Abstract. We establish an analogue of Ratner’s orbit closure theorem for any connected closed subgroup generated by unipotent elements in $\text{SO}(d, 1)$ acting on the space $\Gamma \backslash \text{SO}(d, 1)$, assuming that the associated hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d$ is a convex cocompact manifold with Fuchsian ends.

For dimension $d = 3$, this was proved earlier by McMullen, Mohammadi and Oh. In a higher dimensional case, the possibility of accumulation on closed orbits of intermediate groups causes serious new issues, but in the end, all orbit closures of unipotent flows are relatively homogeneous.

Our results imply the following: for any $k \geq 1$,

1. the closure of any $k$-horosphere in $M$ is a properly immersed submanifold;
2. the closure of any geodesic $(k + 1)$-plane in $M$ is a properly immersed submanifold;
3. any infinite sequence of maximal properly immersed geodesic $(k + 1)$-planes intersecting core $M$ becomes dense in $M$.

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1. Introduction

Let $G$ be a connected simple linear Lie group and $\Gamma < G$ be a discrete subgroup. An element $g \in G$ is called unipotent if all of its eigenvalues are one, and a subgroup of $G$ is called unipotent if all of its elements are unipotent. Let $U$ be a connected unipotent subgroup of $G$, or more generally, any connected closed subgroup of $G$ generated by unipotent elements in it. We are interested in the action of $U$ on the homogeneous space $\Gamma \backslash G$ by right translations.

If the volume of the homogeneous space $\Gamma \backslash G$ is finite, i.e., if $\Gamma$ is a lattice in $G$, then Moore’s ergodicity theorem [32] says that for almost all $x \in \Gamma \backslash G$, $xU$ is dense in $\Gamma \backslash G$. While this theorem does not provide any information for a given point $x$, the celebrated Ratner’s orbit closure theorem [38], which was a conjecture of Raghunathan, states that

\begin{equation}
\text{the closure of every } U\text{-orbit is homogeneous},
\end{equation}

that is, for any $x \in \Gamma \backslash G$, $\overline{xU} = xL$ for some connected closed subgroup $L < G$ containing $U$. Ratner’s proof is based on her classification of all $U$-invariant ergodic probability measures [37] and the work of Dani and Margulis [11] on the non-divergence of unipotent flow. Prior to her work, some important special cases of (1.1) were established by Margulis [22], Dani-Margulis ([9], [10]) and Shah ([44], [43]) by topological methods. This theorem is a fundamental result with numerous applications.

It is natural to ask if there exists a family of homogeneous spaces of infinite volume where an analogous orbit closure theorem holds. When the volume of $\Gamma \backslash G$ is infinite, the geometry of the associated locally symmetric space turns out to play an important role in this question. The first orbit closure theorem in the infinite volume case was established by McMullen, Mohammadi, and Oh ([27], [28]) for a class of homogeneous spaces $\Gamma \backslash \text{SO}(3, 1)$ which arise as the frame bundles of convex cocompact hyperbolic 3-manifolds with Fuchsian ends.

Our goal in this paper is to show that a similar type of orbit closure theorem holds in the higher dimensional analogues of these manifolds.

We present a complete hyperbolic $d$-manifold $M = \Gamma \backslash \mathbb{H}^d$ as the quotient of the hyperbolic space by the action of a discrete subgroup

$$\Gamma < G = \text{SO}^\circ(d, 1) \cong \text{Isom}^+(\mathbb{H}^d)$$

where $\text{SO}^\circ(d, 1)$ denotes the identity component of $\text{SO}(d, 1)$. The geometric boundary of $\mathbb{H}^d$ can be identified with the sphere $S^{d-1}$. The limit set
Λ ⊂ $\mathbb{S}^{d-1}$ of $\Gamma$ is the set of all accumulation points of an orbit $\Gamma x$ in the compactification $\mathbb{H}^d \cup \mathbb{S}^{d-1}$ for $x \in \mathbb{H}^d$.

The convex core of $M$ is a submanifold of $M$ given by the quotient

$$\text{core } M = \Gamma \setminus \text{hull}(\Lambda)$$

where $\text{hull}(\Lambda) \subset \mathbb{H}^d$ is the smallest convex subset containing all geodesics in $\mathbb{H}^d$ connecting points in $\Lambda$. When core $M$ is compact, $M$ is called convex cocompact.

**Figure 1.** A convex cocompact hyperbolic manifold with non-empty Fuchsian ends

**Convex cocompact manifolds with Fuchsian ends.** Following the terminology introduced in [15], we define:

**Definition 1.1.** A convex cocompact hyperbolic $d$-manifold $M$ is said to have Fuchsian ends if core $M$ has non-empty interior and has totally geodesic boundary.

The term *Fuchsian ends* reflects the fact that each component of the boundary of core $M$ is a $(d - 1)$-dimensional closed hyperbolic manifold, and each component of the complement $M \setminus \text{core}(M)$ is diffeomorphic to the product $S \times (0, \infty)$ for some closed hyperbolic $(d - 1)$-manifold $S$ (see Figure 1).

Convex cocompact hyperbolic $d$-manifolds with non-empty Fuchsian ends can also be characterized as convex cocompact hyperbolic manifolds whose limit sets satisfy:

$$\mathbb{S}^{d-1} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

where $B_i$’s are round balls with mutually disjoint closures (see Figure 2). Hence for $d = 2$, any non-elementary convex cocompact hyperbolic surface
has Fushian ends. The double of the core of a convex cocompact hyperbolic $d$-manifold with non-empty Fuchsian ends is a closed hyperbolic $d$-manifold.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Limit set of a convex cocompact hyperbolic 4-manifold with non-empty Fuchsian ends}
\end{figure}

Any convex cocompact hyperbolic manifold with non-empty Fuchsian ends is constructed in the following way. Begin with a closed hyperbolic $d$-manifold $N_0$ with a fixed collection of finitely many, mutually disjoint, properly embedded totally geodesic hypersurfaces. Cut $N_0$ along those hypersurfaces to obtain a compact hyperbolic manifold $W$ with totally geodesic boundary hypersurfaces. There is a canonical procedure of extending each boundary hypersurface to a Fuchsian end, which results in a convex cocompact hyperbolic manifold $M$ (with Fushian ends) which is diffeomorphic to the interior of $W$.

By Mostow rigidity theorem, there are only countably infinitely many convex cocompact hyperbolic manifolds with Fuchsian ends of dimension at least 3. On the other hand, for a fixed closed hyperbolic $d$-manifold $N_0$ with infinitely many properly immersed geodesic hypersurfaces,\footnote{Any closed arithmetic hyperbolic manifold has infinitely many properly immersed geodesic hypersurfaces provided it has at least one. This is due to the presence of Hecke operators [40].} one can produce infinitely many non-isometric convex compact hyperbolic $d$-manifolds with non-empty Fuchsian ends; for each properly immersed geodesic hypersurface $f_i(H^{d-1})$ for a totally geodesic immersion $f_i : H^{d-1} \to N_0$, there is a finite covering $N_i$ of $N_0$ such that $f_i$ lifts to $H^{d-1} \to N_i$ with image $S_i$ being properly imbedded in $N_i$ [18]. Cutting and pasting $N_i$ along $S_i$ as described above produces a hyperbolic manifold $M_i$ with Fushian ends. When the volumes of $S_i$ are distinct, $M_i$'s are not isometric to each other.

**Orbit closures.** In the rest of the introduction, we assume that for $d \geq 2$,

$M$ is a convex cocompact hyperbolic $d$-manifold with Fushian ends.
The homogeneous space $\Gamma \backslash G$ can be regarded as the bundle $F M$ of oriented frames over $M$. Let $A = \{a_t : t \in \mathbb{R}\} < G$ denote the one parameter subgroup of diagonalizable elements whose right translation actions on $\Gamma \backslash G$ correspond to the frame flow. Let $N \simeq \mathbb{R}^{d-1}$ denote the contracting horospherical subgroup:

$$N = \{g \in G : a_{-t} g a_t \to e \text{ as } t \to +\infty\}.$$ 

We denote by $RF M$ the renormalized frame bundle of $M$:

$$RF M := \{x \in \Gamma \backslash G : xA \text{ is bounded}\},$$

and also set

$$RF_+ M := \{x \in \Gamma \backslash G : xA^+ \text{ is bounded}\}$$

where $A^+ = \{a_t : t \geq 0\}$. When Vol$(M) < \infty$, we have

$$RF M = RF_+ M = \Gamma \backslash G.$$ 

In general, $RF M$ projects into core $M$ (but not surjective in general) and $RF_+ M$ projects onto $M$ under the basepoint projection $\Gamma \backslash G \to M$. The sets $RF M$ and $RF_+ M$ are precisely non-wandering sets for the actions of $A$ and $N$ respectively [49].

For a connected closed subgroup $U < N$, we denote by $H(U)$ the smallest closed simple Lie subgroup of $G$ which contains both $U$ and $A$. If $U \simeq \mathbb{R}^k$, then $H(U) \simeq SO^0(k+1,1)$. A connected closed subgroup of $G$ generated by one-parameter unipotent subgroups is, up to conjugation, of the form $U < N$ or $H(U)$ (cf. Corollary 3.8).

We set $F_{H(U)} := RF_+ M \cdot H(U)$, which is a closed subset. It is easy to see that if $x \notin RF_+ M$ (resp. $x \notin F_{H(U)}$), then $xU$ (resp. $xH(U)$) is closed in $\Gamma \backslash G$. On the other hand, $xU$ is dense in $RF_+ M$ for almost all $x \in RF_+ M$, with respect to a unique $N$-invariant locally finite measure, called the Burger-Roblin measure by the work of Mohammadi-Oh [31] for $d = 3$ and Maucourant-Schapira for general $d \geq 3$ [30] (see section 12).

**Orbit closures are relatively homogeneous.** We define the collection $\mathcal{L}_U$ of all subgroups of the form $H(\tilde{U})C$ where $U < \tilde{U} < N$ and $C$ is a closed subgroup of the centralizer of $H(\tilde{U})$ satisfying the following:

$$\mathcal{L}_U := \left\{ L = H(\tilde{U})C : \begin{matrix}
\text{for some } z \in RF_+ M, \ zL \text{ is closed in } \Gamma \backslash G \\
\text{and } \text{Stab}_L(z) \text{ is Zariski dense in } L 
\end{matrix} \right\}.$$

We also define:

$$\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N\}.$$

In view of the previous discussion, the following theorem gives a classification of orbit closures for all connected closed subgroups of $G$ generated by unipotent one-parameter subgroups:

**Theorem 1.2.** Let $M = \Gamma \backslash \mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends, and let $U < N$ be a non-trivial connected closed subgroup.
(1) \((H(U)\)-orbit closures) For any \(x \in RF \cdot H(U)\),

\[
\overline{xH(U)} = xL \cap F_{H(U)}
\]

where \(xL\) is a closed orbit of some \(L \in \mathcal{L}_U\).

(2) \((U\)-orbit closures) For any \(x \in RF_+ M\),

\[
xU = xL \cap RF_+ M
\]

where \(xL\) is a closed orbit of some \(L \in \mathcal{Q}_U\).

(3) (Equidistributions) Let \(x_iL_i v_i\) be a sequence of maximal \(^2\) closed orbits intersecting \(RF M\), where \(x_i \in RF_+ M\), \(L_i \in \mathcal{L}_U\), and \(v_i \in \mathbb{N}\). Suppose either

- \(v_i \to \infty\) modulo \(L_i\); or
- \(v_i\) is bounded modulo \(L_i\) and \(x_iL_i\) are all distinct.

Then

\[
\lim_{i \to \infty} x_iL_i v_i \cap RF_+ M = RF_+ M.
\]

where the limit is taken in the Hausdorff topology on the space of all closed subsets in \(\Gamma \backslash G\).

Remark 1.3. (1) If \(x \in F_{H(U)} - RF \cdot H(U)\), then \(xH(U)\) is contained in an end component of \(M\) under the projection \(\Gamma \backslash G \to M\), and its closure is not relatively homogeneous in \(F_{H(U)}\). More precisely,

\[
\overline{xH(U)} = xLV^+ H(U)
\]

for some \(L \in \mathcal{L}_U\), and some one-parameter semigroup \(V^+ < \mathbb{N}\) (see Theorem 11.5).

(2) If \(M\) has empty ends, i.e., if \(M\) is compact, Theorem 1.2(1) and (2) are special cases of Ratner's theorem [38], also proved by Shah [45] independently, and Theorem 1.2(3) follows from Mozes-Shah equidistribution theorem [33].

Theorem 1.2(1) and (2) can also be presented as follows in a unified manner:

**Corollary 1.4.** Let \(H < G\) be a connected closed subgroup generated by unipotent elements in it. Assume that \(H\) is normalized by \(A\). For any \(x \in RF M\), the closure of \(xH\) is homogeneous in \(RF M\), that is,

\[
\overline{xH} \cap RF M = xL \cap RF M
\]

where \(xL\) is a closed orbit of some \(L \in \mathcal{Q}_U\).

Remark 1.5. If \(\Gamma\) is contained in \(G(\mathbb{Q})\) for some \(\mathbb{Q}\)-structure of \(G\), and \([g]L\) is a closed orbit appearing in Corollary 1.4, then \(L\) is defined by the condition that \(gL^{-1}\) is the smallest connected \(\mathbb{Q}\)-subgroup of \(G\) containing \(gHg^{-1}\).  

\(^2\)This means that if \(x_i' L_i v_i'\) is a closed orbit for some \(x_i' \in RF_+ M\), \(L_i' \in \mathcal{L}_U\) and \(v_i' \in \mathbb{N}\) and \(x_i L_i v_i \subset x_i' L_i v_i' \neq \Gamma \backslash G\), then \(x_i L_i v_i = x_i' L_i v_i'\).
Geodesic planes, horospheres and spheres. We state implications of our main theorems on the closures of geodesic planes and horospheres of the manifold \( M \), as well as on the \( \Gamma \)-orbit closures of spheres in \( \mathbb{S}^{d-1} \).

A geodesic \( k \)-plane \( P \) in \( M \) is the image of a totally geodesic immersion \( f : \mathbb{H}^k \to M \), or equivalently, the image of a geodesic \( k \)-subspace of \( \mathbb{H}^d \) under the covering map \( \mathbb{H}^d \to M \). If \( f \) factors through the covering map \( \mathbb{H}^k \to \Gamma_0 \backslash \mathbb{H}^k \) for a convex cocompact hyperbolic \( k \)-manifold with Fuchsian ends, we call \( P = f(\mathbb{H}^k) \) a convex cocompact geodesic \( k \)-plane with Fuchsian ends.

**Theorem 1.6.** Let \( M = \Gamma \backslash \mathbb{H}^d \) be a convex cocompact hyperbolic manifold with Fuchsian ends, and let \( P \) be a geodesic \( k \)-plane of \( M \) for some \( k \geq 2 \).

1. If \( P \) intersects core \( M \), then \( \overline{P} \) is a properly immersed convex cocompact geodesic \( m \)-plane with Fuchsian ends for some \( m \geq k \).
2. Otherwise, \( P \) is contained in some Fuchsian end \( E = S_0 \times (0, \infty) \) of \( M \), and either \( P \) is properly immersed or \( \overline{P} \) is diffeomorphic to the product \( S \times [0, \infty) \) for a closed geodesic \( m \)-plane \( S \) of \( S_0 \) for some \( k \leq m \leq d - 1 \).

In particular, the closure of a geodesic plane of dimension at least 2 is a properly immersed submanifold of \( M \).

We also obtain:

**Theorem 1.7.**

(1) Any infinite sequence of maximal properly immersed geodesic planes \( P_i \) of \( M \) intersecting core \( M \) becomes dense in \( M \), i.e.,

\[
\lim_{i \to \infty} P_i = M
\]
where the limit is taken in the Hausdorff topology on the space of all closed subsets in \( M \).

(2) There are only countably many properly immersed geodesic planes of dimension at least 2 intersecting core \( M \).

(3) If \( \text{Vol}(M) = \infty \), there are only finitely many maximal properly immersed bounded geodesic planes of dimension at least 2.

In fact, Theorem 1.7(3) holds for any convex cocompact hyperbolic \( d \)-manifold (see Remark 18.3).

A \( k \)-horosphere in \( \mathbb{H}^d \) is a Euclidean sphere of dimension \( k \) which is tangent to a point in \( \mathbb{S}^{d-1} \). A \( k \)-horosphere in \( M \) is simply the image of a \( k \)-horosphere in \( \mathbb{H}^d \) under the covering map \( \mathbb{H}^d \to M = \Gamma \backslash \mathbb{H}^d \).

**Theorem 1.8.** Let \( \chi \) be a \( k \)-horosphere of \( M \) for \( k \geq 1 \). Then either

(1) \( \chi \) is properly immersed; or
(2) \( \overline{\chi} \) is a properly immersed \( m \)-dimensional submanifold, parallel to a convex cocompact geodesic \( m \)-plane of \( M \) with Fuchsian ends for some \( m \geq k + 1 \).
By abuse of notation, let \( \pi \) denote both base point projection maps \( G \to \mathbb{H}^d \) and \( \Gamma \backslash G \to M \) where we consider an element \( g \in G \) as an oriented frame over \( \mathbb{H}^d \).

Let \( H' = SO^o(k+1,1) \) \( SO(d-k-1), 1 \leq k \leq d-2 \). The quotient space \( G/H' \) parametrizes all oriented \( k \)-spheres in \( S^{d-1} \), which we denote by \( C^k \).

For each \( H' \)-orbit \( gH' \subset G \), the image \( \pi(gH') \subset \mathbb{H}^d \) is an oriented geodesic \((k+1)\)-plane and the boundary \( \partial(\pi(gH')) \subset S^{d-1} \) is an oriented \( k \)-sphere.

Passing to the quotient space \( \Gamma \backslash G \), this gives bijections among:

1. the space of all closed \( H' \)-orbits \( xH' \subset \Gamma \backslash G \) for \( x \in RF_M \);
2. the space of all oriented properly immersed geodesic \((k+1)\)-planes \( P \) in \( M \) intersecting core \( M \);
3. the space of all closed \( \Gamma \)-orbits of oriented \( k \)-spheres \( C \in C^k \) with \( \# C \cap \Lambda \geq 2 \)

If \( U := H' \cap N \), then any \( k \)-horosphere in \( M \) is given by \( \pi(xU) \) for some \( x \in \Gamma \backslash G \).

In view of these correspondences, Theorems 1.6, 1.7 and 1.8 follow from Theorem 1.2, Theorem 11.5, and Corollary 5.8.

We also obtain the following description on \( \Gamma \)-orbits of a sphere of any positive dimension.

**Corollary 1.9.** Let \( 1 \leq k \leq d-2 \).

1. Let \( C \in C^k \) with \( \# C \cap \Lambda \geq 2 \). Then there exists a sphere \( S \in C^m \) such that \( \Gamma S \) is closed in \( C^m \) and

\[
\overline{\Gamma C} = \{ D \in C^k : D \cap \Lambda \neq \emptyset, D \subset \Gamma S \}.
\]

2. Let \( C_i \in C^k \) be an infinite sequence of spheres with \( \# C_i \cap \Lambda \geq 2 \) such that \( \Gamma C_i \) is closed in \( C^k \). Assume that \( \Gamma C_i \) is maximal in the sense that there is no proper sphere \( S \subset S^{d-1} \) which properly contains \( C_i \) and that \( \Gamma S \) is closed. Then as \( i \to \infty \),

\[
\lim_{i \to \infty} \Gamma C_i = \{ D \in C^k : D \cap \Lambda \neq \emptyset \}
\]

where the limit is taken in the Hausdorff topology on the space of all closed subsets in \( C^k \).

3. If \( \Lambda \neq S^{d-1} \), there are only finitely many maximal closed \( \Gamma \)-orbits of spheres of positive dimension contained in \( \Lambda \).

**Remark 1.10.** (1) As mentioned before, the main results of this paper for \( d = 3 \) were proved by McMullen, Mohammadi, and Oh ([27], [28]). We refer to [27] for counterexamples to Theorem 1.2 for certain quasi-Fuchsian 3-manifolds.

(2) A convex cocompact hyperbolic 3-manifold with Fuchsian ends (which was referred to as a **rigid acylindrical hyperbolic 3-manifold** in [27]) has a huge deformation space parametrized by the product of the Teichmüller spaces of the boundary components of core \( M \) (cf. [19]). Any convex cocompact acylindrical hyperbolic 3-manifold is a quasi-conformal conjugation of a rigid acylindrical hyperbolic 3-manifold.
An analogue of Theorem 1.2(1) was obtained for all convex cocompact acylindrical hyperbolic 3-manifolds in [29] and for all geometrically finite acylindrical hyperbolic 3-manifolds in [4].

(3) For $d \geq 4$, Kerckhoff and Storm showed that a convex cocompact hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d$ with non-empty Fuchsian ends does not allow any non-trivial deformation, in the sense that the representation of $\Gamma$ into $G$ is infinitesimally rigid [15].

**Remark 1.11.** We discuss an implication of Theorem 1.2(2) on the classification question on all $U$-invariant ergodic locally finite measures on $RF_+ M$. There exists a canonical geometric $U$-invariant measure on each closed orbit $xL$ in Theorem 1.2(2): we write $L = v^{-1}H(\widehat{U})Cv$. As $v$ centralizes $U$, let’s assume $v = e$ without loss of generality. Denoting by $p : L \to H(\widehat{U})$ the canonical projection, the subgroup $p(\text{Stab}_L(x))$ is a convex cocompact Zariski dense subgroup of $H(\widehat{U})$, and hence there exists a unique $\widehat{U}$-invariant locally finite measure on $p(\text{Stab}_L(x)) \backslash H(\widehat{U})$, called the Burger-Roblin measure ([6], [41], [34], [49]). Now its $C$-invariant lift to $(L \cap \text{Stab}_L(x)) \backslash L$ defines a unique $\widehat{U}C$-invariant locally finite measure, say $m^{BR}_{xL}$, whose support is equal to $xL \cap RF_+ M$. Moreover $m^{BR}_{xL}$ is $U$-ergodic (cf. section 12). A natural question is the following:

> is every ergodic $U$-invariant locally finite Borel measure in $RF_+ M$

> proportional to some $m^{BR}_{xL}$?

An affirmative answer would provide an analogue of Ratner’s measure classification [37] in this setup. Theorem 1.2(2) implies that the answer is yes, at least in terms of the support of the measure.

**Remark 1.12.** Ratner’s orbit closure theorem [38] uses her classification of all invariant ergodic measures for unipotent flows [37]. This is based on the following fact: let $\text{Vol}(\Gamma \backslash G) < \infty$, $U = \{u_t\}$ be a one-parameter unipotent subgroup of $G$ and $x \in \Gamma \backslash G$. Let $\mu_T$ be the probability measure supported on the segment $\{xu_t : t \in [-T,T]\}$ obtained by the push-forward of the normalized Lebesgue measure on $[-T,T]$. Then any weak-limit of $\mu_T$, as $T \to \infty$, defines a probability $U$-invariant measure supported on the closure $\overline{xU}$, by a theorem of Dani-Margulis [11].

On the other hand, if $\text{Vol}(\Gamma \backslash G) = \infty$, for almost all $x$, any weak-limit of $\mu_T$ diverges to infinity. Hence it is not immediately clear how the measure classification (even if it existed) would shed any light in the orbit closure classification problem.

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2. Outline of the proof

Topological proofs of orbit closure theorems go back to Margulis’ proof of Oppenheim conjecture [22], and have been extended by Dani-Margulis ([9], [10]) and Shah ([44], and [43]) in certain special cases. Our proofs continue in the same spirit; the only place where a measure is used is in the place where we use the ergodicity of the Burger-Roblin measure to guarantee that there are many generic points for any unipotent one-parameter subgroup of $N$.

We will explain the strategy of our proof with an emphasis on the difference between finite and infinite volume case and the difference between dimension 3 and higher case.

**Thick recurrence of unipotent flows.** Let $U_0 = \{u_t : t \in \mathbb{R}\}$ be a one-parameter unipotent subgroup of $N$. The main obstacle of carrying out unipotent dynamics in a homogeneous space of infinite volume is the scarcity of recurrence of unipotent flow. In a compact homogeneous space, every $U_0$-orbit stays in a compact set for the obvious reason. Already in a noncompact homogeneous space of finite volume, understanding the recurrence of $U_0$-orbit is an important issue. Margulis showed that any $U_0$-orbit is recurrent to a compact subset [20], and Dani-Margulis [11] showed that for any $x \in \Gamma \setminus G$, and for any $\varepsilon > 0$, there exists a compact subset $\Omega \subset \Gamma \setminus G$ such that

$$\ell\{t \in [0,T] : xu_t \in \Omega\} \geq (1-\varepsilon)T$$

for all large $T \gg 1$, where $\ell$ denotes the Lebesgue measure on $\mathbb{R}$. This non-divergence of unipotent flows is an important ingredient of Ratner’s orbit closure theorem [38].

In contrast, when $\Gamma \setminus G$ has infinite volume, for any compact subset $\Omega \subset \Gamma \setminus G$, and for almost all $x$ (with respect to any Borel measure $\mu$ on $\mathbb{R}$),

$$\mu\{t \in [0,T] : xu_t \in \Omega\} = o(T)$$

for all $T \gg 1$ [1].

Nonetheless, the pivotal reason that we can work with convex cocompact hyperbolic manifolds of non-empty Fuchsian ends is the following thick recurrence property that they possess: there exists $k > 1$ such that for any $y \in RF \, M$, the return time

$$T(y) := \{t \in \mathbb{R} : yu_t \in RF \, M\}$$

is $k$-thick, in the sense that for any $\lambda > 0$,

$$T(y) \cap (-k\lambda, -\lambda] \cup [\lambda, k\lambda]) \neq \emptyset.$$  

(2.1)

This recurrence property was first observed by McMullen, Mohammadi and Oh [27] in the case of dimension 3 in order to get an additional invariance of a relative $U_0$-minimal subset with respect to $RF \, M$ by studying the polynomial divergence property of $U_0$-orbits of two nearby $RF \, M$-points.
Beyond $d = 3$. In a higher dimensional case, the possible presence of closed orbits of intermediate subgroups introduces a variety of serious hurdles. Roughly speaking, calling the collection of all such closed orbits as the singular set and its complement as the generic set, one of the main new ingredients of this paper is the avoidance of the singular set along the $k$-th thick recurrence of $U_0$-orbits to $RFM$ for a sequence of $RFM$-points limiting at a generic point. Its analogue in the finite volume case was proved by Dani-Margulis [12] and also independently by Shah [42] based on the linearization methods.

Road map for induction. Theorem 1.2 is proved by the induction argument on the co-dimension of $U$ inside the horospherical subgroup of an intermediate closed orbit. For each $i = 1, 2, 3$, let us say that $(i)_m$ holds, if Theorem 1.2$(i)$ is true for all $U$satisfying $\text{co-dim}_N(U) \leq m$.

Roughly speaking, we show that the validity of $(2)_m$ and $(3)_m$ implies that of $(1)_{m+1}$, the validity of $(1)_{m+1}$, $(2)_m$, and $(3)_m$ implies that of $(2)_{m+1}$ and the validity of $(1)_{m+1}$, $(2)_{m+1}$, and $(3)_m$ implies that of $(3)_{m+1}$. In order to give an outline of the proof of $(1)_{m+1}$, we suppose that $\text{co-dim}_N(U) \leq m + 1$. Let

$$F := RF_+ M \cdot H(U), \quad F^* := \text{Interior}(F), \quad \partial F := F - F^*.$$ 

For any $x \in \partial F \cap RFM$, $xU$ lies in the compact homogeneous space of a subgroup isomorphic to $\text{SO}^0(d - 1, 1)$, and (1) follows from the finite volume case ([38], [45]).

Let $x \in F^* \cap RFM$, and consider

$$X := xH(U) \subset F.$$ 

The strategy in proving $(1)_{m+1}$ for $X$ consists of two steps:

1. (Find) Find a closed $L$-orbit $x_0L$ with $x_0 \in F^* \cap RFM$ such that $x_0L \cap F$ contained in $X$ for some $L \in \mathcal{L}_U$;
2. (Enlarge) If $X \not\subset x_0L \mathcal{C}(H(U))$, then enlarge $x_0L$ to a bigger closed orbit $x_1\hat{L}$ so that $x_1\hat{L} \cap F \subset X$ where $x_1 \in F^* \cap RFM$ and $\hat{L} \in \mathcal{L}_{\hat{U}}$ for some $\hat{U} < N$ containing $L \cap N$ properly.

The enlargement process must end after finitely many steps because of the dimension reason. Finding a closed orbit as in (1) is based on the study of the relative $U$-minimal sets and the unipotent blow up argument using the polynomial divergence of $U$-orbits of nearby $RFM$-points. To explain some ideas behind the enlargement step, suppose that we are given an intermediate closed $L$-orbit with $x_0L \cap F \subset X$, and a one-parameter subgroup $U_0 = \{u_t\}$ of $U$ such that $x_0U_0$ is dense in $x_0L \cap RF_+ M$. Such $L$ turns out to be reductive always, and hence the Lie algebra of $G$ can be decomposed into the $\text{Ad}(L)$-invariant subspaces $l \oplus l^\perp$ where $l$ denotes the

---

3To be precise, we need to carry out the induction argument for orbits contained in a closed orbit $x_0\hat{L}$ and use induction on the co-dimension of $U$ in $\hat{L} \cap N$ (cf. section 14).

4the notation $C(S)$ denotes the identity component of the centralizer of $S$
Lie algebra of $L$. Suppose that we could arrange a sequence $x_0 g_i \to x_0$ in $X$ for some $g_i \to e$ such that writing $g_i = \ell_i r_i$ with $\ell_i \in L$ and $r_i \in \exp(l^\perp)$, the following conditions are satisfied:

- $\ell_i \notin N(U_0)$;
- $x_0 \ell_i \in RF_M$.

Then the $k$-thick return property of $x_0 \ell_i \in RF_M$ along $U_0$ yields a sequence $u_{i_t} \in U_0$ such that

$$x_0 \ell_i u_{i_t} \to x_1 \in RF_M \cap x_0 L \quad \text{and} \quad u_{i_t}^{-1} r_i u_{i_t} \to v$$

for some non-trivial element $v \in N - L$. This gives us a point $x_1 v \in X$.

(2.2) If we could guarantee that $x_1$ is a generic point for $U$ in $x_0 L$, in the sense that its $U$-orbit $x_1 U$ is not contained in a closed orbit of any proper subgroup of $L$, then $x_1 U$ must be equal to $x_0 L \cap RF_+ M$ by the induction hypothesis (2)\textsubscript{m}, since the codimension of $U$ inside $L \cap N$ is at most $m$. Then

$$\overline{x_1 uU} = \overline{x_1 U} v = x_0 L v \cap RF_+ M \subset X.$$  

Using the $A$-invariance of $X$ and the fact that the double coset $AvA$ contains a one-parameter unipotent semigroup $V^+$, we get $x_0 LV^+ \cap RF_+ M \subset X$.

(2.3) Assuming that $x_0 \in F^* \cap RF_+ M$,

we can promote $V^+$ to a one-parameter group $V$, and find an orbit of $\tilde{U} := (L \cap N)V$ of a bigger unipotent subgroup contained in $X$. This enables us to use the induction hypothesis (2)\textsubscript{m} again to complete the enlargement step. Note that if $x_1$ is not generic for $U$ in $x_0 L$, the closure of $x_1 U$ may be stuck in a smaller closed orbit inside $x_0 L$, in which case $\overline{x_1 U} v$ may not be bigger than $x_0 L$ in terms of the dimension, resulting in no progress.

We now explain how we establish (2.2).

**Avoidance of the singular set along the thick return time.** Let $U_0 = \{u_t\}$ be a one parameter unipotent subgroup of $U$. We denote by $\mathcal{J}(U_0)$ the collection of all closed $U_0$-invariant subsets of the form $xL$ where $x \in RF_+ M$ and $L \in \mathcal{Q}_{U_0}$ is a proper connected closed subgroup of $G$. This set is called the singular set for $U_0$. Its complement in $RF_+ M$ is denoted by $\mathcal{G}(U_0)$, and called the set of generic elements of $U_0$. We have

$$\mathcal{J}(U_0) = \bigcup_{H \in \mathcal{H}} \Gamma \backslash \Gamma X(H, U_0)$$

where $\mathcal{H}$ is the countable collection of all proper connected closed subgroups $H$ of $G$ containing a unipotent element such that $\Gamma \backslash \Gamma H$ is closed and $H \cap \Gamma$ is Zariski dense in $H$, and $X(H, U_0) := \{g \in G : gU_0 g^{-1} \subset H\}$ (Proposition
We define $\mathcal{E} = \mathcal{E}_{U_0}$ to be the collection of subsets of $\mathcal{S}(U_0) \cap RF\, M$ of the form
\[
\bigcup \Gamma \backslash \Gamma H_i D_i \cap RF\, M
\]
where $H_i \in \mathcal{H}$ is a finite collection, and $D_i$ is a compact subset of $X(H_i, U_0)$. The following avoidance theorem is one of the main ingredients of this paper: let $k$ be given by (2.1) for $M = \Gamma \backslash \mathbb{H}^d$:

**Theorem 2.1 (Avoidance theorem).** There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$ with
\[
\mathcal{S}(U_0) \cap RF\, M = \bigcup_{j=1}^{\infty} E_j
\]
satisfying the following: for each $j \in \mathbb{N}$ and for any compact subset $F \subset RF\, M - E_{j+1}$, there exists an open neighborhood $O_j = O_j(F)$ of $E_j$ such that for any $x \in F$, the following set
\[
\{ t \in \mathbb{R} : xu_t \in RF\, M - O_j \}
\]
is $2k$-thick.

It is crucial that the thickness size of the set (2.4), which is given by $2k$ here, can be controlled independently of the compact subsets $E_j$ for applications in the orbit closure theorem. If $E_j$ does not intersect any closed orbit of a connected subgroup of $G$, then obtaining $E_{j+1}$ and $O_j$ is much simpler. In general, $E_j$ may intersect infinitely many intermediate closed orbits, and our proof is based on a careful analysis on the graded intersections of those closed orbits and a combinatorial argument, which we call the inductive search argument. This process is quite delicate, compared to the finite volume case treated in ([12], [42]) in which case the set $\{ t : xu_t \in RF\, M \}$, being equal to $\mathbb{R}$, possesses the Lebesgue measure which can be used to measure the time outside of a neighborhood of $E_j$’s.

We deduce the following from Theorem 2.1:

**Theorem 2.2 (Accumulation on a generic point).** Suppose that (2)$_m$ and (3)$_m$ hold in Theorem 1.2. Then the following holds for any connected closed subgroup $U < N$ with $\text{co-dim}_Y(U) = m + 1$: Let $U_0 = \{ u_t : t \in \mathbb{R} \}$ be a one-parameter subgroup of $U$, and let $x_i \in RF\, M$ be a sequence converging to $x_0 \in \mathcal{G}(U_0)$ as $i \to \infty$. Then for any given sequence $T_i \to \infty$,
\[
\limsup_{i \to \infty} \{ x_i u_{t_i} \in RF\, M : T_i \leq |t_i| \leq 2kT_i \}
\]
contains a sequence $\{ y_j : j = 1, 2, \cdots \}$ such that $\limsup_{j \to \infty} y_j U$ contains a point in $\mathcal{G}(U_0)$.

\footnote{Here we allow a constant sequence $y_j = y$ in which case $\limsup_{j \to \infty} y_j U$ is understood as $yU$ and hence $y \in \mathcal{G}(U_0)$.}
Again, it is important that $2k$ is independent of $x_i$ in the above theorem. We prove two independent but related versions of Theorem 2.2 in section 15 depending on the relation of $x_i$ with the set $RF\ M$; we use Proposition 15.1 for the proof of $(1)_{m+1}$ and Proposition 15.2 for the proofs of $(2)_{m+1}$ and $(3)_{m+1}$.

**Comparison with the finite volume case.** We remark that if $\Gamma \backslash G$ is compact, the methods of Dani-Margulis [12] show that if $x_i$ converges to $x \in \mathcal{G}(U_0)$, and $\varepsilon > 0$, then we can find a sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}$, and neighborhoods $\mathcal{O}_j$ of $E_j$ such that $\mathcal{G}(U_0) = \bigcup E_j$, $x_i \notin \bigcup_{j \leq i+1} \mathcal{O}_j$ and for all $i \geq j$ and $T > 0$,

$$\ell\{t \in [0,T] : x_i u_t \in \mathcal{O}_j\} \leq \frac{\varepsilon}{2^j} T$$

and hence

$$\ell\{t \in [0,T] : x_i u_t \in \bigcup_{j \leq i} \mathcal{O}_j\} \leq \varepsilon T.$$  

(2.6)

This in particular implies that the limsup set as in (2.5) always contains an element of $\mathcal{G}(U_0)$, without using the induction hypothesis. This is the reason why $(3)_m$ is not needed in obtaining $(1)_{m+1}$ and $(2)_{m+1}$ in Theorem 19.1 for the finite volume case.\*

In comparison, we are able to get a generic point in Theorem 2.2 only with the help of the induction hypothesis $(2)_m$ and $(3)_m$ and after taking the limsup of the $U$-orbits of all accumulating points from the $2k$-thick sets obtained in Theorem 2.1.

**Generic points in $F^*$ as limits of RF $M$-points.** In the inductive argument, it is important to find a closed orbit $x_0L$ based at a point $x_0 \in F^*$ in order to promote a semi-group $V^+$ to a group $V$ as described following (2.3). Another reason why this is critical is the following: Implementing Theorem 2.2 (more precisely, its versions Theorems 15.1 and 15.2) requires having a sequence of RF $M$-points of $X$ accumulating on a generic point of $x_0L$ with respect to any given one-parameter subgroup $U_0$ of $U$; that is, how do we find a sequence $x_i$ accumulating on a generic point $x$ inside $RF\ M \cap X$? The advantage of having a closed orbit $x_0L$ with $x_0 \in F^* \cap RF_+ M \cap \mathcal{G}(U_0)$ is that $x_0$ can be approximated by a sequence of RF $M$-points in $F^* \cap X$ (Lemmas 8.3 and 8.7).

We also point out that we use the ergodicity theorem obtained in [31] and [30] to guarantee that there are many $U_0$-generic points in any closed orbit $x_0L$ as above.

\*We give a summary of our proof for the case when $\Gamma \backslash G$ is compact and has at least one $SO^\circ(d-1,1)$ closed orbit in the appendix to help readers understand the whole scheme of the proof.
Existence of proper closed orbits in $\Gamma \backslash G$. In our setting, $\Gamma \backslash G$ always contains a closed orbit $xL$ for some $x \in RFM$ and a proper subgroup $L \in L_U$; namely those closed orbits of $SO^0(d-1,1)$ over the boundary of core $M$. This fact was crucially used in deducing $(2)_{m+1}$ from $(1)_{m+1}, (2)_m$ and $(3)_m$.

Organization of the paper.

- In section 3, we set up notations for certain Lie subgroups of $G$, review some basic facts and gather preliminaries about them and geodesic planes of $M$.
- In section 4, for each unipotent subgroup $U$ of $G$, we define the minimal $H(U)$-invariant closed subset $F_{H(U)} \subset \Gamma \backslash G$ containing $RF_+M$ and study its properties for a convex cocompact hyperbolic manifold of non-empty Fuchsian ends.
- In section 5, we define the singular set $\mathcal{S}(U, x_0L)$ for a closed orbit $x_0L \subset \Gamma \backslash G$, and prove a structure theorem and a countability theorem for a general convex cocompact manifold.
- In sections 6, we prove Proposition 6.3, based on a combinatorial lemma 6.4, called Inductive search lemma. This proposition is used in the proof of Theorem 7.13 (Avoidance theorem).
- In section 7, we construct families of triples of intervals which satisfy the hypothesis of Proposition 6.3, by making a careful analysis of the graded intersections of the singular set and the linearization, and prove Theorem 7.13 from which Theorem 2.1 is deduced.
- In section 8, we prove several geometric lemmas which are needed to modify a sequence limiting on a generic point to a sequence of RF $M$-points which still converges to a generic point.
- In sections 9, we study the unipotent blowup lemmas using quasi-regular maps and properties of thick subsets.
- In section 10, we study the translates of relative $U$-minimal sets $Y$ into the orbit closure of an RF $M$ point.
- In section 11, we describe closures of orbits contained in the boundary of $F_{H(U)}$.
- In section 12, we review the ergodicity theorem of [31] and [30] and deduce the density of almost all orbits of a connected unipotent subgroup in $RF_+M$.
- In section 13, the minimality of a horospherical subgroup action is obtained in the presence of compact factors.
- In section 14, we begin to prove Theorem 1.2; the base case $m = 0$ is addressed and the orbit closure of a singular $U$-orbit is classified under the induction hypothesis.
- In section 15 we prove two propositions on how to get additional invariance, which is a main tool in the enlargement step of the proof of Theorem 1.2.
• We prove (1)_{m+1}, (2)_{m+1} and (3)_{m+1} respectively in sections 16, 17 and 18.
• In the appendix, we give an outline of our proof in the case when \( \Gamma \backslash G \) is compact with at least one \( \text{SO}^0(d-1,1) \)-closed orbit.

3. Lie subgroups and geodesic planes

In this section, we fix notation and recall some background about Lie subgroups of \( \text{SO}(d,1) \) as well as some basic notions regarding geodesic planes of a hyperbolic manifold.

Let \( G \) denote the connected simple Lie group \( \text{SO}^0(d,1) \) for \( d \geq 2 \).

As a Lie group, we have \( G \cong \text{Isom}^+(\mathbb{H}^d) \). In order to present a family of subgroups of \( G \) explicitly, we fix a quadratic form \( Q(x_1, \ldots, x_{d+1}) = 2x_1x_{d+1} + x_2^2 + x_3^2 + \cdots + x_d^2 \), and identify \( G = \text{SO}^0(Q) \). The Lie algebra of \( G \) is then given as:

\[
\mathfrak{so}(d,1) = \{ X \in \mathfrak{sl}_{d+1}(\mathbb{R}) : X^t Q + QX = 0 \}
\]

where

\[
Q = \begin{pmatrix}
0 & 0 & 1 \\
0 & \text{Id}_{d-1} & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]

A subset \( S \subset G \) is Zariski closed if \( S \) is defined to be the zero set \( \{ (x_{ij}) \in G : p_1(x_{ij}) = \cdots = p_l(x_{ij}) = 0 \} \) for a finite collection of polynomials with real coefficients in variables \( (x_{ij}) \in M_{d+1}(\mathbb{R}) \). The Zariski closure of a subset \( S \subset G \) means the smallest Zariski closed subset of \( G \) containing \( S \).

