a unique locally finite Borel measure supported on \( \text{RF}_+ M \) (up to a scaling) by ([6], [40], [48]). We won't give an explicit formula of this measure as we will only use the fact that its support is equal to \( \text{RF}_+ M \), together with the following theorem: recall that a locally finite \( U \)-invariant measure \( \mu \) is ergodic if every \( U \)-invariant measurable subset has either zero measure or zero co-measure, and is conservative if for any measurable subset \( S \) with positive measure, \( \int_U 1_S(xu)du = \infty \) for almost all \( x \), where \( du \) denotes the Haar measure on \( U \).
Theorem 10.1 ([32], [31]). Let $U < N$ be a connected subgroup, and let $\Gamma$ be a convex cocompact Zariski dense subgroup of $G$. Then $m^{BR}$ is $U$-ergodic and conservative if $\delta > \text{co-dim}_N(U)$.

Lemma 10.2. Suppose that $\Gamma_1 < \Gamma_2$ are geometrically finite groups of $G$ with $[\Gamma_1 : \Gamma_2] = \infty$. Then $\delta_{\Gamma_1} < \delta_{\Gamma_2}$.

Proof. Note that a geometrically finite group is of divergent type. Hence the claim follows from Proposition 2 of [8] if we check that $\Lambda_{\Gamma_1} \neq \Lambda_{\Gamma_2}$.

If $\Lambda := \Lambda_{\Gamma_1} = \Lambda_{\Gamma_2}$, then their convex hulls are the same, and hence the convex core of $\Gamma_i \setminus \mathbb{H}^d$ is $\Gamma_i \setminus \text{hull}(\Lambda)$, which has finite volume since $\Gamma_i$ are geometrically finite. Since we have a covering map $\Gamma_1 \setminus \text{hull}(\Lambda) \to \Gamma_2 \setminus \text{hull}(\Lambda)$, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. □

Lemma 10.3. If $\Gamma \setminus \mathbb{H}^d$ is a rigid hyperbolic manifold, then $\delta > d - 2$.

Proof. If $\Gamma$ is a lattice, then $\Lambda = S^{d-1}$ and $\delta = d - 1$. If $\Gamma \setminus \mathbb{H}^d$ is rigid with infinite volume, then $\Gamma$ contains a cocompact lattice $\Gamma_0$ in a conjugate of $SO(d-1,1)$ whose limit set is $\partial B_i$ for some $i$. Now $[\Gamma : \Gamma_0] = \infty$; otherwise, $\Lambda = \partial B_i$. Hence $\delta > \delta_{\Gamma_0} = d - 2$ by Lemma 10.2. □

Corollary 10.4. Let $M = \Gamma \setminus \mathbb{H}^d$ be a rigid hyperbolic manifold. Let $U < N$ be any non-trivial connected subgroup. Then for $m^{BR}$-almost every $x \in RF_+ M$,

$$\overline{uU} = RF_+ M.$$  

Proof. It suffices to prove this when $U = \{u_t\}$ is a one-parameter subgroup. By Lemma 10.3 and Theorem 10.1, $m^{BR}$ is $U$-ergodic and conservative. Since $\delta > (d - 1)/2$, there exists a unique positive function $\phi_0 \in L^2(M)$ which is an eigenfunction for the Laplace operator with eigenvalue $\delta(d - 1 - \delta)$, up to a scalar multiple [45]. We may regard $\phi_0$ as a function on $L^2(\Gamma \setminus G)$ which is $K$-invariant. Then $m^{BR}(\phi_0) = ||\phi_0||^2 < \infty$ (cf. [17, Lem 6.7]). Hence, applying the Hopf ratio theorem [1] we get that for almost all $x \in RF_+ M$ and for any continuous function $f \in C(RF_+ M)$ with compact support,

$$\lim_{T \to \infty} \frac{\int_0^T f(xu_t)dt}{\int_0^T \phi_0(xu_t)dt} = \frac{m^{BR}(f)}{||\phi_0||^2}.$$  

Therefore almost all $U$-orbits are dense in $RF_+ M$. □

11. Horospherical action in the presence of a compact factor

Throughout this section, let $M = \Gamma \setminus \mathbb{H}^d$ be a rigid hyperbolic manifold and fix a non-trivial connected subgroup $U$ of $N$. Consider a closed orbit $xL$ for $x \in RF_M$ where $L = H(U)C \in L_U$. The subgroup $U$ is a horospherical subgroup of $H(U)$, which is known to act minimally on $xL \cap RF_+ M$ provided the compact part $C$ is trivial. In this section, we extend the $U$-minimality on $xL$ in the case of a non-trivial compact factor $C$. 
As before, we set
\[ H = H(U), \quad H' = H'(U), \quad \text{and} \quad F^* = F^*_{H(U)}. \]

We let
\[ \pi_1 : H' \to H, \quad \text{and} \quad \pi_2 : H' \to C(H) \]
denote the projections of \( H' = HC(H) \) to \( H \) and to \( C(H) \) respectively.

In the case when \( C = \{e\} \), this following theorem follows from [48] and Corollary 10.4.

**Theorem 11.1.** Let \( X := xL \) be a closed orbit where \( x \in RF M \), and \( L := HC \in L_U \). Then the following holds:

1. \( X \cap RF_+ M \) is \( U \)-minimal.
2. \( X \) is \( H \)-minimal.
3. \( X \cap RF M \) contains a dense \( A \)-orbit.
4. For any non-trivial connected subgroup \( U_0 < U \), for \( m_X^{BR} \)-almost all \( x \in X \),
   \[ xU_0 = X \cap RF_+ M. \]

In order to define \( m_X^{BR} \), choose \( g \in G \) so that \([g] = x\). If we identify \( H \simeq SO^0(k, 1) \), then by Proposition 4.7, \( S := \pi_1(g^{-1} \Gamma g \cap HC) \setminus \mathbb{H}^k \) is a rigid hyperbolic manifold. Now \( \pi_1(g^{-1} \Gamma g \cap HC) \setminus H \) is the frame bundle of \( S \), on which there exists the Burger-Roblin measure as discussed in section 10. In the above statement, the notation \( m_X^{BR} \) means the \( C \)-invariant lift of this measure to \( X = xHC \).

We begin by proving the following proposition.

**Proposition 11.2.** Let \( X = xHC \) be a closed orbit for \( x \in RF M \) and for some closed subgroup \( C < C(H) \). Any \( U \)-minimal set \( Y \) of \( X \) with respect to \( RF M \) such that \( Y \cap RF M \cap F^* \neq \emptyset \) is \( A \)-invariant.

**Proof.** Let \( Y \) be a \( U \)-minimal set of \( X \) with respect to \( RF M \). Let \( y_0 \in Y \cap RF M \cap F^* \). By Corollary 9.3, there exists a sequence \( g_i \to e \in HC - N(U) \) such that \( y_0g_i \in Y \cap RF M \).

Since \( U \) is a horospherical subgroup of \( H \) and \( C \) commutes with \( H \), we can apply Lemma 8.2 to the sequence \( g_i^{-1} \) and the sequence of \( k \)-thick sets \( T_i := \{ u \in U : y_0g_iu \in Y \cap RF M \} \) of \( U \). This gives us sequences \( t_i \to \infty \) in \( T_i \) and \( u_i \in U \) such that as \( i \to \infty \),
\[ u_i^{-1}g_iu_i \to a \]
for some nontrivial element \( a \in A \). Since \( y_0u_i \) converges to some \( y_1 \in Y \cap RF M \) by passing to a subsequence, we have
\[ y_1a = \lim(y_0u_i)(u_i^{-1}g_iu_i) \in Y \]
and hence \( Ya \subset Y \). Since \( a \) can be made arbitrarily close to \( e \) by Lemma 8.2, there exists a subsemigroup \( A_+ \) of \( A \) such that \( YA_+ \subset Y \).

Now choose a sequence \( a_i \to \infty \) in \( A_+ \). Note that \( y_0a_i \in Y \cap RF M \) and \( (y_0a_i)(a_i^{-1}A_+) \subset Y \). Since \( RF M \) is compact, passing to a subsequence,
passing to a subsequence, this case, it suffices to consider the case when \( y \in Y \cap RF M \). Since \( y_2 \in Y \cap RF M \), we have \( Y A \subset Y \).

We now present:

**Proof of Theorem 11.1.** First suppose that \( x \in F^* \). Let \( Y \) be a \( U \)-minimal set of \( X \) with respect to \( RF M \). If \( Y \) were contained in \( \partial F \), then \( Y \subset \partial F \cap RF M \). Since \( \text{Stab}_L(x) \) is Zariski dense in \( L \) by the definition of \( \mathcal{L}_U \), it follows from [5, Lemma 4.13] that \( (X \cap RF_+ M) \) is \( AU \)-minimal. Therefore we have \( \overline{Y A} = X \) and hence \( X \) has to be contained in the closed \( A \)-invariant subset \( \partial F \cap RF M \) as well. Therefore, \( Y \cap RF M \cap F^* \neq \emptyset \). Hence, by Proposition 11.2, \( Y \) is \( A \)-invariant. Therefore the claim (1) follows from the \( AU \)-minimality of \( X \cap RF_+ M \) if \( x \in F^* \). Now suppose \( x \in \partial F \). In this case, it suffices to consider the case when \( U \) is a proper subgroup of \( N \); otherwise \( L = G \) and has no compact factor. Hence we may assume without loss of generality that \( U \subset \mathcal{H} \cap N \), and hence \( X \) is contained in a compact homogeneous space of \( \mathcal{H} = SO^0(d - 1, 1) \), which is the frame bundle of a rigid hyperbolic manifold of finite volume. Therefore the claim (1) follows from the previous case of \( x \in F^* \), since \( F^* = RF M \) in the finite volume case.

Claim (2) follows from (1) since \( RF_+ M H \) is closed, and \( X \subset RF_+ M H \). For the claim (3), it suffices to show that the \( A \) action on \( X \cap RF M \) is topologically transitive (cf. [7]). Let \( x, y \in X \cap RF M \) be arbitrary, and \( O, O' \) be open neighborhoods of \( e \) in \( H \). The set \( UU^t AM_0 \) is a Zariski open neighborhood of \( e \) in \( H \) where \( M_0 = M \cap H \) and \( U^t \) is the expanding horospherical subgroup of \( H \) for the action of \( A \). Choose an open neighborhood \( Q_0 \) of \( e \) in \( U \), and an open neighborhood \( P_0 \) of \( e \) in \( U^t AM_0 \) such that \( Q_0 P_0 \subset O \).

We claim that \( xQ_0 A \cap yO' \neq \emptyset \), which implies \( xO A \cap yO' \neq \emptyset \). Suppose this is not true, then

\[ xQ_0 A \subset \Gamma \backslash G - yO' \]

where the latter is a closed set. Now, choose a sequence \( a_n \in A \) such that \( a_n Q_0 a_n^{-1} \to U \) as \( n \to \infty \), and observe

\[ x a_n^{-1} (a_n Q_0 a_n^{-1}) = x Q_0 a_n^{-1} \subset \Gamma \backslash G - yO'. \]

Passing to a subsequence, \( x a_n^{-1} \to x_0 \) for some \( x_0 \in RF M \), and we obtain that \( x_0 U \) is contained in the closed subset \( \Gamma \backslash G - yO' \). This contradicts the \( U \)-minimality of \( X \cap RF_+ M \), which is claim (1).

For the claim (3), note that by Corollary 10.4, almost all \( U_0 \)-orbits in \( \pi_1 (g^{-1} \Gamma g \cap HC) \backslash H \) are dense. Hence for almost all \( x \), the \( U_0 \)-orbit \( x_0 U \) contains an \( H \)-orbit, and the conclusion follows from the claim (2).

**Proposition 11.3.** For a closed orbit \( xH' \) with \( x = [g] \in F^* \cap RF M \),

\[ \overline{xH} = xHC \]

for some connected closed subgroup \( C < C(H) \). Moreover for \( \Gamma' := g^{-1} \Gamma g \cap H' \), \( C \) is given by the closure \( \overline{\pi_0 (F')} \) and \( HC \) is equal to the identity component of the Zariski closure of \( \Gamma' \). In particular, \( HC \in \mathcal{L}_U \).
Proof. Noting that \( \text{Stab}_{H'}(x) = \Gamma' \), the map

\[
\phi : xH' \to \Gamma'/H'
\]

given by \( xh' \mapsto [h'] \) defines an \( H' \)-equivariant homeomorphism.

By Proposition 4.7, \( \pi_1(\Gamma')/H/(H \cap K) \) is a rigid hyperbolic manifold, and hence \( \pi_1(\Gamma') \) is Zariski dense in \( H \).

We claim that

(11.1) \[
\overline{xH} = xH\pi_2(\Gamma').
\]

Set \( C := \pi_2(\Gamma') \). Let us write \( x = [(h, c)] \) for \( h \in H \) and \( c \in C \). We first consider the case when \( c = e \). For all \( \gamma \in \Gamma' \),

\[
xH = [(e, e)]H = [(e, \pi_2(\gamma))]H = [(e, e)]H_{\pi_2(\gamma)}
\]

and hence \( xH = xH\pi_2(\Gamma') \). It follows that \( xH\pi_2(\Gamma') \subset \overline{xH} \).

To show the other inclusion, let \( (h_0, c_0) \in HC(H) \) be arbitrary. If \( [(h_0, c_0)] \in \overline{xH} = [(e, e)]H \), then there exist sequences \( \gamma_i \in \Gamma' \) and \( h_i \in H \) such that \( \gamma_i(h_i, e) \to (h_0, c_0) \) in \( H' \) as \( i \to \infty \). In particular, \( \pi_2(\gamma_i) \to c_0 \) in \( C(H) \) as \( i \to \infty \) and hence \( c_0 \in \pi_2(\Gamma') \). This proves (11.1) when \( c = e \).

