ORBIT CLOSURES OF UNIPOTENT FLOWS FOR HYPERBOLIC MANIFOLDS WITH FUCHSIAN ENDS

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Abstract. We establish an analogue of Ratner’s orbit closure theorem for any connected closed subgroup generated by unipotent elements in $\text{SO}(d, 1)$ acting on the space $\Gamma \backslash \text{SO}(d, 1)$, assuming that the associated hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d$ is a convex cocompact manifold with Fuchsian ends.

1. Introduction

Let $G$ be a connected simple linear Lie group and $\Gamma < G$ be a discrete subgroup. Let $U$ be a connected closed subgroup of $G$ generated by unipotent elements in it. We are interested in the action of $U$ on the homogeneous space $\Gamma \backslash G$ by right translations. If the volume of the homogeneous space $\Gamma \backslash G$ is finite, the celebrated Ratner’s orbit closure theorem, which was a conjecture of Raghunathan, states that

\[ \text{the closure of every } U\text{-orbit is homogeneous}, \]

that is, for any $x \in \Gamma \backslash G$, $xU = xL$ for some connected closed subgroup $L < G$ containing $U$ [24]. Ratner’s proof is based on her classification of all $U$-invariant ergodic probability measures [23] and the work of Dani and Margulis [10] on the non-divergence of unipotent flow. Prior to her work, some important special cases of (1.1) were established by Margulis [14], Dani-Margulis ([8], [9]) and Shah ([28], [27]) by topological methods. This theorem is a fundamental result with numerous applications.

It is natural to ask if there exists a family of homogeneous spaces of infinite volume where an analogous orbit closure theorem holds. When the volume of $\Gamma \backslash G$ is infinite, the geometry of the associated locally symmetric space turns out to play an important role in this question. The first orbit closure theorem in the infinite volume case was established by McMullen, Mohammadi, and Oh ([16], [17]) for a class of homogeneous spaces $\Gamma \backslash \text{SO}(3, 1)$ which arise as the frame bundles of convex cocompact hyperbolic 3-manifolds with Fuchsian ends.

Our goal in this paper is to show that a similar type of orbit closure theorem holds in the higher dimensional analogues of these manifolds. We present a complete hyperbolic $d$-manifold $M = \Gamma \backslash \mathbb{H}^d$ as the quotient of the hyperbolic space by a discrete subgroup $\Gamma$ of $G = \text{SO}^+(d, 1) \simeq \text{Isom}^+(\mathbb{H}^d)$.

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The geometric boundary of $\mathbb{H}^d$ can be identified with the sphere $S^{d-1}$. The limit set $\Lambda \subset S^{d-1}$ of $\Gamma$ is the set of all accumulation points of an orbit $\Gamma x$ in the compactification $\mathbb{H}^d \cup S^{d-1}$ for $x \in \mathbb{H}^d$. The convex core of $M$ is given by the quotient core $M = \Gamma \setminus \text{hull}(\Lambda)$ where $\text{hull}(\Lambda) \subset \mathbb{H}^d$ is the smallest convex subset containing all geodesics in $\mathbb{H}^d$ connecting points in $\Lambda$. When core $M$ is compact, $M$ is called convex cocompact.

In the rest of the introduction, we assume that $M$ is a convex cocompact hyperbolic $d$-manifold ($d \geq 2$) with Fuchsian ends, that is, core $M$ has non-empty interior and has totally geodesic boundary. The term Fuchsian ends reflects the fact that each component of the boundary of core $M$ is a $(d-1)$-dimensional closed hyperbolic manifold, and each component of the complement $M - \text{core}(M)$ is diffeomorphic to the product $S \times (0, \infty)$ for some closed hyperbolic $(d-1)$-manifold $S$.

The homogeneous space $\Gamma \setminus G$ can be regarded as the bundle $F_M$ of oriented frames over $M$. Let $A = \{a_t : t \in \mathbb{R}\} \subset G$ denote the one parameter subgroup of diagonalizable elements whose right translation actions on $\Gamma \setminus G$ correspond to the frame flow. Let $N = \{g \in G : a_{-t}g a_t \to e \text{ as } t \to +\infty\}$. We denote by $RF_M$ the renormalized frame bundle of $M$:

$$RF_M := \{x \in \Gamma \setminus G : xA \text{ is bounded}\},$$

and also set

$$RF_+ M := \{x \in \Gamma \setminus G : xA^+ \text{ is bounded}\}$$

where $A^+ = \{a_t : t \geq 0\}$. The sets $RF_M$ and $RF_+ M$ are precisely non-wandering sets for the actions of $A$ and $N$ respectively [31].

For a connected closed subgroup $U < N$, we denote by $H(U)$ the smallest closed simple Lie subgroup of $G$ which contains both $U$ and $A$. If $U \simeq \mathbb{R}^k$, then $H(U) \simeq SO_0(k+1,1)$. A connected closed subgroup of $G$ generated by one-parameter unipotent subgroups is, up to conjugation, of the form $U < N$ or $H(U)$ for some $U < N$ (Cor. 2.7).

We set $F_{H(U)} := RF_+ M : H(U)$, which is a closed subset. It is easy to see that if $x \notin RF_+ M$ (resp. $x \notin F_{H(U)}$), then $xU$ (resp. $xH(U)$) is closed in $\Gamma \setminus G$.

**Orbit closures are relatively homogeneous.** We define the following collection of closed connected subgroups of $G$:

$$\mathcal{L}_U := \left\{ L = H(\tilde{U}) C : \text{ for some } z \in RF_+ M, zL \text{ is closed in } \Gamma \setminus G \right\}.$$

where $U < \tilde{U} < N$ and $C$ is a closed subgroup of the centralizer of $H(\tilde{U})$. We also define: $\mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U \text{ and } v \in N\}$. The following theorem gives a classification of orbit closures for all connected closed subgroups of $G$ generated by unipotent one-parameter subgroups:

**Theorem 1.1.** Let $U < N$ be a non-trivial connected closed subgroup.
(1) \((H(U))-\text{orbit closures}\) For any \(x \in RF\ M \cdot H(U)\),

\[
\overline{xH(U)} = xL \cap F_{H(U)}
\]

where \(xL\) is a closed orbit of some \(L \in \mathcal{L}_U\).

(2) \((U)-\text{orbit closures}\) For any \(x \in RF_+ M\),

\[
\overline{xU} = xL \cap RF_+ M
\]

where \(xL\) is a closed orbit of some \(L \in \mathcal{Q}_U\).

(3) \(\text{(Equidistributions)}\) Let \(x_iL_i\) be a sequence of closed orbits intersecting \(RF\ M\), where \(x_i \in RF_+ M\) and \(L_i \in \mathcal{Q}_U\). Assume that no infinite subsequence of \(x_iL_i\) is contained in a subset of the form \(y_0L_0D\) where \(y_0L_0\) is a closed orbit of \(L_0 \in \mathcal{L}_U\) with \(\dim L_0 < \dim G\) and \(D\) is a compact subset of the normalizer \(N(U)\) of \(U\). Then

\[
\lim_{i \to \infty} x_iL_i \cap RF_+ M = RF_+ M.
\]

Remark 1.2. If \(x \in F_{H(U)} - RF\ M \cdot H(U)\), then \(\overline{xH(U)} = xLV^+H(U)\) for some \(L \in \mathcal{L}_U\), and some one-parameter semigroup \(V^+ < N\) (Thm. 10.5).

Theorem 1.1(1) and (2) can be presented as follows in a unified manner:

**Corollary 1.3.** Let \(H < G\) be a connected closed subgroup generated by unipotent elements in it. Assume that \(H\) is normalized by \(A\). For any \(x \in RF\ M\), the closure of \(xH\) is homogeneous in \(RF\ M\), that is,

\[
\overline{xH} \cap RF\ M = xL \cap RF\ M
\]

where \(xL\) is a closed orbit of some \(L \in \mathcal{Q}_U\).

A geodesic plane in \(M\) of dimension \(k\) is the image of a totally geodesic immersion \(f : \mathbb{H}^k \to M\). A horosphere in \(\mathbb{H}^d\) of dimension \(k\) is a Euclidean sphere of dimension \(k\) which is tangent to a point in \(S^{d-1}\). A horosphere in \(M\) is simply the image of a horosphere in \(\mathbb{H}^d\) under the covering map \(\mathbb{H}^d \to M = \Gamma \setminus \mathbb{H}^d\). Theorem 1.1 implies:

**Corollary 1.4.** The closure of a geodesic plane of dimension at least 2 or of a horosphere is a properly immersed submanifold of \(M\) (possibly with boundary).

For \(d = 3\), the main result of this paper was proved earlier in [16] and [17]. In a higher dimensional case, the possibility of accumulation on closed orbits of intermediate subgroups causes serious issues. Calling the collection of all such closed orbits as the singular set and its complement as the generic set, the main achievement of this paper lies in establishing avoidance of the singular set along the \(k\)-thick recurrence of unipotent flows to \(RF\ M\) for a sequence of \(RF\ M\)-points limiting at a generic point (sec. 5-6). Roughly speaking,\(^1\) Theorem 1.1 is proved by induction on the co-dimension of \(U\) in

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\(^1\)To be precise, we need to carry out induction on the co-dimension of \(U\) in \(\hat{L} \cap N\) whenever \(xU\) is contained in a closed orbit \(x_0\hat{L}\) for some \(\hat{L} \in \mathcal{L}_U\) as formulated in Theorem 13.1.
N. For each \( i = 1, 2, 3 \), let us say that \( (i)_m \) holds, if Theorem 1.1(\( i \)) is true for all \( U \) satisfying co-dim\(_N(U) \leq m \). We show that the validity of \( (2)_m \) and \( (3)_m \) implies that of \( (1)_{m+1} \) (sec. 15) the validity of \( (1)_{m+1}, (2)_m \) and \( (3)_m \) implies that of \( (2)_{m+1} \) (sec. 16), and the validity of \( (1)_{m+1}, (2)_{m+1}, \) and \( (3)_m \) implies that of \( (3)_{m+1} \) (sec. 17).

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### 2. Lie Subgroups and Geodesic Planes

Let \( G \) denote the connected simple Lie group \( \text{SO}^0(d, 1) \simeq \text{Isom}^+(\mathbb{H}^d) \) for \( d \geq 2 \). In order to present a family of subgroups of \( G \) explicitly, we fix a quadratic form \( Q(x_1, \cdots, x_{d+1}) = 2x_1x_{d+1} + x_2^2 + x_3^2 + \cdots + x_d^2 \), and identify \( G = \text{SO}^0(Q) \).

**Subgroups of \( G \).** Inside \( G \), we have the following subgroups:

\[
K = \{ g \in G : g^t g = \text{Id}_{d+1} \} \simeq \text{SO}(d),
\]

\[
A = \{ a_s = \text{diag}(e^s, \text{Id}_{d-1}, e^{-s}) : s \in \mathbb{R} \},
\]

\[
M = \text{the centralizer of } A \text{ in } K \simeq \text{SO}(d-1),
\]

\[
N^\pm = \{ \exp u^\pm(x) : x \in \mathbb{R}^{d-1} \},
\]

where \( u^-(x) = \left( \begin{array}{ccc} 0 & x^t \ 0 & -x \ 0 & 0 \end{array} \right) \) and \( u^+(x) = \left( \begin{array}{ccc} 0 & 0 & 0 \ 0 & 0 & 0 \ 0 & -x^t & 0 \end{array} \right) \).

The Lie algebra of \( M \) consists of matrices of the form \( m(C) = \left( \begin{array}{ccc} 0 & 0 & 0 \\
0 & C & 0 \\
0 & 0 & 0 \end{array} \right) \)

where \( C \in M_{d-1}(\mathbb{R}) \) satisfies \( C^t = -C \).

As we will be using the subgroup \( N^- \) frequently, we simply write \( N = N^- \).

We often identify the subgroup \( N^\pm \) with \( \mathbb{R}^{d-1} \) via the map \( \exp u^\pm(x) \mapsto x \). For a connected closed subgroup \( U < N \), we use the notation \( U^\perp \) for the orthogonal complement of \( U \) in \( N \) as a vector subgroup of \( N \), and \( U^t = U^+ \) for the transpose of \( U \). We use the notation \( B_U(r) \) to denote the ball of radius \( r \) centered at 0 in \( U \) for a Euclidean metric on \( N = \mathbb{R}^{d-1} \).

We consider the upper-half space model of \( \mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1} \), so that its boundary is given by \( S^{d-1} = \{ \infty \} \cup \{ \{0\} \times \mathbb{R}^{d-1} \} \). Set \( o = (1, 0, \cdots, 0) \), and fix a standard basis \( e_0, e_1, \cdots, e_{d-1} \) at \( T_o(\mathbb{H}^d) \). The map

\[
g \mapsto (ge_0, \cdots, g e_{d-1})_{g(o)}
\]

gives an identification of \( G \) with the oriented frame bundle \( F \mathbb{H}^d \). The stabilizer of \( o \) and \( e_0 \) in \( G \) are equal to \( K \) and \( M \) respectively, and hence the map (2.1) induces the identifications of the hyperbolic space \( \mathbb{H}^d \) and the unit tangent bundle \( T^1 \mathbb{H}^d \) with \( G/K \) and \( G/M \) respectively. The action of \( G \)
on the hyperbolic space $\mathbb{H}^d = G/K$ extends continuously to the compactification $\mathbb{S}^{d-1} \cup \mathbb{H}^d$. If $g \in G$ corresponds to a frame $(v_0, \cdots, v_{d-1}) \in F \mathbb{H}^d$, we define $g^+, g^- \in \mathbb{S}^{d-1}$ to be the forward and backward end points of the directed geodesic tangent to $v_0$ respectively. The right translation action of $A$ on $G = F \mathbb{H}^d$ defines the frame flow and we have $g^\pm = \lim_{t \to \pm \infty} \pi(ga_t)$ where $\pi : G = F \mathbb{H}^d \to \mathbb{H}^d$ is the basepoint projection.

Note that $g^+ = g(\infty)$ and $g^- = g(0)$. The subgroup $MA$ fixes both points 0 and $\infty$, and $N$ fixes $\infty$, and the restriction of the map $g \mapsto g(0)$ to $N$ defines an isomorphism $N \to \mathbb{R}^{d-1}$ given by $u^-(x) \mapsto x$.

For each non-trivial connected subgroup $U < N$, we denote by $H(U)$ the connected closed subgroup of $G$ generated by $U$ and the transpose of $U$. It is the smallest simple closed Lie subgroup of $G$ containing $A$ and $U$. For a subset $S \subseteq G$, we denote by $N_G(S)$ and $C_G(S)$ the normalizer of $S$ and the centralizer of $S$ respectively. We denote by $N(S)$ and $C(S)$ the identity components of $N_G(S)$ and $C_G(S)$ respectively.

We set $H'(U) := N(H(U)) = H(U)C(H(U))$, which is a connected reductive algebraic subgroup of $G$ with compact center. Fix the standard basis $e_1, \cdots, e_{d-1}$ of $\mathbb{R}^{d-1}$. For $1 \leq k \leq d-1$, define $U_k$ to be the connected subgroup of $N$ spanned by $e_1, \cdots, e_k$. Then $H(U_k) = \langle U_k, U_k^0 \rangle = SO^0(k+1,1)$, $C(H(U_k)) = SO(d-k-1)$ and $N(H(U_k)) = SO^0(k+1,1)SO(d-k-1)$. Since the adjoint action of $M$ on $N$ corresponds to the standard action of $SO(d-1)$ on $\mathbb{R}^{d-1}$, any connected closed subgroup $U < N$ is conjugate to $U_k$ and $H(U)$ is conjugate to $H(U_k)$ by an element of $M$, where $k = \text{dim}(U)$.

We set

\[(2.2) \quad C_1(U) := C(H(U)) = M \cap C(U), \quad \text{and} \quad C_2(U) := M \cap C(U^\perp) \subset H(U).\]

**Lemma 2.1.** We have $N(U) = NAC_1(U)C_2(U)$ and $C(U) = NC_1(U)$.

**Proof.** For the first claim, it suffices to show that for $U = U_k$, $N(U) = NASO(k)SO(d-1-k)$. It is easy to check that $Q := NAC_1(U)C_2(U)$ normalizes $U$. Let $g \in N(U)$. We claim that $g \in Q$. Using the decomposition $G = KAN$, we may assume that $g \in K$. Then $Ug(\infty) = gU(\infty) = g(\infty)$ since $U(\infty) = \infty$. Since $\infty \in \mathbb{S}^{d-1}$ is the unique fixed point of $U$, it follows $g(\infty) = \infty$. As $M = \text{Stab}_K(\infty)$, we get $g \in M$. Now $Ug(0) = Ug(0) = U(0)$. As $U(0) = \mathbb{R}^k$, $g\mathbb{R}^k = \mathbb{R}^k$. Therefore, as $g \in M$, we also have $g\mathbb{R}^{d-1-k} = \mathbb{R}^{d-1-k}$, and consequently $g \in O(k)O(d-1-k)$. This shows that $NASO(k)SO(d-1-k) \subset N(U) \subset NAO(k)O(d-1-k)$. As $N(U)$ is connected, this implies the claim. For the second claim, note first that $NC_1(U) < C(U)$. Now let $g \in C(U)$. Since $C(U) < N(U) = ANC_1(U)C_2(U)$, we can write $g = ac_2nc_1 \in AC_2(U)NC_1(U)$. Since $nc_1$ commutes with $U$, it follows $ac_2 \in C(U)$. Now the adjoint action of $a$ on $U$ is a dilation and the adjoint action of $c_2$ on $U$ is a multiplication by an orthogonal matrix. Therefore we get $a = c_2 = e$, finishing the proof. \(\square\)

Denote by $g = \text{Lie}(G)$ the Lie algebra of $G$. Note that the product $AU^\perp C_2(U)$ is a subgroup of $G$. 


Lemma 2.2. An unbounded one-parameter subsemigroup $S$ of $AU^\perp C_2(U)$ is either \{(v \exp(t\xi_A)v^{-1}) \exp(t\xi_C) : t \geq 0\} or \{\exp(t\xi_V) \exp(t\xi_C) : t \geq 0\} for some $\xi_A \in \text{Lie}(A) - \{0\}$, $\xi_C \in \text{Lie}(C_2(U))$, $v \in U^\perp$, and $\xi_V \in \text{Lie}(U^\perp) - \{0\}$.

Proof. Let $\xi \in \text{Lie}(AU^\perp C_2(U))$ be such that $S = \{\exp(t\xi) : t \geq 0\}$. Write $\xi = \xi_0 + \xi_C$ where $\xi_0 \in \text{Lie}(AU^\perp)$ and $\xi_C \in \text{Lie}(C_2(U))$. Since $AU^\perp$ commutes with $C_2(U)$, $\exp(t\xi) = \exp(t\xi_0) \exp(t\xi_C)$ for any $t \in \mathbb{R}$. Hence we only need to show that either $\xi_0 \in \text{Lie}(U^\perp)$ or

\[ \{\exp(t\xi_0) : t \geq 0\} = \{v \exp(t\xi_A)v^{-1} : t \geq 0\} \]

for some $v \in U^\perp$ and $\xi_A \in \text{Lie}(A)$. Now if $\xi_0 \notin \text{Lie}(U^\perp)$, then writing $\xi_0 = \begin{pmatrix} a & x^t \\ 0 & 0 \end{pmatrix} \in \text{Lie}(AU^\perp)$ with $a \neq 0$, a direct computation shows that $\xi_0 = v\xi_A v^{-1}$ where $\log v = \begin{pmatrix} 0 & -x/a \\ 0 & 0 \end{pmatrix}$ and $\xi_A = \begin{pmatrix} 0 & 0 \\ 0 & -a \end{pmatrix}$. \hfill \Box

A direct computation shows:

Lemma 2.3. If $v_i \to \infty$ in $U^\perp$, then $\limsup_{i \to \infty} v_i A v_i^{-1}$ contains one-parameter subgroup of $U^\perp$.

The complementary subspaces $h_U^\perp$ and $h^\perp$. If $L$ is a reductive Lie subgroup of $G$ with $l = \text{Lie}(L)$, the restriction of the adjoint representation of $G$ to $L$ is completely reducible, and hence there exists an $\text{Ad}(L)$-invariant complementary subspace $l^\perp$ so that $g = l \oplus l^\perp$. It follows from the inverse function theorem that the map $L \times l^\perp \to G$ given by $(g, X) \mapsto g \exp X$ is a local diffeomorphism onto an open neighborhood of $e$ in $G$.

Let $U = U_k$. Denote by $h_U \subset g$ the Lie algebra of $H(U)$, by $u^\perp$ the subspace $\text{Lie}(U^\perp)$, and by $(u^\perp)^t$ its transpose. Then $h_U^\perp$ can be given explicitly as follows:

\[ h_U^\perp = u^\perp \oplus (u^\perp)^t \oplus \mathfrak{m}_0 \]

where $\mathfrak{m}_0 = \{m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & Z \end{pmatrix}, \ Z \in M_{k \times \{a_{-1-k}\}}(\mathbb{R})\}$.

Similarly, setting $h := \text{Lie}(H'(U))$, $h^\perp$ is given by

\[ h^\perp = u^\perp \oplus (u^\perp)^t \oplus \mathfrak{m}'_0 \]

where $\mathfrak{m}'_0 := \{m(C) : C = \begin{pmatrix} 0 & Y \\ -Y^t & 0 \end{pmatrix}\}$.

By Lemma 2.1 and (2.5), we have:

Lemma 2.4. If $r_i \to e$ in $\exp h^\perp - \text{C}(H(U))$, then either $r_i \notin \text{N}(U)$ for all $i$, or $r_i \notin \text{N}(U^\perp)$ for all $i$, by passing to a subsequence.

Definition 2.5. For a connected reductive subgroup $L < G$, denote by $L_{nc}$ the maximal connected normal semisimple subgroup of $L$ with no compact factors.
A connected reductive algebraic subgroup $L$ of $G$ is an almost direct product $L = L_{nc}CT$ where $C$ is a connected semisimple compact normal subgroup of $L$ and $T$ is the central torus of $L$. If $L$ contains a unipotent element, then $L_{nc}$ is non-trivial, and simple, containing a conjugate of $A$, and the center of $L$ is compact.

**Proposition 2.6.** If $L < G$ is a connected reductive algebraic subgroup normalized by $A$ and containing a unipotent element, then $L = H(U)C$ where $U < N$ is a non-trivial connected subgroup and $C$ is a closed subgroup of $C(H(U))$. In particular, $L_{nc}$ and $N(L_{nc})$ are equal to $H(U)$ and $H'(U)$ respectively.

**Proof.** If $L$ is normalized by $A$, then so is $L_{nc}$. Therefore it suffices to prove that a connected non-compact simple Lie subgroup $H < G$ normalized by $A$ is of the form $H = H(U)$ where $U < N$ is a non-trivial connected subgroup.

First, consider the case when $A < H$. Let $\mathfrak{h}$ be the Lie algebra of $H$, and $\mathfrak{a}$ be the Lie algebra of $A$. Since $\mathfrak{h}$ is simple, its root space decomposition for the adjoint action of $\mathfrak{a}$ is of the form $\mathfrak{h} = \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{u}^+ \oplus \mathfrak{u}^-$ where $\mathfrak{u}^\pm$ are the sum of all positive and negative root subspaces respectively and $\mathfrak{z}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$. Since the sum of all negative root subspaces for the adjoint action of $\mathfrak{a}$ on $\mathfrak{g}$ is Lie($N^-$), it follows that $U := \exp(\mathfrak{u}^-) < N^-$ and $H = H(U)$. Now for the general case, $H$ contains a conjugate $gAg^{-1}$ for some $g \in G$. Hence $g^{-1}Hg = H(U)$. Since $H(U)$ contains both $A$ and $g^{-1}Ag$, they must be conjugate within $H(U)$, so $A = h^{-1}g^{-1}Ag$ for some $h \in H(U)$. Hence $gh \in N_G(A) = AM$. Therefore $H = gH(U)g^{-1}$ is equal to $mH(U)m^{-1}$ for some $m \in M$. Since $m$ normalizes $N$ and $mH(U)m^{-1} = H(mUm^{-1})$, the claim follows. \hfill $\Box$

It is easy to deduce the following from the above proposition:

**Corollary 2.7.** Any connected closed subgroup $L$ of $G$ generated by unipotent elements is conjugate to either $U$ or $H(U)$ for some non-trivial connected subgroup $U < N$.

**Totally geodesic immersed planes.** Let $\Gamma$ be a discrete, torsion free, non-elementary, subgroup of $G$, and consider the associated hyperbolic manifold $M = \Gamma \backslash \mathbb{H}^d = \Gamma \backslash G/K$. We refer to [22] for basic properties of hyperbolic manifolds. As in the introduction, we denote by $\Lambda$ the limit set of $\Gamma$ and by $\text{core}(M)$ the convex core of $M$.

We denote by $FM \simeq \Gamma \backslash G$ the bundle of all oriented orthonormal frames over $M$. We denote by $\pi : \Gamma \backslash G \to M = \Gamma \backslash G/K$ the base-point projection. By abuse of notation, we also denote by $\pi : G \to \mathbb{H}^d = G/K$ the base-point projection. For $g \in G$, $[g]$ denotes its image under the covering map $G \to \Gamma \backslash G$.

Fix $1 \leq k \leq d - 2$ and let

$$H = SO^o(k + 1, 1) \quad \text{and} \quad H' = SO^o(k + 1, 1) \cdot SO(d - k - 1).$$

$(2.6)$
Let \( C_0 := \mathbb{R}^k \cup \{\pi\} \) denote the unique oriented \( k \)-sphere in \( S^{d-1} \) stabilized by \( H' \). Then \( \tilde{S}_0 := \text{hull}(C_0) \) is the unique oriented totally geodesic subspace of \( \mathbb{H}^d \) stabilized by \( H' \), and \( \partial \tilde{S}_0 = C_0 \). We note that \( H' \) (resp. \( H \)) consists of all oriented frames \((v_0, \cdots, v_{d-1}) \in G \) (resp. \((v_0, \cdots, v_k, e_{k+1}, \cdots, e_{d-1}) \in G \)) such that the \( k + 1 \)-tuple \((v_0, \cdots, v_k) \) is tangent to \( \tilde{S}_0 \), compatible with the orientation of \( \tilde{S}_0 \). The group \( G \) acts transitively on the space of all oriented \( k \) spheres in \( S^{d-1} \) giving rise to the isomorphisms of \( G/H' \) with \( \mathcal{C}^k \) = the space of all oriented \( k \)-spheres in \( S^{d-1} \) as well as with the space of all oriented totally geodesic \((k+1)\)-planes of \( \mathbb{H}^d \).

We discuss the fundamental group of an immersed geodesic \( k \)-plane \( S \subset M \). Choose a totally geodesic subspace \( \tilde{S} \) of \( \mathbb{H}^d \) which covers \( S \). Then \( \tilde{S} = g \tilde{S}_0 \) for some \( g \in G \), and the stabilizer of \( \tilde{S} \) in \( G \) is equal to \( gH'g^{-1} \). We have \( \Gamma_{\tilde{S}} = \{ \gamma \in \Gamma : \gamma \tilde{S} = \tilde{S} \} = \Gamma \cap gH'g^{-1} \) and get an immersion \( \tilde{f} : \Gamma_{\tilde{S}} \tilde{S} \to M \) with image \( S \). Consider the projection map

\[
(2.7) \quad p \circ gH'g^{-1} \to gHg^{-1}.
\]

Then \( p \) is injective on \( \Gamma_{\tilde{S}} \) and \( \Gamma_{\tilde{S}} \tilde{S} \simeq p(\Gamma_{\tilde{S}})\tilde{S} \) is an isomorphism, since \( g \mathcal{C}(H)g^{-1} \) acts trivially on \( \tilde{S} \). Hence \( \tilde{f} \) gives an immersion

\[
(2.8) \quad f : p(\Gamma_{\tilde{S}}) \tilde{S} \to M
\]

with image \( S \). We say \( S \) properly immersed if \( f \) is a proper map.

The following proposition is standard:

**Proposition 2.8.** Let \( x \in \Gamma \setminus G \), and set \( S := \pi(xH') \subset M \). Then

1. \( xH' \) is closed in \( \Gamma \setminus G \) if and only if \( S \) is properly immersed in \( M \).
2. If \( M \) is convex cocompact and \( S \) is properly immersed, then \( S \) is convex cocompact and \( \partial \tilde{S} \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}})) \) for any geodesic subspace \( \tilde{S} \subset \mathbb{H}^d \) which covers \( S \).

3. Thick return time

In this section, we study the closed \( H(U) \)-invariant subset \( F_{H(U)} := RF_+ \cdot M \cdot H(U) \) when \( M = \Gamma \setminus \mathbb{H}^d \) is a convex cocompact manifold with Fuchsian ends. At the end of the section, we address the global thickness of the return time of any one-parameter subgroup of \( N \) to \( RF M \).

**Definition 3.1.** A convex cocompact hyperbolic manifold \( M = \Gamma \setminus \mathbb{H}^d \) is said to have non-empty **Fuchsian ends** if one of the following equivalent conditions holds:

1. its convex core has non-empty interior and non-empty totally geodesic boundary.
2. \( \Omega := S^{d-1} - \Lambda \) is a dense union \( \bigcup_{i=1}^{\infty} B_i \) of infinitely many round balls with mutually disjoint closures.

In the whole section, let \( M \) be a convex cocompact hyperbolic manifold of non-empty Fuchsian ends.
**Renormalized frame bundle.** The renormalized frame bundle $RF M \subset FM$ is defined as the following $AM$-invariant subset

$$RF M = \{ [g] \in \Gamma \backslash G : g^+ \in \Lambda \} = \{ x \in \Gamma \backslash G : xA \text{ is bounded} \}.$$  

Unless mentioned otherwise\(^2\), we set $A^+ = \{ a_t : t \geq 0 \}$. We define

$$RF_+ M = \{ [g] \in \Gamma \backslash G : g^+ \in \Lambda \} = \{ x \in \Gamma \backslash G : xA^+ \text{ is bounded} \}$$

which is a closed $NAM$-invariant subset. As $\pi(xNA) = \pi(xG) = M$ for any $x \in \Gamma \backslash G$, we have $\pi(RF_+ M) = M$.

It is easy to verify:

**Lemma 3.2.** For $x \in RF_+ M$, $\overline{xA^+}$ meets $RF M$.

**$H(U)$-invariant subsets:** $F_{H(U)}, F_{H(U)}^*, \partial F_{H(U)}$. Fix a non-trivial connected subgroup $U < N$, and consider the associated subgroups $H(U)$ and $H'(U)$ as defined in section 2.

We define

$$(3.1) \quad F_{H(U)} := RF_+ M \cdot H(U).$$

We denote by $F_{H(U)}^*$ the interior of $F_{H(U)}$ and by $\partial F_{H(U)}$ the boundary of $F_{H(U)}$. When there is no room for confusion, we will omit the subscript $H(U)$ and simply write $F, F^*$ and $\partial F$.

The closedness of $F$ is an easy consequence of compactness of the limit set $\Lambda$. It is also $H'(U)$-invariant, since $RF_+ M$ is $M$-invariant and $C(H(U))$ is contained in $M$. For $g \in G$, we denote by $C_g = C_{gH(U)} \subset S^{d-1}$ the sphere given by the boundary of the geodesic plane $\pi(gH(U))$. Then $\text{hull} C_g = \pi(gH(U))$, and $C_g = gH(U)^+ = gH(U)^-$ where $H(U)^\pm = \{ h^\pm : h \in H(U) \}$. It follows that $F = \{ [g] \in \Gamma \backslash G : C_g \cap \Lambda \neq \emptyset \}$.

**Lemma 3.3.** We have $F = \{ x \in \Gamma \backslash G : \pi(xH(U)) \cap \text{core } M \neq \emptyset \}$.

**Proof.** Let $x \in F$. By modifying it using an element of $H(U)$, we may assume that $x \in RF_+ M$. By Lemma 3.2, $\overline{xA^+}$ contains $x_0 \in RF M$. Since $x_0 A$ is bounded, $\pi(x_0 A)$ is a bounded geodesic, and hence $\pi(x_0 A) \subset \pi(xH(U)) \cap \text{core } M$ because $\text{core } M$ contains all bounded geodesics. This proves the inclusion $\subset$. Now suppose $x = [g] \notin F$. Then $C_g \cap \Lambda = \emptyset$, and hence $C_g$ must be contained in a connected component, say $B_i$, of $\Omega$. Hence $\pi(gH(U)) = \text{hull}(C_g)$ is contained in the interior of $\text{hull}(B_i)$, which is disjoint from $\text{hull}(\Lambda)$, by the convexity of $B_i$. Therefore the orbit $\Gamma \pi(gH(U))$ is a closed subset of $\mathbb{H}^d$, disjoint from $\text{hull}(\Lambda)$. This proves $\supset$. \qed

Denote by $M^*$ the interior of the core of $M$. Then $F^* = \{ x \in \Gamma \backslash G : \pi(xH(U)) \cap M^* \neq \emptyset \}$. Note that for $[g] \in F$, $\# C_g \cap \Lambda \geq 2$ and hence

$$(3.2) \quad F^* \subset RF M \cdot H(U).$$

In particular, $RF M \cdot H(U)$ is dense in $F$.