Subgroups of \( G \). Inside \( G \), we have the following subgroups:

\[
K = \{ g \in G : g^t g = \text{Id}_{d+1} \} \cong \text{SO}(d),
\]

\[
A = \left\{ a_s = \begin{pmatrix}
e^s & 0 & 0 \\
0 & \text{Id}_{d-1} & 0 \\
0 & 0 & e^{-s}
\end{pmatrix} : s \in \mathbb{R} \right\}
\]

\[
M = \text{the centralizer of } A \text{ in } K \cong \text{SO}(d-1),
\]

\[
N^- = \{ \exp u^-(x) : x \in \mathbb{R}^{d-1} \},
\]

\[
N^+ = \{ \exp u^+(x) : x \in \mathbb{R}^{d-1} \}
\]

where

\[
u^-(x) = \begin{pmatrix}
0 & x^t & 0 \\
0 & 0 & -x \\
0 & 0 & 0
\end{pmatrix}
\]

and

\[
u^+(x) = \begin{pmatrix}
0 & 0 & 0 \\
x & 0 & 0 \\
0 & -x^t & 0
\end{pmatrix}.
\]

The Lie algebra of \( M \) consists of matrices of the form

\[
m(C) = \begin{pmatrix}
0 & 0 & 0 \\
0 & C & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

where \( C \in M_{d-1}(\mathbb{R}) \) is a skew-symmetric matrix, i.e., \( C^t = -C \).
The subgroups $N^-$ and $N^+$ are respectively the contracting and the expanding horospherical subgroups of $G$ for the action of $A$. We have the Iwasawa decomposition $G = KAN^\pm$. As we will be using the subgroup $N^-$ frequently, we simply write $N = N^-$. We often identify the subgroup $N^\pm$ with $\mathbb{R}^{d-1}$ via the map $\exp u^\pm(x) \mapsto x$. For a connected closed subgroup $U < N$, we use the notation $U^\pm$ for the orthogonal complement of $U$ in $N$ as a vector subgroup of $N$, and $U^t = U^+$ for the transpose of $U$. We use the notation $B_U(r)$ to denote the ball of radius $r$ centered at 0 in $U$ for a Euclidean metric on $N = \mathbb{R}^{d-1}$.

We consider the upper-half space model of $\mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}$, so that its boundary is given by $S^{d-1} = \{ \infty \} \cup \{ 0 \} \times \mathbb{R}^{d-1}$. Set $o = (1, 0, \cdots, 0)$, and fix a standard basis $e_0, e_1, \cdots, e_{d-1}$ at $T_o(\mathbb{H}^d)$. The map

$$g \mapsto (ge_0, \cdots, ge_{d-1})g(o)$$

(3.1)

gives an identification of $G$ with the oriented frame bundle $F \mathbb{H}^d$. The stabilizer of $o$ and $e_0$ in $G$ are equal to $K$ and $M$ respectively, and hence the map (3.1) induces the identifications of the hyperbolic space $\mathbb{H}^d$ and the unit tangent bundle $T^1 \mathbb{H}^d$ with $G/K$ and $G/M$ respectively. The action of $G$ on the hyperbolic space $\mathbb{H}^d = G/K$ extends continuously to the compactification $S^{d-1} \cup \mathbb{H}^d$.

If $g \in G$ corresponds to a frame $(v_0, \cdots, v_{d-1}) \in F \mathbb{H}^d$, we define $g^+, g^- \in S^{d-1}$ to be the forward and backward end points of the directed geodesic tangent to $v_0$ respectively. The right translation action of $A$ on $G = F \mathbb{H}^d$ defines the frame flow and we have

$$g^\pm = \lim_{t \to \pm \infty} \pi(ga_t)$$

where $\pi : G = F \mathbb{H}^d \to \mathbb{H}^d$ is the basepoint projection.

For the identity element $e = \text{Id}, e_{d+1} \in G$, note that $e^+ = \infty$, and $e^- = 0$, and hence $g^+ = g(\infty)$ and $g^- = g(0)$. The subgroup $MA$ fixes both points 0 and $\infty$, and the horospherical subgroup $N$ fixes $\infty$, and the restriction of the map $g \mapsto g(0)$ to $N$ defines an isomorphism $N \to \mathbb{R}^{d-1}$ given by $u^+(x) \mapsto x$.

For each non-trivial connected subgroup $U < N$, we denote by

$$H(U)$$

the connected closed subgroup of $G$ generated by $U$ and the transpose of $U$. It is the smallest simple closed Lie subgroup of $G$ containing $A$ and $U$.

For a subset $S \subset G$, we denote by $N_G(S)$ and $C_G(S)$ the normalizer of $S$ and the centralizer of $S$ respectively. We denote by $N(S)$ and $C(S)$ the identity components of $N_G(S)$ and $C_G(S)$ respectively.

**Example 3.1.** Fix the standard basis $e_1, \cdots, e_{d-1}$ of $\mathbb{R}^{d-1}$. For $1 \leq k \leq d - 1$, define $U_k$ to be the connected subgroup of $N$ spanned by $e_1, \cdots, e_k$. 

The following can be checked directly:

\[ H(U_k) = \langle U_k, U_k^t \rangle = SO^o(k + 1, 1); \]
\[ C(H(U_k)) = SO(d - k - 1); \]
\[ N_G(H(U_k)) = O(k + 1, 1)O(d - k - 1) \cap G; \]
\[ N(H(U_k)) = SO^o(k + 1, 1)SO(d - k - 1). \]

We set

\[ H'(U) := N(H(U)) = H(U)C(H(U)), \]

which is a connected reductive Lie subgroup of \( G \) with compact center.

The adjoint action of \( M \) on \( N \) corresponds to the standard action of \( SO(d - 1) \) on \( \mathbb{R}^{d-1} \). It follows that any connected closed subgroup \( U < N \) and \( H(U) \) are respectively conjugate to \( U_k \) and to \( H(U_k) \) by an element of \( M \), where \( k = \text{dim}(U) \).

We set

\[ (3.2) \ C_1(U) := C(H(U)) = M \cap C(U), \text{ and } \ C_2(U) := M \cap C(U^\perp) \subset H(U). \]

**Lemma 3.2.** We have

\[ N(U) = NAC_1(U)C_2(U) \text{ and } C(U) = NC_1(U). \]

**Proof.** For the first claim, it suffices to show that for \( U = U_k \), \( N(U) = NASO(k)SO(d - 1 - k) \). It is easy to check that \( Q := NAC_1(U)C_2(U) \) normalizes \( U \). Let \( g \in N(U) \). We claim that \( g \in Q \). In view of the Iwasawa decomposition \( G = KAN \), we may assume \( g \in K \). Then \( Ug(\infty) = gU(\infty) = g(\infty) \) since \( U(\infty) = \infty \). Since \( \infty \in S^{d-1} \) is the unique fixed point of \( U \), it follows \( g(\infty) = \infty \). As \( M = \text{Stab}_K(\infty) \), we get \( g \in M \).

It follows that \( g \in M \) fixes \( 0 \in S^{d-1} \), and hence \( gU(0) = Ug(0) = U(0) \). As \( U(0) = R^k \), \( g \) preserves \( R^k = R^k \). Therefore, as \( g \in M \), we also have \( gR^{d-1-k} = R^{d-1-k} \), and consequently \( g \in O(k)O(d - 1 - k) \). This shows that \( NASO(k)SO(d - 1 - k) \subset N(U) \subset NAO(k)O(d - 1 - k) \). As \( N(U) \) is connected, this implies the claim.

For the second claim, note first that \( NC_1(U) < C(U) \). Now let \( g \in C(U) \). Since \( C(U) < N(U) = ANC_1(U)C_2(U) \), we can write \( g = ac_2nc_1 \in AC_1(U)NC_1(U) \). Since \( nc_1 \) commutes with \( U \), it follows \( ac_2 \in C(U) \). Now observe that the adjoint action of \( a \) on \( U \) is a dilation and the adjoint action of \( c_2 \) on \( U \) is a multiplication by an orthogonal matrix. Therefore it follows that \( a = c_2 = e \), finishing the proof. \( \square \)

Denote by \( \mathfrak{g} = \text{Lie}(G) \) the Lie algebra of \( G \). By a one-parameter sub-semigroup of \( G \), we mean a set of the form \( \{ \exp(t\xi) \in G : t \geq 0 \} \) for some non-zero \( \xi \in \mathfrak{g} \). Note that the product \( AU^\perp C_2(U) \) is a subgroup of \( G \).
Lemma 3.3. An unbounded one-parameter semigroup $S$ of $AU^\perp C_2(U)$ is one of the following form:
\[
\{ \exp(t\xi_A)\exp(t\xi_C) : t \geq 0 \}; \\
\{ (v \exp(t\xi_A)v^{-1})\exp(t\xi_C) : t \geq 0 \}; \\
\{ \exp(t\xi_V)\exp(t\xi_C) : t \geq 0 \}
\]
for some $\xi_A \in \text{Lie}(A) - \{0\}, \xi_C \in \text{Lie}(C_2(U)), v \in U^\perp - \{e\}$, and $\xi_V \in \text{Lie}(U^\perp) - \{0\}$.

Proof. Let $\xi \in \text{Lie}(AU^\perp C_2(U))$ be such that $S = \{ \exp(t\xi) : t \geq 0 \}$. Write $\xi = \xi_0 + \xi_C$ where $\xi_0 \in \text{Lie}(AU^\perp)$ and $\xi_C \in \text{Lie}(C_2(U))$. Since $AU^\perp$ commutes with $C_2(U)$, $\exp(t\xi) = \exp(t\xi_0)\exp(t\xi_C)$ for any $t \in \mathbb{R}$. Hence we only need to show that either $\xi_0 \in \text{Lie}(U^\perp)$ or
\[
\{ \exp(t\xi_0) : t \geq 0 \} = \{ v \exp(t\xi_A)v^{-1} : t \geq 0 \}
\]
for some $v \in U^\perp$ and $\xi_A \in \text{Lie}(A)$. Now if $\xi_0 \not\in \text{Lie}(U^\perp)$, then writing
\[
\xi_0 = \begin{pmatrix} a & x^t & 0 \\ 0 & 0_{d-1} & -x \\ 0 & 0 & -a \end{pmatrix} \in \text{Lie}(AU^\perp)
\]
with $a \neq 0$, a direct computation shows that $\xi_0 = v\xi_Av^{-1}$ where
\[
\log v = \begin{pmatrix} 0 & -x^t/a & 0 \\ 0_{d-1} & x/a & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
and $\xi_A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0_{d-1} & 0 \\ 0 & 0 & -a \end{pmatrix}$, proving (3.3). \hfill \Box

Lemma 3.4. If $v_i \to \infty$ in $U^\perp$, then $\lim \inf_{i \to \infty} v_i A_{v_i}^{-1}$ contains one-parameter subgroup $V$ contained in $U^\perp$.

Proof. Writing $v_i = \exp u^-(x_i)$ for $x_i \in \mathbb{R}^{d-1}$, we have
\[
v_i a_s v_i^{-1} = \begin{pmatrix} e^s (1 - e^s)x_i^t - \| (e^s/2 - e^{-s/2}) x_i \|^2/2 \\ 0 \\ (1 - e^{-s}) x_i \\ 0 \\ e^{-s} \end{pmatrix}.
\]
Passing to a subsequence, $x_i/\|x_i\|$ converges to some unit vector $x_0$ as $i \to \infty$. For any $r \in \mathbb{R}$, if we set $s_i := \log(1-r\|x_i\|^{-1})$, then the sequence $v_i a_s v_i^{-1}$ converges to $\exp u^-(rx_0)$. Therefore it suffices to set $V := \{ \exp u^-(rx_0) : s \in \mathbb{R} \} < U^\perp$. \hfill \Box

The complementary subspaces $h^\perp_U$ and $h^\perp$. If $L$ is a reductive Lie subgroup of $G$ with $l = \text{Lie}(L)$, the restriction of the adjoint representation of $G$ to $L$ is completely reducible, and hence there exists an $\text{Ad}(L)$-invariant complementary subspace $l^\perp$ so that
\[
g = l \oplus l^\perp.
It follows from the inverse function theorem that the map \( L \times \mathfrak{l} \to G \) given by \((g, X) \mapsto g \exp X\) is a local diffeomorphism onto an open neighborhood of \( e \) in \( G \).

Let \( U = U_k \). Denote by \( \mathfrak{h}_U \subset \mathfrak{g} \) the Lie algebra of \( H(U) \), by \( \mathfrak{u} \) the subspace \( \text{Lie}(U^\perp) \), and by \((\mathfrak{u}^\perp)^t\) its transpose. Then \( \mathfrak{h}_U^\perp \) can be given explicitly as follows:

\[
\mathfrak{h}_U^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0
\]

where \( \mathfrak{m}_0 \) is given by

\[
\{ m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & Z \end{pmatrix}, Z \in M_{d-1-k}(\mathbb{R}), Z^t = -Z, Y \in M_{k \times (d-1-k)}(\mathbb{R}) \};
\]

to see this, it is enough to check that \( \dim(\mathfrak{g}) = \dim(\mathfrak{h}_U) + \dim(\mathfrak{h}_U^\perp) \) and that \( \mathfrak{h}_U^\perp \) is Ad\((H(U))\)-invariant, which can be done by direct computation.

Similarly, setting \( \mathfrak{h} := \text{Lie}(H'(U)) \), \( \mathfrak{h}^\perp \) is given by

\[
\mathfrak{h}^\perp = \mathfrak{u}^\perp \oplus (\mathfrak{u}^\perp)^t \oplus \mathfrak{m}_0'
\]

where

\[
\mathfrak{m}_0' := \{ m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & 0 \end{pmatrix} \}.
\]

**Lemma 3.5.** If \( r_i \to e \) in \( \exp \mathfrak{h}^\perp \cap C(H(U)) \), then there exists a one-parameter subgroup \( U_0 \) contained in \( U \cup U^+ \) such that \( r_i \notin N(U_0) \) for all \( i \), by passing to a subsequence.

**Proof.** Let \( \{ U_{\pm i}^{(i)} \} \) and \( \{ U_+^{(i)} \} \) be the collections of one-parameter subgroups generating \( U \) and \( U^+ \) respectively. By Lemma 3.2 and (3.5), there exists a neighborhood \( \mathcal{O} \) of \( e \) in \( G \) such that

\[
\bigcap_i N(U_{\pm i}^{(i)}) \cap \exp \mathfrak{h}^\perp \cap \mathcal{O} \subset C(H(U)).
\]

Hence the claim follows. \square

**Reductive subgroups of \( G \).**

**Definition 3.6.** For a connected reductive subgroup \( L < G \), we denote by \( L_{nc} \) the maximal connected normal semisimple subgroup of \( L \) with no compact factors.

A connected reductive subgroup \( L \) of \( G \) is an almost direct product

\[
L = L_{nc}CT
\]

where \( C \) is a connected semisimple compact normal subgroup of \( L \) and \( T \) is the central torus of \( L \). If \( L \) contains a unipotent element, then \( L_{nc} \) is non-trivial, and simple, containing a conjugate of \( A \), and the center of \( L \) is compact.
Proposition 3.7. If \( L < G \) is a connected reductive subgroup normalized by \( A \) and containing a unipotent element, then
\[
L = H(U)C
\]
where \( U < N \) is a non-trivial connected subgroup and \( C \) is a closed subgroup of \( C(H(U)) \). In particular, \( L_{nc} \) and \( N(L_{nc}) \) are equal to \( H(U) \) and \( H'(U) \) respectively.

Proof. If \( L \) is normalized by \( A \), then so is \( L_{nc} \). Therefore it suffices to prove that a connected non-compact simple Lie subgroup \( H < G \) normalized by \( A \) is of the form \( H = H(U) \) where \( U < N \) is a non-trivial connected subgroup.

First, consider the case when \( A < H \). Let \( h \) be the Lie algebra of \( H \), and \( a \) be the Lie algebra of \( A \). Since \( h \) is simple, its root space decomposition for the adjoint action of \( a \) is of the form \( h = z(a) \oplus u^+ \oplus u^- \) where \( u^\pm \) are the sum of all positive and negative root subspaces respectively and \( z(a) \) is the centralizer of \( a \). Since the sum of all negative root subspaces for the adjoint action of \( a \) on \( g \) is \( \text{Lie}(N^-) \), it follows that \( U := \exp(u^-) < N^- \) and \( H = H(U) \).

Now for the general case, \( H \) contains a conjugate \( gAg^{-1} \) for some \( g \in G \). Hence \( g^{-1}Hg = H(U) \). Since \( H(U) \) contains both \( A \) and \( g^{-1}Ag \), they must be conjugate within \( H(U) \), so \( A = h^{-1}g^{-1}Ag \) for some \( h \in H(U) \). Hence \( gh \in N_G(A) = AM \). Therefore \( H = gH(U)g^{-1} \) is equal to \( mH(U)m^{-1} \) for some \( m \in M \). Since \( m \) normalizes \( N \) and \( mH(U)m^{-1} = H(mUm^{-1}) \), the claim follows. \( \square \)

Corollary 3.8. Any connected closed subgroup \( S \) of \( G \) generated by unipotent elements is conjugate to either \( U \) or \( H(U) \) for some non-trivial connected subgroup \( U < N \).

Proof. By the Levi decomposition, \( S = LV \) where \( L \) is reductive and \( V \) is the unipotent radical of \( S \). If \( L \) is trivial, the claim follows since any connected unipotent subgroup can be conjugate into \( N \). Suppose that \( L \) is not-trivial. As \( S \) is generated by unipotent elements, \( L \) does not have any compact factor, and hence up to conjugation, \( L = H(U) \) for some non-trivial \( U < N \) by Proposition 3.7. Unless \( V \) is trivial, the normalizer of \( V \) is contained in a conjugate of \( NAM \), in particular, it cannot contain \( H(U) \). Hence \( V = \{ e \} \). \( \square \)

Totally geodesic immersed planes. Let \( \Gamma \) be a discrete, torsion free, non-elementary, subgroup of \( G \), and consider the associated hyperbolic manifold
\[
M = \Gamma \backslash \mathbb{H}^d = \Gamma \backslash G/K.
\]
We refer to [36] for basic properties of hyperbolic manifolds. As in the introduction, we denote by \( \Lambda \) the limit set of \( \Gamma \) and by \( \text{core}(M) \) the convex core of \( M \). Note that \( \text{core} M \) contains all bounded geodesics in \( M \).
We denote by $F_M \simeq \Gamma\backslash G$ the bundle of all oriented orthonormal frames over $M$. We denote by

$$\pi: \Gamma\backslash G \to M = \Gamma\backslash G/K$$

the base-point projection. By abuse of notation, we also denote by

$$\pi: G \to \mathbb{H}^d = G/K$$

the base-point projection. For $g \in G$, $[g]$ denotes its image under the covering map $G \to \Gamma\backslash G$.

Fix $1 \leq k \leq d-2$ and let

$$H = SO^\circ(k+1,1) \quad \text{and} \quad H' = SO^\circ(k+1,1) \cdot SO(d-k-1).$$

Let $C_0 := \mathbb{R}^k \cup \{\infty\}$ denote the unique oriented $k$-sphere in $S^{d-1}$ stabilized by $H'$. Then $\tilde{S}_0 := \text{hull}(C_0)$ is the unique oriented totally geodesic subspace of $\mathbb{H}^d$ stabilized by $H'$, and $\partial \tilde{S}_0 = C_0$. We note that $H'$ (resp. $H$) consists of all oriented frames $(v_0, \cdots, v_{d-1}) \in G$ (resp. $(v_0, \cdots, v_k, e_{k+1}, \cdots, e_{d-1}) \in G$) such that the $k$-tuple $(v_0, \cdots, v_k)$ is tangent to $\tilde{S}_0$, compatible with the orientation of $\tilde{S}_0$.

The group $G$ acts transitively on the space of all oriented $k$-spheres in $S^{d-1}$ giving rise to the isomorphisms of $G/H'$ with

$$C^k = \text{the space of all oriented } k\text{-spheres in } S^{d-1}$$

and with

$$\text{the space of all oriented totally geodesic } k+1\text{-planes of } \mathbb{H}^d.$$

We discuss the fundamental group of an immersed geodesic $k$-plane $S \subset M$. Choose a totally geodesic subspace $\tilde{S}$ of $\mathbb{H}^d$ which covers $S$. Then $\tilde{S} = g\tilde{S}_0$ for some $g \in G$, and the stabilizer of $\tilde{S}$ in $G$ is equal to $gH'g^{-1}$. We have

$$\Gamma_{\tilde{S}} = \{\gamma \in \Gamma : \gamma\tilde{S} = \tilde{S}\} = \Gamma \cap gH'g^{-1}$$

and get an immersion $\tilde{f}: \Gamma_{\tilde{S}}\backslash \tilde{S} \to M$ with image $S$. Consider the projection map

$$p: gH'g^{-1} \to gHg^{-1}.$$ 

Then $p$ is injective on $\Gamma_{\tilde{S}}$ and

$$\Gamma_{\tilde{S}}\backslash \tilde{S} \simeq p(\Gamma_{\tilde{S}})\backslash \tilde{S}$$

is an isomorphism, since $gC(H)g^{-1}$ acts trivially on $\tilde{S}$. Hence $\tilde{f}$ gives an immersion

$$f: p(\Gamma_{\tilde{S}})\backslash \tilde{S} \to M$$

with image $S$. We say $S$ properly immersed if $f$ is a proper map.

**Proposition 3.9.** Let $x \in \Gamma\backslash G$, and set $S := \pi(xH') \subset M$. Then

1. $xH'$ is closed in $\Gamma\backslash G$ if and only if $S$ is properly immersed in $M$. 

(2) If $M$ is convex cocompact and $S$ is properly immersed, then $S$ is convex cocompact and

$$\partial \tilde{S} \cap \Lambda = \Lambda(p(\tilde{S}))$$

for any geodesic subspace $\tilde{S} \subset \mathbb{H}^d$ which covers $S$.

Proof. Choose a representative $g \in G$ of $x$ and consider the totally geodesic subspace $\tilde{S} := g\tilde{S}_0$. Then $S = \text{Im}(f)$ as $f$ given by (3.11). Now the closedness of $xH'$ in $\Gamma \backslash G$ is equivalent to the properness of the map $(H' \cap g^{-1}\Gamma g) \backslash H' \to \Gamma \backslash G$ induced from map $h \mapsto xh$. This in turn is equivalent to the properness of the induced map $(H' \cap g^{-1}\Gamma g) \backslash H' / (H' \cap K) \to \Gamma \backslash G / K$. If $\Delta$ is the image of $H' \cap g^{-1}\Gamma g$ under the projection map $H' \to H$, then the natural injective map $\Delta \backslash H / H \cap K \to (H' \cap g^{-1}\Gamma g) \backslash H' / H' \cap K$ is an isomorphism. Since $p(\Gamma_{\tilde{S}}) \backslash \tilde{S} = p(\Gamma_{\tilde{S}}) \backslash gH / (H \cap K) \simeq \Delta \backslash H / (H \cap K)$, the first claim follows. The second claim follows from [34, Theorem 4.7].

4. Hyperbolic manifolds with Fuchsian ends and thick return time

In this section, we study the closed $H(U)$-invariant subset $F_{H(U)} := RF_+ M \cdot H(U)$ when $M = \Gamma \backslash \mathbb{H}^d$ is a convex cocompact manifold with Fuchsian ends.

At the end of the section, we address the global thickness of the return time of any unipotent one-parameter subgroup of $N$ to RF $M$.

Definition 4.1. A convex cocompact hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d$ is said to have non-empty Fuchsian ends if one of the following equivalent conditions holds:

1. its convex core has non-empty interior and non-empty totally geodesic boundary.
2. the domain of discontinuity of $\Gamma$

$$\Omega := \mathbb{S}^{d-1} - \Lambda = \bigcup_{i=1}^{\infty} B_i$$

is a dense union of infinitely many round balls with mutually disjoint closures.

In the whole section, let $M$ be a convex cocompact hyperbolic manifold convex cocompact hyperbolic manifolds of non-empty Fuchsian ends.
**Renormalized frame bundle.** The renormalized frame bundle $RF M \subset FM$ is defined as the following $AM$-invariant subset

$$RF M = \{[g] \in \Gamma \backslash G : g^\pm \in \Lambda \} = \{x \in \Gamma \backslash G : xA \text{ is bounded} \}$$

i.e., the closed set consisting of all oriented frames $(v_0, \cdots, v_{d-1})$ such that the complete geodesic through $v_0$ is contained in core $M$.

Similarly, we define

$$RF_+ M = \{[g] \in \Gamma \backslash G : g^+ \in \Lambda \} = \{x \in \Gamma \backslash G : xA^+ \text{ is bounded} \}$$

which is a closed $NAM$-invariant subset (here $A^+ = \{a_t : t \geq 0 \}$). As $\pi(xNA) = \pi(xG) = M$ for any $x \in \Gamma \backslash G$, we have $\pi(RF_+ M) = M$.

**Lemma 4.2.** For $x \in RF_+ M$, $x\bar{A}^+$ meets $RF M$.

**Proof.** Take any sequence $a_i \to \infty$ in $A^+$. Since $xA^+$ is bounded, $xa_i$ converges to some $x_0 \in \overline{xA^+}$ by passing to a subsequence. On the other hand, as $A = \limsup a_i^{-1}A^+$, we have $x_0A \subset \limsup (xa_i)(a_i^{-1}A^+) \subset \overline{xA^+}$. Since $x \in RF_+ M$, $xA^+$ is bounded, so is $x_0A$. Hence $x_0 \in RF M$ as desired. \(\square\)

**$H(U)$-invariant subsets:** $F_{H(U)}, F_{H(U)}^+, \partial F_{H(U)}$. Fix a non-trivial connected subgroup $U < N$, and consider the associated subgroups $H(U)$ and $H'(U)$ as defined in section 3.

We define

$$F_{H(U)} := RF_+ M \cdot H(U). \tag{4.1}$$

The closedness of $F_{H(U)}$ is an easy consequence of compactness of the limit set $\Lambda$. It is also $H'(U)$-invariant, since $RF_+ M$ is $M$-invariant and $C(H(U))$ contained in $M$.

For $g \in G$, we denote by $C_g = C_{gH(U)} \subset \mathbb{S}^{d-1}$ the sphere given by the boundary of the geodesic plane $\pi(gH(U))$. Then hull $C_g = \pi(gH(U))$, and $C_g = gH(U)^+ = gH(U)^-$ where $H(U)^\pm = \{h^\pm : h \in H(U)\}$. It follows that

$$F_{H(U)} = \{[g] \in \Gamma \backslash G : C_g \cap \Lambda \neq \emptyset \}. \tag{4.2}$$

We observe:

**Lemma 4.3.** Fix $x = [g] \in \Gamma \backslash G$. Let $L$ be a closed subgroup of $G$ such that the closure of $\pi(gL)$ in $\mathbb{H}^d \cup \mathbb{S}^{d-1}$ does not meet $\Lambda$. Then the map $L \to xL \subset \Gamma \backslash G$ given by $\ell \mapsto x\ell$ is a proper map, and hence $xL$ is closed.

**Proof.** Suppose that $x\ell_i \to [g_0]$ for some sequence $\ell_i \to \infty \in L$. Then there exists $\gamma_i \in \Gamma$ such that $d(\gamma_i \pi(g\ell_i), \pi(g_0)) = d(\pi(g\ell_i), \gamma_i \pi(g_0)) \to 0$. As $g\ell_i \to \infty$, $\gamma_i \pi(g_0)$ converges to a limit point $\xi \in \Lambda$, after passing to a subsequence. Hence $\pi(gL) \cap \Lambda \neq \emptyset$. \(\square\)

This lemma implies that if $x \notin RF_+ M$ (resp. $x \notin F_{H(U)}$), then $xU$ (resp. $xH(U)$) is closed for any closed subgroup $U < N$. 
Lemma 4.4. When $M$ is a convex cocompact hyperbolic manifold of non-empty Fuchsian ends, we have

$$F_{H(U)} = \{ x \in \Gamma \setminus G : \pi(xH(U)) \cap \text{core } M \neq \emptyset \}. $$

Proof. Denote by $Q$ the subset on the right-hand side of the above equality. To show $F_{H(U)} \subset Q$, let $x \in F_{H(U)}$. By modifying it using an element of $H(U)$, we may assume that $x \in RF_+M$. By Lemma 4.2, $xA^\pm$ contains $x_0 \in RF_M$. Since $x_0A$ is bounded, $\pi(x_0A)$ is a bounded geodesic, and hence

$$\pi(x_0A) \subset \pi(xH(U)) \cap \text{core } M$$

because core $M$ contains all bounded geodesics. Therefore $x \in Q$. To show the other inclusion $Q \subset F_{H(U)}$, we use the hypothesis on $M$. Suppose $x = [g] \notin F_{H(U)}$. Then $C_g \cap \Lambda = \emptyset$, and hence $C_g$ must be contained in a connected component, say $B_i$, of $\Omega$. Hence $\pi(gH(U)) = \text{hull}(C_g)$ is contained in the interior of $\text{hull}(B_i)$, which is disjoint from $\text{hull}(\Lambda)$, by the convexity of $B_i$. Therefore the orbit $\Gamma \pi(gH(U))$ is a closed subset of $\mathbb{H}^d$, disjoint from $\text{hull}(\Lambda)$. Hence $x \notin Q$, proving the claim. \qed

Note also that

$$(4.3) \quad RF M \cdot H(U) = \{ [g] : \#C_g \cap \Lambda \geq 2 \}$$

$$= \{ x \in \Gamma \setminus G : \pi(xH(U)) \cap \text{core } M \neq \emptyset \}. $$

This can be seen using the fact that for any two distinct points $\xi^+, \xi^- \in C_g$, there exists $h \in H(U)$ such that $gh(\infty) = \xi^+$ and $gh(0) = \xi^-$. This fact is clear if $H(U) = H(U_k)$ for some $k$, and a general case follows since $H(U) = mH(U_k)m^{-1}$ for some $m \in M$, and $M$ fixes both 0 and $\infty$.

Denote by $M^*$ the interior of the core of $M$ and by $F_{H(U)}^*$ the interior of $F_{H(U)}$. Then

$$F_{H(U)}^* = \{ x \in \Gamma \setminus G : \pi(xH(U)) \cap M^* \neq \emptyset \}. $$

To see this, note that the right-hand side is equal to

$$\{ [g] \in F_{H(U)} : \text{hull } C_g \cap \text{interior } (\text{hull}(\Lambda)) \neq \emptyset \}$$

$$(4.4) = \{ [g] \in F_{H(U)} : C_g \notin B_i \text{ for any } i \}$$

which can then be seen to be equal to $F_{H(U)}^*$ in view of (4.2). Note that (4.4) implies that for $[g] \in F_{H(U)}$, $\#C_g \cap \Lambda \geq 2$ and hence $F_{H(U)}^* \subset RF M \cdot H(U)$.

Lemma 4.5. We have

$$RF_+M \cap F_{H(U)}^* \subset RF M \cdot U.$$

Proof. Let $y \in RF_+M \cap F_{H(U)}^*$. We need to show that $yU \cap RF M \neq \emptyset$. Choose $g \in G$ so that $[g] = y$. As $y \in RF_+M$, $g^+ = g(\infty) \in \Lambda$, and hence $C_g \cap \Lambda \neq \emptyset$. If $\#C_g \cap \Lambda = 1$, then $C_g$ must be contained in $B_i$ for some $i$, which implies $[g] \notin F_{H(U)}^*$. Therefore $\#C_g \cap \Lambda \geq 2$. We note that $gU(0) \cup \{ g(\infty) \} = C_g$; this is clear when $U = U_k$ for some $k \geq 1$ and $g = e$.
to which a general case is reduced. Hence there exists \( u \in U \) such that \( gu(0) \in \Lambda \). Since \( gu(\infty) = g(\infty) \in \Lambda \), we have \( yu = [g]u \in RF M. \)

We denote by \( \partial F_{H(U)} \) the boundary of \( F_{H(U)} \), that is,

\[
\partial F_{H(U)} = F_{H(U)} - F^*_{H(U)} = \{ [g] \in F_{H(U)} : C_g \subset \overline{B}_i \text{ for some } i \}.
\]

When there is no room for confusion, we will omit the subscript \( H(U) \) and simply write \( F, F^* \) and \( \partial F \).

We call an oriented frame \( g = (v_0, \ldots, v_{d-1}) \in F M = G \) a boundary frame if the first \((d - 1)\) vectors \( v_0, \ldots, v_{d-2} \) are tangent to the boundary of core \( M \). Set

\[
\tilde{H} := H(U_{d-2}) = SO^o(d - 1, 1),
\]

and denote by \( \tilde{\mathcal{V}} \) the one-dimensional subgroup \( \mathbb{R}e_{d-1} \) of \( N = \mathbb{R}^{d-1} \); note that \( \tilde{\mathcal{V}} = (\tilde{H} \cap N)^\perp \).

We denote by \( BF M \) the set of all boundary frames of \( M \); it is a union of compact \( \tilde{H} \)-orbits:

\[
BF M = \bigcup_{i=1}^k z_i \tilde{H}
\]
such that \( \pi(z_i \tilde{H}) = \Gamma \setminus \Gamma \text{ hull}(B_i) \).

**The boundary \( \partial F_{H(U)} \) for \( U \subset \tilde{H} \cap N \).** Suppose that \( U \) is contained in \( \tilde{H} \cap N = \mathbb{R}^{d-2} \). Then there exists a one-parameter semigroup \( \tilde{\mathcal{V}}^+ \) of \( \tilde{\mathcal{V}} \) such that

\[
\partial F = BF M \cdot \tilde{\mathcal{V}}^+ H'(U).
\]

We use the notation \( \tilde{\mathcal{V}}^- = \{ v^{-1} : v \in \tilde{\mathcal{V}}^+ \} \). Note that

\[
(4.5) \quad \partial F \cap RF M = BF M \cdot C(H(U)) \quad \text{and} \quad \partial F \cap RF_+ M = BF M \cdot \tilde{\mathcal{V}}^+ C(H(U)).
\]

For a general proper connected closed subgroup \( U < N, mUm^{-1} \subset \tilde{H} \cap N \) for some \( m \in M \), and

\[
\partial F \cap RF M = BF Mm C(H(U))
\]

where \( BF Mm \) is now a union of finitely many \( m^{-1} \tilde{H}m \)-compact orbits.

**Lemma 4.6.** Let \( U < \tilde{H} \cap N, \ z \in BF M \) and \( v \in \tilde{\mathcal{V}} - \{ e \} \). If \( zv \in RF M \), then \( zv \in F^* \).

**Proof.** Let \( z = [g] \in BF M \). Then \( \partial(\pi(g \tilde{H})) = \partial B_j \) for some \( j \). Let \( v \in \tilde{\mathcal{V}} - \{ e \} \) be such that \( zv \in RF M \). Suppose \( zv \in \partial F_{H(U)} \). Then \( C_{gv} \subset \overline{B}_i \) for some \( i \). Since the sphere \( C_{gv} = \{ gvh(\infty) : h \in H(U) \} \) contains \( g(\infty) \) which belongs to \( \partial B_j \), we have \( i = j \), as \( \overline{B}_i \)'s are mutually disjoint. As \( zv \in RF M \), \( C_{gv} \subset \partial B_j \). Hence \( gvh(U)^+ \subset g \tilde{H}^+ \). It follows that \( gvh(U) \subset g \tilde{H} \), and hence \( vH(U) \cap \tilde{H} \neq \emptyset \), which is a contradiction since \( v \notin \tilde{H} \), and \( H(U) \subset \tilde{H} \). \( \square \)
Properly immersed geodesic planes. Let $H$ and $H'$ be as in (3.9), and $p$ the map in (3.10). In (3.11), if $p(\tilde{\Gamma}_S) \setminus \tilde{S}$ is a convex cocompact hyperbolic $k$-manifold with Fuchsian ends and $f$ is proper, then the image $S = \text{Im}(f)$ is referred to as a properly immersed convex cocompact geodesic $k$-plane of Fuchsian ends.

**Proposition 4.7.** If $xH'$ is closed for $x \in RF M$, then $S = \pi(xH')$ is a properly immersed convex cocompact geodesic plane of Fuchsian ends.

**Proof.** Choose $g \in G$ so that $x = [g]$. Let $\tilde{S}$ and $\Gamma_S$ be as in Proposition 3.9. Set $C = \partial \tilde{S}$. By loc. cit., $S$ is properly immersed, and $C \cap \Lambda = \Lambda(p(\tilde{\Gamma}_S))$. Write
\[(4.6)\quad C - (C \cap \Lambda) = \bigcup_{i \in I}(C \cap B_i)\]
where $I$ is the collection of all $i$ such that $C \cap B_i \neq \emptyset$. If $C \cap \Lambda$ contains a non-empty open subset of $C$, then $p(\Gamma_S)$ is a uniform lattice in $gHg^{-1}$, and hence $S$ is compact. In the other case, $I$ is an infinite set and $\bigcup_{i \in I}(C \cap B_i)$ is dense in $C$; so $S$ is a a convex cocompact hyperbolic submanifold of Fuchsian ends by Definition 4.1(2). □

**Lemma 4.8.** For any sphere $C$ in $S^{d-1}$ with $\#C \cap \Lambda \geq 2$, the intersection $C \cap \Lambda$ is Zariski dense in $C$.

**Proof.** The claim is clear if $C \cap \Lambda$ contains a non-empty open subset of $C$. If not, $C \cap \Lambda$ contains infinitely many $C \cap \partial B_i$’s, each of which is an irreducible co-dimension one real subvariety of $C$. It follows that the Zariski closure of $C \cap \Lambda$ has dimension strictly greater than dim $C - 1$, hence is equal to $C$. □

We let
\[(4.7)\quad \pi_1 : H' \to H, \quad \text{and} \quad \pi_2 : H' \to C(H)\]
denote the canonical projections.

**Proposition 4.9.** Suppose that $xH'$ is closed for $x = [g] \in RF M$, and set $\Gamma' := g^{-1}\Gamma g \cap H'$. Then
\[(4.8)\quad \overline{xH'} = xHC\]
where $C = \overline{\pi_2(\Gamma')}$ and $HC$ is equal to the identity component of the Zariski closure of $\Gamma'$.

**Proof.** Without loss of generality, we may assume $g = e$. As $H'$ is a direct product $H \times C(H)$, we write an element of $H'$ as $(h, c)$ with $h \in H$ and $c \in C(H)$. For all $\gamma \in \Gamma'$,
\[xH = [(e, e)]H = [(e, \pi_2(\gamma))]H = [(e, e)]H\pi_2(\gamma)\]
and hence $xH = xH\pi_2(\Gamma')$. It follows that $xHC \subset \overline{xH'}$.

To show the other inclusion, let $(h_0, c_0) \in H C(H)$ be arbitrary. If $[(h_0, c_0)] \in \overline{xH} = [(e, e)]H$, then there exist sequences $\gamma_i \in \Gamma'$ and $h_i \in H$
such that $\gamma_i(h_i, e) \to (h_0, c_0)$ in $H'$ as $i \to \infty$. In particular, $\pi_2(\gamma_i) \to c_0$ in $C(H)$ as $i \to \infty$ and hence $c_0 \in C = \pi_2(\Gamma')$. This finishes the proof of (4.8).

Let $W$ denote the identity component of the Zariski closure of $\Gamma'$ in $H'$. Since any proper algebraic subgroup of $G$ stabilizes either a point, or a proper sphere in $S^d$, it follows from Proposition 3.9 and Lemma 4.8 that $\pi_1(\Gamma')$ is Zariski dense in $H$; so $\pi_1(W) = H$. So the quotient $W/H'$ is compact. This implies that $W$ contains a maximal real-split connected solvable subgroup, say, $P$ of $H'$. Now $H \cap W$ is a normal subgroup of $H$, and hence $H < W$. Hence $W = H\pi_2(W)$. As any compact linear group is algebraic, $C$ is algebraic and hence $C = \pi_2(W) = \pi_2(\Gamma')$. Therefore $W = HC$, finishing the proof.

**Global thickness of the return time to $RF_M$.** We recall the various notions of thick subsets of $\mathbb{R}$, following [27] and [29].

**Definition 4.10.** Fix $k > 1$.

- A closed subset $T \subset \mathbb{R}$ is locally $k$-thick at $t$ if for any $\lambda > 0$,
  $$T \cap (t \pm [\lambda, k\lambda]) \neq \emptyset.$$
- A closed subset $T \subset \mathbb{R}$ is $k$-thick if $T$ is locally $k$-thick at 0.
- A closed subset $T \subset \mathbb{R}$ is $k$-thick at $\infty$ if
  $$T \cap ([\lambda, k\lambda]) \neq \emptyset$$
  for all sufficiently large $\lambda \gg 1$.
- A closed subset $T \subset \mathbb{R}$ is globally $k$-thick if $T \neq \emptyset$ and $T$ is locally $k$-thick at every $t \in T$.

We will frequently use the fact that if $T_i$ is a sequence of $k$-thick subsets, then $\limsup T_i$ is also $k$-thick, and that if $T$ is $k$-thick, so is $-T$.

The following proposition shows that $RF_M$ has a thick return property under the action of any one-dimensional subgroup $U$ of $N$.

**Proposition 4.11.** There exists a constant $k > 1$ depending only on $M$ such that for any one-parameter unipotent subgroup $U = \{u_t : t \in \mathbb{R}\}$ of $N^\pm$, and any $y \in RF_M$,

$$T(y) := \{t \in \mathbb{R} : yu_t \in RF_M\}$$

is globally $k$-thick.

**Proof.** We first prove the case when $U < N$. Let $s \in T(y)$ be arbitrary. To show that $T(y)$ is locally $k$-thick at $s$, by replacing $y$ with $yu_s \in RF_M$, we may assume $s = 0$. Note that

$$\eta := \inf_{i \neq j} d(\text{hull}B_i, \text{hull}B_j)$$

is strictly positive because $2\eta$ is bounded below by the shortest length of a closed geodesic in the hyperbolic double of core $M$, which is a closed hyperbolic manifold. We may assume that $y = [g]$ where $g(\infty) = \infty$ and $g(0) = 0$. 


As \( y \in RF M \), this implies that \( 0, \infty \in \Lambda \). Since \( gu_t(\infty) = g(\infty) \in \Lambda \), we have

\[
T(y) = \{ t \in \mathbb{R} : gu_t(0) \in \Lambda \}.
\]

Choose \( k > 1 \) so that

\[
d(\text{hull}([-k, -1]), \text{hull}(1, k)) = 2\eta
\]

where \( d \) is the hyperbolic distance in the upper half plane \( \mathbb{H}^2 \) with \( \partial \mathbb{H}^2 = \mathbb{R} \cup \{\infty\} \). For any \( w \in \mathbb{R}^{d-1} \), denoting by \( d_w \) the hyperbolic distance of the plane above the line \( \mathbb{R}w \), we have that

\[
d_w(\text{hull}([-kt, -t] \cdot w), \text{hull}(t, kt) \cdot w)
\]

is independent of \( w \in \mathbb{R}^{d-1} \) and \( t > 0 \), because both the dilation centered at \( 0 \) and the \((d-2)\)-dimensional rotation with respect to the vertical axis above \( 0 \) are hyperbolic isometries.

Suppose that \( T(y) \) is not locally \( k \)-thick at \( 0 \). Then there exist \( w \in U \) and \( t > 0 \) such that

\[
([-kt, -t] \cdot w \cup [t, kt] \cdot w) \cap \Lambda = \emptyset.
\]

Since each component of \( \Omega \) is convex and \( 0 \not\in \Omega \), it follows that \([-kt, -t] \cdot w \) and \([t, kt] \cdot w \) lie in distinct components of \( \Omega \), say \( B_i \) and \( B_j \), \((i \neq j) \). But this yields

\[
\eta/2 = d_w(\text{hull}([-kt, -t] \cdot w), \text{hull}(t, kt) \cdot w) \\
\geq d(\text{hull}B_i, \text{hull}B_j) \geq \eta
\]

which is a contradiction. The case of \( U < N^+ \) is proved similarly, just replacing the role of \( g^+ \) and \( g^- \) in the above arguments.

\[\square\]

5. Structure of singular sets

Let \( \Gamma < G = \text{SO}^o(d, 1) \) be a convex cocompact torsion-free Zariski-dense subgroup. Let \( U < G \) be a connected closed subgroup of \( G \) generated by unipotent elements in it.

In this section, we define the singular set \( S(U) \) associated to \( U \) and study its structural property. The singular set \( S(U) \) is defined so that it contains all closed orbits of intermediate subgroups between \( U \) and \( G \).