Now consider a general \( x = [(h, c)] \). For \( y := [(h, e)] \), we have proved:

\[
\overline{yH} = yH\pi_2((h, e)^{-1}\Gamma(h, e) \cap H').
\]

Therefore we have

\[
\overline{xH} = \overline{yH}\pi_2((h, c)^{-1}\Gamma(h, c) \cap H')
\]

This finishes the proof of (11.1).

Let \( \Gamma'^z \) denote the identity component of the Zariski closure of \( \Gamma' \) in \( H' \). Since \( HC \) is the identity component of an algebraic subgroup, \( \Gamma'^z \) is contained in \( HC \) and the quotient \( \Gamma'^z/HC \) is compact, since \( \pi_1(\Gamma') \) is Zariski dense in \( H \). The compactness of the quotient \( \Gamma'^z/HC \) implies that \( \Gamma'^z \) contains a maximal real-split connected solvable subgroup of \( HC \), which is a conjugate of \( AU \). Without loss of generality, we assume that \( AU < \Gamma'^z \).

Let

\[
L := H \cap \Gamma'^z.
\]

Then \( L \) is a normal subgroup of \( H \), as \( \pi_1(\Gamma'^z) = \pi_1(\Gamma')^z = H \). Since \( AU < L \) and \( H \) is simple, we conclude that \( L = H \). Together with \( \pi_2(\Gamma') = \pi_2(\Gamma') = C \), this implies that \( \Gamma'^z = HC \). This proves the claim. \( \square \)
12. Limits of RF $M$-points in $F^*$ and generic points

We collect all geometric lemmas which are needed in modifying a sequence limiting on a generic point in $\mathcal{G}(U)$ to a sequence of RF $M$-points whose limit still remains inside $\mathcal{G}(U)$.

Throughout the section, assume that $\Gamma \backslash \mathbb{H}^d$ is a rigid hyperbolic manifold. Recall that $\Lambda \subset S^{d-1}$ denotes the limit set of $\Gamma$.

**Lemma 12.1.** Let $C_n \to C$ be a sequence of convergent circles in $S^{d-1}$. If $C \not\subset \Lambda$ and $\# C \cap \Lambda \geq 2$, then

$$\# \liminf C_n \cap \Lambda \geq 2.$$

**Proof.** Without loss of generality, we assume that $\infty \not\in \Lambda$ and hence we may consider $\Lambda$ as a subset of the Euclidean space $\mathbb{R}^{d-1}$. This implies that there is a uniform upper bound for the diameter of a component of $\Omega = \mathbb{R}^{d-1} - \Lambda$. It then implies that there are only finitely many components of $\Omega$ whose diameter is bounded from below by a fixed positive number; this follows from the fact that $\Gamma B$ is closed for each component $B$ of $\Omega$, and that there are only finitely many $\Gamma$-orbits of components of $\Omega$.

Let $\delta = 0.5 \text{ diam}(C)$ so that we may assume $\text{diam}(C_n) > \delta$ for sufficiently large $n \gg 1$. It suffices to show that $C_n \cap \Lambda$ contains two points $\xi_n, \xi'_n$ with $d(\xi_n, \xi'_n) \geq \varepsilon_0$ for all sufficiently large $n$, where $\varepsilon_0$ is a uniform positive constant.

Suppose that the claim is not true. Then for any $\varepsilon > 0$, there exists an interval $I_n \subset C_n$ such that $\text{diam}(I_n) \leq \varepsilon$ and $C_n - I_n \subset \Omega$ for some infinite sequence of $n$’s. Since $C_n - I_n$ is connected, there exists a component $B_n$ of $\Omega$ such that $C_n \subset \mathcal{N}_\varepsilon(B_n)$, where $\mathcal{N}_\varepsilon(B_n)$ denotes the $\varepsilon$-neighborhood of $B_n$. In particular, we have $\text{diam}(B_n) + \varepsilon > \delta$. Taking $\varepsilon$ small enough, this means that $\text{diam}(B_n) > 0.5\delta$. Hence, by passing to a subsequence, we have $C_n \subset \mathcal{N}_\varepsilon(B)$ for some fixed component $B$ of $\Omega$. It implies that $C \subset \mathcal{N}_\varepsilon(B)$. Since this holds for any small $\varepsilon > 0$, we conclude that $C \subset \overline{B}$. This means either $C \subset \Lambda$ or $\# C \cap \Lambda = 1$, yielding a contradiction. $\square$

**Lemma 12.2.** Let $U < N$ be a connected subgroup with dimension $m \geq 1$, and let $U^-_1, \cdots, U^-_m$ be one-parameter subgroups generating $U$. Let $U^+_i$ and $U^-_i$ be the transposes of $U$ and $U^-_i$ respectively for each $i = 1, \cdots, m$. Consider a closed orbit $yL$ where $L \in \mathcal{L}_U$ is of the form $L = H(\bar{U})C$ for a subgroup $\bar{U}$ containing $U$ properly and

$$y \in F^*_{H(U)} \cap RF M \cap \bigcap_{i=1}^m \mathcal{G}(U^+_i, yL).$$

If a sequence $x_n$ converges to $y$ in $yL$, then we can find a sequence $h_n \to h$ in $H(\bar{U})$ so that

$$x_nh_n \in RF M \quad \text{and} \quad yh \in \bigcap_{i=1}^m \mathcal{G}(U^+_i, yL).$$
Proof. Let \( k := \dim(\tilde{U}) > m \). Let \( g_n \to g_0 \) be a convergent sequence in \( G \) so that \( x_n = [g_n] \) and \( y = [g_0] \). Let \( \pi : G \to \mathbb{R}^d \) be the basepoint projection, and set \( S_n := \partial \pi(g_nH(U)) \) and \( S_0 := \partial \pi(g_0H(U)) \). Then \( S_n \) converges to \( S_0 \) in the space \( C^m \) of oriented spheres of dimension \( m \) in \( \mathbb{S}^{d-1} \) as \( n \to \infty \), and \( S_n, S_0 \) are all contained in \( \partial \pi(g_0L) \).

Since \( yL \) is closed, the \( \Gamma \)-orbit of \( \partial \pi(g_0L) \) is closed in \( \mathcal{O}^k \). Since \( y \in F^+_{H(U)} \), it follows that \( \partial \pi(g_0L) \cap \Omega = \bigcup_{j=1}^{\infty} D_j \) for some non-empty collection of round balls \( D_j \) contained in \( \partial \pi(g_0L) \).

Note that for all sufficiently large \( n \), each \( S_n \) should intersect \( \Lambda \) in at least two points, otherwise \( S_0 \subset \partial D_j \) for some component \( D_j \), which contradicts \( y \notin \bigcup_{i=1}^{m} \mathcal{G}(U_i^\pm, yL) \).

It follows from Corollary 5.8 that there are only countably many spheres in \( \mathbb{S}^{d-1} \) of varying dimensions, which meet \( \Lambda \) and whose \( \Gamma \)-orbits are closed. Let \( Q \) be the collection of all such spheres of dimension not greater than \( k \), properly contained in the sphere \( \partial \pi(g_0L) \). Since \( y \in \bigcap_{i=1}^{m} \mathcal{G}(U_i^\pm, yL) \), either \( S_0 = \partial \pi(g_0L) \), or \( \Gamma S_0 \) is not closed in \( C^m \). In any case, \( S_0 \notin Q \).

We can find a circle \( C_0 \subset S_0 \) such that \( \#(C_0 \cap \Lambda) \geq 2 \), and \( C_0 \nsubseteq S' \) for any \( S' \in Q \). In particular, \( C_0 \cap \Lambda \) is a globally \( k \)-thick subset of \( C_0 \), and \( C_0 \) is not contained in \( \overline{B} \) for any component \( B \) of \( \Omega \). Since \( S_n \to S_0 \) as \( n \to \infty \), we can then find a sequence of circles \( C_n \subset S_n \) converging to \( C_0 \) as \( n \to \infty \).

Let

\[
E := \liminf_{n \to \infty} C_n \cap \Lambda \subset C_0.
\]

Then by Lemma 12.1, \( \# C_n \cap \Lambda \geq 2 \) for all large enough \( n \) and \( \# E \geq 2 \).

Since \( C_0 \nsubseteq \Lambda \), we may choose a point \( c_0 \in C_0 - \Lambda \), and consider \( E \) as a subset of \( \mathbb{R} = C_0 - \{c_0\} \). Viewed this way, \( E \) is a globally \( k \)-thick set, as the liminf of a sequence of globally \( k \)-thick subsets is globally \( k \)-thick, provided it has at least 2 points. In particular it is an uncountable subset of \( C_0 \cap \Lambda \).

We claim that there exists a pair of points \( \xi^-, \xi^+ \in C_0 \cap \Lambda \) with the following property:

- If \( g \in G \) is a frame whose first vector is tangent to the geodesic \([\xi^-, \xi^+]\), and \([g] \in yL \), then

\[
[g] \in \bigcap_{i=1}^{m} \mathcal{G}(U_i^\pm, yL) .
\]

To see this, observe that each sphere \( S' \in Q \) meets \( C_0 \) in no more than two points, since \( C_0 \nsubseteq S' \). Let \( \Psi \) be the union of all possible intersection points of \( C \) and spheres in \( Q \). Note that \( \Psi \) is countable. By the cardinality reason, we can choose distinct points \( \xi^-, \xi^+ \) from \( E - \Psi \). Now let \( g \in G \) be a frame whose first vector is tangent to the geodesic \([\xi^-, \xi^+]\) and \([g] \in yL \).

We claim that \([g] \in \bigcap \mathcal{G}(U_i^\pm, yL) \). Assume to the contrary that \([g] \in \mathcal{G}(U_i^\pm, yL) \) for some \( i \). We will assume \([g] \in \mathcal{G}(U_i^+, yL) \), as the case when \([g] \in \mathcal{G}(U_i^-, yL) \) can be dealt similarly, by changing the role of \( g^- \) and \( g^+ \) below.
Now by Proposition 5.12, there exists $L_0 \in \mathcal{L}_{U_\alpha^-}$ such that $(L_0)_{\alpha C}$ is properly contained in $L_{\alpha C}$, and $\alpha \in N(U_\alpha^-) \cap L$ such that $[g]_{\alpha L_0}$ is closed.

Consequently, $\partial \pi(g_\alpha L_0)$ is an $m$-sphere properly contained in $\partial \pi(g_0 L)$, whose $\Gamma$-orbit is closed in $\mathcal{C}^m$. Hence, $\partial \pi(g_\alpha L_0) \in Q$, and in particular, $(g_\alpha)^- \in Q$. On the other hand, since $N(U_\alpha^-) < AN^- M$, we have $(g_\alpha)^- = g^- = \xi^-$ and therefore $\xi^- \in \Psi$, a contradiction.

Now choose $\xi^-, \xi^+$ as above, and a frame in $g_0 H(U)$ whose first vector $v_1$ is tangent to the geodesic $[\xi^-, \xi^+]$, which we write as $g_0 h$ for some $h \in H(U)$.

By definition, there exists $\xi^-_n, \xi^+_n \in C_n \cap \Lambda$ such that $\xi^-_n \rightarrow \xi^-$, and $\xi^+_n \rightarrow \xi^+$ as $n \rightarrow \infty$. Choose a vector $v_1^{(n)}$ which is tangent to the geodesic $[\xi^-_n, \xi^+_n]$. We then extend $v_1^{(n)}$ to a frame $g_n h_n \in g_n H(U)$ so that $g_n h_n$ converges to $gh$ as $n \rightarrow \infty$. This completes the proof.

We will need the following lemma later.

**Lemma 12.3.** [29, Lemma 4.2]. Let $\chi$ be a horocycle in $\mathbb{H}^2$, resting at $p \in \partial \mathbb{H}^2$. Let $\gamma$ be a geodesic joining $\xi, \xi' \in \partial \mathbb{H}^2$, and $\delta$ be another geodesic joining $\xi$ and $p$. Let $q = \delta \cap \chi$. If $d(\chi, \gamma) < R - 1$, then $d(q, \gamma) < R$ for sufficiently large $R$'s.

**Lemma 12.4.** Let $U < H \cap N$ be a non-trivial connected subgroup. If $x_n \in RF M \cdot U$, $x \in RF M$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then passing to a subsequence, there exists $u_n \in U$ such that $x_n u_n \in RF M$ and at least one of the following holds:

1. $u_n \rightarrow e$ and hence $x_n u_n \rightarrow x$, or
2. $x = zc$ for some $z \in BF M$ with $c \in C(H(U))$, and $x_n u_n$ accumulates on $z H c$.

**Proof.** If $x_n$ belongs to $RF M$ for infinitely many $n$, the lemma is trivial. Let us assume $x_n \notin RF M$ for all $n$. Choose $g_n \rightarrow g$ in $G$ so that $x_n = [g_n]$, $x = [g]$.

Recall that $G$ acts on $\mathbb{S}^{d-1} = \mathbb{R}^{d-1} \cup \{\infty\}$, where $\infty$ is fixed by $N$. By the hypothesis, $g_n(\infty), g(\infty)$ and $g(0)$ are all contained in $\Lambda$, but $g_n(0) \in \Omega$. For each $n$, choose an element $u_n \in U$ so that $\|u_n\|$ is the minimum of $\|u\|$ for all $u$ satisfying $g_n u(0) = (g_n u)^- \in \Lambda$. Consider

$$\alpha := \limsup_n \|u_n\|.$$ 

If $\alpha = 0$, then we are in case (1).

Hence we will assume $0 < \alpha \leq \infty$, and let $C_n = \partial \pi(g_n H(U))$ and $C = \partial \pi(g H(U))$ so that $C_n \rightarrow C$ in $\mathcal{C}^{\dim(U)}$ as $n \rightarrow \infty$. Note that $g_n(B_U(\alpha))$ is an open ball in $C_n$, which is contained in $\Omega$, and meets $\Lambda$ at one of its boundary. Since $g_n(B_U(\alpha))$ is connected, it is contained in some component $B_i$ of $\Omega$, which is a round ball.