\(^2\)At certain places, we use notation $A^+$ for any subsemigroup of $A$
Lemma 3.4. We have $RF_{+}M \cap F^{*} \subset RFM \cdot U$.

Proof. Let $y = [g] \in RF_{+}M \cap F^{*}$. We need to show that $yU \cap RFM \neq \emptyset$. As $y \in RF_{+}M$, $g^{\pm} = g(\infty) \in \Lambda$, and hence $C_{g} \cap \Lambda \neq \emptyset$. If $\#C_{g} \cap \Lambda = 1$, then $C_{g}$ must be contained in $\overline{B_{i}}$ for some $i$, which implies $[g] \notin F_{H(U)}^{*}$. Therefore $\#C_{g} \cap \Lambda \geq 2$. We note that $gU(0) \cup \{g(\infty)\} = C_{g}$; this is clear when $U = U_{k}$ for some $k \geq 1$ and $g = e$, to which a general case is reduced. Hence there exists $u \in U$ such that $gu(0) \in \Lambda$. Since $gu(\infty) = g(\infty) \in \Lambda$, we have $yu = [g]u \in RFM$. \qed

We call an oriented frame $g = (v_{0}, \ldots, v_{d-1}) \in FM = G$ a boundary frame if the first $(d - 1)$ vectors $v_{0}, \ldots, v_{d-2}$ are tangent to the boundary of core $M$. Set $\tilde{H} := H(U_{d-2}) = SO^{*}(d - 1, 1)$, and denote by $\tilde{V}$ the one-dimensional subgroup $\mathbb{R}e_{d-1}$ of $N = \mathbb{R}^{d-1}$; note that $\tilde{V} = (\tilde{H} \cap N)^{\perp}$. We denote by $BFM$ the set of all boundary frames of $M$; it is a union of compact $\tilde{H}$-orbits: $BFM = \bigcup_{i=1}^{k} z_{i} \tilde{H}$ such that $\pi(z_{i}H) = \Gamma \backslash \Gamma$ hull$(B_{i})$ for some component $B_{i}$ of $\Omega$.

The boundary $\partial F$ for $U < \tilde{H} \cap N$. Suppose that $U$ is contained in $\tilde{H} \cap N = \mathbb{R}^{d-2}$. Then there exists a one-parameter semigroup $V^{+}$ of $\tilde{V}$ such that $\partial F = BM \cdot V^{+} \cdot H'(U)$. We use the notation $V^{-} = \{v^{-1} : v \in V^{+}\}$. Note that

\begin{equation}
\partial F \cap RFM = BM \cdot C(H(U)); \partial F \cap RF_{+}M = BM \cdot V^{+} \cdot C(H(U)).
\end{equation}

For a general proper connected closed subgroup $U < N$, $mUm^{-1} \subset \tilde{H} \cap N$ for some $m \in M$, and $\partial F \cap RFM = BMm C(H(U))$ where $BFM$ is now a union of finitely many $m^{-1}\tilde{H}m$-compact orbits.

Lemma 3.5. Let $U < \tilde{H} \cap N$, $z \in BM$ and $v \in \tilde{V} - \{e\}$. If $zv \in RFM$, then $zv \in F^{*}$.

Proof. Let $z = [g] \in BM$. Then $\partial(\pi(g\tilde{H})) = \partial B_{j}$ for some $j$. Let $v \in \tilde{V} - \{e\}$ be such that $zv \in RFM$. Suppose $zv \in \partial F_{H(U)}$. Then $C_{g}v \subset \overline{B_{i}}$ for some $i$. Since the sphere $C_{g}v = \{gvh(\infty) : h \in H(U)\}$ contains $g(\infty)$ which belongs to $\partial B_{j}$, we have $i = j$, as $\overline{B_{i}}$’s are mutually disjoint. As $zv \in RFM$, $C_{g}v \subset \partial B_{j}$. Hence $gvH(U)^{+} \subset g\tilde{H}$, and hence $vH(U) \cap \tilde{H} \neq \emptyset$, which is a contradiction since $v \notin \tilde{H}$, and $H(U) \subset \tilde{H}$. \qed

Properly immersed geodesic planes. Let $H = H(U_{k})$ and $H' = H'(U_{k})$ be as in (2.6), and $p$ be the map in (2.7). In (2.8), if $p(\Gamma \xi) \backslash S$ is a convex cocompact hyperbolic $k$-manifold with Fuchsian ends and $f$ is proper, then the image $S = \text{Im}(f)$ is referred to as a properly immersed convex cocompact geodesic $k$-plane of Fuchsian ends.

Proposition 3.6. If $xH'$ is closed for $x \in RFM$, then $S = \pi(xH')$ is a properly immersed convex cocompact geodesic plane with (possibly empty) Fuchsian ends.
Proof. Choose \( g \in G \) so that \( x = [g] \). Let \( \tilde{S} \) and \( \Gamma_{\tilde{S}} \) be as in Proposition 2.8. Set \( C = \partial \tilde{S} \). By loc. cit., \( S \) is properly immersed, and \( C \cap \Lambda = \Lambda(p(\Gamma_{\tilde{S}})) \). Write \( C - (C \cap \Lambda) = \bigcup_{i \in I}(C \cap B_i) \) where \( I \) is the collection of all \( i \) such that \( C \cap B_i \neq \emptyset \). If \( C \cap \Lambda \) contains a non-empty open subset of \( C \), then the limit set of \( p(\Gamma_{\tilde{S}}) \) has Hausdorff dimension equal to the dimension of \( C \). So \( p(\Gamma_{\tilde{S}}) \) is a uniform lattice in \( gHg^{-1} \), and hence \( S \) is compact. In the other case, \( I \) is an infinite set and \( \bigcup_{i \in I}(C \cap B_i) \) is dense in \( C \); so \( S \) is a a convex cocompact hyperbolic submanifold of Fuchsian ends by Definition 3.1(2). \( \square \)

**Lemma 3.7.** For a sphere \( C \subset S^{d-1} \) with \( \# C \cap \Lambda \geq 2 \), the intersection \( C \cap \Lambda \) is Zariski dense in \( C \).

Proof. The claim is clear if \( C \cap \Lambda \) contains a non-empty open subset of \( C \). If not, \( C \cap \Lambda \) contains infinitely many \( C \cap \partial B_i \)'s, each of which is an irreducible co-dimension one real subvariety of \( C \). It follows that the Zariski closure of \( C \cap \Lambda \) has dimension strictly greater than \( \dim C - 1 \), hence is equal to \( C \). \( \square \)

We let \( \pi_1 : H' \to H \) and \( \pi_2 : H' \to C(H) \) denote the canonical projections.

**Proposition 3.8.** Suppose that \( xH' \) is closed for \( x = [g] \in RF M \), and set \( \Gamma' := g^{-1} \Gamma g \cap H' \). Then \( \overline{xH} = xHC \) where \( C = \pi_2(\Gamma') \) and \( HC \) is equal to the identity component of the Zariski closure of \( \Gamma' \).

Proof. Without loss of generality, we may assume \( g = e \). As \( H' \) is a direct product \( H \times C(H) \), we write an element of \( H' \) as \((h, c)\) with \( h \in H \) and \( c \in C(H) \). For all \( \gamma \in \Gamma' \), \( xH = [(e, e)]H = [(e, \pi_2(\gamma))]H = [(e, e)]H\pi_2(\gamma) \) and hence \( xH = xH\pi_2(\Gamma') \). It follows that \( xHC \subset \overline{xH} \). To show the other inclusion, let \((h_0, c_0) \in H \ C(H) \) be arbitrary. If \([h_0, c_0] \in \overline{xH} = [(e, e)]H \), then there exist sequences \( \gamma_i \in \Gamma' \) and \( h_i \in H \) such that \( \gamma_i(h_i, e) \to (h_0, c_0) \) in \( H' \) as \( i \to \infty \). In particular, \( \pi_2(\gamma_i) \to c_0 \) in \( C(H) \) as \( i \to \infty \) and hence \( c_0 \in C = \overline{\pi_2(\Gamma')} \). This proves \( \overline{xH} = xHC \). Let \( W \) denote the identity component of the Zariski closure of \( \Gamma' \) in \( H' \). Since any proper algebraic subgroup of \( G \) stabilizes either a point, or a proper sphere in \( S^{d-1} \), it follows from Proposition 2.8 and Lemma 3.7 that \( \pi_1(\Gamma') \) is Zariski dense in \( H \); so \( \pi_1(W) = H \). So the quotient \( W \backslash H' \) is compact. This implies that \( W \) contains a maximal real-split connected solvable subgroup, say, \( P \) of \( H' \).

Now \( H \cap W \) is a normal subgroup of \( H \), as \( \pi_1(W) = H \). Since \( P < H \cap W \) and \( H \) is simple, we conclude that \( H \cap W = H \), i.e., \( H < W \). Hence \( W = H\pi_2(W) \). As any compact linear group is algebraic, \( C \) is algebraic and hence \( C = \pi_2(W) = \pi_2(\Gamma') \). Therefore \( W = HC \), finishing the proof. \( \square \)

**Global thickness of the return time to RF M.** We recall the various notions of thick subsets of \( \mathbb{R} \), following [16] and [18].

**Definition 3.9.** Fix \( k > 1 \).

- A closed subset \( \mathcal{T} \subset \mathbb{R} \) is locally \( k \)-thick at \( t \) if for any \( \lambda > 0 \), \( \mathcal{T} \cap (t \pm [\lambda, k\lambda]) \neq \emptyset \).
• A closed subset $\mathcal{T} \subset \mathbb{R}$ is $k$-thick if $\mathcal{T}$ is locally $k$-thick at 0.
• A closed subset $\mathcal{T} \subset \mathbb{R}$ is $k$-thick at $\infty$ if $\mathcal{T} \cap (\pm [\lambda, k\lambda]) \neq \emptyset$ for all sufficiently large $\lambda \gg 1$.
• A closed subset $\mathcal{T} \subset \mathbb{R}$ is globally $k$-thick if $\mathcal{T} \neq \emptyset$ and $\mathcal{T}$ is locally $k$-thick at every $t \in \mathcal{T}$.

We will frequently use the fact that if $T_i$ is a sequence of $k$-thick subsets, then $\limsup T_i$ is also $k$-thick, and that if $\mathcal{T}$ is $k$-thick, so is $-\mathcal{T}$.

**Proposition 3.10.** There exists a constant $k > 1$ depending only on the systole of the double of core $M$ such that for any one-parameter subgroup $U = \{u_t : t \in \mathbb{R}\}$ of $N^\pm$, and any $y \in RF M$, $T(y) := \{t \in \mathbb{R} : yu_t \in RF M\}$ is globally $k$-thick.

**Proof.** Let $\eta > 0$ be the systole of the hyperbolic double of core $M$, which is a closed hyperbolic manifold. Let $k > 1$ be given by the equation

$$d(\text{hull}([-k, -1]), \text{hull}([1, k])) = \eta/4$$

where $d$ is the hyperbolic distance in the upper half plane $\mathbb{H}^2$. Note that

$$\inf_{i \neq j} d(\text{hull}B_i, \text{hull}B_j) \geq \eta/2$$

as the geodesic realizing this distance is either a closed geodesic or half of a closed geodesic in the double of core $M$.

We first prove the case when $U < N$. Let $s \in T(y)$ be arbitrary. To show that $T(y)$ is locally $k$-thick at $s$, we may assume that $s = 0$, by replacing $y$ with $yu_s \in RF M$. We may also assume that $y = [g]$ where $g(\infty) = \infty$ and $g(0) = 0$. As $y \in RF M$, this implies that $0, \infty \in \Lambda$. Since $gu_t(\infty) = g(\infty) \in \Lambda$, we have $T(y) = \{t \in \mathbb{R} : gu_t(0) \in \Lambda\}$. Suppose that $T(y)$ is not locally $k$-thick at 0. Then there exist $w \in U$ and $t > 0$ such that $([-kt, -t] \cdot w \cup [t, kt] \cdot w) \cap \Lambda = \emptyset$. Since each component of $\Omega$ is convex and $0 \not\in \Omega$, it follows that $[-kt, -t] \cdot w$ and $[t, kt] \cdot w$ lie in distinct components of $\Omega$, say $B_i$ and $B_j$, ($i \neq j$). But this yields

$$d_w(\text{hull}([-kt, -t] \cdot w), \text{hull}([t, kt] \cdot w)) \geq d(\text{hull}B_i, \text{hull}B_j) \geq \eta/2$$

where $d_w$ denotes the hyperbolic distance of the plane above the line $\mathbb{R}w$. Observe that the distance in (3.6) is independent of $w \in \mathbb{R}^{d-1}$ and $t > 0$, because both the dilation centered at 0 and the $(d-2)$-dimensional rotation with respect to the vertical axis above 0 are hyperbolic isometries. Therefore, we get a contradiction to (3.4). The case of $U < N^+$ is proved similarly, just replacing the role of $g^+$ and $g^-$ in the above arguments. $\square$

4. Structure of singular sets

Let $\Gamma < G = \text{SO}^0(d, 1)$ be a convex cocompact torsion-free Zariski-dense subgroup. Let $U < G$ be a connected closed subgroup of $G$ generated by unipotent elements in it. In this section, we define the singular set $\mathcal{S}(U)$ associated to $U$ and study its structural property.
**Definition 4.1.** (Singular set). We set

\[ \mathcal{I}(U) = \left\{ x \in \Gamma \setminus G : \begin{array}{c}
\text{there exists a proper connected} \\
\text{closed subgroup } W \supset U \text{ such that } xW \\
\text{is closed and } \text{Stab}_W(x) \text{ is Zariski dense in } W. 
\end{array} \right\}. \]

**Definition 4.2.** We denote by \( \mathcal{H} \) the collection of all proper connected closed subgroups \( H < G \) containing a unipotent element such that \( \Gamma \setminus \Gamma H \) is closed, and \( H \cap \Gamma \) is Zariski dense in \( H \).

**Proposition 4.3.** Any \( H \in \mathcal{H} \) is a reductive subgroup of \( G \), and hence is of the form \( gH(U)Cg^{-1} \) for some connected subgroup \( U < N \), a closed subgroup \( C < C(H(U)) \) and \( g \in G \) such that \( [g] \in RF M \).

**Proof.** Suppose \( H \in \mathcal{H} \) is not reductive. Then its unipotent radical is non-trivial, which we can assume to be a subgroup \( U \) of \( N \), up to conjugation. Now we write \( H = H_{nc}CTU \) where \( C \) is a connected semisimple compact subgroup and \( T \) is a torus centralizing \( H_{nc} \). As \( H \) is contained in \( N(U) = NA C_1(U) C_2(U) \), which does not contain any non-compact simple Lie subgroup, it follows that \( H_{nc} \) is trivial. Now if \( T \) were compact, then \( H \cap \Gamma \) would consist of parabolic elements, which is a contradiction as \( \Gamma \) is convex cocompact. Hence \( T \) is non-compact. Write \( T = T_0S \) where \( S \) is a split torus and \( T_0 \) is compact. Then \( T_0 \) is equal to a conjugate of \( A \), say, \( g^{-1}Ag \) for some \( g \in G \). As \( T_0 \) normalizes \( U \), and \( N(U) \) fixes \( x \), we deduce that \( g(x) \) is either either \( 0 \) or \( \infty \). Since \( Stab_G(\infty) = NAM \), \( g(\infty) = \infty \) implies \( g \in NAM \), and \( g(\infty) = 0 \) implies \( jg \in NAM \) where \( j \in G \) is an element of order 2 such that \( j(0) = \infty \). In either case, \( T_0 = v^{-1}Av \) for some \( v \in N \).

By replacing \( H \) with \( vHv^{-1} \), we may assume that \( T_0 = A \). Since \( CS \) is a compact subgroup commuting with \( A \), \( CS \subset M \). Therefore \( H \) is of the form \( M_0A \) where \( M_0 \) is a closed subgroup of \( M \cap N(U) \); note that we used the fact that \( v \) commutes with \( U \). Now the commutator subgroup \([H, H]\) is equal to \([M_0, M_0]U\). Since \([H \cap \Gamma, H \cap \Gamma]\) must be Zariski dense in \([H, H]\), we deduce that \( \Gamma \) contains an element \( m_0u \in M_0U \) with \( u \) non-trivial. Since \( m_0u \) is a parabolic element of \( \Gamma \), this is a contradiction to the assumption that \( \Gamma \) is convex cocompact. This proves that \( H \) is reductive.

By Proposition 2.6, \( H \) is of the form \( gH(U)Cg^{-1} \) for some \( g \in G \) and \( C < C(H(U)) \). For some \( m \in M \) and \( 1 \leq k \leq d - 2 \), \( H(U) = mH(U_k)m^{-1} \). Hence \( \Gamma \setminus \Gamma gmH(U_k)C_0 \) is closed where \( C_0 = m^{-1}Cm \). By Proposition 2.8, the boundary of the geodesic plane \( \pi(gmH(U_k)) \) contains uncountably many points of \( \Lambda \), since \((gm)H(U_k)C_0(gm)^{-1} \cap \Gamma \) is Zariski dense in \((gm)H(U_k)C_0(gm)^{-1} \). Using two such limit points, we can find an element \( h \in H(U_k) \) such that \((gmh)_{\pm} \in \Lambda \). Since \((gmh)^{-1} = (gmh)_{\pm} \) and \( mh^{-1} \in H(U) \), it follows that \([g]H(U) \cap RF M \neq \emptyset \), and hence we can take \([g] \in RF M \) by modifying it with an element of \( H(U) \) if necessary. This finishes the proof. \( \Box \)

Therefore, for each \( H \in \mathcal{H} \), the non-compact semisimple part \( H_{nc} \) of \( H \) is well defined.
Proposition 4.4. For any $H \in \mathcal{H}$, $H \cap \Gamma$ is finitely generated and $[N_G(H_{nc}) \cap \Gamma : H \cap \Gamma] < \infty$.

Proof. Let $p$ denote the projection $N_G(H_{nc}) \to H_{nc}$. Note that $p$ is an injective map on $N_G(H_{nc}) \cap \Gamma$, as $\Gamma$ is torsion free and the kernel of $p$ is a compact subgroup. It follows from Proposition 4.3 that $H_{nc}$ is co-compact in $N_G(H_{nc})$. Since $H \in \mathcal{H}$, the orbit $[e]H$ is closed and hence $[e]N_G(H_{nc})$ is closed. It follows that both $p(H \cap \Gamma)$ and $p(N_G(H_{nc}) \cap \Gamma)$ are convex cocompact Zariski dense subgroups of $H_{nc}$ by Proposition 2.8. As any convex cocompact subgroup is finitely generated [3], $p(H \cap \Gamma)$ is finitely generated. Hence $H \cap \Gamma$ is finitely generated by the injectivity of $p|_{H \cap \Gamma}$. Since $p(H \cap \Gamma)$ is a normal subgroup of $p(N_G(H_{nc}) \cap \Gamma)$, it follows that $p(H \cap \Gamma)$ has finite index in $p(N_G(H_{nc}) \cap \Gamma)$ by Lemma 4.5 below. Since $p|_{N_G(H_{nc}) \cap \Gamma}$ is injective, it follows that $H \cap \Gamma$ has finite index in $N_G(H_{nc}) \cap \Gamma$. □

Lemma 4.5. Let $\Gamma_1$ and $\Gamma_2$ be non-elementary convex cocompact subgroups of $G$. If $\Gamma_2$ is a normal subgroup of $\Gamma_1$, then $[\Gamma_1 : \Gamma_2] < \infty$.

Proof. Let $\Lambda_i$ be the limit set of $\Gamma_i$ for $i = 1, 2$. Since $\Gamma_2 \subset \Gamma_1$, $\Lambda_2 \subset \Lambda_1$. As $\Gamma_2$ is normalized by $\Gamma_1$, $\Lambda_2$ is $\Gamma_1$-invariant. Since $\Gamma_1$ is non-elementary, $\Lambda_1$ is a minimal $\Gamma_1$-invariant closed subset. Hence $\Lambda_1 = \Lambda_2$. Let $M_i := \Gamma_i \setminus \mathbb{H}^d$. Then the convex core of $M_1$ is equal to $\Gamma_1 \setminus \text{hull}(\Lambda_2)$ and covered by core $M_2 = \Gamma_2 \setminus \text{hull}(\Lambda_2)$. Since core $M_2$ is compact, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. □

Definition 4.6 (Definition of $\mathcal{H}^*$).

(4.1) \[ \mathcal{H}^* := \{N_G(H_{nc}) : H \in \mathcal{H}\}. \]

Corollary 4.7 (Countability). The collection $\mathcal{H}$ is countable, and the map $H \to N_G(H_{nc})$ defines a bijection between $\mathcal{H}$ and $\mathcal{H}^*$.

Proof. As $\Gamma$ is convex cocompact, it is finitely generated. Therefore there are only countably many finitely generated subgroups of $\Gamma$. By Proposition 4.4, there are only countably many possible $H \cap \Gamma$ for $H \in \mathcal{H}$. Since $H$ is determined by $H \cap \Gamma$, being its Zariski closure, the first claim follows. Since $H \cap \Gamma$ has finite index in $N_G(H_{nc}) \cap \Gamma$ by Proposition 4.4, $H$ is determined as the identity component of the Zariski closure of $N_G(H_{nc}) \cap \Gamma$. This proves the second claim. □

In the case of a convex cocompact hyperbolic manifold of Fuchsian ends, there is a one to one correspondence between $\mathcal{H}$ and the collection of all closed $H'_U$-orbits of points in $\text{RF} M$ for $U < N$: if $H \in \mathcal{H}$, then $H = gH(U)Cg^{-1}$ for some $U < N$ and $g \in G$ with $[g] \in \text{RF} M$ and $[g]H'(U)$ is closed. Conversely, if $[g]H'(U)$ is closed for some $[g] \in \text{RF} M$, then the identity component of the Zariski closure of $\Gamma \cap qH'(U)g^{-1}$ is given by $gH(U)Cg^{-1}$ for some closed subgroup $C < C(H(U))$ by Proposition 3.8, and hence $gH(U)Cg^{-1} \in \mathcal{H}$. Moreover, since the normalizer of $H(U)C$ is contained in $H'(U)$ if $g_1H(U)Cg_1^{-1} = g_2H(U)Cg_2^{-1}$, then $g_2^{-1}g_1 \in H'(U)$, so $[g_1]H'(U) = [g_2]H'(U)$. Therefore Corollary 4.7 implies the following corollary by Propositions 2.8 and 3.8.
Corollary 4.8. Let $M$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Then

1. there are only countably many properly immersed geodesic planes of dimension at least 2 intersecting core $M$.
2. For each $1 \leq m \leq d-2$, there are only countably many spheres $S \subset S^{d-1}$ of dimension $m$, such that $\#S \cap \Lambda \geq 2$ and $\Gamma S$ is closed in the space $\mathbb{C}^m$.

Remark 4.9. In (2), we may replace the condition $\#S \cap \Lambda \geq 2$ with $\#S \cap \Lambda \geq 1$, because if $\#S \cap \Lambda = 1$, then $\Gamma S$ is not closed (see Remark 10.6).

For a subgroup $H < G$, define

\[ X(H, U) := \{ g \in G : gUg^{-1} \subset H \}. \tag{4.2} \]

Note that $X(H, U)$ is left-$N_G(H)$ and right-$N_G(U)$-invariant, and for any $g \in G$, $X(gHg^{-1}, U) = gX(H, U)$. For $H \in \mathcal{H}$ and any connected unipotent subgroup $U < G$, observe that $X(H, U) = X(H_{nc}, U) = X(N_G(H_{nc}), U)$; this follows since any unipotent element of $N_G(H_{nc})$ is contained in $H_{nc}$.

Proposition 4.10. We have $\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}^*} \Gamma \backslash X(H, U)$.

Proof. If $x = [g] \in \mathcal{S}(U)$, then there exists a proper connected closed subgroup $W$ of $G$ containing $U$ such that $[g]W$ is closed and Stab$_W(x)$ is Zariski dense in $W$. This means $H := gWg^{-1} \in \mathcal{H}$ and $g \in X(H, U)$. Since $X(H, U) = X(N_G(H_{nc}), U)$, and $N_G(H_{nc}) \in \mathcal{H}^*$, this proves the inclusion $\subset$. Conversely, let $g \in X(N_G(H_{nc}), U)$ for some $H \in \mathcal{H}$. Set $W := g^{-1}Hg$. Then $U \subset W$, $[g]W = \Gamma Hg$ is closed and Stab$_W([g]) = g^{-1}(\Gamma \cap H)g$ is Zariski dense in $W$. Hence $[g] \in \mathcal{S}(U)$. \hfill $\square$

Singular subset of a closed orbit. Let $L < G$ be a connected reductive subgroup of $G$ containing unipotent elements. For a closed orbit $x_0L$ of $x_0 \in RF M$, and a connected subgroup $U_0 < L \cap N$, we define the singular set $\mathcal{S}(U_0, x_0L)$ by the following:

\[ \left\{ \begin{array}{l}
\text{there exists a connected closed subgroup } W < L, \\
x \in x_0L : \text{ containing } U_0 \text{ such that } \dim W_{nc} < \dim L_{nc}, \\
xW \text{ is closed and Stab}_W(x) \text{ is Zariski dense in } W
\end{array} \right\}. \tag{4.3} \]

It follows from Proposition 4.10 and Proposition 4.3 that the subgroup $W$ in the definition 4.1 is conjugate to $H(\tilde{U})C$ for some $\tilde{U} < N$. Hence $W$ being a proper subgroup of $G$ is same as requiring $\dim W_{nc} < \dim G$. Therefore $\mathcal{S}(U_0) = \mathcal{S}(U_0, \Gamma \backslash G)$ and $\mathcal{S}(U_0, x_0L) = x_0L \cap \bigcup \Gamma \backslash X(H, U_0)$ where the union is taken over all subgroups $H \in \mathcal{H}^*$ such that $H$ is a subgroup of $g_0Lg_0^{-1}$ with $\dim H_{nc} < \dim L_{nc}$ and $x_0 = [g_0]$. Equivalently,

\[ \mathcal{S}(U_0, x_0L) = \bigcup_{W \in \mathcal{H}^*} x_0(L \cap X(W, U_0)) \tag{4.4} \]
where \( \mathcal{H}_{\mathfrak{sp}} \) consists of all subgroups of the form \( W = g_0^{-1} H g_0 \cap L \) for some \( H \in \mathcal{H}^* \) and \( \dim W_{nc} < \dim L_{nc} \). Then the generic set \( \mathcal{S}(U_0, x_0L) \) is defined by

\[
(4.5) \quad \mathcal{S}(U_0, x_0L) := (x_0L \cap RF_+ M) - \mathcal{S}(U_0, x_0L).
\]

**Definition of \( \mathcal{L}_U \) and \( \mathcal{Q}_U \).** Fix a non-trivial connected closed subgroup \( U < N \). We define the collection \( \mathcal{L}_U \) of all subgroups of the form \( H(\hat{U})C \) where \( U < \hat{U} < N \) and \( C \) is a closed subgroup of \( C(H(\hat{U})) \) satisfying the following:

\[
(4.6) \quad \mathcal{L}_U := \left\{ L = H(\hat{U})C : \text{ for some } [g] \in RF_+ M, [g]L \text{ is closed in } \Gamma \backslash G \text{ and } L \cap g^{-1} \Gamma g \text{ is Zariski dense in } L \right\}.
\]

Observe that for \( L = H(\hat{U})C \neq G \), the condition \( L \in \mathcal{L}_U \) with \([g]L \) closed is equivalent to the condition that \( gLg^{-1} \in \mathcal{H} \).

**Lemma 4.11.** Let \( L_1 \) and \( L_2 \) be members of \( \mathcal{L}_U \) such that \( xL_1 \) and \( xL_2 \) are closed for some \( x \in RF M \). If \((L_1)_{nc} = (L_2)_{nc} \), then \( L_1 = L_2 \).

**Proof.** If \( L_1 \) or \( L_2 \) is equal to \( G \), then the claim is trivial. Suppose that both \( L_1 \) and \( L_2 \) are proper subgroups of \( G \). If \( x = [g] \), then both subgroups \( H_1 := gL_1g^{-1} \) and \( H_2 := gL_2g^{-1} \) belong to \( \mathcal{H} \). Since \((H_1)_{nc} = (H_2)_{nc} \), we have \( H_1 = H_2 \) by Corollary 4.7. Hence \( L_1 = L_2 \). \( \square \)

We also define

\[
(4.7) \quad \mathcal{Q}_U := \{vLv^{-1} : L \in \mathcal{L}_U, v \in N(U)\}.
\]

Since \( N(U) = AN C_1(U) C_2(U) \) by Lemma 2.1, and the collection \( \mathcal{L}_U \) is invariant under a conjugation by an element of \( AU C_1(U) C_2(U) \), we have

\[
(4.8) \quad \mathcal{Q}_U = \{vLv^{-1} : L \in \mathcal{L}_U, v \in U^\perp\}.
\]

**Lemma 4.12.** For \( U_0 < U < N \), we have \( X(H(U), U_0) = N_G(H(U)) N_G(U_0) \).

**Proof.** Without loss of generality, we may assume \( U = U_{\ell} \) and \( U_0 = U_\ell \) with \( 1 \leq \ell \leq m \leq d - 1 \). Set \( H = H(U_m) \). If \( m = d - 1 \), then \( H = G \), and the statement is trivial. Assume \( m \leq d - 2 \) below. We will prove the inclusion \( X(H, U_0) \subset N_G(H)N_G(U_0) \), as the other one is clear. Let \( g \in X(H, U_0) \) be arbitrary. By multiplying \( g \) by an element of \( N_G(H) \) on the left as well as by an element of \( N_G(U_0) \) on the right, we will reduce \( g \) to an element of \( N_G(U_0) \), which implies the claim. In view of the Iwasawa decomposition \( G = KAN \), since \( AN < N_G(U_0) \), we may assume that \( g = k \in K \). As \( k \in X(H, U_0) \), we have \( kU_0k^{-1} \subset H \). Hence there exists \( w \in K \cap H \) such that \( kU_0k^{-1} = wkU_0w^{-1} \). Since \( w^{-1}kU_0 = U_0w^{-1}k \), we deduce \( w^{-1}k(\infty) = U_0(w^{-1}k(\infty)) \). Since \( \infty \in S^{d-1} \) is the unique fixed point of \( U_0 \), \( w^{-1}k(\infty) = \infty \). Hence \( w^{-1}k \in K \cap (MAN) = M \). Since \( w \in H \), we may now assume that \( k \in M \). From \( kU_0 \subset Hk \), we get \( kU_0(0) \subset Hk(0) = H(0) \) and hence \( \langle ke_1, \ldots, ke_\ell \rangle \subset \langle e_1, \ldots, e_\ell \rangle \). By considering the action of \( H \cap K \) on space of \( \ell \)-tuples of orthonormal vectors in the subspace \( \langle e_1, \ldots, e_\ell \rangle \), we
Lemma 4.14. If \( k \in C_{1}(U_{0}) \), or \( \omega \in C_{1}(U_{0}) \) where \( \omega \in M \) is an involution which fixes all \( e_{i}, i \neq \ell, \ell + 1 \) and \( \omega(e_{i}) = -e_{i} \) for \( i = \ell, \ell + 1 \). As \( N_{G}(U_{0}) \) contains \( C_{1}(U_{0}) \) and \( \omega \), the proof is complete. \( \square \)

Proposition 4.13. Consider a closed orbit \( x_{0}L \) for \( L \in Q_{U} \) and \( x_{0} \in RF \ M \). If \( x \in \mathcal{S}(U_{0}, x_{0}L) \) for a connected closed subgroup \( U_{0} \subset U \), then there exists a subgroup \( Q \in Q_{U_{0}} \) such that \( \dim Q_{nc} < \dim L_{nc} \). \( xQ \) is closed and \( \overline{xU_{0}} \subset xQ \).