**Definition 5.1** (Singular set). We set

\[
S(U) = \left\{ x \in \Gamma \backslash G : \begin{array}{l}
\text{there exists a proper connected} \\
\text{closed subgroup} W \supset U \text{ such that} xW \\
\text{is closed and Stab}_W(x) \text{ is Zariski dense in} W.
\end{array} \right\}.
\]

**Definition 5.2** (Definition of \( \mathcal{H} \)). We denote by \( \mathcal{H} \) the collection of all proper connected closed subgroups \( H < G \) containing a unipotent element such that

- \( \Gamma \backslash \Gamma H \) is closed, and
- \( H \cap \Gamma \) is Zariski dense in \( H \).
Proposition 5.3. If $H \in \mathcal{H}$, then $H$ is a reductive subgroup of $G$, and hence is of the form $gH(U)Cg^{-1}$ for some connected subgroup $U < N$, a closed subgroup $C < C(H(U))$ and $g \in G$ such that $[g] \in RF M$.

Proof. In order to prove that $H$ is reductive, suppose not. Then its unipotent radical is non-trivial, which we can assume to be a subgroup $U$ of $N$, up to a conjugation. Now we write $H = H_{nc}CTU$ where $C$ is a connected semisimple compact subgroup and $T$ is a torus centralizing $H_{nc}C$. As $H$ is contained in $N(U) = NAC_1(U)C_2(U)$, which does not contain any non-compact simple Lie subgroup, it follows that $H_{nc}$ is trivial. Now if $T$ were compact, then $H \cap \Gamma$ would consist of parabolic elements, which is a contradiction as $\Gamma$ is convex cocompact. Hence $\Gamma$ is non-compact. Write $T = T_0S$ where $S$ is a split torus and $T_0$ is compact. Then $T_0$ is equal to a conjugate of $A$, say, $g^{-1}Ag$ for some $g \in G$. As $T_0$ normalizes $U$, and $(N(U)$ fixes $\infty$, we deduce that $g(\infty)$ is either $\infty$ or $0$. Since $\text{Stab}_G(\infty) = NAM$, $g(\infty) = \infty$ implies $g \in NAM$, and $g(\infty) = 0$ implies $jg \in NAM$ where $j \in G$ is an element of order $2$ such that $j(0) = \infty$. In either case, $T_0 = v^{-1}Av$ for some $v \in N$. By replacing $H$ with $vHv^{-1}$, we may assume that $T_0 = A$. Since $CS$ is a compact subgroup commuting with $A$, $CS \subset M$. Therefore $H$ is of the form $M_0AU$ where $M_0$ is a closed subgroup of $M \cap N(U)$; note that we used the fact that $v$ commutes with $U$. Now the commutator subgroup $[H, H]$ is equal to $[M_0, M_0]U$. Since $[H \cap \Gamma, H \cap \Gamma]$ must be Zariski dense in $[H, H]$, we deduce that $\Gamma$ contains an element $m_0u \in M_0U$ with $u$ non-trivial. Since $m_0u$ is a parabolic element of $\Gamma$, this is a contradiction to the assumption that $\Gamma$ is convex cocompact. This proves that $H$ is reductive.

By Proposition 3.7, $H$ is of the form $gH(U)Cg^{-1}$ for some $g \in G$ and $C < C(H(U))$. For some $m \in M$ and $1 \leq k \leq d - 2$, $H(U) = mH(U_k)m^{-1}$. Hence $\Gamma \backslash \Gamma gH(U_k)C_0$ is closed where $C_0 = m^{-1}Cm$. By Proposition 3.9, the boundary of the geodesic plane $\pi(gmH(U_k))$ contains uncountably many points of $\Lambda$, since $(gmH(U_k)C_0(gm)^{-1} \cap \Gamma$ is Zariski dense in $(gmH(U_k)C_0(gm)^{-1}$. Using two such limit points, we can find an element $\tilde{h} \in H(U_k)$ such that $\tilde{g}(\tilde{h}) \in \Lambda$. Since $(gmhm^{-1})^{\pm} = (gm)^{\pm}$ and $mhm^{-1} \in H(U)$, it follows that $[g]H(U) \cap RF M \neq \emptyset$, and hence we can take $[g] \in RF M$ by modifying it with an element of $H(U)$ if necessary. This finishes the proof.

Therefore, for each $H \in \mathcal{H}$, the non-compact semisimple part $H_{nc}$ of $H$ is well defined.

Proposition 5.4. If $H \in \mathcal{H}$, then

- $H \cap \Gamma$ is finitely generated;
- $[N_G(H_{nc}) \cap \Gamma; H \cap \Gamma] < \infty$.

Proof. Let $p$ denote the projection map $N_G(H_{nc}) \to H_{nc}$. Note that $p$ is an injective map on $N_G(H_{nc}) \cap \Gamma$, as $\Gamma$ is torsion free and the kernel of $p$ is a compact subgroup.
It follows from Proposition 5.3 that $H_{nc}$ is co-compact in $N_G(H_{nc})$. Since $H \in \mathcal{H}$, the orbit $[e]H$ is closed and hence $[e]N_G(H_{nc})$ is closed. It follows that both $p(H \cap \Gamma)$ and $p(N_G(H_{nc}) \cap \Gamma)$ are convex cocompact Zariski dense subgroups of $H_{nc}$ by Proposition 3.9. As any convex cocompact subgroup is finitely generated [3], $p(H \cap \Gamma)$ is finitely generated. Hence $H \cap \Gamma$ is finitely generated by the injectivity of $p|_{H \cap \Gamma}$.

Since $p(H \cap \Gamma)$ is a normal subgroup of $p(N_G(H_{nc}) \cap \Gamma)$, it follows that $p(H \cap \Gamma)$ has finite index in $p(N_G(H_{nc}) \cap \Gamma)$ by Lemma 5.5 below. Since $p|_{N_G(H_{nc}) \cap \Gamma}$ is injective, it follows that $H \cap \Gamma$ has finite index in $N_G(H_{nc}) \cap \Gamma$. ⊓⊔

**Lemma 5.5.** Let $\Gamma_1$ and $\Gamma_2$ be non-elementary convex cocompact subgroups of $G$. If $\Gamma_2$ is a normal subgroup of $\Gamma_1$, then $[\Gamma_1 : \Gamma_2] < \infty$.

**Proof.** Let $A_i$ be the limit set of $\Gamma_i$ for $i = 1, 2$. Since $\Gamma_2 < \Gamma_1$, $A_2 \subset A_1$. As $\Gamma_2$ is normalized by $\Gamma_1$, $A_2 \Gamma_1$ is $\Gamma_1$-invariant. Since $\Gamma_1$ is non-elementary, $A_1$ is a minimal $\Gamma_1$-invariant closed subset. Hence $A_1 = A_2$. Let $M_i := \Gamma_i \setminus \text{Hull}(A_2)$. Then the convex core of $M_1$ is equal to $\Gamma_1 \setminus \text{hull}(A_2)$ and covered by core $M_2 = \Gamma_2 \\text{hull}(A_2)$. Since core $M_2$ is compact, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. ⊓⊔

**Definition 5.6** (Definition of $\mathcal{H}$*).

(5.1) \[ \mathcal{H}^* := \{ N_G(H_{nc}) : H \in \mathcal{H} \}. \]

**Corollary 5.7** (Countability). The collection $\mathcal{H}$ is countable, and the map $H \rightarrow N_G(H_{nc})$ defines a bijection between $\mathcal{H}$ and $\mathcal{H}^*$.

**Proof.** As $\Gamma$ is convex cocompact, it is finitely generated. Therefore there are only countably many finitely generated subgroups of $\Gamma$. By Proposition 5.4, there are only countably many possible $H \cap \Gamma$ for $H \in \mathcal{H}$. Since $H$ is determined by $H \cap \Gamma$, being its Zariski closure, the first claim follows.

Since $H \cap \Gamma$ has finite index in $N_G(H_{nc}) \cap \Gamma$ by Proposition 5.4, $H$ is determined as the identity component of the Zariski closure of $N_G(H_{nc}) \cap \Gamma$. This proves the second claim. ⊓⊔

In the case of a convex cocompact hyperbolic manifold of Fuchsian ends, there is a one to one correspondence between $\mathcal{H}$ and the collection of all closed $H'(U)$-orbits of points in $RF_M$ for $U < N$: if $H \in \mathcal{H}$, then $H = gH(U)Cg^{-1}$ for some $U < N$ and $g \in G$ with $[g] \in RF_M$ and $[g]H'(U)$ is closed. Conversely, if $[g]H'(U)$ is closed for some $[g] \in RF_M$, then the identity component of the Zariski closure of $\Gamma \cap gH'(U)g^{-1}$ is given by $gH(U)Cg^{-1}$ for some closed subgroup $C < C(H(U))$ by Proposition 4.9, and hence $gH(U)Cg^{-1} \in \mathcal{H}$. Moreover, since the normalizer of $H(U)C$ is contained in $H'(U)$, if $g_1H(U)Cg_1^{-1} = g_2H(U)Cg_2^{-1}$, then $g_2^{-1}g_1 \in H'(U)$, so $[g_1]H'(U) = [g_2]H'(U)$. Therefore Corollary 5.7 implies the following corollary by Propositions 3.9 and 4.9.

**Corollary 5.8.** Let $M$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Then
there are only countably many properly immersed geodesic planes of dimension at least 2 intersecting core $M$.

(2) For each $1 \leq m \leq d - 2$, there are only countably many spheres $S \subset S^{d-1}$ of dimension $m$, such that $\#S \cap \Lambda \geq 2$ and $\Gamma S$ is closed in the space $C^m$.

Remark 5.9. In (2), we may replace the condition $\#S \cap \Lambda \geq 2$ with $\#S \cap \Lambda \geq 1$, because if $\#S \cap \Lambda = 1$, then $\Gamma S$ is not closed (see Remark 11.6).

For a subgroup $H < G$, define

$$X(H, U) := \{ g \in G : gUg^{-1} \subset H \}.$$  

Note that $X(H, U)$ is left-$\mathcal{N}_G(H)$ and right-$\mathcal{N}_G(U)$-invariant, and for any $g \in G$,

$$X(gHg^{-1}, U) = gX(H, U).$$

For $H \in \mathcal{K}$ and any unipotent subgroup $U < G$, observe that

$$X(H, U) = X(H_{nc}, U) = X(\mathcal{N}_G(H_{nc}), U);$$

this follows since any unipotent element of $\mathcal{N}_G(H_{nc})$ is contained in $H_{nc}$.

**Proposition 5.10.** We have

$$\mathcal{J}(U) = \bigcup_{H \in \mathcal{K}^*} \Gamma \backslash \Gamma X(H, U).$$

Proof. If $x = [g] \in \mathcal{J}(U)$, then there exists a proper connected closed subgroup $W$ of $G$ containing $U$ such that $[g]W$ is closed and $\text{Stab}_W(x)$ is Zariski dense in $W$. This means $H := gWg^{-1} \in \mathcal{K}$ and $g \in X(H, U)$. Since $X(H, U) = X(\mathcal{N}_G(H_{nc}), U)$, and $\mathcal{N}_G(H_{nc}) \in \mathcal{K}^*$, this proves the inclusion $\subset$.

Conversely, let $g \in X(\mathcal{N}_G(H_{nc}), U)$ for some $H \in \mathcal{K}$. Set $W := g^{-1}H g$. Then $U \subset W$, $[g]W = \Gamma H g$ is closed and $\text{Stab}_W([g]) = g^{-1}(\Gamma \cap H)g$ is Zariski dense in $W$. Hence $[g] \in \mathcal{J}(U)$. \hfill $\square$

**Singular subset of a closed orbit.** Let $L < G$ be a connected reductive subgroup of $G$ containing unipotent elements. For a closed orbit $x_0L$ of $x_0 \in RF M$, and a connected subgroup $U_0 < L \cap N$, we define the singular set $\mathcal{J}(U_0, x_0L)$ by

$$\mathcal{J}(U_0, x_0L) = \{ x \in x_0L : \text{ containing } U_0 \text{ such that } \dim W_{nc} < \dim L_{nc}, xW \text{ is closed and } \text{Stab}_W(x) \text{ is Zariski dense in } W \}. $$

It follows from Proposition 5.10 and Proposition 5.3 that the subgroup $W$ in the definition 5.1 is conjugate to $H(\hat{U})C$ for some $\hat{U} < N$. Hence $W$ being a proper subgroup of $G$ is same as requiring $\dim W_{nc} < \dim G$. Therefore $\mathcal{J}(U_0) = \mathcal{J}(U_0, \Gamma \backslash G)$ and

$$\mathcal{J}(U_0, x_0L) = x_0L \cap \bigcup_{H \in \mathcal{K}^*_{x_0L}} \Gamma \backslash \Gamma X(H, U_0).$$
where $\mathcal{H}_{x_0 L}$ consists of all subgroups $H \in \mathcal{H}^*$ such that $H$ is a subgroup of $g_0 L g_0^{-1}$ with $\dim H_{nc} < \dim L_{nc}$ and $x_0 = [g_0]$.

Then the generic set $\mathcal{G}(U_0, x_0 L)$ is defined by

\begin{equation}
\mathcal{G}(U_0, x_0 L) := (x_0 L \cap \text{RF}_+ M) - \mathcal{G}(U_0, x_0 L).
\end{equation}

**Definition of $\mathcal{L}_U$ and $Q_U$.** For a connected subgroup $U < N$, we define the collection $\mathcal{L}_U$ of all subgroups of the form $H(\hat{U})C$ where $U < \hat{U} < N$ and $C$ is a closed subgroup of $C(H(\hat{U}))$ satisfying the following:

\begin{equation}
\mathcal{L}_U := \left\{ L = H(\hat{U})C : \begin{array}{ll}
\text{for some } [g] \in \text{RF}_+ M, [g]L \text{ is closed in } \Gamma \backslash G \\
\text{and } L \cap g^{-1} \Gamma g \text{ is Zariski dense in } L
\end{array} \right\}.
\end{equation}

Observe that for $L = H(\hat{U})C \neq G$, the condition $L \in \mathcal{L}_U$ with $[g]L$ closed is equivalent to the condition that

$$gLg^{-1} \in \mathcal{H}.$$  

**Lemma 5.11.** Let $L_1$ and $L_2$ be members of $\mathcal{L}_U$ such that $xL_1$ and $xL_2$ are closed for some $x \in \text{RF}_+ M$. If $(L_1)_{nc} = (L_2)_{nc}$, then $L_1 = L_2$.

**Proof.** If $L_1$ or $L_2$ is equal to $G$, then the claim is trivial. Suppose that both $L_1$ and $L_2$ are proper subgroups of $G$. If $x = [g]$, then both subgroups $H_1 := gL_1 g^{-1}$ and $H_2 := gL_2 g^{-1}$ belong to $\mathcal{H}$. Since $(H_1)_{nc} = (H_2)_{nc}$, we have $H_1 = H_2$ by Corollary 5.7. Hence $L_1 = L_2$. \hfill \Box

We also define

\begin{equation}
Q_U := \{vLxv^{-1} : L \in \mathcal{L}_U, v \in N(U)\}.
\end{equation}

Since $N(U) = AN C_1(U) C_2(U)$ by Lemma 3.2, and the collection $\mathcal{L}_U$ is invariant under a conjugation by an element of $AU C_1(U) C_2(U)$, we have

\begin{equation}
Q_U = \{vLxv^{-1} : L \in \mathcal{L}_U, v \in U^\perp\}.
\end{equation}

**Lemma 5.12.** For $U_0 < U < N$, we have

$$X(H(U), U_0) = N_G(H(U)) N_G(U_0).$$

**Proof.** Without loss of generality, we may assume $U = U_m$ and $U_0 = U_\ell$ with $1 \leq \ell \leq m \leq d - 1$. Set $H = H(U_m)$. If $m = d - 1$, then $H = G$, and the statement is trivial. Assume $m \leq d - 2$ below. We will prove the inclusion $X(H, U_0) \subset N_G(H) N_G(U_0)$, as the other one is clear. Let $g \in X(H, U_0)$ be arbitrary. By multiplying $g$ by an element of $N_G(H)$ on the left as well as by an element of $N_G(U_0)$ on the right, we will reduce $g$ to an element of $N_G(U_0)$, which implies the claim.

In view of the Iwasawa decomposition $G = KAN$, since $AN < N_G(U_0)$, we may assume that $g = k \in K$. As $k \in X(H, U_0)$, we have

\begin{equation}
kU_\ell \subset Hk.
\end{equation}

It follows that $k(\infty) = kU_\ell (\infty) \subset \partial \mathbb{H}^{m+1}$, since $\pi(H'k) = \mathbb{H}^{m+1}$, and that $\langle ke_1, \ldots, ke_\ell \rangle \subset \langle e_0, e_1, \ldots, e_m \rangle$ as subspaces of $T_a \mathbb{H}^d$. Since the compact
subgroup $K_0 := K \cap H$ acts transitively on $\partial \mathbb{H}^{m+1}$, by multiplying an element of $K_0$ to $k$ on the left, we may assume that $k(\infty) = \infty$. Now that $k$ fixes the vertical geodesic joining $o \in \mathbb{H}^d$ and $\infty$, it follows $k \in M$ and hence
\[
\langle ke_1, \ldots, ke_\ell \rangle \subset \langle e_1, \ldots, e_m \rangle.
\]

By considering the action of $K_0 \cap M$ on space of $\ell$-tuples of orthonormal vectors in the subspace $(e_1, \ldots, e_m)$, we may assume $ke_1 = e_1, \ldots, ke_\ell = e_\ell$, and $ke_\ell = \pm e_\ell$ after multiplying an element of $K_0 \cap M$ to $k$ on the left. This implies that $k \in C_1(U_0)$, or $k\omega \in C_1(U_0)$ where $\omega \in M$ is an involution which fixes all $e_i$, $i \neq \ell, \ell + 1$ and $\omega(e_i) = -e_i$ for $i = \ell, \ell + 1$. As $N_G(U_0)$ contains $C_1(U_0)$ and $\omega$, the proof is complete.

**Proposition 5.13.** Consider a closed orbit $x_0L$ for $L \in Q_U$ and $x_0 \in RF M$. Let $U_0$ be a connected subgroup of $U$. If $x \in \mathcal{S}(U_0, x_0L)$, then there exists a subgroup $Q \subset Q_{U_0}$ such that
- $\dim Q_{nc} < \dim L_{nc}$;
- $xQ$ is closed;
- $\overline{xU_0} \subset xQ$.

**Proof.** If $x = [g] \in \mathcal{S}(U_0, x_0L)$, then $g \in X(H, U_0)$ for some $H \in \mathcal{H}$ such that $\dim H_{nc} < \dim L_{nc}$, then $\overline{xU_0} \subset x(g^{-1}Hg)$. By Proposition 5.3, $H = gH(\widehat{U})Cq^{-1}$ for some $U_0 \subset \widehat{U} \subset L \cap N$ and some $[g] \in RF M$. Note that $q^{-1}g \in X(H(\widehat{U}), U_0)$.

By Lemma 5.12, we have
\[
q^{-1}g \in N_G(H(\widehat{U}))N_G(U_0).
\]
Hence $g^{-1}Hg = vH(\widehat{U})Cv^{-1}$ for some $v \in N_G(U_0)$, and $\overline{xU_0} \subset xvH(\widehat{U})Cv^{-1}$. It suffices to set $Q := vH(\widehat{U})Cv^{-1}$.

**Lemma 5.14.** Let $x_0L$ be a closed orbit of $L \in L_U$ with $x_0 \in RF M$. If $U$ is a proper subgroup of $L \cap N$, then $\mathcal{S}(U, x_0L) \cdot H(U) \cap F_{H(U)}$ is a proper subset of $x_0L \cap F_{H(U)}$.

**Proof.** Without loss of generality, we may assume that $x_0 = e$, and that $\Gamma$ is contained in $L$, by replacing $\Gamma$ by $L \cap \Gamma$. For the projection map $\pi : L \to \Gamma \setminus L$, the pre-image $\pi^{-1}(\mathcal{S}(U, x_0L) \cdot H(U))$ is a countable union $\gamma X(H, U)H(U)$ where $\gamma \in \Gamma$ and $H \in \mathcal{H}_{x_0L}$. Since $\gamma X(H, U)H(U)$ is a proper algebraic subvariety of $L$, and $\pi^{-1}(F_{H(U)}^* \cap x_0L)$ is an open subset of $L$, it follows from the Baire category theorem that there exists $\ell \in \pi^{-1}(F_{H(U)}^* \cap x_0L)$ such that $[g] \notin \mathcal{S}(U, xL) \cdot H(U)$. This proves the claim.

### 6. Inductive search lemma

In this section, we prove a combinatorial lemma 6.4, which we call an inductive search lemma, and use it to prove Proposition 6.3 on the thickness of a certain subset of $\mathbb{R}$, constructed by the intersection of a global thick
subset $T$ and finite families of triples of subsets of $\mathbb{R}$ with controlled regularity, degree and the multiplicity with respect to $T$. This proposition will be used in the proof of the avoidance theorem 7.13 in the next section.

**Definition 6.1.** Let $J^* \subset I$ be a pair of open subsets of $\mathbb{R}$.

- The degree of $(I, J^*)$ is defined to be the minimal integer $\delta \in \mathbb{N}$ such that for each connected component $I^o$ of $I$, the number of connected components of $J^*$ contained in $I^o$ is bounded by $\delta$.
- For $\beta > 0$, the pair $(I, J^*)$ is said to be $\beta$-regular if for any connected component $I^o$ of $I$, and any component $J^\circ \cap I^o$, \[ J^\circ \pm \beta \cdot |J^\circ| \subset I^o \] where $|J^\circ|$ denotes the length of $J^\circ$.

**Definition 6.2.** Let $X$ be a family of countably many triples $(I, J^*, J')$ of open subsets of $\mathbb{R}$ such that $I \supset J^* \supset J'$.

- Given $\beta > 0$ and $\delta \in \mathbb{N}$, we say that $X$ is $\beta$-regular of degree $\delta$ if for every triple $(I, J^*, J') \in X$, the pair $(I, J^*)$ is $\beta$-regular with degree at most $\delta$.
- Given a subset $T \subset \mathbb{R}$, we say that $X$ is of $T$-multiplicity free if for any two distinct triples $(I_1, J^*_1, J'_1)$ and $(I_2, J^*_2, J'_2)$ of $X$, we have $I_1 \cap J'_2 \cap T = \emptyset$.

For a family $X = \{(I_\lambda, J^*_\lambda, J'_\lambda) : \lambda \in \Lambda\}$, we will use the notation
\[
I(X) := \bigcup_{\lambda \in \Lambda} I_\lambda, \quad J^*(X) := \bigcup_{\lambda \in \Lambda} J^*_\lambda \quad \text{and} \quad J'(X) := \bigcup_{\lambda \in \Lambda} J'_\lambda.
\]

The goal of this section is to prove:

**Proposition 6.3** (Thickness of $T - J'(X)$). Given $n, k, \delta \in \mathbb{N}$, there exists a positive number $\beta_0 = \beta_0(n, k, \delta)$ for which the following holds: let $T \subset \mathbb{R}$ be a globally $k$-thick set, and let $X_1, \ldots, X_\ell$, $\ell \leq n$, be $\beta_0$-regular families of degree $\delta$ and of $T$-multiplicity free. Let $X = \bigcup_{\ell=1}^\ell X_\ell$. If $0 \in T - I(X)$, then $T - J'(X)$ is a $2k$-thick set.

We prove Proposition 6.3 using the inductive search lemma 6.4. The case of $n = 1$ and $\delta = 1$ is easy. As the formulation of the lemma is rather complicated in a general case, we first explain a simpler case of $n = 2$ and $\delta = 1$ in order to motivate the statement.

For simplicity, let us show that $T - (J'(X_1) \cup J'(X_2))$ is 4$k$-thick instead of 2$k$-thick, given that $X_1$ and $X_2$ are $8k^2$-regular families of degree 1, and of $T$-multiplicity free. For any $r > 0$, we need to find a point \[ t \in \pm(r, 4kr) \cap \left(T - J'(X)\right) \] where $X = X_1 \cup X_2$. 

First, we know that there exists \( t_1 \in \pm (2r, 2kr) \cap T \), as \( T \) is locally \( k \)-thick at 0. If \( t_1 \not\in J'(\mathcal{X}_1) \cup J'(\mathcal{X}_2) \), then we are done. So we assume that \( t_1 \in J'(\mathcal{X}_1) \). Our strategy is then to search for a sequence in \( T \) of length at most 4, starting with \( t_1 \), say \((t_1, t_2, t_3, t_4)\) such that

\[
 \frac{|t_{i-1}|}{\sqrt{2}} \leq |t_i| \leq \sqrt{2}|t_{i-1}|
\]

for each \( i = 2, 3, 4 \), and the last element \( t_4 \) does not belong to \( J'(\mathcal{X}) \). This will imply

\[
|t_1|/2 \leq |t_4| \leq 2|t_1|
\]

as desired, because \( 2r \leq |t_1| \leq 2kr \).

We next sketch how we find \( t_2 \) from \( t_1 \) and so on. Let \( t_1 \in J'_1 \) where \((I_1, J'_1, J'_1) \in \mathcal{X}_1 \). Since \( T \) is locally \( k \)-thick at \( t_1 \), there exists

\[
 t_2 \in (t_1 \pm (|J'_1|, k|J'_1|)) \cap T.
\]

We will refer to \( t_1 \) as a pivot for searching \( t_2 \) in (6.1), as \( t_2 \) was found in a symmetric interval around \( t_1 \). Note that \( t_2 \in I_1 - J'_1 \) as \((I_1, J'_1) \) is \( k \)-regular. This implies that \( t_2 \not\in J'(\mathcal{X}_1) \) as the family \( \mathcal{X}_1 \) is of \( T \)-multiplicity free.

Now we will assume \( t_2 \in J'_2 \) for some triple \((I_2, J'_2, J'_2) \in \mathcal{X}_2 \), since otherwise, \( t_2 \not\in J'(\mathcal{X}) \) and we are done (cf. Figure 3).

![Figure 3](image)

**Figure 3.** \( J^* \subset I \) drawn on one of the two copies of \( \mathbb{R} \), according to where \((I, J^*, J') \) belongs.

To search for the next point \( t_3 \in T \), we choose our pivot between two candidates \( t_1 \) and \( t_2 \) as follows: we will choose \( t_1 \) if \( |J'_1| \geq |J'_2| \), and \( t_2 \) otherwise. Without loss of generality, we will assume \( |J'_1| \geq |J'_2| \). Then we can find

\[
 t_3 \in (t_1 \pm 2k(|J'_1|, k|J'_1|)) \cap T,
\]
because $T$ is locally $k$-thick at $t_1$. Note that $t_3 \in I_1 - J_1^*$ as the pair $(I_1, J_1^*)$ is $2k^2$-regular. This implies $t_3 \notin J'(X_1)$ as $X_1$ is of $T$-multiplicity free. Now we can assume that $t_3 \in J_3^*$ for some $(I_3, J_3^*, J_3^*) \in X_2$, otherwise we are done. One can check that $J_3^*$ cannot coincide with $J_2^*$. We claim that $|J_1^*| \geq |J_3^*|$ (cf. Figure 4).

![Figure 4](image.png)

Figure 4. $|J_1^*| \geq |J_3^*|$, otherwise $t_2$ would also be in $I_3$.

Suppose not, i.e. $|J_3^*| > |J_1^*|$. Then we would have $|t_2 - t_1| < k|J_3^*|$ and $|t_1 - t_3| < 2k^2|J_3^*|$, which implies that $t_2 \in I_3$, as the pair $(I_3, J_3^*)$ is $(2k^2 + k)$-regular. This is a contradiction as $X_2$ is $T$-multiplicity free and hence $J_2^* \cap I_3 \cap T = \emptyset$.

Finally, we will choose $t_3$ as a pivot and search for $t_4$. By the local $k$-thickness of $T$ at $t_3$, we can find

$$t_4 \in (t_3 \pm (|J_3^*|, k|J_3^*|)) \cap T.$$

Since the pair $(I_3, J_3^*)$ is $k$-regular, we have $t_4 \in I_3 - J_3^*$. From the fact that the pair $(I_1, J_1^*)$ is $(2k^2 + k)$-regular, one can check that $t_4 \in I_1 - J_1^*$. As a result, $t_4 \in (I_1 - J_1^*) \cup (I_3 - J_3^*)$ and hence $t_4 \notin J'(X)$ (cf. Figure 5).

It remains to check that $|t_{i-1}|/\sqrt{2} \leq |t_i| \leq \sqrt{2}|t_{i-1}|$ for each $i = 2, 3, 4$. This does not necessarily hold for the current sequence, but will hold after passing to a subsequence where $t_{i-1}$ becomes a pivot for searching $t_i$ for all $i$. In the previous case, $(t_1, t_3, t_4)$ will be such a subsequence, as $t_2$ was not a pivot for searching $t_3$.

It follows from the $\beta$-regularity of $(I_{i-1}, J_{i-1}^*)$ that $|\beta|J_{i-1}^*| > 0$, as $t_{i-1} \in J_{i-1}^*$ and $0 \notin I_{i-1}$. On the other hand, observe that

$$t_i \in t_{i-1} \pm C_i(|J_{i-1}^*|, k|J_{i-1}^*|) \cap T.$$
for some $C_i \leq 2k^2$. This gives us the desired upper bound for $|t_i/t_{i-1}|$, as

$$|t_i| < |t_{i-1}| + C|J^*_i| \leq (1 + C_i/\beta)|t_{i-1}|$$

and $1 + C_i/\beta \leq \sqrt{2}$. The lower bound is obtained similarly, completing the proof for $n = 2$ and $\delta = 1$.

The general case reduces to the case of $\delta = 1$, by replacing $m$ by $m\delta$, and the previous argument suggests that we look for a sequence that is almost geometric in a sense that the ratio $|t_i|/|t_{i-1}|$ is coarsely a constant, which finally lands on $T - J'(\mathcal{X})$ in a time controlled by $n$.

The following lemma gives an inductive argument for the search of $t' \in T - J'(\mathcal{X})$ of size comparable to $|t|$ for any given $t \in T \cap J'(\mathcal{X})$. That is to say, for given $n \in \mathbb{N}$, if we choose $\beta$ large enough, then for any $\beta$-regular family $\mathcal{X}_1, \ldots, \mathcal{X}_n$ starting from an arbitrary $t_1 \in T$, we can always find such a sequence in $T$ of length at most $2^n$.

**Lemma 6.4** (Inductive search lemma). Let $k > 1$, $n \in \mathbb{N}$ and $\varepsilon > 0$ be fixed. There exists $\beta = \beta(n, k, \varepsilon) > 0$ for which the following holds: Let $T \subset \mathbb{R}$ be a globally $k$-thick set, and let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ be $\beta$-regular families of countably many triples $(I_\lambda, J^*_\lambda, J'_\lambda)$ with degree 1, and of $T$-multiplicity free. Set $\mathcal{X} = \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_n$, and assume $0 \notin I(\mathcal{X})$. For any $t \in T \cap J'(\mathcal{X})$ and any $1 \leq r \leq n$, we can find a sequence $(I_1, J^*_1, J'_1), \ldots, (I_{m-1}, J^*_{m-1}, J'_{m-1}) \in \mathcal{X}$ for some $2 \leq m \leq 2^r$ with $J^*_1, \ldots, J^*_{m-1}$ all distinct, and a sequence

$$t = t_1 \in T \cap J'_1, t_2 \in T \cap J'_2, \ldots, t_{m-1} \in T \cap J'_{m-1}, t_m \in T$$

which satisfy the following conditions:
(1) Either $t_m \notin J'(\mathcal{X})$, or $t_m \in J'_m$ for some $(I_m, J'_m, J^*_m) \in \mathcal{X}$ where $J'_m$ is distinct from $J^*_1, \ldots, J^*_{m-1}$, and $\{(I_i, J_i^*): 1 \leq i \leq m\}$ intersect at least $r + 1$ number of $\mathcal{X}_i$'s.

(2) For any $2 \leq j \leq m$, there exists an increasing sequence $1 = j_0 < j_1 < \cdots < j_a = j$ such that for each $1 \leq b \leq a$,

$$|t_{j_b} - t_{j_{b-1}}| < (4k)^{r+1} - 1)k|J^*_{j_{b-1}}|;$$

(3) For each $1 \leq i \leq m$,

$$(1 - \varepsilon)^{m-1}|t_1| \leq |t_i| \leq (1 + \varepsilon)^{m-1}|t_1|.$$ 

In particular, for any $t \in \mathcal{T} \cap J'(\mathcal{X})$, there exists $t' \in \mathcal{T} - J'(\mathcal{X})$ such that

$$|t - t'| \leq (1 - \varepsilon)^{2^{n-1}-1}|t| \leq (1 + \varepsilon)^{2^{n-1}}|t|.$$ 

Proof. We set

$$\beta = \beta(n, k, \varepsilon) = 2^{3n+5k^3n+3}\varepsilon^{-1}.$$ 

Consider the increasing sequence $Q(r) := (4k)^{r+1} - 1$ for $r \in \mathbb{N}$. Note that $Q(1) \geq 2$ and $Q(r + 1) \geq 4Q(r)k + 1$.

Moreover we can check that

$$\beta > \max((Q(n) + 4Q(n - 1))k + 1, (Q(n + 1) + (2n + 2)Q(n))k, Q(n)\varepsilon^{-1}).$$

We proceed by induction on $r$.

First consider the case when $r = 1$. There exists $(I_1, J_1^*, J'_1) \in \mathcal{X}$ such that $t_1 := t \in J'_1 \cap \mathcal{T}$. As $\mathcal{T}$ is globally $k$-thick, we can choose

$$t_2 \in (t_1 \pm Q(1)(|J_1^*|, k|J_1^*|)) \cap \mathcal{T}.$$ 

We claim that $t_1, t_2$ is our desired sequence with $m = 2$.

In the case when $t_2 \in J'(\mathcal{X})$, there exists $(I_2, J_2^*, J'_2) \in \mathcal{X}$ such that $t_2 \in J'_2$.

We check:

(1): If $t_2 \in J'(\mathcal{X})$, then $t_2 \in J'_2 - J_2^*$ implies that $J_2^*$ and $J_2^*$ are distinct.

Since $\beta > Q(1)k$, by the $\beta$-regularity of $(I_1, J_1^*)$, we have $t_2 \in I_1$. Now $\{(I_1, J_1^*, J'_1), (I_2, J_2^*, J'_2)\}$ intersect two of $\mathcal{X}_i$’s.

(2): By (6.3), $|t_1 - t_2| < Q(1)k|J_1^*| = (4k)^2 - 1)k|J_1^*|$. 

(3): Note that $0 \notin I_1$, since $0 \notin I(\mathcal{X})$. By the $\beta$-regularity of $(I_1, J_1^*)$, we have $t_1 \pm \beta|J_1^*| \subset I_1$. Since $0 \notin I_1$ and $\beta > \varepsilon^{-1}Q(1)k$, we have

$$|t_1| - \varepsilon^{-1}Q(1)k|J_1^*| > 0.$$ 

On the other hand, by (6.3),

$$|t_2 - t_1| \leq Q(1)k|J_1^*| \leq \varepsilon|t_1|.$$ 

In particular,

$$|t_2| < |t_1| + |t_2 - t_1| < |t_1| + Q(1)k|J_1^*| < (1 + \varepsilon)|t_1|$$

and

$$|t_2| > |t_1| - |t_2 - t_1| > |t_1| - Q(1)k|J_1^*| > (1 - \varepsilon)|t_1|.$$ 

This proves the base case of $r = 1$. 

Next, assume the induction hypothesis for \( r \). Hence we have a sequence
\[
t_1 = t \in J_1', \; t_2 \in J_2', \ldots, \; t_{m-1} \in J_{m-1}', \; \text{and} \; t_m
\]
in \( T \) with \( m \leq 2^r \) together with \( \{(I_i, J_i^*, J_i') : 1 \leq i \leq m-1\} \) satisfying the three conditions listed in the lemma.

If \( t_m \notin J'(\mathcal{X}) \), the same sequence would satisfy the hypothesis for \( r + 1 \) and we are done. Now we assume that \( t_m \in J_m' \) for some \( (I_m, J_m^*, J_m') \in \mathcal{X} \), and that \( \{(I_i, J_i^*, J_i') : 1 \leq i \leq m\} \) intersect at least \( (r + 1) \) numbers of \( \mathcal{X}_i \)’s. We may assume that they intersect exactly \( (r + 1) \)-number of \( \mathcal{X}_i \)’s, which we may label as \( \mathcal{X}_1', \ldots, \mathcal{X}_{r+1}' \), since if they intersect more than \( (r + 1) \) of them, we are already done.

Choose a largest interval \( J_{x'}^* \) among \( J_1^*, \ldots, J_m^* \). Again using the global \( k \)-thickness of \( T \), we can choose
\[
(6.4) \quad s_1 \in \left( t_\ell \pm Q(r + 1)(|J_{x'}^*|, k|J_{x'}^*|) \right) \cap T.
\]

First, consider the case when \( s_1 \notin J'(\mathcal{X}) \). We will show that the points \( t_1, \ldots, t_m, s_1 \) give the desired sequence. Indeed, the conditions (1) and (2) are immediate. To show (3), since \( \beta > \varepsilon^{-1} Q(r + 1)k \) and \( 0 \notin I_\ell \), by applying the \( \beta \)-regularity to the pair \( (I_\ell, J_{x'}^*) \), we have
\[
|t_\ell| - \varepsilon^{-1} Q(r + 1)k|J_{x'}^*| > 0.
\]
It follows that
\[
\begin{align*}
|s_1| & \leq |t_\ell| + |s_1 - t_\ell| < |t_\ell| + Q(r + 1)k|J_{x'}^*| < (1 + \varepsilon)|t_\ell| \leq (1 + \varepsilon)^m|t_1|; \\
|s_1| & \geq |t_\ell| - |s_1 - t_\ell| > |t_\ell| + Q(r + 1)k|J_{x'}^*| > (1 - \varepsilon)|t_\ell| \geq (1 - \varepsilon)^m|t_1|.
\end{align*}
\]
This proves (3).

For the rest of the proof, we now assume that \( s_1 \in J'(\mathcal{X}) \). Apply the induction hypothesis for \( r \) to \( s_1 \in T \cap J'(\mathcal{X}) \) to obtain a sequence \( \{(I_j, J_j^*, J_j') \in \mathcal{X} : 1 \leq j \leq m' - 1\} \) with \( m' \leq 2^r \) and
\[
s_1 \in J_1^* \cap T, \; s_2 \in J_2^* \cap T, \ldots, \; s_{m'-1} \in J_{m'-1}^* \cap T, \; \text{and} \; s_{m'} \in T
\]
satisfying the corresponding condition. We claim that the sequence
\[
(6.5) \quad t_1, \ldots, t_m, s_1, \ldots, s_{m'}
\]
of length \( m + m' \leq 2^{r+1} \) satisfies the conditions of the lemma for \( r + 1 \).

We first claim that \( J_i^* \) and \( J_j^* \) are distinct for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq m' - 1 \). To see that, let \( J_q^* \) be the largest one among \( J_i^* \)'s, and \( J_q^* \) be the larger one between \( J_i^* \) and \( J_j^* \).

By the induction hypothesis, we have
\[
\begin{align*}
\max_{1 \leq i,j \leq m} |t_i - t_j| & < 2Q(r)k|J_{x'}^*| \quad \text{and} \\
\max_{1 \leq i,j \leq m'} |s_i - s_j| & < 2Q(r)k|J_q^*|.
\end{align*}
\]
Now for \( t_i \in J^*_i (1 \leq i \leq m) \) and \( s_j \in \tilde{J}^*_j (1 \leq j < m') \), we estimate:

\[
|s_j - t_i| \geq |s_1 - t_\ell| - |t_\ell - t_i| - |s_1 - s_j| \\
> Q(r + 1)|J^*_\ell| - 2Q(r)k|J^*_s| - 2Q(r)k|\tilde{J}^*_q| \\
\geq (Q(r + 1) - 4Q(r)k)|J^*_\ell| \\
\geq |J^*_m|.
\]

This in particular means that \( s_j \notin J^*_i \) and \( t_i \notin \tilde{J}^*_j \). Hence

\[ J^*_i \neq \tilde{J}^*_j. \]

We now begin checking the conditions (1), (2) and (3).

(1): If \( s_{m'} \notin J^*(X) \), there is nothing to check. Now assume \( s_{m'} \in \tilde{J}^*_{m'} \) for some \( (I_{m'}, \tilde{J}^*_{m'}, J^*_{m'}) \in \mathcal{X} \). By induction hypothesis for \( r \) on the sequence \( (s_1, \ldots, s_{m'}) \), we can choose \( \tilde{J}^*_{m'} \) to be distinct from other \( \tilde{J}^*_j \)'s, and \( \{(I_j, \tilde{J}^*_j, J^*_j) : 1 \leq j \leq m'\} \) would intersect at least \( (r + 1) \) numbers of \( X_i \)'s. Observe that in the estimate (6.6), there is no harm in allowing \( j = m' \) in addition to \( j < m' \). This shows that \( J^*_m \) is also distinct from all \( J^*_i \)'s.

Therefore, unless the following inclusion

\[
(6.7) \quad \{(I_j, \tilde{J}^*_j, J^*_j) : 1 \leq j \leq m'\} \subset \mathcal{X_1} \cup \cdots \cup \mathcal{X}_{r+1},
\]

holds, we are done. Suppose that (6.7) holds. We will deduce a contradiction. Without loss of generality, we assume that

\[
(I_\ell, J^*_\ell, J^*_\ell) \in \mathcal{X}_{r+1}.
\]

Claim: We have

\[
(6.8) \quad |J^*_\ell| = \max_{1 \leq i \leq m, 1 \leq j \leq m'}(|J^*_i|, |\tilde{J}^*_j|).
\]

Recall that \( |J^*_\ell| \) was chosen to be maximal among \( |J^*_1|, \ldots, |J^*_m| \). Hence, if the claim does not hold, then we can take \( j \) to be the least number such that \( |J^*_j| > |J^*_\ell| \). Then by induction hypothesis for (2), there exists an increasing sequence \( 1 = j_0 < j_1 < \cdots < j_a = j \) such that

\[
|s_{j_a} - s_{j_{a-1}}| < Q(r)k|\tilde{J}^*_{j_{a-1}}| \\
\vdots \\
|s_{j_2} - s_{j_1}| < Q(r)k|J^*_{j_{0}}|.
\]

In particular all of the right hand sides in the above inequalities are less than or equal to \( Q(r)k|J^*_\ell| \), by the choice of \( j = j_a \). Therefore, we get

\[
|t_\ell - s_j| \leq |t_\ell - s_{j_0}| + |s_{j_0} - s_{j_1}| + \cdots + |s_{j_{a-1}} - s_{j_a}| \\
\leq Q(r + 1)k|J^*_\ell| + Q(r)k|J^*_\ell| + \cdots + Q(r)k|J^*_\ell| \\
\leq (Q(r + 1) + 2'Q(r))k|J^*_\ell|.
\]

Now as the collection \( \{(I_i, \tilde{J}^*_i, J^*_i) : 1 \leq i \leq m\} \) intersects \( (r + 1) \) families \( \mathcal{X}_1, \cdots, \mathcal{X}_{r+1} \) and \( (I_j, \tilde{J}^*_j, J^*_j) \) belongs to one of these families, there exists a
triple \((I_t, J_t^*, J_t')\) that belongs to the same family as \((\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j')\). Recall that the induction hypothesis for \(t_1, \cdots, t_m\) gives us
\[
|t_\ell - t_i| \leq 2Q(r)k|J^*_\ell|.
\]
Since \(\beta > (Q(r+1) + (2^r + 2)Q(r))k\), we have
\[
|t_i - s_j| \leq |t_\ell - t_i| + |t_\ell - s_j| \\
\leq (Q(r+1) + (2^r + 2)Q(r))k|J^*_\ell| \\
\leq \beta|J^*_j|.
\]
Applying the \(\beta\)-regularity to the pair \((\tilde{I}_j, \tilde{J}_j^*)\), we conclude that
\[
t_i \in \tilde{I}_j \cap J_t^* \cap T.
\]
Since \(J_t^*\) and \(\tilde{I}_j\) belong to the same family, this is a contradiction. This proves the claim (6.8).