On the other hand, since $g_n B_U(\alpha) \rightarrow g B_U(\alpha)$ as $n \rightarrow \infty$, its diameter in $\mathbb{S}^{d-1}$ is bounded below by some positive number. Hence, passing to a subsequence, we may assume that $g_n B_U(\alpha)$ are all contained in the same
component, say $B$. Since $g(0) \in \Lambda$, we conclude that $gB_U(\alpha)$ is contained in $\partial B$, and hence so is $C$.

This implies that the element $x$ as given in the hypothesis is of the form $x =zc$ for some boundary frame $z \in BF$ and $c \in C(H(U))$. We proceed to show that $x_nu_n$ accumulates on $z\tilde{H}c$. Since $c \in C(U)$, replacing $x$ with $xc^{-1}$, and $x_n$ with $x_ne^{-1}$, we may assume $c = e$. Now that $x_n \to z$ as $n \to \infty$ and $z\tilde{H}$ is compact, its enough to show that the basepoints of the sequence $x_nu_n$ in $M$ goes arbitrarily close to the boundary of core $M$.

Let $C_n \cap \partial B = D_n$, and $P_n := \hull(D_n)$. We set $H_n := \hull(C_n)$, $H := \hull(C)$ and $H' := \hull(\partial B)$. Then $H_n \cap H' = P_n$. Let $\varepsilon > 0$ be arbitrary, and $N_\varepsilon(H')$ be an $\varepsilon$-neighborhood of $H'$ in $\mathbb{H}^d$. Let $d_{H_n}(\cdot, \cdot)$ denote the hyperbolic distance in $H_n$. Note that there exists $R_n > 0$ such that

$$N_\varepsilon(H') \cap H_n = \{p \in H_n : d_{H_n}(p, P_n) < R_n\}.$$ 

This is because, $N_\varepsilon(H') \cap H_n$ is convex, and invariant under family of isometries, whose translation axis are contained in $P_n$. As $C_n \to C$ as $n \to \infty$, and $C \subset \partial B$, it follows that $R_n \to \infty$ as $n \to \infty$.

Let $\chi_n := \pi(g_nU)$, and $\chi := \pi(gU)$ be horospheres in $H_n$ and $H$ respectively.

We claim that $d_{H_n}(P_n, \chi_n)$ is uniformly bounded from above. To see this, we only need to consider those $P_n$’s which are disjoint from $\chi_n$, as $d_{H_n}(P_n, \chi_n) = 0$ otherwise. Let us write $C_n - D_n = E_1 \cup E_2$, where $E_1$ is a connected components of $C_n - D_n$ meeting $B$, and $E_2$ is the other component. Then the diameter of $E_1$ with respect to the spherical metric on $S^{d-1}$ is bounded below by some positive constant, as it is eventually greater than the half of the diameter of $gB_U(\alpha)$. On the other hand, since $\chi_n$ converges to $\chi$, the condition that $P_n \cap \chi_n = \emptyset$ implies that the diameter of $E_2$ with respect to the spherical metric on $S^{d-1}$ is also bounded below by some positive constant. Now that $\chi_n \to \chi$, $C_n \to C$ as $n \to \infty$ and each component of $C_n - D_n$ has diameter bounded below by a definite constant, it follows that $d_{H_n}(P_n, \chi_n)$ is bounded from above.

Since $R_n \to \infty$, we deduce that $d_{H_n}(P_n, \chi_n) < R_n - 1$ for all sufficiently large $n$. Applying Lemma 12.3, we have $d_{H_n}(\pi(g_nu_n), P_n) < R_n$, and hence $\pi(g_nu_n) \in N_\varepsilon(H') \cap H_n$, for all sufficiently large $n$. As $\varepsilon$ was arbitrary, the proof is complete. \hfill \Box

**Obtaining limits in $F^\ast$.** As before, we denote by $\pi$ for the basepoint projection maps $FM \to M$ and $G \to \mathbb{H}^d$. Recall the notation: $\Omega = \bigcup_{i=1}^\infty B_i$ where $B_i$’s are components of $\Omega$. For $\varepsilon > 0$, we set

$$\text{(12.1)} \ \ \ \ \ \text{core}_\varepsilon(M) = \{x \in \Gamma \setminus G : \pi(x) \in \text{core } M \ \text{and } d(\pi(x), \partial \text{ core } M) \geq \varepsilon\},$$

and choose

$$\text{(12.2)} \ \ \ \ \ \varepsilon_0 = \frac{1}{3} \inf_{i \neq j} d(\hull(B_i), \hull(B_j)).$$

Note that $\varepsilon_0$ is a positive constant only depending on $M$. 

Lemma 12.5. There exists $\varepsilon > 0$ such that for any $x \in RF M$, and for any one-parameter subsemigroup $V^+$ of $N$, if $\pi(xV) \not\subset \partial \text{core } M$, and

$$\limsup (xV^+ \cap RF M) \neq \emptyset,$$

then

$$\limsup (xV^+ \cap \text{core}_x(M)) \neq \emptyset.$$  

Proof. We will check that $\varepsilon_0$ given in (12.2) works. Let $x = [g]$, and set $o = (0, \ldots, 0, 1) \in \mathbb{H}^d$. We may assume $g = (e_1, \ldots, e_d)_o \in F \mathbb{H}^d$ where $e_i$ are standard basis vectors in $T_o \mathbb{H}^d \simeq \mathbb{R}^d$. Note $gV^+$ is a translation of the frame $\tilde{x}$ along a horizontal ray emanating from $o$ along $V^+$-direction. By the hypothesis,

$$\limsup xV^+ \cap RF M \neq \emptyset,$$

and $\pi(gV^+) \not\subset \partial \text{hull}(B_i)$ for any $i$. Write $V^+ = \{v_t : t \geq 0\}$. Now for each $i$,

$$\tau_i = \{t \geq 0 : d(\pi(gv_t), \text{hull}(B_i)) \leq \varepsilon\}$$

is a closed bounded set. By the choice of $\varepsilon = \varepsilon_0$, they are pairwise disjoint. We will consider $t_i := \sup\{t : v_t \in \tau_i\}$, only for nonempty $\tau_i$'s. By reordering them, we may assume that $t_1 < t_2 < \cdots$. Noting that $t_i - t_{i-1} \geq \varepsilon$, the claim follows. \hfill $\square$

13. Orbit closure theorems: beginning of the induction

In the rest of the paper, let $M = \Gamma \backslash \mathbb{H}^d$ be a rigid hyperbolic $d$-manifold, and $G = \text{SO}^0(d, 1)$. Let $U < N$ be a non-trivial connected subgroup, and $H(U)$ be its associated simple Lie subgroup of $G$.

Recall from (5.7) that $\mathcal{L}_U$ denotes the collection of all subgroups of the form $H(\tilde{U})C$ where $U < \tilde{U} < N$ and $C$ is a closed subgroup of $C(H(\tilde{U}))$ satisfying the following:

$$\mathcal{L}_U := \left\{L = H(\tilde{U})C : \text{ for some } [g] \in \text{RF } M, [g]L \text{ is closed in } \Gamma \backslash G \text{ and } L \cap g^{-1} \Gamma g \text{ is Zariski dense in } L \right\}.$$  

Recall the notation $\mathcal{Q}_U$:

$$\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U, v \in N(U)\}$$

$$= \{vLv^{-1} : L \in \mathcal{L}_U, v \in U^\perp\}$$

(see Lemma 5.11). Note that the group $G$ belongs both to $\mathcal{L}_U$ and $\mathcal{Q}_U$.

In the rest of the paper, we prove the following theorem:

Theorem 13.1. Let $x \in \text{RF } M$. We have the following:

(1) We have

$$\overline{xH(U)} = xL \cap F_{H(U)}$$

where $xL$ is closed for some $L \in \mathcal{L}_U$. 
(2) If $\overline{xU}$ is contained in a closed orbit $x\hat{L}$ for some $\hat{L} \in Q_U$, then
\[ \overline{xU} = xL \cap RF_+ M \]
where $xL$ is closed for some subgroup $L < \hat{L}$ contained in $Q_U$. Moreover,
\[ \overline{xAU} = xL \cap RF_+ M \]
for some $L \in \mathcal{L}_U$.

(3) If $x\hat{L}$ is closed for some $\hat{L} \in \mathcal{L}_U$, and $y_i L_i v_i \subset x\hat{L}$ is a sequence of closed orbits where $y_i \in RF M$, $L_i \in \mathcal{L}_U$, and $v_i \in \hat{L} \cap N$ satisfy either of the following:
- $v_i \to \infty$ modulo $L_i$
- $v_i$ is bounded modulo $L_i$ and $y_i L_i$ are all distinct,
then
\[ \limsup_{i \to \infty} (y_i L_i v_i \cap RF_+ M) = x\hat{L} \cap RF_+ M. \]

Remark 13.2. We remark the following:
- If $x \in \partial F_H(U) \cap RF M$, then $xH(U)$ lies in a compact homogeneous space of a subgroup isomorphic to $SO(d-1,1)$ and hence Theorem 13.1 (1) and (2) follows from the work of Ratner and Shah (cf. Theorem 15.4), and (3) follows from the work of Mozes-Shah [31]. So the main new case is when $x \in F^*_H(U)$.
- If $x \in RF_+ M \cap F^*_H(U)$, then its $U$-orbit contains an RF $M$ point, hence (2) holds for every $x \in RF_+ M$.

We will prove (1), (2), and (3) of Theorem 13.1 by induction on the co-dimension of $U$ in $N$ and the co-dimension of $U$ in $\hat{L} \cap N$, respectively.

For simplicity, let us say $(1)_m$ holds, if (1) is true for all $U$ satisfying $\text{co-dim}_N(U) \leq m$. We will say $(2)_m$ holds, if (2) is true for all $U$ and $\hat{L}$ satisfying $\text{co-dim}_N(\hat{L} \cap N)(U) \leq m$ and similarly for $(3)_m$.

We begin with the following observation:

**Singular $U$-orbits under the induction hypothesis.** Recall the notation $\mathcal{H}(U, x\hat{L})$ and $\mathcal{G}(U, x\hat{L})$ from (5.6).

**Lemma 13.3.** Suppose that $(2)_m$ is true and that for $x \in RF M$, $xU$ is contained in a closed orbit $x\hat{L}$ for some $\hat{L} \in \mathcal{L}_U$. Then we have:

1. If $\text{co-dim}_N(\hat{L} \cap N)(U) \leq m + 1$, then for any $x_0 \in \mathcal{H}(U, x\hat{L})$,
\[ \overline{x_0U} = x_0L \cap RF_+ M. \]
for some closed orbit $x_0L$ where $L \in Q_U$ is contained in $\hat{L}$ and $\dim L_{nc} < \dim \hat{L}_{nc}$.
2. If $\text{co-dim}_N(\hat{L} \cap N)(U) \leq m$, then for any $x_0 \in \mathcal{G}(U, x\hat{L})$,
\[ \overline{x_0U} = x_0\hat{L} \cap RF_+ M. \]
Proof. Suppose that co-dim$_{L \cap N}(U) \leq m + 1$ and that $x_0 \in \mathcal{G}(U, x\hat{L})$. By Proposition 5.12, we get
\[
\overline{x_0U} \subset x_0Q
\]
for some closed orbit $x_0Q$ where $Q \in Q_U$ satisfies dim $Q_{nc} < \dim \hat{L}_{nc}$.

Now $Q = vL_0v^{-1}$ for some $L_0 \in L_U$ and $v \in U^\perp$. We have $x_0Uv = x_0vU \subset x_0vL_0$. Since co-dim$_{N \cap L_0}(U) = \text{co-dim}_{N \cap Q}(U) \leq m$, by applying (2)$_m$, we get
\[
x_0vU = x_0vL_0 \cap \text{RF}_+ M
\]
for some closed orbit $x_0vL$ where $L \in Q_U$ is contained in $L_0$. Therefore
\[
x_0U = x_0vLv^{-1} \cap \text{RF}_+ M.
\]
As $vLv^{-1} \in Q_U$ and dim $L_{nc} \leq \text{dim } Q_{nc} < \dim \hat{L}_{nc}$, the claim (1) is proved.

To prove (2), suppose that co-dim$_{L \cap N}(U) \leq m$ and that $x_0 \in \mathcal{G}(U, x\hat{L})$.

By (2)$_m$, $\overline{x_0U} = x_0L \cap \text{RF}_+ M$ for some closed orbit $x_0L$ with $L \in Q_U$ such that $L \subset \hat{L}$. Since $x_0 \in \mathcal{G}(U, x\hat{L})$, we have dim $L_{nc} = \dim \hat{L}_{nc}$.

Since $L \subset \hat{L}$, $L \cap N$ is a horospherical subgroup of $\hat{L}$. By Theorem 11.1, $L \cap N$ acts minimally on $x\hat{L}$, and hence $L = \hat{L}$. \qed

Base case of $m = 0$. Note that for the base cases (1)$_0$, and (3)$_0$ are trivial, and that (2)$_0$ follows from Theorem 11.1 (or from Lemma 13.3(2) for $m = 0$).

Assuming that (2)$_m$, and (3)$_m$ are true, we will first show that (1)$_{m+1}$ is true in section 15. After that, assuming that (1)$_{m+1}$, (2)$_m$, and (3)$_m$ hold, we will show that (2)$_{m+1}$ and (3)$_{m+1}$ hold in sections 16 and 17 respectively.

14. Generic points and additional invariance

The main goal of this section is to prove Propositions 14.2 and 14.3, which will be later used in order to get additional invariance using a sequence converging to a generic point of an intermediate closed orbit. The results in this section are main tools in the enlargement steps of the proof of Theorem 13.1.

We first prove the following Theorem 14.1 using Theorem 7.14; in fact, we won’t be using this theorem directly, but the proof of this theorem points to important ideas on how we use the avoidance theorems of section 7 and the induction hypothesis to get a generic point as a limit along the thick recurrence time.