Proof. If \( x = [g] \in \mathcal{S}(U_{0}, x_{0}L) \), then \( g \in X(H, U_{0}) \) for some \( H \in \mathcal{H} \) such that \( \dim H_{nc} < \dim L_{nc} \). Then \( \overline{xU_{0}} \subset x(g^{-1}Hg) \). By Proposition 4.3, \( H = qH(\tilde{U})C_{q}^{-1} \) for some \( U_{0} < \tilde{U} < L \cap N \) and some \( [q] \in RF \ M \). Note that \( q^{-1}g \in X(H(\tilde{U}), U_{0}) \). By Lemma 4.12, we have \( q^{-1}g \in N_{G}(H(\tilde{U}))N_{G}(U_{0}) \). Hence \( g^{-1}Hg = vH(\tilde{U})C_{v}^{-1} \) for some \( v \in N_{G}(U_{0}) \), and \( \overline{xU_{0}} \subset xvH(\tilde{U})C_{v}^{-1} \). It suffices to set \( Q := vH(\tilde{U})C_{v}^{-1} \). \( \square \)

Lemma 4.14. Let \( L = H(\tilde{U})C \) for a connected closed subgroup \( \tilde{U} \subset N \) and closed subgroup \( C < C(H(\tilde{U})) \). Let \( W = g^{-1}H(\tilde{U})C_{0}g \) be a subgroup of \( L \) where \( g \in L \), \( \tilde{U} \) is a proper connected closed subgroup of \( \tilde{U} \) and \( C_{0} \) is a closed subgroup of \( H(\tilde{U}) \). Then for any non-trivial closed connected subgroup \( U \subset \tilde{U} \), \( (L \cap X(W, U))H(U) \) is a nowhere dense subset of \( L \).

Proof. Write \( g = hc \in H(\tilde{U})C \). Note that

\[
\begin{align*}
L \cap X(W, U) &= L \cap X(g^{-1}H(\tilde{U})g, U) = L \cap X(h^{-1}H(\tilde{U})h, U) \\
&= h(L \cap X(H(\tilde{U}), U)) = h(H(\tilde{U}) \cap X(H(\tilde{U}), U))C.
\end{align*}
\]

Hence it suffices to show that \( (H(\tilde{U}) \cap X(H(\tilde{U}), U))H(U) \) is a nowhere dense subset of \( H(\tilde{U}) \). Without loss of generality, we may now assume \( H(\tilde{U}) = G \). We observe that using Lemma 4.12,

\[
X(H(\tilde{U}), U)H(U) = N_{G}(H(\tilde{U}))N_{G}(U)H(U) = H(\tilde{U})C_{1}(\tilde{U})AN_{C_{1}(U)}C_{2}(U)H(U) = (K \cap H(\tilde{U}))U^{-1}H'(U).
\]

Let \( \dim \tilde{U} = m \) and \( \dim U = k \). Then \( 1 \leq k \leq m < d - 1 = \dim N \). Now, if we view the subset \( (K \cap H(\tilde{U}))U^{-1}H'(U)/H'(U) \) in the space \( C^{k} = G/H'(U) \), this set is contained in the set of all spheres \( C \in C^{k} \) which are tangent to the \( m \)-sphere given by \( S_{0} := (K \cap H(\tilde{U}))(\infty) \). Since \( m < d - 1 \), it follows that \( X(H(\tilde{U}), U)H(U)/H'(U) \) is a nowhere dense subset of \( C^{k} \), and hence \( X(H(\tilde{U}), U)H(U) \) is a nowhere dense subset of \( G \). \( \square \)

Lemma 4.15. Let \( x_{0}\tilde{L} \) be a closed orbit of \( \tilde{L} \in L_{U} \) with \( x_{0} \in RF \ M \). If \( U \) is a proper subgroup of \( \tilde{L} \cap N \), then \( \mathcal{S}(U, x_{0}\tilde{L}) \cdot H(U) \cap F \) is a proper subset of \( x_{0}\tilde{L} \cap F \).

Proof. Choose \( g_{0} \in G \) so that \( x_{0} = [g_{0}] \). Let \( p : G \to \Gamma \backslash G \) be the canonical projection map. Then \( p^{-1}(\mathcal{S}(U, x_{0}\tilde{L}) \cdot H(U)) \) is a countable union \( \gamma g_{0}(\tilde{L} \cap \\)
Theorem that $F \dim(U) 18$

Let $F$ be a countable union of nowhere dense subsets of $x_0L$. Since $F^* \cap x_0L$ is an open subset of $x_0L$, it follows from the Baire category theorem that $F^* \cap x_0L \not\subset \mathcal{N}(U, x_0L) \cdot H(U)$. This proves the claim. □

The following geometric property of a convex cocompact hyperbolic manifold of Fuchsian ends is one of its key features which is needed in the proof of our main theorems stated in the introduction.

**Proposition 4.16.** Let $M$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $x_0\hat{L}$ be a closed orbit of $\hat{L} \in \mathcal{L}_U$ with $x_0 \in RM$ and with $\dim(\hat{L} \cap N) \geq 2$. Either $x_0\hat{L}$ is compact or $\mathcal{N}(U, x_0\hat{L})$ contains a compact orbit $zL_0$ with $L_0 \in \mathcal{L}_U$.

**Proof.** Write $\hat{L} = H(\hat{U})C$ for a connected closed subgroup $U < \hat{U} < N$. Since $x_0\hat{L}$ is closed, $\pi(x_0\hat{L}) = \pi(x_0H'(\hat{U}))$ is a properly immersed convex cocompact geodesic plane of dimension at least 3 with Fuchsian ends by Proposition 3.6. Suppose that $x_0L$ is not compact. Then $\pi(x_0L)$ has non-empty Fuchsian ends. This means that there exist a co-dimension one subgroup $U_0$ of $U$ and $z \in \hat{L}$ such that $zH'(U_0)$ is compact and $\pi(zH'(U_0))$ is a component of the core of $\pi(x_0\hat{L})$. By Proposition 3.8, there exists a closed subgroup $C_0 < C(H(U_0)) \cap \hat{L}$ such that $H(U_0)C_0 \in \mathcal{L}_{U_0}$ and $zH(U_0)C_0$ is compact. Let $m \in M \cap \hat{L}$ be an element such that $U \subset m^{-1}U_0m$. Then $zm(m^{-1}H(U_0)C_0m)$ is a compact orbit contained in $\mathcal{N}(U, x_0\hat{L})$ and $m^{-1}H(U_0)C_0m \in \mathcal{L}_U$, finishing the proof. □

5. **Inductive Search Lemma**

In this section, we prove a combinatorial lemma 5.4, which we call an inductive search lemma, and use it to prove Proposition 5.3. This proposition will be used in the proof of the avoidance theorem 6.13 in the next section.

**Definition 5.1.** Let $J^* \subset I$ be a pair of open subsets of $\mathbb{R}$.

- The degree of $(I, J^*)$ is defined to be the minimal integer $\delta \in \mathbb{N}$ such that for each connected component $I^0$ of $I$, the number of connected components of $J^*$ contained in $I^0$ is bounded by $\delta$.
- For $\beta > 0$, the pair $(I, J^*)$ is said to be $\beta$-regular if for any connected component $I^0$ of $I$, and any component $J^0$ of $J^* \cap I^0$, $J^0 \pm \beta \cdot |J^0| \subset I^0$ where $|J^0|$ denotes the length of $J^0$.

**Definition 5.2.** Let $\mathcal{X}$ be a family of countably many triples $(I, J^*, J')$ of open subsets of $\mathbb{R}$ such that $I \supset J^* \supset J'$.

- Given $\beta > 0$ and $\delta \in \mathbb{N}$, we say that $\mathcal{X}$ is $\beta$-regular of degree $\delta$ if for every triple $(I, J^*, J') \in \mathcal{X}$, the pair $(I, J^*)$ is $\beta$-regular with degree at most $\delta$. 

• Given a subset $T \subset \mathbb{R}$, we say that $X$ is of $T$-multiplicity free if for any two distinct triples $(I_1, J^*_1, J'_1)$ and $(I_2, J^*_2, J'_2)$ of $X$, we have $I_1 \cap J'_2 \cap T = \emptyset$.

For a family $X = \{(I_\lambda, J^*_\lambda, J'_\lambda) : \lambda \in \Lambda\}$, we will use the notation 

$$I(X) := \bigcup_{\lambda \in \Lambda} I_\lambda, \quad J^*(X) := \bigcup_{\lambda \in \Lambda} J^*_\lambda \quad \text{and} \quad J'(X) := \bigcup_{\lambda \in \Lambda} J'_\lambda.$$ 

The goal of this section is to prove:

**Proposition 5.3 (Thickness of $T - J'(X)$).** Given $n, k, \delta \in \mathbb{N}$, there exists a positive number $\beta_0 = \beta_0(n, k, \delta)$ for which the following holds: let $T \subset \mathbb{R}$ be a globally $k$-thick set, and let $X_1, \ldots, X_{\ell} \leq n$, be $\beta_0$-regular families of degree $\delta$ and of $T$-multiplicity free. Let $\mathcal{X} = \bigcup_{i=1}^{\ell} X_i$. If $0 \in T - I(\mathcal{X})$, then $T - J'(\mathcal{X})$ is a $2k$-thick set.

The general case reduces to the case of $\delta = 1$, by replacing $m$ by $m \delta$. Roughly speaking, the following lemma gives an inductive argument for the search of a sequence of $t_i$'s which is almost geometric in a sense that the ratio $|t_i|/|t_{i-1}|$ is coarsely a constant and which lands on $T - J'(\mathcal{X})$ in a time controlled by $n$.

**Lemma 5.4 (Inductive search lemma).** Let $k > 1$, $n \in \mathbb{N}$ and $0 < \varepsilon < 1$ be fixed. There exists $\beta = \beta(n, k, \varepsilon) > 0$ for which the following holds: Let $T \subset \mathbb{R}$ be a globally $k$-thick set, and let $X_1, \ldots, X_n$ be $\beta$-regular families of countably many triples $(I_\lambda, J^*_\lambda, J'_\lambda)$ with degree 1, and of $T$-multiplicity free. Set $\mathcal{X} = X_1 \cup \cdots \cup X_n$, and assume $0 \notin I(\mathcal{X})$. For any $t \in T \cap J'(\mathcal{X})$ and any $1 \leq r \leq n$, we can find distinct triples $(I_1, J^*_1, J'_1), \ldots, (I_{m-1}, J^*_m, J'_m) \in \mathcal{X}$ with $2 \leq m \leq 2^r$, and a sequence of pivots $t = t_1 \in T \cap J'_1, t_2 \in T \cap J'_2, \ldots, t_{m-1} \in T \cap J'_m, t_m \in T$ which satisfy the following conditions:

1. either $t_m \notin J'(\mathcal{X})$, or $t_m \in J'_m$ for some $(I_m, J^*_m, J'_m) \in \mathcal{X}$, which is distinct from $(I_i, J^*_i, J'_i)$ for all $1 \leq i < m - 1$, and the collection $\{(I_i, J^*_i, J'_i) : 1 \leq i \leq m\}$ intersects at least $(r + 1)$ number of $X_i$'s;

2. for all $1 \leq i \leq j \leq m$, $|t_i - t_j| \leq 2((4k)^r - 1)k \max_{1 \leq p \leq j-1} |J'_p|$;

3. for each $1 \leq i \leq m$, $(1 - \varepsilon)^{i-1}|t_1| \leq |t_i| \leq (1 + \varepsilon)^{i-1}|t_1|$. 

In particular, for any $t \in T \cap J'(\mathcal{X})$, there exists $t' \in T - J'(\mathcal{X})$ such that 

$$(1 - \varepsilon)^{2^{n-1}}|t| \leq |t'| \leq (1 + \varepsilon)^{2^{n-1}}|t|.$$ 

**Proof.** We set \n
$$(5.1) \quad \beta = \beta(n, k, \varepsilon) = (4k)^{n+1}\varepsilon^{-1}.$$ 

Consider the increasing sequence $Q(r) := (4k)^r - 1$ for $r \in \mathbb{N}$. Note that $Q(1) \geq 2$ and $Q(r + 1) \geq 4Q(r)k + 1$. Moreover we check that $\beta > \max((Q(n) + 4Q(n-1))k, Q(n)k\varepsilon^{-1})$. We proceed by induction on $r$. First consider the case when $r = 1$. There exists $(I_1, J^*_1, J'_1) \in \mathcal{X}$ such that $t_1 := t \in J'_1 \cap T$. As $T$ is globally $k$-thick, we can choose 

$$(5.2) \quad t_2 \in (t_1 \pm Q(1)(|J^*_1|, k|J'_1|)) \cap T.$$
We claim that $t_1, t_2$ is our desired sequence with $m = 2$. In the case when $t_2 \in J'(X)$, there exists $(I_2, J_2^*, J_2^*) \in X$ such that $t_2 \in J_2^*$. We check:

1: If $t_2 \in J'(X)$, then $t_2 \in J_2^* - J_2^*$ implies that $J_2^*$ and $J_2^*$ are distinct. Hence $(I_1, J_1^*, J_1^*)$ and $(I_2, J_2^*, J_2^*)$ are distinct as well. Since $\beta > Q(1)k$, by the $\beta$-regularity of $(I_1, J_1^*)$, we have $t_2 \in I_1$. By the $T$-multiplicity free condition, $(I_1, J_1^*, J_1^*)$ and $(I_2, J_2^*, J_2^*)$ don’t belong to the same family, that is, $\{(I_1, J_1^*, J_1^*), (I_2, J_2^*, J_2^*)\}$ intersects two of $X_i$’s.

2: By (5.2), $|t_1 - t_2| < Q(1)k|J_1^*| = (4k - 1)k|J_1^*|$. Hence $(\beta > \varepsilon)$ is, $I_1$ in $X$.

3: Note that $0 \notin I_1$, since $0 \notin I(X)$. By the $\beta$-regularity of $(I_1, J_1^*)$, we have $t_1 \pm \beta|J_1^*| \subset I_1$. Since $0 \notin I_1$ and $\beta > \varepsilon |Q(1)k|$, we have $|t_1| - \varepsilon |Q(1)k|J_1^*| > 0$. On the other hand, by (5.2), $|t_2 - t_1| < Q(1)k|J_1^*| \leq \varepsilon |t_1|$. In particular, $|t_2| < |t_1| + |t_2 - t_1| < |t_1| + Q(1)k|J_1^*| < (1 + \varepsilon)|t_1|$. Hence we have a sequence $t_1(= t) \in J_1^*, t_2 \in J_2^*, \ldots, t_m \in J_m^*$, and $t_m$ in $T$ with $m \leq 2^r$ together with $\{(I_i, J_i^*, J_i^*) : 1 \leq i \leq m - 1\}$ satisfying the three conditions listed in the lemma. If $t_m \notin J'(X)$, the same sequence would satisfy the hypothesis for $r + 1$ and we are done. Now we assume that $t_m \in J_m^*$ for some $(I_m, J_m^*, J_m^*) \in X$ and that $\{(I_i, J_i^*, J_i^*) : 1 \leq i \leq m\}$ intersect at least $(r + 1)$ numbers of $X_i$’s. We may assume that they intersect exactly $(r + 1)$-number of $X_i$’s, which we may label as $X_1, \ldots, X_{r + 1}$, since if they intersect more than $(r + 1)$ of them, we are already done. Choose a largest interval $J_i^*$ among $J_1^*, \ldots, J_m^*$. Again using the global $k$-thickness of $T$, we can choose

$$s_1 \in (t_1 \pm Q(r + 1)(|J_i^*|, k|J_i^*|)) \cap T.$$

First, consider the case when $s_1 \notin J'(X)$. We will show that the points $t_1, \ldots, t_m, s_1$ give the desired sequence. Indeed, the condition (1) is immediate. For (2), observe that by the induction hypothesis for $r$, we have $|s_1 - t_i| \leq |t_i - t_i| + |t_i - t_i| \leq (Q(r + 1) + 2Q(r)k)|J_i^*|$ for all $1 \leq i \leq m$. The conclusion follows as $Q(r + 1) > 2Q(r)$. To show (3), since $\beta > \varepsilon |Q(r + 1)k|$, and $0 \notin I_1$, by applying the $\beta$-regularity to the pair $(I_1, J_1^*)$, we have $|t_1 - \varepsilon |Q(r + 1)k|J_1^*| > 0$. It follows that

$$|s_1| \leq |t_1| + |s_1 - t_1| < |t_1| + Q(r + 1)k|J_1^*| < (1 + \varepsilon)|t_1| \leq (1 + \varepsilon)^m |t_1|;$$

$$|s_1| \geq |t_1| - |s_1 - t_1| > |t_1| - Q(r + 1)k|J_1^*| > (1 - \varepsilon)|t_1| \geq (1 - \varepsilon)^m |t_1|.$$ 

This proves (3).

For the rest of the proof, we now assume that $s_1 \in J'(X)$. Apply the induction hypothesis for $r$ to $s_1 \in T \cap J'(X)$ to obtain a sequence $\{(I_j, J_j^*, J_j^*) \in X : 1 \leq j \leq m' - 1\}$ with $m' \leq 2^r$ and $s_1 \in J_1^* \cap T$, $s_2 \in J_2^* \cap T$, $\ldots, s_{m' - 1} \in J_{m' - 1}^* \cap T$, and $s_{m'} \in T$. Set $q_0$ to be the smallest $1 \leq q \leq m' - 1$ satisfying

$$n = \{(I_j, J_j^*, J_j^*) : 1 \leq j \leq q\} \subset X_1 \cup \cdots \cup X_{r + 1}.$$
if it exists, and \( q_0 := m' \) otherwise. We claim that the sequence
\[
(5.5) \quad t_1, \cdots, t_m, s_1, \cdots, s_{q_0}
\]
of length \( m + q_0 \leq 2^{r+1} \) satisfies the conditions of the lemma for \( r + 1 \).

**Claim:** We have
\[
(5.6) \quad |J^*_j| = \max_{1 \leq i \leq m, 1 \leq j \leq q_0 - 1} (|J^*_i|, |\tilde{J}^*_j|).
\]
Recall that \( |J^*_i| \) was chosen to be maximal among \( |J^*_1|, \cdots, |J^*_m| \). Hence, if the claim does not hold, then we can take \( j \) to be the least number such that \( |J^*_j| > |J^*_j| \). Then by the induction hypothesis for (2),
\[
|t_\ell - s_j| \leq |t_\ell - s_1| + |s_1 - s_j| \leq Q(r + 1)k|J^*_j| + 2Q(r)k \max_{1 \leq i \leq j - 1} |\tilde{J}^*_i| + (Q(r + 1) + 2Q(r))k|J^*_j|.
\]
Now the collection \( \{(I_i, J^*_i, J'_i) : 1 \leq i \leq m \} \) intersects \( (r + 1) \) families \( X_1, \cdots, X_{r+1} \) and \( (\tilde{I}_j, \tilde{J}^*_j, \tilde{J}'_j) \) belongs to one of these families, as \( j \leq q_0 - 1 \). Hence there exists a triple \( (I_i, J^*_i, J'_i) \) that belongs to the same family as \( (\tilde{I}_j, \tilde{J}^*_j, \tilde{J}'_j) \). Recall that the induction hypothesis for \( t_1, \cdots, t_m \) gives us
\[
|t_\ell - t_\ell| \leq 2Q(r)k|J^*_i|. \quad \text{Since } \beta > (Q(r + 1) + 4Q(r))k, \text{ we have}
\]
\[
|t_i - s_j| \leq |t_i - t_\ell| + |t_\ell - s_j| \leq (Q(r + 1) + 4Q(r))k|J^*_j| < \beta|\tilde{J}^*_j|.
\]
Applying the \( \beta \)-regularity to the pair \( (\tilde{I}_j, \tilde{J}^*_j) \), we conclude that \( t_i \in \tilde{I}_j \cap J'_j \cap \tilde{J}'_j \). Since \( (\tilde{I}_j, \tilde{J}^*_j, \tilde{J}'_j) \) and \( (I_i, J^*_i, J'_i) \) belong to the same family which is \( \mathcal{T} \)-multiplicity free, they are equal to each other. This is a contradiction since \( |\tilde{J}^*_j| > |J^*_i| \geq |J^*_i| \), proving the claim (5.6).

We next prove that \( (I_i, J^*_i, J'_i) \) and \( (\tilde{I}_j, \tilde{J}^*_j, \tilde{J}'_j) \) are distinct for all \( 1 \leq i \leq m \) and \( 1 \leq j \leq q_0 - 1 \). It suffices to check that \( J^*_i \) and \( \tilde{J}^*_j \) are distinct. Note that we have
\[
\max_{1 \leq i, j \leq m} |t_i - t_j| < 2Q(r)k|J^*_i| \quad \text{and} \quad \max_{1 \leq i, j \leq q_0} |s_i - s_j| < 2Q(r)k|J^*_j|
\]
by the induction hypothesis together with claim (5.6). Now for \( t_i \in J^*_i (1 \leq i \leq m) \) and \( s_j \in \tilde{J}^*_j (1 \leq j < q_0) \), we estimate:
\[
(5.7) \quad |s_j - t_i| \geq |s_1 - t_\ell| - |t_\ell - t_i| - |s_1 - s_j|
\]
\[
> Q(r + 1)|J^*_i| - 2Q(r)k|J^*_i| - 2Q(r)k|\tilde{J}^*_j|
\]
\[
= (Q(r + 1) - 4Q(r)k)|J^*_i| \geq |J^*_i|.
\]
This in particular means that \( s_j \notin J^*_i \) and \( t_i \notin \tilde{J}^*_j \). Hence \( J^*_i \neq \tilde{J}^*_j \).

We now begin checking the conditions (1), (2) and (3).

1: If \( s_{q_0} \notin J(\mathcal{X}) \), there is nothing to check.

Now assume that \( s_{q_0} \in \tilde{J}^*_{q_0} \) for some \( (\tilde{I}_{q_0}, \tilde{J}^*_{q_0}, \tilde{J}'_{q_0}) \in \mathcal{X} \). If \( q_0 < m' \), then again there is nothing to prove, as the union
\[
(5.8) \quad \{(I_i, J^*_i, J'_i) : 1 \leq i \leq m\} \cup \{(\tilde{I}_j, \tilde{J}^*_j, \tilde{J}'_j) : 1 \leq j \leq q_0\}
\]
intersects a family other than \( X_1, \cdots, X_{r+1} \). Hence we will assume \( q_0 = m' \).

By the induction hypothesis for \( r \) on the sequence \( (s_1, \cdots, s_{m'}) \), the family
\{ (\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j^*_{\ell} ) : 1 \leq j \leq m' \} consists of pairwise distinct triples intersecting at least \((r + 1)\) numbers of \(X_i\)’s. Observe that in the estimate (5.7), there is no harm in allowing \(j = q_0\) in addition to \(j < q_0\). This shows that \(\tilde{J}_{m'}^*\) is also distinct from all \(J_i^*\)’s. Hence the the triples in (5.8) are all distinct.

Now, unless the following inclusion
\[
(5.9) \quad \{ (\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j^*_{\ell} ) : 1 \leq j \leq m' \} \subseteq X_1 \cup \cdots \cup X_{r+1}
\]
holds, we are done. Suppose that (5.9) holds. We will deduce a contradiction. Without loss of generality, we assume that \((I_\ell, J_\ell^*, J_\ell^*_{\ell}) \in X_{r+1}\). We now claim that the following inclusion holds:
\[
(5.10) \quad \{ (\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j^*_{\ell} ) : 1 \leq j \leq m' \} \subseteq X_1 \cup \cdots \cup X_r.
\]
Note that this gives the desired contradiction, since \{ \((\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j^*_{\ell} ) : 1 \leq j \leq m' \) \} must intersect at least \((r + 1)\) number of \(X_i\) by the induction hypothesis. In order to prove the inclusion (5.10), suppose on the contrary that \((\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j^*_{\ell} ) \notin X_{r+1}\) for some \(1 \leq j \leq m'\). Using \(\beta > (Q(r + 1) + 2Q(r))k\) and (5.6), we deduce \(|t_\ell - s_j| \leq |t_\ell - s_1| + |s_1 - s_j| \leq Q(r + 1)k|J_\ell^*| + 2Q(r)k|J_\ell^*| < \beta|J_\ell^*|\) where we used the induction hypothesis for the sequence \((s_1, \cdots, s_m')\) in the second line, to estimate the term \(|s_1 - s_j|\).

Next, applying the \(\beta\)-regularity to the pair \((I_\ell, J_\ell^*)\), we conclude that \(s_j \in I_\ell\). Since \(s_j \in \tilde{J}_j^*\), it follows that \(I_\ell \cap \tilde{J}_j^* \cap T \neq \emptyset\). This contradicts the condition that \(X_{r+1}\) is of \(T\)-multiplicity free, as both \((\tilde{I}_j, \tilde{J}_j^*, \tilde{J}_j^*_{\ell})\) and \((I_\ell, J_\ell^*, J_\ell^*_{\ell})\) belong to the same family \(X_{r+1}\). This completes the proof of (1).

(2): For \(1 \leq i \leq m\) and \(1 \leq j \leq q_0\), observe that
\[
|t_i - s_j| \leq |t_i - t_\ell| + |t_\ell - s_1| + |s_1 - s_j|
\leq 2Q(r)k|J_\ell^*| + Q(r + 1)k|J_\ell^*| + 2Q(r)k|J_\ell^*| < 2Q(r + 1)k|J_\ell^*|
\]
as \(Q(r + 1) > 4Q(r)\). Hence we get the desired result by (5.6).

(3): We already have observed that the inequality \(\beta > \varepsilon^{-1}Q(r + 1)k\) implies that \((1 - \varepsilon)^m|t_1| \leq |s_1| \leq (1 + \varepsilon)^m|t_1|\). Combining this with the induction hypothesis, we deduce that \((1 - \varepsilon)^m+i|t_1| \leq |s_i| \leq (1 + \varepsilon)^m+i|t_1|\) for all \(1 \leq i \leq q_0\). Finally, the last statement of the lemma is obtained from the case \(r = n\), since there are only \(n\)-number of \(X_i\)’s; hence the second possibility of (1) cannot arise for \(r = n\).

Proof of Proposition 5.3. We may assume that \(X_i\)’s are all of degree 1, by replacing each \(X_i\)’s with \(\delta\)-number of families associated to it.

We set \(\beta_0(n, k, 1) = (4k)^{n+1}\varepsilon^{-1}\) where \(\varepsilon\) satisfies \((\frac{1+\varepsilon}{1-\varepsilon})^{2^{n-1}} \leq 2\). Note that \(\beta_0(n, k, 1)\) is equal to the number given in (5.1). We may assume \(x = 0\) without loss of generality. Let \(\lambda > 0\). We need to find a point
\[
(5.11) \quad t' \in \left( [-2k\lambda, -\lambda] \cup [\lambda, 2k\lambda] \right) \cap \left( T - \bigcup_{i \in \Lambda} J'(X_i) \right).
\]
Choose \(s > 0\) such that \((1 - \varepsilon)^-(2^{n-1}) \lambda \leq s \leq 2(1 + \varepsilon)^-(2^{n-1}) \lambda\). Since \(T\) is globally \(k\)-thick, there exists \(t \in ((-ks, -s] \cup [s, ks]) \cap T\). If \(t \not\in \bigcup_{i=1}^n J'(X_i)\),
then by choosing $t' = t$, we are done. Now suppose $t \in \bigcup_{i=1}^n J'(\mathcal{X}_i)$. Since $0 \notin \bigcup_{i=1}^n I(\mathcal{X}_i)$, by applying Lemma 5.4 to $t \in \mathcal{T} \cap (\bigcup_{i=1}^n J'(\mathcal{X}_i))$, we obtain $t' \in \mathcal{T} - \bigcup_{i=1}^n J'(\mathcal{X}_i)$ such that $(1 - \varepsilon)2^{n-1}|t| \leq |t'| \leq (1 + \varepsilon)2^{n-1}|t|$. Note that $|t'| \leq (1 + \varepsilon)2^{n-1}|t| \leq (1 + \varepsilon)2^{n-1}ks \leq 2k\lambda$. Similarly, we have $|t'| \geq (1 - \varepsilon)2^{n-1}|t| \geq (1 - \varepsilon)2^{n-1}s \geq \lambda$. This completes the proof since $t'$ satisfies (5.11).

6. Avoidance of the singular set

Let $\Gamma < G$ be a convex cocompact non-elementary subgroup and let $U = \{u_i\} < N$ be a one-parameter subgroup. Let $\mathcal{S}(U), \mathcal{G}(U), X(H,U)$, and $\mathcal{H}^*$ be as defined in section 4. In particular, $\mathcal{S}(U)$ is a countable union: $\mathcal{S}(U) = \bigcup_{H \in \mathcal{H}^*} \Gamma \backslash \Gamma X(H,U)$. The main goal of this section is to prove the avoidance Theorem 6.13 for any convex cocompact hyperbolic manifold with Fuchsian ends. For this, we extend the linearization method developed by Dani and Margulis [11] to our setting. Via a careful analysis of the graded self-intersections of the union $\bigcup_i \Gamma \backslash \Gamma H_i D_i \cap RF M$ for finitely many groups $H_i \in \mathcal{H}^*$ and compact subsets $D_i \subset X(H_i,U)$, we construct families of triples of subsets of $\mathbb{R}$ satisfying the conditions of Proposition 5.3 relative to the global $k$-thick subset of the return time to $RF M$ under $U$ given in Proposition 3.10.

**Linearization.** Let $H \in \mathcal{H}^*$. Then $H$ is reductive, algebraic, and is equal to $N_G(H)$ by Proposition 4.3. There exists an $\mathbb{R}$-regular representation $\rho_H : G \to \text{GL}(V_H)$ with a point $p_H \in V_H$, such that $H = \text{Stab}_G(p_H)$ and the orbit $p_H \Gamma$ is Zariski closed [2, Theorem 3.5]. Since $\Gamma \backslash \Gamma H$ is closed, it follows that $p_H \Gamma$ is a closed (and hence discrete) subset of $V_H$.

Let $\eta_H : G \to V_H$ denote the orbit map defined by $\eta_H(g) = p_H g$ for all $g \in G$. As $H$ and $U$ are algebraic subgroups, the set $X(H,U) = \{g \in G : g U g^{-1} \subset H\}$ is Zariski closed in $G$. Since $p_H G$ is Zariski closed in $V_H$, it follows that $A_H := p_H X(H,U)$ is Zariski closed in $V_H$ and $X(H,U) = \eta_H(A_H)$.

Following [13], for given $C > 0$ and $\alpha > 0$, a function $f : \mathbb{R} \to \mathbb{R}$ is called $(C,\alpha)$-good if for any interval $I \subset \mathbb{R}$ and $\varepsilon > 0$, we have

$$\ell\{t \in I : |f(t)| \leq \varepsilon\} \leq C \cdot \left(\frac{\varepsilon}{\sup_{t \in I}|f(t)|}\right)^\alpha \ell(I)$$

where $\ell$ is a Lebesgue measure on $\mathbb{R}$.

**Lemma 6.1.** For given $C > 1$ and $\alpha > 0$, consider functions $p_1, p_2, \ldots, p_k : \mathbb{R} \to \mathbb{R}$ satisfying the $(C,\alpha)$-good property. For $0 < \delta < 1$, set

$$I = \{t \in \mathbb{R}: \max_i |p_i(t)| < 1\} \quad \text{and} \quad J(\delta) = \{t \in \mathbb{R}: \max_i |p_i(t)| < \delta\}.$$ 

For any $\beta > 1$, there exists $\delta = \delta(C,\alpha,\beta) > 0$ such that the pair $(I, J(\delta))$ is $\beta$-regular (see Def. 5.2).
Proof. We prove that the conclusion holds for \( \delta := (1 + \beta)|C|^{-1/\alpha} \). First, note that the function \( q(t) := \max_i |p_i(t)| \) also has the \((C, \alpha)\)-good property. Let \( J' = (a, b) \) be a component of \( J(\delta) \), and \( I' \) be the component of \( I \) containing \( J' \). Note that \( I' \) is an open interval and \((a, \infty) \cap I' = (a, c)\) for some \( b \leq c \leq \infty \). We claim

\[
(6.1) \quad J' + \beta|J'| \subset (a, \infty) \cap I' \subset I'.
\]

We may assume that \( c < \infty \); otherwise the inclusion is trivial. We claim that \( q(c) = 1 \). Since \( \{t \in \mathbb{R} : q(t) < 1\} \) is open and \( c \) is the boundary point of \( I' \), we have \( q(c) \geq 1 \). If \( q(c) \) were strictly bigger than 1, since \( \{t \in \mathbb{R} : q(t) > 1\} \) is open, \( I' \) would be disjoint from an open interval around \( c \), which is impossible. Hence \( q(c) = 1 \). Now that \( \sup \{q(t) : t \in (a, \infty) \cap I'\} = q(c) = 1 \), by applying the \((C, \alpha)\)-good property of \( q \) on the interval \((a, \infty) \cap I' \), we get

\[
\ell(J') \leq \ell(t \in (a, \infty) \cap I' : |q(t)| \leq \delta) \leq C\delta^\alpha \cdot \ell((a, \infty) \cap I').
\]

Now as \( J' = (a, b) \) and \((a, \infty) \cap I' \) are nested intervals with one common endpoint, it follows from the equality \( C\delta^\alpha = 1/(1 + \beta) \) that \( J' + \beta|J'| \subset (a, \infty) \cap I' \subset I' \), proving (6.1). Similarly, applying the \((C, \alpha)\)-good property of \( q \) on \((0, \infty) \cap I' \), we deduce that \( J' - \beta|J'| \subset I' \). This proves that \((I, J(\delta))\) is \( \beta \)-regular.