We now claim that the following inclusion holds:
\[
(6.9) \quad \{(\tilde{I}_j, \tilde{J}_j^*, J_t') : 1 \leq j \leq m'\} \subset \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_r.
\]
Note that this gives the desired contradiction, since \(\{(\tilde{I}_j, \tilde{J}_j^*, J_t') : 1 \leq j \leq m'\}\) must intersect at least \((r+1)\) number of \(\mathcal{X}_i\) by induction hypothesis. In order to prove the inclusion (6.9), suppose on the contrary that \(J_t^*\) belongs to \(J^*(\mathcal{X}_{r+1})\) for some \(1 \leq j \leq m'\). Using \(\beta > (Q(r+1) + 2Q(r))k\) and (6.8), we deduce
\[
|t_\ell - s_j| \leq |t_\ell - s_1| + |s_1 - s_j| \\
\leq Q(r+1)k|J^*_\ell| + 2Q(r)k|J^*_\ell| \\
< \beta|J^*_j|
\]
where we used the induction hypothesis for the sequence \((s_1, \cdots, s_{m'})\) in the second line, to estimate the term \(|s_1 - s_j|\).

Next, applying the \(\beta\)-regularity to the pair \((I_\ell, J^*_\ell)\), we conclude that \(s_j \in I_\ell\). Since \(s_j \in \tilde{J}_j^*\), it follows that \(I_\ell \cap J_t^* \cap T \neq \emptyset\). This contradicts the condition that \(\mathcal{X}_{r+1}\) is of \(T\)-multiplicity free, as both \((\tilde{I}_j, \tilde{J}_j^*, J_t')\) and \((I_\ell, J^*_\ell, J_t')\) belong to the same family \(\mathcal{X}_{r+1}\). This completes the proof of (1).

(2): Let \(1 \leq j \leq m'\) be arbitrary. First, recall from (6.4) that we chose \(s_1\) such that
\[
|s_1 - t_\ell| \leq Q(r+1)k|J^*_\ell|
\]
in the choice of the sequence \(t_1, \cdots, t_m, s_1, \cdots, s_{m'}\) in (6.5).

Secondly, by the induction hypothesis on the sequence \(s_1, \cdots, s_{m'}\), there exists a sequence \(1 = j_0 < j_1 < \cdots < j_a = j\) such that for each \(1 \leq b \leq a\),
\[
|s_{j_b} - s_{j_{b-1}}| < Q(r)k|J^*_{j_{b-1}}|.
\]
Lastly, also by the induction hypothesis on the sequence \(t_1, \cdots, t_m\), there exists \(1 = j'_0 < j'_1 < \cdots < j'_d = \ell\) such that for each \(1 \leq b' \leq d'\),
\[
|t_{j'_b} - t_{j'_{b-1}}| < Q(r)k|J^*_{j'_{b-1}}|.
\]
These three observations prove (2) for \( r+1 \), by concatenating the two index sets 

\[ 1 = j_0 < j'_1 < \cdots < j'_a = \ell \text{ for } t \text{ and } 1 = j_0 < j_1 < \cdots < j_a = j \text{ for } s. \]

(3): We already have observed that the inequality \( \beta > \varepsilon^{-1}Q(r+1)k \) implies that

\[(1 - \varepsilon)^m|t_1| \leq |s_1| \leq (1 + \varepsilon)^m|t_1|.
\]

Combining this with the induction hypothesis, we deduce that

\[(1 - \varepsilon)^{m'-1}|s_1| \leq |s_i| \leq (1 + \varepsilon)^{m'-1}|s_1|\]

for all \( 1 \leq i \leq m' \).

Finally, the last statement of the lemma is obtained from the case \( r = n \), since there are only \( n \)-number of \( X_i \)'s; hence the second possibility of (1) cannot arise for \( r = n \). \( \square \)

**Proof of Proposition 6.3.** We may assume that \( X_i \)'s are all of degree 1, by replacing each \( X_i \)'s with \( \delta \)-number of families associated to it.

We set

\[ \beta_0(n, k, 1) = 2^{3n+5}k^{n+3}\varepsilon^{-1} \]

where \( \varepsilon \) satisfies \( \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{2n-1} \leq 2 \).

Note that \( \beta_0(n, k, 1) \) is equal to the number given in (6.2). We may assume \( x = 0 \) without loss of generality. Let \( \lambda > 0 \). We need to find a point

\[(6.10) \quad t' \in \left( [-2k\lambda, -\lambda] \cup [\lambda, 2k\lambda] \right) \cap \left( T - \bigcup_{i \in \Lambda} J'(X_i) \right). \]

Choose \( s > 0 \) such that

\[(6.11) \quad (1 - \varepsilon)^{-2n-1}\lambda \leq s \leq 2(1 + \varepsilon)^{-2n-1}\lambda. \]

Since \( T \) is globally \( k \)-thick, there exists

\[ t \in \left( [-ks, -s] \cup [s, ks] \right) \cap T. \]

If \( t \not\in \bigcup_{i=1}^n J'(X_i) \), then by choosing \( t' = t \), we are done. Now suppose \( t \in \bigcup_{i=1}^n J'(X_i) \).

Since \( 0 \not\in \bigcup_{i=1}^n I(X_i) \), by applying Lemma 6.4 to \( t \in T \cap (\bigcup_{i=1}^n J'(X_i)) \), we obtain \( t' \in T - \bigcup_{i=1}^n J'(X_i) \) such that

\[(1 - \varepsilon)^{2n-1}|t| \leq |t'| \leq (1 + \varepsilon)^{2n-1}|t|. \]

Note that

\[ |t'| \leq (1 + \varepsilon)^{2n-1}|t| \leq (1 + \varepsilon)^{2n-1}ks \leq 2k\lambda. \]

Similarly, we have

\[ |t'| \geq (1 - \varepsilon)^{2n-1}|t| \geq (1 - \varepsilon)^{2n-1}s \geq \lambda. \]

This completes the proof since \( t' \) satisfies (6.10).
7. Avoidance of the singular set

Let \( G = \text{SO}^\circ(d, 1) \), \( \Gamma < G \) be a convex cocompact non-elementary subgroup and let
\[
U = \{ u_t \} < N
\]
be a one-parameter unipotent subgroup. Let \( \mathcal{S}(U) \), \( \mathcal{G}(U) X(H, U) \), and \( \mathcal{H}^* \) be as defined in section 5. In particular, \( \mathcal{S}(U) \) is a countable union:
\[
\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}^*} \Gamma \backslash \Gamma X(H, U).
\]

The main goal of this section is to prove the avoidance Theorem 7.13 for any convex cocompact hyperbolic manifold with Fuchsian ends. For this, we extend the linearization method developed by Dani and Margulis [12] to our setting. Via a careful analysis of the graded self-intersections of the union
\[
\bigcup_i \Gamma \backslash \Gamma H_i D_i \cap RF M
\]
for finitely many groups \( H_i \in \mathcal{H}^* \) and compact subsets \( D_i \subset X(H_i, U) \), we construct families of triples of subsets of \( \mathbb{R} \) satisfying the conditions of Proposition 6.3 relative to the global \( k \)-thick subset of the return time to \( RF M \) under \( U \) given in Proposition 4.11.

**Linearization.** Let \( H \in \mathcal{H}^* \). Then \( H \) is reductive, algebraic, and is equal to \( N_G(H) \) by Proposition 5.3 and (3.1). There exists an \( \mathbb{R} \)-regular representation \( \rho_H : G \to \text{GL}(V_H) \) with a point \( p_H \in V_H \), such that \( H = \text{Stab}_G(p_H) \) and the orbit \( p_H G \) is Zariski closed [2, Theorem 3.5]. Since \( \Gamma \backslash \Gamma H \) is closed, it follows that \( p_H \Gamma \) is a closed (and hence discrete) subset of \( V_H \).

Let \( \eta_H : G \to V_H \) denote the orbit map defined by
\[
\eta_H(g) = p_H g \quad \text{for all } g \in G.
\]

As \( H \) and \( U \) are algebraic subgroups, the set \( X(H, U) = \{ g \in G : gUg^{-1} \subset H \} \) is Zariski closed in \( G \). Since \( p_H G \) is Zariski closed in \( V_H \), it follows that
\[
A_H := p_H X(H, U)
\]
is Zariski closed in \( V_H \) and \( X(H, U) = \eta_H^{-1}(A_H) \).

Following [17], for given \( C > 0 \) and \( \alpha > 0 \), a function \( f : \mathbb{R} \to \mathbb{R} \) is called \((C, \alpha)\)-good if for any interval \( I \subset \mathbb{R} \) and \( \varepsilon > 0 \), we have
\[
\ell\{ t \in I : |f(t)| \leq \varepsilon \} \leq C \cdot \left( \frac{\varepsilon}{\sup_{t \in I} |f(t)|} \right)^\alpha \cdot \ell(I)
\]
where \( \ell \) is a Lebesgue measure on \( \mathbb{R} \).

**Lemma 7.1.** For given \( C > 1 \) and \( \alpha > 0 \), consider functions \( p_1, p_2, \ldots, p_k : \mathbb{R} \to \mathbb{R} \) satisfying the \((C, \alpha)\)-good property. For \( 0 < \delta < 1 \), set
\[
I = \{ t \in \mathbb{R} : \max \limits_i |p_i(t)| < 1 \} \quad \text{and} \quad J(\delta) = \{ t \in \mathbb{R} : \max \limits_i |p_i(t)| < \delta \}.
\]
For any \( \beta > 1 \), there exists \( \delta = \delta(C, \alpha, \beta) > 0 \) such that the pair \( (I, J(\delta)) \) is \( \beta \)-regular (see Def. 6.2).
Proof. We prove that the conclusion holds for 
\[ \delta := \left( \frac{1}{1 + |v| c} \right)^{1/\alpha} \].

First, note that the function \( q(t) := \max_i |p_i(t)| \) also has the \((C, \alpha)\)-good property. Let \( J' = (a, b) \) be a component of \( J(\delta) \), and \( I' \) be the component of \( I \) containing \( J' \). Note that \( I' \) is an open interval and \((a, \infty) \cap I' = (a, c)\) for some \( b \leq c \leq \infty \). We claim
\[ J' + \beta |J'| \subset (a, \infty) \cap I' \subset I'. \]

We may assume that \( c < \infty \); otherwise the inclusion is trivial. We claim that \( q(c) = 1 \).

Since \( \{ t \in \mathbb{R} : q(t) < 1 \} \) is open and \( c \) is the boundary point of \( I' \), \( q(c) \geq 1 \). If \( q(c) \) were strictly bigger than 1, since \( \{ t \in \mathbb{R} : q(t) > 1 \} \) is open, \( I' \) would be disjoint from an open interval around \( c \), which is impossible. Hence \( q(c) = 1 \).

Now that \( \sup\{q(t) : t \in (a, \infty) \cap I'\} = q(c) = 1 \), by applying the \((C, \alpha)\)-good property of \( q \) on the interval \((a, \infty) \cap I'\), we get
\[ \ell(J') \leq \ell(\{ t \in (a, \infty) \cap I' : |q(t)| \leq \delta \}) \]
\[ \leq C \delta^\alpha \cdot \ell((a, \infty) \cap I'). \]

Now as \( J' = (a, b) \) and \((a, \infty) \cap I'\) are nested intervals with one common endpoint, it follows from the inequality \( C \delta^\alpha < 1/(1 + \beta) \) that
\[ J' + \beta |J'| \subset (a, \infty) \cap I' \subset I', \]
proving (7.1). Similarly, applying the \((C, \alpha)\)-good property of \( q \) on \((-\infty, b) \cap I'\), we deduce that
\[ J' - \beta |J'| \subset I'. \]

This proves that \((I, J(\delta))\) is \(\beta\)-regular. \(\square\)

**Proposition 7.2.** Let \( V \) be a finite dimensional real vector space, \( \theta \in \mathbb{R}[V] \) be a polynomial and \( A = \{ v \in V : \theta(v) = 0 \} \). Then for any compact subset \( D \subset A \) and any \( \beta > 0 \), there exists a compact neighborhood \( D' \subset A \) of \( D \) which has a \(\beta\)-regular size with respect to \( D \) in the following sense: for any neighborhood \( \Phi \) of \( D' \), there exists a neighborhood \( \Psi \subset \Phi \) of \( D \) such that for any \( q \in V - \Phi \) and for any one-parameter unipotent subgroup \( \{ u_t \} \subset \mathrm{GL}(V) \), the pair \((I(q), J(q))\) is \(\beta\)-regular where
\[ I(q) = \{ t \in \mathbb{R} : qu_t \in \Phi \} \text{ and } J(q) = \{ t \in \mathbb{R} : qu_t \in \Psi \}. \]

Furthermore, the degree of \((I(q), J(q))\) is at most \((\deg \theta + 2) \cdot \dim V\).

**Proof.** Choose a norm on \( V \) so that \( \|v\|^2 \) is a polynomial function on \( V \). Since \( D \) is compact, we can find \( R > 0 \) such that
\[ D \subset \{ v \in V : \|v\| < R \}. \]

Then we set
\[ D' = \{ v \in V : \theta(v) = 0, \|v\| < R/\sqrt{\delta} \}, \]
where $0 < \delta < 1$ is to be specified later. Note that if $\Phi$ is a neighborhood of $D'$, there exists $0 < \eta < 1$ such that
\[
\{ v \in V : \theta(v) < \eta, \|v\| < (R + \eta)/\sqrt{\delta} \} \subset \Phi.
\]
We will take $\Psi$ to be
\[
\Psi = \{ v \in V : \theta(v) < \eta \delta, \|v\| < (R + \eta)/\sqrt{\delta} \}.
\]
Set
\[
\tilde{I}(q) = \{ t \in \mathbb{R} : \theta(qu_t) < \eta, \|qu_t\| < (R + \eta)/\sqrt{\delta} \}.
\]
Since $\tilde{I}(q) \subset I(q)$ for $0 < \delta < 1$, it suffices to find $\delta$ (and hence $D'$ and $\Psi$) so that the pair $(\tilde{I}(q), J(q))$ is $\beta$-regular.

If we set
\[
\psi_1(t) := \frac{\theta(qu_t)}{\eta} \quad \text{and} \quad \psi_2(t) := \left( \frac{\|qu_t\| \sqrt{\delta}}{R + \eta} \right)^2,
\]
then
\[
\tilde{I}(q) = \{ t \in \mathbb{R} : \max(\psi_1(t), \psi_2(t)) < 1 \};
\]
\[
J(q) = \{ t \in \mathbb{R} : \max(\psi_1(t), \psi_2(t)) < \delta \}.
\]
As $\psi_1$ and $\psi_2$ are polynomials, they have the $(C, \alpha)$-property for an appropriate choice of $C$ and $\alpha$. Therefore by applying Lemma 7.1, by choosing $\delta$ small enough, we can make the pair $(\tilde{I}(q), J(q))$ $\beta$-regular for any $\beta > 0$.

Note that the degrees of $\psi_1$ and $\psi_2$ are bounded by $\deg \theta \cdot \dim V$ and $2 \dim V$ respectively. Therefore $J(q)$ cannot have more than $(\deg \theta + 2) \cdot \dim V$ number of components. Hence the proof is complete. \qed

**Collection $\mathcal{E}_U$.** Recall the collection $\mathcal{H}^*$ and the singular set:
\[
\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}^*} \Gamma \setminus \Gamma X(H, U).
\]

**Definition 7.3.** We define $\mathcal{E} = \mathcal{E}_U$ to be the collection of all compact subsets of $\mathcal{S}(U) \cap \text{RF} \ M$ which can be written as
\[
E = \bigcup_{i \in \Lambda} \Gamma \setminus \Gamma H_i D_i \cap \text{RF} \ M \tag{7.2}
\]
where $\{ H_i \in \mathcal{H}^* : i \in \Lambda \}$ is a finite collection and $D_i \subset X(H_i, U)$ is a compact subset. In this expression, we always use the minimal index set $\Lambda$ for $E$. When $E$ is of the form (7.2), we will say that $E$ is associated to the family $\{ H_i : i \in \Lambda \}$.

**Remark 7.4.** We note that $E$ can also be expressed as $\bigcup_{i \in \Lambda} \Gamma \setminus \Gamma H_i D_i \cap \text{RF} \ M$ where $H_i \in \mathcal{H}$ is a finite collection, and $D_i \subset X(H_i, U)$ is a compact subset which is left $C(H_i)$-invariant.

**Lemma 7.5.** In the expression (7.2) for $E \in \mathcal{E}$, the collection $\{ H_i : i \in \Lambda \}$ is not redundant, in the sense that
• no $\gamma H_j \gamma^{-1}$ is contained in $H_i$ for all triples $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$ except for the trivial cases of $i = j$ and $\gamma \in H_i$.

Proof. Observe that if $\gamma H_j \gamma^{-1} \subset H_i$ for some $\gamma \in \Gamma$, then $\Gamma H_j D_j \subset \Gamma H_i \gamma D_j$, and hence by replacing $D_i$ by a new compact subset $D_i \cup \gamma D_j \subset X(H_i, U)$, we may remove $j$ from the index subset $\Lambda$. This contradicts the minimality of $\Lambda$. \hfill \Box

Observe that for any subgroups $H_1, H_2$ of $G$, and $g \in G$,

$$X(H_1 \cap gH_2g^{-1}, U) = X(H_1, U) \cap X(gH_2g^{-1}, U) = X(H_1, U) \cap gX(H_2, U).$$

Note that for $D_i \subset X(H_i, U)$, and $\gamma \in \Gamma$, the intersection $H_1 D_1 \cap \gamma H_2 D_2$ only depends on the $(\Gamma \cap H_1, \Gamma \cap H_2)$-double coset of $\gamma$.

**Proposition 7.6.** Let $H_1, H_2 \in \mathcal{K}^\ast$. Then for any compact subset $D_i \subset X(H_i, U)$ for $i = 1, 2$ and a compact subset $K \subset \Gamma \setminus G$, there exists a finite set $\Delta \subset (H_1 \cap \Gamma \setminus \Gamma \cap H_2)$ such that

$$\{ K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) \}_{\gamma \in \Gamma} = \{ K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) \}_{\gamma \in \Delta}$$

where the latter set consists of distinct elements.

Moreover for each $\gamma \in \Delta$, there exists a compact subset $C_0 \subset H_1 D_1 \cap \gamma H_2 D_2 \subset X(H_1 \cap \gamma H_2 \gamma^{-1}, U)$ such that

$$K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) = \Gamma \setminus \Gamma C_0.$$  

Proof. For simplicity, write $\eta H_i = \eta_i$ and $p_i = p_{H_i}$. Let $K_0 \subset G$ be a compact set such that $K = \Gamma \setminus \Gamma K_0$. We fix $\gamma \in \Gamma$, and define for any $\gamma' \in \Gamma$,

$$K_{\gamma'} = \{ g \in K_0 : \gamma' g \in H_1 D_1 \cap \gamma H_1 D_2 \}.$$ 

We check that

$$K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) = \Gamma \setminus \Gamma (\cup_{\gamma' \in \Gamma} K_{\gamma'}).$$

If this set is non-empty, then $K_{\gamma'} \neq \emptyset$ for some $\gamma' \in \Gamma$ and

$$p_1 \gamma' g \in p_1 D_1, \quad p_2 \gamma^{-1} \gamma' g \in p_2 D_2$$

for some $g \in K_0$. In particular,

$$(7.3) \quad p_1 \gamma' \in p_1 DK_0^{-1}, \quad p_2 \gamma^{-1} \in p_2 DK_0^{-1} \gamma'^{-1}.$$ 

As $p_1 \Gamma$ is discrete, and $p_1 D_1 K_0^{-1}$ is compact, the first condition of (7.3) implies that there exists a finite set $\Delta_0 \subset G$ such that $\gamma' \in (H_1 \cap \Gamma) \Delta_0$. Writing $\gamma' = h \delta_0$ where $h \in H_1 \cap \Gamma$, and $\delta_0 \in \Delta_0$, the second condition of (7.3) implies

$$p_2 \gamma^{-1} h \in p_2 D_2 K_0^{-1} \delta_0^{-1}.$$ 

As $p_2 D_2 K_0^{-1} \delta_0^{-1}$ is compact and $p_2 \Gamma$ is discrete, there exists a finite set $\Delta \subset G$ such that $\gamma^{-1} h \in (H_2 \cap \Gamma) \Delta$. Hence, if $K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) \neq \emptyset$, then $\gamma \in (H_1 \cap \Gamma) \Delta(H_2 \cap \Gamma)$. This completes the proof of the first claim.

For the second claim, it suffices to set $C_0 := \bigcup_{\gamma' \in \Delta} K_{\gamma'}$. \hfill \Box
Proposition 7.7. Let \( H_1, H_2 \in \mathcal{H}^* \) be such that \( H_1 \cap H_2 \) contains a unipo
tent element. Then there exists a unique smallest connected closed subgroup,
say \( H_0 \), of \( H_1 \cap H_2 \) containing all unipotent elements of \( H_1 \cap H_2 \) such that
\( \Gamma \setminus \Gamma H_0 \) is closed. Moreover, \( H_0 \in \mathcal{H} \).

Proof. The orbit \( \Gamma \setminus \Gamma(H_1 \cap H_2) \) is closed \([42, \text{Lemma 2.2}]\). Hence such \( H_0 \)
e
exists. We need to show that \( \Gamma \cap H_0 \) is Zariski dense in \( H_0 \). Let \( L \) be the
subgroup of \( H_0 \) generated by all unipotent elements in \( H_0 \). Note that \( L \) is a
normal subgroup of \( H_0 \) and hence \( (H_0 \cap \Gamma)L \subset H_0 \). If \( F \) is the
identity component of the closure of \( (H_0 \cap \Gamma)L \), then \( \Gamma \setminus \Gamma F \) is closed. By
the minimality assumption on \( H_0 \), we have \( F = H_0 \). Hence \( (H_0 \cap \Gamma)L = H_0 \); so \([e]L = [e]H_0 \). We can then apply \([42, \text{Corollary 2.12}]\) and deduce the Zariski
density of \( H_0 \cap \Gamma \) in \( H_0 \).

Corollary 7.8. Let \( H_1, H_2 \in \mathcal{H}^* \) and \( \gamma \in \Gamma \) be satisfying that \( X(H_1 \cap \gamma H_2 \gamma^{-1}, U) \neq \emptyset \). Then there exists a subgroup \( H \in \mathcal{H}^* \) contained in \( H_1 \cap \gamma H_2 \gamma^{-1} \) such that for any compact subsets \( D_i \subset X(H_i, U) \), \( i = 1, 2 \), there exists a compact subset \( D_0 \subset X(H, U) \) such that
\[
K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) = K \cap \Gamma \setminus \Gamma HD_0.
\]

Proof. Let \( F \in \mathcal{H}^* \) be given by Proposition 7.7 for the subgroup \( H_1 \cap \gamma H_2 \gamma^{-1} \). Set \( H := N_G(F) \in \mathcal{H}^* \). Note that \( X(H_1 \cap \gamma H_2 \gamma^{-1}, U) = X(H, U) \). Hence, by the second claim of Proposition 7.6, there exists a compact subset \( D_0 \subset H_1 D_1 \cap \gamma H_2 D_2 \) such that
\[
K \cap \Gamma \setminus \Gamma(H_1 D_1 \cap \gamma H_2 D_2) = \Gamma \setminus \Gamma D_0.
\]

We claim that
\[
\Gamma \setminus \Gamma D_0 = K \cap \Gamma \setminus \Gamma HD_0.
\]

The inclusion \( \subset \) is clear. Let \( g := hd \in HD_0 \) with \( h \in H \) and \( d \in D_0 \), and
\( [g] \in K \). Then by the condition on \( D_0 \), we have \( g \in H_1 D_1 \) and \( \gamma^{-1}g \in H_2 D_2 \).
Therefore \( g \in H_1 D_1 \cap \gamma H_2 D_2 \). By (7.4), this proves the inclusion \( \supset \). \( \square \)

Definition 7.9 (Self-intersection operator on \( \mathcal{E}_U \)). We define an operator
\[
s : \mathcal{E}_U \cup \{\emptyset\} \to \mathcal{E}_U \cup \{\emptyset\}
\]
as follows: we set \( s(\emptyset) = \emptyset \). For any
\[
E = \bigcup_{i \in \Lambda} \Gamma \Gamma H_i D_i \cap \mathcal{R} F M \in \mathcal{E}_U,
\]
we define
\[
s(E) := \bigcup_{i, j \in \Lambda} \bigcup_{\gamma_{ij} \in \Gamma} \Gamma \Gamma(H_i D_i \cap \gamma_{ij} H_j D_j) \cap \mathcal{R} F M
\]
where \( \gamma_{ij} \in \Gamma \) ranges over all elements of \( \Gamma \) satisfying
\[
\dim(H_i \cap \gamma_{ij} H_j \gamma_{ij}^{-1})_{nc} < \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}.
\]

By Proposition 7.6 and Corollary 7.8, we have:
Corollary 7.10.  
1. For $E \in \mathcal{E}_U$, we have $s(E) \in \mathcal{E}_U$.
2. For $E_1, E_2 \in \mathcal{E}_U$, we have $E_1 \cap E_2 \in \mathcal{E}_U$.

Hence for $E \in \mathcal{E}_U$ as in (7.5), $s(E)$ is of the form
$$s(E) = \bigcup_{i \in \Lambda'} \Gamma \setminus \Gamma H_i D_i \cap RF M$$
where $\Lambda'$ is a (minimal) finite index set, $H_i \in \mathcal{H}$ with $X(H_i, U) \neq \emptyset$ and
$$\max\{\dim(H_i)_{nc} : i \in \Lambda'\} \leq \max\{\dim(H_i)_{nc} : i \in \Lambda\}.$$ Hence, $s$ maps $\mathcal{E}_U$ to $\mathcal{E}_U \cup \{\emptyset\}$ and for any $E \in \mathcal{E}_U$,
$$s^{\dim G}(E) = \emptyset.$$

Definition 7.11. For a compact subset $K \subset \Gamma \setminus G$ and $E \in \mathcal{E}_U$, we say that $K$ does not have any self-intersection point of $E$, or simply that $K$ is $E$-self intersection-free, if
$$K \cap s(E) = \emptyset.$$

Proposition 7.12. Let $E = \bigcup_{i \in \Lambda} \Gamma \setminus \Gamma H_i D_i \cap RF M \in \mathcal{E}$ where $D_i \subset X(H_i, U)$ is a compact subset and $\Lambda$ is a finite subset. Let $K \subset RF M$ be a compact subset which is $E$-self intersection-free. Then there exists a collection of open neighborhoods $\Omega_i$ of $D_i$, $i \in \Lambda$, such that for $O := \bigcup_{i \in \Lambda} \Gamma \setminus \Gamma H_i \Omega_i$, the compact subset $K$ is $O$-self intersection free, in the sense that, if
$$K \cap \Gamma \setminus \Gamma (H_i \Omega_i \cap \gamma H_j \Omega_j) \neq \emptyset$$
for some $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$, then $i = j$ and $\gamma \in H_i \cap \Gamma$.

Proof. For each $k \in \mathbb{N}$ and $i \in \Lambda$, let $\Omega_i(k)$ be the $1/k$-neighborhood of the compact subset $D_i$. Since $\Lambda$ is finite, if the proposition does not hold, by passing to a subsequence, there exist $i, j \in \Lambda$ and a sequence $\gamma_k \in \Gamma$ such that
$$K \cap \Gamma \setminus \Gamma (H_i \Omega_i(k) \cap \gamma_k H_j \Omega_j(k)) \neq \emptyset$$
and
$$(i, j, \gamma_k) \notin \{(i, i, \gamma) : i \in \Lambda, \gamma \in H_i \cap \Gamma\}.$$ 

Hence there exist $g_k = h_k w_k \in H_i \Omega_i(k)$ and $g'_k = h'_k w'_k \in H_j \Omega_j(k)$ such that $g_k = \gamma_k g'_k$ where $[g_k] \in K$.

Now as $k \to \infty$, we have $w_k \to w \in D_i$ and $w'_k \to w' \in D_j$. There exists $\delta_k \in \Gamma$ such that $\delta_k g_k \in \bar{K}$ where $\bar{K}$ is a compact subset of $G$ such that $K = \Gamma \setminus \Gamma \bar{K}$, so the sequence $\delta_k g_k$ converges to $g_0$ as $k \to \infty$. Since $\Gamma H_i$ and $\Gamma H_j$ are closed, we have $\delta_k h_k \to \delta_0 h_i$ and $\delta_k \gamma_k b'_k \to \delta_0 \gamma_j$ where $\delta_0, \delta'_0 \in \Gamma$, $h_i \in H_i$ and $h_j \in H_j$. As $\Gamma[H_i]$ and $\Gamma[H_j]$ are discrete in the spaces $G/H_i$ and $G/H_j$ respectively, we have
$$\delta_0^{-1} \delta_k \in H_i \quad \text{and} \quad (\delta'_0)^{-1} \delta_k \gamma_k \in H_j$$
for all sufficiently large $k$.

Therefore $g_0 = \delta_0 h_i w = \delta'_0 h_j w' \in \delta_0 (H_i D_i \cap \delta_0^{-1} \delta'_0 H_j D_j)$ and $[g_0] \in K$. 

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Hence
\[ K \cap \Gamma \setminus \Gamma(D_i \cap \delta_i^{-1}\delta_j D_j) \neq \emptyset. \]

Set \( \delta := \delta_i^{-1}\delta_j \in \Gamma. \)

Since \( K \cap s(E) = \emptyset, \) this implies that \( \Gamma \setminus \Gamma(D_i \cap \delta H_j D_j) \not\subset s(E). \) By the definition of \( s(E), \)
\[ \dim(H_i \cap \delta H_j \delta^{-1})_{nc} = \min \{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}. \]

Since \( H_i = N_G(H_i) = N_G((H_i)_{nc}), \) and similarly for \( H_j, \) we have \( H_i \cap \delta H_j \delta^{-1} \)

is either \( H_i \) or \( \delta H_j \delta^{-1}, \) that is, \( \delta H_j \delta^{-1} \subset H_i \) or \( H_i \subset \delta H_j \delta^{-1}. \) By Lemma 7.5, this implies that \( i = j \) and \( \delta \in N_G(H_i) \cap \Gamma. \) It follows from (7.7) that
\[ \gamma_k \in N_G(H_i) \cap \Gamma = H_i \cap \Gamma \]

for all large \( k. \) This is a contradiction to (7.6), completing the proof. \( \square \)

In the rest of this section, we assume that \( M = \Gamma \setminus \mathbb{H}^d \) is a convex co-compact hyperbolic manifold with Fuchsian ends, and let \( k \) be as given by Proposition 4.11.

**Theorem 7.13** (Avoidance theorem I). Let \( U = \{u_1\} < N \) be a one-parameter subgroup. For any \( E \in \mathcal{E}_U, \) there exists \( E' \in \mathcal{E}_U \) such that the following holds: If \( F \subset RF M \) is a compact set disjoint from \( E', \) then there exists a neighborhood \( O^\circ \) of \( E \) such that for all \( x \in F, \) the following set
\[ \{t \in \mathbb{R} : xu_t \in RF M - O^\circ\} \]

is \( 2k \)-thick. Moreover, if \( E \) is associated to \( \{H_i : i \in \Lambda\}, \) then \( E' \) is also associated to the same family \( \{H_i : i \in \Lambda\} \) in the sense of Definition 7.3.

**Proof.** \( \blacklozenge \) 1. The constant \( \beta_0: \) We write \( \mathcal{H}^* = \{H_i\}. \) For simplicity, set \( V_i = V_{H_i} \) and \( p_i = p_{H_i}. \) Let \( \theta_i \) be the defining polynomial of the algebraic variety \( A_{H_i}. \)

Set
\[ m := \dim(G); \text{ and} \]
\[ \delta := \max_{H_i \in \mathcal{H}^*} (\deg \theta_i + 2) \dim V_i. \]

Note that if \( H_i \) is conjugate to \( H_j, \) then \( \theta_i \) and \( \theta_j \) have same degree and \( \dim V_i = \dim V_j. \) Since there are only finitely many conjugacy classes in \( \mathcal{H}^* \)
by Proposition 5.3, the constant \( \delta \) is finite. Now let
\[ \beta_0 := \beta_0(m\delta, k, 1) = 2^{3m\delta + 3} k^{m\delta + 3} \varepsilon^{-1} \]
be given as in Proposition 6.3 where \( \varepsilon = \varepsilon_{m\delta} \) satisfies \( \left( \frac{1+\varepsilon}{1-\varepsilon} \right)^{2m\delta - 1} \leq 2. \)

\( \blacklozenge \) 2. Definition of \( E_n \) and \( E'_n: \) We write
\[ E = \bigcup_{i \in \Lambda} \Gamma \setminus \Gamma D_i \cap RF M \]

for some finite minimal set \( \Lambda. \) Set
\[ \ell := \max_{i \in \Lambda} \dim(H_i)_{nc}. \]
We define $E_n, E'_n \in \mathcal{E}_U$ for all $1 \leq n \leq \ell$ inductively as follows: set
\[ E_\ell := E \quad \text{and} \quad \Lambda_\ell := \Lambda_0. \]

For each $i \in \Lambda_\ell$, let $D'_i$ be a compact subset of $X(H_i, U)$ containing $D_i$ such that $p_iD'_i$ has a $\beta_0$-regular relative size with respect to $p_iD_i$ as in Proposition 7.2. Set
\[ E'_\ell := \bigcup_{i \in \Lambda_\ell} \Gamma \setminus \Gamma H_i D'_i \cap \text{RF} M. \]

Suppose that $E_{n+1}, E'_{n+1} \in \mathcal{E}_U$ are given for $n \geq 1$. Then, define
\[ E_n := E \cap s(E'_{n+1}). \]

Then by Corollary 7.10, $E_n$ belongs to $\mathcal{E}_U$ and hence can be written as
\[ E_n = \bigcup_{i \in \Lambda_n} \Gamma \setminus \Gamma H_i D_i \cap \text{RF} M \]
where $D_i$ is a compact subset of $X(H_i, U)$, so that $\Lambda_n$ is a minimal index set.

For each $i \in \Lambda_n$, let $D'_i$ be a compact subset of $X(H_i, U)$ containing $D_i$ such that $p_iD'_i$ has a $\beta_0$-regular relative size with respect to $p_iD_i$ as in Proposition 7.2. Set
\[ E'_n := \bigcup_{i \in \Lambda_n} \Gamma \setminus \Gamma H_i D'_i \cap \text{RF} M. \]

Hence we get a sequence of compact (possibly empty) subsets of $E$:
\[ E_1, E_2, \ldots, E_{\ell-1}, E_\ell = E, \]
and a sequence of compact sets
\[ E'_1, E'_2, \ldots, E'_{\ell-1}, E'_\ell = E'. \]

Note that $s(E_1) = s(E'_1) = \emptyset$ by the dimension reason.  

\section*{3. Outline of the plan:}

Let $F \subset \text{RF} M$ be a compact set disjoint from $E'$. For $x \in F$, we set
\[ T'(x) := \{ t \in \mathbb{R} : xu_t \in \text{RF} M \} \]
which is a globally $k$-thick set by Proposition 4.11. We will construct
\begin{itemize}
  \item a neighborhood $\mathcal{O}'$ of $E'$ disjoint from $F$, and
  \item a neighborhood $\mathcal{O}'^\circ$ of $E$
\end{itemize}
such that for any $x \in \text{RF} M - \mathcal{O}'$, we have
\[ \{ t \in \mathbb{R} : xu_t \in \text{RF} M - \mathcal{O}'^\circ \} = T'(x) - J'(\mathcal{X}) \]
where $\mathcal{X} = \mathcal{X}(x)$ is the union of $\ell \leq m$ number of $\beta_0$-regular families $\mathcal{X}_i$ of triples $(I(q), J^*(q), J'(q))$ of subsets of $\mathbb{R}$ with degree $\delta$ and of $T'(x)$-multiplicity free. Once we do that, the theorem is a consequence of Proposition 6.3. Construction of such $\mathcal{O}'$ and $\mathcal{O}'^\circ$ requires an inductive process on $E_n$'s.

\footnote{In fact $E_{\ell-i} = \emptyset$ for all $i \geq d - 1$, but we won’t use this information.}
4. **Inductive construction of $K_n$, $O'_{n+1}$, $O_{n+1}$, and $O^*_{n+1}$**: Let

$$K_0 := RF M.$$ 

For each $i \in \Lambda_1$, there exists a neighborhood $\Omega'_i$ of $D'_i$ such that for

$$O'_1 := \bigcup_{i \in \Lambda_1} \Gamma \setminus \Gamma H_i \Omega'_i,$$

the compact subset $K_0$ is $O'_1$-self intersection free by Lemma 7.12, since $s(E'_2) = \emptyset$.

By Proposition 7.2, there exists a neighborhood $\Omega_i$ of $D_i$ such that the pair $(I(q), J(q))$ is $\beta_0$-regular for all $q \in V_i \setminus \Gamma H_i \Omega'_i$ where

$$I(q) = \{ t \in \mathbb{R} : qu_t \in p_i \Omega'_i \} \quad \text{and} \quad J(q) = \{ t \in \mathbb{R} : qu_t \in p_i \Omega_i \}.$$ 

Set

$$O_1 := \bigcup_{i \in \Lambda_1} \Gamma \setminus \Gamma H_i \Omega_i.$$ 

Since $E_1 = \bigcup_{i \in \Lambda_1} \Gamma \setminus \Gamma H_i D_i \cap RF M$, $O_1$ is a neighborhood of $E_1 = s(E'_2) \cap E$. Now the compact subset $s(E'_2) - O_1$ is contained in $s(E'_2) - E$, which is relatively open in $s(E'_2)$. Therefore we can take a neighborhood $O'_1$ of $s(E'_2) - O_1$ so that

$$\overline{O'_1 \cap E} = \emptyset.$$ 

We will now define the following quadruple $K_n, O'_{n+1}, O_{n+1}$ and $O^*_{n+1}$ for each $1 \leq n \leq \ell - 1$ inductively:

- a compact subset $K_n = K_{n-1} - (O_n \cup O^*_n) \subset RF M$,
- a neighborhood $O'_{n+1}$ of $E'_{n+1}$,
- a neighborhood $O_{n+1}$ of $E_{n+1}$ and
- a neighborhood $O^*_{n+1}$ of $s(E'_{n+2}) - O_{n+1}$ such that

$$\overline{O^*_{n+1} \cap E} = \emptyset.$$ 

Assume that the sets $K_{n-1}$, $O'_{n}$, $O_n$ and $O^*_n$ are defined. We define

$$K_n := K_{n-1} - (O_n \cup O^*_n).$$ 

For each $i \in \Lambda_{n+1}$, let $\Omega'_i$ be a neighborhood of $D'_i$ in $G$ such that for

$$O'_{n+1} := \bigcup_{i \in \Lambda_{n+1}} \Gamma \setminus \Gamma H_i \Omega'_i, \quad K_n \setminus O'_{n+1}$$

is $O'_{n+1}$-self intersection free. Since $O_n \cup O^*_n$ is a neighborhood of $s(E'_{n+1})$, which is the set of all self-intersection points of $E'_{n+1}$, such collection of $\Omega'_i$, $i \in \Lambda_{n+1}$ exists by Lemma 7.12.

Since $F \subset RF M$ is compact and disjoint from $E'$, we can also assume $\Gamma \setminus \Gamma H_i \Omega'_i$ is disjoint from $F$, by shrinking $\Omega'_i$ if necessary. More precisely, writing $F = \Gamma \setminus \Gamma \tilde{F}$ for some compact subset $\tilde{F} \subset G$, this can be achieved by choosing a neighborhood $\Omega'_i$ of $D'_i$ so that $p_i \Omega'_i$ is disjoint from $p_i \Gamma \tilde{F}$; and this is possible since $p_i \Gamma \tilde{F}$ is a closed set disjoint from a compact subset $p_i D'_i$. 

After choosing $\Omega'_i$ for each $i \in \Lambda_{n+1}$, define the following neighborhood of $E'_{n+1}$:

$$O'_{n+1} := \bigcup_{i \in \Lambda_{n+1}} \Gamma \setminus \Gamma H_i \Omega'_i.$$  

We will next define $O_{n+1}$. By Lemma 7.2, there exists a neighborhood $\Omega_i$ of $D_i$ such that the pair $(I(q), J(q))$ is $\beta_0$-regular for all $q \in V_i - p_i \Omega'_i$ where

$$I(q) = \{ t \in \mathbb{R} : q u_t \in p_i \Omega'_i \} \text{ and } J(q) = \{ t \in \mathbb{R} : q u_t \in p_i \Omega_i \}.$$  

We then define the following neighborhood of $E_{n+1} = s(E_{n+2}') \cap E$:

$$O_{n+1} := \bigcup_{i \in \Lambda_{n+1}} \Gamma \setminus \Gamma H_i \Omega_i.$$  

Now since the compact subset $s(E_{n+2}') - O_{n+1}$ is contained in the set $s(E_{n+2}') - E$, which is relatively open inside $s(E_{n+2}')$, we can take a neighborhood $O^*_{n+1}$ of $s(E_{n+2}') - O_{n+1}$ so that

$$O^*_{n+1} \cap E = \emptyset.$$  

This finishes the inductive construction.

\section{5. Definition of $O'$ and $O^*$:}

We define:

$$O' := \bigcup_{n=1}^{\ell} O'_n, \quad O := \bigcup_{n=1}^{\ell} O_n, \quad O^* := \bigcup_{n=1}^{\ell} O^*_n.$$  

Note that $O'$ and $O$ are neighborhoods of $E'$ and $E$ respectively. Since $E \cap O^* = \emptyset$, the following defines a neighborhood of $E$:

$$O^0 := O - O^*.$$  

\section{6. Construction of $\beta_0$-regular families of $T(x)$-multiplicity free:}

Fix $x \in F \subset RF M - O'$. Choose a representative $g \in G$ of $x$.

For each $q \in \bigcup_{i \in \Lambda_n} p_i \Gamma g$, we define the following subsets of $\mathbb{R}$:

- $I(q) = \{ t : qu_t \in O'_n \}$ and
- $J(q) = \{ t : qu_t \in O_n \}.$

In general, $I(q)$'s have high multiplicity among $q$'s in $\bigcup_{i \in \Lambda_n} p_i \Gamma g$, but the following subset $I'(q)$'s will be multiplicity-free, and this is is why we defined $K_{n-1}$ as carefully as above.