Let $k \geq 1$ be as given by Proposition 4.9.

**Theorem 14.1** (Accumulation on generic points). Suppose that (2)$_m$ and (3)$_m$ are true. Let $U < N$ be a connected subgroup. Suppose that $x\hat{L}$ is a closed orbit for some $x \in \text{RF} M$, and $\hat{L} \in L_U$, and that co-dim$_{L \cap N}(U) \leq m + 1$.

Let $U_0 = \{u_t : t \in \mathbb{R}\}$ be a one-parameter subgroup of $U$. Let $x \in \mathcal{G}(U_0, x\hat{L})$ and $x_i \in \text{RF} M$ be a sequence converging to $x$ as $i \to \infty$. Then
for any given sequence \( T_i \to \infty \),
\[
\limsup_{i \to \infty} \{ x_i u_{t_i} \in RF M : T_i \leq |t_i| \leq 2kT_i \}
\]
contains a sequence \( \{ y_N \} \) such that
\[
\limsup_{N \to \infty} y_N U = x\hat{L} \cap RF_+ M.
\]

In particular, \( \limsup_{N \to \infty} y_N U \) contains a dense \( U_0 \)-orbit in \( x\hat{L} \cap RF_+ M \).

**Proof.** Let \( E_N, O_N \) and \( i_N, N \in \mathbb{N} \), be sequences given by Theorem 7.14 for \( x\hat{L} \) with respect to \( U_0 \). Since \( \{ t \in \mathbb{R} : x_i u_{t_i} \in RF M - O_N \} \) is 2k-thick for all \( i \geq i_N \), we can find a sequence
\[
t_i \in \mathbb{R} \cap ([-2kT_i, -T_i] \cup [T_i, 2kT_i])
\]
such that \( x_i u_{t_i} \in RF M - O_N \) for all \( i \geq i_N \). Hence, by passing to a subsequence, \( x_i u_{t_i} \) converges to some \( y_N \in RF M - O_N \) as \( i \to \infty \). If \( y_N \in \mathcal{G}(U, x\hat{L}) \) for some \( N \), then \((2)_m \) and Lemma 13.3(2) imply that \( y_N U = \overline{y_N U} = x\hat{L} \cap RF_+ M \), proving the claim.

Now, we assume that \( y_N \in \mathcal{G}(U, x\hat{L}) \) for all \( N \). Then by \((2)_m \) and Lemma 13.3(1)
\[
\overline{y_N U} = y_N L_N \cap RF_+ M
\]
for some closed \( y_N L_N \) where \( L_N \in Q_U \) is contained in \( \hat{L} \) and \( \dim(L_N)_{nc} < \dim \hat{L}_{nc} \).

Write \( L_N = v_N^{-1} L'_N v_N \) for \( L'_N \in L_U \) and \( v_N \in U^\perp \). We claim that the sequence \( y_N L_N = (y_N v_N^{-1}) L'_N v_N \) satisfies the hypothesis of \((3)_m \). If not, by passing to a subsequence, \( y_N v_N^{-1} L'_N \) is a constant sequence, say, \( \Gamma g L \) for \([g] \in RF M \) and \( L \in L_U \), and \( v_N \) is bounded modulo \( L \). Now \( H := g L g^{-1} \in \mathcal{K} \), and since \( v_N \in (\hat{L} \cap U)^\perp \), \( v_N U v_N^{-1} \subset L \). So all \( g v_N \) are contained in a compact subset \( D \subset X(H, U) \). Therefore \( y_N L_N \subset \Gamma \backslash \Gamma H D \in E_U \). Hence all \( y_N L_N \) must be contained in some \( E_j \), which is a contradiction since \( y_N \notin E_N \) for all \( N \in \mathbb{N} \).

Hence applying \((3)_m \) to the sequence \( y_N L_N \), we get
\[
\limsup y_N U = \limsup y_N U = \limsup(y_N L_N \cap RF_+ M) = x\hat{L} \cap RF_+ M.
\]
The last claim follows from Theorem 11.1(4).

**Proposition 14.2** (Additional invariance I). Suppose that \((2)_m \) and \((3)_m \) are true. Suppose that \( x\hat{L} \) is a closed orbit for some \( x \in RF M \), and \( \hat{L} \in L_U \), and that \( \text{co-dim}_{E \cap N}(U) \leq m + 1 \). Let \( X \) be a closed \( U \)-invariant subset of \( x\hat{L} \). Let \( U_0 = \{ u_i \} \) be a one-parameter subgroup of \( U \), and let \( L \in L_{U_0} \). Suppose that \( X \) contains \( x_0 L \cap RF_+ M \) for some closed orbit \( x_0 L \) of \( x_0 \in \mathcal{G}(U_0, x_0 L) \cap RF M \). Let \( y_i \to x_0 \) in \( X \) where \( y_i := x_0 e^{t_i} r_i \) with \( x_0 e^{t_i} \in x_0 L \cap RF M \) and \( r_i \in \exp_{\perp} - N(U_0) \), where \( I \) is \( \text{Ad}(L) \) invariant.

\footnote{Here we allow the constant sequence \( y_N = y \) in which case \( \limsup_{N \to \infty} y_N U \) is understood as \( \overline{y U} \).}
complementary subspace to \( l = \text{Lie}(L) \) in \( g \). Then there exists a sequence \( v_M \to \infty \) in \((N \cap L)^\perp\) such that

\[
(x_0L \cap RF^+M)v_M \subset X.
\]

**Proof.** Let \( M \) be an arbitrary positive integer. Since \( x_i := x_0\ell_i \to x_0 \) in \( x_0L \cap RF^+M \) and \( x_0 \in \mathcal{G}(U_0, x_0L) \), we have sequences \( E_N, \mathcal{O}_N \) and \( i_N \), given by Thoerem 7.14 for \( x_0L \) with respect to \( U_0 \) so that for each \( N \),

\[
T_i = \{ t \in \mathbb{R} : x_iu_t \in RF^+M - \mathcal{O}_N \},
\]
is a \( 2k \)-thick subset for all large \( i \geq i_N \).

Applying Lemma 8.3 to \( T_i \), we can find a sequence \( u_{t_i} \in U_0, i \geq i_N \) satisfying that as \( i \to \infty \),

- \( x_0\ell_t u_{t_i} \to y_N \in (RF^+M \cap x_0L) - \mathcal{O}_N \).
- \( u_{t_i}^{-1}r_i u_{t_i} \to v_N \in (L \cap N)^\perp \) with \( M \leq \|v_N\| \leq c_kM \)

where \( c_k \) is a constant from Lemma 8.3.

If some \( y_N \) belongs to \( \mathcal{G}(U, x_0L) \), then as \( y_iu_{t_i} = (x_0\ell_t u_{t_i}) (u_{t_i}^{-1}r_i u_{t_i}) \),

\[
X \supset \overline{y_Nv_NU} = \overline{y_NU} v_N = (x_0L \cap RF^+M)v_N,
\]
and hence the claim follows as \( M \) is arbitrary. Suppose that \( y_N \notin \mathcal{G}(U, x_0L) \) for all \( N \) after passing to a subsequence. Then by (2)\(_m\),

\[
\overline{y_NU} = y_NL_N \cap RF^+M
\]
for some \( L_N \in \mathcal{Q}_U \). Since \( y_N \notin \mathcal{G}_N \), the sequence \( y_NL_N \) satisfies the hypothesis of (3)\(_m\) (cf. proof of Theorem 14.1). Hence

\[
\limsup \overline{y_NU} = \limsup (y_NL_N \cap RF^+M) = x_0L \cap RF^+M.
\]

Hence for some \( u_N \in U \), \( y_Nu_N \) converges to \( x_0 \). As \( M \leq \|v_N\| \leq c_kM \), the sequence \( v_N \) converges to some \( v_M \in (L \cap N)^\perp \) as \( N \to \infty \), after passing to a subsequence. Therefore

\[
X \supset \limsup \overline{y_Nv_NU} = \limsup \overline{y_NU} v_N \supset \overline{x_0U} v_M = (x_0L \cap RF^+M)v_M.
\]

\[\square\]

Note that in the above proposition, \( y_i = x_0\ell_ir_i \) is not necessarily in \( RF^+M \), and hence we cannot apply the avoidance Theorem 7.14 to the sequence \( x_i \) directly. We instead applied it to the sequence \( x_0\ell_i \).

In the proposition below, we will consider a sequence \( x_i \to y \) inside \( RF^+M \), and apply the avoidance theorem 7.14 to the sequence \( x_i \).

**Proposition 14.3** (Additional invariance II). Assume (2)\(_m\) and (3)\(_m\) are true. Let \( \text{co-dim}_N(U) \leq (m + 1) \), and \( X \) be a closed \( U \)-invariant set. Let \( U_0 < U \) be a one-parameter subgroup, and \( L \in \mathcal{Q}_U \). Let \( x_0L \) be a closed orbit such that \( x_0L \cap RF^+M \subset X \) and \( x_0 \in RF^+M \cap \mathcal{G}(U_0, x_0L) \). Assume
Theorem 14.1). Consequently, for some subsequence, \(y\) such that
\[
(x_0L \cap RF_+)v_M \subset X, \\
(x_0L \cap RF) v_M \cap RF \neq \emptyset.
\]

Proof. Let \(E_N \subset E\) be a sequence of increasing compact sets such that
\[
\mathcal{S}(U_0, x_0L) \cap RF M = \bigcup_{N=1}^{\infty} E_N
\]
as given by Corollary 7.13. Fix \(N, M \in \mathbb{N}\). Now that \(x_i \to x_0\) in \(RF M\) as \(i \to \infty\), and \(x_0 \notin \mathcal{S}(U_0, x_0L)\), it follows \(x_i \notin \Gamma \cap E\) for sufficiently large \(i\)'s. Let \(x_i = x_0g_i\) for \(g_i \to e\) in \(G\). Since \(L\) is reductive, we can write \(g_i = h_i r_i\) where \(h_i \to e\) in \(L\) and \(r_i \to e\) in \(exp L\) as \(i \to \infty\). Write \(\tilde{U} = L \cap N\). Define
\[
T_i = \{ t \in \mathbb{R} : x_i u_t \in RF M - E_N \cdot B(c_kM) \},
\]
with
\[
B(c_kM) = \{ v \in \tilde{U}^\perp : \|v\| \leq c_k \cdot M \}
\]
where \(c_k\) is a constant from Lemma 8.3. Note that \(x_i \notin E_N \cdot B(c_kM)\) for sufficiently large \(i\)'s. Hence, by Corollary 7.13, \(T_i\) is a \(2k\)-thick set for sufficiently large \(i\)'s. Applying Lemma 8.3 to \(T_i\), and \(r_i \to e\), we can find \(u_{t_i} \in T_i\) such that \(u_{t_i}^{-1} r_i u_{t_i} \to v_N\) for some \(v_N \in \tilde{U}^\perp\), with \(M \leq \|v_N\| \leq c_k \cdot M\).

Passing to a subsequence, there exists \(x_N \in RF M - E_N \cdot B(c_kM)\) such that \(x_i u_{t_i} \to x_N\) as \(i \to \infty\). On the other hand, since
\[
x_i u_{t_i} = x_0 h_i u_{t_i} (u_{t_i}^{-1} r_i u_{t_i}),
\]
\(x_0 h_i u_{t_i}\) converges to \(y_N := x_N v_N^{-1}\). Since \(\|v_N\| \leq c_k \cdot M\), it follows \(y_N \notin E_N\).

Now we repeat this for all \(N \in \mathbb{N}\). Note that \(x_N = y_N v_N \in \tilde{xU}\), and hence \(y_N U v_N \subset \tilde{xU}\).

Since \(y_N L\) is closed and \(\text{co-dim}_U U \leq m\), by (2) \(m\), it follows
\[
\bar{y_N U} = y_N L \cap RF_+ M
\]
for some \(L_N \in Q_U\) with \(y_N L_N \subset x_0L\). Observe that by passing to a subsequence, \(y_N L_N\) satisfies the hypothesis of (3) \(m\) inside \(yL\) (cf. proof of Theorem 14.1). Consequently,
\[
\limsup_{N \to \infty} (y_N L_N \cap RF_+ M) = x_0L \cap RF_+ M.
\]

Therefore, passing to a limit, we can find \(v \in \tilde{U}^\perp\) with \(M \leq \|v\| \leq c_k \cdot M\), such that
\[
(x_0L \cap RF_+ M)v \subset X.
\]
As \(M\) is arbitrary, the proof is complete. \(\square\)
15. $H(U)$-orbit closures: proof of (1)$_{m+1}$

In the rest of the paper, we fix a connected closed subgroup $U < N$, and set

$$H = H(U), \quad H' = H'(U), \quad F = F_{H(U)} \quad F^* = F_{H(U)}^* \quad \text{and} \quad \partial F = \partial F_{H(U)}.$$ 

In this section, we fix $m \in \mathbb{N} \cup \{0\}$ and assume that

$$1 \leq \text{co-dim}_N(U) = (m + 1).$$

Without loss of generality, we may assume

$$U < \bar{N} \cap H$$

using a conjugation by an element of $M$. The last assumption is for a simpler geometric description of the boundary of $F$:

$$\partial F \cap RF M = BF_M C(H').$$

If $x \in \partial F = BF_M C(H)$, the claim follows from Ratner’s theorem. So we assume $x \in F^* \cap RF M$.

The following proposition says that the $H'$-orbit classification gives the $H$-orbit classification.