Proposition 6.2. Let \( V \) be a finite dimensional real vector space, \( \theta \in \mathbb{R}[V] \) be a polynomial and \( A = \{v \in V : \theta(v) = 0\} \). Then for any compact subset \( D \subset A \) and any \( \beta > 0 \), there exists a compact neighborhood \( D' \subset A \) of \( D \) which has a \( \beta \)-regular size with respect to \( D \) in the following sense: for any neighborhood \( \Phi \) of \( D' \), there exists a neighborhood \( \Psi \subset \Phi \) of \( D \) such that for any \( q \in V - \Phi \) and for any one-parameter unipotent subgroup \( \{u_t\} \subset \text{GL}(V) \), the pair \((I(q), J(q))\) is \( \beta \)-regular where \( I(q) = \{t \in \mathbb{R} : q u_t \in \Phi \} \) and \( J(q) = \{t \in \mathbb{R} : q u_t \in \Psi \} \). Furthermore, the degree of \((I(q), J(q))\) is at most \((\deg \theta + 2) \cdot \dim V \).

Proof. Choose a norm on \( V \) so that \( \|\cdot\|^2 \) is a polynomial function on \( V \). Since \( D \) is compact, we can find \( R > 0 \) such that \( D \subset \{v \in V : \|v\| < R\} \). Then we set \( D' = \{v \in V : \theta(v) = 0, \|v\| < R/\sqrt{\delta}\} \), where \( 0 < \delta < 1 \) is to be specified later. Note that if \( \Phi \) is a neighborhood of \( D' \), there exists \( 0 < \eta < 1 \) such that \( \{v \in V : \theta(v) < \eta, \|v\| < (R + \eta)/\sqrt{\delta}\} \subset \Phi \). We set \( \Psi := \{v \in V : \theta(v) < \eta\delta, \|v\| < (R + \eta)/\sqrt{\delta}\} \) and \( I(q) := \{t \in \mathbb{R} : \theta(q u_t) < \eta, \|q u_t\| < (R + \eta)/\sqrt{\delta}\} \). Since \( I(q) \subset I(q) \) for \( 0 < \delta < 1 \), it suffices to find \( \delta \) (and hence \( D' \) and \( \Psi \)) so that the pair \((I(q), J(q))\) is \( \beta \)-regular. If we set \( \psi_1(t) := \frac{\theta(q u_t)}{\eta} \) and \( \psi_2(t) := \left(\frac{\|q u_t\|\sqrt{\delta}}{R + \eta}\right)^2 \), then \( I(q) = \{\max(\psi_1(t), \psi_2(t)) < 1\} \) and \( J(q) = \{\max(\psi_1(t), \psi_2(t)) < \delta\} \). As \( \psi_1 \) and \( \psi_2 \) are polynomials, they have the \((C, \alpha)\)-property for an appropriate choice of \( C \) and \( \alpha \). Therefore by applying Lemma 6.1, by choosing \( \delta \) small enough, we can make the pair \((I(q), J(q))\) \( \beta \)-regular for any \( \beta > 0 \). Note that the degrees of \( \psi_1 \) and \( \psi_2 \) are bounded by \( \deg \theta \cdot \dim V \) and \( 2 \dim V \) respectively. Therefore \( J(q) \) cannot
Lemma 6.5. which is left $C(\cdot)$

where $H$ family for Remark 6.4 \{ is complete. \hfill □

(E.2)

We define Definition 6.3.

Definition 6.3. We define $E = E_U$ to be the collection of all compact subsets of $\mathcal{S}(U) \cap RF M$ which can be written as

\[ E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i D_i \cap RF M \]

where $\{ H_i \in \mathcal{H}^* : i \in \Lambda \}$ is a finite collection and $D_i \subset X(H_i, U)$ is a compact subset. In this expression, we always use the minimal index set $\Lambda$ for $E$. When $E$ is of the form (6.2), we will say that $E$ is associated to the family $\{ H_i : i \in \Lambda \}$.

Remark 6.4. We note that $E$ can also be expressed as $\bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i D_i \cap RF M$ where $H_i \in \mathcal{H}$ is a finite collection, and $D_i \subset X(H_i, U)$ is a compact subset which is left $C(H_i)$-invariant.

Lemma 6.5. In the expression (6.2) for $E \in E$, the collection $\{ H_i : i \in \Lambda \}$ is not redundant, in the sense that no $\gamma H_j \gamma^{-1}$ is equal to $H_i$ for all triples $(i, j, \gamma) \in \Lambda \times \Lambda \times \Gamma$ except for the trivial cases of $i = j$ and $\gamma \in H_i$.

Proof. Observe that if $\gamma H_j \gamma^{-1} = H_i$ for some $\gamma \in \Gamma$, then $\Gamma H_j D_j = \Gamma H_i \gamma D_j$, and hence by replacing $D_j$ by $D_j \cup \gamma D_j \subset X(H_i, U)$, we may remove $j$ from the index subset $\Lambda$. This contradicts the minimality of $\Lambda$. \hfill □

Note that for $D_i \subset X(H_i, U)$, and $\gamma \in \Gamma$, the intersection $H_1 D_1 \cap \gamma H_2 D_2$ only depends on the $(\Gamma \cap H_1, \Gamma \cap H_2)$-double coset of $\gamma$.

Proposition 6.6. Let $H_1, H_2 \in \mathcal{H}^*$. Then for any compact subset $D_i \subset X(H_i, U)$ for $i = 1, 2$ and a compact subset $K \subset \Gamma \backslash G$, there exists a finite set $\Delta \subset (H_1 \cap \Gamma) \backslash \Gamma / (H_2 \cap \Gamma)$ such that

\[ \bigcup \{ K \cap \Gamma \backslash \Gamma (H_1 D_1 \cap \gamma H_2 D_2) \}_{\gamma \in \Gamma} = \bigcup \{ K \cap \Gamma \backslash \Gamma (H_1 D_1 \cap \gamma H_2 D_2) \}_{\gamma \in \Delta} \]

where the latter set consists of distinct elements.

Moreover for each $\gamma \in \Delta$, there exists a compact subset $C_0 \subset H_1 D_1 \cap \gamma H_2 D_2 \subset X(H_1 \cap \gamma H_2 \gamma^{-1}, U)$ such that $K \cap \Gamma \backslash \Gamma (H_1 D_1 \cap \gamma H_2 D_2) = \gamma C_0$.

Proof. For simplicity, write $\eta_{H_i} = \eta_i$ and $p_i = p_{H_i}$. Let $K_0 \subset G$ be a compact set such that $K = \Gamma \backslash \Gamma K_0$. We fix $\gamma \in \Gamma$, and define for any $\gamma' \in \Gamma$, $K_{\gamma'} = \{ g \in K_0 : \gamma' g \in H_1 D_1 \cap \gamma H_2 D_2 \}$. We check that $K \cap \Gamma \backslash \Gamma (H_1 D_1 \cap \gamma H_2 D_2) = \gamma C_0$. If this set is non-empty, then $K_{\gamma'} \neq \emptyset$ for some $\gamma' \in \Gamma$ and $p_1 \gamma' g \in p_1 D_1$, and $p_2 \gamma^{-1} \gamma' g \in p_2 D_2$ for some $g \in K_0$. In particular,

\[ p_1 \gamma' \in p_1 D K_0^{-1}, \quad p_2 \gamma^{-1} \in p_2 D K_0^{-1} \gamma' \gamma^{-1}. \]

As $p_1 \Gamma$ is discrete, and $p_1 D_1 K_0^{-1}$ is compact, the first condition of (6.3) implies that there exists a finite set $\Delta_0 \subset G$ such that $\gamma' \in (H_1 \cap \Gamma) \Delta_0$. Writing $\gamma' = h \delta_0$ where $h \in H_1 \cap \Gamma$, and $\delta_0 \in \Delta_0$, the second condition of (6.3) implies $p_2 \gamma^{-1} h \in p_2 D_2 K_0^{-1} \delta_0^{-1}$. As $p_2 D_2 K_0^{-1} \Delta_0^{-1}$ is compact and $p_2 \Gamma$
is discrete, there exists a finite set $\Delta \subset G$ such that $\gamma^{-1}h \in (H_2 \cap \Gamma)\Delta$. Hence, if $K \cap \Gamma \Gamma(H_1D_1 \cap \gamma H_2D_2) \neq \emptyset$, then $\gamma \in (H_1 \cap \Gamma)\Delta(H_2 \cap \Gamma)$. This completes the proof of the first claim. For the second claim, it suffices to set $C_0 := \bigcup_{\gamma \in \Delta} K\gamma$. □

**Proposition 6.7.** Let $H_1, H_2 \in \mathcal{H}^*$ be such that $H_1 \cap H_2$ contains a unipotent element. Then there exists a unique smallest connected closed subgroup, say $H_0$, of $H_1 \cap H_2$ containing all unipotent elements of $H_1 \cap H_2$ such that $\Gamma \backslash \Gamma H_0$ is closed. Moreover, $H_0 \in \mathcal{H}$.

**Proof.** The orbit $\Gamma \backslash \Gamma(H_1 \cap H_2)$ is closed [26, Lem. 2.2]. Hence such $H_0$ exists. We need to show that $\Gamma \cap H_0$ is Zariski dense in $H_0$. Let $L$ be the subgroup of $H_0$ generated by all unipotent elements in $H_0$. Note that $L$ is a normal subgroup of $H_0$ and hence $(H_0 \cap \Gamma)L$ is a subgroup of $H_0$. If $F$ is the identity component of the closure of $(H_0 \cap \Gamma)L$, then $\Gamma \backslash \Gamma F$ is closed. By the minimality assumption on $H_0$, we have $F = H_0$. Hence $(H_0 \cap \Gamma)L = H_0$; so $[e]L = [e]H_0$. We can then apply [26, Cor. 2.12] and deduce the Zariski density of $H_0 \cap \Gamma$ in $H_0$. □

**Corollary 6.8.** Let $H_1, H_2 \in \mathcal{H}^*$ and $\gamma \in \Gamma$ be satisfying that $X(H_1 \cap \gamma H_2\gamma^{-1}, U) \neq \emptyset$. Then there exists a subgroup $H \in \mathcal{H}^*$ contained in $H_1 \cap \gamma H_2\gamma^{-1}$ such that for any compact subsets $D_i \subset X(H_i, U)$, $i = 1, 2$, there exists a compact subset $D_0 \subset X(H, U)$ such that $K \cap \Gamma \backslash \Gamma(H_1D_1 \cap \gamma H_2D_2) = K \cap \Gamma \backslash \Gamma HD_0$.

**Proof.** Let $F \in \mathcal{H}$ be given by Proposition 6.7 for the subgroup $H_1 \cap \gamma H_2\gamma^{-1}$. Set $H := N_G(F_{nc}) \in \mathcal{H}^*$. Note that $X(H_1 \cap \gamma H_2\gamma^{-1}, U) = X(H, U)$. Hence, by the second claim of Proposition 6.6, there exists a compact subset $D_0 \subset H_1D_1 \cap \gamma H_2D_2$ such that

$$(6.4) \quad K \cap \Gamma \backslash \Gamma(H_1D_1 \cap \gamma H_2D_2) = \Gamma \backslash \Gamma D_0.$$ 

We claim that $\Gamma \backslash \Gamma D_0 = K \cap \Gamma \backslash \Gamma HD_0$. The inclusion $\subset$ is clear. Let $g := hd \in HD_0$ with $h \in H$ and $d \in D_0$, and $[g] \in K$. Then by the condition on $D_0$, we have $g \in H_1D_1$ and $\gamma^{-1}g \in H_2D_2$. Therefore $g \in H_1D_1 \cap \gamma H_2D_2$. By (6.4), this proves the inclusion $\supset$. □

**Definition 6.9** (Self-intersection operator on $\mathcal{E}_U$). We define an operator $s : \mathcal{E}_U \cup \{\emptyset\} \to \mathcal{E}_U \cup \{\emptyset\}$ as follows: we set $s(\emptyset) = \emptyset$. For any

$$(6.5) \quad E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i D_i \cap \mathcal{R}F M \in \mathcal{E}_U,$$

we define

$$s(E) := \bigcup_{i \in \Lambda} \bigcup_{\gamma \in \Gamma} \Gamma \backslash \Gamma(H_i D_i \cap \gamma H_j D_j) \cap \mathcal{R}F M$$

where $\gamma_{ij} \in \Gamma$ ranges over all elements of $\Gamma$ satisfying $\dim(H_i \cap \gamma_{ij} H_j \gamma_{ij}^{-1})_{nc} < \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}$.

By Proposition 6.6 and Corollary 6.8, we have:
Corollary 6.10.  

(1) For $E \in \mathcal{E}_U$, we have $s(E) \in \mathcal{E}_U$.

(2) For $E_1, E_2 \in \mathcal{E}_U$, we have $E_1 \cap E_2 \in \mathcal{E}_U$.

Hence for $E \in \mathcal{E}_U$, as in (6.5), $s(E)$ is of the form $s(E) = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i D_i \cap RF M$ where $\Lambda'$ is a (minimal) finite index set, $H_i \in \mathcal{H}$ with $X(H_i,U) \neq \emptyset$ and

$$\max\{\dim(H_i)_{nc} : i \in \Lambda'\} < \max\{\dim(H_i)_{nc} : i \in \Lambda\}.$$ 

Hence, $s$ maps $\mathcal{E}_U$ to $\mathcal{E}_U \cup \{\emptyset\}$ and for any $E \in \mathcal{E}_U$, $s^{\dim(G)}(E) = \emptyset$.

Definition 6.11. For a compact subset $K \subset \Gamma \backslash G$ and $E \in \mathcal{E}_U$, we say that $K$ does not have any self-intersection point of $E$, or simply say that $K$ is $E$-self intersection-free, if $K \cap s(E) = \emptyset$.

Proposition 6.12. Let $E = \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i D_i \cap RF M \in \mathcal{E}$ where $D_i \subset X(H_i,U)$ is a compact subset and $\Lambda$ is a finite subset. Let $K \subset RF M$ be a compact subset which is $E$-self intersection-free. Then there exists a collection of open neighborhoods $\Omega_i$ of $D_i$, $i \in \Lambda$, such that for $\mathcal{O} := \bigcup_{i \in \Lambda} \Gamma \backslash \Gamma H_i \Omega_i$, the compact subset $K$ is $\mathcal{O}$-self intersection free, in the sense that, if $\dim H_i = \dim H_j$ and $\Gamma \cap \Gamma \backslash \Gamma (H_i \Omega_i \cap \gamma H_j \Omega_j) \neq \emptyset$ for some $(i,j,\gamma) \in \Lambda \times \Lambda \times \Gamma$, then $i = j$ and $\gamma \in H_i \cap \Gamma$.

Proof. For each $k \in \mathbb{N}$ and $i \in \Lambda$, let $\Omega_i(k)$ be the $1/k$-neighborhood of the compact subset $D_i$. Since $\Lambda$ is finite, if the proposition does not hold, by passing to a subsequence, there exist $i,j \in \Lambda$ with $\dim H_i = \dim H_j$ and a sequence $\gamma_k \in \Gamma$ such that $\Gamma \cap \Gamma \backslash \Gamma (H_i \Omega_i(k) \cap \gamma_k H_j \Omega_j(k)) \neq \emptyset$ and

$$(6.6) \quad (i,j,\gamma_k) \notin \{(i,i,\gamma) : i \in \Lambda, \gamma \in H_i \cap \Gamma\}.$$ 

Hence there exist $g_k = h_k w_k \in H_i \Omega_i(k)$ and $g_k' = h_k' w_k' \in H_j \Omega_j(k)$ such that $g_k = \gamma_k g_k'$ where $[g_k] \in K$. Now as $k \to \infty$, we have $w_k \to w \in D_i$ and $w_k' \to w' \in D_j$. There exists $\delta_k \in \Gamma$ such that $\delta_k g_k \in \tilde{K}$ where $\tilde{K}$ is a compact subset of $G$ such that $\Gamma \cap \Gamma \backslash \Gamma \tilde{K}$, so the sequence $\delta_k g_k$ converges to $g_0$ as $k \to \infty$. Since $\Gamma H_i$ and $\Gamma H_j$ are closed, we have $\delta_k h_k \to \delta_0 h_i$ and $\delta_k \gamma_k h_k \to \delta_0 \gamma_j h_j$ where $\delta_0, \delta_0' \in \Gamma$, $h_i \in H_i$ and $h_j \in H_j$. As $\Gamma[H_i]$ and $\Gamma[H_j]$ are discrete in the spaces $G/H_i$ and $G/H_j$ respectively, we have

$$(6.7) \quad \delta_0^{-1} \delta_k \in H_i \quad \text{and} \quad (\delta_0')^{-1} \delta_k \gamma_k \in H_j$$ 

for all sufficiently large $k$. Therefore $g_0 = \delta_0 h_i w = \delta_0' h_j w' \in \delta_0(H_i D_i \cap \delta_0^{-1} \delta_0' H_j D_j)$ and $[g_0] \in K$. Hence $K \cap \Gamma \backslash \Gamma (H_i D_i \cap \delta_0^{-1} \delta_0' H_j D_j) \neq \emptyset$. Set $\delta := \delta_0^{-1} \delta_0' \in \Gamma$. Since $K \cap s(E) = \emptyset$, this implies that $RF M \cap \Gamma \backslash \Gamma (H_i D_i \cap \delta H_j D_j) \not\subset s(E)$. By the definition of $s(E)$, $\dim(H_i \cap \delta H_j \delta^{-1})_{nc} = \min\{\dim(H_i)_{nc}, \dim(H_j)_{nc}\}$. Since $H_i = N_G(H_i) = N_G((H_i)_{nc})$, and similarly for $H_j$, we have $H_i \cap \delta H_j \delta^{-1}$ is either $H_i$ or $\delta H_j \delta^{-1}$. Since $\dim H_i = \dim H_j$, $\delta H_j \delta^{-1} = H_i$ or $H_i = \delta H_j \delta^{-1}$. By Lemma 6.5, this implies that $i = j$ and $\delta \in N_G(H_i) \cap \Gamma$. It follows from (6.7) that $\gamma_k \in N_G(H_i) \cap \Gamma = H_i \cap \Gamma$ for all large $k$. This is a contradiction to (6.6), completing the proof.

In the rest of this section, we assume that $M = \Gamma \backslash \mathbb{H}^d$ is a convex co-compact hyperbolic manifold with Fuchsian ends, and let $k$ be as given by Proposition 3.10.
Theorem 6.13 (Avoidance theorem I). Let $U = \{u_i\} < N$ be a one-parameter subgroup. For any $E \in \mathcal{E}_U$, there exists $E' \in \mathcal{E}_U$ such that the following holds: If $F \subset RF M$ is a compact set disjoint from $E'$, then there exists a neighborhood $O^\circ$ of $E$ such that for all $x \in F$, the set $\{t \in \mathbb{R} : xu_t \in RF M - O^\circ\}$ is $2k$-thick. Moreover, if $E$ is associated to $\{H_i : i \in \Lambda\}$, then $E'$ is also associated to the same family $\{H_i : i \in \Lambda\}$ in the sense of Definition 6.3.

Proof. \(\triangledown\) 1. The constant $\beta_0$: We write $\mathcal{H}^* = \{H_i\}$. For simplicity, set $V_i = V_{H_i}$ and $p_i = p_{H_i}$. Let $\theta_i$ be the defining polynomial of the algebraic variety $A_{H_i}$. Set $m := \dim (G)^2$ and $\delta := \max_{H_i \in \mathcal{H}^*} (\deg \theta_i + 2) \dim V_i$.

Note that if $H_i$ is conjugate to $H_j$, then $\theta_i$ and $\theta_j$ have same degree and $\dim V_i = \dim V_j$. Since there are only finitely many conjugacy classes in $\mathcal{H}^*$ by Proposition 4.3, the constant $\delta$ is finite. Now let $\beta_0 := \beta_0(m\delta, k, 1) = (4k)^m\delta + 1 - 1$ be given as in Proposition 5.3 where $\varepsilon = \varepsilon_{m\delta}$ satisfies $\left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{2m\delta - 1} \leq 2$.

\(\triangledown\) 2. Definition of $E_n$ and $E'_n$: We write $E = \bigcup_{i \in \Lambda_0} \Gamma \setminus H_i D_i \cap RF M$ for some finite minimal set $\Lambda_0$. Set $\ell := \max_{i \in \Lambda_0} \dim(H_i)$$_{nc}$. We define $E_n, E'_n \in \mathcal{E}_U$ for all $1 \leq n \leq \ell$ inductively as follows: set $E_0 := E$ and $\Lambda_\ell := \Lambda_0$. For each $i \in \Lambda_\ell$, let $D'_i$ be a compact subset of $X(H_i, U)$ containing $D_i$ such that $p_i D'_i$ has a $\beta_0$-regular size with respect to $p_i D_i$ as in Proposition 6.2. Set $E'_n := \bigcup_{i \in \Lambda_n} \Gamma \setminus H_i D'_i \cap RF M$. Suppose that $E_{n+1}, E'_{n+1} \in \mathcal{E}_U$ are given for $n \geq 1$. Then, define $E_n := E \cap s(E'_{n+1})$. Then by Corollary 6.10, $E_n$ belongs to $\mathcal{E}_U$ and hence can be written as $E_n = \bigcup_{i \in \Lambda_n} \Gamma \setminus H_i D_i \cap RF M$ where $D_i$ is a compact subset of $X(H_i, U)$, so that $\Lambda_n$ is a minimal index set. For each $i \in \Lambda_n$, let $D'_i$ be a compact subset of $X(H_i, U)$ containing $D_i$ such that $p_i D'_i$ has a $\beta_0$-regular size with respect to $p_i D_i$ as in Proposition 6.2. Set $E'_n := \bigcup_{i \in \Lambda_n} \Gamma \setminus H_i D'_i \cap RF M$. Hence we get a sequence of compact (possibly empty) subsets of $E$: $E_1, E_2, \cdots, E_{\ell-1}, E_\ell = E$, and a sequence of compact sets $E'_1, E'_2, \cdots, E'_{\ell-1}, E'_\ell = E'$. Note that $s(E_1) = s(E'_1) = \emptyset$ by the dimension reason.

\(\triangledown\) 3. Outline of the plan: Let $F \subset RF M$ be a compact set disjoint from $E'$. For $x \in F$, we set $T(x) := \{t \in \mathbb{R} : xu_t \in RF M\}$ which is a globally $k$-thick set by Proposition 3.10. We will construct

- a neighborhood $O'$ of $E'$ disjoint from $F$, and
- a neighborhood $O^\circ$ of $E$

such that for any $x \in RF M - O'$, we have $\{t \in \mathbb{R} : xu_t \in RF M - O^\circ\} \subset T(x) - J'(X')$ where $X' = X(x)$ is the union of at most $m$-number of $\beta_0$-regular families $X'_i$ of triples $(I(q), J^*(q), J'(q))$ of subsets of $\mathbb{R}$ with degree $\delta$ and of $T(x)$-multiplicity free. Once we do that, the theorem is a consequence of Proposition 5.3. Construction of such $O'$ and $O^\circ$ requires an inductive process on $E_n$’s.

\[\text{In fact } E_{\ell-1} = \emptyset \text{ for all } i \geq d - 1, \text{ but we won't use this information}\]
4. Inductive construction of $K_n$, $O'_{n+1}$, $O_{n+1}$, and $O^*_{n+1}$: Let $K_0 := RF M$. For each $i \in A_1$, there exists a neighborhood $\Omega'_i$ of $D'_i$ such that for $O'_1 := \bigcup_{i \in A_1} \Gamma \setminus \Gamma H_i \Omega'_i$, the compact subset $K_0$ is $O'_1$-self intersection free by Lemma 6.12, since $s(E'_1) = \emptyset$. By Proposition 6.2, there exists a neighborhood $\Omega_i$ of $D_i$ such that the pair $(I(q), J(q))$ is $\beta_0$-regular for all $q \in V_i - p_i \Omega'_i$, where

\begin{equation}
I(q) = \{ t \in \mathbb{R} : qu_t \in p_i \Omega'_i \} \text{ and } J(q) = \{ t \in \mathbb{R} : qu_t \in p_i \Omega_i \}.
\end{equation}

Set $O_1 := \bigcup_{i \in A_1} \Gamma \setminus \Gamma H_i \Omega_i$. Since $E_1 = \bigcup_{i \in A_1} \Gamma \setminus \Gamma H_i D_i \cap RF M$, $O_1$ is a neighborhood of $E_1 = s(E'_2) \cap E$. Now the compact subset $s(E'_2) - O_1$ is contained in $s(E'_2) - E$, which is relatively open in $s(E'_2)$. Therefore we can take a neighborhood $O'_1$ of $s(E'_2) - O_1$ so that $\overline{O'_1} \cap E = \emptyset$.

We will now define the following quadruple $K_n, O'_{n+1}, O_{n+1}$ and $O^*_{n+1}$ for each $1 \leq n \leq \ell - 1$ inductively:

- a compact subset $K_n = K_{n-1} - (O_n \cup O^*_n) \subset RF M$,
- a neighborhood $O'_{n+1}$ of $E'_{n+1}$,
- a neighborhood $O_{n+1}$ of $E_{n+1}$ and
- a neighborhood $O^*_{n+1}$ of $s(E'_{n+2}) - O_{n+1}$ such that $\overline{O^*_{n+1}} \cap E = \emptyset$.

Assume that the sets $K_{n-1}$, $O_n$, $O^*_n$ and $O^*_n$ are defined. We define $K_n := K_{n-1} - (O_n \cup O^*_n) = RF M - \bigcup_{i=1}^n (O_i \cup O^*_i)$. For each $i \in A_{n+1}$, let $\Omega'_i$ be a neighborhood of $D'_i$ in $G$ such that for $O'_{n+1} := \bigcup_{i \in A_{n+1}} \Gamma \setminus \Gamma H_i \Omega'_i$, $K_n$ is $O'_{n+1}$-self intersection free. Since $O_n \cup O^*_n$ is a neighborhood of $s(E'_{n+1})$, which is the set of all self-intersection points of $E'_{n+1}$, such collection of $O'_i$, $i \in A_{n+1}$ exists by Lemma 6.12.

Since $F \subset RF M$ is compact and disjoint from $E'$, we can also assume $\Gamma \setminus \Gamma H_i \Omega'_i$ is disjoint from $F$, by shrinking $\Omega'_i$ if necessary. More precisely, writing $F = \Gamma \setminus \Gamma \tilde{F}$ for some compact subset $\tilde{F} \subset G$, this can be achieved by choosing a neighborhood $\Omega'_i$ of $D'_i$ so that $p_i \Omega'_i$ is disjoint from $p_i \Gamma \tilde{F}$; and this is possible since $p_i \Gamma \tilde{F}$ is a closed set disjoint from a compact subset $p_i D'_i$. After choosing $\Omega'_i$ for each $i \in A_{n+1}$, define the following neighborhood of $E'_{n+1}$, $O'_{n+1} := \bigcup_{i \in A_{n+1}} \Gamma \setminus \Gamma H_i \Omega'_i$.

We will next define $O_{n+1}$. By Lemma 6.2, there exists a neighborhood $\Omega_i$ of $D_i$ such that the pair $(I(q), J(q))$ is $\beta_0$-regular for all $q \in V_i - p_i \Omega'_i$ where $I(q) = \{ t \in \mathbb{R} : qu_t \in p_i \Omega'_i \}$ and $J(q) = \{ t \in \mathbb{R} : qu_t \in p_i \Omega_i \}$. We then define the following neighborhood of $E_{n+1} = s(E'_{n+2}) \cap E$: $O_{n+1} := \bigcup_{i \in A_{n+1}} \Gamma \setminus \Gamma H_i \Omega_i$. Since the compact subset $s(E'_{n+2}) - O_{n+1}$ is contained in the set $s(E'_{n+2}) - E$, which is relatively open inside $s(E'_{n+2})$, we can take a neighborhood $O^*_{n+1}$ of $s(E'_{n+2}) - O_{n+1}$ so that $\overline{O^*_{n+1}} \cap E = \emptyset$. This finishes the inductive construction.

5. Definition of $O'$ and $O^*$: We define: $O' := \bigcup_{n=1}^\ell O'_n$, $O := \bigcup_{n=1}^\ell O_n$, $O^* := \bigcup_{n=1}^\ell O^*_n$. Note that $O'$ and $O$ are neighborhoods of $E'$ and $E$ respectively. Since $E \cap O^* = \emptyset$, the following defines a neighborhood of $E$:

\begin{equation}
O^\circ := O - O^*.
\end{equation}
6. Construction of $\beta_0$-regular families of $T(x)$-multiplicity free:

Fix $x \in F \subset RF M - O'$. Choose a representative $g \in G$ of $x$. We write each $\Lambda_n$ as the disjoint union $\Lambda_n = \bigcup_{j \in \theta_n} \Lambda_{n,j}$ where $\Lambda_{n,j} = \{ i \in \Lambda_n : \dim H_i = j \}$ and $\theta_n = \{ j : \Lambda_{n,j} \neq \emptyset \}$. Note that $\# \theta_n < \dim G$.

Fix $1 \leq n \leq \ell, j \in \theta_n$ and $i \in \Lambda_{n,j}$. For each $q \in p_i \Gamma g$, we define $I(q) := \{ t : qu_t \in p_i \Omega_t \}$ and $J(q) := \{ t : qu_t \in p_i \Omega_t \}$. In general, $I(q)$’s have high multiplicity among $q$’s in $\bigcup_{i \in \Lambda_{n,j}} p_i \Gamma g$, but the following subset $I'(q)$’s will be multiplicity-free, and this is why we defined $K_{n-1}$ as carefully as above:

- $I'(q) := \{ t : \text{for some } a \geq 0, [t, t+a] \subset I(q) \text{ and } xu_{t+a} \in K_{n-1} \}$;
- $J'(q) := I'(q) \cap J(q)$;
- $J''(q) := \{ t \in J(q) : xu_t \in K_{n-1} \}$.

Observe that $I'(q)$ and $J''(q)$ are unions of finitely many intervals, $J'(q) \subset T(x)$ and that $J''(q) \subset J'(q) \subset I'(q)$. Now, for each $1 \leq n \leq \ell$ and $j \in \theta_n$, define the family

$$X_{n,j} = \{(I(q), J'(q), J''(q)) : q \in \bigcup_{i \in \Lambda_{n,j}} p_i \Gamma g \}.$$  

We claim that each $X_{n,j}$ is a $\beta_0$-regular family with degree at most $\delta$ and $T(x)$-multiplicity free. Note for each $q \in p_i \Gamma g$, the number of connected components of $J''(q)$ is less than or equal to that of $J(q)$. Now that $J''(q) \subset J'(q) \subset J'(q) \subset I'(q)$, it follows that $X_{n,j}$’s are $\beta_0$-regular families with degree at most $\delta$.

We now claim that $X_{n,j}$ has $T(x)$-multiplicity free, that is, for any distinct indices $q_1, q_2 \in \bigcup_{i \in \Lambda_{n,j}} p_i \Gamma g$ of $X_{n,j}$, $I(q_1) \cap J'(q_2) = \emptyset$. We first show that $I'(q_1) \cap I'(q_2) = \emptyset$. Suppose not. Then there exists $t \in I'(q_1) \cap I'(q_2)$ for some $q_1 = p_i \gamma_1 g$ and $q_2 = p_k \gamma_2 g$, where $i, k \in \Lambda_{n,j}$. Then for some $a \geq 0$, we have $[t, t+a] \subset I(q_1) \cap I(q_2)$ and $xu_{t+a} \in K_{n-1}$. In particular, $xu_{t+a} \in \Gamma \cap [\gamma_1^{-1} H, \Omega_{i}] \cap \gamma_2^{-1} H, \Omega_{k}] \cap K_{n-1}$. Since $K_{n-1}$ is $\Omega_i$-$\Omega_k$-self intersection free, and $\dim H_i = \dim H_k = k$, we deduce from Proposition 6.12 that this may happen only when $i = k$, and $\gamma_1 \gamma_2^{-1} \in H_i \cap \Gamma$. Hence we have $q_1 = q_2$. This shows that $I'(q)$’s are pairwise disjoint. Now suppose that there exists an element $t \in I(q_1) \cap J'(q_2)$. Then by the disjointness of $I'(q_1)$ and $I'(q_2)$, it follows that $t \in (I(q_1) - I'(q_1)) \cap J'(q_2)$. By the definition of $I'(q_1)$, we have $xu_t \notin K_{n-1}$. This contradicts the assumption that $t \in J'(q_2)$.