For each $q \in \bigcup_{i \in \Lambda_n} p_i \Gamma g$, we define the following subsets of $I(q)$:

- $I'(q) := \{ t : \text{ for some } a \geq 0, \ [t, t + a] \subset I(q) \text{ and } xu_{t+a} \in K_{n-1} \}$;
- $J^*(q) := I'(q) \cap J(q)$;
- $J'(q) := \{ t \in J(q) : xu_t \in K_{n-1} \}.$

Observe that $I'(q)$ and $J^*(q)$ are unions of finitely many intervals, $J'(q) \subset T(x)$ and that

$$J'(q) \subset J^*(q) \subset I'(q).$$

Now, for each $n = 1, \ldots, \ell$, define the family

$$\mathcal{X}_n = \{ (I(q), J^*(q), J'(q)) : q \in \bigcup_{i \in \Lambda_n} p_i \Gamma g \}.$$
We claim that each $X_n$ is a $\beta_0$-regular family with degree at most $\delta$ and $T(x)$-multiplicity free.

Note for each $q \in p_i \Gamma g$, the number of connected components of $J^*(q)$ is less than or equal to that of $J(q)$. Now that $J^*(q) \subset J(q)$ and all the pairs $(I(q), J(q))$ are $\beta_0$-regular pairs of degree at most $\delta$, it follows that $X_n$'s are $\beta_0$-regular families with degree at most $\delta$.

We now claim that $X_n$ has $T(x)$-multiplicity free, that is, for any distinct indices $q_1, q_2 \in \bigcup_{i \in \Lambda_n} p_i \Gamma g$ of $X_n$,

$$I(q_1) \cap J'(q_2) = \emptyset.$$ 

We first show that

$$I'(q_1) \cap I'(q_2) = \emptyset.$$ 

Suppose not. Then there exists $t \in I'(q_1) \cap I'(q_2)$ for some $q_1 = p_i \gamma_1 g$ and $q_2 = p_j \gamma_2 g$. Then for some $a \geq 0$, we have $[t, t + a] \subset I(q_1) \cap I(q_2)$ and $xu_{t + a} \in K_{n-1}$. In particular,

$$xu_{t + a} \in \Gamma \setminus \Gamma(H_1^{-1} \Omega_i \cup \gamma_2 H_2 \Omega_j) \cap K_{n-1}.$$ 

Since $K_{n-1}$ is $O'$-self intersection free, we deduce from Proposition 7.12 that this may happen only when $i = j$, and $\gamma_1 \gamma_2^{-1} \in H_i \cap \Gamma$. Hence we have

$$q_1 = q_2.$$ 

This shows that $I'(q)$'s are pairwise disjoint. Now suppose that there exists an element $t \in I(q_1) \cap J'(q_2)$. Then by the disjointness of $I'(q_1)$ and $I'(q_2)$, it follows that

$$t \in (I(q_1) - I'(q_1)) \cap J'(q_2).$$ 

By the definition of $I'(q_1)$, we have $xu_t \notin K_{n-1}$. This contradicts the assumption that $t \in J'(q_2)$.

7. Completing the proof: In view of Proposition 6.3, it remains to check that the condition $t \in T(x) - J'(X)$ implies that $xu_t \notin O^\circ$ where $O^\circ$ is given in (7.9).

Write the neighborhood $O^\circ$ as the disjoint union

$$O^\circ = \bigcup_{n=1}^\ell (O_n - (\bigcup_{i \leq n-1} O_i \cup O^*))$$

Let $n \leq \ell$ be such that

$$xu_t \in O_n - (\bigcup_{i=1}^{n-1} O_i \cup O^*).$$

If $xu_t \in T(x) - J'(X)$, then $xu_t \notin K_{n-1}$, that is,

$$xu_t \in \bigcup_{i=1}^{n-1} O_i \cup O^*_i.$$ 

This is a contradiction, since $\bigcup_{i=1}^\ell O^*_i \subset O^*$. 

\qed
As $\mathcal{H}^*$ is countable and $X(H_i, U)$ is $\sigma$-compact, the intersection $\mathcal{H}(U) \cap \text{RF } M$ can be exhausted by the union of the increasing sequence of $E_j \in \mathcal{E}_U$'s. Therefore, we deduce:

**Corollary 7.14.** There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots \in \mathcal{E}_U$ with $\mathcal{H}(U) \cap \text{RF } M = \bigcup_{j=1}^{\infty} E_j$ which satisfies the following: for each $j \in \mathbb{N}$, there exists an open neighborhood $O_j$ of $E_j$ such that for all $x \in \text{RF } M - E_{j+1}$,

$$\{t \in \mathbb{R} : xu_t \in \text{RF } M - O_j\}$$

is a $2k$-thick set for all $j \in \mathbb{N}$.

**Corollary 7.15.** Let $x_i \in \text{RF } M$ be a sequence converging to $x \in \mathcal{H}(U) \cap \text{RF } M$. Then for each $j \in \mathbb{N}$, there exist a neighborhood $O_j$ of $E_j$ and $i_j \geq 1$ such that

$$\{t \in \mathbb{R} : x_i u_t \in \text{RF } M - O_j\}$$

is $2k$-thick for all $i \geq i_j$.

**Proof.** We fix $j \in \mathbb{N}$. Then there exists $i_j \in \mathbb{N}$ such that $x_i \notin E_{j+1}$ for all $i \geq i_j$. Applying Proposition 7.13 to a compact subset $F = \{x_i : i \geq i_j\}$ of $\text{RF } M$, we obtain a neighborhood $O_j$ of $E_j$ such that

$$\{t \in \mathbb{R} : x_i u_t \in \text{RF } M - O_j\}$$

is $2k$-thick for all $i \geq i_j$. □

Indeed we will apply Corollary 7.15 for the sequence $\{x_i\}$ contained in a closed orbit $x_0L$ of a proper connected closed subgroup $L < G$, which can be proved in the same way:

**Theorem 7.16 (Avoidance Theorem II).** Consider a closed orbit $x_0L$ for some $x_0 \in \text{RF } M$ and $L \in \mathcal{Q}_U$. There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots \in \mathcal{E}_U$ with $\mathcal{H}(U, x_0L) \cap \text{RF } M = \bigcup_{j=1}^{\infty} E_j$, which satisfies the following: if $x_i \to x$ in $\text{RF } M \cap x_0L$ with $x \in \mathcal{H}(U, x_0L)$, then for each $j \in \mathbb{N}$, there exist $i_j \geq 1$ and an open neighborhood $O_j \subset x_0L$ of $E_j$ such that

$$\{t \in \mathbb{R} : x_i u_t \in \text{RF } M - O_j\}$$

is a $2k$-thick set for all $i \geq i_j$.

8. LIMITS OF RF M-POINTS IN F* AND GENERIC POINTS

We collect some geometric lemmas which are needed in modifying a sequence limiting on an RF M point (resp. limiting on a point in $\text{RF } M \cap \mathcal{H}(U)$) to a sequence of RF M-points (resp. whose limit still remains inside $\mathcal{H}(U)$).

Throughout the section, assume that $M = \Gamma \backslash \mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends. Recall that $\Lambda \subset S^{d-1}$ denotes the limit set of $\Gamma$. 
Lemma 8.1. Let $C_n \to C$ be a sequence of convergent circles in $\mathbb{S}^{d-1}$. If $C \not\subset \overline{B}$ for any component $B$ of $\Omega$, then
\[
\# \lim \sup C_n \cap \Lambda \geq 2.
\]

Proof. Without loss of generality, we may assume that $\infty \notin \Lambda$ and hence consider $\Lambda$ as a subset of the Euclidean space $\mathbb{R}^{d-1}$. Note that there is one component, say, $B_1$ of $\Omega$ which contains $\infty$ and all other components of $\Omega$ are contained in the complement of $B_1$, which is a (bounded) round ball in $\mathbb{R}^{d-1}$. It follows that there are only finitely many components of $\Omega$ whose diameters are bounded from below by a fixed positive number; this follows from the fact that $\Gamma B$ is closed for each component $B$ of $\Omega$, and that there are only finitely many $\Gamma$-orbits of components of $\Omega$.

Let $\delta = 0.5 \text{diam}(C)$ so that we may assume $\text{diam}(C_n) > \delta$ for all sufficiently large $n \gg 1$. It suffices to show that there exists $\varepsilon_0 > 0$ such that $C_n \cap \Lambda$ contains $\xi_n, \xi'_n$ with $d(\xi_n, \xi'_n) \geq \varepsilon_0$ for all sufficiently large $n$. Suppose not. Then for any $\varepsilon > 0$, there exists an interval $I_n \subset C_n$ such that diam$(I_n) \leq \varepsilon$ and $C_n - I_n \subset \Omega$ for some infinite sequence of $n$'s. Since $C_n - I_n$ is connected, there exists a component $B_n$ of $\Omega$ such that $C_n \subset N_{\varepsilon}(B_n)$, where $N_{\varepsilon}(B_n)$ denotes the $\varepsilon$-neighborhood of $B_n$. In particular, we have diam$(B_n) + \varepsilon > \delta$. Taking $\varepsilon$ smaller than $0.5\delta$, this means that diam$(B_n) > 0.5\delta$. On the other hand, there are only finitely many components of $\Omega$ whose diameters are greater than $0.5\delta$, say $B_1, \ldots, B_k$. Let $\varepsilon_0 > 0$ be such that $N_{\varepsilon_0}(B_1), \ldots, N_{\varepsilon_0}(B_k)$ are all disjoint. Then by passing to a subsequence, there exists $B_i$ such that $C_n \subset N_{\varepsilon_0}(B_i)$ for all small $0 < \varepsilon < \varepsilon_0$ and all $n \geq 1$; hence $C \subset N_{\varepsilon_0}(B_i)$. Since this holds for all sufficiently small $\varepsilon > 0$, we get that $C \subset \overline{B_i}$, yielding a contradiction to the hypothesis on $C$. \hfill \square

In the next two lemmas, we set $U^- = U$ and $U^+ = U^t$.

Lemma 8.2. Let $U < N$ be a connected closed subgroup. Let $[g]L$ be a closed orbit for some $L \in \mathcal{L}_U$ and $[g] \in \mathcal{R}F M$. Let $S_0$ and $S^*$ denote the boundaries of $\pi(gH(U))$ and $\pi(gL)$ respectively. If $S$ is a sphere such that $S_0 \subset S \subset S^*$ and $\Gamma S$ is closed, then $[g] \in \mathcal{I}(U^\pm, [g]L)$.

Proof. Write $L = H(\overline{U})C \in \mathcal{L}_U$. Since $S_0 \subset S \subset S^*$, there exists a connected proper subgroup $\overline{U}$ of $\overline{U}$, containing $U$ such that $S$ is the boundary of $\pi(gH(\overline{U}))$. Since $\Gamma S$ is closed, $[g]H'(\overline{U})$ is closed by Proposition 3.9. Now the claim follows from Proposition 4.9 and the definition of $\mathcal{I}(U^\pm, [g]L)$. \hfill \square

Lemma 8.3. Let $U < N$ be a connected closed subgroup with dimension $m \geq 1$, and let $U_{+}^{(1)}, \ldots, U_{+}^{(m)}$ be one-parameter subgroups generating $U^\pm$. Consider a closed orbit $yL$ where $L \in \mathcal{L}_U$ and
\[
y \in F_{H(U)} \cap \mathcal{R}F M \cap \bigcap_{i=1}^{m} \mathcal{I}(U_{\pm}^{(i)}, yL).
\]
If \( x_n \to y \) in \( yL \), then, by passing to a subsequence, there exists a sequence \( h_n \to h \) in \( H(U) \) so that

\[
x_nh_n \in RF\;M \cap yL \quad \text{and} \quad yh \in RF\;M \cap \bigcap_{i=1}^{m} \mathcal{G}(U_{\pm}^{(i)},yL).
\]

**Proof.** Let \( S^* \) denote the boundary of \( \pi(g_0L) \). Let \( \mathcal{Q} \) be the collection of all spheres \( S \subseteq S^* \) such that \( S \cap \Lambda \neq \emptyset \) and \( \Gamma S \) is closed in \( C^{\dim S} \). By Corollary 5.8 and Remark 5.9, \( \mathcal{Q} \) is countable.

Choose a sequence \( g_n \to g_0 \) in \( G \) as \( n \to \infty \), so that \( x_n = [g_n] \) and \( y = [g_0] \). Let \( S_n \) and \( S_0 \) denote the boundaries of \( \pi(g_nH(U)) \) and \( \pi(g_0H(U)) \) respectively so that \( S_n \to S_0 \) in \( C^m \) as \( n \to \infty \).

We will choose a circle \( C_0 \subset S_0 \) and a sequence of circles \( C_n \subset S_n \) so that \( C_n \to C_0 \) and \( \lim\sup(C_n \cap \Lambda) \) contains two distinct points outside of \( \cup_{S \in \mathcal{Q}} S \).

If \( m = 1 \), we set \( C_0 = S_0 \). When \( m \geq 2 \), we choose a circle \( C_0 \subset S_0 \) as follows. Note that \( S_0 \) is not contained in any sphere in \( \mathcal{Q} \) by the assumption on \( y \) and Lemma 8.2. Hence for any \( S \in \mathcal{Q} \), \( S_0 \cap S \) is a proper sub-sphere of \( S_0 \). Since \( y \in F_{H(U)}^* \), for any component \( B_i \) of \( \Omega \), \( S_0 \not\ subsets B_i \) and hence \( S_0 \cap \partial B_i \) is a proper sub-sphere of \( S_0 \). Choose a circle \( C_0 \subset S_0 \) such that \( \{g_0^i, g_0^j\} \subset C_0 \cap \Lambda \), \( C_0 \not\subset S \) for any \( S \in \mathcal{Q} \), and \( C_0 \not\subset \partial B_i \cap S_0 \) for all \( i \). This is possible, since \( \mathcal{Q} \) is countable.

Since \( S_n \to S_0 \), we can find a sequence of circles \( C_n \subset S_n \) such that \( C_n \to C_0 \). We claim that \( \lim\sup_n(C_n \cap \Lambda) \) is uncountable. Since \( \#(C_0 \cap \Lambda) \geq 2 \) and \( C_0 \not\subset \partial B_i \), \( C_0 \not\subset B_i \) for all \( i \). Therefore, by Lemma 8.1, for any infinite subsequence \( C_{n_k} \) of \( C_n \),

\[
\# \lim\sup_k(C_{n_k} \cap \Lambda) \geq 2.
\]

By passing to a subsequence, we can find two distinct points \( \xi, \xi' \in C_0 \cap \Lambda \) which converge to two distinct points \( \xi, \xi' \) of \( C_0 \cap \Lambda \) respectively as \( n \to \infty \). Choose a sequence \( p_n \to p \in G \) such that \( p_n^+ = \xi_n \), \( p_n^- = \xi_n' \), \( p^+ = \xi \) and \( p^- = \xi' \). The set \( T_n = \{ t : [p_n]u_t \in RF\;M \} \) is a global \( k \)-thick subset, and hence \( T := \lim\sup_n T_n \) is a global \( k \)-thick subset contained in the set \( \{ t : [p]u_t \in RF\;M \} \). Then \( C_n \cap \Lambda \) converges, in the Hausdorff topology, to a compact subset \( L \subset C_0 \cap \Lambda \) homeomorphic to the one-point compactification of \( T \). Therefore \( L \) is uncountable, so is \( \lim\sup_n(C_n \cap \Lambda) \), proving the claim.

Let \( \Psi := \cup_{S \in \mathcal{Q}} C_0 \cap S \), i.e., the union of all possible intersection points of \( C_0 \) and spheres in \( \mathcal{Q} \). Since \( C_0 \not\subset S \) for any \( S \in \mathcal{Q} \), \( \#(C_0 \cap S) \leq 2 \). Hence \( \Psi \) is countable, and hence \( \lim\sup_n(C_n \cap \Lambda) - \Psi \) is uncountable. Note that this works for any infinite subsequence of \( C_n \)'s. Therefore we can choose sequences \( \xi_n^-, \xi_n^+ \in C_0 \cap \Lambda \) converging to distinct points \( \xi^-, \xi^+ \) of \( (C_0 \cap \Lambda) - \Psi \) respectively, by passing to a subsequence.

As \( \xi^-, \xi^+ \in C_0 \) and \( C_0 \subset S_0 \), there exists a frame \( g_0h = (v_0, \cdots, v_{d-1}) \in g_0H(U) \) whose first vector \( v_0 \) is tangent to the geodesic \( [\xi^-, \xi^+] \).
Setting $g := g_0 h$, we claim that

$$[g] \in \bigcap_i \mathcal{G}(U^{(i)}_\pm, y L).$$

Suppose that $[g] \in \mathcal{G}(U^{(i)}_\pm, y L)$ for some $i$. We will assume $[g] \in \mathcal{G}(U^{(i)}_+, y L)$, as the case when $[g] \in \mathcal{G}(U^{(i)}_-, y L)$ can be dealt similarly, by changing the role of $g^-$ and $g^+$ below. For simplicity, set $U^{(i)} := U^{(i)}_+$. Now by Proposition 5.13, there exist $L_0 \in L_{U^{(i)}}$ and $\alpha \in N \cap L$ such that $(L_0)_{nc} \leq L_{nc}$ and $[g] \alpha L_0$ is closed. Let $S$ denote the boundary of $\pi(g \alpha L_0)$. Since $\alpha \in N \cap L$, we have $(g \alpha)^+ = g^+ = \xi^+ \in S \cap \Lambda \cap C_0$. Since $S \subset S^*$, $S \cap \Lambda \neq \emptyset$ and $\Gamma S$ is closed, we have $S \in \mathcal{Q}$. It follows that $\xi^+ \in \Psi$, contradicting the choice of $\xi^+$. This proves the claim.

Now choose a vector $v_0^{(n)}$ which is tangent to the geodesic $[\xi^-, \xi^+]$. We then extend $v_0^{(n)}$ to a frame $g_n h_n \in g_n H(U)$ so that $g_n h_n$ converges to $g = g_0 h$ as $n \to \infty$. Since $\{\xi^+_n\} \subset \Lambda$, we have $[g_n h_n] \in RF M$. This completes the proof. \hfill \Box

We will need the following lemma later.

**Lemma 8.4.** Let $k \geq 1$. Let $\chi$ be a $k$-horosphere in $\mathbb{H}^{k+1}$ resting at $p \in \partial \mathbb{H}^{k+1}$, and $\mathcal{P}$ be a geodesic $k$-plane in $\mathbb{H}^{k+1}$. Let $\xi \in \partial \mathcal{P}$, $\delta$ be a geodesic joining $\xi$ and $\partial \mathcal{P}$, and $q = \delta \cap \chi$. There exists $R_0 > 1$ such that for any $R > R_0$, if $d(\chi, \mathcal{P}) < R - 1$, then $d(q, \mathcal{P}) < R$.

**Proof.** For $k = 1$, this is shown in [28, Lemma 4.2]. Now let $k \geq 2$. Consider a geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^{k+1}$ which passes through $q$ and orthogonal to $\mathcal{P}$. Then $\chi \cap \mathbb{H}^2$ and $\mathcal{P} \cap \mathbb{H}^2$ are a horocycle and a geodesic in $\mathbb{H}^2$ respectively. As $d_{\mathbb{H}^{k+1}}(\chi, \mathcal{P}) = d_{\mathbb{H}^2}(\chi \cap \mathbb{H}^2, \mathcal{P} \cap \mathbb{H}^2)$ and $d_{\mathbb{H}^{k+1}}(q, \mathcal{P}) = d_{\mathbb{H}^2}(q, \mathcal{P} \cap \mathbb{H}^2)$, the conclusion follows from the case $k = 1$. \hfill \Box

**Lemma 8.5.** Let $U < \hat{H} \cap N$ be a non-trivial connected closed subgroup. If the boundary of $\pi(g H(U))$ is contained in $\partial B$ for some component $B$ of $\Omega$, then $[g] \in BF M \cdot C(H(U))$.

**Proof.** As $U$ is equal to $m U_k m^{-1}$ for some $m \in \hat{H} \cap M$ and $1 \leq k \leq d - 2$, the general case is easily reduced to the case when $U = U_k$. Since $g = (v_0, \ldots, v_d)$ has its first $(k + 1)$-vectors tangent to the geodesic $(k + 1)$-plane $\pi(g H(U_k))$ and $\partial(\pi(g H(U_k))) \subset \partial B$, we can use an element $c \in C(H(U_k)) = SO(d - k - 2)$ to modify the next $(d - k - 2)$-vectors so that $g c$ has its first $(d - 1)$-vectors tangent to $\text{hull}(\partial B)$. Then $[g c] \in BF M$, proving the claim. \hfill \Box

**Lemma 8.6.** Let $U < \hat{H} \cap N$ be a non-trivial connected closed subgroup. If $x_n \in RF M \cdot U$ is a sequence converging to some $x \in RF M$, then passing to a subsequence, there exists $u_n \in U$ such that $x_n u_n \in RF M$ and at least one of the following holds:

1. $u_n \to e$ and hence $x_n u_n \to x$, or
(2) \( x = zc \) for some \( z \in BF M \) with \( c \in C(H(U)) \), and \( x_n u_n \) accumulates on \( zHc \).

**Proof.** If \( x_n \) belongs to \( RF M \) for infinitely many \( n \), we simply take \( u_n = e \). So assume that \( x_n \not\in RF M \) for all \( n \). Choose a sequence \( g_n \to g_0 \) in \( G \) so that \( x_n = [g_n] \) and \( x = [g_0] \).

As \( x \in RF M \), we have \( \{g_0(0), g_0(\infty)\} \subset \Lambda \). As \( x_n \in RF_+ M - RF M \), we have \( g_n(\infty) \in \Lambda \) and \( g_n(0) \in \Omega \). For each \( n \), choose an element \( u_n \in U \) so that \( 0 < \alpha_n := \|u_n\| \leq \infty \) is the minimum of \( \|u\| \) for all \( u \in U \) satisfying \( g_n u(0) \in \Lambda \). Set

\[
\alpha := \limsup_n \alpha_n.
\]

If \( \alpha = 0 \), then we are in case (1). Hence we will assume \( 0 < \alpha \leq \infty \). Let \( C_n \) denote the boundary of \( \pi(g_n H(U)) \) and \( C_0 \) the boundary of \( \pi(g_0 H(U)) \). Then \( C_n \to C_0 \) in \( C^{dim} U \). Recall that \( B_U(r) \) denotes the ball of radius \( r \) centered at \( 0 \) inside \( U \). Set

\[
B_n := g_n B_U(\alpha_n)(0) \quad \text{and} \quad B_0 := g_0 B_U(\alpha)(0).
\]

Then \( B_n \subset C_n \cap \Omega \), and \( \partial B_n \cap \Lambda \neq \emptyset \) by the choice of \( u_n \).

By passing to a subsequence, we have \( \alpha_n \to \alpha \) and \( B_n \to B_0 \) as \( n \to \infty \) and hence the diameter of \( B_n \) in \( S^{d-1} \) is bounded below by some positive number. Hence, passing to a subsequence, we may assume that \( B_n \) are all contained in the same component, say \( B \) of \( \Omega \). Consequently, \( B_0 \subset \overline{B} \).

We claim that \( \#\overline{B_0} \cap \partial B \geq 2 \). First note that \( g_0(0) \in \Lambda \). If \( \alpha = \infty \), then \( g_n u_n(0) \to g_0(\infty) \in \Lambda \cap \overline{B_0} \). If \( \alpha < \infty \), then \( u_n \) converges to some \( u \in U \), passing to a subsequence, and \( u \neq e \), as \( \alpha > 0 \). Now, \( g_n u_n(0) \to g_0 u(0) \in \Lambda \cap \overline{B_0} \). Since \( \Lambda \cap \overline{B} \subset \partial B \), this proves the claim.

Therefore \( B_0 \) is contained in \( \partial B \), and hence so is \( C_0 \). By Lemma 8.5, this implies that \( x = zc \) for some \( z \in BF M \) and \( c \in C(H(U)) \). We proceed to show that \( x_n u_n \) accumulates on \( zHc \). Since \( c \in C(H(U)) \), we may assume \( c = e \) by replacing \( x \) with \( x - c x^{-1} \), and \( x_n \) with \( x_n - x \).

We claim that \( \pi(g_n u_n) \) goes arbitrarily close to the plane \( \pi(g_0 \hat{H}) \) as \( n \to \infty \). Since \( x \hat{H} = [g_0] \hat{H} \) is compact, \( g_n u_n \in g_0 \hat{H} \) and \( \pi(g_n \hat{H}) \) is a geodesic plane nearly parallel to \( \pi(g_0 \hat{H}) \). For all large \( n \), this claim implies that \( [g_n] u_n \) accumulates on \( z \hat{H} \), completing the proof.

Now, to prove the claim, let \( D_n := C_n \cap \partial B \) and \( P_n := \text{hull}(D_n) \). Let \( k = \text{dim} U \). Since \( C_n \) is a \( k \)-sphere meeting the \((d - 2)\)-sphere \( \partial B \subset S^{d-1} \), and \( C_n \not\subset \partial B \), it follows that \( D_n \) is a \((k - 1)\)-sphere. We set \( \mathcal{H}_n := \text{hull}(C_n) \), \( \mathcal{H}_0 := \text{hull}(C_0) \) and \( \mathcal{H} := \text{hull}(\partial B) = \pi(g_0 \hat{H}) \). Then \( \mathcal{H}_n \cap \mathcal{H} = \mathcal{P}_n \). Let \( \varepsilon > 0 \) be arbitrary, and \( \mathcal{N}_\varepsilon(\mathcal{H}) \) denote the \( \varepsilon \)-neighborhood of \( \mathcal{H} \) in \( \mathbb{H}^d \). Letting \( d_{\mathcal{H}_n}(\cdot, \cdot) \) denote the hyperbolic distance in \( \mathcal{H}_n \), we may write

\[
\mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n = \{ p \in \mathcal{H}_n : d_{\mathcal{H}_n}(p, \mathcal{P}_n) < R_n \}
\]

for some \( R_n > 0 \). This is because \( \mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n \) is convex and invariant under family of isometries, whose axes of translation and rotation are contained in \( \mathcal{P}_n \). As \( C_n \to C_0 \subset \partial B \) as \( n \to \infty \), it follows that \( R_n \to \infty \) as \( n \to \infty \). Let
\(\chi_n := \pi(g_n U)\), and \(\chi_0 := \pi(g_0 U)\), which are \(k\)-horospheres contained in \(H_n\) and \(H_0\) respectively.

We next show that there is a uniform upper bound for \(d_{H_n}(P_n, \chi_n), n \in \mathbb{N}\). To see this, we only need to consider those \(P_n\)’s which are disjoint from \(\chi_n\), as \(d_{H_n}(P_n, \chi_n) = 0\) otherwise. Since \(\chi_n \to \chi_0\) and \(C_n \to C_0\) as \(n \to \infty\), it suffices to check that the diameters of \(D_n\) with respect to the spherical metric on \(S^{d-1}\) have a uniform positive lower bound. Let us write \(C_n - D_n = E_n \cup E'_n\), where \(E_n\) is a connected component of \(C_n - D_n\) meeting \(B\), and \(E'_n\) is the other component. Since \(C_n \to C_0\) as \(n \to \infty\), a uniform lower bound for both \(\text{diam}(E_n)\) and \(\text{diam}(E'_n)\) will give a uniform upper bound for \(\text{diam}(D_n)\).

Since \(B_n \subset E_n\), \(\text{diam}(E_n) > \text{diam}(B)/2\) for all sufficiently large \(n\). On the other hand, note that \(\chi_n \subset H_n\) is a horosphere resting at a point in \(E'_n\). Since \(\chi_n\) converges to \(\chi\), the condition that \(P_n \cap \chi_n = \emptyset\) implies that \(\text{diam}(E'_n)\) is also bounded below by some positive constant.

Since \(R_n \to \infty\), we conclude that \(d_{H_n}(P_n, \chi_n) < R_n - 1\) for all sufficiently large \(n\). Applying Lemma 8.4 to \(H^{k+1} = H_n\), \(\chi = \chi_n\), \(P = \mathcal{P}_n\), \(\xi = g^+_n\) and \(q = \pi(g_n u_n)\), we have
\[
d_{H_n}(\pi(g_n u_n), \mathcal{P}_n) < R_n
\]
and hence \(\pi(g_n u_n) \in N_\varepsilon(\mathcal{H}) \cap H_n\), for all sufficiently large \(n\). As \(\varepsilon\) was arbitrary, the proof is complete.

**Lemma 8.7.** Let \(U < N\) be a non-trivial connected closed subgroup. If \(x_n \to x\) in \(F^* \cap RF_+ M\), and \(x \in F^* \cap RF M\), then there exists \(u_n \to e\) in \(U\) such that \(x_n u_n \in RF M\); in particular, \(x_n u_n \to x\) in \(F^* \cap RF M\).

**Proof.** The general case easily reduces to the case when \(U < H \cap N\). Then the claim follows from Lemma 8.6 and Lemma 4.5.

**Obtaining limits in \(F^*\).**

**Lemma 8.8.** Let \(x \in RF M\), and \(V = \{v_t : t \in \mathbb{R}\} < N\) be a one-parameter subgroup. If \(\pi(x V) \nsubseteq \partial \text{core} M\), and \(x v_{t_i} \in RF M\) for some sequence \(t_i \to +\infty\), then there exists a sequence \(s_i \to +\infty\) such that \(x v_{s_i}\) converges to a point in \(F^*\), by passing to a subsequence.

**Proof.** For \(\varepsilon > 0\), we set
\[
\text{core}_\varepsilon(M) := \{x \in \Gamma \setminus G : \pi(x) \in \text{core} M \text{ and } d(\pi(x), \partial \text{core} M) \geq \varepsilon\}.
\]
We note that \(\text{core}_\varepsilon(M)\) is a closed bounded subset of \(F^*\). Hence it suffices to show that there exists a sequence \(s_i \to +\infty\) such that \(x v_{s_i} \in \text{core}_{\varepsilon_0}(M)\) where \(\varepsilon_0 = \eta/3\) with \(\eta\) given in (4.9). Let \(x = [g]\), and set \(o = (1, 0, \ldots, 0) \in \mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1}\). We may assume \(g = (e_0, \ldots, e_{d-1})_o \in F \mathbb{H}^d\) where \(e_i\) are standard basis vectors in \(T_0 \mathbb{H}^d \simeq \mathbb{R}^d\). Note that for \(V^+ = \{v_t : t > 0\}\), \(g V^+\) is a translation of the frame \(g\) along a horizontal ray emanating from \(o\) along the \(V^+\)-direction. By the choice of \(\varepsilon_0\), the \(\varepsilon_0\)-neighborhoods of hull \(B_i\)'s are mutually disjoint. For each \(i\), set \(s_i := t_i\) if \(x v_{t_i} \in \text{core}_{\varepsilon_0}(M)\).
Otherwise, there exists a unique $j$ such that $d(\pi(gv_{t_i}), \text{hull } B_j) < \varepsilon_0$. If $\pi(gV_{[t_i, \infty)})$ were contained in the $\varepsilon_0$-neighborhood of $\text{hull } B_j$, then the unique geodesic 2-plane which contains $\pi(gV_{[t_i, \infty)})$ must lie in $\partial \text{hull } B_j$, and hence $\pi(xV) \subset \partial \text{core}(M)$; this contradicts the hypothesis. Therefore there exists $t_i < s_i < \infty$ such that $d(\pi(gv_{s_i}), \text{hull } B_j) = \varepsilon_0$. The sequence $s_i$ satisfies the claim. □

9. Limits of unipotent blowups

In this section, we assume that $M$ is a convex cocompact hyperbolic manifold with Fuchsian ends and fix $k > 1$ as given by Proposition 4.11.

In the whole section, we fix a non-trivial connected subgroup $U < N$.

For a given sequence $g_i \to e$, and a sequence of $k$-thick subsets $T_i$ of a one-parameter unipotent subgroup $U_0 < U$, we study the following set

$$\limsup T_i g_i U$$

under certain conditions on the sequence $g_i$. The basic tool used here is the so-called quasi-regular map associated to the sequence $g_i$ introduced in the work of Margulis-Tomanov [25] to study the object $\limsup U_0 g_i U$ in the finite volume case. For our application, we need a somewhat more precise information on the shape of the set $\limsup U_0 g_i U$ as well as $\limsup T_i g_i U$ than discussed in [25].

Let $U^\perp$ denote the orthogonal complement of $U$ in $N \simeq \mathbb{R}^{d-1}$ as defined in section 3. Recall from (3.2) that

$$N(U) = AN C_1(U) C_2(U)$$

where $C_1(U) = C(H(U))$ and $C_2(U) = H(U) \cap M \cap C(U^\perp)$. Since $N(U)$ is the identity component of $N_G(U)$, for a sequence $g_i \to e$, the condition $g_i \in N_G(U)$ means $g_i \in N(U)$ for all sufficiently large $i \gg 1$. Note that the product $AU^\perp C_2(U)$ is a connected subgroup of $G$, since $C_2(U)$ commutes with $U^\perp$, and $A$ normalizes $U^\perp C_2(U)$.

**Lemma 9.1.** For a given sequence $g_i \to e$ in $G - N(U)$, there exists a one-parameter subgroup $U_0 < U$ such that the following holds; for any given sequence of $k$-thick subsets $T_i \subset U_0$, there exist sequences $t_i \in T_i$, and $u_i \in U$ such that as $i \to \infty$,

$$u_i g_i u_i \to \alpha$$

for some non-trivial element $\alpha \in AU^\perp C_2(U) - C_2(U)$. Moreover, $\alpha$ can be made arbitrarily close to $e$.

**Proof.** Set $L := AU^\perp MN^\perp$. Note that

$$N(U) \cap L = AU^\perp C_1(U) C_2(U)$$

and that the product map from $U \times L$ to $G$ is a diffeomorphism onto a Zariski open neighborhood of $e$ in $G$.

Following [25], we will construct a quasi-regular map

$$\psi : U \to N(U) \cap L$$
associated to the sequence \( g_i \).

Except for a Zariski closed subset of \( U \), the product \( g_i u \) can be written as an element of \( UL \) in a unique way. We denote by \( \psi_i(u) \in L \) its \( L \)-component so that

\[ g_i u \in U \psi_i(u). \]

By Chevalley’s theorem, there exists an \( \mathbb{R} \)-regular representation \( G \to \text{GL}(W) \) with a distinguished point \( p \in W \) such that \( U = \text{Stab}_G(p) \). Then \( pG \) is locally closed, and

\[ N_G(U) = \{ g \in G : pgu = pg \text{ for all } u \in U \}. \]

As \( U \) is a connected unipotent subgroup of \( G \), isomorphic to \( \mathbb{R}^m \) for some \( 1 \leq m \leq d - 1 \), the map \( \tilde{\phi}_i : U \to W \) defined by

\[ \tilde{\phi}_i(u) = pg_i u \]

is a polynomial map in \( m \)-variables of degree uniformly bounded for all \( i \), and \( \tilde{\phi}_i(e) \) converges to \( p \) as \( i \to \infty \).

As \( g_i \notin N_G(U) \), \( \tilde{\phi}_i \) is non-constant. Denote by \( B(p, r) \) the ball of radius \( r \) centered at \( p \), fixing a norm \( \| \cdot \| \) on \( W \).

Since \( pG \) is open in its closure, we can find \( \lambda_0 > 0 \) such that

\[ B(p, \lambda_0) \cap pG = pG. \]

Without loss of generality, we may assume that \( \lambda_0 = 2 \) by renormalizing the norm. Now define

\[ \lambda_i := \sup \{ \lambda \geq 0 : \tilde{\phi}_i(B_U(\lambda)) \subset B(p, 2) \}. \]

Note that \( \lambda_i < \infty \) as \( \phi_i \) is nonconstant, and \( \lambda_i \to \infty \) as \( i \to \infty \), as \( g_i \to e \).

We define \( \phi_i : U \to W \) by

\[ \phi_i(u) := \tilde{\phi}_i(\lambda_i u). \]

This forms an equi-continuous family of polynomials on \( U = \mathbb{R}^m \).

Therefore, after passing to a subsequence, \( \phi_i \) converges to a non-constant polynomial \( \phi \) uniformly on every compact subset of \( U \). Moreover \( \sup \{ \| \phi(u) - p \| : u \in B_U(1) \} = 1 \), \( \phi(B_U(1)) \subset pL \), and \( \phi(0) = p \).

Now the following map \( \psi \) defines a non-constant rational map defined on a Zariski open dense neighborhood of \( \mathcal{U} \) of \( e \) in \( U \):

\[ \psi := \rho_L^{-1} \circ \phi \]

where \( \rho_L \) is the restriction to \( L \) of the orbit map \( g \mapsto p.g \).

We have \( \psi(e) = e \) and

\[ \psi(u) = \lim \psi_i(\lambda_i u) \]

where the convergence is uniform on compact subsets of \( \mathcal{U} \) and

\[ \psi(u) \in L \cap \mathcal{N}(U) = AU^\perp C_1(U) C_2(U). \]
Since $\psi$ is non-constant, there exists a one-parameter subgroup $U_0 < U$ such that $\psi|_{U_0}$ is non-constant. Now let $T_i$ be a sequence of $k$-thick sets in $U_0 \simeq \mathbb{R}$. Then $T_i/\lambda_i$ is also a $k$-thick set, and so is

$$T_\infty := \limsup_{i \to \infty} (T_i/\lambda_i) \subset U_0.$$  

Finally, for all $t \in T_\infty$, there exists a sequence $t_i \in T_i$ such that $t_i/\lambda_i \to t$ as $i \to \infty$ (by passing to a subsequence). Since $\psi \circ \lambda_i \to \psi$ uniformly on compact subsets,

$$\psi(t) = \lim_{i \to \infty} (\psi \circ \lambda_i)(t_i/\lambda_i) = \lim_{i \to \infty} \psi_i(t_i).$$

By the definition of $\psi_i$, this means that there exists $u_i \in U$ such that

$$\psi(t) = \lim_{i \to \infty} u_ig_iu_{t_i}.$$

Since $\psi|_{U_0}$ is a non-constant continuous map, and an uncountable set $T_\infty$ accumulates on 0, the image $\psi(T_\infty)$ contains a non-trivial element $\alpha$ of $AU^\perp C_1(U)C_2(U)$ which can be taken arbitrarily close to $e$.

We now claim that if $\alpha$ is sufficiently close to $e$, then it belongs to $AU^\perp C_2(U)$. Consider $H'(U) := H(U)C_1(U)$, and let $h$ denote its Lie algebra. Now for all $i$ large enough, using the decomposition $g = h \oplus h^\perp$ in (3.5), we can write $g_i = c_id_ir_i$ where $c_i \in C_1(U), d_i \in H(U)$ and $r_i \in \exp h^\perp$.

Now since $c_i$ commutes with $U$, we can write

$$u_ig_iu_{t_i} = (u_igt_i)c_i(u_{t_i}^{-1}d_iu_{t_i})(u_{t_i}^{-1}r_iu_{t_i}).$$

On the other hand, we have

$$\lim_{i} pu_igt_i = \lim_{i} pc_i(u_{t_i}^{-1}d_iu_{t_i})(u_{t_i}^{-1}r_iu_{t_i}) = pc.$$  

Since $c_i \to e$, $u_igt_i^{-1} \to H(U)$, and $u_igt_i^{-1} \in \exp h^\perp$, it follows that both sequences $u_igt_i^{-1}$ and $u_igt_i^{-1}$ must converge, say to $h \in H(U)$ and to $q \in \exp h^\perp$, respectively.

Hence $\alpha = hq$ by replacing $h$ by $uh$ for some $u \in U$. On the other hand, we can write $\alpha = avc_1c_2 \in AU^\perp C_1(U)C_2(U)$.

So $hq = avc_1c_2$. Note that $c := c_1c_2 \in C(H(U)H(U) = H'(U)$. We get

(9.2)  

$$(a^{-1}hc^{-1})(qc^{-1}) = v.$$  

Now, when $\alpha$ is sufficiently close to $e$, all elements appearing in (9.2) are also close to $e$.

Recall that the map $H'(U) \times h^\perp \to G$ given by $(h',X) \to h'\exp X$ is a local diffeomorphism onto a neighborhood of $e$. Since $(a^{-1}hc^{-1}) \in H'(U)$, and $cqc^{-1}, v \in \exp h^\perp$, we have $a^{-1}hc^{-1} = e$ and $cqc^{-1} = v$ for $\alpha$ sufficiently small. In particular,

$$a^{-1}hc_2^{-1} = c_1^{-1} \in H(U) \cap C(H(U)) = \{e\}.$$  

Hence $c_1 = e$. It follows that $\alpha \in AU^\perp C_2(U)$, as desired.

We further claim that we can choose $\alpha$ outside of $C_2(U)$. As $C_2(U)$ is a compact subgroup, we can choose a $C_2(U)$-invariant Euclidean norm $\|\cdot\|$ on
Lemma 9.2. Let \( \alpha = \psi(t) \in C_2(U) \) for some \( t \in T_\infty \subset U_0 \), then \( t \) is one of finitely many solutions of the polynomial equation \( \|\phi(t)\|^2 = \|p\|^2 \). Therefore, except for finitely many \( t \in T_\infty \), \( \alpha = \psi(t) \in AU^+C_2(U) - C_2(U) \). This finishes the proof.

The following lemma is similar to Lemma 9.1, but here we consider the case when \( U \) is the whole horospherical subgroup \( N \). In this restrictive case, the limiting element can be taken inside \( A \).

Lemma 9.2. Let \( T_i \subset N \) be a sequence of \( k \)-thick subsets in the sense that for any one-parameter unipotent subgroup \( U_0 < N \), \( T_i \cap U_0 \) is a \( k \)-thick subset of \( U_0 \subset \mathbb{R} \). For any sequence \( g_i \to e \) in \( G - N_G(N) \), there exist \( t_i \to \infty \) in \( T_i \) and \( u_i \in N \) such that

\[ u_ig_iu_i \to a \]

for some non-trivial element \( a \in A \). Moreover, \( a \) can be chosen to be arbitrarily close to \( e \).

Proof. We first consider the case when \( g_i \) belongs to the opposite horospherical subgroup \( N^+ \). We will use the notations \( u^+ \) and \( u^- \) defined in Section 3. Write \( g_i = \exp u^+(w_i) \) for some \( w_i \in \mathbb{R}^{d-1} \). For \( x \in \mathbb{R}^{d-1} \), set \( u_x := \exp u^-(x) \in N \).

Let \( \varepsilon > 0 \) be arbitrary. Since \( T_i \) is a \( k \)-thick subset of \( N \), there exists \( \alpha_i \in \mathbb{R} \) such that \( \alpha_i w_i \in T_i \) and

\[ \varepsilon < \frac{|\alpha_i|\|w_i\|^2}{2} < k\varepsilon. \]

Setting \( u_{x_i} := u_{\alpha_i w_i} \in T_i \) and \( y_i := -\alpha_i w_i \left( 1 + \frac{\alpha_i\|w_i\|^2}{2} \right)^{-1} \), we compute:

\[
\begin{pmatrix}
\left( 1 + \frac{\alpha_i\|w_i\|^2}{2} \right)^{-2} & 0 & 0 \\
0 & \left( 1 + \frac{\alpha_i\|w_i\|^2}{2} \right)^{-1} & w_i \\
\left( 1 + \frac{\alpha_i\|w_i\|^2}{2} \right)^{-1} & 0 & \left( 1 + \frac{\alpha_i\|w_i\|^2}{2} \right)^2
\end{pmatrix}
\]

The condition for the size of \( \alpha_i \) guarantees that, by passing to a subsequence, the sequence \( u_{x_i}g_iu_{y_i} \) converges to an element \( \text{diag}(\alpha, I_{d-1}, \alpha^{-1}) \in A \), for \( \alpha \in [(1-\varepsilon)^{-2}, (1-k\varepsilon)^{-2}] \cup [(1+k\varepsilon)^{-2}, (1+\varepsilon)^{-2}] \) as \( i \to \infty \). This proves the claim when \( g_i \in N^+ \).