**Proposition 15.1.** Let $x \in RF M$, and assume that there exists $\bar{U} < N$ containing $U$ such that $xH'(\bar{U})$ is closed, and

$$\bar{xH'} = xH(\bar{U}) \cdot C(H) \cap RF_+ M \cdot H.$$ 

Then there exists a closed subgroup $C < C(H)$ such that

$$\overline{xH} = xH(\bar{U})C \cap RF_+ M \cdot H.$$ 

**Proof.** By Proposition 11.3 and Theorem 11.1, there exists $C < C(H)$ such that

$$X := xH(\bar{U})C \cap RF_+ M \cdot H(\bar{U})$$

is $H(\bar{U})$-minimal. In particular,

$$\overline{xH} \subset xH(\bar{U})C \cap RF_+ M \cdot H.$$ 

Now, by Theorem 11.1, there exists $y \in xH(\bar{U})C \cap RF_+ M \cdot H(\bar{U})$ such that

$$\overline{yA} = X \cap RF M.$$ 

Since

$$\overline{xH} \cdot C(H) = \overline{xH'} = xH(\bar{U}) \cdot C(H) \cap RF_+ M \cdot H,$$

there exists $c \in C(H)$ such that $yc \in \overline{xH}$. Since $\overline{ycA} = \overline{ycA} \subset \overline{xH}$, and $(X \cap RF M) \cdot H = X \cap RF M \cdot H$, it follows

$$(X \cap RF M \cdot H)c \subset \overline{xH} \subset X \cap RF_+ M \cdot H.$$ 

Note that the closure of $X \cap RF M \cdot H$ is $X \cap RF_+ M \cdot H$ by (4.3). This forces $X \cap RF_+ M \cdot H$ to be $c$-invariant, and $\overline{xH} = X \cap RF_+ M \cdot H$ as desired. \qed
By Proposition 15.1, it suffices show that
\[(15.1) \quad \overline{xH'} = xL \subset \Gamma \cap F \]
for some closed orbit \(xL\) for \(L \in \mathcal{L}_U\).

Set
\[X = \overline{xH'}\]
and assume that \(X \neq xH'\).

The following proposition says that \(xH'\) is not closed in \(F^*\) either.

**Proposition 15.2.** Let \(x \in F^*\). If \(xH'\) is closed in \(F^*\), then it is closed in \(\Gamma \setminus G\).

**Proof.** Suppose that \(xH'\) is closed in \(F^*\), but not closed. Then \(\overline{xH'} \supset \overline{yH'}\)
for some \(y \in (F - F^*) \cap \mathcal{R}_\Gamma M\). We note that \(\overline{yA}\) contains an RF \(M\) point. Hence, by \((4.2)\), \(\overline{xH'}\) contains \(z \in \mathcal{B} \cap \mathcal{B} M\). Let \(xw_n \to z = [g_0]\) for \(w_n \in H'\).

Then there exists \(\gamma_n \in \Gamma\) such that \(\gamma_n gw_n \to g_0\) for some \(x = [g]\). Then the sphere \(C_n = \partial \pi(\gamma_n gH')\) converges to a sphere \(C = \partial \pi(g_0 H')\) contained in \(\partial B\), where \(B = \partial \pi(g_0 H)\) is a component of \(\Omega\).

Let \(B'\) be an open round ball such that \(\mathbb{S}^{d-1} \setminus \partial B = B \cup B'\). We note that \(\Gamma_{B'} := \text{Stab}(B')\) is cocompact in \(\text{Stab}(B') \simeq \hat{H}\). Next, choose a compact fundamental domain \(K \subset B'\) for the action of \(\Gamma_{B'}\). Since \(x \in F^*, C_n\) meets \(B'\). So there exists \(\delta_n \in \Gamma_{B'}\) such that \(\delta_n C_n \cap K \neq \emptyset\). Let \(p_n \in \delta_n C_n \cap K\), and passing to a subsequence, we may assume that \(p_n\) converges to some \(p \in K\) as \(n \to \infty\).

Since \(\delta_n C_n \to D\) as \(n \to \infty\), considering \(B'\) as a hyperbolic ball model, the geodesic curvature of \(\delta_n C_n \cap B'\) must tend to \(1\). It follows that \(\delta_n C_n\) converges to a sphere \(D\) contained in \(\overline{B'}\) which is tangent at a point in \(\partial B\), which passes through \(p\). Note that \(D\) does not lie in \(\Gamma C\), because every circle in \(\Gamma C\) intersects \(\partial B\) in at least two points. This contradicts that \(xH'\) is closed in \(F^*\).

By Proposition 15.2, \(xH'\) is not closed in \(F^*\) and in particular,
\[(15.2) \quad (X - xH') \cap (F^* \cap \mathcal{R} M) \neq \emptyset.\]

Roughly speaking, our strategy in proving \((1)_{m+1}\) is first to find a closed \(L\)-orbit \(x_0 L\) such that \(x_0 L \cap F\) contained in \(X\) for some \(L \in \mathcal{L}_U\).

If \(X \neq x_0 L \cap F\), then we enlarge \(x_0 L\) to a bigger closed orbit \(x_1 L\) for some \(\hat{L} \in \mathcal{L}_{\hat{U}}\) for some \(\hat{U}\) properly containing \(U\) such that \(x_1 L \cap F\) is contained in \(X\).

It is in the enlargement step where the avoidance theorem 7.14 is a crucial ingredient of the arguments. Implementing the avoidance theorems requires knowing that \(\mathcal{R} M\)-points of \(X\) accumulates on a generic point of \(x_0 L\) with respect to any given one-parameter subgroup \(U_0\) of \(U\). A priori there is no reason that this has to be the case, so to bypass the difficulty, we find a closed orbit \(x_0 L\) with a basepoint \(x_0 \in F^*\), and enlarge it to a bigger closed orbit, again based at a point in \(F^*\).
The advantage of having a closed orbit $xL$ with $x \in F^*$ is that any generic $U_0$ point in $xL \cap RF M$ can be approximated by a sequence of $RF M$-point in $F^* \cap X$ by Lemma 12.2.

The enlargement process must end after finitely many steps because of the dimension reasons.

**Finding a closed orbit of $L \in \mathcal{L}_U$ in $X$.**

**Lemma 15.3** (Moving from $Q_U$ to $\mathcal{L}_U$). Assume (2)$_m$ is true. If $x_0L \cap RF_+ M \subset X$ for some closed orbit $x_0L$ with $x_0 \in RF M$, and $L \in Q_U$, then

$$x_1 \bar{L} \cap RF_+ M \subset X$$

for some closed orbit $x_1L$ with $x_1 \in RF M$, and $\bar{L} \in \mathcal{L}_U$.

**Proof.** We may assume that $L_{nc}$ has the maximal dimension among those $L \in Q_U$ such that $x_0L \cap RF_+ M \subset X$ for some $x_0 \in RF M$. Write $L = v^{-1}\hat{L}v$ for some $\hat{L} \in \mathcal{L}_U$ and $v \in (\hat{L} \cap N)^\perp$. We will prove $v = e$, which would finish the proof.

By contradiction, assume $v \neq e$. Let $\hat{L}_{nc} = H(\hat{U})$. Note $x_0v^{-1}\hat{U}Av \subset x_1\hat{L} \cap RF_+ M$, as $A\hat{U} < L$, and hence $x_0v^{-1}\hat{U}AvA \subset X$ from the hypothesis together with the fact $X$ is $A$-invariant.

Let $V^+$ be a one-parameter subgroup in $\hat{U}^\perp$ containing $v$. Since $V^+ \subset AvA$, we have $x_0v^{-1}V^+\hat{U} \subset X$ as well. Let $a_n \in A$ be elements such that $a_n^{-1}v^{-1}V^+a_n \to V$ as $n \to \infty$ and note that

$$x_0a_n(a_n^{-1}v^{-1}\hat{U}V^+a_n) = x_0v^{-1}\hat{U}V^+a_n \subset X.$$ 

Since $x_0a_n \in RF M$ and RF $M$ is compact, $x_0a_n \to x_1$ for some $x_1 \in RF M$ as $n \to \infty$ after passing to a subsequence. As a result, we obtain $x_1\hat{U}V \subset X$.

Since co-dim$_N(\hat{U}V) \leq m$, by (2)$_m$, there exists $\tilde{L} \in Q_{\hat{U}V}$ such that $x_1\tilde{L} \subset X$, which is a contradiction. 

By the definition of $BF M$, $x\hat{H}$ is a compact homogeneous space of $\hat{H}$ when $x \in BF M \cdot C(H)$. Therefore, the following is a special case for Ratner’s theorem [38], which was also proved by Shah independently [44].

**Theorem 15.4.** If $x \in BF M \cdot C(H)$, then $\overline{xU} = xL$ for some $L \in Q_U$ contained in $\hat{H}$, and $\overline{xH} = xL'$ for some $L' \in \mathcal{L}_U$ contained in $\hat{H}$.

Given the induction hypothesis, the next proposition tells us that $\overline{xH'}$ contains a closed orbit of a subgroup in $\mathcal{L}_U$, whenever $xH'$ is not closed.

**Proposition 15.5.** If (2)$_m$ is true, then

$$x_0L \cap RF_+ M \subset X$$

for some closed orbit $x_0L$ with $x_0 \in RF M \cap F^*$ and $L \in \mathcal{L}_U$.

**Proof.** Let $R = X \cap RF M \cap F^*$. We divide the proof into two cases depending on the compactness of $R$. 
Case 1: $R$ is compact. Thanks to (2)$_m$ and Lemma 15.3, it is enough to show that $X$ contains an orbit $z\hat{U}$ for some $\hat{U}$ properly containing $U$ and $z \in R$. By Proposition 9.7, it suffices to find a $U$-minimal subset $Y \subset X$ relative to $R$ and $y \in Y \cap R$ such that $X - yH'$ is not closed; this implies that $yg_n \in X$ for some $g_n \to e$ in $G - H'$.

If $xH'$ is not locally closed, then take any $U$-minimal subset $Y$ of $X$ relative to $R$. If $Y \cap R \subset xH'$, then choose any $y \in Y \cap R$. Then $X - yH' = X - xH'$ cannot be closed, as $xH'$ is not locally closed. If $Y \cap R \not\subset xH'$, then choose $y \in (Y \cap R) - xH'$. Then $X - yH'$ contains $xH'$ and hence cannot be closed.

If $xH'$ is locally closed, then $X - xH'$ is a closed $H'$-invariant subset which intersects $R$ non-trivially. So we can take a $U$-minimal subset $Y \subset X - xH'$ relative to $R$. Take any $y \in Y \cap R$. Then $X - yH'$ is not closed.

Case 2: $R$ is non-compact. Then $R$ accumulates on $Z \cap RF M = BF M \cdot C(H)$. Hence, there exists $x_n \in R$, $z \in BF M \cdot C(H)$ such that $x_n \to z$ as $n \to \infty$.

We claim $X$ contains $z_1v$ where $z_1 \in BF M \cdot C(H)$ and $v$ is a nontrivial element in $V := (H \cap N)^\perp$. Write $x_n = zh_n r_n$ for some $h_n \in H$ and $r_n \in \exp(\mathfrak{h}^\perp)$, where $\mathfrak{h}^\perp$ denotes the $\text{Ad}(H)$-complementary subspace to $\text{Lie}(H)$ in $\mathfrak{g}$. If $r_n \in N(U) = AN C_1(U) C_2(U)$ for some $n$, then the $(H \cap N)^\perp$-component of $r_n$ should be non-trivial since $x_n \in F^*$ and $z \in BF M C(H)$. This implies the claim.

Now suppose $r_n \not\in N(U)$ for all $n$. Then $r_n \not\in N(U_0)$ for some one-parameter subgroup $U_0 = \{ut\}$ inside $U$. Applying Lemma 8.3, with

$$T(x_n) := \{t \in \mathbb{R} : x_n u_t \in RF M\},$$

we obtain $z_1v \in X \cap RF M$ where $z_1 \in BF M \cdot C(H)$ and $v \in V$.

We now claim there exists $z_2 \in BF M \cdot C(H)$ and $V^+ \not\subset \exp(\mathfrak{h}^\perp)$ such that $z_2 V^+ \subset X$. Now by Theorem 15.4, $z_1 \hat{U}$ contains $z_1 \alpha^{-1} A \alpha$ for some $\alpha \in N$. Let $z_2 = z_1 \alpha^{-1} \in BF M \cdot C(H)$. We choose $V^+$ to be the semigroup of $V$ given by $\{\exp(t \log \alpha) : t \geq 0\}$.

Then $z_2 \in BF M \cdot C(H)$, $\log V^+ \not\subset \mathfrak{h}^\perp$, and $z_2 v \in RF M$. It follows that $\limsup (z_2 V^+ \cap RF M) \neq \emptyset$. Hence by Lemma 12.5, there exists $v_n \to \infty$ in $V^+$ such that $z_2 v_n \to y_1$ for some $y_1 \in F^*$. Since

$$(z_2 v_n)(v_n^{-1} V^+) = z_2 V^+ \subset X,$$

we get $y_1 A \hat{U} \subset X$. By (2)$_m$, $\overline{y_1 AUV} = y_1 \hat{L}$ for some $L \in \mathcal{L}_{UV}$. This completes the proof. □

Enlarging a closed orbit of $L \in \mathcal{L}_U$ in $X$. The goal of this subsection is to prove the following proposition which says that if $X = xH'$ contains $x_0 L \cap F^*$ properly for some $L \in \mathcal{L}_U$, then it contains a closed orbit of $\hat{L} \in \mathcal{L}_{\hat{U}}$ where $\dim \hat{U} > \dim(U)$.

Lemma 15.6. Let $L \in \mathcal{L}_U$, and $x_0 L$ be a closed orbit with $x_0 \in F^*$. Consider a sequence $v_n \to \infty$ in $(L \cap N)^\perp$. If $X$ is a closed $U$-invariant subset
containing $\bigcup_n (x_0 L \cap RF_+ M)v_n$, then

$$X \supset y\hat{U}$$

for some $y \in F^*$, and for some connected subgroup $\hat{U} < N$ with $\dim \hat{U} > \dim(U)$.