7. Completing the proof: Let $X := \bigcup_{1 \leq \ell \leq \ell, i \in \theta_n} X_{n,j}$. In view of Proposition 5.3, it remains to check that the condition $t \in T(x) - J'(X)$ implies that $xu_t \notin \Omega^o$ where $\Omega^o$ is given in (6.9). Suppose that there exists $t \in T(x) - J'(X)$ such that $xu_t \in \Omega^o$. Write the neighborhood $\Omega^o$ as the disjoint union $\Omega^o = \bigcup_{n=1}^\ell (O_n - (\bigcup_{l=1}^{n-1} O_l \cup O^*) \cup O^*)$. Let $n \leq \ell$ be such that $xu_t \in O_n - (\bigcup_{l=1}^{n-1} O_l \cup O^*)$. Since $t \in T(x) - J'(X)$, we have $xu_t \in RF M - K_{n-1}$. Since $K_{n-1} = RF M - \bigcup_{l=1}^n (O_l \cup O^*)$, $xu_t \in \bigcup_{l=1}^n O_l \cup O^*$. This is a contradiction, since $\bigcup_{l=1}^n O^*_l \subset O^*$. \qed
As $\mathcal{K}^*$ is countable and $X(H_{\mathcal{U}}) \subset \sigma$-compact, the intersection $\mathcal{J}(U) \cap RF\ M$ can be exhausted by the union of the increasing sequence of $E_j \in \mathcal{E}_U$'s. Therefore, we deduce:

**Corollary 6.14.** There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ with $\mathcal{J}(U) \cap RF\ M = \bigcup_{j=1}^{\infty} E_j$ which satisfies the following: Let $x_i \in RF\ M$ be a sequence converging to $x \in \mathcal{J}(U) \cap RF\ M$. Then for each $j \in \mathbb{N}$, there exist a neighborhood $O_j$ of $E_j$ and $i_j \geq 1$ such that $\{ t \in \mathbb{R} : x_iu_t \in RF\ M - O_j \}$ is $2k$-thick for all $i \geq i_j$.

*Proof.* For each $j \geq 1$, we may assume $E_{j+1} \supset E'_j$ where $E'_j$ is given by Theorem 6.13. For each $j \geq 1$, there exists $i_j \in \mathbb{N}$ such that $x_i \not\in E_{j+1}$ for all $i \geq i_j$. Applying Proposition 6.13 to a compact subset $F = \{ x_i : i \geq i_j \}$ of $RF\ M$, we obtain a neighborhood $O_j$ of $E_j$ such that $\{ t \in \mathbb{R} : x_iu_t \in RF\ M - O_j \}$ is $2k$-thick for all $i \geq i_j$. □

Indeed we will apply Corollary 6.14 for the sequence $\{x_i\}$ contained in a closed orbit $x_0L$ of a proper connected subgroup $L < G$, which can be proved in the same way:

**Theorem 6.15** (Avoidance Theorem II). Consider a closed orbit $x_0L$ for some $x_0 \in RF\ M$ and $L \in \mathcal{Q}_U$. There exists an increasing sequence of compact subsets $E_1 \subset E_2 \subset \cdots$ in $\mathcal{E}_U$ with $\mathcal{J}(U, x_0L) \cap RF\ M = \bigcup_{j=1}^{\infty} E_j$, which satisfies the following: if $x_i \to x$ in $RF\ M \cap x_0L$ with $x \in \mathcal{J}(U, x_0L)$, then for each $j \in \mathbb{N}$, there exist $i_j \geq 1$ and an open neighborhood $O_j \subset x_0L$ of $E_j$ such that $\{ t \in \mathbb{R} : x_iu_t \in RF\ M - O_j \}$ is a $2k$-thick set for all $i \geq i_j$.

7. **LIMITS OF RF M-POINTS IN F* AND GENERIC POINTS**

In the rest of the paper, let $M = \Gamma \setminus \mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Recall that $\Lambda \subset \mathbb{S}^{d-1}$ denotes the limit set of $\Gamma$. In this section, we collect some geometric lemmas which are needed in modifying a sequence limiting on an RF M point (resp. limiting on a point in $RF\ M \cap \mathcal{J}(U)$) to a sequence of RF M-points (resp. whose limit still remains inside $\mathcal{J}(U)$).

**Lemma 7.1.** Let $C_n \to C$ be a sequence of convergent circles in $\mathbb{S}^{d-1}$. If $C \not\subset B$ for any component $B$ of $\Omega$, then $\# \limsup_{n \to \infty} C_n \cap \Lambda \geq 2$.

*Proof.* Without loss of generality, we may assume that $\infty \not\in \Lambda$ and hence consider $\Lambda$ as a subset of the Euclidean space $\mathbb{R}^{d-1}$. Note that there is one component, say, $B_1$ of $\Omega$ which contains $\infty$ and all other components of $\Omega$ are contained in the complement of $B_1$, which is a (bounded) round ball in $\mathbb{R}^{d-1}$. It follows that there are only finitely many components of $\Omega$ whose diameters are bounded from below by a fixed positive number; this follows from the fact that $\Gamma B$ is closed for each component $B$ of $\Omega$, and that there are only finitely many $\Gamma$-orbits of components of $\Omega$. 


Let $\delta = \text{diam}(C)/2$ so that we may assume $\text{diam}(C_n) > \delta$ for all sufficiently large $n \gg 1$. It suffices to show that there exists $\varepsilon_0 > 0$ such that $C_n \cap \Lambda$ contains $\xi_n, \xi'_n$ with $d(\xi_n, \xi'_n) \geq \varepsilon_0$ for all sufficiently large $n$. Suppose not. Then for any $\varepsilon > 0$, there exists an interval $I_n \subset C_n$ such that $\text{diam}(I_n) \leq \varepsilon$ and $C_n - I_n \subset \Omega$ for some infinite sequence of $n$'s. Since $C_n - I_n$ is connected, there exists a component $B_n$ of $\Omega$ such that $C_n \subset \mathcal{N}_\varepsilon(B_n)$, where $\mathcal{N}_\varepsilon(B_n)$ denotes the $\varepsilon$-neighborhood of $B_n$. In particular, we have $\text{diam}(B_n) + \varepsilon > \delta$. Taking $\varepsilon$ smaller than $0.5\delta$, this means that $\text{diam}(B_n) > \delta/2$. On the other hand, there are only finitely many components of $\Omega$ whose diameters are greater than $0.5\delta$, say $B_1, \cdots, B_\ell$. Let $\varepsilon_0 > 0$ be such that $\mathcal{N}_{\varepsilon_0}(B_1), \cdots, \mathcal{N}_{\varepsilon_0}(B_\ell)$ are all disjoint. Then by passing to a subsequence, there exists $B_i$ such that $C_n \subset \mathcal{N}_\varepsilon(B_i)$ for all small $0 < \varepsilon < \varepsilon_0$ and $n \geq 1$; hence $C \subset \mathcal{N}_\varepsilon(B_i)$. Since this holds for all sufficiently small $\varepsilon > 0$, we get that $C \subset B_i$, yielding a contradiction. \qed

In the next two lemmas, we set $U^- = U$ and $U^+ = U^t$.

**Lemma 7.2.** Let $U < N$ be a connected closed subgroup. Let $[g]L$ be a closed orbit for some $L \in \mathcal{L}_U$ and $[g] \in RF M$. Let $S_0$ and $S^*$ denote the boundaries of $\pi(gH(U))$ and $\pi(gL)$ respectively. If $S$ is a sphere such that $S_0 \subset S \subset \subset S^*$ and $\Gamma S$ is closed, then $[g] \in \mathcal{I}(U^*, [g]L)$.

**Proof.** Write $L = H(\tilde{U})C \subset \mathcal{L}_U$. Since $S_0 \subset S \subset \subset S^*$, there exists a connected proper subgroup $\tilde{U}$ of $\tilde{U}$, containing $U$ such that $S$ is the boundary of $\pi(gH(\tilde{U}))$. Since $\Gamma S$ is closed, $[g]H'(\tilde{U})$ is closed by Proposition 2.8. Now the claim follows from Proposition 3.8 and the definition of $\mathcal{I}(U^*, [g]L)$. \qed

**Lemma 7.3.** Let $U < N$ be a connected closed subgroup with dimension $m \geq 1$, and let $U^1_\pm, \cdots, U^m_\pm$ be one-parameter subgroups generating $U^\pm$. Consider a closed orbit $yL$ where $L \in \mathcal{L}_U$ and $y \in F^\prime_{H(U)} \cap RF M \cap \bigcap_{i=1}^m \mathcal{I}(U^i_\pm, yL)$. If $x_n \to y$ in $yL$, then, by passing to a subsequence, there exists a sequence $h_n \to h$ in $H(U)$ so that $x_nh_n \in RF M \cap yL$ and $yh \in RF M \cap \bigcap_{i=1}^m \mathcal{I}(U^i_\pm, yL)$.

**Proof.** Let $S^*$ denote the boundary of $\pi(g_0L)$. Let $\mathcal{Q}$ be the collection of all spheres $S \subset S^*$ such that $S \cap \Lambda \neq \emptyset$ and $\Gamma S$ is closed in $\mathcal{C}^{\dim S}$. By Corollary 4.8 and Remark 4.9, $\mathcal{Q}$ is countable. Choose a sequence $g_n \to g_0$ in $G$ as $n \to \infty$, so that $x_n = [g_n]$ and $y = [g_0]$. Let $S_n$ and $S_0$ denote the boundaries of $\pi(g_nH(U))$ and $\pi(g_0H(U))$ respectively so that $S_n \to S_0$ in $\mathcal{C}^m$ as $n \to \infty$. We will choose a circle $C_0 \subset S_0$ and a sequence of circles $C_n \subset S_n$ so that $C_n \to C_0$ and $\lim \sup (C_n \cap \Lambda)$ contains two distinct points outside of $\bigcup_{S \in \mathcal{Q}} S$. If $m = 1$, we set $C_0 = S_0$. When $m \geq 2$, we choose a circle $C_0 \subset S_0$ as follows. Note that $S_0$ is not contained in any sphere in $\mathcal{Q}$ by the assumption on $y$ and Lemma 7.2. Hence for any $S \in \mathcal{Q}$, $S_0 \cap S$ is a proper sub-sphere of $S_0$. Since $y \in F^\prime_{H(U)}$, for any component $B_i$ of $\Omega$, $S_0 \not\subset \partial B_i$ and hence $S_0 \cap \partial B_i$ is a proper sub-sphere of $S_0$. Choose a
circle $C_0 \subset S_0$ such that $\{g_0^+, g_0^-\} \subset C_0 \cap \Lambda$, $C_0 \not\subset S$ for any $S \in Q$, and $C_0 \not\subset \partial B_i \cap S_0$ for all $i$. This is possible, since $Q$ is countable. Since $S_0 \to S_0$, we can find a sequence of circles $C_n \subset S_0$ such that $C_n \to C_0$. We claim that $\limsup_n (C_n \cap \Lambda)$ is uncountable. Since $\#C_0 \cap \Lambda \geq 2$ and $C_0 \not\subset \partial B_i$, $C_0 \not\subset B_i$ for all $i$. Therefore, by Lemma 7.1, for any infinite subsequence $C_{n_k}$ of $C_n$, $\# \limsup_k (C_{n_k} \cap \Lambda) \geq 2$. By passing to a subsequence, we can find two distinct points $\xi_n, \xi'_n \in C_n \cap \Lambda$ which converge to two distinct points $\xi, \xi'$ of $C_0 \cap \Lambda$ respectively as $n \to \infty$. Choose a sequence $p_n \to p \in G$ such that $p_n^+ = \xi_n, p_n^- = \xi'_n, p^+ = \xi$ and $p^- = \xi'$. The set $T_n = \{t : [p_n]u_t \in RF M\}$ is a global $k$-thick subset, and hence $T := \limsup_n T_n$ is a global $k$-thick subset contained in the set $\{t : [p]u_t \in RF M\}$. Then $C_n \cap \Lambda$ converges, in the Hausdorff topology, to a compact subset $L \subset C_0 \cap \Lambda$ homeomorphic to the one-point compactification of $T$. Therefore $L$ is uncountable, so is $\lim\sup_n (C_n \cap \Lambda)$, proving the claim.

Let $\Psi := \bigcup_{S \in Q} C_0 \cap S$, i.e., the union of all possible intersection points of $C_0$ and spheres in $Q$. Since $C_0 \not\subset S$ for any $S \in Q$, $\#C_0 \cap S \leq 2$. Hence $\Psi$ is countable, and hence $\lim\sup_n (C_n \cap \Lambda) - \Psi$ is uncountable. Note that this works for any infinite subsequence of $C_n$’s. Therefore we can choose sequences $\xi_n^-, \xi_n^+ \subset C_n \cap \Lambda$ converging to distinct points $\xi^-, \xi^+$ of $(C_0 \cap \Lambda) - \Psi$ respectively, by passing to a subsequence. As $\xi^-, \xi^+ \subset C_0$ and $C_0 \subset S_0$, there exists a frame $g_0 h = (v_0, \cdots, v_{d-1}) \in g_0 H(U)$ whose first vector $v_0$ is tangent to the geodesic $[\xi^-, \xi^+]$. Setting $g := g_0 h$, we claim that $[g] \in \bigcap_i \mathcal{U}(U^{(i)}_{\pm}, yL)$. Suppose that $[g] \in \mathcal{U}(U^{(i)}_{\pm}, yL)$, as the case when $[g] \in \mathcal{U}(U^{(i)}_{+}, yL)$ can be dealt similarly, by changing the role of $g^-$ and $g^+$ below. For simplicity, set $U^{(i)} := U^{(i)}_{\pm}$. Now by Proposition 4.13, there exist $L_0 \in L_{U^{(i)}}$ and $\alpha \in N \cap L$ such that $(L_0)_{nc} \leq L_{nc}$ and $[g]a\alpha L_0$ is closed. Let $S$ denote the boundary of $\pi(g_0 L_0)$. Since $\alpha \in N \cap L$, we have $(g^+) = g^+ = \xi^+ \subset S \cap \Lambda \cap C_0$. Since $S \not\lor S^+$, $S \cap \Lambda \not= \emptyset$ and $\Gamma S$ is closed, we have $S \subset Q$. It follows that $\xi^+ \in \Psi$, contradicting the choice of $\xi^+$. This proves the claim.

Now choose a vector $v_0^{(n)}$ which is tangent to the geodesic $[\xi_n^-, \xi_n^+]$. We then extend $v_0^{(n)}$ to a frame $g_n h_n \subset g_n H(U)$ so that $g_n h_n$ converges to $g = g_0 h$ as $n \to \infty$. Since $\{\xi_n^+\} \subset \Lambda$, we have $[g_n h_n] \in RF M$. This completes the proof.\[\square\]

We will need the following lemma later.

**Lemma 7.4.** Let $k \geq 1$. Let $\chi$ be a $k$-horosphere in $\mathbb{H}^{k+1}$ resting at $p \in \partial\mathbb{H}^{k+1}$, and $\mathcal{P}$ be a geodesic $k$-plane in $\mathbb{H}^{k+1}$. Let $\xi \in \partial\mathcal{P}$, $\delta$ be a geodesic joining $\xi$ and $p$, and $q = \delta \cap \chi$. There exists $R_0 > 1$ such that for any $R > R_0$, if $d(\chi, \mathcal{P}) < R - 1$, then $d(q, \mathcal{P}) < R$.

**Proof.** For $k = 1$, this is shown in [17, Lem. 4.2]. Now let $k \geq 2$. Consider a geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^{k+1}$ which passes through $q$ and orthogonal to $\mathcal{P}$. Then $\chi \cap \mathbb{H}^2$ and $\mathcal{P} \cap \mathbb{H}^2$ are a horocycle and a geodesic in $\mathbb{H}^2$ respectively.
As \( d_{B^k+1}(\chi, \mathcal{P}) = d_{B^2}(\chi \cap \mathbb{H}^2, \mathcal{P} \cap \mathbb{H}^2) \) and \( d_{B^k+1}(q, \mathcal{P}) = d_{B^2}(q, \mathcal{P} \cap \mathbb{H}^2) \), the conclusion follows from the case \( k = 1 \).

\[ \text{□} \]

**Lemma 7.5.** Let \( U < \tilde{H} \cap N \) be a non-trivial connected closed subgroup. If the boundary of \( \pi(gH(U)) \) is contained in \( \partial B \) for some component \( B \) of \( \Omega \), then \( [g] \in BF M \cdot C(H(U)) \).

**Proof.** As \( U \) is equal to \( mU_km^{-1} \) for some \( m \in \tilde{H} \cap M \) and \( 1 \leq k \leq d - 2 \), the general case is easily reduced to the case when \( U = U_k \). Since \( g = (v_0, \cdots, v_d) \) has its first \( (k+1) \)-vectors tangent to the geodesic \( (k+1) \)-plane \( \pi(gH(U_k)) \) and \( \partial(\pi(gH(U_k))) \subset \partial B \), we can use an element \( c \in C(H(U_k)) \) = \( SO(d - k - 2) \) to modify the next \( (d - k - 2) \)-vectors so that \( gc \) has its first \( (d - 1) \)-vectors tangent to \( hull(\partial B) \). Then \( [gc] \in BF M \), proving the claim. \[ \text{□} \]

**Lemma 7.6.** Let \( U < \tilde{H} \cap N \) be a non-trivial connected closed subgroup. If \( x_n \in RF M \cdot U \) is a sequence converging to some \( x \in RF M \), then passing to a subsequence, there exists \( u_n \in U \) such that \( x_nu_n \in RF M \) and at least one of the following holds:

1. \( u_n \to e \) and hence \( x_nu_n \to x \), or
2. \( x = zc \) for some \( z \in BF M \) with \( c \in C(H(U)) \), and \( x_nu_n \) accumulates on \( \tilde{H}c \).

**Proof.** If \( x_n \) belongs to \( RF M \) for infinitely many \( n \), we simply take \( u_n = e \). So assume that \( x_n \notin RF M \) for all \( n \). Choose a sequence \( g_n \to g_0 \) in \( G \) so that \( x_n = [g_n] \) and \( x = [g_0] \). As \( x \in RF M \), we have \( \{g_0(0), g_0(\infty)\} \subset \Lambda \). As \( x_n \in RF^+ \cdot M - RF M \), we have \( g_n(\infty) \in \Lambda \) and \( g_n(0) \in \Omega \). For each \( n \), choose an element \( u_n \in U \) so that \( 0 < \alpha_n := \|u_n\| \leq \infty \) is the minimum of \( \|u\| \) for all \( u \in U \) satisfying \( g_nu(0) \in \Lambda \). Set \( \alpha := \limsup \alpha_n \). If \( \alpha = 0 \), then we are in case (1). Hence we will assume \( 0 < \alpha \leq \infty \). Let \( C_n \) denote the boundary of \( \pi(g_nH(U)) \) and \( C_0 \) the boundary of \( \pi(g_0H(U)) \). Then \( C_n \to C_0 \) in \( C^{dim U} \). Recall that \( B_U(r) \) denotes the ball of radius \( r \) centered at \( 0 \) inside \( U \). Set \( \mathcal{B}_n := g_nB_U(\alpha_n)(0) \) and \( \mathcal{B}_0 := g_0B_U(\alpha)(0) \). Then \( \mathcal{B}_n \subset C_n \cap \Omega \), and \( \partial \mathcal{B}_n \cap \Lambda \neq \emptyset \) by the choice of \( u_n \). By passing to a subsequence, we have \( \alpha_n \to \alpha \) and \( \mathcal{B}_n \to \mathcal{B}_0 \) as \( n \to \infty \) and hence the diameter of \( \mathcal{B}_n \) in \( S^{d-1} \) is bounded below by some positive number. Hence, passing to a subsequence, we may assume that \( \mathcal{B}_n \) are all contained in the same component, say \( B \) of \( \Omega \). Consequently, \( \mathcal{B}_0 \subset \overline{B} \).

We claim that \( \#\overline{\mathcal{B}_0} \cap \partial B \geq 2 \). First note that \( g_0(0) \in \Lambda \). If \( \alpha = \infty \), then \( g_nu_n(0) \to g_0(\infty) \in \Lambda \cap \overline{\mathcal{B}_0} \). If \( \alpha < \infty \), then \( u_n \) converges to some \( u \in U \), passing to a subsequence, and \( u \neq e \), as \( \alpha > 0 \). Now, \( g_nu_n(0) \to g_0u(0) \in \Lambda \cap \overline{\mathcal{B}_0} \). Since \( \Lambda \cap \overline{\mathcal{B}} \subset \partial B \), this proves the claim.

Therefore \( \mathcal{B}_0 \) is contained in \( \partial B \), and hence so is \( C_0 \). By Lemma 7.5, this implies that \( x = zc \) for some \( z \in BF M \) and \( c \in C(H(U)) \). We proceed to show that \( x_nu_n \) accumulates on \( z\tilde{H}c \). Since \( c \in C(H(U)) \), we may assume \( c = e \) by replacing \( x \) with \( xc^{-1} \), and \( x_n \) with \( x_n e^{-1} \).
We claim that \( \pi(g_n u_n) \) goes arbitrarily close to the plane \( \pi(g_0 \hat{H}) \) as \( n \to \infty \). Since \( x \hat{H} = [g_0] \hat{H} \) is compact, \( g_n u_n \in g_0 \hat{H} \) and \( \pi(g_n \hat{H}) \) is a geodesic plane nearly parallel to \( \pi(g_0 \hat{H}) \) for all large \( n \), this claim implies that \( [g_n] u_n \) accumulates on \( z \hat{H} \), completing the proof.

Now, to prove the claim, let \( D_n := C_n \cap \partial B \), and \( \mathcal{P}_n := \text{hull}(D_n) \). Let \( k = \text{dim} \, U \). Since \( C_n \) is a \( k \)-sphere meeting the \((d - 2)\)-sphere \( \partial B \subset \mathbb{S}^{d-1} \), and \( C_n \not\subset \partial B \), it follows that \( D_n \) is a \((k - 1)\)-sphere. We set \( \mathcal{H}_n := \text{hull}(C_n) \), \( \mathcal{H}_0 := \text{hull}(C_0) \) and \( \mathcal{H} := \text{hull}(\partial B) = \pi(g_0 \hat{H}) \). Then \( \mathcal{H}_n \cap \mathcal{H} = \mathcal{P}_n \). Let \( \varepsilon > 0 \) be arbitrary, and \( \mathcal{N}_\varepsilon(\mathcal{H}) \) denote the \( \varepsilon \)-neighborhood of \( \mathcal{H} \) in \( \mathbb{H}^d \).

Letting \( d_{\mathcal{H}_n}(\cdot, \cdot) \) denote the hyperbolic distance in \( \mathcal{H}_n \), we may write
\[
\mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n = \{ p \in \mathcal{H}_n : d_{\mathcal{H}_n}(p, \mathcal{P}_n) < R_n \}
\]
for some \( R_n > 0 \). This is because \( \mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n \) is convex and invariant under family of isometries, whose axes of translation and rotation are contained in \( \mathcal{P}_n \). As \( C_n \to C_0 \subset \partial B \) as \( n \to \infty \), it follows that \( R_n \to \infty \) as \( n \to \infty \). Let \( \chi_n := \pi(g_n U) \), and \( \chi_0 := \pi(g_0 U) \), which are \( k \)-horospheres contained in \( \mathcal{H}_n \) and \( \mathcal{H}_0 \) respectively.

We next show that there is a uniform upper bound for \( d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n), n \in \mathbb{N} \). To see this, we only need to consider those \( \mathcal{P}_n \)'s which are disjoint from \( \chi_n \), as \( d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n) = 0 \) otherwise. Since \( \chi_n \to \chi_0 \) and \( C_n \to C_0 \) as \( n \to \infty \), it suffices to check that the diameters of \( D_n \) with respect to the spherical metric on \( \mathbb{S}^{d-1} \) have a uniform positive lower bound. Let us write \( C_n - D_n = E_n \cup E'_n \), where \( E_n \) is a connected component of \( C_n - D_n \) meeting \( B \), and \( E'_n \) is the other component. Since \( C_n \to C_0 \) as \( n \to \infty \), a uniform lower bound for both \( \text{diam}(E_n) \) and \( \text{diam}(E'_n) \) will give a uniform upper bound for \( \text{diam}(D_n) \). Since \( B_n \subset E_n \), \( \text{diam}(E'_n) > \text{diam}(B_n)/2 \) for all sufficiently large \( n \). On the other hand, note that \( \chi_n \subset \mathcal{H}_n \) is a horosphere resting at a point in \( E'_n \). Since \( \chi_n \) converges to \( \chi \), the condition that \( \mathcal{P}_n \cap \chi_n = \emptyset \) implies that \( \text{diam}(E'_n) \) is also bounded below by some positive constant. Since \( R_n \to \infty \), we conclude that \( d_{\mathcal{H}_n}(\mathcal{P}_n, \chi_n) < R_n - 1 \) for all sufficiently large \( n \). Applying Lemma 7.4 to \( \mathbb{H}^{k+1} = \mathcal{H}_n \), \( \chi = \chi_n \), \( \mathcal{P} = \mathcal{P}_n \), \( \xi = g_n^+ \) and \( q = \pi(g_n u_n) \), we have \( d_{\mathcal{H}_n}(\pi(g_n u_n), \mathcal{P}_n) < R_n \) and hence \( \pi(g_n u_n) \in \mathcal{N}_\varepsilon(\mathcal{H}) \cap \mathcal{H}_n \), for all sufficiently large \( n \). As \( \varepsilon > 0 \) was arbitrary, this proves that \( \pi(g_n u_n) \) goes arbitrarily close to \( \pi(g_0 \hat{H}) \) as \( n \to \infty \). This finishes the proof. \( \square \)

**Lemma 7.7.** Let \( U < N \) be a non-trivial connected closed subgroup. If \( x_n \to x \) in \( F^* \cap \text{RF}_+ M \), and \( x \in F^* \cap \text{RF}_+ M \), then there exists \( u_n \to e \) in \( U \) such that \( x_n u_n \in \text{RF}_+ M \); in particular, \( x_n u_n \to x \) in \( F^* \cap \text{RF}_+ M \).

**Proof.** The general case easily reduces to the case when \( U < \hat{H} \cap N \). Then the claim follows from Lemma 7.6 and Lemma 3.4. \( \square \)

**Obtaining limits in \( F^* \).** For \( \varepsilon > 0 \), we set
\[
(7.1) \quad \text{core}_\varepsilon(M) := \left\{ x \in \Gamma\backslash G : \pi(x) \in \text{core} \, M \text{ and } d(\pi(x), \partial \text{ core} \, M) \geq \varepsilon \right\}.
\]
We note that \( \text{core}_\varepsilon(M) \) is a compact of \( F^* \) for all sufficiently large \( \varepsilon > 0 \).
Lemma 7.8. Let \( x \in RF_M \), and \( V = \{ v_t : t \in \mathbb{R} \} \) be a one-parameter subgroup. If \( \pi(xV) \not\subset \partial \text{core} \, M \), and \( xv_t \in RF_M \) for some sequence \( t_i \to +\infty \), then there exists a sequence \( s_i \to +\infty \) such that \( xv_{s_i} \) converges to a point in \( F^* \).

Proof. It suffices to show that there exists \( s_i \to +\infty \) such that \( xv_{s_i} \in \text{core}_{\eta/3}(M) \) where \( \eta \) is as given in (3.5). Let \( x = [g] \), and set \( o = (1, 0, \ldots , 0) \in \mathbb{H}^d = \mathbb{R}^+ \times \mathbb{R}^{d-1} \). We may assume \( g = (e_0, \ldots , e_{d-1})_0 \in F \mathbb{H}^d \) where \( e_i \) are standard basis vectors in \( T_o \mathbb{H}^d \simeq \mathbb{R}^d \). Note that for \( V^+ = \{ v_t : t > 0 \} \), \( gV^+ \) is a translation of the frame \( g \) along a horizontal ray emanating from \( o \) along the \( V^+ \)-direction. By the definition of \( \eta \), the \( \eta/3 \)-neighborhoods of \( \text{hull} \, B_j \)'s are mutually disjoint. For each \( i \), set \( s_i := t_i \) if \( xv_{t_i} \in \text{core}_{\eta/3}(M) \). Otherwise, there exists a unique \( j \) such that \( d(\pi(gv_{s_i}), \text{hull} \, B_j) < \eta/3 \). If \( \pi(gv_{[t_i, \infty)}) \) were contained in the \( \eta/3 \)-neighborhood of \( \text{hull} \, B_j \), then the unique geodesic 2-plane which contains \( \pi(gv_{[t_i, \infty)}) \) must lie in \( \partial \text{hull} \, B_j \), and hence \( \pi(xV) \subset \partial \text{core}(M) \); this contradicts the hypothesis. Therefore there exists \( t_i < s_i < \infty \) such that \( d(\pi(gv_{s_i}), \text{hull} \, B_j) = \eta/3 \), as desired. \( \square \)

Lemma 7.9. Let \( x_nL_nv_n \) be a sequence of closed orbits with \( x_n \in RF_+M \), \( L_n \in \mathcal{L}_U \) and \( v_n \in (L_n \cap N) \). Suppose that either

1. \( x_n \in F^* \) for all \( n \); or
2. \( x_nL_nv_n \cap RF_+M \cap F^* \neq \emptyset \) for all \( n \).

Then \( F^* \cap \limsup\sup (x_nL_nv_n \cap RF_+M) \neq \emptyset \).

Proof. We claim that if \( x_n \in F^* \), then \( x_nL_nv_n \cap RF_+M \cap F^* \neq \emptyset \), that is, the hypothesis (1) implies (2). Suppose not. Then, since \( A \subset L_n \), \( (x_nAv_nA \cap RF_+M) \subset RF_+M - F^* \). Since the set \( RF_+M - F^* \) is a closed \( A \)-invariant set and \( e \in \overline{Av_nA} \), we would have \( x_n \in RF_+M - F^* \), yielding a contradiction. It follows from the claim that there exists \( z_n \in x_nL_n \cap RF_+M \) such that \( \pi(z_nv_nU) \not\subset \partial \text{core}(M) \) for all \( n \). In particular, there exists \( u_n \in U \) such that \( z_nv_nu_n \in \text{core}_{\eta/3}(M) \). Since \( \text{core}_{\eta/3}(M) \) is a compact subset of \( F^* \), \( z_nv_nu_n = z_nv_nv_n \) converges to a point in \( F^* \), finishing the proof. \( \square \)

Lemma 7.10. Let \( x_0L \) be a closed orbit with \( x_0 \in RF_M \) and \( L \in \mathcal{L}_U \). Suppose that \( E \) is a closed \( U \)-invariant subset containing \( x_0Lv_n \cap RF_+M \) for some sequence \( v_n \to \infty \) in \( (L \cap N) \). If \( x_0 \in F^* \) or \( x_0Lv_n \cap RF_+M \cap F^* \neq \emptyset \) for all \( n \), then there exist \( y \in RF_M \cap F^* \) and a one parameter subgroup \( V \subset (L \cap N) \) such that \( E \supset y(L \cap N)V \).

Proof. Note that \( (x_0Lv_n \cap RF_+M)(v_n^{-1}Av_n) \subset E \). By Lemma 7.9, there exists \( y \in F^* \cap \limsup\sup (x_0Lv_n \cap RF_+M) \). Since \( y \in F^* \cap RF_+M \subset RF_M \cdot U \), we may assume \( y \in F^* \cap RF_M \) by modifying \( y \) using an element of \( U \). Note that \( \liminf\sup (x_0Lv_n \cap RF_+M) \supset y(L \cap N) \), passing to a subsequence. Since \( \limsup\sup (v_n^{-1}Av_n) \) contains a one-parameter subgroup \( V \subset (L \cap N) \) by Lemma 2.3, we obtain that \( y(L \cap N)V \subset E \). \( \square \)
Lemma 7.11. If \( yLv_0 \cap RF M \cap F^* \neq \emptyset \) for some \( v_0 \in N \) and \( L \in L_U \), then \( yLv \cap F^* \cap RF M \neq \emptyset \) for all \( v \in Av_0A \).

Proof. Let \( y_0 := yLv_0 \in yLv \cap F^* \cap RF M \), and \( v = av_0b \in Av_0A \). Then \((ya^{-1})v = yLv_0b \in F^* \cap RF M \) as \( F^* \cap RF M \) is \( A \)-invariant. Since \( ya^{-1}v \in yLv \), the claim is proved. \( \square \)

Lemma 7.12. Let \( x_0L \) be a closed orbit with \( x_0 \in RF M \) and \( L \in L_U \). Suppose that \( E \) is a closed \( AU \)-invariant subset containing \( x_0Lv \cap RF_+ M \) for some non-trivial element \( v \in (L \cap N)^\perp \). If \( x_0 \in F^* \) or \( x_0Lv \cap RF M \cap F^* \neq \emptyset \), then there exist \( y \in F^* \cap RF M \) and a one parameter subgroup \( V \subset (L \cap N)^\perp \) such that \( E \supseteq y(L \cap N)V.A \).