Since the product map \( A \times M \times N^+ \times N \to G \) is a diffeomorphism onto a Zariski-open neighborhood of \( e \) in \( G \), we can write \( g_i = a_i m_i u_i^+u_i^- \) for some \( a_i \in A, m_i \in M, u_i^+ \in N^+ \) and \( u_i^- \in N \) all of which converge to \( e \) as \( i \to \infty \).

By the previous case, we can find \( u_{t_i} \in T_i \) and \( u_i \in N \) such that \( u_iu_{t_i}^+u_i^- \) converges to a non-trivial element \( a \in A \). Let \( \tilde{u}_i := (a_i m_i)u_i(a_i m_i)^{-1} \in N \). Then \( \tilde{u}_ig_iu_i = a_i m_i u_i^+u_i^- u_i = a_i m_i (u_iu_{t_i}^+u_i^-)u_i^- \to a \) as \( i \to \infty \), proving the claim.
Lemma 9.3. Let $L$ be any connected reductive subgroup of $G$ normalized by $A$. Let $U_0 < L \cap N$ be a one-parameter unipotent subgroup of $L$. Let $T_i \subset U_0$ be a sequence of $k$-thick subsets. For a given sequence $r_i \to e$ in $\exp(L^\perp) - N(U_0)$, there exists a sequence $t_i \in T_i$ such that as $i \to \infty$,

$$u_{t_i}^{-1}r_iu_{t_i} \to v$$

for some non-trivial element $v \in (L \cap N)^\perp$, and $v$ can be chosen arbitrarily close to $e$. Moreover, for all $n$ large enough, we can make $v$ so that

$$n \leq \|v\| \leq 2k^2 n.$$  

Proof. Without loss of generality, by Proposition 3.7, we may assume that $L_{nc} = H(U)$ for $U = U_k = \mathbb{R}^k$ some $k \geq 1$ and $U_0 := \mathbb{R}e_1$. We write $r_i = \exp(q_i)$ where $q_i \to 0$ in $L^\perp$.

Using the notations introduced in section 3 and setting $u^\perp = \text{Lie}(U^\perp) = \mathbb{R}^{d-1-k}$, we can write

$$q_i = u^-(x_i) + u^+(y_i) + m(C_i)$$

where $x_i \in u^\perp$, $y_i \in (u^\perp)^\perp$, and $C_i = \begin{pmatrix} 0_k & B_i \\ -B_i^t & A_i \end{pmatrix}$ is a skew symmetric matrix, all of which converge to 0 as $i \to \infty$. We consider $U_0 = \mathbb{R}e_1$ as $\{u_s = se_1 \in \mathbb{R}^{d-1}\}$ and define the map $\psi_i : \mathbb{R} \to L^\perp$ by

$$\psi_i(s) = u_s^{-1}q_iu_s \quad \text{for all } s \in \mathbb{R} ;$$

this is well-defined since $L^\perp$ is $\text{Ad}(L)$-invariant. Then a direct computation shows

$$\psi_i(s) = u^-(x_i + sB_i^te_1 + s^2y_i/2) + u^+(y_i) + m(\tilde{C}_i)$$

where $\tilde{C}_i$ is a skew-symmetric matrix of the form

$$\tilde{C}_i = \begin{pmatrix} 0_k & B_i + se_1y_i^t \\ -B_i^t - sy_i^te_1^t & A_i \end{pmatrix}.$$  

Since $r_i \notin N(U_0)$, it follows that either $y_i \neq 0$ or $y_i = 0$ and $B_i^te_1 \neq 0$. Hence $\psi_i$ is a non-constant polynomial of degree at most 2, and $\psi_i(0) \to 0$. Let $\lambda_i \in \mathbb{R}$ be defined by

$$\lambda_i = \sup \{ \lambda > 0 : |\psi_i[\lambda] \leq 1 \}.$$  

Then $0 < \lambda_i < \infty$ and $\lambda_i \to \infty$.

Now the rescaled polynomials $\phi_i = \psi_i \circ \lambda_i : \mathbb{R} \to L^\perp$ form an equicontinuous family of polynomials of degree at most 2 and $\lim_{i \to \infty} \phi_i(0) = 0$. Therefore $\phi_i$ converges to a polynomial $\phi : \mathbb{R} \to L^\perp$ uniformly on compact subsets. Since $\phi(0) = 0$ and $\sup \{|\phi(\lambda)| : \lambda \in [-1, 1]\} = 1$, $\phi$ is a non-constant polynomial. From (9.3), it can be easily seen that $\text{Im}(\phi)$ is contained $\text{Lie}(N) \cap L^\perp$, by considering the two cases of $y_i \neq 0$, and $y_i = 0$ and $B_i^te_1 \neq 0$ separately.
For a given sequence $T_i$ of $k$-thick subsets of $U_0$, set
\[ T_\infty := \limsup_{i \to \infty} (T_i/\lambda_i), \]
which is also a $k$-thick subset of $U_0$.

Let $s \in T_\infty$. By passing to a subsequence, there exists $t_i \in T_i$ such that
\[ t_i/\lambda_i \to s \text{ as } i \to \infty. \]
As $\phi_i \to \phi$ uniformly on compact subsets, it follows that
\[ \phi(s) = \lim_{i \to \infty} \psi_i(\lambda_i \cdot t_i/\lambda_i) = \lim_{i \to \infty} u_i^{-1} q_i u_i. \]

Since $T_\infty$ accumulates on 0, so does $\phi(T_\infty)$. Taking the exponential map
to each side of the above, the first part of the lemma follows.

The second part of the lemma holds by applying Lemma 9.4 below for
the non-constant polynomial $p(s) = \|\phi(s)\|^2$ of degree at most 4.

**Lemma 9.4.** If $p \in \mathbb{R}[s]$ is a polynomial of degree $\delta \geq 1$ and $T \subset \mathbb{R}$ is a $k$-thick subset, then $p(T)$ is $2k^\delta$-thick at $\infty$.

**Proof.** Let $C$ be the coefficient of $s^\delta$ term of the polynomial $p$. Then there
exists $s_0 > 1$ such that $1/\sqrt{2} \leq |p(s)/|s|^\delta| \leq \sqrt{2}$ for all $|s| > s_0$. Let $r > \frac{|C|s_0^\delta}{\sqrt{2}}$. Since $T$ is $k$-thick, there exists $t \in T$ such that $(\sqrt{2}r/|C|)^{1/\delta} < |t| < k(\sqrt{2}r/|C|)^{1/\delta}$.
We compute that $r \leq |p(t)| \leq 2k^\delta r$, proving the claim. \qed

10. Translates of relative $U$-minimal sets

Assume that $M$ is a convex cocompact hyperbolic manifold with Fuchsian ends and fix $k > 1$ as given by Proposition 4.11.

In this section, we fix a non-trivial connected closed subgroup $U < N$. Unless mentioned otherwise, we let $R$ be a compact $A$-invariant subset of
$RF_M$ such that for every $x \in R$, and for any one-parameter subgroup $U_0 = \{u_t\}$ of $U$, the following set
\[ \{t \in \mathbb{R} : xu_t \in R\} \]
is $k$-thick. In practice, $R$ will be either $RF_M$ or of the form $RF_M \cap F^*_H(U) \cap X$
when it is compact) for a closed $H(U)$-invariant subset $X$.

The main aim of this section is to prove Propositions 10.6 and 10.9 using
the results of section 9.

**Definition 10.1.**
- A $U$-invariant closed subset $Y \subset \Gamma \backslash G$ is $U$-minimal if $yU$ is dense in $Y$ for any $y \in Y$.
- A $U$-invariant closed subset $Y \subset \Gamma \backslash G$ is $U$-minimal with respect to $R$ if $Y \cap R \neq \emptyset$ and for any $y \in Y \cap R$, $yU$ is dense in $Y$.

A $U$-minimal subset may not exist, but a $U$-minimal subset with respect
to a compact subset $R$ always exists by Zorn’s lemma. In this section, we
study how to find an additional invariance of $Y$ beyond $U$ under certain
conditions.

We will use the following lemma:
Lemma 10.2. Let $Y \subset \Gamma\backslash G$ be a $U$-minimal subset with respect to $R$. For any $y \in Y \cap R$, there exists a sequence $u_n \rightarrow \infty$ in $U$ such that $yu_n \rightarrow y$.

Proof. The set $Z := \{z \in Y : yu_n \rightarrow z \text{ for some } u_n \rightarrow \infty \text{ in } U\}$ is $U$-invariant and closed. By the assumption on $R$, there exists $u_n \rightarrow \infty$ in $U$ such that $yu_n \in Y \cap R$. Since $Y \cap R$ is compact, $yu_n$ converges to some $z \in Y \cap R$, by passing to a subsequence. Hence $Z$ intersects $Y \cap R$ non-trivially. Therefore $Z = Y$, by the $U$-minimality of $Y$ with respect to $R$. □

A subset $S$ of a topological space is said to be locally closed if $S$ is open in its closure $\overline{S}$.

Lemma 10.3. Let $Y$ be a $U$-minimal subset of $\Gamma\backslash G$ with respect to $R$, and $S$ be a closed subgroup of $N(U)$ containing $U$. For any $y_0 \in Y \cap R$, the orbit $y_0S$ is not locally closed.

Proof. Suppose that $y_0S$ is locally closed for some $y_0 \in Y \cap R$. Since $Y$ is $U$-minimal with respect to $R$, there exists $u_n \rightarrow \infty$ in $U$ such that $y_0u_n \rightarrow y_0$ by Lemma 10.2. We may assume that $y_0 = [e]$ without loss of generality. Since $y_0S$ is locally closed, $y_0S$ is homeomorphic to $(S \cap \Gamma)\backslash S$ (cf. [50, Theorem 2.1.14]). Therefore there exists $\delta_n \in S \cap \Gamma$ such that $\delta_nu_n \rightarrow e$ as $n \rightarrow \infty$.

Since $N(U) = ANC_1(U)C_2(U)$, writing $\delta_n = a_n r_n$ for $a_n \in A$ and $r_n \in NC_1(U)C_2(U)$, it follows that $a_n \rightarrow e$. On the other hand, note that $a_n$ is non-trivial as $\Gamma$ does not contain any elliptic or parabolic element. This is a contradiction, as there exists a positive lower bound for the translation lengths of elements of $\Gamma$, which is given by the minimal length of a closed geodesic in $M$. □

In the rest of this section, we use the following notation:

$$H = H(U), \quad H' = H'(U), \quad F^* = F^*_H(U).$$

Lemma 10.4. For every $U$-minimal subset $Y \subset \Gamma\backslash G$ with respect to $RF M$ such that $Y \cap RF M \cap F^* \neq \emptyset$, and for any $y_0 \in Y \cap RF M \cap F^*$, there exists a sequence $g_n \rightarrow e$ in $G - N(U)$ such that $y_0g_n \in Y \cap RF M$ for all $n$.

Proof. Let $y_0 \in Y \cap RF M \cap F^*$. As $Y = \overline{y_0U}, Y \subset RF_+ M$. Using Lemma 4.5 and the fact that $F^*$ is open, we get that there exists an open neighborhood $O$ of $e$ such that

$$y_0O \subset Y \cap F^* \subset Y \cap RF M \cdot U. \quad (10.1)$$

Without loss of generality, we may assume that the map $g \mapsto y_0g \in \Gamma\backslash G$ is injective on $O$, by shrinking $O$ if necessary. We claim that there exists $g_n \rightarrow e$ in $G - N(U)$ such that $y_0g_n \in Y \cap F^*$. Suppose not. Then there exists a neighborhood $O' \subset O$ of $e$ such that

$$y_0O' \cap Y \subset y_0N(U). \quad (10.2)$$

Set

$$S := \{g \in N(U) : Yg = Y\}$$
which is a closed subgroup of $N(U)$ containing $U$. We will show that $y_0S$ is
locally closed; this contradicts Lemma 10.3. We first claim that

(10.3) \[ y_0O' \cap Y \subset y_0S. \]

If $g \in O'$ such that $y_0g \in Y$, then $g \in N(U)$. Therefore $\overline{y_0gU} = y_0Ug = Yg \subset Y$. Moreover, $Yg \cap RF M \neq \emptyset$ by (10.1). Hence $Yg = Y$, proving that
$g \in S$.

Now, (10.3) implies that $y_0S$ is open in $Y$. On the other hand, since
$U \subset S$, we get $Y = \overline{y_0S}$. Therefore, $y_0S$ is locally closed.

Hence we have $g_n \to e$ in $G - N(U)$ such that $y_0g_n \in Y \cap F^*$. Since
$y_0g_n \in F^* \cap RF, M$ converges to $y_0 \in RF M \cap F^*$, by Lemma 8.7, there
exists a sequence $u_n \to e$ in $U$ such that $y_0g_nu_n \in RF M$. Therefore, by
replacing $g_n$ with $g_nu_n$, this finishes the proof. \qed

We record the following basic lemma:

**Lemma 10.5.** Let $Y$ be a $U$-minimal subset with respect to $R$, and let $W$
be a connected closed subgroup of $N(U)$. Suppose that there exists a se-
quency $\alpha_i \to e$ in $W$ such that $Y\alpha_i \subset Y$. Then there exists a one-parameter
semigroup $S < W$ such that $YS \subset Y$.

**Proof.** The set $S_0 = \{g \in W : Yg \subset Y\}$ is a closed
subsemigroup of $W$. Write $\alpha_i = \exp \xi_i$ for some $\xi_i \in \text{Lie}(W)$. Then the sequence $v_i := \|\xi_i\|^{-1}\xi_i$ of
unit vectors has a limit, say, $v$. It suffices to note that $S := \{\exp(tv) : t \geq 0\}$
is contained in the closure of the semigroup generated by $\alpha_i$’s. \qed

**Proposition 10.6** (Translate of $Y$ inside of $Y$). Let $Y$ be a $U$-minimal set
of $\Gamma \backslash G$ with respect to $RF M$ such that $Y \cap RF M \cap F^* \neq \emptyset$.

Then there exists an unbounded one-parameter subsemigroup $S$ inside the
subgroup $AU^\perp C_2(U)$ such that

\[ YS \subset Y. \]

**Proof.** Choose $y_0 \in Y \cap RF M \cap F^*$. By Lemma 10.4, there exists $g_i \to e$
in $G - N(U)$ such that $y_0g_i \in Y \cap RF M$. Let $U_0 = \{u_i\}$ be a one-parameter
subgroup of $U$ as given by Lemma 9.1, with respect to the sequence $g_i$.

Let

\[ T_i := \{u_t \in U_0 : y_0g_iu_t \in Y \cap RF M\} \]

which is a $k$-thick subset of $U_0$.

Hence by Lemma 9.1, there exists sequences $u_{t_i} \to \infty$ in $T_i$, and $u_i \in U$
such that

\[ \lim_{i \to \infty} u_i g_i u_{t_i} = \alpha \]

for some non-trivial $\alpha \in AU^\perp C_2(U) - C_2(U)$. Note that $y_0g_iu_{t_i} \in Y \cap RF M$
converges to some $y_1 \in Y \cap RF M$ by passing to a subsequence. Hence as
$i \to \infty$,

\[ y_0 u_{t_i}^{-1} = y_0 g_i u_{t_i} (u_i g_i u_{t_i})^{-1} \to y_1 \alpha^{-1}. \]
So $y_1\alpha^{-1} \in Y$, and hence $Y\alpha^{-1} \subset Y$, since $y_1 \in Y \cap \text{RF} \ M$. Since $\alpha$ can be made arbitrarily close to $e$ in Lemma 9.1, the claim follows from Lemma 10.5.

**Proposition 10.7** (Translate of $Y$ inside of $X$). Let $X$ be a closed $H'$-invariant set such that $X \cap R \neq \emptyset$. Let $Y \subset X$ be a $U$-minimal subset with respect to $R$, and assume that there exists $y \in Y \cap R$ and a sequence $g_n \to e$ in $G - H'$ such that $yg_n \in X$ for all $n$. Then there exists some non-trivial $v \in U^\perp$ such that

$$Yv \subset X.$$ 

**Proof.** Let $\mathfrak{h}$ denote the Lie algebra of $H'$. We may write $g_n = r_n h_n$ where $h_n \in H'$ and $r_n \in \exp \mathfrak{h}^\perp$. By replacing $g_n$ with $g_n h_n^{-1}$, we may assume $g_n = r_n$. If $r_n \in U^\perp$ for some $n$, then the claim follows since $y_0 r_n \in X$ and hence $Yr_n \subset X$. Hence we assume that $r_n \notin U^\perp$ for all $n$. We have from (3.5)

$$\mathfrak{h}^\perp \cap \text{Lie}(N(U)) = \text{Lie} U^\perp.$$

Hence $r_n \notin N(U)$ for all $n$. Therefore there exists a one-parameter subgroup $U_0 = \{u_t\} < U$ such that $r_n \notin N(U_0)$. Let

$$T = \{t \in \mathbb{R} : yu_t \in R\}.$$ 

Since $y \in R$, it follows that $T$ is a $k$-thick subset of $\mathbb{R}$ by the assumption on $R$. Hence, by Lemma 9.3, there exists $t_n \in T$ such that $u_{t_n}^{-1} r_n u_{t_n} \to v$ for some non-trivial $v \in U^\perp$. Observe

$$(yu_{t_n})(u_{t_n}^{-1} r_n u_{t_n}) = yr_n u_{t_n} \in X.$$ 

Passing to a subsequence, $yu_{t_n} \to y_0$ for some $y_0 \in Y \cap R$, and hence $y_0 v \in X$. It follows $Yv \subset X$. 

For a one-parameter subgroup $V = \{v_t : t \in \mathbb{R}\}$ and a subset $I \subset \mathbb{R}$, the notation $V_I$ means the subset $\{v_t : t \in I\}$.

**Lemma 10.8.** Let $X$ be a closed $AU$-invariant set of $\Gamma \setminus G$, and $V$ be a one-parameter subgroup of $U^\perp$. Assume that $R : = X \cap \text{RF} \ M \cap F^*$ is non-empty and compact. If $x_0 V_I \subset X$ for some $x_0 \in R$ and a closed interval $I$ containing 0, then $X$ contains a $V$-orbit of a point in $R$.

**Proof.** Choose a sequence $a_n \in A$ such that $\limsup a_n V_I a_n^{-1}$ contains a semigroup $V^+$ of $V$ as $n \to \infty$. Then

$$(x_0 a_n^{-1})(a_n V_I a_n^{-1}) = x_0 V_I a_n^{-1} \subset X.$$ 

By passing to a subsequence, we have $x_0 a_n^{-1}$ converges to some $x_1 \in \text{RF} \ M$; so $x_1 V^+ \subset X$. Since $R$ is compact, so is $x_0 A \cap F^*$, which implies that $x_1 \in \overline{x_0 A} \cap F^*$.

Since $x_1$ belongs to the open set $F^*$, it follows $x_1 v_s \in F^*$ for all sufficiently small $s \in \mathbb{R}$. By Lemma 4.5, this implies that $x_1 v_s U \cap \text{RF} \ M \neq \emptyset$ for some $s > 0$ with $v_s \in V^+$. Note that

$$(x_1 v_s U)(v_s^{-1} V^+) = x_1 U V^+ \subset X.$$
Choose \( x_2 \in x_1 v_s U \cap RF M \subset X \cap RF M \cap F^* \). Then \( x_2 (v_s^{-1} V^+) \subset X \). Similarly as before, let \( a_n \in A \) be a sequence such that \( \lim \sup_n a_n (v_s^{-1} V^+) = V \) and such that \( x_2 a_n^{-1} \) converges to some \( x_3 \in R \). From

\[
(x_2 a_n^{-1}) (a_n v_s^{-1} V^+) a_n^{-1} = x_2 v_s^{-1} V^+ a_n^{-1} \subset X,
\]

we conclude that \( x_3 V \subset X \). This finishes the proof. \( \square \)

**Proposition 10.9.** Let \( X \) be a closed \( H' \)-invariant set. Assume that \( R := X \cap F^* \cap RF M \) is a non-empty compact set, and let \( Y \subset X \) be a \( U \)-minimal set with respect to \( R \). Suppose that there exist \( y \in Y \cap R \) and a sequence \( g_n \to e \) in \( G - H' \) such that \( yg_n \in X \) for all \( n \geq 1 \). Then there exists \( z \in R \) and a non-trivial connected closed subgroup \( V < U^\perp \) such that

\[
zV \subset X.
\]

**Proof.** By Lemma 10.8, it suffices to find \( x_0 \in R \), a one-parameter subgroup \( V < U^\perp \) and \( V_I \subset U^\perp \) for some interval \( I < \mathbb{R} \) containing 0 such that \( x_0 V_I \subset X \). It follows from Proposition 10.7 that there exists \( v_0 \in U^\perp \) such that \( Y v_0 \subset X \). By Proposition 10.6, there exists an unbounded one-parameter subgroup \( S \) inside the subgroup \( AU^\perp C_2(U) \) such that \( YS \subset Y \). By Lemma 3.3, \( S \) is either of the form

1. \( S = \{ \exp(t \xi_V) \exp(t \xi_C) : t \geq 0 \} \), or
2. \( S = \{ (v \exp(t \xi_A) v^{-1}) \exp(t \xi_C) : t \geq 0 \} \)

for some \( \xi_A \in \text{Lie}(A) - \{0\}, \xi_C \in \text{Lie}(C_2(U)), \xi_V \in \text{Lie}(V) - \{0\} \), and \( v \in U^\perp \). Since \( X \) is \( H'(U) \)-invariant, we may assume \( YS \subset X \) with \( \xi_C = 0 \).

In case (1), the claim is immediate. Hence we assume that we are in case (2). First suppose \( v = e \). So \( YA^+ \subset Y \). By the compactness and the \( A \)-invariance of \( R \), we get \( YA \subset Y \). Then we have \( YA v_0 \subset Y v_0 \subset X \). Then by the \( A \)-invariance of \( X \), we have \( Y (Av_0 A) \subset X \). Since \( Av_0 A \) contains some \( V^+, YV_I \subset X \) for some interval \( I \) containing 0 as desired.

Next suppose \( v \neq e \). Then since \( YA \subset X \), we have

\[
Y (v^{-1} A^+ v) A \subset YA \subset X.
\]

We will take \( V = \exp \mathbb{R} (\log v) \). Now note that \( (v^{-1} A^+ v) A \) contains \( V_I \) for some interval \( I \) containing 0 for any subsemigroup \( A^+ \) of \( A \). Hence \( YV_I \subset X \) also holds for this case. \( \square \)

11. Closures of orbits inside \( \partial F \)

Throughout the section, assume that \( M = \Gamma \backslash \mathbb{H}^d \) is a convex cocompact hyperbolic manifold with Fuchsian ends.

Let \( U \) be a connected closed subgroup of \( \tilde{H} \cap N \). Then \( \partial F_{H(U)} = BF M \cdot \tilde{V}^+ \cdot H'(U) \) and \( \partial F_{H(U)} \cap RF M = BF M \cdot C(H(U)) \).

The following theorem are special cases of Ratner’s theorem [38], which were also proved by Shah independently [45]:

**Theorem 11.1.** If \( x = x_0 c \in BF M \cdot C(H(U)) \) with \( z \in BF M \) and \( c \in C(H(U)) \). Then
(1) $\overline{xU} = xL$ for some $L \in Q_U$ contained in $c^{-1}\tilde{H}c$;
(2) $\overline{xH(U)} = xL$ for some $L \in L_U$ contained in $c^{-1}\tilde{H}c$, and for any $y \in \mathcal{G}(U, xL)$, $\overline{yU} = xL$;
(3) $\overline{xAU} = \overline{xH(U)}$.

Proof. Since $x$ is contained in the compact homogeneous space $xc^{-1}\tilde{H}c$, we only need to discuss the proof of (3). We show that $\overline{xAU} = xL$ where $L$ is given by (2). If $U = L \cap N$, then the claim follows from Theorem 13.1. Suppose that $U$ is a proper subgroup of $L \cap N$. Since $\overline{xAU}(K \cap H(U)) = \overline{xH(U)} = xL$ and $\mathcal{G}(U, xL) \cdot (K \cap H(U))$ is a proper subset of $xL$ (cf. Lemma 5.14), there exists $y \in \overline{xAU} \cap \mathcal{G}(U, xL)$. Hence (3) follows from (2). □

Lemma 11.2. Let $V^+ \subset N$ be a one-parameter semigroup which is not contained in $\tilde{H}$. Then $V^+H(U)$ is a closed subset of $G$.

Proof. Since the product map $A \times N \to AN$ is a diffeomorphism and $AN$ is closed, the product subset $A\mathcal{U}$ is closed in $G$ for any closed subset $W$ of $N$. Hence $A\mathcal{U}V^+$ is a closed subset of $AN$. We use Iwasawa decompositions $H(U) = UA(K \cap H(U))$, and the fact that $AV^+ = V^+A$ in order to write $V^+H(U) = A\mathcal{U}V^+(K \cap H(U))$. Hence the conclusion follows from compactness of $K \cap H(U)$.

□

Lemma 11.3. Let $V^+ \subset N$ be a one-parameter semigroup which is not contained in $\tilde{H}$. If $g_i \in \tilde{H}$ is a sequence such that $g_iv_ih_i$ converges for some $v_i \in V^+$ and $h_i \in H(U)$ as $i \to \infty$, then, after passing to a subsequence, there exists $p_i \in AU$ such that $g_ip_i$ converges to an element of $\tilde{H}$ as $i \to \infty$.

Proof. We write $g_i = \tilde{k}_i\tilde{a}_i\tilde{n}_i \in (K \cap \tilde{H})A(N \cap \tilde{H})$ and $h_i = u_ia_ik_i \in UA(K \cap H(U))$. Since $K \cap \tilde{H}$ and $K \cap H(U)$ are compact, we may assume without loss of generality that $\tilde{k}_i = k_i = e$ for all $i$. Observe that

$$g_iv_ih_i = \tilde{a}_i\tilde{n}_iv_iu_ia_i = \tilde{a}_i\tilde{n}_ia_i(a_i^{-1}\tilde{n}_iu_ia_i)(a_i^{-1}v_i a_i)$$

where $\tilde{a}_ia_i \in A$, $a_i^{-1}\tilde{n}_iu_ia_i \in N \cap \tilde{H}$, and $a_i^{-1}v_i a_i \in V^+$. Since $g_iv_ih_i$ converges as $i \to \infty$ and the product map $A \times (N \cap \tilde{H}) \times V^+ \to G$ is an injective proper map, it follows that all three sequences $\tilde{a}_ia_i, a_i^{-1}\tilde{n}_iu_ia_i$ and $a_i^{-1}v_i a_i$ converge as $i \to \infty$. Noting that

$$g_iu_ia_i = \tilde{a}_i\tilde{n}_iu_ia_i = \tilde{a}_i\tilde{n}_ia_i(a_i^{-1}\tilde{n}_iu_ia_i),$$

it remains to set $p_i := u_ia_i \in AU$ to finish the proof. □

For $z \in BF M$, $\pi(z\tilde{H}V^+\tilde{H}) = \pi(z\tilde{H}\tilde{V}^+)$ is the closure of a Fuchsian end, of the form $\mathcal{S}_0 \times [0, \infty)$ where $\mathcal{S}_0 = \pi(z\tilde{H})$.

Lemma 11.4. Let $z \in BF M$. Let $zL$ be a closed orbit contained in $BF M$ for some $L \in L_U$ contained in $\tilde{H}$, and $V^+ \subset N$ be a one-parameter semigroup such that $\tilde{H}V^+ = \tilde{H}\tilde{V}^+$. Then both $zLV^+H(U)$ and $zLV^+$ are closed.
Proof. Without loss of generality, we assume \( z = [e] \). Let \( B \) denote the component of \( \Omega \) such that \( \partial B = \pi(\tilde{H}) \) for the projection map \( \pi : G \to \mathbb{H}^d \). Since \( \tilde{H}V^+ = \tilde{H}\mathbb{V}^+ \), we have \( \pi(\tilde{H}V^+\tilde{H}) = \text{hull}(\tilde{B}) \). Note that if \( \gamma((\text{hull}(B)) \cap \text{hull}(B) \neq \emptyset \) for \( \gamma \in \Gamma \), then \( \gamma \in \tilde{H} \cap \Gamma = \text{Stab}(B) \).

Suppose that \( \gamma_i \ell_i v_i h_i \to g \) in \( G \) where \( \gamma_i \in \Gamma \), \( \ell_i \in L \), \( v_i \in V^+ \) and \( h_i \in H(U) \). Since \( \pi(\gamma_i \ell_i v_i h_i) \in \Gamma \text{ hull}(\tilde{B}) \), and \( \Gamma \text{ hull}(\tilde{B}) \) is a closed subset of \( \mathbb{H}^d \), we have \( \pi(g) \in \Gamma \text{ hull}(\tilde{B}) \). Without loss of generality, we may assume \( \pi(g) \in \text{hull}(\tilde{B}) \) by replacing \( \gamma_i \) by \( \gamma \gamma_i \) for some \( \gamma \in \Gamma \) if necessary.

We claim that by passing to a subsequence, \( \gamma_i \in \tilde{H} \cap \Gamma \).

Let \( \mathcal{O} \) be a neighborhood of \( \pi(g) \) such that
\[
\mathcal{O} \cap \Gamma \text{ hull}(\tilde{B}) \subset \text{hull}(\tilde{B});
\]
such \( \mathcal{O} \) exists since \( d((\text{hull}(\gamma B), \text{hull}(B)) \geq \eta \) for all \( \gamma \in \Gamma - (\tilde{H} \cap \Gamma) \) where \( \eta > 0 \) is given in (4.9). By passing to a subsequence, we may assume \( \pi(\gamma_i \ell_i v_i h_i) \in \mathcal{O} \). Since \( \pi(\ell_i v_i h_i) \in \text{hull}(\tilde{B}) \) for all \( i \), it follows that \( \pi(\gamma_i \ell_i v_i h_i) \in \text{hull}(\tilde{B}) \) for all \( n \). Therefore \( \gamma_i \in \tilde{H} \cap \Gamma \).

Applying Lemma 11.3 to the sequence \( (\gamma_i \ell_i v_i h_i) \to g \), there exists \( p_i \in AU \) such that \( \gamma_i \ell_i p_i \to h \) in \( \tilde{H} \) as \( i \to \infty \). Since \( \Gamma L \) is closed, we have \( h \in \Gamma L \).

Since \( p_i^{-1} v_i h_i \in AUV^+ H(U) = V^+ H(U) \) and
\[
\lim_{i \to \infty} p_i^{-1} v_i h_i = h^{-1} g,
\]
we have \( h^{-1} g \in V^+ H(U) \) by Lemma 11.2. Therefore, \( g = h(h^{-1} g) \in \Gamma L V^+ H(U) \). This proves that \( \Gamma L V^+ H(U) \) is closed.

Note that in the above argument, if \( h_i = e \) for all \( i \), then \( h^{-1} g = \lim p_i^{-1} v_i \in AUV^+ \). Hence \( g = h(h^{-1} g) \in \Gamma L AUV^+ = \Gamma L V^+ \). This proves that \( \Gamma L V^+ \) is closed. \( \square \)

**Theorem 11.5.** Let \( x \in RF_+ M - RF M \cdot H(U) \). Then there exist a compact orbit \( zL \subset BF M \) with \( L \in \mathcal{L}_U \), an element \( c \in C(H(U)) \) and a one-parameter semigroup \( V^+ \subset N \) with \( \tilde{H} V^+ = \tilde{H} \mathbb{V}^+ \) such that
\[
\begin{align*}
(1) & \quad xH(U) = zL V^+ H(U) c; \\
(2) & \quad xA U = zL V^+ c.
\end{align*}
\]
Moreover the closure of the geodesic plane \( \pi(xH(U)) \) is diffeomorphic to a properly immersed submanifold \( S \times [0,\infty) \) where \( S = \pi(zL) \) is a compact geodesic plane inside \( BF M \).

**Proof.** The condition on \( x \) implies that \( x \in (RF_+ M \cap \partial F_{H(U)}) - BF M \cdot C(H(U)) \). We write \( x = z_0 v c \) for some non-trivial \( v \in \mathbb{V}^+ \), \( z_0 \in BF M \) and \( c \in C(H(U)) \). Without loss of generality, we may assume \( c = e \). By Theorem 11.1, \( z_0 U = z_0 v_0^{-1} L v_0 \) where \( L \in \mathcal{L}_U \) is contained in \( \tilde{H} \) and \( v_0 \in \tilde{H} \cap N \). Hence \( xH(U) \) contains \( zL(v_0 v)H(U) \) for \( z := z_0 v_0^{-1} \in BF M \). Set \( V^+ := \{ \exp(t \log(v_0)) : t \geq 0 \} \).
Note that $V^+$ is contained in $A(v_0v)A \cup \{e\}$, and hence
$$zL \cup zL v_0 v H(U) = zL V^+ H(U)$$
and $\hat{H}V^+ = \hat{H}V$ since $v \neq e$.

Since $\overline{xH(U) z} \subset zL v_0 v H(U)$, and $zL$ lies in the closure of $zL v_0 v H(U)$, the claim (1) follows since $zL V^+ H(U)$ is closed by Lemma 11.4. For the claim (2), note that $\overline{x A U} \subset \overline{z_0 U v A} = zL V^+$. By Lemma 11.4, $zL V^+$ is $AU$-invariant and closed. Since $x \in zL V^+$, we conclude $\overline{x A U} = zL V^+$.

To see the last claim, observe that $\pi(zL V^+ H(U)) = \pi(zL V^+ A U) = \pi(zL V^+)$ since $V^+ A U = A U V^+$, and $A U < L$. Since $\hat{H}V^+ = \hat{H}V^+$, and $\pi(zL)$ is a compact geodesic plane (without boundary) in $\pi(zH)$, we get $\pi(z\hat{H} V^+) \simeq \pi(zH) \times [0, \infty)$ and $\pi(zL V^+) \simeq \pi(zL) \times [0, \infty)$.

**Remark 11.6.** An immediate consequence of Theorem 11.5 is that if $P \subset M$ is a geodesic plane such that $P \cap core M = \emptyset$ but $\overline{P} \cap core M \neq \emptyset$, then $P$ is not properly immersed in $M$ and $\overline{P}$ is a properly immersed submanifold with non-empty boundary.

### 12. Density of almost all $U$-orbits

Let $\Gamma < G = \text{SO}^0(d, 1)$ be a Zariski dense convex cocompact subgroup. The action of $N$ on $RF_+ M$ is minimal, and hence any $N$-orbit is dense in $RF_+ M$ [49]. Given a non-trivial connected closed subgroup $U$ of $N$, there exists a dense $U$-orbit in $RF_+ M$ [30]. In this section, we deduce from [31] and [30] that almost every $U$-orbit is dense in $RF_+ M$ with respect to the Burger-Roblin measure in the case of a convex cocompact hyperbolic manifold with Fuchsian ends (Corollary 12.4).

The critical exponent $\delta = \delta_\Gamma$ of $\Gamma$ is defined to be the infimum $s \geq 0$ such that the Poincare series $\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma(o))}$ converges for any $o \in \mathbb{H}^d$. It is known that $\delta$ is equal to the Hausdorff dimension of the limit set $\Lambda$ and $\delta = d - 1$ if and only if $\Gamma$ is a lattice in $G$ [46].

Denote by $m^{BR}_N$ the $N$-invariant Burger-Roblin measure supported on $RF_+ M$; it is characterized as a unique locally finite Borel measure supported on $RF_+ M$ (up to a scaling) by ([6], [41], [49]). We won’t give an explicit formula of this measure as we will only use the fact that its support is equal to $RF_+ M$, together with the following theorem: recall that a locally finite $U$-invariant measure $\mu$ is ergodic if every $U$-invariant measurable subset has either zero measure or zero co-measure, and is conservative if for any measurable subset $S$ with positive measure, $\int_U 1_S(xu) du = \infty$ for $\mu$-almost all $x$, where $du$ denotes the Haar measure on $U$.

**Theorem 12.1** ([31], [30]). Let $U < N$ be a connected closed subgroup, and let $\Gamma$ be a convex cocompact Zariski dense subgroup of $G$. Then $m^{BR}_N$ is $U$-ergodic and conservative if $\delta > \text{co-dim}_N(U)$.

**Lemma 12.2.** Suppose that $\Gamma_1 < \Gamma_2$ are convex cocompact subgroups of $G$ with $[\Gamma_1 : \Gamma_2] = \infty$. Then $\delta_{\Gamma_1} < \delta_{\Gamma_2}$.
Proof. Note that a convex cocompact subgroup is of divergent type ([46], [41]). Hence the claim follows from [8, Proposition 9] if we check that \( \Lambda_{\Gamma_1} \neq \Lambda_{\Gamma_2} \).

If \( \Lambda := \Lambda_{\Gamma_1} = \Lambda_{\Gamma_2} \), then their convex hulls are the same, and hence the convex core of the manifold \( \Gamma_1 \setminus H^d \) is equal to \( \Gamma_2 \setminus \text{hull}(\Lambda) \), which is compact. Since we have a covering map \( \Gamma_1 \setminus \text{hull}(\Lambda) \to \Gamma_2 \setminus \text{hull}(\Lambda) \), it follows that \( [\Gamma_1 : \Gamma_2] < \infty \). \( \square \)

Lemma 12.3. If \( \Gamma \setminus H^d \) is a convex cocompact hyperbolic manifold with Fuchsian ends, then \( \delta > \frac{d}{2} \).

Proof. If \( \Gamma \) is a lattice, then \( \Lambda = S^d \) and \( \delta = d - 1 \). If \( \Gamma \setminus H^d \) is a convex cocompact hyperbolic manifold with non-empty Fuchsian ends, then \( \Gamma \) contains a cocompact lattice \( \Gamma_0 \) in a conjugate of \( \text{SO}(d-1,1) \) whose limit set is \( \partial B_i \) for some \( i \). Now \( [\Gamma : \Gamma_0] = \infty \); otherwise, \( \Lambda = \partial B_i \). Hence \( \delta > \delta_{\Gamma_0} = d - 2 \) by Lemma 12.2. \( \square \)

Corollary 12.4. Let \( M = \Gamma \setminus H^d \) be a convex cocompact hyperbolic manifold with Fuchsian ends. Let \( U < N \) be any non-trivial connected closed subgroup. Then for \( m^{BR} \)-almost every \( x \in RF_+ M \),

\[ xU = RF_+ M. \]

Proof. Without loss of generality, we may assume that \( U = \{u_t\} \) is a one-parameter subgroup. By Lemma 12.3 and Theorem 12.1, \( m^{BR} \) is \( U \)-ergodic and conservative. Since \( \delta > (d-1)/2 \), there exists a unique function \( \phi_0 \in L^2(M) \) which is an eigenfunction for the Laplace operator with eigenvalue \( \delta(d-1-\delta) \), up to a scalar multiple [46]. Moreover \( \phi_0 \) is positive. We may regard \( \phi_0 \) as a function on \( L^2(\Gamma \setminus G) \) which is \( K \)-invariant. Then \( m^{BR}(\phi_0) = ||\phi_0||^2 < \infty \) (cf. [16, Lem 6.7]). Hence, applying the Hopf ratio theorem [1] we get that for almost all \( x \in RF_+ M \) and for any continuous function \( f \) on \( RF_+ M \) with compact support,

\[ \lim_{T \to \infty} \frac{\int_0^T f(xu_t)dt}{\int_0^T \phi_0(xu_t)dt} = \frac{m^{BR}(f)}{||\phi_0||^2}. \]

Therefore almost all \( U \)-orbits are dense in \( \text{supp}(m^{BR}) = RF_+ M \). \( \square \)

13. Horospherical action in the presence of a compact factor

Let \( M = \Gamma \setminus H^d \) be a convex cocompact hyperbolic manifold with Fuchsian ends and fix a non-trivial connected closed subgroup \( U \) of \( N \). Consider a closed orbit \( xL \) for \( x \in RF_+ M \) where \( L \in \mathcal{Q}_U \). The subgroup \( U = L \cap N \) is a horospherical subgroup of \( L \), which is known to act minimally on \( xL \cap RF_+ M \) provided \( L = L_{nc} \). In this section, we extend the \( U \)-minimality on \( xL \) in the case when \( L \) has a compact factor.

Theorem 13.1. Let \( X := xL \) be a closed orbit where \( x \in RF_+ M \), and \( L \in \mathcal{Q}_U \). Let \( U := L \cap N \). Then the following holds:
(1) $X \cap RF_+ M$ is $U$-minimal.
(2) $X$ is $L_{\text{ac}}$-minimal.
(3) If $L \in \mathcal{L}_U$ and $x \in RF M$, then $X \cap RF M$ contains a dense $A$-orbit.
(4) For any non-trivial connected closed subgroup $U_0 < U$, for $m^{BR}_X$-almost all $x \in X$,

$$xU_0 = X \cap RF_+ M.$$ 

The subgroup $L \in \mathcal{Q}_U$ is of the form $v^{-1}H(U)Cv$ where $H(U)C \in \mathcal{L}_U$ and $v \in N$. A general case can be easily reduced to the case where $L \in \mathcal{L}_U$.

In the following, we assume $L = H(U)C \in \mathcal{L}_U$. As before, we set

$$H = H(U), \ H' = H'(U), \ \text{and} \ \ F^* = F^*_{H(U)}$$

and let $\pi_1 : H' \to H$ and $\pi_2 : H' \to C(H)$ be the canonical projections.

In order to define $m^{BR}_X$, choose $g \in G$ so that $[g] = x$. If we identify $H \simeq \text{SO}^+(k,1)$, then by Proposition 4.9, $X := \pi_1(g^{-1}Tg \cap HC) \setminus \mathbb{H}^k$ is a convex cocompact hyperbolic manifold with Fuchsian ends. Now $\pi_1(g^{-1}Tg \cap HC) \setminus H$ is the frame bundle of $S$, on which there exists the Burger-Roblin measure as discussed in section 12. In the above statement, the notation $m^{BR}_X$ means the $C$-invariant lift of this measure to $X = xHC$.

We first prove the following, which is a more concrete version of Proposition 10.6 in the case at hand:

**Proposition 13.2.** Let $X$ be as in Theorem 13.1. Any $U$-minimal set $Y$ of $X$ with respect to $RF M$ such that $Y \cap RF M \cap F^* \neq \emptyset$ is $A$-invariant.

**Proof.** Let $Y$ be a $U$-minimal set of $X$ with respect to $RF M$. Let $y_0 \in Y \cap RF M \cap F^*$. By Lemma 10.4, there exists a sequence $g_i \to e$ in $HC \setminus N(U)$ such that $y_0g_i \in Y \cap RF M$ for all $i \geq 1$.