Proof. We claim that

$$\limsup_{n \to \infty} (x_0 L \cap RF_+ M)v_n \cap \text{core}_e(M) \neq \emptyset$$

where $\text{core}_e(M)$ is defined as in (12.1). It suffices to show that

$$x_nv_nU \cap \text{core}_e(M) \neq \emptyset$$

for some $x_n \in x_0 L \cap RF_+ M$. By Lemma 12.5, this follows if $\pi(x_nv_n U) \not\subset \partial \text{core}_e(M)$ and

$$\limsup_{n \to \infty} x_nv_nU \cap RF M \neq \emptyset.$$

Note that these conditions are equivalent to $x_nv_n \in RF_+ M \cap F^*$, or in the case at hand, to $x_nv_n \in F^*$. Now assume to the contrary, that

$$(x_0 L \cap RF_+ M)v_n \subset RF_+ M - F^*$$

for some $n$. Then since the set $RF_+ M - F^*$ is a closed $A$-invariant set and $e \in \limsup_{n \to \infty} Av_n A$, we would have

$$x_0 L \cap RF_+ M \subset RF_+ M - F^*,$$

a contradiction, as $x_0 \in F^*$. \qed

Corollary 15.7. Let $x_0 L$ be a closed orbit with $x_0 \in F^*$ and $L \in \mathcal{L}_U$. Suppose that $X$ is either

1. a closed $U$-invariant subset containing $(x_0 L \cap RF_+ M)v_n$ for some sequence $v_n \in U^\perp$ tending to $\infty$, or
2. a closed $AU$-invariant subset containing $(x_0 L \cap RF_+ M)v$ for some non-trivial element $v \in (L \cap N)^\perp$.

Then $X$ contains $yUV$ for some $y \in F^*$, and for some one parameter subgroup $V \subset (L \cap N)^\perp$.

Proof. To prove the first case, note

$$(x_0 L \cap RF_+ M)v_n(v_n^{-1}Av_n) \subset X.$$
Proposition 15.8. Assume that (2)\textsubscript{m}, and (3)\textsubscript{m} are true. Suppose that there exists a closed orbit $x_0L$ for some $x_0 \in RF \cap F^*$ and $L \in \mathcal{L}_U$ such that
\begin{equation}
(15.3) \quad x_0L \cap RF_+ M \subset X \quad \text{and} \quad X \neq x_0L \cdot C(H) \cap F.
\end{equation}
Then there exists a closed orbit $x_1\hat{L}$ for some $x_1 \in RF \cap F^*$, and $\hat{L} \in \mathcal{L}_{\hat{U}}$ for some $\hat{U} < N$ with $\dim \hat{U} > \dim(U)$ such that
\[x_1\hat{L} \cap RF_+ M \subset X.\]

Proof. Note that if $X \subset x_0L \cdot C(H)$, then $X = x_0L \cdot C(H) \cap F$. Therefore we assume that $X \not\subset x_0L \cdot C(H)$. First note that the hypothesis implies that $L \neq G$, and hence $\text{co-dim}_{L \cap N}(U) \leq m$. Let $U_1, \cdots, U_\ell$ be one-parameter subgroups of $U$ generating $U$. Similarly, let $U^-_1, \cdots, U^-_\ell$ be one-parameter subgroups generating $U^-$. By Theorem 11.1,
\[\bigcap_{i=1}^\ell \mathcal{G}(U^\pm_i, x_0L) \neq \emptyset.\]
Without loss of generality, we can assume $x_0 \in \bigcap_{i=1}^\ell \mathcal{G}(U^\pm_i, x_0L)$.

Let us write $L = H(\hat{U})C$ for some closed subgroup $C$ of $C(H(\hat{U}))$. Notice from the hypothesis that we have
\[(x_0L \cap RF_+ M) \cdot H' \subset X.\]

Observe that (15.3) implies that $x \not\in x_0L \cdot H' = x_0L \cdot C(H)$.

Since $C < C(H)$, we have $x \not\in x_0H(\hat{U})$. Now choose $w_i \in H'$ such that $xw_i \to x_0$, as $i \to \infty$. Then there exists $g_i \to e$ in $G - LH' = G - L \cdot C(H)$ such that
\[xw_i = x_0g_i.\]
Let us write $g_i = \ell_ir_i$ where $\ell_i \in L$, and $r_i \in \exp(\mathfrak{l}^\perp)$. In particular, $r_i \not\in C(H)$.

Let $x_i = x_0\ell_i$, so that $x_ir_i \in X$.

We now break the remaining proof into several steps.

**Step 1.** We can assume that $x_i \in RF \cap x_0L$, $r_i \not\in C(H)$, and $x_ir_i \in X$.

Since $x_0 \in F^*$, by Lemma 12.2, we can find $w'_i \to w' \in H$ such that $x_0\ell_iw'_i \in RF \cap \mathcal{G}(U^\pm_i, x_0L)$ and
\[\overline{x_0w'U} = x_0L \cap RF_+ M.\]
Writing $x'_i = x_0\ell_iw'_i$ and $r'_i = w'^{-1}_i r_i w'_i$, we have
\[x'_ir'_i = xw_iw'_i \in X,\]
where $x'_i \to x_0w'$ in $x_0L \cap RF M$, and $r'_i \to e$ in $\exp(\mathfrak{l}^\perp)$. Since $F^*$ is $H'$-invariant, we have $x_0w' \in F^*$. Since $F^*$ is open and $x_0w' \in F^*$, it follows that $x'_i \in X \cap RF M \cap F^*$ for sufficiently large $i$'s. Note that $r'_i \not\in C(H)$, as $r_i \not\in C(H)$. This proves the first step.
Step 2. There exists a one-parameter subgroup $U_0$ among $U^\pm_1, \cdots, U^\pm_\ell$ such that $r_i \not\in N(U_0)$ for all $i$’s after passing to a subsequence.

Note that
\[ \exp(h^\perp_k) \cap N(U^+_k) \cap N(U^-_k) \subset C(H(U_k)), \]
if the claim does not hold, then
\[ r_i \in \bigcap_{k=1}^\ell C(H(U_k)) = C(H). \]

Without loss of generality, we may assume $r_i \not\in N(U_0)$ for a one-parameter subgroup $U_0$ inside $U$.

Step 3. We claim that
\[(x_0L \cap RF^+ M)v \subset X \]
for some nontrivial $v \in \tilde{U}^\perp$. Since $x_0 \in \mathcal{G}(U_0, x_0L)$ and as we are assuming that $(2)_m$, and $(3)_m$ hold, we may apply Additional invariance lemma 14.2 to the sequence $x_0l_i r_i \to x_0$ and conclude
\[(x_0L \cap RF^+ M)v \subset X \]
for some nontrivial $v \in \tilde{U}^\perp$.

Step 4. We claim that
\[ x_2 \tilde{U} \subset X \]
for some $x_2 \in RF^+ M \cap F^*$ and $\tilde{U}$ properly containing $U$.

This follows from Corollary 15.7 (2).

Since co-dim$_N(\tilde{U}) \leq m$, the last claim of Step (4) together with $(2)_m$ finishes the proof of the proposition.

We will use the following lemma in the next proposition to deal with the case when $x_0L \subset BF^+ M$.

Proof of $(1)_{m+1}$. Combining Propositions 15.5 and 15.8, we now prove:

Theorem 15.9. If $(2)_m$, and $(3)_m$ are true, then $(1)_{m+1}$ is true.

Proof. Recall that we only need to consider the case $X = \bar{xH'}$ where $x \in F^*$ and $xH'$ is not closed in $F^*$.

By Proposition 15.5, there exists $x_0 \in F^*$, and $L \in \mathcal{L}_U$ such that $x_0L$ is closed, and
\[ x_0L \cap RF^+ M \subset X. \]
Since $X$ is $H'$-invariant, it follows
\[ (x_0L \cap RF^+ M) \cdot H' \subset X. \]

Note that $(x_0L \cap RF^+ M) \cdot H' = x_0L \cdot C(H) \cap F$ is a closed set. We may assume the inclusion in (15.4) is proper, otherwise we have nothing to prove. Then by Proposition 15.8, there exists $\tilde{L} \in \mathcal{L}_{\tilde{U}}$ for some $\tilde{U}$ properly containing $U$, and a closed orbit $x_1\tilde{L}$ with $x_1 \in F^*$ such that
\[ x_1\tilde{L} \cap RF^+ M \subset X. \]
By changing $x_1$ within $x_1\hat{L} \cap \text{RF}_+ M$ if necessary, we may assume $x_1 \in F^*$. If
\[(x_1\hat{L} \cap \text{RF}_+ M) \cdot C(H) \neq X,\]
then we can apply Proposition 15.8 on
\[x_1\hat{L} \cap \text{RF}_+ M \subset X,\]
as $\mathcal{L}_{\hat{U}} \subset \mathcal{L}_U$. Continuing in this fashion, the process terminates in a finite step for a dimension reason, and hence
\[X = (x_1\hat{L} \cap \text{RF}_+ M) \cdot H' = x_1\hat{L} \cdot C(H) \cap F\]
for some $\hat{L} \in \mathcal{L}_U$, completing the proof. \hfill \Box

16. $U$-orbit closures: proof of $(2)_{m+1}$

In this section, we fix a closed orbit $x_0\hat{L}$ for $x_0 \in F^*$ and $\hat{L} \in \mathcal{Q}_U$. Let $U < \hat{L} \cap \hat{N}$ be a connected closed subgroup with $1 \leq \text{co-dim}_{\hat{L} \cap \hat{N}} U \leq m + 1$.

Since $v\hat{L}v^{-1} \in L_U$ for some $v \in N(U)$, by replacing $x$ by $xv$, we may assume without loss of generality that $\hat{L} \in \mathcal{L}_U$. Moreover we may assume that $U \subset \hat{L} \cap \hat{N} \cap \hat{H}$, using an element $m$ of $\mathcal{M}$.

We fix $x \in \text{RF}_M \cap x_0\hat{L} \cap F^*$, and set
\[(16.1) \quad X := x\overline{U} \quad \text{and assume that } X \neq x_0\hat{L} \cap \text{RF}_+ M.\]

Lemma 16.1. Assume that $(1)_{m+1}$ and $(2)_m$ hold. Then
\[\overline{xAU} \cap \mathcal{J}(U, x_0\hat{L}) \neq \emptyset.\]

Proof. Case 1: $\hat{L} = G$. Note that $\mathcal{J}(U) \neq \emptyset$, as it contains a compact $H$-orbit, say, $z\hat{H}$. Since $(1)_{m+1}$ is true, we have
\[x\overline{H} = xQ \cap F\]
for some $Q \in \mathcal{L}_U$. If $Q = G$, i.e., $x\overline{H} = F$, then, for $K_0 := K \cap H$, $x\overline{H} = xAU K_0 = F$ and it contains $z\hat{H}$. Since $K_0 \subset \hat{H}$, it follows that $x\overline{AU} \supset z\hat{H}$, proving the claim. If $Q \neq G$, then $xQ \subset \mathcal{J}(U)$.

Case 2: $\hat{L} \neq G$. In this case, $\text{co-dim}_{\hat{L} \cap \hat{N}} U \leq m$. Hence by $(2)_m,$
\[x\overline{U} = xQ \cap \text{RF}_+ M\]
for some $Q \subset \hat{L}$ contained in $\mathcal{L}_U$. By the hypothesis (16.1) and Lemma 5.10, $\dim Q_{nc} < \dim \hat{L}_{nc}$. Therefore $x \subset \mathcal{J}(U, x_0\hat{L})$. \hfill \Box

Lemma 16.2. Assume that $(1)_{m+1}$ and $(2)_m$ hold. Then
\[\overline{xU} \cap \mathcal{J}(U, x_0\hat{L}) \cap F^* \neq \emptyset.\]

Proof. Step 1. We claim that $\overline{xU} \cap \mathcal{J}(U, x_0\hat{L}) \neq \emptyset$.

First, note $x_0\hat{L} \cap \text{RF}_+ M \subset F^* \subset \mathcal{J}(U, x_0\hat{L})$. Hence we will assume $\overline{xU} \subset F^*$, otherwise the conclusion is trivial. Since $x \in F^*$, $xU$ meets $\text{RF}_M$. Let $Y \subset X$ be a $U$-minimal set with respect to $\text{RF}_M$.

Since $Y \subset F^*$, by Proposition 9.4, there exists an unbounded one-parameter semigroup $L$ inside $AU^+ C_2(U) \cap \hat{L}$ such that $YL \subset Y$. By modifying $L$, we get $YL \subset X(C_2(U) \cap \hat{L})$ for an unbounded one parameter semigroup $L$. 

inside $A(U \cap \hat{L})$. In view of Lemma 8.4, either $L = v^{-1}A^+v$, or $L = V^+$ for some one-parameter semigroup $A^+ \subset A$, $V^+ \subset U \cap \hat{L}$ and $v \in U \cap \hat{L}$ possibly allowing $v = e$. Note it suffices to show that $X(NC_2(U) \cap \hat{L})$ meets $\mathcal{Y}(U, x_0 \hat{L})$, as $\mathcal{Y}(U, x_0 \hat{L})$ is $NC_2(U) \cap \hat{L}$-invariant.

If $L = v^{-1}A^+v$ then $Yv^{-1}A^+ \subset XV^{-1}(C_2(U) \cap \hat{L})$. Let $y \in Y$. We may assume $yv^{-1} \in F^*$, otherwise we have nothing to prove, as $x_0 \hat{L} \cap RF_+ M - F^* \subset \mathcal{Y}(U, x_0 \hat{L})$. Then, replacing $y$ with an element in $yU$ if necessary, we may assume $yv^{-1} \in RF M$. Choose $a_n \to \infty$ in $A^+$. Since $(yv^{-1}a_n)(a_n^{-1}A^+) \subset XV^{-1}(C_2(U) \cap \hat{L})$,

passing to a limit, we get $y_0A^+ \subset \overline{xv^{-1}U}$ for some $y_0 \in RF M$. Hence, $y_0A \subset XV^{-1}(C_2(U) \cap \hat{L})$. Now we get the conclusion, as $y_0AU \subset XV^{-1}(C_2(U) \cap \hat{L})$ and $y_0AU$ meets $\mathcal{Y}(U, x_0 \hat{L})$ by (1)$_{m+1}$ and Lemma 16.1.