Proof. Since \( X \) is \( A \)-invariant, we get \((x_0L \cap RF_+ M)AvA \subset E \). Choose a sequence \( v_n := a_nv_n^{-1} \in AvA \) tending to \( \infty \). Note that either \( x_0 \in F^* \) or for all \( n \), \( x_0Lv_n \cap RF M \cap F^* \neq \emptyset \) by Lemma 7.11. Therefore the claim follows from Lemma 7.10. \( \square \)

8. Limits of unipotent blowups

Fix \( k > 1 \) as given by Proposition 3.10. In the whole section, we fix a non-trivial connected subgroup \( U < N \). For a given sequence \( g_i \to e \), and a sequence of \( k \)-thick subsets \( T_i \) of a one-parameter subgroup \( U_0 < U \), we study the set \( \limsup T_i g_i U \) under certain conditions on the sequence \( g_i \).

The basic tool used here is the so-called quasi-regular map associated to the sequence \( g_i \) introduced in the work of Margulis-Tomanov [15] to study the object \( \limsup U_0g_i U \) in the finite volume case. For our application, we need a somewhat more precise information on the shape of the set \( \limsup U_0g_i U \) as well as \( \limsup T_i g_i U \) than discussed in [15].

Let \( U^\perp \) denote the orthogonal complement of \( U \) in \( N \simeq \mathbb{R}^{d-1} \) as defined in section 2. Recall from (2.2) that \( N(U) = ANC_1(U) \cap C_2(U) \) where \( C_1(U) = C(H(U)) \) and \( C_2(U) = H(U) \cap M \cap C(U) \). Since \( N(U) \) is the identity component of \( NC(U) \), for a sequence \( g_i \to e \), the condition \( g_i \in NC(U) \) means \( g_i \in N(U) \) for all sufficiently large \( i \gg 1 \). Note that the product \( AU^\perp C_2(U) \) is a connected subgroup of \( G \), since \( C_2(U) \) commutes with \( U^\perp \), and \( A \) normalizes \( U^\perp C_2(U) \).

Lemma 8.1. For a given sequence \( g_i \to e \) in \( G - N(U) \), there exists a one-parameter subgroup \( U_0 < U \) such that the following holds; for any given sequence of \( k \)-thick subsets \( T_i \subset U_0 \), there exist sequences \( t_i \in T_i \), and \( u_i \in U \) such that as \( i \to \infty \), \( u_i g_i u_i^{-1} \to \alpha \) for some non-trivial element \( \alpha \in AU^\perp C_2(U) - C_2(U) \). Moreover, \( \alpha \) can be made arbitrarily close to \( e \).

Proof. Set \( L := AU^\perp MN^\perp \). Note that \( N(U) \cap L = AU^\perp C_1(U) \cap C_2(U) \) and that the product map from \( U \times L \) to \( G \) is a diffeomorphism onto a Zariski open neighborhood of \( e \) in \( G \). Following [15], we will construct a quasi-regular map \( \psi : U \to N(U) \cap L \) associated to the sequence \( g_i \). Except for a Zariski closed subset of \( U \), the product \( g_i u \) can be written as an element of
UL in a unique way. We denote by $\psi_i(u) \in L$ its $L$-component so that $g_i u \in U \psi_i(u)$. By Chevalley’s theorem, there exists an $\mathbb{R}$-regular representation $G \to \text{GL}(W)$ with a distinguished point $p \in W$ such that $U = \text{Stab}_G(p)$. Then $pG$ is locally closed, and $N_G(U) = \{g \in G : pg u = pg \text{ for all } u \in U\}$.

For each $i$, the map $\hat{\phi}_i : U \to W$ defined by $\hat{\phi}_i(u) = pg_i u$ is a polynomial map in $U = \mathbb{R}^m$ of degree uniformly bounded, and $\hat{\phi}_i(e)$ converges to $p$ as $i \to \infty$. As $g_i \not\in N_G(U)$, $\hat{\phi}_i$ is non-constant. Denote by $B(p, r)$ the ball of radius $r$ centered at $p$, fixing a norm $\| \cdot \|$ on $W$. Since $pG$ is open in its closure, we can find $\lambda_0 > 0$ such that

\begin{equation}
B(p, \lambda_0) \cap \overline{pG} \subset pG.
\end{equation}

Without loss of generality, we may assume that $\lambda_0 = 2$ by renormalizing the norm. Now define $\lambda_i := \sup\{\lambda \geq 0 : \hat{\phi}_i(B_U(\lambda)) \subset B(p, 2)\}$. Note that $\lambda_i < \infty$ as $\hat{\phi}_i$ is nonconstant, and $\lambda_i \to \infty$ as $i \to \infty$, as $g_i \to e$. We define $\phi_i : U \to W$ by $\phi_i(u) := \hat{\phi}_i(\lambda_i u)$. This forms an equi-continuous family of polynomials on $U$. Therefore, after passing to a subsequence, $\phi_i$ converges to a non-constant polynomial $\phi$ uniformly on every compact subset of $U$.

Moreover $\sup\{\|\phi(u) - p\| : u \in B_U(1)\} = 1$, $\phi(B_U(1)) \subset pL$, and $\phi(0) = p$. Now the following map $\psi$ defines a non-constant rational map defined on a Zariski open dense neighborhood of $U$ of $e$ in $U$: $\psi := \rho_L^{-1} \circ \phi$ where $\rho_L$ is the restriction to $L$ of the orbit map $g \mapsto p.g$. We have $\psi(e) = e$ and $\psi(u) = \lim_i \psi_i(\lambda_i u)$ where the convergence is uniform on compact subsets of $U$ and $\psi(u) \in L \cap \overline{N(U)} = AU^\perp C_1(U) C_2(U)$. Since $\psi$ is non-constant, there exists a one-parameter subgroup $U_0 < U$ such that $\psi|_{U_0}$ is non-constant.

Now let $T_i$ be a sequence of $k$-thick sets in $U_0 \simeq \mathbb{R}$. Then $T_i/\lambda_i$ is also a $k$-thick set, and so is $T_\infty := \limsup_{i \to \infty} (T_i/\lambda_i) \subset U_0$. Finally, for all $t \in T_\infty$, there exists a sequence $t_i \in T_i$ such that $t_i/\lambda_i \to t$ as $i \to \infty$ (by passing to a subsequence). Since $\psi_i \circ \lambda_i \to \psi$ uniformly on compact subsets, $\psi(t) = \lim_{i \to \infty} (\psi_i \circ \lambda_i)(t_i/\lambda_i) = \lim_{i \to \infty} \psi_i(t_i)$. By the definition of $\psi_i$, this means that there exists $u_i \in U$ such that $\psi(t) = \lim_{i \to \infty} u_i g_i u_{t_i}$. Since $\psi|_{U_0}$ is a non-constant continuous map, and an uncountable set $T_\infty$ accumulates on 0, the image $\psi(T_\infty)$ contains a non-trivial element $\alpha$ of $AU^\perp C_1(U) C_2(U)$ which can be taken arbitrarily close to $e$.

We now claim that if $\alpha$ is sufficiently close to $e$, then it belongs to $AU^\perp C_2(U)$. Consider $H'(U) := H(U) C_1(U)$, and let $\mathfrak{h}$ denote its Lie algebra. Now for all $i$ large enough, using the decomposition $g = \mathfrak{g} \oplus \mathfrak{h}^\perp$ in (2.5), we can write $g_i = c_i d_i r_i$ where $c_i \in C_1(U), d_i \in H(U)$ and $r_i \in \exp \mathfrak{h}^\perp$. Since $c_i$ commutes with $U$, we can write $u_i g_i u_{t_i} = (u_i u_{t_i}) c_i (u_i^{-1} d_i u_{t_i})(u_i^{-1} r_i u_{t_i})$.

On the other hand, we have $\lim_{i \to \infty} p u_i g_i u_{t_i} = \lim_{i \to \infty} p c_i (u_i^{-1} d_i u_{t_i})(u_i^{-1} r_i u_{t_i}) = p\alpha$. Since $c_i \to e$, $u_i d_i u_{t_i}^{-1} \in H(U)$, and $u_i r_i u_{t_i}^{-1} \in \exp \mathfrak{h}^\perp$, it follows that both sequences $u_i d_i u_{t_i}^{-1}$ and $u_i r_i u_{t_i}^{-1}$ must converge, say to $h \in H(U)$ and to $q \in \exp \mathfrak{h}^\perp$, respectively. Hence $\alpha = h q$ by replacing $h$ by $u h$ for some $u \in U$. On the other hand, we can write $\alpha = av c_1 c_2 \in AU^\perp C_1(U) C_2(U)$. 


So $hq = avc_1c_2$. Note that $c := c_1c_2 \in C(H(U))H(U) = H'(U)$. We get
\[(a^{-1}hc^{-1})(cqc^{-1}) = v.
\]
Now, when $\alpha$ is sufficiently close to $e$, all elements appearing in (8.2) are also close to $e$. Recall that the map $H'(U) \times h^+ \to G$ given by $(h', X) \to h' \exp X$ is a local diffeomorphism onto a neighborhood of $e$. Since $(a^{-1}hc^{-1}) \in H'(U)$, and $cqc^{-1}, v \in \exp h^+$, we have $a^{-1}hc^{-1} = e$ and $cqc^{-1} = v$ for $\alpha$ sufficiently small. In particular, $a^{-1}hc_2^{-1} = c_1^{-1} \in H(U) \cap C(H(U)) = \{e\}$. Hence $c_1 = e$. It follows that $\alpha \in AU^+ C_2(U)$, as desired.

We further claim that we can choose $\alpha$ outside of $C_2(U)$. As $C_2(U)$ is a compact subgroup, we can choose a $C_2(U)$-invariant Euclidean norm $\|\cdot\|$ on $W$. If $\alpha = \psi(t) \in C_2(U)$ for some $t \in T_\infty \subset U_0$, then $t$ is one of finitely many solutions of the polynomial equation $\|\phi(t)\|^2 = \|p\|^2$. Therefore, except for finitely many $t \in T_\infty$, $\alpha = \psi(t) \in AU^+ C_2(U) - C_2(U)$. This finishes the proof. \hfill \Box

The following lemma is similar to Lemma 8.1, but here we consider the case when $U$ is the whole horospherical subgroup $N$. In this restrictive case, the limiting element can be taken inside $A$.

**Lemma 8.2.** Let $T_i \subset N$ be a sequence of $k$-thick subsets in the sense that for any one-parameter subgroup $U_0 < N$, $T_i \cap U_0$ is a $k$-thick subset of $U_0 \simeq \mathbb{R}$. For any sequence $g_i \to e$ in $G - N_G(N)$, there exist $t_i \to \infty$ in $T_i$ and $u_i \in N$ such that $u_i g_i u_{t_i} \to a$ for some non-trivial element $a \in A$. Moreover, $a$ can be chosen to be arbitrarily close to $e$.

**Proof.** We first consider the case when $g_i$ belongs to the opposite horospherical subgroup $N^+$. We will use the notations $u^+$ and $u^-$ defined in Section 2. Write $g_i = \exp u^+(w_i)$ for some $w_i \in \mathbb{R}^{d-1}$. For $x \in \mathbb{R}^{d-1}$, set $u_x := \exp u^-(x) \in N$. Let $\varepsilon > 0$ be arbitrary. Since $T_i$ is a $k$-thick subset of $N$, there exists $\alpha_i \in \mathbb{R}$ such that $\alpha_i w_i \in T_i$ and $\varepsilon < \frac{|\alpha|\|w_i\|^2}{2} < k\varepsilon$. Setting $u_{x_i} := u_{\alpha_i w_i} \in T_i$ and $y_i := -\alpha_i w_i \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right)^{-1}$, we compute:

$$u_{y_i} g_i u_{x_i} = \begin{pmatrix} 1 + \frac{\alpha_i\|w_i\|^2}{2} & 0 & 0 \\ \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right)^{-1} w_i & 1 & 0 \\ -\frac{\|w_i\|^2}{2} & \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right) w_i^t \left(1 + \frac{\alpha_i\|w_i\|^2}{2}\right)^2 \end{pmatrix}.$$  

The condition for the size of $\alpha_i$ guarantees that, by passing to a subsequence, the sequence $u_{x_i} g_i u_{y_i}$ converges to an element $\text{diag}(\alpha, 1_{d-1}, \alpha^{-1}) \in A$ for $\alpha \in [(1 - \varepsilon)^{-2}, (1 - k\varepsilon)^{-2}] \cup [(1 + k\varepsilon)^{-2}, (1 + \varepsilon)^{-2}]$ as $i \to \infty$. This proves the claim when $g_i \in N^+$. Since the product map $A \times M \times N^+ \times N \to G$ is a diffeomorphism onto a Zariski-open neighborhood of $e$ in $G$, we can write $g_i = \alpha_i m_i u_i^+ u_{t_i}^-$ for some $\alpha_i \in A$, $m_i \in M$, $u_i^+ \in N^+$ and $u_i^- \in N$ all of which converge to $e$ as $i \to \infty$. By the previous case, we can find $u_{t_i} \in T_i$ and $u_i \in N$ such that $u_i u_i^+ u_{t_i}$ converges to a non-trivial element.
Lemma 8.3. Let $L$ be any connected reductive subgroup of $G$ normalized by $A$. Let $U_0$ be a one-parameter subgroup of $L \cap N$. Let $T_i \subset U_0$ be a sequence of $k$-thick subsets. For a given sequence $r_i \to e$ in $\exp(1-)-N(U_0)$, there exists a sequence $t_i \in T_i$ such that as $i \to \infty$, $u_i^{-1}r_iu_i \to v$ for some non-trivial element $v \in (L \cap N) \setminus \{0\}$, and $v$ can be chosen arbitrarily close to $e$. Moreover, for all $n$ large enough, we can make $v$ so that $n \leq |v| \leq 2k^2n$.

Proof. Without loss of generality, by Proposition 2.6, we may assume that $L_{nc} = H(U)$ for $U = U_k = \mathbb{R}^k$ some $k \geq 1$ and $U_0 := \mathbb{R}e_1$. We write $r_i = \exp(q_i)$ where $q_i \to 0$ in $\mathbb{R}^k$. Using the notations introduced in section 2 and setting $u^\perp = \text{Lie}(U^\perp) = \mathbb{R}^{d-1-k}$, we can write $q_i = u^-(x_i) + u^+(y_i) + m(C_i)$ where $x_i \in u^\perp$, $y_i \in (u^\perp)_t$, and $C_i = \left(\begin{smallmatrix} 0_k & B_i \\ -A_i & 1 \end{smallmatrix} \right)$ is a skew symmetric matrix, all of which converge to 0 as $i \to \infty$. We consider $U_0 = \mathbb{R}e_1$ as $\{u_s = se_1 \in \mathbb{R}^{d-1}\}$ and define the map $\psi_i : \mathbb{R} \to t^\perp$ by $\psi_i(s) = u_s^{-1}q_iu_s$ for all $s \in \mathbb{R}$. This is well-defined since $t^\perp$ is $\text{Ad}(L)$-invariant. Then a direct computation shows

$$\psi_i(s) = u^-(x_i + sB_i^t e_1 + s^2 y_i/2) + u^+(y_i) + m(\tilde{C}_i)$$

where $\tilde{C}_i$ is a skew-symmetric matrix of the form $\tilde{C}_i = \left(\begin{smallmatrix} 0_k & B_{i,s} + se_1y_i \\ -B_i^t e_1 & A_i \end{smallmatrix} \right)$.

Since $r_i \notin N(U_0)$, it follows that either $y_i \neq 0$ or $y_i = 0$ and $B_i^t e_1 \neq 0$. Hence $\psi_i$ is a non-constant polynomial of degree at most 2, and $\psi_i(0) \to 0$. Let $\lambda_1 \in \mathbb{R}$ be defined by $\lambda_1 = \sup\{\lambda > 0 : |\psi_1[-\lambda, \lambda]| \leq 1\}$. Then $0 < \lambda_1 < \infty$ and $\lambda_1 \to \infty$. Now the rescaled polynomials $\phi_i = \psi_i \circ \lambda_i : \mathbb{R} \to t^\perp$ form an equicontinuous family of polynomials of degree at most 2 and $\lim_{i \to \infty} \phi_i(0) = 0$. Therefore $\phi_i$ converges to a polynomial $\phi : \mathbb{R} \to t^\perp$ uniformly on compact subsets. Since $\phi(0) = 0$ and $\sup\{|\phi(\lambda)| : \lambda \in [-1,1]\} = 1$, $\phi$ is a non-constant polynomial. From (8.3), it can be easily seen that $\text{Im}(\phi)$ is contained $\text{Lie}(N) \cap t^\perp$, by considering the two cases of $y_i \neq 0$, and $y_i = 0$ and $B_i^t e_1 \neq 0$ separately. For a sequence $T_i$ of $k$-thick subsets of $U_0$, set $T_\infty := \limsup_{i \to \infty}(T_i/\lambda_1)$, which is a $k$-thick subset of $U_0$.

Let $s \in T_\infty$. By passing to a subsequence, there exists $t_i \in T_i$ such that $t_i/\lambda_i \to s$ as $i \to \infty$. As $\phi_i \to \phi$ uniformly on compact subsets, it follows that $\phi(s) = \lim_{i \to \infty} \psi_i(t_i/\lambda_i) = \lim_{i \to \infty} u_i^{-1}q_iu_i$. Since $T_\infty$ accumulates on 0, so does $\phi(T_\infty)$. Taking the exponential map to each side of the above, the first part of the lemma follows.

The second part of the lemma holds by applying Lemma 8.4 below for the non-constant polynomial $p(s) = ||\phi(s)||^2$ of degree at most 4.

Lemma 8.4. If $p \in \mathbb{R}[s]$ is a polynomial of degree $\delta \geq 1$ and $T \subset \mathbb{R}$ is a $k$-thick subset, then $p(T)$ is $2k^\delta$-thick at $\infty$.

Proof. Let $C$ be the coefficient of $s^\delta$ term of the polynomial $p$. Then there exists $s_0 > 1$ such that $\frac{1}{\sqrt{2}} \leq \frac{|p(s)|}{|Cs^\delta|} \leq \sqrt{2}$ for all $|s| > s_0$. Let $r > \frac{|C|s^\delta}{\sqrt{2}}$. Since
T is $k$-thick, there exists $t \in T$ such that $(\sqrt{2r/|C|})^{1/\delta} < |t| < k(\sqrt{2r/|C|})^{1/\delta}$. We compute that $r \leq |p(t)| \leq 2k^2 r$, proving the claim. \hfill \Box

9. Translates of relative $U$-minimal sets

Fix $k > 1$ as given by Proposition 3.10. In this section, we fix a non-trivial connected closed subgroup $U < \mathcal{N}$. Unless mentioned otherwise, we let $R$ be a compact $A$-invariant subset of $RF M$ such that for every $x \in R$, and for any one-parameter subgroup $U_0 = \{u_t\}$ of $U$, the set $\{t \in \mathbb{R} : xu_t \in R\}$ is $k$-thick. In practice, $R$ will be either $RF M$ or a compact subset of the form $RF M \cap F_{H(U)}^*$ for a closed $H(U)$-invariant subset $X$. The main aim of this section is to prove Propositions 9.6 and 9.9 using the results of section 8.

Definition 9.1. A $U$-invariant closed subset $Y \subset \Gamma \backslash G$ is $U$-minimal with respect to $R$ if $Y \cap R \neq \emptyset$ and for any $y \in Y \cap R$, $yU$ is dense in $Y$.

In this section, we study how to find an additional invariance of $Y$ beyond $U$ under certain conditions.

Lemma 9.2. Let $Y \subset \Gamma \backslash G$ be a $U$-minimal subset with respect to $R$. For any $y \in Y \cap R$, there exists a sequence $u_n \to \infty$ in $U$ such that $yu_n \to y$.

Proof. The set $Z := \{z \in Y : yu_n \to z \text{ for some } u_n \to \infty \text{ in } U\}$ is $U$-invariant and closed. The hypothesis on $Y$ implies that $Z = Y$. \hfill \Box

A subset $S$ of a topological space is said to be locally closed if $S$ is open in its closure $\overline{S}$.

Lemma 9.3. Let $Y$ be a $U$-minimal subset of $\Gamma \backslash G$ with respect to $R$, and $S$ be a closed subgroup of $N(U)$ containing $U$. For any $y_0 \in Y \cap R$, the orbit $y_0 S$ is not locally closed.

Proof. Suppose that $y_0 S$ is locally closed for some $y_0 \in Y \cap R$. Since $Y$ is $U$-minimal with respect to $R$, there exists $u_n \to \infty$ in $U$ such that $y_0 u_n \to y_0$ by Lemma 9.2. We may assume that $y_0 = [e]$ without loss of generality. Since $y_0 S$ is locally closed, $y_0 S$ is homeomorphic to $(S \cap \Gamma) \backslash S$ (cf. [32, Thm. 2.1.14]). Therefore there exists $\delta_n \in S \cap \Gamma$ such that $\delta_u u_n \to e$ as $n \to \infty$. Since $N(U) = ANC_1(U) C_2(U)$, writing $\delta_n = a_n r_n$ for $a_n \in A$ and $r_n \in N C_1(U) C_2(U)$, it follows that $a_n \to e$. On the other hand, note that $a_n$ is non-trivial as $\Gamma$ does not contain any elliptic or parabolic element. This is a contradiction, as there exists a positive lower bound for the translation lengths of elements of $\Gamma$, which is given by the minimal length of a closed geodesic in $M$. \hfill \Box

In the rest of this section, we use the following notation: $H = H(U), H' = H'(U), F^* = F_{H(U)}^*.$

Lemma 9.4. For every $U$-minimal subset $Y \subset \Gamma \backslash G$ with respect to $RF M$ such that $Y \cap F^* \cap RF M \neq \emptyset$, and for any $y_0 \in Y \cap F^* \cap RF M$, there exists a sequence $g_n \to e$ in $G - N(U)$ such that $y_0 g_n \in Y \cap RF M$ for all $n$. 


Proof. Let $y_0 \in Y \cap F^* \cap RF \cdot M$. As $Y = \overline{y_0U}$, $Y \subset RF \cdot M$. Using Lemma 3.4 and the fact that $F^*$ is open, we get that there exists an open neighborhood $O$ of $e$ such that

\[
(9.1) \quad y_0O \subset Y \cap F^* \subset Y \cap RF \cdot M \cdot U.
\]

Without loss of generality, we may assume that the map $g \mapsto y_0g \in \Gamma \setminus G$ is injective on $O$, by shrinking $O$ if necessary. We claim that there exists $g_n \to e$ in $G - N(U)$ such that $y_0g_n \in Y \cap F^*$. Suppose not. Then there exists a neighborhood $O' \subset O$ of $e$ such that $y_0O' \cap Y \subset y_0N(U)$. Set $S := \{g \in N(U) : Yg = Y\}$ which is a closed subgroup of $N(U)$ containing $U$. We will show that $y_0S$ is locally closed; this contradicts Lemma 9.3. We first claim that

\[
(9.2) \quad y_0O' \cap Y \subset y_0S.
\]

If $g \in O'$ such that $y_0g \in Y$, then $g \in N(U)$. Therefore $\overline{y_0gU} = \overline{y_0U} = Yg \subset Y$. Moreover, $Yg \cap RF \cdot M \neq \emptyset$ by (9.1). Hence $Yg = Y$, proving that $g \in S$. Now, (9.2) implies that $y_0S$ is open in $Y$. On the other hand, since $U \subset S$, we get $Y = \overline{y_0S}$. Therefore, $y_0S$ is locally closed.

Hence we have $g_n \to e$ in $G - N(U)$ such that $y_0g_n \in Y \cap F^*$. Since $y_0g_n \in F^* \cap RF \cdot M$ converges to $y_0 \in F^* \cap RF \cdot M$, by Lemma 7.7, there exists a sequence $u_n \to e$ in $U$ such that $y_0g_nu_n \in RF \cdot M$. Therefore, by replacing $g_n$ with $g_nu_n$, this finishes the proof.

Lemma 9.5. Let $Y$ be a $U$-minimal subset with respect to $R$, and let $W$ be a connected closed subgroup of $N(U)$. If there exists a sequence $\alpha_i \to e$ in $W$ such that $Y\alpha_i \subset Y$, then there exists a one-parameter subsemigroup $S < W$ such that $YS \subset Y$. Moreover if $W_0$ is a compact Lie subgroup of $W$ and $\alpha_i \in W - W_0$ for all $i$, then $S$ can be taken so that $S \not\subset W_0$.

Proof. The set $S_0 = \{g \in W : Yg \subset Y\}$ is a closed subsemigroup of $W$. Write $\alpha_i = \exp \xi_i$ for some $\xi_i \in \text{Lie}(W)$. Then the sequence $v_i := \|\xi_i\|^{-1}\xi_i$ of unit vectors has a limit, say, $v$. It suffices to note that $S := \{\exp(tv) : t \geq 0\}$ is contained in the closure of the subsemigroup generated by $\alpha_i$’s. Now suppose that $\alpha_i \in W - W_0$. Set $M_0 := \{g \in W_0 : Yg = Y\}$. This is a closed Lie subgroup of $W_0$. Write $\text{Lie} W = m_0 \oplus m_0^\perp$ where $m_0 = \text{Lie} M_0$. By modifying $\alpha_i$ by elements of $M_0$, we may assume $\alpha_i = \exp \xi_i$ for $\xi_i \to 0$ in $m_0^\perp$. Letting $v \in m_0^\perp$ be a limit of $\xi_i/\|\xi_i\|$, it remains to check $v \notin W_0$. Suppose not. Since $W_0$ is compact, we have $\{\exp tv : t \geq 0\} = \exp \mathbb{R}v$. Hence for all $t \geq 0$, $Y \exp tv \subset Y$ as well as $Y \exp(-tv) \subset Y$. Therefore $Y \exp tv = Y$. Hence $\exp v \in M_0$. This is a contradiction, since $v \in m_0^\perp$.

\[\Box\]

Proposition 9.6 (Translate of $Y$ inside of $Y$). If $Y$ is a $U$-minimal set of $\Gamma \setminus G$ with respect to $RF \cdot M$ such that $Y \cap F^* \cap RF \cdot M \neq \emptyset$, then there exists an unbounded one-parameter subsemigroup $S$ inside the subgroup $AU^\perp \cdot C_2(U)$ such that $YS \subset Y$. 

Proof. Choose \( y_0 \in Y \cap RFM \cap F^* \). By Lemma 9.4, there exists \( g_t \to e \) in \( G - N(U) \) such that \( y_0g_t \in Y \cap RFM \). Let \( U_0 = \{ u_t \} \) be a one-parameter subgroup of \( U \) as given by Lemma 8.1, with respect to the sequence \( g_t \).

Let \( T_i := \{ u_t \in U_0 : y_0g_tu_t \in Y \cap RFM \} \) which is a \( k \)-thick subset of \( U_0 \). By Lemma 8.1, there exist sequences \( u_t \to \infty \) in \( T_i \), and \( u_t \in U \) such that \( u_tg_tu_t \to \alpha \) for some element \( \alpha \in AU^1C2(U) - C2(U) \). Note that \( y_0g_tu_t \in Y \cap RFM \) converges to some \( y_1 \in Y \cap RFM \). By passing to a subsequence. Hence as \( i \to \infty \), \( y_0u_t^{-1} = y_0g_tu_t(u_i g_i u_i)^{-1} \to y_1 \).

So \( y_1 \alpha^{-1} \in Y \), and hence \( Y \alpha^{-1} \subset Y \), since \( y_1 \in Y \cap RFM \). Since \( \alpha \) can be made arbitrarily close to \( e \) in Lemma 8.1, the claim follows from Lemma 9.5.

\[\Box\]

Proposition 9.7 (Translate of \( Y \) inside of \( X \)). Let \( X \) be a closed \( H' \)-invariant set such that \( X \cap R \neq \emptyset \). Let \( Y \subset X \) be a \( U \)-minimal subset with respect to \( R \), and assume that there exists \( y \in Y \cap R \) and a sequence \( g_n \to e \) in \( G - H' \) such that \( yg_n \in X \) for all \( n \). Then there exists some non-trivial \( v \in U^\perp \) such that \( Yv \subset X \).

Proof. Let \( h := \text{Lie} H' \). We may write \( g_n = r_n h_n \) where \( h_n \in H^' \) and \( r_n \in \exp h^\perp \). By replacing \( g_n \) with \( g_n h_n^{-1} \), we may assume \( g_n = r_n \). If \( r_n \in U^\perp \) for some \( n \), then the claim follows since \( y_0r_n \in X \) and hence \( Yr_n \subset X \). Hence we assume that \( r_n \notin U^\perp \) for all \( n \). We have from (2.5) \( h^\perp \cap \text{Lie}(N(U)) = \text{Lie} U^\perp \). Hence \( r_n \notin N(U) \) for all \( n \). Therefore there exists a one-parameter subgroup \( U_0 = \{ u_t \} < U \) such that \( r_n \notin N(U_0) \). Let \( T = \{ t \in \mathbb{R} : yu_t \in R \} \). Since \( y \in R \), it follows that \( T \) is a \( k \)-thick subset of \( \mathbb{R} \) by the assumption on \( R \). Hence, by Lemma 8.3, there exists \( t_n \in T \) such that \( u_t^{-1}r_n u_n \to v \) for some non-trivial \( v \in U^\perp \). Observe \( (yu_t)(u_t^{-1}r_n u_n) = yu_nu_n \in X \). Passing to a subsequence, \( yu_n \to y_0 \) for some \( y_0 \in Y \cap R \), and hence \( y_0v \in X \). It follows \( Yv \subset X \).

For a one-parameter subgroup \( V = \{ v_t : t \in \mathbb{R} \} \) and a subset \( I \subset \mathbb{R} \), the notation \( V_I \) means the subset \( \{ v_t : t \in I \} \).

Lemma 9.8. Let \( X \) be a closed \( AU \)-invariant set of \( \Gamma \backslash G \), and \( V \) be a one-parameter subgroup of \( U^\perp \). Assume that \( R := X \cap RFM \cap F^* \) is non-empty and compact. If \( x_0 V_I \subset X \) for some \( x_0 \in R \) and a closed interval \( I \) containing 0, then \( X \) contains a \( V \)-orbit of a point in \( R \).

Proof. Choose a sequence \( a_n \in A \) such that \( \lim inf_{n \to \infty} a_n V_I a_n^{-1} \) contains a subsemigroup \( V^+ \) of \( V \) as \( n \to \infty \). Then \( (x_0 a_n^{-1})(a_n V_I a_n^{-1}) = x_0 V_I a_n^{-1} \subset X \). By passing to a subsequence, we have \( x_0 a_n^{-1} \) converges to some \( x_1 \in RFM \); so \( x_1 V^+ \subset X \). Since \( R \) is compact, so is \( \overline{x_0 A} \cap F^* \), which implies that \( x_1 \in \overline{x_0 A} \cap F^* \). Since \( x_1 \) belongs to the open set \( F^* \), it follows \( x_1 v_s \in F^* \) for all sufficiently small \( s \in \mathbb{R} \). By Lemma 3.4, this implies that \( x_1 v_s U \cap RFM \neq \emptyset \) for some \( s > 0 \) with \( v_s \in V^+ \). Note that \( (x_1 v_s U)(v_s^{-1} V^+) = x_1 U V^+ \subset X \). Choose \( x_2 \in x_1 v_s U \cap RFM \subset X \cap RFM \cap F^* \). Then \( x_2 (v_s^{-1} V^+) \subset X \). Similarly as before, let \( a_n \in A \) be a sequence such
that \( \liminf_{n \to \infty} a_n (v_n^{-1} V^+) a_n^{-1} = V \) and such that \( x_2 a_n^{-1} \) converges to some \( x_3 \in R \). From \((x_2 a_n^{-1})(a_n v_n^{-1} V^+ a_n^{-1}) = x_2 v_n^{-1} V^+ a_n^{-1} \subset X \), we conclude that \( x_3 V \subset X \). This finishes the proof. \( \square \)

**Proposition 9.9.** Let \( X \) be a closed \( H' \)-invariant set. Assume that \( R := X \cap F^* \cap RF M \) is a non-empty compact set, and let \( Y \subset X \) be a \( U \)-minimal subset with respect to \( R \). Suppose that there exists \( y \in Y \cap R \) such that \( X - yH' \) is not closed. Then there exist an element \( z \in R \) and a non-trivial connected closed subgroup \( V < U^\perp \) such that \( zUV \subset X \).