Since $U$ is a horospherical subgroup of $H$ and $C$ commutes with $H$, we can apply Lemma 9.2 to the sequence $g_i^{-1}$ and the sequence of $k$-thick sets $T_i := \{u \in U : y_0g_iu \in Y \cap RF M\}$ of $U$. This gives us sequences $u_{t_i} \to \infty$ in $T_i$ and $u_i \in U$ such that as $i \to \infty$,

$$u_{t_i}^{-1}g_iu_i \to a$$

for some non-trivial element $a \in A$. Since $y_0u_{t_i}$ converges to some $y_i \in Y \cap RF M$ by passing to a subsequence, we have

$$y_1a = \lim(y_0u_{t_i})(u_{t_i}^{-1}g_iu_i) \in Y.$$ 

Since $y_1U = Y$, we get $Ya \subset Y$. Since $a$ can be made arbitrarily close to $e$ by Lemma 9.2, there exists a subgroup $A_+ \subset A$ such that $YA_+ \subset Y$ by Lemma 10.5. Moreover, for any $a \in A_+$, $Ya \cap RF M \neq \emptyset$ as $RF M$ is $A$-invariant. Therefore, $Ya = Y$. It follows that $Ya^{-1} = Y$ as well. Hence $Y$ is $A$-invariant. □

We now present:
Proof of Theorem 13.1. First suppose that \( xL \cap F^* \neq \emptyset \). We may then assume \( x \in F^* \cap RF M \). Let \( Y \) be a \( U \)-minimal set of \( X \) with respect to \( RF M \). If \( Y \) were contained in \( \partial F \), then \( Y \subset \partial F \cap RF M \). Since \( Stab_L(x) \) is Zariski dense in \( L \) by the definition of \( L_U \), it follows from [5, Lemma 4.13] that \( X \cap RF_+ M \) is \( AU \)-minimal. Therefore we have \( YA = X \cap RF_+ M \) and hence \( X \) has to be contained in the closed \( A \)-invariant subset \( \partial F \cap RF M \) as well, yielding a contradiction. Therefore, \( Y \cap RF M \cap F^* \neq \emptyset \). Hence, by Proposition 13.2, \( Y \) is \( A \)-invariant. Therefore the claim (1) follows from the \( AU \)-minimality of \( X \cap RF_+ M \) if \( x \in F^* \). Now suppose \( xL \subset \partial F \). In this case, it suffices to consider the case when \( U \) is a proper subgroup of \( N \); otherwise \( L = G \) and has no compact factor. Hence we may assume without loss of generality that \( U \subset \hat{H} \cap N \). As \( xL \) is closed, Theorem 11.5 implies that \( xL \subset BF M \cdot C(H(U)) \). Hence by modifying \( x \) by an element of \( C(H(U)) \), we may assume that \( X \) is contained in a compact homogeneous space of \( \hat{H} = SO^0(d-1,1) \), which is the frame bundle of a convex cocompact hyperbolic manifold with empty Fuchsian ends. Therefore the claim (1) follows from the previous case of \( x \in F^* \), since \( F^* = RF M \) in the finite volume case.

Claim (2) follows from (1) since \( RF_+ M H \) is closed, and \( X \subset RF_+ M H \).

For the claim (3), it suffices to show that the \( A \)-action on \( X \cap RF M \) is topologically transitive (cf. [7]). Let \( x, y \in X \cap RF M \) be arbitrary, and \( O, O' \) be open neighborhoods of \( e \) in \( H \). The set \( UU^t A(M \cap H) \) is a Zariski open neighborhood of \( e \) in \( H \) where \( U^t \) is the expanding horospherical subgroup of \( H \) for the action of \( A \). Choose an open neighborhood \( Q_0 \) of \( e \) in \( U \), and an open neighborhood \( P_0 \) of \( e \) in \( U^t A(M \cap H) \) such that \( Q_0P_0 \subset O \).

We claim that \( xQ_0A \cap yO' \neq \emptyset \), which implies \( xO A \cap yO' \neq \emptyset \). Suppose that this is not true. Then

\[
xQ_0A \subset \Gamma \backslash G - yO'
\]

where the latter is a closed set. Now, choose a sequence \( a_n \in A \) such that \( a_nQ_0a_n^{-1} \to U \) as \( n \to \infty \), and observe

\[
xa_n^{-1}(a_nQ_0a_n^{-1}) = xQ_0a_n^{-1} \subset \Gamma \backslash G - yO'.
\]

Passing to a subsequence, \( xa_n^{-1} \to x_0 \) for some \( x_0 \in RF M \), and we obtain that \( x_0U \) is contained in the closed subset \( \Gamma \backslash G - yO' \). This contradicts the \( U \)-minimality of \( X \cap RF_+ M \), which is claim (1). This proves (3).

For the claim (4), note that by Corollary 12.4, almost all \( \bar{U}_0 \)-orbits in \( \pi_1(g^{-1}\Gamma g \cap HC) \backslash H \) are dense in the corresponding \( RF_+ M \)-set. It follows that for almost all \( x \), the closure \( \bar{xU_0} \) contains a \( U \)-orbit of \( X \). Hence (4) follows from the claim (1).
14. Orbit closure theorems: beginning of the induction

In the rest of the paper, let \( M = \Gamma \setminus \mathbb{H}^d \) be a convex cocompact hyperbolic \( d \)-manifold with Fuchsian ends, and \( G = \text{SO}^o(d, 1) \). Let \( U < N \) be a non-trivial connected proper closed subgroup, and \( H(U) \) be its associated simple Lie subgroup of \( G \).

Let \( \mathcal{L}_U \) and \( \mathcal{Q}_U \) be as defined in (5.7) and (5.8). The remainder of the paper is devoted to the proof of Theorem 1.2 which we restate as follows:

**Theorem 14.1.**

1. For any \( x \in \text{RF}^M \), 
   \[
   \overline{xH(U)} = xL \cap \text{RF}_M
   \]
   where \( xL \) is a closed orbit of some \( L \in \mathcal{L}_U \).

2. Let \( x_0 \hat{L} \) be a closed orbit for some \( \hat{L} \in \mathcal{L}_U \) and \( x_0 \in \text{RF}^M \).
   
   (a) For any \( x \in x_0 \hat{L} \cap \text{RF}_M \), 
   \[
   \overline{xU} = xL \cap \text{RF}_M
   \]
   where \( xL \) is a closed orbit of some \( L \in \mathcal{Q}_U \).

   (b) For any \( x \in x_0 \hat{L} \cap \text{RF}_M \), 
   \[
   \overline{xAU} = xL \cap \text{RF}_M
   \]
   where \( xL \) is a closed orbit of some \( L \in \mathcal{L}_U \).

3. Let \( x_0 \hat{L} \) be a closed orbit for some \( \hat{L} \in \mathcal{L}_U \) and \( x_0 \in \text{RF}^M \). Let \( y_iL_i v_i \subset x_0 \hat{L} \) be a sequence of maximal closed orbits such that \( y_iL_i v_i \cap \text{RF}_M \neq \emptyset \), where \( y_i \in \text{RF}_M \), \( L_i \in \mathcal{L}_U \), and \( v_i \in \hat{L} \cap N \). Suppose either
   - \( v_i \to \infty \) modulo \( L_i \) or
   - \( v_i \) is bounded modulo \( L_i \) and \( y_iL_i \) are all distinct,

   Then
   \[
   \lim_{i \to \infty} (y_iL_i v_i \cap \text{RF}_M) = x_0 \hat{L} \cap \text{RF}_M
   \]
   where the limit is taken in the Hausdorff topology on the space of all closed subsets in \( \Gamma \setminus G \).

We will prove (1), (2), and (3) of Theorem 14.1 by induction on the co-dimension of \( U \) in \( N \) and the co-dimension of \( U \) in \( \hat{L} \cap N \), respectively.

For simplicity, let us say \((1)_m\) holds, if (1) is true for all \( U \) satisfying co-dim\(_N(U) \leq m \). We will call \((2)_m\) (resp. \((2.a)_m\), \((2.b)_m\)) holds, if (2) (resp. \((a)\) of (2), \((b)\) of (2)) is true for all \( U \) and \( \hat{L} \) satisfying co-dim\(_{\hat{L} \cap N}(U) \leq m \) and similarly for \((3)_m\).

**Base case of** \( m = 0 \). Note that the bases cases \((1)_0\), and \((3)_0\) are trivial, and that \((2)_0\) follows from Theorem 13.1.

We will deduce \((1)_{m+1}\) from \((2)_m\) and \((3)_m\) in section 16, and \((2)_{m+1}\) from \((1)_{m+1}\), \((2)_m\), and \((3)_m\) in section 17, and finally deduce \((3)_{m+1}\) from \((1)_{m+1}\), \((2)_{m+1}\) and \((3)_m\) in section 18.
Remark 14.2. When co-dim_{L\cap N}(U) \geq 1 and \hat{L} \in L_U, we may assume without loss of generality that
\[ U \subset \hat{L} \cap N \cap \hat{H} \]
by replacing U and \hat{L} by their conjugates using an element \( m \in M \).

Remark 14.3. In the case when \( x \in \partial F_H(U) \), Theorem 14.1 (1) and (2) follow from Theorem 11.1, and if \( x_0 \in \partial F_H(U) \), (3) follows from the work of Mozes-Shah [30]. So the main new cases of Theorem 14.1 are when \( x, x_0 \in F^*_H(U) \).

We will use following observation:

**Singular U-orbits under the induction hypothesis.** Recall the notation \( \mathcal{S}(U, x\hat{L}) \) and \( \mathcal{G}(U, x\hat{L}) \) from (5.6).

**Lemma 14.4.** Suppose that (2.a)\(_m\) is true and that for \( x \in RF_M \), \( xU \) is contained in a closed orbit \( x\hat{L} \) for some \( \hat{L} \in L_U \).

1. If co-dim_{L\cap N}(U) \leq m + 1, then for any \( x_0 \in \mathcal{S}(U, x\hat{L}) \cap RF_+ M \),
   \[ \overline{x_0U} = x_0L \cap RF_+ M \]
   where \( x_0L \) is a closed orbit of some subgroup \( L < \hat{L} \) contained in \( Q_U \), satisfying \( \dim L_{nc} < \dim \hat{L}_{nc} \).

2. If co-dim_{L\cap N}(U) \leq m, then for any \( x_0 \in \mathcal{G}(U, x\hat{L}) \),
   \[ \overline{x_0U} = x_0\hat{L} \cap RF_+ M. \]

**Proof.** Suppose that co-dim_{L\cap N}(U) \leq m + 1 and that \( x_0 \in \mathcal{S}(U, x\hat{L}) \cap RF_+ M \). By Proposition 5.13, we get
\[ \overline{x_0U} \subset x_0Q \]
for some closed orbit \( x_0Q \) where \( Q \in Q_U \) satisfies \( \dim Q_{nc} < \dim \hat{L}_{nc} \).

Now \( Q = vL_0v^{-1} \) for some \( L_0 \in L_U \) and \( v \in U^+ \). We have \( x_0Uv = x_0vU \subset x_0vL_0 \). Since co-dim_{N\cap L_0}(U) = co-dim_{N\cap Q}(U) \leq m, by applying (2)\(_m\), we get
\[ \overline{x_0vU} = x_0vL \cap RF_+ M \]
for some closed orbit \( x_0vL \) where \( L \in Q_U \) is contained in \( L_0 \). Therefore
\[ \overline{x_0U} = x_0vLv^{-1} \cap RF_+ M. \]
As \( vLv^{-1} \in Q_U \) and \( \dim L_{nc} \leq \dim Q_{nc} < \dim \hat{L}_{nc} \), the claim (1) is proved.

To prove (2), note that by (2)\(_m\), we get \( \overline{x_0U} = x_0L \cap RF_+ M \) for some closed orbit \( x_0L \) with \( L \in Q_U \) such that \( L \subset \hat{L} \). Since \( x_0 \in \mathcal{G}(U, x\hat{L}) \), we have \( \dim L_{nc} = \dim \hat{L}_{nc} \).

Since \( L \subset \hat{L} \), \( L \cap N \) is a horospherical subgroup of \( \hat{L} \). By Theorem 13.1, \( L \cap N \) acts minimally on \( x\hat{L} \), and hence \( L = \hat{L} \).  \( \square \)
15. Generic points and additional invariance

The primary goal of this section is to prove Propositions 15.1 and 15.2 in obtaining additional invariances using a sequence converging to a generic point of an intermediate closed orbit; the main ingredient is Theorem 7.16 (Avoidance theorem II). The results in this section are main tools in the enlargement steps of the proof of Theorem 14.1.

In this section, we let \( U < N \) be a non-trivial connected closed subgroup. We suppose that

- \( (2)_m \) and \( (3)_m \) are true;
- \( x \hat{L} \) is a closed orbit for some \( x \in RF M \), and \( \hat{L} \in L_U \);
- \( \text{co-dim}_{\hat{L} \cap N}(U) \leq m + 1 \).

We let \( \{U^i\} \) be a collection of one-parameter subgroups generating \( U \).

In the next two propositions, we let \( X \) be a closed \( U \)-invariant subset of \( x_0 \hat{L} \) such that

\[ X \supset xL \cap RF_+ M \]

for some closed orbit \( xL \) where \( L \in Q_U \) is a proper subgroup of \( \hat{L} \) and \( x \in \bigcap_i \mathcal{G}(U^i, xL) \cap RF M \).

Proposition 15.1 (Additional invariance I). Suppose that there exists a sequence \( x_i \to x_0 \) in \( X \) where \( x_i = x \ell_ir_i \) with \( x \ell_i \in x_0L \cap RF M \) and \( r_i \in \exp t_{-1} - N(U) \).

Then there exists a sequence \( v_j \to \infty \) in \( (L \cap N)^\perp \) such that

\[ xLv_j \cap RF_+ M \subset X. \]

Proof. Since \( r_i \notin N(U) \), we can fix a one-parameter subgroup \( U_0 = \{u_t : t \in \mathbb{R}\} \) in the family \( \{U^i\} \) such that \( r_i \notin N(U_0) \) by passing to a subsequence.

Let \( E_j \), \( j \in \mathbb{N} \), be a sequence of compact subsets in \( \mathcal{G}(U_0, xL) \cap RF M \) given by Theorem 7.16. Set \( z_i := x \ell_i \in xL \cap RF M \). Fix \( j \in \mathbb{N} \) and \( n \gg 1 \). Since \( z_i \to x \) and \( x \in \mathcal{G}(U_0, xL) \), there exist \( i_j \geq 1 \) and an open neighborhood \( O_j \) of \( E_j \) such that for each \( i \geq i_j \), the set

\[ T_i = \{t \in \mathbb{R} : z_iu_t \in RF M - O_j\}, \]

is \( 2k \)-thick by loc. cit. We apply Lemma 9.3 to the sequence \( T_i \). We can find a sequence \( t_i = t_i(n) \in T_i \), \( i \geq i_j \) and elements \( y_j = y_j(n), v_j = v_j(n) \) satisfying that as \( i \to \infty \),

- \( z_iu_t \to y_j \in (RF M \cap xL) - O_j \),
- \( u_{t_i}^{-1}r_iz_iu_{t_i} \to v_j \in (L \cap N)^\perp \) with \( n \leq \|v_j\| \leq (2k^2)n \).

So as \( i \to \infty \),

\[ x_iu_{t_i} = z_ii_ri_iu_{t_i} \to y_jv_j \text{ in } X. \]

Note that since \( L \) is a proper subgroup of \( \hat{L} \), we have \( \text{co-dim}_{L \cap N}(U) \leq m \) by Lemma 5.11.
Hence the claim follows if \( y_j(n) \in \mathcal{G}(U, xL) \) for an infinite subsequence of \( n \).

Now we may suppose that for all \( n \geq 1 \) and \( j \geq 1 \), \( y_j(n) \in \mathcal{G}(U, xL) \cap RF_+ M \), after passing to a subsequence. Fix \( n \), and set \( y_j = y_j(n) \) and \( v_j = v_j(n) \). Then by (2)_m, we have

\[
\overline{y_j U} = y_j L_j \cap RF_+ M
\]

(15.1)

for some closed \( y_j L_j \) where \( L_j \in \mathcal{Q}_U \) is contained in \( \hat{L} \) and \( \dim(L_j)_nc < \dim \hat{L}_nc \).

Write \( L_j = w_j^{-1}L'_j w_j \) for \( L'_j \in \mathcal{L}_U \) and \( w_j \in U^\perp \). We claim that the sequence \( y_j L_j = y_j w_j^{-1}L'_j w_j \) satisfies the hypothesis of (3)_m. If not, by passing to a subsequence, \( y_j w_j^{-1}L'_j \) is a constant sequence, say, \( \Gamma gL \) for \( [g] \in RF M \) and \( L \in \mathcal{L}_U \), and \( w_j \) is bounded modulo \( L \). Now \( H := g L^{-1} \in \mathcal{H} \), and since \( w_j \in N, U = w_j U w_j^{-1} \subset L \). So all \( g w_j \) are contained in a compact subset of \( X(H, U) \), say \( D \). Therefore for each \( j \),

\[
y_j L_j = \Gamma \setminus \Gamma H gw_j \subset \Gamma \setminus \Gamma H D \in \mathcal{E}_U.
\]

Hence all \( y_j L_j \) must be contained in some \( E_{j_0} \), which is a contradiction since \( y_j \notin \mathcal{O}_j \) and hence \( y_j \notin E_j \) for each \( j \in \mathbb{N} \).

Since \( y_j \in RF M \), by (3)_m, we have

\[
\limsup_j y_j L_j \cap RF_+ M = xL \cap RF_+ M.
\]

Therefore for each fixed \( n \gg 1 \) and \( y_j = y_j(n) \),

\[
\limsup_j \overline{y_j U} = xL \cap RF_+ M.
\]

By passing to a subsequence, there exists \( u_j \in U \) such that \( y_j u_j \) converges to \( x \). As \( n \leq \|v_j(n)\| \leq (2k^2)n \), the sequence \( v_j(n) \) converges to some \( v_n \in (L \cap N)^\perp \) as \( j \to \infty \), after passing to a subsequence. Therefore

\[
\limsup_j \overline{y_j(n) v_j(n) U} = \limsup_j \overline{y_j(n) U v_j(n)} \supset xU v_n = xL v_n \cap RF_+ M
\]

where the last equality follows from Lemma 14.4(2), since \( \co-dim_{L \cap N}(U) \leq m \). 

Note that in the above proposition, \( y_i = x \ell_i r_i \) is not necessarily in \( RF M \), and hence we cannot apply the avoidance theorem 7.16 to the sequence \( y_i \) directly. We instead applied it to the sequence \( x \ell_i \).

In the proposition below, we will consider a sequence \( x_i \to y \) inside \( RF M \), and apply Theorem 7.16 to the sequence \( x_i \).
Proposition 15.2 (Additional invariance II). Suppose that there exists a sequence \( x_i \in X \cap RF M \cdot xL \cdot N(\U) \), converging to \( x \) as \( i \to \infty \). Then there exists a sequence \( v_j \to \infty \) in \((N \cap L)\perp\) such that
\[
xLv_j \cap RF_+ M \subset X \quad \text{and} \quad xLv_j \cap RF M \neq \emptyset.
\]
The same works for a sequence \( x_i \in RF M \cdot xL \cdot N(\U) \) such that \( \limsup x_i U \subset X \).

Proof. Let \( x_i \in RF M \cdot xL \cdot N(\U) \) be a sequence converging to \( x \) such that \( \limsup x_i U \subset X \). Write \( x_i = xg_i \) for \( g_i \to e \) in \( \hat{L} \). Since \( L \) is reductive, we can write \( g_i = \ell_i r_i \) where \( \ell_i \to e \) in \( L \) and \( r_i \to e \) in \( \exp L\perp \) as \( i \to \infty \). By the assumption on \( x_i \), there exists a one-parameter subgroup \( U_0 = \{ u_t : t \in \mathbb{R} \} \) among \( U(4) \) such that \( r_i \notin N(U_0) \) by passing to a subsequence.

For \( R > 0 \), we set \( B(R) := \{ v \in (L \cap N)\perp \cap \hat{L} : \| v \| \leq R \} \). Fix \( j \) and \( n \in \mathbb{N} \). Let \( E_j, O_j \) be given by Theorem 7.16 for \( xL \) with respect to \( U_0 \). Then \( E_j \) is of the form
\[
E_j = \bigcup_{i \in \Lambda_j} \Gamma \cap \Gamma H_i D_i \cap RF M
\]
where \( H_i \in \mathcal{H}^* \) satisfies \( \dim(H_i)_{\text{nc}} < \dim L_{\text{nc}} \) and \( D_i \) is a compact subset of \( X(H_i, U_0) \cap L \).

As \( B(2k^2n) \subset C(U_0) \), we have \( D_j^* := D_j B(2k^2n) \) is a compact subset of \( X(H_i, U_0) \). Hence the following set
\[
\tilde{E}_j := \bigcup_{i \in \Lambda_j} \Gamma \cap \Gamma H_i D_i^* \cap RF M
\]
belongs to \( \mathcal{E}_{U_0} \) and is associated to the family \( \{ H_i : i \in \Lambda_j \} \), as defined in (7.3).

Let \( \tilde{E}_j' \in \mathcal{E}_{U_0} \) be a compact subset given by Theorem 7.13, which is also associated to the same family \( \{ H_i : i \in \Lambda_j \} \). Note that for any \( z \in \tilde{E}_j' \), the closure \( \overline{\mathcal{U}_0} \) is contained in \( \Gamma \cap \Gamma H_i D_i^* \) for some \( i \in \Lambda_j \). In particular, \( \tilde{E}_j' \) is a compact subset disjoint from \( \mathcal{F}(U_0, xL) \). Since \( x_i \to x \) and \( x \in \mathcal{F}(U_0, xL) \), there exists \( i_j \geq 1 \) such that \( x_i \notin \tilde{E}_j' \) for all \( i \geq i_j \). By Theorem 7.13, there exists a neighborhood \( \tilde{O}_j \) of \( \tilde{E}_j \) such that for each \( i \geq i_j \), the set
\[
T_i = \{ t \in \mathbb{R} : x_i u_t \in RF M - \tilde{O}_j \}
\]
is \( 2k \)-thick.

Applying Lemma 9.3 to \( T_i \), and \( r_i \to e \), we can find \( t_i = t_i(n) \in T_j \) such that \( u_{t_i}^{-1} r_i u_{t_i} \to v_j \) for some \( v_j = v_j(n) \in (L \cap N)\perp \), with \( n \leq \| v_j \| \leq 2k^2 \cdot n \).

Passing to a subsequence, \( x_i u_{t_i} \) converges to some \( x_j = x_j(n) \in RF M - \tilde{O}_j \) as \( i \to \infty \). Set
\[
z_i := x\ell_i, \quad \text{and} \quad O_j := \tilde{O}_j B(2k^2n) \cap xL.
\]
Since \( x_i u_{t_i} = z_i u_{t_i} (u_{t_i}^{-1} r_i u_{t_i}) \), we have
\[
z_i u_{t_i} \to y_j \in (RF_+ M \cap xL) - O_j
\]
where $y_j = y_j(n) := x_j v_j^{-1}$.

We check that $E_j \subset O_j$ as $B(2k^2 n) B(2k^2 n)$ contains $e$. It follows that

$$y_j \notin E_j.$$ 

Note that $x_j U = y_j U v_j \subset X$, and hence $y_j U v_j \cap RF_M \neq \emptyset$. Since $v_j = v_j(n) \in B(2k^2 n)$ are bounded for all $j$ (fixed), it follows that $\operatorname{lim sup}(y_j U \cap RF_O M) \neq \emptyset$.

Given these, we can now repeat verbatim the proof of Proposition 15.1 to complete the proof. □

Theorem 2.2 in the introduction can be proved similarly to the proof of Proposition 15.1.

**Proof of Theorem 2.2** Let $E_j$, $j \in \mathbb{N}$, be a sequence of compact subsets of $\mathcal{S}(U_0) \cap RF_M$ given by Theorem 7.16. Fix $j \in \mathbb{N}$. Then there exist $i_j \geq 1$ and a neighborhood $O_j$ of $E_j$ such that

$$\{t \in \mathbb{R} : x_i u_t \in RF_M - O_j \}$$

is $2k$-thick for all $i \geq i_j$. Hence we can find a sequence $t_i \in [-2k T_i, -T_i] \cup [T_i, 2k T_i]$ such that $x_i u_{t_i} \in RF_M - O_j$ for all $i \geq i_j$. Hence, by passing to a subsequence, $x_i u_{t_i}$ converges to some $y_j \in RF_M - O_j$ as $i \to \infty$. If $y_j \in \mathcal{G}(U)$ for some $j$, then (2)m and Lemma 14.4(1) imply that $y_j U = RF_M$, proving the claim.

Now, we assume that $y_j \in \mathcal{S}(U, x \hat{L})$ for all $j$. Then by (2)m and Lemma 14.4(1), we have

$$\overline{y_j U} = y_j L_j \cap RF_+ M$$

for some closed $y_j L_j$ where $L_j \in \mathcal{Q}_U$ is a proper subgroup of $G$.

Similarly to the proof of Theorem 15.1, we can show that the sequence $y_j L_j$ satisfies the hypothesis of $(3)_m$. Hence, by applying $(3)_m$ to the sequence $y_j L_j$, we get

$$\operatorname{lim sup}(y_j L_j \cap RF_+ M) = RF_+ M.$$ 

Therefore $\operatorname{lim sup} y_j U = \operatorname{lim sup} \overline{y_j U} = RF_+ M$. This, together with Theorem 13.1(4), finishes the proof.

**16. $H(U)$-orbit closures: proof of $(1)_{m+1}$**

In the rest of the paper, we fix a non-trivial connected proper subgroup $U < N$. Without loss of generality, we may assume

$$U < N \cap \hat{H}$$

using a conjugation by an element of $M$. We set

$$H = H(U), \ H' = H'(U), \ F = F_{H(U)}, \ F^* = F^*_{H(U)}, \ and \ \partial F = \partial F_{H(U)}.$$ 

Thanks to the assumption $U < N \cap \hat{H}$, we get

$$\partial F \cap RF_M = BF M \cdot C(H).$$

We will be using the following observation:
Lemma 16.1. For \( i = 1, 2 \), let \( x_iL_i \) be a closed orbit where \( L_1 \in Q_U \), \( L_2 \in L_U \) and \( x_i \in RF M \). If \( x_1L_1 \cap RF M \subset x_2L_2 \), then \( L_1 \subset L_2 \) and \( x_1L_1 \subset x_2L_2 \).

Proof. By Theorem 13.1, we can assume that \( x_1U = x_1L_1 \cap RF_+ M \). Since \( L_2 \) contains \( H \), we get that \( x_1L_1 \cap RF M \cdot H \subset x_2L_2 \).

Suppose that \( x_1L_1 \cap F^* \neq \emptyset \). We may assume \( x_1 \in F^* \). Since \( F^* \subset RF MH \), we have \( x_1L_1 \cap F^* \subset x_2L_2 \). Since \( F^* \) is open, there exist \( g_1, g_2 \in G \) such that \([g_i] = x_i\), and \( g_1L_1 \cap O \subset g_2L_2 \) for some open neighborhood \( O \) of \( g_1 \). It follows that \( L_1 \cap g_1^{-1}O \subset g_1^{-1}g_2L_2 \). Since \( e \in g_1^{-1}g_2L_2 \), \( g_1^{-1}g_2L_2 = L_2 \). Since \( L_1 \) is generated by \( L_1 \cap g_1^{-1}O \), we deduce \( L_1 \subset L_2 \). Since \( x_1L_1 \cap x_2L_2 \neq \emptyset \), it follows that \( x_1L_1 \subset x_2L_2 \).

Now consider the case when \( x_1L_1 \cap F^* = \emptyset \). In this case, \( x_1L_1 \cap RF M \subset RF M \cap \partial F \). As \( x_1 \) is contained in \( BF M \cdot C(H) \), so is \( x_1U \). Therefore \( x_1L_1 \subset RF M \), and hence the claim follows.

The following proposition says that the classification of \( H' \)-orbit closures yields the classification of \( H \)-orbit closures:

Proposition 16.2. Let \( x \in RF M \), and assume that there exists \( \tilde{U} < N \) containing \( U \) such that \( xH'(\tilde{U}) \) is closed, and

\[
\overline{xH'} = xH(\tilde{U}) \cdot C(H) \cap F.
\]

Then there exists a closed subgroup \( C < C(H(\tilde{U})) \) such that

\[
\overline{xH'} = xH(\tilde{U})C \subset F.
\]

Proof. By Proposition 4.9 and Theorem 13.1(2), there exists a closed subgroup \( C < C(H(\tilde{U})) \) such that \( X := xH(\tilde{U})C \) is a closed \( H(\tilde{U}) \)-minimal subset. In particular, \( \overline{xH'} \subset X \cap F \). Now, by Theorem 13.1(3), there exists \( y \in X \) such that \( yA = X \cap RF M \). Since \( C \subset C(H(\tilde{U})) \subset C(H) \) and

\[
\overline{xH'} \cdot C(H) = \overline{xH'} = xH(\tilde{U}) \cdot C(H) \cap F,
\]

there exists \( c_0 \in C(H) \) such that \( yc_0 \in \overline{xH'} \). Since \( yA c_0 = yc_0A \subset \overline{xH'} \) and \( c_0 \in C(H) \), it follows \( Xc_0 \cap RF M \subset \overline{xH'} \subset X \). Applying Lemma 16.1, we get \( Xc_0 = \overline{xH'} = X \).

In the rest of this section, fix \( m \in \mathbb{N} \cup \{0\} \) and assume that

\[
1 \leq \text{co-dim}_N(U) = m + 1.
\]

In order to study the closure of \( xH(U) \), we assume

\[
x \in F^* \cap RF M
\]

in view of Theorem 11.1.

By Proposition 16.2, it suffices to show that

\[
(16.1) \quad \overline{xH'} = xLC(H) \cap F
\]

for some closed orbit \( xL \) of \( L \in L_U \).
In the rest of this section, we set \( X = \overline{xH'} \) and assume that \( xH' \) is not closed, i.e., \( X \neq xH' \).

The following proposition says that \( xH' \) is not closed in \( F^* \) either.

**Proposition 16.3.** Let \( x \in F^* \cap RF M \). If \( xH' \) is closed in \( F^* \), then it is closed in \( \Gamma \backslash G \).

**Proof.** Suppose that \( xH' \) is closed as a subset of \( F^* \), but not closed in \( \Gamma \backslash G \). Then \( \overline{xH'} \supset \overline{yH'} \) for some \( y \in \partial F \cap RF_+ M \). We note that \( \overline{yA} \) contains an \( RF M \) point by Lemma 4.2. Hence, \( \overline{xH'} \) contains \( z \in BF M \); so there exists a sequence \( w_n \in H' \) such that \( xw_n \to z \). Choose \( g_0, g \in G \) so that \( z = [g_0] \) and \( x = [g] \). Then there exists \( \gamma_n \in \Gamma \) such that \( \gamma_ngw_n \to g_0 \). Let \( B \) be a component of \( \Omega \) whose convex hull is \( \pi(g_0 \hat{H}) \). Then the sequence of spheres \( C_n := \partial \pi(\gamma_ngH') \) converges to the sphere \( C := \partial \pi(g_0H') \) contained in \( \partial B \).

Let \( B' \) be an open round ball such that \( S^{d-1}_\Gamma \) is a cocompact lattice in \( Stab_G(B') \simeq \hat{H} \). Next, choose a compact fundamental domain \( K \subset B' \) for the action of \( \Gamma B' \). Since \( x \in F^* \), \( C_n \) meets \( B' \). So there exists \( \delta_n \in \Gamma B' \) such that \( \delta_nC_n \cap K \neq \emptyset \). Choose \( p_n \in \delta_nC_n \cap K \) for each \( n \), and passing to a subsequence, we may assume that \( p_n \) converges to some \( p \in K \) as \( n \to \infty \).

Since \( C_n \) converges to \( C \) and \( C \subset \partial B' \) as \( n \to \infty \), considering \( B' \) as a hyperbolic ball model, the geodesic curvatures of \( \delta_nC_n \cap B' \) must tend to 1 as \( n \to \infty \). It follows that \( \delta_nC_n \) converges to a sphere \( D \) contained in \( B' \) which is tangent at a point in \( \partial B' \) and passes through \( p \). Note that \( D \) does not lie in \( \Gamma C \); because every circle in \( \Gamma C \) intersects \( \partial B' \) in at least two points, and \( \#D \cap \partial B' = 1 \). It follows that if \( g_1 \in G \) is such that \( D \) is the boundary of \( \pi(g_1H') \), then \( [g_1] \in F^* \) (cf. (4.4)) and \( [g_1] \in \overline{xH'} - xH' \). This contradicts the assumption that \( xH' \) is closed in \( F^* \). \( \square \)

By Proposition 16.3, \( xH' \) is not closed in \( F^* \) and in particular,

\[
(16.2) \quad (X - xH') \cap (F^* \cap RF M) \neq \emptyset.
\]

Roughly speaking, our strategy in proving (1)_{m+1} is first to find a closed \( L \)-orbit \( x_0L \) such that \( x_0L \cap F \) is contained in \( X \) for some \( L \in \mathcal{L}_U \).

If \( X \neq x_0L \cap F \), then we enlarge \( x_0L \) to a bigger closed orbit \( x_1\hat{L} \) for some \( \hat{L} \in \mathcal{L}_G \) for some \( \hat{U} \) properly containing \( U \), such that \( x_1\hat{L} \cap F \) is contained in \( X \).

It is in the enlargement step where the additional invariance Proposition 15.1 is a crucial ingredient of the arguments. In order to find a sequence \( x_i \) accumulating on a generic point of \( x_0L \) satisfying the hypothesis of the proposition, we find a closed orbit \( x_0L \) with a base point \( x_0 \) in \( F^* \), and enlarge it to a bigger closed orbit, again based at a point in \( F^* \). The advantage of having a closed orbit \( xL \) with \( x \in F^* \) is that any \( U_0 \)-generic point in \( xL \cap RF M \) can be approximated by a sequence of \( RF M \)-points in \( F^* \cap xL \) by Lemma 8.3. The enlargement process must end after finitely many steps because of the dimension reasons.
Finding a closed orbit of $L \in \mathcal{L}_U$ in $X$.

Lemma 16.4 (Moving from $\mathcal{Q}_U$ to $\mathcal{L}_U$). Assume that (2)$_m$ is true. If $x_0L \cap RF_+ M \subset X$ for some closed orbit $x_0L$ with $x_0 \in RF M$, and $L \in \mathcal{Q}_U$, then
\[ x_1\tilde{L} \cap RF_+ M \subset X \]
for some closed orbit $x_1\tilde{L}$ with $x_1 \in RF M$, and $\tilde{L} \in \mathcal{L}_U$ with $\dim \hat{L}_{nc} \geq \dim L_{nc}$.

Proof. We may assume that $L_{nc}$ has the maximal dimension among those $L \in \mathcal{Q}_U$ such that $x_0L$ is closed and $x_0L \cap RF_+ M \subset X$ for some $x_0 \in RF M$. By Lemma 5.9, we can write $L = v^{-1}\tilde{L}v$ for some $\tilde{L} \in \mathcal{L}_U$ and $v \in (L \cap N)^\perp$. We claim that $v = e$, which would finish the proof. Suppose that $v \neq e$. We have $L_{nc} = H(\tilde{L})$ for some $\tilde{L} < N$. Note that $x_0v^{-1}UAv \subset x_0L \cap RF_+ M$, as $U \subset \tilde{L}$, and hence $x_0v^{-1}UAv \subset X$ from the hypothesis together with the fact that $X$ is $A$-invariant.

Let $V^+$ be a one-parameter subsemigroup in $\hat{U}^+ \cap AvA$ containing $v$. Then we have $x_0v^{-1}V^+ \tilde{U} \subset X$ as well. Let $V$ be the one-parameter subgroup containing $V^+$. Since $v^{-1}V^+$ is an open neighborhood of $e$ in $V$, there exists $a_n \in A$ such that $a_n^{-1}v^{-1}V^+ a_n \to V$ as $n \to \infty$. Note that
\[ x_0a_n(a_n^{-1}v^{-1}V^+ a_n) = x_0v^{-1}U^+ a_n \subset X. \]
Since $x_0a_n \in RF M$ and $RF M$ is compact, the sequence $x_0a_n$ converges to some $x_1 \in RF M$ as $n \to \infty$ after passing to a subsequence. As a result, we obtain $x_1\tilde{V} \subset X$.

Since co-$\dim_N(\tilde{U}V) \leq m$, by (2)$_m$, there exists $\tilde{L} \in \mathcal{Q}_{\tilde{U}V}$ such that $x_1\tilde{L}$ is closed and $x_1\tilde{L} \cap RF_+ M \subset X$, which is a contradiction to the maximality assumption on $L_{nc} = H(\tilde{L})$. \qed

Given the induction hypothesis, the next proposition tells us that $x\overline{H}'$ contains a closed orbit of a subgroup in $\mathcal{L}_U$ based at a point of $F^*$, whenever $xH'$ is not closed.

Proposition 16.5. If (2)$_m$ is true, then there exists a closed orbit $x_0L$ with $x_0 \in RF M \cap F^*$ and $L \in \mathcal{L}_U$ such that
\[ x_0L \cap RF_+ M \subset X. \]

Proof. Let $R := X \cap RF M \cap F^*$. We divide the proof into two cases depending on the compactness of $R$.

Case 1: $R$ is compact. Thanks to (2)$_m$ and Lemma 16.4, it is enough to show that $X$ contains an orbit $z\hat{U}$ for some $\hat{U}$ properly containing $U$ and $z \in R$. By Proposition 10.9, it suffices to find a $U$-minimal subset $Y \subset X$ with respect to $R$ and a point $y \in Y \cap R$ such that $X - yH'$ is not closed; this implies that $yg_n \in X$ for some $g_n \to e$ in $G - H'$.

If $xH'$ is not locally closed, then take any $U$-minimal subset $Y$ of $X$ with respect to $R$. If $Y \cap R \subset xH'$, then choose any $y \in Y \cap R$. Then
$X - yH' = X - xH'$ cannot be closed, as $xH'$ is not locally closed. If $Y \cap R \notin xH'$, then choose $y \in (Y \cap R) - xH'$. Then $X - yH'$ contains $xH'$ and hence cannot be closed.

If $xH'$ is locally closed, then $X - xH'$ is a closed $H'$-invariant subset which intersects $R$ non-trivially. So we can take a $U$-minimal subset $Y \subset X - xH'$ with respect to $R$. Take any $y \in Y \cap R$. Then $X - yH'$ is not closed.

**Case 2**: $R$ is non-compact. In this case, $R$ accumulates on $\partial F \cap RF M = BF M \cdot C(H)$. Hence, there exists $x_n \in R$ and $z \in BF M \cdot C(H)$ such that $x_n \to z$ as $n \to \infty$. We may assume $z \in BF M$ without loss of generality, since $R$ is $C(H)$-invariant. We claim that $X$ contains $z_1 v$ where $z_1 \in BF M$ and $v$ is a non-trivial element in $V = (\hat{H} \cap N)^\perp$. Write $x_n = zh_n r_n$ for some $h_n \in \hat{H}$ and $r_n \in \exp \hat{h}^\perp$, where $\hat{h}^\perp$ denotes the $\text{Ad}(\hat{H})$-complementary subspace to $\text{Lie}(\hat{H})$ in $g$. Since $x_n \in F^*$ and $z \in BF M$, it follows that $r_n \notin C(H(U))$ for all large $n$. By (3.2) and (3.5), we have

$$N(U) \cap \exp \hat{h}^\perp \subset \hat{V} C(H).$$

Therefore, if $r_n \in N(U)$ for some $n$, then the $\hat{V}$-component of $r_n$ should be non-trivial, and hence the claim follows.

Now suppose that $r_n \notin N(U)$ for all $n$. Then there exists a one-parameter subgroup $U_0 = \{u_t\} \subset U$ such that $r_n \notin N(U_0)$. Applying Lemma 9.3, with a sequence of $k$-thick subsets

$$T(x_n) := \{t \in \mathbb{R} : x_n u_t \in RF M\},$$

we get a sequence $t_n \in T(x_n)$ such that $u_{t_n}^{-1} r_n u_{t_n}$ converges to a non-trivial element $v \in \hat{V}$.

Since $zh_n u_{t_n} \in z\hat{H}$ and $z\hat{H}$ is compact, the sequence $zh_n u_{t_n}$ converges to some $z_1 \in z\hat{H}$, after passing to a subsequence. Then

$$z_1 v = \lim (zh_n u_{t_n})(u_{t_n}^{-1} r_n u_{t_n}) \in X \cap RF M,$$  

proving the claim.

Now by Theorem 11.1, $z_2 u = z_1 (\alpha^{-1} L \alpha)$ for some $L \in \mathcal{L}_U$ and $\alpha \in (L \cap N)^\perp \cap \hat{H}$. In particular, $z_2 u$ contains $z_1 \alpha^{-1} A \alpha$.

Setting $z_2 := z_2 \alpha^{-1} \in z_2 \hat{H}$, we get from $X \supset z_2 \hat{V} v A$ that

$$X \supset z_2 (A v A).$$

Now, $A v A$ contains a one-parameter semigroup $V^+$ not contained in $\hat{H}$. By (16.3), we may apply Lemma 8.8. Hence there exists $v_n \to \infty$ in $V^+$ such that $z_2 v_n$ converges to some $y \in F^*$. Since

$$(z_2 v_n)(v_n^{-1} V^+) = z_2 V^+ \subset X,$$

we get $y AUV \subset X$. By (2), $y AUV = y \hat{L}$ for some $\hat{L} \in \mathcal{L}_{UV}$. This finishes the proof.
Enlarging a closed orbit of \( L \in \mathcal{L}_U \) in \( X \). The goal of this subsection is to prove the following proposition 16.10 which says that if \( X = X_{HF} \) contains \( x_0L \cap F \) properly for some \( L \in \mathcal{L}_U \) and \( x_0 \in F^* \), then it contains a closed orbit of \( L \in \mathcal{L}_U \) where \( \dim \bar{U} > \dim (L \cap N) \).

**Lemma 16.6.** Let \( x_nL_nv_n \) be a sequence of closed orbits with \( x_n \in RF_+M \), \( L_n \in \mathcal{L}_U \) and \( v_n \in (L_n \cap N)^\perp \). Suppose either of the following:

- \( x_n \in F^* \) for all \( n \);
- \( x_nL_nv_n \cap RF_+M \cap F^* \neq \emptyset \) for all \( n \).

Then

\[
F^* \cap \limsup_n(x_nL_nv_n \cap RF_+M) \neq \emptyset.
\]

**Proof.** We claim that \( x_nL_nv_n \cap RF_+M \) intersects \( F^* \) for all large \( n \). We only need to consider the case when \( x_n \in F^* \) for all \( n \). Suppose not. By passing to a subsequence, we then have \( (x_nL_nv_n \cap RF_+M) \subseteq RF_+M - F^* \) for all \( n \). Since the set \( RF_+M - F^* \) is a closed \( A \)-invariant set and \( e \in Av_nA \), we would have \( x_n \in RF_+M - F^* \), a contradiction.

There exists \( z_n \in x_nL_n \cap RF_+M \) such that \( \pi(z_nv_nU) \not\subseteq \partial \text{core}(M) \) for all large \( n \). In particular, there exists \( u_n \in U \) such that \( z_nv_nu_n \in \text{core}_{n/3}(M) \) (cf. the proof of Lemma 8.8). Since \( \text{core}_{n/3}(M) \) is a compact subset of \( F^* \), the claim follows.

**Lemma 16.7.** Let \( x_0L \) be a closed orbit with \( x_0 \in RFM \) and \( L \in \mathcal{L}_U \).

Suppose that \( X \) contains a closed \( U \)-invariant subset containing \( x_0Lv_n \cap RF_+M \) for some sequence \( v_n \to \infty \) in \( (L \cap N)^\perp \). If \( x_0 \in F^* \) or \( x_0Lv_n \cap RF_+M \cap F^* \neq \emptyset \) for all \( n \), then there exist \( y \in RFM \cap F^* \) and a one parameter subgroup \( \bar{V} \subset (L \cap N)^\perp \) such that

\[
X \supset y(L \cap N)V.
\]

**Proof.** Note that

\[
(x_0Lv_n \cap RF_+M)(v_n^{-1}Av_n) \subset X.
\]

By Lemma 16.6, there exists

\[
y \in F^* \cap \limsup_{n \to \infty}(x_0Lv_n \cap RF_+M).
\]

Since \( y \in F^* \cap RF_+M \subseteq RFM \cdot U \), we may assume \( y \in F^* \cap RFM \) by modifying \( y \) using an element of \( U \).