Next, assume $L = V^+$, so that $YV^+ \subset X(C_2(U) \cap \hat{L})$. Let $v_n \to \infty$ in $V^+$. Since $X \subset F^*$ from the beginning, we have $Yv_n \subset F^*$. Together with the fact $Yv_n$ is $U$-invariant, this implies $Yv_n$ meets $RF M$. Note $Yv_n(v_n^{-1}V^+) \subset X(C_2(U) \cap \hat{L})$. Choose $y_n \in Yv_n \cap RF M$. As $RF M$ is compact, $y_n \to y_0$ for some $y_0 \in RF M$ as $n \to \infty$, and hence $y_0UV \subset X(C_2(U) \cap \hat{L})$. Since co-dim$_Y(\mathcal{Y}U) \leq m$, the conclusion follows from (2)$_m$.

**Step 2.** We now claim that $\overline{xU} \cap \mathcal{Y}(U, x_0 \hat{L}) \cap F^* \neq \emptyset$.

By the previous step, there exists $y \in \overline{xU} \cap \mathcal{Y}(U, x_0 \hat{L})$. Hence, there exists $L \in \mathcal{Q}_U$ contained in $\hat{L}$ such that $yL$ is closed. By the hypothesis 16.1, $L_{nc}$ is a proper subgroup of $\hat{L}_{nc}$, and hence co-dim$_MD(M \cap N) \leq m$. Hence by (2)$_m$, we may assume that $yL \cap RF_+ M \subset \overline{xU}$ We can further assume that $L_{nc}$ is maximal among those $L \in \mathcal{Q}_U$ such that $yL$ is closed, and $yL \cap RF_+ M \subset \overline{xU}$.

If $yL \cap F^* \neq \emptyset$, then the claim follows easily. Now suppose that $yL \cap F^* = \emptyset$. Let $\hat{U} = L \cap N$. Since $y\hat{U} \subset RF_+ M - F^*$, it follows $y\hat{U} \subset BF M \cdot N(U)$. In particular, $y\hat{U}$ is compact and $yL \cap RF_+ M = yL$.

Let $U_1, \ldots, U_\ell$ be one-parameter subgroups generating $\hat{U}$. We may assume $y \in \cap_{i=1}^\ell \mathcal{Y}(U_i, yL)$ by Corollary 10.4. Since $y \in \overline{xU}$, there exists $u_n \in U$ such that $xu_n \to y$ as $n \to \infty$. Write $xu_n = y\ell_n r_n$ for some $\ell_n \in L$ and $r_n \in \exp(1^+) \cap \hat{L}$ converging to $e$.

Suppose $r_n \in N(U)$ for some $n$. Then since $\overline{xU} = \overline{y\ell_n r_n U} = \overline{y\ell_n U} r_n$ and $yL$ is closed, $x \in \mathcal{Y}(U, x_0 \hat{L}) \cap F^*$, a contradiction.

Now assume $r_n \notin N(U)$. Note that

$$\bigcap_{i=1}^\ell N(U_i) \subset N(U).$$

Hence there exists $U_0$ among $U_1, \ldots, U_\ell$ such that $r_n \notin N(U_0)$. By the additional invariance II proposition 14.3 to the sequence $xu_n \to y$, we obtain $v_k \to \infty$ in $(L \cap N)^-$ such that $yLv_k \subset X$, and $yLv_k \cap RF M \neq \emptyset$. Note $yLv_k(v_k^{-1}Av_k) \subset X$ and $\limsup_k v_k^{-1}Av_k$ contains some nontrivial subgroup.
V ⊂ (L ∩ N) ⊥ . Hence by passing to a subsequence, we get y_1UV ⊂ X for some y_1 ∈ lim sup_{k}(y_{L_{V_k}} ∩ RF M).

Since co-dim_N(UV) ≤ m, there exist ˜L ∈ Q_{UV} and a closed orbit y_1 ˜L such that y_1 ˜L ∩ RF_+ M ⊂ X by (2)_m. This contradicts the maximality assumption on L_{nc}, completing the proof. □

**Lemma 16.3.** Assume (1)_{m+1}, (2)_{m}, (3)_{m} are true. Then (2)_{m+1} is true.

**Proof.** We need to prove that \( \overline{xU} = xL' \cap RF_+ M \) for some \( L' \in Q_U \) contained in ˜L. Then by Lemma 16.2, (2)_{m} and the hypothesis 16.1, there exists \( y \in F^* \) and \( L \in Q_U \) contained in ˜L such that \( yL \) is closed, \( \overline{xU} \) contains \( yL \cap RF_+ M \) and \( L_{nc} \neq ˜L_{nc} \). Hence co-dim_{L ∩ N} U ≤ m.

First, suppose \( xU \subset yL C(H) \). Then there exists \( c \in C(H) \) such that \( xc \in yL \), and \( \overline{xU} = \overline{xcU}c^{-1} \) as \( c \) normalizes \( U \). On the other hand, as co-dim_{L ∩ N}(U) ≤ m, (2)_{m} gives us the exact description of \( \overline{xcU} \). This in turn implies \( \overline{xU} = xL' \cap RF_+ M \) for some \( L' \in Q_U \).

Hence we will assume \( \overline{xU} \nsubseteq yL C(H) \). It suffices to show that \( \overline{xU} \) contains \( y_0L \cap RF_+ M \) for some \( L \in Q_U \), where \( \overline{U} \) properly contains \( U \), and \( y_0 \in RF_+ M \cap F^* \).

By Theorem 11.1, we can assume that \( y \in \bigcap_{i=1}^\ell i \mathcal{H}(U_i, yL) \cap RF M \cap F^* \), where \( U_1, \cdots, U_\ell \) are one-parameter subgroups generating \( U \). As \( y \in \overline{xU} \), there exists \( u_i \in U \) such that \( xu_i \to y \) as \( i \to \infty \). Since \( y \in F^* \), we can assume \( xu_i \in RF M \) after possibly modifying \( u_i \) by Lemma 12.4. We will write \( xu_i = y\ell_i r_i \) where \( \ell_i \in L \) and \( r_i \in \exp(1) \cap \overline{L} - C(H) \).

If \( r_i \in N(U) \) for some \( i \), then \( X = xu_i U = y\ell_i U r_i \). Since \( y\ell_i U \subset yL \), and co-dim_{L ∩ N}(U) ≤ m, \( y\ell_i U = y\ell_i L' \cap RF_+ M \) for some \( L' \in Q_U \). This implies that \( \overline{xU} \) is of the desired form as in the statement of (2)_{m+1}.

Therefore we assume that \( r_i \notin N(U) \) for all \( i \). Hence there exists \( U_0 \) among \( U_1, \cdots, U_\ell \) such that \( r_i \notin N(U_0) \) for all \( i \), by passing to a subsequence. Since \( y_0 \in \mathcal{H}(U_0, yL) \) and \( xu_i \in X \cap RF M - yL \cdot N(U_0) \), we can apply Linearization lemma II to the sequence \( xu_i \to y_0 \) and obtain \( v_N \to \infty \) in \( (L ∩ N) ^{\perp} \) such that

\[ (y_0L \cap RF_+ M)v_N \subset X. \]

We claim that \( (y_0L \cap RF_+ M)v_N \nsubseteq BF M \cdot C(H(U)) \) for any \( N \). If not, the \( A \)-invariance of \( BF M \) implies that

\[ (y_0L \cap RF_+ M) \subset BF M \cdot C(H(U)), \]

contradicting \( y_0 \in F^* \).

Now by Lemma 15.7, there exists \( V \subset (L ∩ N) ^{\perp} \) such that \( y_1UV \subset X \) for some \( y_1 \in F^* \). Hence, the conclusion follows from (2)_{m}. □

**17. Topological equidistribution: proof of (3)_{m+1}**

In this section, we prove (3)_{m+1}. Let \( x\hat{L} \) be a closed orbit for \( x \in RF M \) and \( \hat{L} \in L_U \) such that co-dim_{L}(U) = (m + 1). Let \( y_1 \in RF_+ M, L_i \in L_U, \)
$y_i L_i v_i \subset x \hat{L}$ be closed for some $v_i \in (L_i \cap N) \perp \cap \hat{L}$, satisfying the hypothesis of (3)$_{m+1}$:

- $v_i \to \infty$, or
- $v_i$ is bounded and $y_i L_i$ are all distinct.

**Lemma 17.1.** Assume that (1)$_{m+1}$ and (3)$_m$ are true.

If

$$E := \limsup_{i \to \infty} (y_i L_i v_i \cap RF_+ M)$$

is non-empty, then $E$ contains a $\hat{U}$-orbit of a point in $RF_+ M$, for some $\hat{U}$ properly containing $U$.

**Proof.** We remark that the condition $E \neq \emptyset$ is redundant if $v_i$’s are all bounded. Let us proceed case by case.

**Case 1:** $v_i \to \infty$ as $i \to \infty$.

Since $AU \leq L_i$ for all $i$, it follows

$$(y_i L_i \cap RF_+ M) v_i (v_i^{-1} A U v_i) \subset E.$$ As $E \neq \emptyset$, there exists $y_\infty \in \limsup_i (y_i L_i \cap RF_+ M) v_i$. Clearly, $y_\infty \in RF_+ M$. Also, note $\limsup_i (v_i^{-1} A U v_i)$ contains $A \hat{U}$ for some $\hat{U}$ properly containing $U$. Therefore, we get the conclusion $y_\infty \hat{U} \subset E$ after passing to a limit.

**Case 2:** $v_i$ is bounded and $y_i L_i$ are all distinct.

Passing to a subsequence, $v_i \to v$ for some $v \in V$ as $i \to \infty$. Since

$$\limsup_{i} (y_i L_i \cap RF_+ M) v = \limsup_{i} (y_i L_i \cap RF_+ M) v$$

for this subsequence, it is enough to consider the case when $v_i = v$ for all $i \in \mathbb{N}$. Moreover we can assume that $v = e$ as $RF_+ M$ is $N$-invariant. In particular, $E$ becomes a closed $AU$-invariant set.

Recall $H < L_i$. If there exists infinitely many $L_i$’s contained in $L_{\hat{U}}$ for some $\hat{U}$ properly containing $U$ for infinitely many $i$’s, the conclusion is straightforward. Hence, we may assume $L_i = HC_i$ for some $C_i < C(H)$.

As $E$ is closed and $AU$-invariant, by Lemma 16.1, there exists $y \in RF M$ and $L \in \mathcal{L}_U$ with $L < \hat{L}$ such that $y L \subset x \hat{L}$ is closed and $y L \subset E$.

Then we may assume $L$ is of the form $L = HC$, otherwise the conclusion follows immediately.

By definition, there exists $y_i^* \in y_i L_i \cap RF_+ M$ such that $y_i^* \to y$ as $i \to \infty$. Note that $y_i^* U$ meets $RF M$, and hence by Lemma 12.4, there exists $u_i \in U$ such that $y_i^* u_i$ converges to a point of $y L$ in $RF M$. For simplicity, we will rename this new point of $y L$ as $y$, and also set $y_i = y_i^* u_i$, so that $y_i \to y$ in $RF M$ as $i \to \infty$.

Since $y_i L_i$ are all distinct, $y_i \not\subset y L$ for sufficiently large $i$’s. Hence, $y_i = y_{\ell_i} r_i$ for some $\ell_i \to e$ in $L$ and nontrivial $r_i \to e$ in $\exp(I^\perp)$.

If $r_i \in N$ for all $i$, write $r_i = v_i$, and we have

$$\overline{y_{\ell_i} U v_i} = \overline{y_{\ell_i} v_i U} \subset E.$$
Since \( \text{co-dim}_{L} (U) = 0 \),

\[
\overline{y M U} := \overline{y L} \cap \text{RF}_+ M
\]

for some \( y_i \in \text{RF}_+ M \) and \( L \) with \( y_i L \subset y L \), by Proposition 11.3 and Theorem 11.1. Let \( V_i := \exp(\mathbb{R}^+ \log(v_i)) \). Observing both \( L \) and \( E \) are \( A \)-invariant, and \( V^+_i \subset Av_i A \), one can check

\[
(y_i L \cap \text{RF}_+ M) V^+_i \subset E.
\]

Fix any \( i \), and note that there exists \( v_M \to \infty \) as \( M \to \infty \) in \( V^+_i \) such that \( (y_i L \cap \text{RF}_+ M) v_M \) meets \( \text{RF}_+ M \). Since \( v_M^{-1} Au M \) converges to \( Au^* \) for some \( U^* \) properly containing \( U \) as \( M \to \infty \), we get the conclusion.

If \( r_i \not\in N \) for all \( i \), let \( M \in \mathbb{N} \) be arbitrary. We apply Lemma 8.3 to the sequence of \( k \)-thick subsets

\[
T_i = \{ t \in \mathbb{R} : y_i u t \in \text{RF}_+ M \},
\]

with \( r_i \to 0 \) to get \( v_M \in N \) with \( M \leq \| v_M \| \leq M + 1 \) and \( t_i \in T_i \) such that \( u_{t_i}^{-1} r_i u_{t_i} \to v_M \) as \( i \to \infty \). Passing to a subsequence, \( y_i u t_i \to y M \in \text{RF}_+ M \). Note that

\[
y_i u t_i = y t_i u t_i (u_{t_i}^{-1} r_i u_{t_i}) \in E.
\]

As a result, the sequence \( y_i u t_i \) converges to \( y M v_M^{-1} \in y L \cap \text{RF}_+ M \) and \( y M \in E \). Because \( E \) is \( U \)-invariant, \( \overline{y M U} = \overline{y M v_M^{-1} U v_M} \subset E \). Now that \( \text{co-dim}_{L} (U) = 0 \), using Proposition 11.3 and that \( v_M \) can be made arbitrarily large by letting \( M \to \infty \), same argument as in the previous case when \( r_i \in N \) implies that \( E \) contains a \( U^* \) orbit for some \( U^* \) containing \( U \) properly. \( \square \)

**Proposition 17.2.** If \( (1)_{m+1}, (2)_{m+1}, \) and \( (3)_m \) are true, then \( (3)_{m+1} \) is true.

**Proof.** Let \( x \tilde{L} \) and \( L_i \in \mathcal{L}_U \) be so that \( y L_i v_i = x_i (v_i^{-1} L_i v_i) \) be as in the hypothesis of \( (3)_{m+1} \), and assume \( \text{co-dim}_{L} (U) = (m + 1) \). Let

\[
E := \limsup_{i \to \infty} (y L_i v_i \cap \text{RF}_+ M).
\]

Then \( E \) contains a \( \tilde{U} \)-orbit of a point in \( \text{RF}_+ M \) for some \( \tilde{U} \) properly containing \( U \), by \( (1)_{m+1} \) together with Lemma 17.1. Fix \( \tilde{U} \) of maximal dimension satisfying the above condition.