**Proof.** Since \( X - yH' \) is not closed, there exists a sequence \( g_n \to e \) in \( G - H' \) such that \( yg_n \in X \) for all \( n \geq 1 \). By Lemma 9.8, it suffices to find \( x_0 \in R \) and a one-parameter subgroup \( V < U^\perp \) such that \( x_0 V I \subset X \) for some interval \( I < \mathbb{R} \) containing 0. It follows from Propositions 9.6 and 9.7 that \( Yv_0 \subset X \) and \( YS \subset Y \) where \( v_0 \subset U^\perp - \{ e \} \) and \( S \) is an unbounded one-parameter subgroup of \( AU^\perp C_2(U) \). By Lemma 2.2, \( S \) is either of the form

\[
\begin{align*}
(1) & \quad S = \{ \exp(t_\xi V) \exp(t_\xi C) : t \geq 0 \}, \\
(2) & \quad S = \{ (v \exp(t_\xi A) v^{-1}) \exp(t_\xi C) : t \geq 0 \}
\end{align*}
\]

for some \( \xi_A \in \text{Lie}(A) - \{0\} \), \( \xi_C \in \text{Lie}(C_2(U)) \), \( \xi_V \in \text{Lie}(V) - \{0\} \), and \( v \subset U^\perp \).

**Case (1):** Since \( X \) is \( H'(U) \)-invariant, we may assume \( YS \subset X \) with \( \xi_C = 0 \); so the claim follows.

**Case (2):** Set \( Y_0 := Y C_2(U) \). It is easy to check that \( Y \) is a \( U C_2(U) \)-minimal subset of \( X \) with respect to \( R \). First suppose that \( v = e \). Let \( A^+ := \{ \exp(t_\xi A) : t \geq 0 \} \). Since \( YS \subset Y \) and \( \xi_C \in \text{Lie}(C_2(U)) \), it follows that \( Y_0 A^+ \subset Y_0 \). Choose \( y \in Y \cap R \), and let \( a_n \to \infty \) be a sequence in \( A^+ \). Since \( R \) is compact and \( A \)-invariant, \( ya_n \) converges to some \( z_0 \in R \) by passing to a subsequence. Since \( Y_0 A^+ \subset Y_0 \), we have \( z_0 \in Y_0 \cap R \). Since \( \liminf a_{-n} A^+ = A \), we get \( z_0 A \subset Y_0 \). Since \( z_0 A U C_2(U) = z_0 U C_2(U) A \), \( Y_0 \) is \( U C_2(U) \)-minimal with respect to \( R \), we obtain \( Y_0 A \subset Y_0 \). Since \( v_0 \) commutes with \( C_2(U) \), we also get \( Y_0 v_0 \subset X \). Therefore \( Y_0 v_0 \subset X \) and \( Y_0 A v_0 \subset Y_0 v_0 \subset X \).

By the \( A \)-invariance of \( X \), it follows \( Y_0 (A v_0 A) \subset X \). Since \( A v_0 A \) contains some \( V^+ \), the claim follows. Next suppose \( v \neq e \). Since \( C_2(U) \) commutes with \( v \), it follows that \( Y_0 v A^+ v^{-1} \subset Y_0 \). Since \( X \) is \( A \)-invariant, we get \( Y_0 (v A^+ v^{-1}) A \subset Y_0 A \subset X \). Set \( V := \exp(\log v) \). Since \( v A^+ v^{-1} A \) contains \( V_I \) for some interval \( I \) containing 0 for any subsemigroup \( A^+ \) of \( A \), we get \( Y_0 V_I \subset X \), finishing the proof. \( \square \)

10. Closures of Orbits inside \( \partial F \) and Non-Homogeneity

Let \( U \) be a connected closed subgroup of \( \tilde{H} \cap N \) and set \( H := H(U) \) as before. Then \( \partial F = BF M \cdot V^+ \cdot H'(U) \) and \( \partial F \cap RF M = BF M \cdot C(H(U)) \). In this section, we classify closures of \( xH(U) \) and \( xAU \) for \( x \in \partial F - RF M \) (Thm. 10.5); they are never homogeneous.

**Theorem 10.1.** If \( x = zc \in BF M \cdot C(H(U)) \) with \( z \in BF M \) and \( c \in C(H(U)) \). Then

\[
(1) \quad \bar{xU} = xL \text{ for some } L \in \mathcal{Q}_U \text{ contained in } c^{-1} \tilde{H} c;
\]
\( (2) \ \overline{xH(U)} = xL \) for some \( L \in \mathcal{L}_U \) contained in \( c^{-1}Hc \), and for any \( y \in \mathcal{G}(U,xL) \), \( \overline{yU} = xL \);
\( (3) \ \overline{xAU} = xH(U) \).

**Proof.** Since \( x \) is contained in the compact homogeneous space \( xc^{-1}Hc \), the claims (1) and (2) are special cases of Ratner’s theorem [24], which were also proved by Shah independently [29]. So we only need to discuss the proof of (3). We show that \( \overline{xAU} = xL \) where \( L \) is given by (2). If \( U = L \cap N \), then the claim follows from Theorem 12.1. Suppose that \( U \) is a proper subgroup of \( L \cap N \). Since \( \overline{xAU} \) is closed, the product subset \( \overline{xAU} \) is a proper subset of \( xL \) (cf. Lemma 4.15), there exists \( y \in \overline{xAU} \cap \mathcal{G}(U,xL) \). Hence (3) follows from (2). \( \square \)

**Lemma 10.2.** Let \( V^+ \subset N \) be a one-parameter subsemigroup which is not contained in \( \hat{H} \). Then \( V^+H(U) \) is a closed subset of \( G \).

**Proof.** Since the product map \( A \times N \to AN \) is a diffeomorphism and \( AN \) is closed, the product subset \( AW \) is closed in \( G \) for any closed subset \( W \) of \( N \). Hence \( AUV^+ \) is a closed subset of \( AN \). We use Iwasawa decompositions \( H(U) = U(A(K \cap H(U))) \), and the fact that \( AV^+ = V+A \) in order to write \( V^+H(U) = AUV^+(K \cap H(U)) \). Hence the conclusion follows from compactness of \( K \cap H(U) \). \( \square \)

**Lemma 10.3.** Let \( V^+ \subset N \) be as in Lemma 10.2. If \( g_i \in \hat{H} \) is a sequence such that \( g_i v_i h_i \) converges for some \( v_i \in V^+ \) and \( h_i \in H(U) \) as \( i \to \infty \), then, after passing to a subsequence, there exists \( p_i \in AU \) such that \( g_i p_i \) converges to an element of \( \hat{H} \) as \( i \to \infty \).

**Proof.** We write \( g_i = \tilde{k}_i \tilde{a}_i \tilde{n}_i \in (K \cap \hat{H}) A(N \cap \hat{H}) \) and \( h_i = u_i a_i k_i \in U A(K \cap H(U)) \). Since \( K \cap \hat{H} \) and \( K \cap H(U) \) are compact, we may assume without loss of generality that \( \tilde{k}_i = k_i = e \) for all \( i \). Observe that \( g_i v_i h_i = \tilde{a}_i \tilde{n}_i v_i u_i a_i = a_i a_i (a_i^{-1} \tilde{n}_i u_i a_i) (a_i^{-1} v_i a_i) \) where \( a_i , a_i^{-1} \tilde{n}_i u_i a_i \in N \cap \hat{H} \), and \( a_i^{-1} v_i a_i \in V^+ \). Since \( g_i v_i h_i \) converges as \( i \to \infty \) and the product map \( A \times (N \cap \hat{H}) \times V^+ \to G \) is an injective proper map, it follows that all three sequences \( a_i a_i , a_i^{-1} \tilde{n}_i u_i a_i \) and \( a_i^{-1} v_i a_i \) are convergent as \( i \to \infty \). Noting that \( g_i u_i a_i = \tilde{a}_i \tilde{n}_i u_i a_i = \tilde{a}_i a_i (a_i^{-1} \tilde{n}_i u_i a_i) \), it remains to set \( p_i := u_i a_i \) to finish the proof. \( \square \)

For \( z \in BF \), \( \pi(z\hat{H}^+\hat{H}) = \pi(z\hat{H}^+) \) is the closure of a Fuchsian end, of the form \( S_0 \times [0, \infty) \) where \( S_0 = \pi(z\hat{H}) \).

**Lemma 10.4.** Let \( zL \) be a closed orbit contained in \( BF \) for some \( L \in \mathcal{L}_U \) contained in \( \hat{H} \), and \( V^+ \subset N \) be a one-parameter subsemigroup such that \( \hat{H}^+ = \hat{H}^\perp \). Then both \( zL^+ H(U) \) and \( zLV^+ \) are closed.

**Proof.** Without loss of generality, we assume \( z = \lfloor z \rfloor \). Let \( B \) denote the component of \( \Omega \) such that \( \text{hull}(\partial B) = \pi(\hat{H}) \) for the projection map \( \pi : G \to \mathbb{H}^d \). Since \( \hat{H}^+ = \hat{H}^\perp \), we have \( \pi(\hat{H}^+ \hat{H}) = \text{hull} B \). Note that if \( \gamma(\text{hull}(B)) \cap \text{hull}(B) \neq \emptyset \) for \( \gamma \in \Gamma \), then \( \gamma \in \hat{H} \cap \Gamma = \text{Stab}_\Gamma(B) \). Suppose
that $\gamma_i \ell_i v_i h_i$ converges to some element $g \in G$ where $\gamma_i \in \Gamma$, $\ell_i \in L$, $v_i \in V^+$ and \( h_i \in H(U). \) Since $\pi(\gamma_i \ell_i v_i h_i) \in \Gamma \text{hull } B$, and $\Gamma \text{hull } B$ is a closed subset of $\mathbb{H}^d$, we have $\pi(g) \in \Gamma \text{hull } B$. Without loss of generality, we may assume $\pi(g) \in \text{hull } B$ by replacing $\gamma_i$ by $\gamma \gamma_i$ for some $\gamma \in \Gamma$ if necessary.

We claim that by passing to a subsequence, $\gamma_i \in \mathcal{H} \cap \Gamma$. Let $\mathcal{O}$ be a neighborhood of $\pi(g)$ such that $\mathcal{O} \cap \Gamma \text{hull } B \subset \text{hull } B$; such $\mathcal{O}$ exists, since $d(\text{hull}(\gamma B), \text{hull}(B)) \geq \eta$ for all $\gamma \in \Gamma - (\mathcal{H} \cap \Gamma)$ where $\eta > 0$ is given in (3.5). By passing to a subsequence, we may assume that $\pi(\gamma_i \ell_i v_i h_i) \in \mathcal{O}$. Since $\pi(\ell_i v_i h_i) \in \text{hull } B$ for all $i$, it follows that $\pi(\gamma_i \ell_i v_i h_i) \in \text{hull } B$ for all $n$.

Therefore $\gamma_i \in \mathcal{H} \cap \Gamma$. Applying Lemma 10.3 to the sequence $(\gamma_i \ell_i v_i h_i) \to g$, there exists $p_i \in AU$ such that $\gamma_i \ell_i v_i h_i \to h$ in $\mathcal{H}$ as $i \to \infty$. Since $\Gamma L$ is closed, we have $h \in \Gamma L$. Since $p_i^{-1}v_i h_i \in AUV^+H(U) = V^+H(U)$ and $\lim_{i \to \infty} p_i^{-1}v_i h_i = h^{-1}g$, we have $h^{-1}g \in V^+H(U)$ by Lemma 10.2. Therefore, $g = h(h^{-1}g) \in \Gamma V^+H(U)$. This proves that $\Gamma V^+H(U)$ is closed. Note that in the above argument, if $h_i = e$ for all $i$, then $h^{-1}g = \lim p_i^{-1}v_i \in AU$. Hence $g = h(h^{-1}g) \in \Gamma LAU = \Gamma V^+$. This proves that $\Gamma V^+$ is closed. \( \square \)

Note that $x \in RF_+M - RF \cdot H(U)$ if and only if $x \in (RF_+M \cap \partial F(H(U)) - BF \cdot C(H(U))$.  

**Theorem 10.5.** Let $x \in RF_+M - RF \cdot H(U)$. Then there exist a compact orbit $zL \subset BF M$ with $L \in \mathcal{L}_U$, an element $c \in C(H(U))$ and a one-parameter subsemigroup $V^+ \subset N$ with $H^+ = HV^+$ such that $\pi(H(U)) = zL + H(U)c$ and $\pi AU = zL^+c$. Moreover the closure of the geodesic plane $\pi(zL(U))$ is diffeomorphic to a properly immersed submanifold $S \times [0,\infty)$ where $S = \pi(zL)$ is a compact geodesic plane inside $BF M$.

**Proof.** We write $x = zv_0c$ for some non-trivial $v \in V^+$, $v_0 \in BF M$ and $c \in C(H(U))$. Without loss of generality, we may assume $c = e$. By Theorem 10.1, $\bar{zU} = zv_0^{-1}L v_0$ where $L \in \mathcal{L}_U$ is contained in $H$ and $v_0 \in H \cap N$. Hence $\bar{zU}$ contains $zL(v_0v)H(U)$ for $z := zv_0^{-1} \in BF M$. Set $V^+ := \{ \exp(t(\log(v_0))) : t \geq 0 \}$. Note that $V^+$ is contained in $A(v_0)vA \cup \{ e \}$, and hence $zL \cup zL v_0 H(U) = zLv_0 H(U)$ and $Hv^+ = Hv^+$ since $v \neq e$.

Since $\bar{zU}$ contains $zL \cup zL(v_0v)H(U)$, and $zL$ lies in the closure of $zL(v_0v)H(U)$, the claim (1) follows since $zLv^+H(U)$ is closed by Lemma 10.4. For the claim (2), note that $xAU = zv_0^{-1}UvA = zL^+$. By Lemma 10.4, $zL^+$ is $AU$-invariant and closed. Since $x \in zL^+$, we conclude $\bar{xAU} = zL^+$. 

To see the last claim, observe that $\pi(zL^+H(U)) = \pi(zL^+AU) = \pi(zL^+)$ since $V^+AU = AUV^+$, and $AU \subset L$. Since $HV^+ = HV^+$, and $\pi(zL)$ is a compact geodesic plane (without boundary) in $\pi(\mathcal{H})$, we get $\pi(zL^+) \simeq \pi(zL) \times [0,\infty)$ and $\pi(zL^+) \simeq \pi(zL) \times [0,\infty)$. \( \square \)

**Remark 10.6.** An immediate consequence of Theorem 10.5 is that if $P \subset M$ is a geodesic plane such that $P \cap \text{core } M = \emptyset$ but $\overline{P} \cap \text{core } M \neq \emptyset$, then $P$...
is not properly immersed in $M$ and $\mathcal{F}$ is a properly immersed submanifold with non-empty boundary.

11. Density of almost all $U$-orbits

Let $\Gamma < G = \text{SO}^0(d,1)$ be a Zariski dense convex cocompact subgroup. The action of $N$ on $RF_+ M$ is minimal, and hence any $N$-orbit is dense in $RF_+ M$ [31]. Given a non-trivial connected closed subgroup $U$ of $N$, there exists a dense $U$-orbit in $RF_+ M$ [19]. In this section, we deduce from [20] and [19] that almost every $U$-orbit is dense in $RF_+ M$ with respect to the Burger-Roblin measure in the case of a convex cocompact hyperbolic manifold with Fuchsian ends (Cor. 11.4).

The critical exponent $\delta = \delta_\Gamma$ of $\Gamma$ is defined to be the infimum $s \geq 0$ such that the Poincaré series $\sum_{\gamma \in \Gamma} e^{-sd(o,\gamma(o))}$ converges for any $o \in \mathbb{H}^d$. It is known that $\delta$ is equal to the Hausdorff dimension of the limit set $\Lambda$ and $\delta = d-1$ if and only if $\Gamma$ is a lattice in $G$ [30]. Denote by $m^{\text{BR}}$ the $N$-invariant Burger-Roblin measure supported on $RF_+ M$; it is characterized as a unique locally finite Borel measure supported on $RF_+ M$ (up to a scaling) by ([5], [25], [31]). We won’t give an explicit formula of this measure as we will only use the fact that its support is equal to $RF_+ M$, together with the following theorem: recall that a locally finite $U$-invariant measure $\mu$ is ergodic if every $U$-invariant measurable subset has either zero measure or zero co-measure, and is conservative if for any measurable subset $S$ with positive measure, $\int_{U} 1_S(xu) du = \infty$ for $\mu$-almost all $x$, where $du$ denotes the Haar measure on $U$.

**Theorem 11.1** ([20], [19]). Let $U < N$ be a connected closed subgroup, and let $\Gamma$ be a convex cocompact Zariski dense subgroup of $G$. Then $m^{\text{BR}}$ is $U$-ergodic and conservative if $\delta > \text{co-dim}_N(U)$.

**Lemma 11.2.** Suppose that $\Gamma_1 \triangleleft \Gamma_2$ are convex cocompact subgroups of $G$ with $[\Gamma_1 : \Gamma_2] = \infty$. Then $\delta_\Gamma_1 < \delta_\Gamma_2$.

**Proof.** Note that a convex cocompact subgroup is of divergent type ([30], [25]). Hence the claim follows from [7, Proposition 9] if we check that $\Lambda_{\Gamma_1} \neq \Lambda_{\Gamma_2}$. If $\Lambda :\Lambda_{\Gamma_1} = \Lambda_{\Gamma_2}$, then their convex hulls are the same, and hence the convex core of the manifold $\Gamma_1 \backslash \mathbb{H}^d$ is equal to $\Gamma_2 \backslash \text{hull}(\Lambda)$, which is compact. Since we have a covering map $\Gamma_1 \backslash \text{hull}(\Lambda) \to \Gamma_2 \backslash \text{hull}(\Lambda)$, it follows that $[\Gamma_1 : \Gamma_2] < \infty$. \hfill \Box

**Lemma 11.3.** If $\Gamma \backslash \mathbb{H}^d$ is a convex cocompact hyperbolic manifold with Fuchsian ends, then $\delta > d-2$.

**Proof.** If $\Gamma$ is a lattice, then $\Lambda = S^{d-1}$ and $\delta = d-1$. If $\Gamma \backslash \mathbb{H}^d$ is a convex cocompact hyperbolic manifold with non-empty Fuchsian ends, then $\Gamma$ contains a cocompact lattice $\Gamma_0$ in a conjugate of $\text{SO}(d-1,1)$ whose limit set is equal to $\partial B_i$ for some $i$. Now $[\Gamma : \Gamma_0] = \infty$; otherwise, $\Lambda = \partial B_i$. Hence $\delta > \delta_{\Gamma_0} = d-2$ by Lemma 11.2. \hfill \Box
Corollary 11.4. Let $M = \Gamma \backslash \mathbb{H}^d$ be a convex cocompact hyperbolic manifold with Fuchsian ends. Let $U < N$ be any non-trivial connected closed subgroup. Then for $m^{\text{BR}}$-almost every $x \in RF_+ M$, $\overline{xU} = RF_+ M$.

Proof. Without loss of generality, we may assume that $U = \{ u_1 \}$ is a one-parameter subgroup. By Lemma 11.3 and Theorem 11.1, $m^{\text{BR}}$ is $U$-ergodic and conservative. Since $\delta > (d - 1)/2$, there exists a unique function $\phi_0 \in L^2(M)$ which is an eigenfunction for the Laplace operator with eigenvalue $\delta(d - 1 - \delta)$, up to a scalar multiple [30]. Moreover $\phi_0$ is positive. We may regard $\phi_0$ as a function on $L^2(\Gamma \backslash G)$ which is $K$-invariant. Then $m^{\text{BR}}(\phi_0) = \| \phi_0 \|^2 < \infty$ (cf. [12, Lem 6.7]). Hence, applying the Hopf ratio theorem [1] we get that for almost all $x \in RF_+ M$ and for any continuous function $f$ on $RF_+ M$ with compact support, $\lim_{T \to \infty} \int_0^T f(xu_t) dt = \int_0^{\infty} f(xu_t) dt = m^{\text{BR}}(f) \| \phi_0 \|^2$. Therefore almost all $U$-orbits are dense in $\text{supp}(m^{\text{BR}}) = RF_+ M$. $\square$

Since $F^*_H(U) \cap RF_+ M$ is a non-empty open subset, it follows that almost all $U$-orbits in $F^*_H(U) \cap RF_+ M$ are dense in $RF_+ M$.

12. Horospherical action in the presence of a compact factor

Fix a non-trivial connected closed subgroup $U$ of $N$. Consider a closed orbit $xL$ for $x \in RF M$ where $L \in Q_U$. The subgroup $U = L \cap N$ is a horospherical subgroup of $L$, which is known to act minimally on $xL \cap RF_+ M$ provided $L = L_{nc}$. In this section, we extend the $U$-minimality on $xL$ in the case when $L$ has a compact factor.

Theorem 12.1. Let $X := xL$ be a closed orbit where $x \in RF_+ M$, and $L \in Q_U$. Let $U := L \cap N$. Then the following holds:

1. $X \cap RF_+ M$ is $U$-minimal.
2. $X$ is $L_{nc}$-minimal.
3. If $L \in Q_U$ and $x \in RF M$, then $X \cap RF M$ contains a dense $A$-orbit.
4. For any non-trivial connected closed subgroup $U_0 < U$, for $m^{\text{BR}}_X$-almost all $x \in X$, $\overline{xU_0} = X \cap RF_+ M$.

The subgroup $L \in Q_U$ is of the form $v^{-1}H(U)Cv$ where $H(U)C \in \mathcal{L}_U$ and $v \in N$. A general case can be easily reduced to the case where $L \in \mathcal{L}_U$. In the following, we assume $L = H(U)C \in \mathcal{L}_U$. As before, we set $H = H(U)$, $H' = H'(U)$, and $F^* = F^*_{H(U)}$ and let $\pi_1 : H' \to H$ and $\pi_2 : H' \to C(H)$ be the canonical projections. In order to define $m^{\text{BR}}_X$, choose $g \in G$ so that $[g] = x$. If we identify $H \simeq SO(k,1)$, then by Proposition 3.8, $S := \pi_1(g^{-1}Hg \cap HC) \backslash \mathbb{H}^k$ is a convex cocompact hyperbolic manifold with Fuchsian ends. Now $\pi_1(g^{-1}Hg \cap HC) \backslash H$ is the frame bundle of $S$, on which there exists the Burger-Roblin measure as discussed in section 11. In the above statement, the notation $m^{\text{BR}}_X$ means the $C$-invariant lift of this measure to $X = xHC$. 
We first prove the following, which is a more concrete version of Proposition 9.6 in the case at hand:

**Proposition 12.2.** Let $X$ be as in Theorem 12.1. Any $U$-minimal set $Y$ of $X$ with respect to $RFM$ such that $Y \cap F^* \cap RFM \neq \emptyset$ is $A$-invariant.

**Proof.** Let $Y$ be a $U$-minimal set of $X$ with respect to $RFM$. Let $y_0 \in Y \cap F^* \cap RFM$. By Lemma 9.4, there exists a sequence $g_i \to e$ in $HC \setminus \mathbb{N}(U)$ such that $y_0 g_i \in Y \cap RFM$ for all $i \geq 1$. Since $U$ is a horospherical subgroup of $H$ and $C$ commutes with $H$, we can apply Lemma 8.2 to the sequence $g_i^{-1}$ and the sequence of $k$-thick sets $T_i := \{u \in U : y_0 g_i u \in Y \cap RFM\}$ of $U$. This gives us sequences $u_i \to \infty$ in $T_i$ and $u_i \in U$ such that as $i \to \infty$, $u_i^{-1} g_i u_i \to a$ for some non-trivial element $a \in A$. Since $y_0 u_i$ converges to some $y_1 \in Y \cap RFM$ by passing to a subsequence, we have $y_1 a = \lim(y_0 u_i)(u_i^{-1} g_i u_i) \in Y$. Since $\overline{FY} = Y$, we get $Y a \subset Y$. Since $a$ can be made arbitrarily close to $e$ by Lemma 8.2, there exists a subsemigroup $A_+$ of $A$ such that $Y A_+ \subset Y$ by Lemma 9.5. Moreover, for any $a \in A_+$, $Y a \cap RFM \neq \emptyset$ as $RFM$ is $A$-invariant. Therefore, $Y a = Y$. It follows that $Y a^{-1} = Y$ as well. Hence $Y$ is $A$-invariant. \hfill $\Box$

**Proof of Theorem 12.1.** First suppose that $xL \cap F^* \neq \emptyset$. We may then assume $x \in F^* \cap RFM$. Let $Y$ be a $U$-minimal set of $X$ with respect to $RFM$. If $Y$ were contained in $\partial F \cap RFM$, then $Y \subset \partial F \cap RFM$. Since $\text{Stab}_L(x)$ is Zariski dense in $L$ by the definition of $L_U$, it follows from [4, Lemma 4.13] that $X \cap RF_+ M$ is $AU$-minimal. Therefore we have $\overline{Y A} = X \cap RF_+ M$ and hence $X$ has to be contained in the closed $A$-invariant subset $\partial F \cap RFM$ as well, yielding a contradiction. Therefore, $Y \cap F^* \cap RFM \neq \emptyset$.

Hence, by Proposition 12.2, $Y$ is $A$-invariant. Therefore the claim (1) follows from the $AU$-minimality of $X \cap RF_+ M$ if $x \in F^*$. Now suppose $xL \subset \partial F$. In this case, it suffices to consider the case when $U$ is a proper subgroup of $N$; otherwise $L = G$ and has no compact factor. Hence we may assume without loss of generality that $U \subset \hat{H} \cap N$. As $xL$ is closed, Theorem 10.5 implies that $xL \subset BF M \cdot C(H(U))$. Hence by modifying $x$ by an element of $C(H(U))$, we may assume that $X$ is contained in a compact homogeneous space of $\hat{H} = \text{SO}^+(d - 1, 1)$, which is the frame bundle of a convex cocompact hyperbolic manifold with empty Fuchsian ends. Therefore the claim (1) follows from the previous case of $x \in F^*$, since $F^* = RFM$ in the finite volume case.

Claim (2) follows from (1) since $RF_+ MH$ is closed, and $X \subset RF_+ MH$.

For the claim (3), it suffices to show that the $A$ action on $X \cap RFM$ is topologically transitive (cf. [6]). Let $x, y \in X \cap RFM$ be arbitrary, and $O$, $O'$ be open neighborhoods of $e$ in $H$. The set $UU^t A(M \cap H)$ is a Zariski open neighborhood of $e$ in $H$ where $U^t$ is the expanding horospherical subgroup of $H$ for the action of $A$. Choose an open neighborhood $Q_0$ of $e$ in $U$, and an open neighborhood $P_0$ of $e$ in $U^t A(M \cap H)$ such that $Q_0 P_0 \subset O$. 
We claim that $xQ_0A \cap yO' \neq \emptyset$, which implies $xO'A \cap yO' \neq \emptyset$. Suppose that this is not true. Then $xQ_0A \subset \Gamma \setminus G - yO'$ where the latter is a closed set. Now, choose a sequence $a_n \in A$ such that $a_nQ_0a_n^{-1} \to U$ as $n \to \infty$, and observe $xa_n^{-1}(a_nQ_0a_n^{-1}) = xQ_0a_n^{-1} \subset \Gamma \setminus G - yO'$. Passing to a subsequence, $xa_n^{-1} \to x_0$ for some $x_0 \in RF_M$, and we obtain that $x_0U$ is contained in the closed subset $\Gamma \setminus G - yO'$. This contradicts the $U$-minimality of $X \cap RF_+M$, which is claim (1). This proves (3). For the claim (4), note that by Corollary 11.4, almost all $U_0$-orbits in $\pi_1(g^{-1}\Gamma g \cap HC) \setminus H$ are dense in the corresponding $RF_+M$-set. It follows that for almost all $x$, the closure $xU_0$ contains a $U$-orbit of $X$. Hence (4) follows from the claim (1).

13. Orbit closure theorems: beginning of the induction

Let $G = SO^0(d,1)$ and $U < N$ be a non-trivial connected proper closed subgroup, and $H(U)$ be its associated simple Lie subgroup of $G$. Let $L_U$ and $Q_U$ be as defined in (4.6) and (4.7). The remainder of the paper is devoted to the proof of the next theorem from which Theorem 1.1 follows:

**Theorem 13.1.**

1. For any $x \in RF_M$, $xH(U) = xL \cap F_{H(U)}$ where $xL$ is a closed orbit of some $L \in L_U$.
2. Let $x_0\hat{L}$ be a closed orbit for some $\hat{L} \in L_U$ and $x_0 \in RF_M$.
   a. For any $x \in x_0\hat{L} \cap RF_+M$, $xU = xL \cap RF_+M$ where $xL$ is a closed orbit of some $L \in Q_U$.
   b. For any $x \in x_0\hat{L} \cap RF_M$, $x\tilde{A}U = xL \cap RF_+M$ where $xL$ is a closed orbit of some $L \in L_U$.
3. Let $x_0\hat{L}$ be a closed orbit for some $\hat{L} \in L_U$ and $x_0 \in RF_M$. Let $x_iL_i \subset x_0\hat{L}$ be a sequence of closed orbits intersecting $RF_M$ where $x_i \in RF_+M$, $L_i \in Q_U$. Assume that no infinite subsequence of $x_iL_i$ is contained in a subset of the form $y_0L_0D$ where $y_0L_0$ is a closed orbit of $L_0 \in L_U$ with $\dim L_0 < \dim \hat{L}$ and $D \subset N(U)$ is a compact subset. Then $\lim_{i \to \infty} (x_iL_i \cap RF_+M) = x_0L \cap RF_+M$.

We will prove (1), (2), and (3) of Theorem 13.1 by induction on the co-dimension of $U$ in $N$ and the co-dimension of $U$ in $\hat{L} \cap N$, respectively.

For simplicity, let us say (1)$_m$ holds, if (1) is true for all $U$ satisfying $\text{co-dim}_N(U) \leq m$. We will say (2)$_m$ (resp. (2)$_a$$_m$, (2)$_b$$_m$) holds, if (2) (resp. (a) of (2), (b) of (2)) is true for all $U$ and $\hat{L}$ satisfying $\text{co-dim}_{\hat{L} \cap N}(U) \leq m$ and similarly for (3)$_m$.

**Base case of $m = 0$.** Note that the bases cases (1)$_0$, and (3)$_0$ are trivial, and that (2)$_0$ follows from Theorem 12.1. We will deduce (1)$_{m+1}$ from (2)$_m$ and (3)$_m$ in section 15, and (2)$_{m+1}$ from (1)$_{m+1}$, (2)$_m$, and (3)$_m$ in section 16, and finally deduce (3)$_{m+1}$ from (1)$_{m+1}$, (2)$_{m+1}$ and (3)$_m$ in section 17.

**Remark 13.2.** When $\text{co-dim}_{\hat{L} \cap N}(U) \geq 1$ and $\hat{L} \in L_U$, we may assume without loss of generality that $U \subset \hat{L} \cap N \cap H$ by replacing $U$ and $\hat{L}$ by their conjugates using an element $m \in M$. 
Lemma 13.4. Suppose that $x \in \partial F_{H(U)}$, Theorem 13.1 (1) and (2) follow from Theorem 10.1, and if $x_0 \in \partial F_{H(U)}$, (3) follows from the work of Mozes-Shah [19]. So the main new cases of Theorem 13.1 are when $x, x_0 \in F^*_U$.

**Singular $U$-orbits under the induction hypothesis.** Recall the notation $\mathcal{S}(U, x\hat{L})$ and $\mathcal{G}(U, x\hat{L})$ from (4.5).

**Lemma 13.4.** Suppose that $(2.a)_m$ is true and that for $x \in RF M$, $xU$ is contained in a closed orbit $x\hat{L}$ for some $\hat{L} \in \mathcal{L}_U$.

1. If $\text{co-dim}_{\mathcal{L}_U}(U) \leq m + 1$, then for any $x_0 \in \mathcal{S}(U, x\hat{L}) \cap RF_+ M$, $\overline{x_0U} = x_0L \cap RF_+ M$ where $x_0L$ is a closed orbit of some subgroup $L < \hat{L}$ contained in $Q_U$, satisfying $\dim L_{nc} < \dim \hat{L}_{nc}$.

2. If $\text{co-dim}_{\mathcal{L}_U}(U) \leq m$, then for any $x_0 \in \mathcal{G}(U, x\hat{L})$, $\overline{x_0U} = x_0\hat{L} \cap RF_+ M$.