Note that \( \liminf_{n \to \infty}(x_0Lv_n \cap RF_+M) \supset y(L \cap N) \). Since \( \liminf_{n \to \infty}(v_n^{-1}Av_n) \) contains a one-parameter subgroup \( V \subset (L \cap N)^\perp \), passing to a limit, by Lemma 3.4, we obtain that \( y(L \cap N)V \subset X \).

**Lemma 16.8.** If \( yLv_0 \cap RFM \cap F^* \neq \emptyset \) for some \( v_0 \in N \) and \( L \in \mathcal{L}_U \), then \( yLv \cap RFM \cap F^* \neq \emptyset \) for all \( v \in Av_0A \).

**Proof.** Let \( y_0 := yLv_0 \in yLv_0 \cap RFM \cap F^* \), and \( v = av_0b \in Av_0A \). Then \( (y\alpha^{-1})v = yLv_0b \in RFM \cap F^* \) as \( RFM \cap F^* \) is \( A \)-invariant. Since \( y\alpha^{-1}v \in yLv \), the claim is proved.
Lemma 16.9. Let \( x_0L \) be a closed orbit with \( x_0 \in RF M \) and \( L \in \mathcal{L}_U \). Suppose that \( X \) is a closed \( AU \)-invariant subset containing \( x_0L \cap RF M \) for some non-trivial element \( v \in (L \cap N)^\perp \). If \( x_0 \in F^* \) or \( x_0L \cap RF M \cap F^* \neq \emptyset \), then there exist \( y \in RF M \cap F^* \) and a one parameter subgroup \( V \subset (L \cap N)^\perp \) such that
\[
X \supseteq y(L \cap N)V.
\]

Proof. Since \( X \) is \( A \)-invariant, we get
\[
(x_0L \cap RF M)AvA \subset X.
\]
Choose a sequence \( v_n := a_nva_n^{-1} \in AvA \) tending to \( \infty \). Note that either \( x_0 \in F^* \) or for all \( n \), \( x_0L \cap RF M \cap F^* \neq \emptyset \) by Lemma 16.8. Therefore the claim follows from Lemma 16.7.

Proposition 16.10. Assume that \((2)_m \) and \((3)_m \) are true. Suppose that there exists a closed orbit \( x_0L \) for some \( x_0 \in RF M \cap F^* \) and \( L \in \mathcal{L}_U \) such that
\[
(x_0L \cap RF M) \subset X \text{ and } X \neq x_0L \cdot C(H) \cap F.
\]
Then there exists a closed orbit \( x_1 \tilde{L} \) for some \( x_1 \in RF M \cap F^* \), and \( \tilde{L} \in \mathcal{L}_{\tilde{U}} \) for some \( \tilde{U} < N \) with \( \dim \tilde{U} > \dim (L \cap N) \) such that
\[
x_1 \tilde{L} \cap RF M \subset X.
\]

Proof. Note that if \( X \subset x_0L \cdot C(H) \), then \( X = x_0L \cdot C(H) \cap F \). Therefore we assume that \( X \not\subset x_0L \cdot C(H) \). First note that the hypothesis implies that \( L \neq G \), and hence co-dim_{\mathcal{L}_U}(U) \leq m \). Let \( U_{(1)}^+, \ldots, U_{(\ell)}^+ \) be one-parameter subgroups generating \( U \). Similarly, let \( U_{+}^{(1)}, \ldots, U_{+}^{(\ell)} \) be one-parameter subgroups generating \( U^+ \). By Theorem 13.1,
\[
\bigcap_{i=1}^{\ell} \mathcal{G}(U_{+}^{(i)}, x_0L) \neq \emptyset.
\]
Therefore without loss of generality, we can assume \( x_0 \in \bigcap_{i=1}^{\ell} \mathcal{G}(U_{+}^{(i)}, x_0L) \).

Let us write \( L = H(\hat{U})C \) for some \( \hat{U} < N \) and a closed subgroup \( C \) of \( C(H(\hat{U})) \). Note from the hypothesis that we have
\[
(x_0L \cap RF M) \cdot H' \subset X.
\]
Observe that (16.4) implies that \( x \not\in x_0L \cdot H' = x_0L \cdot C(H) \). Since \( C < C(H) \), we have \( x \not\in x_0H(\hat{U}) \).

Now choose a sequence \( w_i \in H' \) such that \( xw_i \to x_0 \), as \( i \to \infty \). Write \( xw_i = x_0g_i \) where \( g_i \to e \) in \( G - LH' \). Let us write \( g_i = \ell_ir_i \) where \( \ell_i \in L \), and \( r_i \in \exp H' \). In particular, \( r_i \not\in C(H) \).

Let \( x_i = x_0\ell_i \), so that \( x_i r_i \in X \).
We now break the remaining proof into several steps.

Step 1. We can assume that \( x_i \in RF M \cap x_0L, r_i \not\in C(H) \), and \( x_i r_i \in X \).
Since \( x_0 \in F^* \), by Lemma 8.3, we can find \( w'_i \to w' \in H \) such that \( x_0 \ell_i w'_i \in RF M \), and \( x_0 w' \in \bigcap_{i=1}^{\ell} \mathcal{G}(U_{\pm}^{(i)}, x_0 L) \); hence

\[
\overline{x_0 w' U} = x_0 L \cap RF_+ M.
\]

Writing \( x'_i = x_0 \ell_i w'_i \) and \( r'_i = w'^{-1}_i r_i w'_i \), we have

\[
x'_i r'_i = x w_i w'_i \in X,
\]

where \( x'_i \to x_0 w' \) in \( x_0 L \cap RF M \), and \( r'_i \to e \) in \( \exp \perp \). Since \( F^* \) is \( H' \)-invariant, we have \( x_0 w' \in F^* \). Since \( F^* \) is open and \( x_0 w' \in F^* \), it follows that \( x'_i \in X \cap RF M \cap F^* \) for sufficiently large \( i \).

Note that \( r'_i \not\in C(H) \), as \( r_i \not\in C(H) \). This proves the first step.

**Step 2.** There exists a one-parameter subgroup \( U_0 \) among \( U_{\pm}^{(i)} \), \( i = 1, \ldots, \ell \) such that \( r_i \not\in N(U_0) \) for all \( i \)'s after passing to a subsequence.

This follows from Lemma 3.5, up to switching the roles of \( U \) and \( U^+ \).

**Step 3.** We claim that there exists a non-trivial element \( v \in \hat{U} \perp \) such that

\[
x_0 L v \cap RF_+ M \subset X.
\]

Note that \( x_0 \in \mathcal{G}(U_0, x_0 L) \) and \( x_i \to x_0 \) in \( RF M \cap x_0 L \). As we are assuming that \((2)_m\), and \((3)_m\) hold, we may apply Proposition 15.1 to the sequence \( x_0 \ell_i r_i \to x_0 \) to deduce the claim.

**Step 4.** We claim that there exist \( x_2 \in RF M \cap F^* \) and a connected closed subgroup \( \hat{U} < N \) properly containing \( L \cap N \) such that

\[
x_2 \hat{U} \subset X.
\]

This follows from Lemma 16.9, since \( x_0 \in F^* \cap RF M \).

Since \( \text{co-dim}_N(\hat{U}) \leq m \), the last claim of Step (4) together with \((2)_m\) finishes the proof of the proposition. \( \square \)

**Proof of \((1)_{m+1}\).** Combining Propositions 16.5 and 16.10, we now prove:

**Theorem 16.11.** If \((2)_m\) and \((3)_m\) are true, then \((1)_{m+1}\) is true.

**Proof.** Recall that we only need to consider the case \( X = \overline{\{xH\}} \) where \( x \in F^* \) and \( xH' \) is not closed in \( F^* \). By Proposition 16.5, there exists \( x_0 \in F^* \) and \( L \in \mathcal{L}_U \) such that \( x_0 L \) is closed and

\[
x_0 L \cap RF_+ M \subset X.
\]

Since \( X \) is \( H' \)-invariant, it follows

\[
(16.5) \quad (x_0 L \cap RF_+ M) \cdot H' \subset X.
\]

Note that \((x_0 L \cap RF_+ M) \cdot H' = x_0 L \cdot C(H) \cap F\) is a closed set. We may assume the inclusion in \((16.5)\) is proper, otherwise we have nothing further to prove. Then by Proposition 16.10, there exists \( \hat{L} \in \mathcal{L}_{\hat{U}} \) for some \( \hat{U} < N \) properly containing \( L \cap N \), and a closed orbit \( x_1 \hat{L} \) with \( x_1 \in F^* \cap RF M \) such that \( x_1 \hat{L} \cap RF_+ M \subset X \).
If
\[(x_1 \hat{L} \cap RF^+ M) \cdot C(H) \neq X,\]
then we can apply Proposition 16.10 on
\[x_1 \hat{L} \cap RF^+ M \subset X,\]
as \(L_U \subset L_U\). Continuing in this fashion, the process terminates in a finite step for a dimension reason, and hence
\[X = (x_1 \hat{L} \cap RF^+ M) \cdot H' = x_1 \hat{L} \cdot C(H) \cap F\]
for some \(\hat{L} \in L_U\), completing the proof. \(\Box\)

17. \(U\) AND \(AU\)-ORBIT CLOSURES: PROOF OF \((2)_{m+1}\)

In this section, we fix a closed orbit \(x_0 \hat{L}\) for \(x_0 \in F^+\) and \(\hat{L} \in Q_U\). Let \(U \subset \hat{L} \cap N\) be a connected closed subgroup with co-dim \(L_U \cap N U \leq m + 1\).

Since \(v \hat{L}v^{-1} \in L_U\) for some \(v \in N(U)\), by replacing \(x\) by \(xv\), we may assume without loss of generality that \(\hat{L} \in L_U\). Moreover, by replacing \(U\) and \(L\) by their conjugates using an element \(m \in M\), we may assume that \(U \subset \hat{L} \cap \tilde{H} \cap N\).

If \(x \in RF^+ M \cap \partial F\) (resp. if \(x \in RF M \cap \partial F\)), then \((2.a)\) (resp. \((2.b)\)) follows from Theorem 11.1.

In the rest of this section, we fix \(x \in RF M \cap x_0 \hat{L} \cap F^+\), and set
\[(17.1) \quad X := xU\]
and assume that \(X \neq x_0 \hat{L} \cap RF^+ M\).

**Lemma 17.1.** Assume that \((1)_{m+1}\) and \((2)_{m}\) hold. Then
\[xU \cap \mathcal{J}(U, x_0 \hat{L}) \neq \emptyset.\]

**Proof.** **Case 1:** \(\hat{L} = G\). Note that \(\mathcal{J}(U) \neq \emptyset\), as it contains a compact \(H\)-orbit, say, \(z \hat{H}\). Since \((1)_{m+1}\) is true, we have
\[xH = xQ \cap F\]
for some \(Q \in L_U\) such that \(xQ\) is closed. If \(Q = G\), i.e., if \(xH = F\), then \(xH = xAU(K \cap H) = F\), since \(H = AU(K \cap H)\). In particular, \(xH\) contains \(z\). Since \(K \cap H \subset \hat{H}\), it follows that \(xAU\) intersects \(z \hat{H}\), proving the claim. If \(Q \neq G\), then \(xQ \subset \mathcal{J}(U)\).

**Case 2:** \(\hat{L} \neq G\). In this case, co-dim \(L_U \cap N\) \(U \leq m\). Hence by \((2)_{m}\), \(xAU = xQ \cap RF^+ M\) where \(xQ\) is closed for some \(Q < \hat{L}\) contained in \(L_U\). By the hypothesis \((17.1)\) and Lemma 5.11, dim \(Q_{nc}\) \(<\) dim \(\hat{L}_{nc}\). Therefore \(x \in \mathcal{I}(U, x_0 \hat{L})\). \(\Box\)

**Lemma 17.2.** Assume that \((1)_{m+1}\) and \((2)_{m}\) hold. Then
\[xU \cap \mathcal{J}(U, x_0 \hat{L}) \neq \emptyset.\]
Assume that Lemma 17.3.

Proof. Since (17.2) 

\[(x_0 \hat{L} \cap RF_+ M) - F^* \subset \mathcal{I}(U, x_0 \hat{L}),\]

it suffices to consider the case when \(X := \overline{xU} \subset F^*\). Since \(x \in F^*, xU\) meets RF \(M\). Let \(Y \subset X\) be a \(U\)-minimal set with respect to RF \(M\).

Since \(Y \subset F^*\), by Proposition 10.6, there exists an unbounded one-parameter semigroup \(S\) inside \(AU^\perp C_2(U) \cap \hat{L}\) such that \(YS \subset Y\). In view of Lemma 3.3, we could remove \(C_2(U)\)-component of \(S\) so that \(S\) is either of the following:

- \(v^{-1}A^+ v\) for a one-parameter semigroup \(A^+ \subset A\) and \(v \in U^\perp \cap \hat{L}\);
- \(V^+\) for a one-parameter semigroup \(V^+ \subset U^\perp \cap \hat{L}\).

and

\[YS \subset X(C_2(U) \cap \hat{L}).\]

Since \(\mathcal{I}(U, x_0 \hat{L})\) is invariant by \(NC_2(U) \cap \hat{L}\), it suffices to show that

\[X(nc_2(U) \cap \hat{L}) \cap \mathcal{I}(U, x_0 \hat{L}) \neq \emptyset.\]

If \(S = v^{-1}A^+ v\), then \(Yv^{-1}A^+ \subset Xv^{-1}(C_2(U) \cap \hat{L})\). Choose \(y \in Y\). We may assume that \(yv^{-1} \in F^*\) by (17.2). Then, replacing \(y\) with an element in \(yU\) if necessary, we may assume \(yv^{-1} \in RF M \cap F^*\). Choose a sequence \(a_n \rightarrow \infty\) in \(A^+\). Then \(yv^{-1}a_n\) converges to some \(y_0 \in RF M\) by passing to a subsequence. Since \(\lim \sup a_n^{-1}A^+ = A\), and

\[(yv^{-1}a_n)(a_n^{-1}A^+) \subset Xv^{-1}(C_2(U) \cap \hat{L}),\]

we obtain that

\[y_0A \subset Xv^{-1}(C_2(U) \cap \hat{L}).\]

Since \(\overline{y_0AU} \subset Xv^{-1}(C_2(U) \cap \hat{L})\) and \(\overline{y_0AU}\) meets \(\mathcal{I}(U, x_0 \hat{L})\) by Lemma 17.1, the claim follows.

Next, assume that \(S = V^+\), so that \(YV^+ \subset XC_2(U) \cap \hat{L}\). Let \(v_n \rightarrow \infty\) be a sequence in \(V^+\). We have \(Yv_n \subset X \cap F^*\). Together with the fact \(Yv_n\) is \(U\)-invariant, this implies \(Yv_n\) meets RF \(M\). Note that

\[Yv_n(v_n^{-1}V^+) \subset X(C_2(U) \cap \hat{L}).\]

Choose \(y_n \in Yv_n \cap RF M\). As RF \(M\) is compact, \(y_n\) converges to some \(y_0 \in RF M\), by passing to a subsequence, and hence

\[y_0UV \subset X(C_2(U) \cap \hat{L}).\]

Since co-dim\(_N\)(UV) ≤ \(m\), the conclusion follows from (2)\(_m\). \(\square\)

**Lemma 17.3.** Assume that (1)\(_{m+1}\) and (2)\(_m\) hold. Then

\[\overline{xU} \cap \mathcal{I}(U, x_0 \hat{L}) \cap F^* \neq \emptyset.\]

Proof. By Lemma 17.2, there exists \(y \in \overline{xU} \cap \mathcal{I}(U, x_0 \hat{L})\). Hence by (2)\(_m\),

\[\overline{yU} = yL \cap RF_+ M \subset \overline{xU}\]
for some \( L \in Q_U \) properly contained in \( \hat{L} \). Consider the collection of all subgroups \( L \in Q_U \) such that \( yL \subset xU \) for some \( y \in RF_+ M \). Choose \( L \) from this collection such that \( L_{ne} \) has maximal dimension.

If \( yL \cap F^* \neq \emptyset \), then the claim follows.

Now consider the case when \( yL \subset \partial F \). As \( y \in RF_+ M \cap \partial F \), \( y = zv_0c_0 \) for some \( z \in BF M \), \( v_0 \in \hat{V}^+ \) and \( c_0 \in C(H) \).

Since \( y \in xU \), there exists \( u_n \in U \) such that \( xu_n \) converges to \( y \) as \( n \to \infty \). Set

\[
x_n := xu_n c_0^{-1} v_0^{-1}.
\]

Hence \( x_n \to z \). As \( v_0 \in \hat{V}^+ \) and hence \( v_0^{-1} \in \hat{V}^- \), we have \( x_n \in F^* \cap RF_+ M \subset RF M \cdot U \). By Lemma 8.6, we may modify \( x_n \) by elements of \( U \) so that \( x_n \in RF M \) and \( x_n \) converges to some \( z_0 \in z\hat{H} \). Write \( x_n = z_0 \ell_n r_n \) for some \( \ell_n \in \hat{H} \) and \( r_n \in \exp \hat{h}^+ \) converging to \( e \). Since \( x_n \in F^* \), we have \( r_n \neq e \).

By Theorem 11.1, we have \( z_0 \ell_n U = z_0 \ell_n L_n \) for some \( L_n \in Q_U \) contained in \( \hat{H} \).

Suppose that \( r_n \in N(U) \) for some \( n \). Then

\[
\overline{xU} = z_0 \ell_n r_n v_0 c_0 U = z_0 \ell_n U (r_n v_0 c_0) = z_0 \ell_n L_n (r_n v_0 c_0).
\]

As \( \overline{xU} \neq x_0 \hat{L} \) by the hypothesis, it follows that \( x \in \mathcal{J}(U, x_0 \hat{L}) \cap F^* \), proving the claim.

Now assume \( r_n \notin N(U) \) for all \( n \). Then there exists a one-parameter subgroup \( U_0 < U \) such that \( r_n \notin N(U_0) \) for all \( n \), by passing to a subsequence.

By Lemma 9.3, we can find \( u_t \to \infty \) in \( U_0 \) so that \( x_n u_t \in RF M \) and \( u_t^{-1} r_n u_t \) converges to a non-trivial element \( v \in V \), whose size is strictly bigger than \( \|v_0\| \). As \( z_0 \ell_n u_t \) is contained in the compact subset \( z_0 \hat{H} \), we may assume that \( z_0 \ell_n u_t \) converges to some \( z_1 \in z_0 \ell_n L_n \). Hence

\[
x_n u_t = z_0 \ell_n u_t (u_t^{-1} r_n u_t) \to z_1 v \in RF M.
\]

Since \( z_1 \in BF M \) and \( z_1 v \in RF M \), we have \( v \in \hat{V}^- \).

By Theorem 11.1, \( z_1 U = z_1 Q_1 \) for some \( Q_1 \in Q_U \). Hence

\[
\overline{xU} \supset z_1 Q_1 (v v_0) c_0.
\]

Since the size of \( v \) is larger than the size of \( v_0 \), then \( v v_0 \) is a non-trivial element of \( \hat{V}^- \). Since \( z_1 Q_1 \subset BF M \), the closed orbit \( z_1 Q_1 (v v_0) c_0 \) meets \( F^* \). Hence the claim follows.

**Theorem 17.4.** Assume that \((1)_{m+1}, (2)_m, \text{ and } (3)_m \) are true. Then \((2)_{m+1} \) is true.

**Proof.** We first show \((2.a)_{m+1} \) holds. That is, we claim that \( X = xL' \cap RF_+ M \) for some \( L' \in Q_U \) contained in \( \hat{L} \). By Lemma 17.3 and \((2)_m \), there exists a closed orbit \( yL \) with \( y \in F^* \) and \( L \in Q_U \) such that

\[
\overline{xU} \supset yL \cap RF_+ M.
\]
and $L_{nc} \neq \hat{L}_{nc}$. Note that co-dim$_{L \cap N} U \leq m$. It suffices to show that $\overline{xU}$ contains a subset of the form $y_0 \hat{L} \cap \text{RF}_+ M$ for some $\hat{L} \in \mathcal{Q}_U$, where $\hat{U}$ properly contains $U$, and $y_0 \in \text{RF}_+ M \cap F^*$; this process will end after finitely many repetitions.

By Theorem 13.1, we can assume that

$$y \in \bigcap_{i=1}^{\ell} \mathcal{G}(U^{(i)}, yL) \cap \text{RF} M \cap F^*$$

where $U^{(1)}, \ldots, U^{(\ell)}$ are one-parameter subgroups generating $U$. As $y \in \overline{xU}$, there exists $u_i \in U$ such that $xu_i \to y$ as $i \to \infty$. Since $y \in F^*$, we can assume $xu_i \in \text{RF}_+ M$ after possibly modifying $u_i$ by Lemma 8.6. We will write $xu_i = y \ell_i r_i$ where $\ell_i \in L$ and $r_i \in \exp L \cap \hat{L}$.

If $r_i \in N(U)$ for some $i$, then $y \ell_i \in \text{RF}_+ M$ and $X = \overline{xu_i U} = \overline{y \ell_i U r_i}$. Since $y \ell_i U \subset yL$, and co-dim$_{L \cap N}(U) \leq m$, we have

$$X = \overline{y \ell_i U r_i} = \overline{y \ell_i L' r_i} \cap \text{RF}_+ M$$

for some $L' \in \mathcal{Q}_U$, proving the claim.

Therefore we assume that $r_i \notin N(U)$ for all $i$. Hence there exists $U_0$ among $U^{(1)}, \ldots, U^{(\ell)}$ such that $r_i \notin N(U_0)$ for all $i$, by passing to a subsequence. Since $y \in \mathcal{G}(U_0, yL)$ and $xu_i \in X \cap \text{RF} M - yL \cdot N(U_0)$, we can apply Proposition 15.2 to the sequence $xu_i \to y$ and obtain a sequence $v_j \to \infty$ in $(L \cap N)\perp$ such that

$$(yL \cap \text{RF}_+ M)v_j \subset X.$$

Since $y \in F^*$, by Lemma 16.7, there exists a one-parameter subgroup $V \subset (L \cap N)\perp$ such that $y_1(L \cap N)V \subset X$ for some $y_1 \in F^* \cap \text{RF} M$. Hence, the conclusion follows from $(2)_m$.

Now we show $(2.b)_{m+1}$. Consider the closure $\overline{xAU}$.

By $(1)_{m+1}$, we have $\overline{xH} = xL \cap F$ for some $L \in \mathcal{L}_U$ contained in $\hat{L}$. Hence $\overline{xAU} \subset xL \cap \text{RF}_+ M$. It suffices to show that

$$(17.3) \quad \overline{xAU} = xL \cap \text{RF}_+ M.$$

If $U = L \cap N$, then $\overline{xU} = xL \cap \text{RF}_+ M$ by Theorem 13.1, which implies $(17.3)$. So, suppose that $U$ is a proper closed subgroup of $L \cap N$. Since $\overline{xAU}(K \cap H) = \overline{xH} = xL \cap F$, it follows from Lemma 5.14 that we can choose $y \in \overline{xAU} \cap \mathcal{G}(U, xL)$. By $(2.a)_{m+1}$ and Lemma 14.4, we have $\overline{xU} = xL \cap \text{RF}_+ M$, finishing the proof.

18. Topological equidistribution: proof of $(3)_{m+1}$

In this section, we prove $(3)_{m+1}$. Let $U < N$ be a non-trivial connected closed subgroup. Let $x_0 \hat{L}$ be a closed orbit for $x_0 \in \text{RF} M \cap F^*$ and $\hat{L} \in \mathcal{L}_U$ such that co-dim$_{\mathcal{L}^{(U)}_L}(U) = m+1$. As before we may assume that $U \subset \hat{L} \cap H \cap N$.

Let $y_i L_i v_i \subset x_0 \hat{L}$ be a sequence of maximal closed orbits intersecting $\text{RF} M$ where $y_i \in \text{RF}_+ M$, $L_i \in \mathcal{L}_U$, and $v_i \in (L_i \cap N)\perp \cap \hat{L}$. Suppose that
\[ \bullet \ v_i \to \infty, \] or
\[ \bullet \ v_i \text{ is bounded and } y_i L_i \text{ are all distinct.} \]

In order to show \( \lim_{i \to \infty} (y_i L_i v_i \cap RF_+ M) = RF_+ M \), by passing to a subsequence, we may assume that \( y_i L_i v_i \cap RF_+ M \) converges to a closed subset \( E \) in the Hausdorff topology of closed subsets of \( RF_+ M \). Then
\[ E := \limsup_{i \to \infty} (y_i L_i v_i \cap RF_+ M). \]
To prove (3)\(_{m+1}\), it suffices to show that \( E = RF_+ M \).

We note that by (3)\(_{m}\), we may assume that \((L_i)_{nc} = H(U)\) for all \(i\).

This in particular implies that each \( y_i L_i v_i \cap RF_+ M \) is \( U \)-minimal by Theorem 13.1.

**Lemma 18.1.** Assume that (1)\(_{m+1}\), (2)\(_{m+1}\) and (3)\(_{m}\) are true. Then
\[ E \supset yL \cap RF_+ M \]
for some closed orbit \( yL \) where \( y \in RF \cap F^* \) and \( L \in Q_U \) satisfies that \( L \cap N \) contains \( U \) properly.

**Proof.** By (2)\(_{m}\), it suffices to show that there exist \( y_0 \in F^* \) and \( \hat{U} < N \) properly containing \( U \) such that
\[ E \supset y_0 \hat{U}. \]

If \( y_i L_i v_i \in \partial F \), then, since \( y_i L_i v_i \cap RF \neq \emptyset \), \( y_i L_i v_i \) must be equal to \( z_i \hat{H}c \) for some \( z_i \in BF \) and \( c \in C(H) \) by the maximality assumption. By the assumption on \( y_i L_i v_i \)'s, it follows that we may assume that for all \(i\),
\[ y_i L_i v_i \cap RF_+ M \cap F^* \neq \emptyset. \]

Since \( AU < L_i \) for all \(i\), it follows that \( (y_i L_i v_i \cap RF_+ M)(v_i^{-1} AU v_i) \subset E \).
By Lemma 16.6, there exists \( y_0 \in \limsup_i (y_i L_i v_i \cap RF_+ M) \cap F^* \). Hence
\[ y_0 \limsup_i (v_i^{-1} AU v_i) \subset E. \tag{18.1} \]

If \( v_i \to \infty \), then \( \limsup_i (v_i^{-1} AU v_i) \) contains \( \hat{U} \) for some \( \hat{U} \) properly containing \( U \) by Lemma 3.4. Therefore, we get the conclusion \( y_0 \hat{U} \subset E \) from (18.1).

Now suppose that \( v_i \) converges to some \( v \in N \cap \hat{L} \) and that \( y_i L_i \) are all distinct. Then (18.1) gives
\[ y_0 v^{-1} AU v \subset E. \]

Then by (2)\(_{m+1}\), \( y_0 v^{-1} AU \) is of the form \( y_0 v^{-1} L_0 \cap RF_+ M \) for some \( L_0 \in \mathcal{L}_U \). Hence
\[ E \supset y_0 L \cap RF_+ M \tag{18.2} \]
where \( L := v^{-1} L_0 v \). If \( L \cap N \) contains \( U \) properly, this proves the claim. So we suppose that \((L_0)_{nc} = H(U)\).
By Theorem 13.1, we can assume that \( y_0 \in \bigcap_{i=1}^{k} G(U^{(i)}, y_0 L) \cap RF M \cap F^* \), where \( U^{(1)}, \ldots, U^{(t)} \) are one-parameter subgroups generating \( U \). By replacing \( y_i \) by an element of \( y_i L \cap RF_+ M \), we may assume that \( y_i v_i \to y_0 \). Furthermore, as \( y_0 \in F^* \cap RF M \) and for all \( i \) sufficiently large, \( y_i v_i \in F^* \cap RF_+ M \subset RF M \cdot U \) (as \( F^* \) is open), we can also assume \( y_i v_i \in RF M \) by Lemma 8.7. Hence we may write

\[
y_i v_i = y_0 \ell_i r_i
\]

for some \( \ell_i \to e \) in \( L \) and non-trivial \( r_i \to e \) in \( \exp L \).

Suppose that \( r_i \) belongs to \( N(U) \) for infinitely many \( i \). Then

\[
y_i L_i v_i \cap RF_+ M = y_i v_i U \cap RF_+ M = y_0 \ell_i U r_i \cap RF_+ M = y_0 L r_i \cap RF_+ M.
\]

Hence \( y_i L_i v_i r_i^{-1} \cap RF_+ M = y_0 L \cap RF_+ M \). In particular, \( y_i L_i v_i r_i^{-1} \cap RF M \) contains \( y_0 \) and is contained in \( y_0 L \). By Lemma 16.1, this implies that \( y_i L_i v_i \subset y_0 L r_i \). By the maximality of the orbits \( y_i L_i v_i \), we get \( y_i L_i v_i = y_0 L r_i \). As \( r_i \to e \), this contradicts the hypothesis on \( y_i L_i v_i \)'s.

Therefore \( r_i \not\in N(U) \) for all \( i \) but finitely many. We may now apply Proposition 15.2 and Lemma 16.7 to deduce that \( E \) contains an orbit \( z_0 U \) for some unipotent subgroup \( \hat{U} \) containing \( U \) properly and for some \( z_0 \in RF_+ M \) such that \( z_0 \in F^* \). This proves the claim. \( \square \)

**Theorem 18.2.** If \( (1)_{m+1}, (2)_{m+1}, \) and \( (3)_{m} \) are true, then \( (3)_{m+1} \) is true.

**Proof.** We claim that

\[
(18.3) \quad x_0 \hat{L} \cap RF_+ M = E.
\]

By Lemmas 18.1, we can take a maximal \( \hat{U} \) such that \( E \supset y \hat{U} \) for some \( y \in F^* \cap RF M \). By \( (2)_{m} \), we get a closed orbit \( yL \) for some \( L \in Q_{\hat{U}} \) such that

\[
(18.4) \quad yL \cap RF_+ M \subset E.
\]

If \( L = \hat{L} \), then the claim \( (18.3) \) is clear. Now suppose that \( L \) is a proper subgroup of \( \hat{L} \). This implies that \( L \cap N \) is a proper subgroup of \( \hat{L} \cap N \), since \( \hat{L} \cap N \) acts minimally on \( x_0 \hat{L} \cap RF_+ M \).

By Theorem 13.1, we can assume that \( y \in \bigcap_{i=1}^{t} G(U^{(i)}, y L) \cap RF M \), where \( U^{(1)}, \ldots, U^{(t)} \) are one-parameter subgroups generating \( U \).

As \( y \in E \), there exists a sequence \( x_i \in y_i L_i v_i \cap RF_+ M \) converging to \( y \), by passing to a subsequence. We claim that

\[
x_i \not\in yLN(U).
\]

Suppose not, i.e., \( x_i = y \ell_i r_i \) for some \( \ell_i \in L \) and \( r_i \in N(U) \). Then

\[
y_i L_i v_i \cap RF_+ M = \overline{z_i U} = \overline{y \ell_i U r_i} \subset y L r_i.
\]
By the maximality assumption on \( y_i L_i v_i \), we have \( v_i^{-1} L_i v_i = r_i^{-1} L r_i \). On the other hand, the dimension of \((L_i)_{nc}\) is strictly smaller than the dimension of \( L_{nc} \), since \((L_i)_{nc} = H(U) \) and \( H(\tilde{U}) < L_{nc} \), yielding a contradiction.

Hence \( x_i \notin y L(N(U)) \). Since \( U = v_i^{-1} L_i v_i \cap N \), we have \( x_i \in RF M \cdot U \). By Lemma 8.7, by replacing \( x_i \) with \( x_i u_i \) for some \( u_i \to e \) in \( U \), we may assume \( x_i \in RF M \). We can apply Proposition 15.2 and Lemma 16.9 and deduce that \( E \) contains \( y_1 \tilde{U} \) for some \( y_1 \in F^* \cap RF M \). This is a contradiction to the maximality assumption on \( \tilde{U} \).

\[ \square \]

**Proof of Theorem 1.7.** We explain how to deduce this theorem from Theorem 14.1(3). For (1), we may first assume that \( P_i \) have all same dimension so that \( P_i = \pi(x_i H') \) for some closed orbit \( x_i H' \) where \( H' = H'(U) \) and \( x_i \in RF M \). Then there exists \( L_i \in L_U \) such that \( x_i L_i \) is closed by Proposition 4.9. Since \( P_i \) is maximal, \( x_i L_i \) is a maximal closed orbit satisfying the hypothesis of (3). Hence \( \lim(x_i L_i \cap RF_+ M) = RF_+ M \). Since \( \pi(RF_+ M) = \Gamma \setminus \mathbb{H}^d \), the claim follows. (2) follows from Corollary 5.8. For (3), if there are infinitely many bounded properly immersed \( P_i \)'s, then \( \lim P_i = M \) by (1). On the other hand, \( P_i \subset \text{core}(M) \); because any bounded \( H' \) orbit should be inside \( RF M \). Since \( \text{core}(M) \) is a proper closed subset of \( M \), as \( \text{Vol}(M) = \infty \), this gives a contradiction.

**Remark 18.3.** In fact, when \( M \) is any convex cocompact hyperbolic manifold of infinite volume, there are only finitely many bounded maximal closed \( H' \)-orbits, and hence only finitely many maximal properly immersed bounded geodesic planes. The reason is that if not, we will be having infinitely many maximal closed orbits \( x_i L_i \) contained in \( RF M \) for some \( L_i \in L_U \), and for any \( U \)-invariant subset \( E \) contained in \( RF M \), the 1-thickness for points in \( E \) holds for any one-parameter subgroup of \( U \) for the trivial reason, which makes our proof of Theorem 14.1 work with no change (in fact, much simpler) for a general \( M \).

### 19. Appendix: Orbit closures for \( \Gamma \setminus G \) compact case

In this section we give an outline of the proof of the orbit closure theorem for the actions of \( H(U) \) and \( U \), assuming that \( \Gamma \setminus G \) is compact and there exists at least one closed orbit of \( SO^0(d - 1, 1) \). We hope that giving an outline of the proof of Theorem 14.1 in this special case will help readers understand the whole scheme of the proof better and see the differences with the infinite volume case more clearly.

Note that in the case at hand,

\[ RF M = F_{H(U)}^* = RF_+ M = \Gamma \setminus G. \]

Without loss of generality, we assume that \( U \subset SO^0(d - 1, 1) \cap N \).

**Theorem 19.1.** Let \( x \in \Gamma \setminus G \).

1. We have \( \overline{xH(U)} = xL \)
for some $L \in \mathcal{L}_U$.

(2) If $\bar{x}U$ is contained in a closed orbit $x\tilde{L}$ for some $\tilde{L} \in \mathcal{Q}_U$, then

$$\bar{x}U = xL$$

for some $L \in \mathcal{Q}_U$.

In the case when $\Gamma \setminus G$ is compact, we don’t need the topological equidistribution statement, which is Theorem 14.1(3) to run the induction argument, thanks to (2).

The base case $(2)_0$ follows from a special case of Theorem 13.1.

For $m \geq 0$, we will show that $(2)_m$ implies $(1)_{m+1}$, and that $(1)_{m+1}$ and $(2)_m$ together imply $(2)_{m+1}$.

\textbf{Remark 19.2.} Theorem 19.1 is proved by Shah [45] by topological arguments. Our proof presented in this appendix is somewhat different from Shah’s in that we prove that $(1)_m$ implies $(2)_m$ using the existence of a closed $\text{SO}^0(d-1,1)$-orbit, while he shows that $(2)_m$ implies $(1)_m$.

\textbf{Proof of $(1)_{m+1}$.} Let $U \subset N$ have co-dimension $m+1$ in $N$. We assume $U \neq N$. By Proposition 16.2, it suffices to show that $X := xH'(U) = xL \cap C(H(U))$ for some $L \in \mathcal{L}_U$. Assume that $xH'(U)$ is not closed.

\textbf{Step 1: Find a closed orbit inside $X$.} Propositions 10.6, 10.7, and 10.9 imply:

\textbf{Proposition 19.3.} If $Y \subset X$ is a $U$-minimal set such that $yg_n \in X$ for some $y \in Y$ and a sequence $g_n \to e$ in $G - H'(U)$, then $zU \subset X$ for some connected closed subgroup $zU < N$ containing $U$ properly.

Since the co-dimension of $\hat{U} := UV$ in Proposition 19.3 is less than $m+1$, by $(2)_m$ and Lemma 16.4, it suffices to find $Y$ satisfying this proposition. If $xH'(U)$ is not locally closed, then any $U$-minimal subset $Y \subset X$ does the job. If $xH'(U)$ is locally closed, then any $U$-minimal subset $Y$ of $X - xH'(U)$ does the job; note that the set $X - xH'(U)$ is a compact $H'(U)$-invariant subset and hence contains a $U$-minimal subset.

Hence $X$ contains a closed orbit $zL$. We may assume that $X \neq zL \cap C(H(U))$; otherwise, we are done.

\textbf{Step 2: Enlarge a closed orbit inside $X$.} Since $zL$ is compact, by Theorem 13.1, we can assume that $zU^{(i)}_0$ is dense in $zL$ where $U^{(1)}_0, \ldots, U^{(k)}_0$ are one-parameter subgroups of $U^\pm$ generating $U^\pm$. Note that there exists $g_i \to e$ in $G - L \cap C(H(U))$ such that $zg_i \in X$. We can write $g_i = \ell_i r_i$ where $r_i \in \text{exp} \mathfrak{l}^1$ and $\ell_i \in L$. Then $r_i \notin C(H(U))$. Since $\bigcap_{i=1}^k (N(U^{(i)}_0) \cap N(U^{(i)}_0)) \subset C(H(U))$, we have $r_i \notin N(U_0)$ where $U_0$ is one of the subgroups $U^{(i)}_\pm$. If $U_0 \in \{U^{(i)}_\pm\}$, then replace $U$ by $U^+$. 
Fix any $k > 1$. Applying (2.6) to the sequence $z_i := z \ell_i \to z$, the set

\[(19.1) \quad T(z_i) := \{ t \in \mathbb{R} : z_i u_t \in \Gamma \setminus G - \bigcup_{j=1}^{i} O_j \} \]

is a $k$-thick subset (take $0 < \varepsilon < 1 - 1/k$).

By Lemma 9.3, there exists $t_i \in T(z_i)$ such that $u_i^{-1} r_i u_i$ converges to a non-trivial element $v \in U^\perp$. Now the sequence $z_i u_t$ converges to $z_0 \in \mathcal{G}(U_0, z \mathring{L})$. Since $z g_i u_t$ converges to $z_0 v$, we deduce

$$z L v = z_0 v U_0 \subset X$$

and hence $z L V^+ \subset z L (A v A) \subset X$ where $V^+ = \exp(\mathbb{R}^+ \log v)$. Take any sequence $v_i \to \infty$ in $V^+$ such that $z v_i$ converges to some $x_0$. Then $x_0 V \subset \limsup(z v_i)(v_i^{-1} V^+) \subset X$ and hence $X$ contains $x_0(L \cap N)V$. By the induction hypothesis (2) and Lemma 16.4, $X$ contains a closed orbit of $\hat{L}$ for some $\hat{L} \in \mathcal{L}_U$.

**Proof of (2)$_{m+1}$.** Set $X := \overline{\mathcal{G} U}$. We assume that $X \neq x_0 \hat{L}$. Since the co-dimension of $U$ in $\hat{L} \cap N$ is at least 1, we may assume without loss of generality that $U < N \cap SO^0(d - 1, 1)$ using conjugation by an element of $M$.

**Step 1: Find a closed orbit inside $X$.** It follows from (1)$_{m+1}$, (2)$_{m}$, the hypothesis on the existence of a closed $L_0 := SO^0(d - 1, 1)$-orbit, and the cocompactness of $AU$ in $H'(U)$ that any $AU$-orbit closure intersects $\mathcal{G}(U, x_0 \hat{L})$ (cf. proof of Lemma 17.1).

We claim that $X$ intersects $\mathcal{G}(U, x_0 \hat{L})$. Since $\mathcal{G}(U, x_0 \hat{L})$ is $N C_2(U)$-invariant, it suffices to show $X N C_2(U)$ intersects $\mathcal{G}(U, x_0 \hat{L})$. Let $Y \subset X$ be a $U$-minimal subset. Then there exists a one-parameter subgroup $L \subset AU^\perp C_2(U)$ such that $Y g = Y$ for all $g \in L$ by Lemma 10.6. Strictly speaking, the cited lemma gives $Y g \subset Y'$ for $g$ in a semigroup $L$, but in the case at hand, $Y g \subset Y$ implies $Y g = Y$, since $Y g$ is $U$-minimal again, and hence $Y g^{-1} = Y$ as well.

In view of Lemma 3.3, we get $Y A \subset X N C_2(U)$ or $Y v A \subset X N C_2(U)$ for some $v \in N$. In either case, $X N C_2(U)$ contains an $AU$-orbit and hence intersects $\mathcal{G}(U, x_0 \hat{L})$. Hence the claim follows.

Since $X$ intersects $\mathcal{G}(U, x_0 \hat{L})$, by applying (2)$_m$, $X$ contains $z L$ for some $L \in \mathcal{Q}_U$.

**Step 2: Enlarge a closed orbit inside $X$.** Suppose $L \neq G$ and $X \neq z L$. It suffices to show that $X$ contains a closed orbit $y \hat{L}$ for some $\hat{L} \in \mathcal{L}_U$ for some $\hat{U}$ properly containing $U$. We may assume $X \not\subset z L C(H(U))$; otherwise, the claim follows from (2)$_m$. We may assume $z \in \bigcap_{i=1}^{\ell} \mathcal{G}(U^{(i)}, y L)$ where $U^{(i)}$’s are one-parameter generating subgroups of $U$. Take a sequence $x u_i \to z$ where $u_i \in U$, and write $x u_i = z \ell_i r_i$ where $\ell_i \in L$ and $r_i \in \exp(1^{-})$. The case of $r_i \in N(U)$ for some $i$ follows from (2)$_m$ (cf. Proof of Lemma
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17.4). Hence we may assume \( r_i \notin N(U) \), and by passing to a subsequence, \( r_i \notin N(U_0) \) for some \( U_0 \in \{ U^{(i)} \} \).

Fix any \( k > 1 \). Then \( T(z_i) \) as in (19.1) is a \( k \)-thick subset. We now repeat the same argument of Step (2) in the proof of (1)\(_{m+1}\). By Lemma 9.3, there exists \( t_i \in T(z_i) \) such that \( u_{t_i}^{-1} r_i u_{t_i} \) converges to a non-trivial element \( v \in U^\perp \). Now the sequence \( z_i u_{t_i} \) converges to \( z_0 \in G(U_0, zL) \). Hence \( X \supset \overline{z_0 U_0 v} = zLv \).

Moreover, by Lemma 9.3, such \( v \) can be made of arbitrarily large size, so we get \( X \supset zLv_j \) for a sequence \( v_j \in U^\perp \) tending to \( \infty \). The set \( \limsup v_j^{-1} Av_j \) contains a one-parameter subgroup \( V \subset U^\perp \) by Lemma 3.4. Hence if \( y \) is a limit of \( zv_j \), then

\[
X \supset \limsup(zLv_j) \supset y(L \cap N) \limsup(v_j^{-1} Av_j) \supset y(L \cap N)V.
\]

Hence \( X \) contains \( y(L \cap N)V \), and hence the claim follows from (2)\(_m\).

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