Since \( \text{co-dim}_{L} (\tilde{U}) \leq m \), by \( (2)_{m} \), there exist \( y \in \text{RF}_+ M, L \in \mathcal{L}_U, v \in (L \cap N)^{-1} \cap \tilde{L} \) and a closed \( y L v \subset x \tilde{L} \), such that

\[
y L v \cap \text{RF}_+ M \subset E.
\]

We claim that \( y \tilde{L} \cap \text{RF}_+ M \subset E \). This implies that

\[
E = y \tilde{L} \cap \text{RF}_+ M
\]

For the claim, we can assume \( v = e \) in \( (17.1) \). Moreover by \( (2)_{m} \) together with Lemma 10.4, after reducing \( L_i \), we may assume

\[
\overline{y U} = y L \cap \text{RF}_+ M.
\]
In case when $yL \subset \partial F$, we may further assume

$$yH \cap \mathcal{F}(U, yL) = \emptyset.$$  

Suppose that $L \neq \tilde{L}$. Then since both $L, \tilde{L} \in \mathcal{L}_U$, the non-compact part of $L$ is strictly smaller than the non-compact part of $\tilde{L}$. The inclusion in (17.1) should be strict, otherwise $y_iL_i v_i \subset yL$ for this $L \in \mathcal{L}_U$, contradicting the hypothesis in (3)$_{m+1}$.

We claim that for sufficiently large $i$'s, we have

$$y_i L_i v_i \cap \text{RF}_+ M = \emptyset.$$  

To see this, observe that $\text{co-dim}_{L_i}(U) \leq m$ and hence by (2)$_m$, there exists $z_i \in y_i L_i v_i \cap \text{RF}_+ M$ such that $z_i \tilde{U} = y_i L_i v_i \cap \text{RF}_+ M$. Then $z_i \notin yL$, otherwise it would mean $y_i L_i v_i \subset yL$, contradicting the hypothesis. Therefore

$$y_i L_i v_i \cap \text{RF}_+ M = \overline{\lim}_{i \to \infty} (y_i L_i v_i - yL) \cap \text{RF}_+ M.$$  

The other inclusion is clear as $y_i L_i v_i$ is closed, proving the claim. Now,

$$E = \limsup_{i \to \infty} y_i L_i v_i \cap \text{RF}_+ M$$  

Combining this with (17.1), we deduce that there exists $x_i \to y$ in $E$ as $i \to \infty$ such that $x_i \notin yL$. By replacing $x_i$ with $x_i u_i$ for some $u_i \in U$ and modifying $y \in yL$ if necessary, we have $x_i \to y$ in $\text{RF}_M$ by Lemma 12.4. Due to our choice of original $y$, we still have equation (17.1) and (17.2) with $v = e$, even after the modification.

Since $E$ is closed and $U$-invariant, applying Lemma 14.3 to the sequence $x_i \to y$ in $E \cap \text{RF}_M$, we can find $v_i \to \infty$ in $(L \cap N)_{\perp}$ and $y_i \in yL \cap \text{RF}_+ M$ such that $y_i v_i \in E \cap \text{RF}_M$ and

$$(yL \cap \text{RF}_+ M) v_i \subset E.$$  

Passing to a subsequence, $y_i v_i \to y' \in E \cap \text{RF}_M$. Writing $L_{nc} = H(\tilde{U})$, we have

$$y_i v_i (v_i^{-1}A \tilde{U} v_i) \in E,$$

passing to a limit we obtain $y' A \tilde{U} \subset E$ for some $\tilde{U}$ properly containing $\tilde{U}$. Since $\text{co-dim}_{L}(\tilde{U}) \leq m$, this would contradict the maximality of $L$ by (2)$_m$, completing the proof. $\square$
Appendix: Orbit closures for $\Gamma \backslash G$ compact case

In this section we give an outline of the proof of the orbit closure theorem for the actions of $H'(U)$ and $U$, assuming that $\Gamma \backslash G$ is compact and there exists at least one closed orbit $z_0 \in SO(d-1,1)$. The $H(U)$-orbit closure classification follows rather easily from the classification of $H'(U)$-orbits by Proposition 15.1.

Of course, this case is a special case of Ratner’s theorem, which was also proved by Shah [44]. We hope that giving an outline of the proof of Theorem 13.1 in this special case will help readers understand the whole scheme of the proof better and see the differences with the infinite volume case more clearly.

Note that in the case at hand, $RF M = F^*_{H(U)} = RF_+ M = \Gamma \backslash G$.

Without loss of generality, we assume that $U \subset SO(d-1,1) \cap N$.

**Theorem 17.3.** Let $x \in \Gamma \backslash G$.

1. We have $xH(U) = xL$ for some $L \in \mathcal{L}_U$.

2. If $xU$ is contained in a closed orbit $x\hat{L}$ for some $\hat{L} \in \mathcal{Q}_U$, then $xU = xL$ where $xL$ is closed for some $L \in \mathcal{Q}_U$.

In the case when $\Gamma \backslash G$ is compact, we don’t need the topological equidistribution statement, which is Theorem 13.1(3) to run the induction argument, thanks to (2.6).

The base case (2)$_0$ follows from a special case of Theorem 11.1.

For $m \geq 0$, we will show that (2)$_m$ implies (1)$_{m+1}$, and that (1)$_{m+1}$ and (2)$_m$ together imply (2)$_{m+1}$.

**Proof of (1)$_{m+1}$.** Let $U < N$ have co-dimension $m+1$ in $N$. By Proposition 15.1, it suffices to show that $X := xH'(U) = xLC(H(U))$ for some $L \in \mathcal{L}_U$. Assume that $xH'(U)$ is not closed.

**Step 1: Find a closed orbit inside $X$.** Propositions 9.4, 9.5, and 9.7 imply:

**Proposition 17.4.** If $Y \subset X$ is $U$-minimal such that for some $y \in Y$, $yg_n \in X$ for a sequence $g_n \to e$ in $G - H'(U)$, then $zU \subset X$ for some connected subgroup $\hat{U}$ containing $U$ properly.

Since the co-dimension of $\hat{U} := UV$ in Proposition 17.4 is less than $m+1$, by (2)$_m$ and Lemma 15.3, it suffices to find $Y$ satisfying this proposition. If $xH'(U)$ is not locally closed, then any $U$-minimal subset $Y \subset X$ does the job. If $xH'(U)$ is locally closed, then any $U$-minimal subset $Y$ of $X - xH'(U)$
Page does the job; note that the set $X - xH'(U)$ is a compact $H'(U)$-invariant subset and hence contains a $U$-minimal subset.

Hence $X$ contains a closed orbit $zL$. We may assume that $X \neq zLC(H(U))$; otherwise, we are done.

**Step 2: Enlarge a closed orbit inside $X$.** Since $zL$ is compact, by Theorem 11.1, we can assume that $zU_i^{\pm}$ is dense in $zL$ where $U_1^{\pm}, \ldots, U_k^{\pm}$ are one-parameter subgroups of $U$ generating $U$. By Corollary 9.3, there exists $g_i \to e$ in $G - LC(H(U))$ such that $zg_i \in X$. We can write $g_i = t_i r_i$ where $r_i \in \exp(t_i)$ and $t_i \to L$. Then $r_i \notin C(H(U))$. Since $\bigcap_{i=1}^k (N(U_i^{+}) \cap N(U_i^{-})) \subset C(H(U))$, $r_i \notin N(U_0)$ for one of the subgroups $U_i^{\pm}$, which we denote by $U_0$. If $U_0 \notin \{U_i^\pm\}$, then replace $U$ by $U^{-}$.

Fix any $k > 1$. Applying (2.6) to the sequence $z_i := zl_i \to z$, the set

$$T(z_i) := \{ t \in \mathbb{R} : z_i u_t \in \Gamma \setminus G - \bigcup_{j=1}^i \mathcal{O}_j \}$$

is a $k$-thick subset (take $0 < \varepsilon < k - 1$).

By Lemma 8.3, there exists $t_i \in T_i$ such that $u_i^{-1} r_i u_t$ converges to a non-trivial element $v \in U^\perp$. Now the sequence $z_i u_{t_i}$ converges to $z_0 \in \mathcal{H}(U, zL)$. Since $zg_i u_{t_i}$ converges to $z_0 v$, we deduce

$$zL v = z_0 v U_0 \subset X$$

and hence $zL V^+ \subset zL(AvA) \subset X$ where $V^+ = \exp(\mathbb{R}_+ \log v)$. Take any sequence $v_i \to \infty$ in $V^+$ such that $zv_i$ converges to some $x_0$. Then $x_0 V \subset \limsup(zv_i)(v_i^{-1} V^+) \subset X$ and hence $X$ contains $x_0 U$ where $U = UV$. By the induction hypothesis (2)$_m$ and Lemma 15.3, $X$ contains a closed orby of $\hat{U}$ for some $\hat{L} \in \mathcal{L}_U$.

**Proof of** (2)$_{m+1}$. Set $X := \overline{x_0 U}$. We assume that $X \neq x_0 \hat{U}$. Since the co-dimension of $U$ in $\hat{L} \cap N$ is at least 1, we may assume without loss of generality that $U < N \cap SO(d - 1, 1)$ using conjugation.

**Step 1: Find a closed orbit inside $X$.** It now follows from (1)$_{m+1}$, (2)$_m$, the hypothesis on the existence of a closed $L_0 := SO(d - 1, 1)$-orbit, and the cocompactness of $AU$ in $H'(U)$ that any $\overline{AU}$-orbit closure intersects $\mathcal{H}(U, x_0 \hat{L})$ (cf. proof of Lemma 16.1).

We claim that $X$ intersects $\mathcal{H}(U, x_0 \hat{L})$. Since $\mathcal{H}(U, x_0 \hat{L})$ is $NC_2(U)$-invariant, it suffices to show $XN C_2(U)$ intersects $\mathcal{H}(U, x_0 \hat{L})$. Let $Y \subset X$ be a $U$-minimal subset. Then $Y g = Y$ for all $g \in L$ for some one-parameter subgroup $L < AU^{\perp} C_2(U)$ by Lemma 9.4; strictly speaking, the cited lemma gives $Y g \subset Y$ for $g$ in a semigroup $L$, but in the case at hand, $Y g \subset Y$ implies $Y g = Y$, since $Y g$ is $U$-minimal again, and hence $Y g^{-1} = Y$ as well.

In view of Lemma 8.4, we get $YA \subset XN C_2(U)$ or $Y vA \subset XN C_2(U)$ for some $v \in N$. In either case, $XN C_2(U)$ contains an $AU$-orbit and hence intersects $\mathcal{H}(U, x_0 \hat{L})$. Hence the claim follows.
Since $X$ intersects $\mathcal{I}(U, x_0\hat{L})$, by applying (2)$_m$, $X$ contains $zL$ for some $L \in Q_U$.

**Step 2: Enlarge a closed orbit inside $X$.** Suppose $L \neq G$ and $X \neq zL$. It suffices to show that $X$ contains a closed orbit $y\hat{L}$ for some $\hat{L} \in \mathcal{L}_{\hat{U}}$ for some $\hat{U}$ properly containing $U$. We may assume $X \nsubseteq zLC(H(U))$; otherwise, the claim follows from (2)$_m$. We may assume $z \in \cap_{i=1}^t \mathcal{I}(U_i, yL)$ where $U_i$'s are one-parameter generating subgroups of $U$. Take a sequence $\lim_{n \to \infty} x_{u_i} = z$ where $u_i \in U$, and write $x_{u_i} = z\ell_i r_i$ where $\ell_i \in L$ and $r_i = \exp q_i$ for $q_i \in l^\perp$. The case of $r_i \in N(U)$ for some $i$ follows from (2)$_m$ (cf. Proof of Lemma 16.3). Hence we may assume $r_i \nsubseteq N(U_0)$ for some $U_0 \in \{U_i\}$.

Fix any $k > 1$. Then $T(z_{i})$ as in (17.3) is a $k$-thick subset. We now repeat the same argument of Step (2) in the proof of (1)$_{m+1}$. By Lemma 8.3, there exists $t_i \in T_i$ such that $u_i^{-1}r_i u_{t_i}$ converges to a non-trivial element $v \in U^\perp$. Now the sequence $z_i u_{t_i}$ converges to $z_0 \in \mathcal{I}(U_0, zL)$. Hence $X \ni z_0 U_0 v = zLv$.

Moreover, by Lemma 8.3, such $v$ can be made of arbitrarily large size, so we get $X \ni zLv_N$ for a sequence $v_N \in U^\perp$ tending to $\infty$. The set $\limsup v_N^{-1}Av_N$ contains a one-parameter subgroup $V \subset U^\perp$. Hence if $y$ is a limit of $zv_N$, then $\limsup(zAv_N) = \limsup zv_N(v_N^{-1}Av_N) \supset yV$.

Hence $X$ contains $yUV$, and the claim follows from (2)$_m$ now.

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