**Proof.** Suppose that $\text{co-dim}_{\mathcal{L}_U}(U) \leq m + 1$ and that $x_0 \in \mathcal{S}(U, x\hat{L}) \cap RF_+ M$. By Proposition 4.13, we get $\overline{x_0U} \subset x_0Q$ for some closed orbit $x_0Q$ where $Q \in Q_U$ satisfies $\dim Q_{nc} < \dim \hat{L}_{nc}$. Now $Q = vL_0v^{-1}$ for some $L_0 \in \mathcal{L}_U$ and $v \in U^\perp$. We have $x_0Uv = x_0vU \subset x_0vL_0$. Since $\text{co-dim}_{\mathcal{N}\cap Q}(U) = \text{co-dim}_{\mathcal{N}\cap Q}(U) \leq m$, by applying $(2)_m$, we get $\overline{x_0vU} = x_0vL \cap RF_+ M$ for some closed orbit $x_0vL$ where $L \in Q_U$ is contained in $L_0$. Therefore $\overline{x_0U} = x_0vL_0^{-1} \cap RF_+ M$. As $vL_0^{-1} \in Q_U$ and $\dim L_{nc} \leq \dim Q_{nc} < \dim \hat{L}_{nc}$, the claim (1) is proved.

To prove (2), note that by $(2.a)_m$, we get $\overline{x_0U} = x_0L \cap RF_+ M$ for some closed orbit $x_0L$ with $L \in Q_U$ such that $L \subset \hat{L}$. Since $x_0 \in \mathcal{G}(U, x\hat{L})$, we have $\dim L_{nc} = \dim \hat{L}_{nc}$. Since $L \subset \hat{L}$, $L \cap N$ is a horospherical subgroup of $\hat{L}$. By Theorem 12.1, $L \cap N$ acts minimally on $x\hat{L}$, and hence $L = \hat{L}$. 

### 14. Uniform recurrence and additional invariance

The primary goal of this section is to prove Propositions 14.1 and 14.2 in obtaining additional invariances using a sequence converging to a generic point of an intermediate closed orbit; the main ingredient is Theorem 6.15 (Avoidance theorem II).

In this section, we let $U < N$ be a non-trivial connected closed subgroup. We suppose that

- $(2)_m$ and $(3)_m$ are true;
- $x\hat{L}$ is a closed orbit for some $x \in RF M$, and $\hat{L} \in \mathcal{L}_U$;
- $\text{co-dim}_{\mathcal{L}_U}(U) \leq m + 1$.

We let $\{U^{(i)}\}$ be a collection of one-parameter subgroups generating $U$.

In the next two propositions, we let $X$ be a closed $U$-invariant subset of $x_0\hat{L}$ such that $X \supset xL \cap RF_+ M$ for some closed orbit $xL$ where $L \in Q_U$ is a proper subgroup of $\hat{L}$ and $x \in \bigcap_i \mathcal{G}(U^{(i)}, xL) \cap RF M$.

**Proposition 14.1.** (Additional invariance I). Suppose that there exists a sequence $x_i \to x$ in $X$ where $x_i = x\ell_ir_i$ with $x\ell_i \in xL \cap RF M$ and $r_i \in ...
exp t\(y\) - N(U). Then there exists a sequence \(v_n \to \infty\) in \((L \cap N)^\perp\) such that \(xLv_n \cap RF_+ M \subset X\).

**Proof.** Since \(r_i \notin N(U)\), we can fix a one-parameter subgroup \(U_0 = \{u_t : t \in \mathbb{R}\}\) in the family \(\{U^{(i)}\}\) such that \(r_i \notin N(U_0)\) by passing to a subsequence.

Let \(E_j, j \in \mathbb{N}\), be a sequence of compact subsets in \(\mathcal{S}(U_0, xL) \cap RF M\) given by Theorem 6.15. Set \(z_i = x\ell_i \in xL \cap RF M\). Fix \(j \in \mathbb{N}\) and \(n \gg 1\). Since \(z_i \to x\) and \(x \in \mathcal{S}(U_0, xL)\), there exist \(i_j \geq 1\) and an open neighborhood \(\mathcal{O}_j\) of \(E_j\) such that for each \(i \geq i_j\), the set \(T_i = \{t \in \mathbb{R} : z_iu_t \in RF(M - \mathcal{O}_j)\}\), is \(2k\)-thick by loc. cit. We apply Lemma 8.3 to the sequence \(T_i\). We can find a sequence \(t_i = t_i(n) \in T_i\), \(i \geq i_j\) and elements \(y_j = y_j(n), v_j = v_j(n)\) satisfying that as \(i \to \infty\),

- \(z_iu_{t_i} \to y_j \in (RF(M \cap xL) - \mathcal{O}_j)\);
- \(u_{t_i}^{-1}r_iu_{t_i} \to v_j \in (L \cap N)^\perp\) with \(n \leq \|v_j\| \leq (2k^2)n\)

So as \(i \to \infty\), \(x_iu_{t_i} = z_i r_i u_{t_i} \to y_j v_j \) in \(X\). Note that since \(L\) is a proper subgroup of \(\hat{L}\), we have \(\dim_{\hat{L} \cap N}(U) \leq m\) by Lemma 4.11. If \(y_j\) belongs to \(\mathcal{S}(U, xL)\), then \(\overline{y_jUv_j} = xL \cap RF_+ M\) by Lemma 13.4(2), and hence \(X \supset \overline{y_jUv_j} = \overline{y_jv_j} = xLv_j \cap RF_+ M\). Hence the claim follows if \(y_j(n) \in \mathcal{S}(U, xL)\) for an infinite subsequence of \(n\)'s. Now we may suppose that for all \(n \geq 1\) and \(j \geq 1\), \(y_j(n) \in \mathcal{S}(U, xL) \cap RF_+ M\), after passing to a subsequence. Fix \(n\), and set \(y_j = y_j(n)\) and \(v_j = v_j(n)\). Then, since \(\dim_{\hat{L} \cap N} U \leq m\), by (2)\(m\), we have

\[
(14.1) \quad \overline{y_jU} = y_jL_j \cap RF_+ M
\]

for some closed \(y_jL_j\) where \(L_j \in \mathcal{Q}_U\) is contained in \(\hat{L}\) and \(\dim(L_j)_{\text{nc}} < \dim \hat{L}_{\text{nc}}\). Write \(L_j = w_j^{-1}L_j'w_j\) for \(L_j' \in \mathcal{L}_U\) and \(w_j \in U^\perp\). We claim that the sequence \(y_jL_j = y_jw_j^{-1}L_j'w_j\) satisfies the hypothesis of (3)\(m\). It follows from the condition \(y_j \in (RF(M \cap xL) - \mathcal{O}_j\) for all \(j\) that no infinite subsequence of \(y_jL_j\) is contained in a subset of the form \(y_0L_0D \subset \mathcal{S}(U, xL)\) where \(y_0L_0\) is closed, \(L_0 \subset \mathcal{Q}_U\) and \(D \subset N(U)\) is a compact subset. Hence, by (3)\(m\), we have \(\limsup y_jL_j \cap RF_+ M = xL \cap RF_+ M\). Therefore for each fixed \(n \gg 1\) and \(y_j = y_j(n)\), \(\limsup y_jU = xLv_j \cap RF_+ M\). By passing to a subsequence, there exists \(u_j \in U\) such that \(y_ju_j \) converges to \(x\). As \(n \leq \|v_j(n)\| \leq (2k^2)n\), the sequence \(v_j(n)\) converges to some \(v_n \in (L \cap N)^\perp\) as \(j \to \infty\), after passing to a subsequence. Therefore \(\limsup y_j(n)v_j(n)U = \limsup y_jUv_j(n) \supset \overline{xUv_n} = xLv_n \cap RF_+ M\) where the last equality follows from Lemma 13.4(2), since \(\co-dim_{\hat{L} \cap N}(U) \leq m\).

Note that in the above proposition, \(y_i = x\ell_i r_i\) is not necessarily in \(RF M\), and hence we cannot apply the avoidance theorem 6.15 to the sequence \(y_i\) directly. We instead applied it to the sequence \(x\ell_i\).

In the proposition below, we will consider a sequence \(x_i \to y\) inside \(RF M\), and apply Theorem 6.15 to the sequence \(x_i\).
Let $\partial F \frac{\partial F}{x} L$ of generality, we may assume $U < N$. Then there exists a one-parameter subgroup $U = \{ u_t : t \in \mathbb{R} \}$ among $U^{(i)}$ such that $r_i \not\in N(U_0)$ by passing to a subsequence.

For $R > 0$, we set $B(R) := \{ v \in (L \cap N)^{+} \cap \tilde{L} : \| v \| \leq R \}$. Fix $j$ and $n \in N$. Let $E_j, O_j$ be given by Theorem 6.15 for $x L$ with respect to $U_0$. Then $E_j$ is of the form $E_j = \bigcup_{i \in \Lambda_j} \Gamma \Gamma H_i D_0 \cap RF M$ where $H_i \in \mathcal{H}^*$ satisfies $\dim(H_i)_{nc} < \dim L_{nc}$ and $D_i$ is a compact subset of $X(H_i, U_0) \cap L$. As $B(2k^2 n) \subset C(U_0)$, we have $D_j^* := D_j B(2k^2 n)$ is a compact subset of $X(H_i, U_0)$. Hence the following subset $\tilde{E}_j := \bigcup_{i \in \Lambda_j} \Gamma \Gamma H_i D_j^* \cap RF M$ belongs to $\mathcal{E}_{U_0}$ and is associated to the family $\{ H_i : i \in \Lambda_j \}$, as defined in (6.3).

Let $\tilde{E}_j^o \subset \mathcal{E}_{U_0}$ be a compact subset given by Theorem 6.13, which is also associated to the same family $\{ H_i : i \in \Lambda_j \}$. Note that for any $z \in \tilde{E}_j^o$, the closure $\mathcal{U}_0$ is contained in $\Gamma \Gamma H_i D_j^*_i$ for some $i \in \Lambda_j$. In particular, $\tilde{E}_j^o$ is a compact subset disjoint from $\mathcal{G}(U_0, x L)$. Since $x_i \to x$ and $x \in \mathcal{G}(U_0, x L)$, there exists $i_j \geq 1$ such that $x_i \not\in \tilde{E}_j^o$ for all $i \geq j$. By Theorem 6.13, there exists a neighborhood $\mathcal{O}_j$ of $\mathcal{E}_j$ such that for each $i \geq i_j$, the set $T_i = \{ t \in \mathbb{R} : x_i u_t \in RF M - \mathcal{O}_j \}$ is $2k$-thick. Applying Lemma 8.3 to $T_i$, and $r_i \to e$, we can find $t_i = t_i(n) \in T_i$ such that $u_i r_i u_t \to v_j$ for some $v_j = v_j(n) \in (L \cap N)^{+}$, with $n \leq \| v_j \| \leq 2k^2 \cdot n$. Passing to a subsequence, $x_i u_t$ converges to some $\tilde{x}_j(n) \in RF M - \mathcal{O}_j$ as $i \to \infty$. Set $z_i := x_i t_i$, and $O_j := \mathcal{O}_j B(2k^2 n) \cap x L$. Since $z_i u_t = z_i u_t (u_i^{1} r_i u_t)$, we have $z_i u_t \to y_j \in (RF^+ M \cap x L) - O_j$ where $y_j = y_j(n) := \tilde{x}_j(n) u_j^{-1}$. We check that $E_j \subset O_j$ as $B(2k^2 n) B(2k^2 n)$ contains $e$. It follows that $y_j \not\in \tilde{E}_j$. Since $\tilde{x}_j(n) \in \mathcal{U}_0 v_j \subset X$, we have $\mathcal{U}_0 v_j \cap RF M \neq \emptyset$. Given these, we can now repeat verbatim the proof of Proposition 14.1 to complete the proof. \hfill $\Box$

15. $H(U)$-ORBIT CLOSURES: PROOF OF $(1)_{m+1}$

We fix a non-trivial connected proper subgroup $U < N$. Without loss of generality, we may assume $U < N \cap H$ using a conjugation by an element of $M$. We set $H = H(U)$, $H' = H'(U)$, $F = F_{H(U)}$, $F' = F_{H'(U)}$, and $\partial F = \partial F_{H(U)}$. By the assumption $U < N \cap H$, we have $\partial F \cap RF M = BF M \cdot C(H)$.

**Lemma 15.1.** Let $x_1 L_1$ and $x_2 L_2$ be closed orbits where $x_1, x_2 \in RF M$, $L_1 \in \mathcal{Q}_U$ and $L_2 \in \mathcal{L}_U$. If $x_1 L_1 \cap RF M \subset x_2 L_2$, then $L_1 \subset L_2$ and $x_1 L_1 \subset x_2 L_2$. 

Proof. Since $L_2$ contains $H$, we get that $x_1 L_1 \cap RF M \cdot H \subset x_2 L_2$. Suppose that $x_1 L_1 \cap F^* \neq \emptyset$. We may assume $x_1 \in F^*$. Since $F^* \subset RF MH$, we have $x_1 L_1 \cap F^* \subset x_2 L_2$. Since $F^*$ is open, there exist $g_1, g_2 \in G$ such that $[g_i] = x_i$, and $g_1 L_1 \cap O \subset g_2 L_2$ for some open neighborhood $O$ of $g_1$. It follows that $L_1 \cap g_1^{-1} O \subset g_1^{-1} g_2 L_2$. Since $e \in g_1^{-1} g_2 L_2$, we have $g_1^{-1} g_2 L_2 = L_2$. Since $L_1$ is topologically generated by $L_1 \cap g_1^{-1} O$, we deduce $L_1 \subset L_2$. Since $x_1 L_1 \cap x_2 L_2 \neq \emptyset$, it follows that $x_1 L_1 \subset x_2 L_2$.

Now consider the case when $x_1 L_1 \cap F^* = \emptyset$. In this case, $x_1 L_1 \cap RF M \subset RF M \cap \partial F$. By Theorem 12.1(4), we can assume that $x_1 U = x_1 L_1 \cap RF^+ M$. As $x_1$ is contained in $BF M \cdot C(H)$, so is $x_1 U$. It follows that $x_1 L_1$ is compact and hence is contained in $RF M$. Hence the hypothesis implies that $x_1 L_1 \subset x_2 L_2$, which then implies $L_1 \subset L_2$ by the same argument in the previous case. □

Lemma 15.2. Let $y_1 L_1$ and $y_2 L_2$ be closed orbits where $y_1 \in RF M$, $y_2 \in RF^+ M$, $L_1 \in \mathcal{Q} U$ and $L_2 \in \mathcal{L} U$. If $y_1 L_1 \subset y_2 L_2 D$ for some subset $D \subset N(U)$, then there exists $d \in D$ such that $L_1 \subset d^{-1} L_2 d$ and $y_1 L_1 \subset y_2 L_2 d$.

Proof. By Theorem 12.1(4), we may assume $y_1 U = y_1 L_1 \cap RF^+ M$. By the assumption, $y_1 = y_2 \ell_2 d$ for some $\ell_2 \in L_2$ and $d \in D$. Since $y_2 \ell_2 = y_1 d^{-1}$ and $N(U)$ preserves $RF^+ M$, $y_2 \ell_2 \in RF^+ M$. Hence we may replace $y_2$ by $y_2 \ell_2$, and hence assume that $y_1 = y_2 d$. Since $y_1 L_1 \cap RF^+ M = y_2 d U = y_2 U d \subset y_2 L_2 d$, and $F^* \subset RF^+ MH$, we get $y_1 L_1 d^{-1} \cap F^* \subset y_2 L_2$.

If $y_1 L_1 d^{-1} \cap F^* \neq \emptyset$, using the openness of $F^*$, the conclusion follows as in the first part of the proof of Lemma 15.1. Now consider the case when $y_1 L_1 d^{-1} \cap F^* = \emptyset$. In particular, $y_2 = y_1 d^{-1}$ belongs to $RF^+ M - F^* \subset BF M \cdot N(U)$ by (3.3). It follows from Theorem 10.1 that $y_2 U = y_2 L_2'$ for some $L_2' \in \mathcal{Q} U$ contained in $L_2$. In view of (15.1), we get $y_1 L_1 \cap RF^+ M = y_1 d^{-1} L_2' d$. Therefore $d^{-1} L_2' d \subset L_1$. Since $y_1 L_1 \cap RF^+ M$ is $A(L_1 \cap N)$-invariant, it follows that $d^{-1} L_2' d \subset L_U$ and $d^{-1} L_2' d \cap N = L_1 \cap N$. As a result, $(L_1)_{nc} = d^{-1} (L_2')_{nc} d$. By Lemma 4.11, we get that $L_1 = d^{-1} L_2' d \subset d^{-1} L_2 d$ and that $y_1 L_1 = y_2 L_2 d \subset y_2 L_2 d$. □

The following proposition says that the classification of $H'$-orbit closures yields the classification of $H$-orbit closures:

Proposition 15.3. Let $x \in RF M$, and assume that there exists $U < \bar{U} < N$ such that $x H'(\bar{U})$ is closed, and $\bar{x} H = x H(\bar{U}) \cdot C(H) \cap F$. Then there exists a closed subgroup $C < C(H(\bar{U}))$ such that $\bar{x} H = x H(\bar{U}) C \cap F$.

Proof. By Proposition 3.8 and Theorem 12.1(2), there exists a closed subgroup $C < C(H(\bar{U}))$ such that $H(\bar{U}) C \subset L_U$ and $X \ := x H(\bar{U}) C$ is a closed $H(\bar{U})$-minimal subset. In particular, $\bar{x} H \subset X \cap F$. Now, by Theorem 12.1(3), there exists $y \in X$ such that $y A = X \cap RF M$. Since $C$ is contained in $C(H)$ and $\bar{x} H \cdot C(H) = \bar{x} H = x H(\bar{U}) \cdot C(H) \cap F$, there exists $c_0 \in C(H)$
such that $y c_0 \in \overline{x H}$. Since $\overline{y A c_0} = \overline{y c_0 A} \subset \overline{x H}$ and $c_0 \in C(H)$, it follows $X c_0 \cap R F M \subset \overline{x H} \subset X$. Applying Lemma 15.1, we get $X c_0 = \overline{x H} = X$. □

In the rest of this section, fix $m \in \mathbb{N} \cup \{0\}$ and assume that $1 \leq \text{co-dim}_N(U) = m + 1$. In order to describe the closure of $x H(U)$, in view of Theorem 10.1, we assume that $x \in F^* \cap R F M$. By Proposition 15.3, it suffices to show that

$$L = \overline{x H'} = x L C(H) \cap F$$

for some closed orbit $x L$ for some $L \in \mathcal{L}_U$.

In the rest of this section, we set $X := \overline{x H'}$ and assume that $x H'$ is not closed, i.e., $X \neq x H'$. We also assume that $(2)_m$ holds in the entire section.

**Lemma 15.4** (Moving from $Q_U$ to $L_U$). If $x_0 L \cap R F_+ M \subset X$ for some closed orbit $x_0 L$ with $x_0 \in R F M$, and $L \in Q_U \setminus \mathcal{L}_U$, then $x_1 \tilde{L} \cap R F_+ M \subset X$ for some closed orbit $x_1 \tilde{L}$ with $x_1 \in R F M$, and $\tilde{L} \in \mathcal{L}_U$ with $\text{dim}(\tilde{L} \cap N) > \text{dim}(L \cap N)$. Moreover, $x_1$ can be taken to be any element of the set $\limsup_{t \to +\infty} x_0 u a_{-t}$ for any $u \in U$.

**Proof.** By (4.8), we can write $L = v^{-1} \tilde{L} v$ for some $\tilde{L} \in \mathcal{L}_U$ and $v \in (\tilde{L} \cap N)^\perp$. As $L \notin \mathcal{L}_U$, we have $v \neq e$. Set $\tilde{U} := \tilde{L} \cap N$. Note that $x_0 v^{-1} \tilde{U} A v \subset x_0 L \cap R F_+ M$, as $\tilde{U} A \subset \tilde{L}$. Since $X$ is $A$-invariant, $x_0 v^{-1} \tilde{U} A v A \subset X$. Let $V^+$ be the unipotent one-parameter subgroup contained in $A v A$, and let $V$ be the one-parameter subgroup containing $V^+$. Then $x_0 v^{-1} V^+ \tilde{U} \subset X$. Since $x_0 A \subset R F M$ and $R F M$ is compact, $\limsup_{t \to +\infty} x_0 u a_{-t}$ is not empty. Now let $x_1$ be any limit of $x_0 u a_{-t_n}$ for some sequence $t_n \to \infty$ and $u \in U$. Since $v^{-1} V^+$ is an open neighborhood of $e$ in $V$, $\liminf_{n \to \infty} a_{t_n} v^{-1} V^+ a_{-t_n} = V$. Note that as $u \in \tilde{U}$, $x_0 u a_{-t_n} (a_{t_n} v^{-1} \tilde{U} V^+ a_{-t_n}) = x_0 v^{-1} \tilde{U} V^+ a_{-t_n} \subset X$. As a result, we obtain that $x_1 \tilde{U} V \subset X$ and hence $x_1 \tilde{U} V A \subset X$. Since $\text{co-dim}_N(\tilde{U} V) \leq m$, the claim follows from by $(2)_m$. □

**Proposition 15.5.** If $R := X \cap F^* \cap R F M$ accumulates on $\partial F$, i.e., there exists $x_n \in R$ converging to a point in $\partial F$, then $X \supset x_0 L \cap R F_+ M$ for some closed orbit $x_0 L$ with $x_0 \in F^* \cap R F M$ and $L \in \mathcal{L}_U$ such that $\dim(L \cap N) > \dim(U)$.

**Proof.** There exists $x_n \in R$ which converges to some $z \in B F M \cdot C(H)$ as $n \to \infty$. We may assume $z \in B F M$ without loss of generality, since $R$ is $C(H)$-invariant. We claim that $X \cap R$ contains $z_1 v$ where $z_1 \in B F M$ and $v \in \tilde{V} \setminus \{e\}$. Write $x_n = z h_n r_n$ for some $h_n \in \tilde{H}$ and $r_n \in \exp \tilde{h}^\perp$, where $\tilde{h}^\perp$ denotes the $\text{Ad}(\tilde{H})$-complementary subspace to $\text{Lie}(\tilde{H})$ in $\mathfrak{g}$. Since $x_n \in F^*$ and $z \in B F M$, it follows that $r_n \notin C(H)$ for all large $n$. By (2.1) and (2.5), we have $N(U) \cap \exp(\tilde{h}^\perp \cap \mathcal{O}) \subset \tilde{V} \cap C(H)$ for a small neighborhood $\mathcal{O}$ of 0 in $\mathfrak{g}$. Therefore, if $r_n \in N(U)$ for some $n$, then the $\tilde{V}$-component of $r_n$ should be non-trivial. Hence by Theorem 10.1, $X \supset z h_n \tilde{U} r_n = z h_n L r_n$ for some $L \in Q_U$ contained in $\tilde{H}$. Note that $x_n = z h_n r_n \in F^*$ and that $r_n^{-1} L r_n \in Q_U \setminus \mathcal{L}_U$, since $r_n \in \tilde{V} \setminus \{e\}$. Hence the claim follows from
Lemma 15.4. Now suppose that \( r_n \not\in \mathcal{N}(U) \) for all \( n \). Then there exists a one-parameter subgroup \( U_0 = \{ u_t \} < U \) such that \( r_n \not\in \mathcal{N}(U_0) \). Applying Lemma 8.3, with a sequence of \( k \)-thick subsets \( T(x_n) := \{ t \in \mathbb{R} : x_n u_t \in RF M \} \), we get a sequence \( t_n \in T(x_n) \) such that \( u_{t_n}^{-1} r_n u_{t_n} \) converges to non-trivial element \( v \in \overline{V} \). Since \( z u_{t_n} u_n \in \overline{H} \) and \( \overline{H} \) is compact, the sequence \( z u_{t_n} u_n \) converges to some \( z_1 \in \overline{H} \), after passing to a subsequence. Then
\[
(15.3) \quad z_1 v = \lim (z u_{t_n} u_n) (u_{t_n}^{-1} r_n u_{t_n}) \in X \cap RF M.
\]
Since \( z_1 \in BF M \) and \( v \in \overline{V} - \{ e \} \), \( z_1 v \in RF M \) implies that \( z_1 v \in F^* \), and hence \( z_1 v \in R \). This proves the claim.

Now by Theorem 10.1, \( z_1 U = z_1 L \) for some \( L \in \mathcal{Q}_U \) contained in \( \overline{H} \), and hence \( X \supseteq z_1 v U = z_1 v U = (z_1 v)(v^{-1} L v) \). Since \( v \in \overline{V} - \{ e \} \), \( v^{-1} L v \not\in \mathcal{L}_U \).

Therefore, by Lemma 15.4, it suffices to prove that there exists \( u \in U \) such that
\[
(15.4) \quad (F^* \cap RF M) \cap \limsup_{t \to +\infty} z_1 u v a_{-t} \neq \emptyset.
\]

Let \( g_1 \in G \) be such that \( z_1 = [g_1] \), and set \( A_{(-\infty, -t]} := \{ a_{-s} : s \geq t \} \) for \( t > 0 \). Since \( z_1 v u \in F^* \cap RF M \), the sphere \((gvU)^{-} \cup g^+ \) intersects \( \Lambda - \bigcup_i \overline{B}_i \) non-trivially. Let \( u \in U \) be an element such that \( (gvu)^{-} \) is contained in \( \Lambda - \bigcup_i \overline{B}_i \). As \( z_1 v u \in RF M \), \( \pi(zvu A_{(-\infty, -t]}) \) is contained in the \( \varepsilon \)-neighborhood of hull \( B_j \)'s are mutually disjoint. If (15.4) does not hold for \( z_1 u v \), then there exists \( t > 1 \) such that the geodesic ray \( \pi(z_1 u v A_{(-\infty, -t]}) \) is contained in the \( \varepsilon \)-neighborhood of \( \partial \) core \( M \) (cf. proof of Lemma 7.8).

As \( \pi(g_1 u v A_{(-\infty, -t]}) \) is connected, there exists \( B_j \) such that \( \pi(g_1 u v A_{(-\infty, -t]}) \) is contained in the \( \varepsilon \)-neighborhood of hull \( B_j \). This implies that \( (g_1 u v)^{-} \in \partial B_j \), yielding a contradiction. This proves (15.4).

**Proposition 15.6.** The orbit \( xH' \) is not closed in \( F^* \).

**Proof.** Suppose that \( xH' \) is closed in \( F^* \). Since we are assuming that \( xH' \) is not closed in \( F \), \( xH' \) contains some point \( y \in \partial F \). Since \( \partial F = BF M \hat{V}^+ C(H) \), we may assume \( y \in BF M \hat{V}^+ \). Write \( y = zv \) where \( z \in BF M \) and \( v \in \hat{V}^+ \). If \( v \neq e \), \( zvH' \) intersects \( BF M \) by Theorem 10.5. Therefore \( zvH' \) always contains a point of \( BF M \), say \( z \). Let \( x_n \in xH' \) be a sequence converging to a point \( z \). Since \( xH' \subset F^* \), there exist \( k_n \in H \cap K \) converging to some \( k \in H \cap K \) such that \( x_n k_n \in xH' \cap RF_+ M \) and \( x_n k_n \to zk \). Then \( zk \in BF M \cdot H' = BF M \cdot C(H) \). Since \( x_n k_n \in RF M \cdot U \) by Lemma 3.4, there exists \( u_n \in U \) such that \( x_n k_n u_n \) belongs to \( RF M \) and converges to a point in \( \partial F \) by Lemma 7.6. Hence \( X \cap F^* \cap RF M \) accumulates on \( \partial F \). Now the claim follows from Proposition 15.5.

This proposition implies that
\[
(15.5) \quad (X - xH') \cap (F^* \cap RF M) \neq \emptyset.
\]

Roughly speaking, our strategy in proving (1)\(_{m+1} \) is first to find a closed \( L \)-orbit \( x_0 L \) such that \( x_0 L \cap F \) is contained in \( X \) for some \( L \in \mathcal{L}_U \). If
$X \neq x_0L \cdot C(H) \cap F$, then we enlarge $x_0L$ to a bigger closed orbit $x_1\hat{L}$ for some $\hat{L} \in \mathcal{L}_0$, for some $\hat{U}$ properly containing $U$, such that $x_1\hat{L} \cap F$ is contained in $X$.

It is in the enlargement step where Proposition 14.1 (Additional invariance I) is a crucial ingredient of the arguments. In order to find a sequence $x_i$ accumulating on a generic point of $x_0L$ satisfying the hypothesis of the proposition, we find a closed orbit $x_0L$ with a base point $x_0$ in $F^* \cap RFM$, and enlarge it to a bigger closed orbit, again based at a point in $F^* \cap RFM$.

The advantage of having a closed orbit $xL$ with $x \in F^* \cap RFM$ is that any $U_0$-generic point in $xL \cap RFM$ can be approximated by a sequence of $RFM$-points in $F^* \cap xL$ by Lemma 7.3. The enlargement process must end after finitely many steps because of dimension reason.

**Finding a closed orbit of $L \in \mathcal{L}_U$ in $X$.**

**Proposition 15.7.** There exists a closed orbit $x_0L$ with $x_0 \in F^* \cap RFM$ and $L \in \mathcal{L}_U$ such that $x_0L \cap RF_+M \subset X$.

**Proof.** Let $R := X \cap F^* \cap RFM$. If $R$ is non-compact, the claim follows from Proposition 15.5. Now suppose that $R$ is compact. By (2.a)$_m$, it is enough to show that $X$ contains an orbit $zU$, and hence $zU \cdot A$, for some $\hat{U} \subset N$ properly containing $U$ and $z \in R$. By Proposition 9.9, it suffices to find a $U$-minimal subset $Y \subset X$ with respect to $R$ and a point $y \in Y \cap R$ such that $X - yH'$ is not closed. If $xH'$ is not locally closed, then take any $U$-minimal subset $Y$ of $X$ with respect to $R$. If $Y \cap R \subset xH'$, then choose any $y \in Y \cap R$. Then $X - yH' = X - xH'$ cannot be closed, as $xH'$ is not locally closed.

If $Y \cap R \not\subset xH'$, then choose $y \in (Y \cap R) - xH'$. Then $X - yH'$ contains $xH'$ and hence cannot be closed. If $xH'$ is locally closed, then $X - xH'$ is a closed $H'$-invariant subset which intersects $R$ non-trivially. So we can take a $U$-minimal subset $Y \subset X - xH'$ with respect to $R$. Take any $y \in Y \cap R$. Then $X - yH'$ is not closed.

**Enlarging a closed orbit of $L \in \mathcal{L}_U$ in $X$.**

**Proposition 15.8.** Assume that (3)$_m$ holds as well. Suppose that there exists a closed orbit $x_0L$ for some $x_0 \in F^* \cap RFM$ and $L \in \mathcal{L}_U$ such that

$$(15.6) \quad x_0L \cap RF_+M \subset X \text{ and } X \neq x_0L \cdot C(H) \cap F.$$ 

Then there exists a closed orbit $x_1\hat{L}$ for some $x_1 \in F^* \cap RFM$, and $\hat{L} \in \mathcal{L}_0$ for some $\hat{U} \subset N$ with $\dim \hat{U} > \dim(L \cap N)$ such that $x_1\hat{L} \cap RF_+M \subset X$.

**Proof.** Note that if $X \subset x_0L \cdot C(H)$, then $X = x_0L \cdot C(H) \cap F$. Therefore we assume that $X \not\subset x_0L \cdot C(H)$. First note that the hypothesis implies that $L \neq G$, and hence co-$\dim_{L \cap N}(U) \leq m$. Let $U_1^{(1)}, \ldots, U_{\ell}^{(1)}$ be one-parameter subgroups generating $U$. Similarly, let $U_1^{(1)}, \ldots, U_{\ell}^{(1)}$ be one-parameter subgroups generating $U^+$. By Theorem 12.1, $\bigcap_{i=1}^{\ell} \mathcal{G}(U_\pm^{(i)}, x_0L) \neq$
\[ \emptyset. \] Therefore without loss of generality, we can assume
\[ x_0 \in \bigcap_{i=1}^{\ell} \mathcal{G}(U^{(i)} \pm, x_0 L). \]

Let us write \( L = H(\hat{U})C \) for some \( \hat{U} < N \) and a closed subgroup \( C \) of \( C(H(\hat{U})) \). Note from the hypothesis that we have \( (x_0 L \cap RF_+ M) \cdot H' \subset X \). Observe that (15.6) implies that \( x \not\in x_0 L \cdot H' = x_0 L \cdot C(H) \). Since \( C < C(H) \), we have \( x \not\in x_0 H(U) \). Now choose a sequence \( w_i \in H' \) such that \( xw_i \to x_0 \), as \( i \to \infty \). Write \( xw_i = x_0 g_i \) where \( g_i \to e \) in \( G - LH' \). Let us write \( g_i = \ell_i r_i \) where \( \ell_i \in L \), and \( r_i \in \exp L^+ \). In particular, \( r_i \not\in C(H) \). Let \( x_i = x_0 \ell_i \), so that \( x_i r_i \in X \).

We claim that we can assume that \( x_i \in RF M \cap x_0 L, r_i \not\in C(H) \), and \( x_i r_i \in X \). Since \( x_0 \in F^* \), by Lemma 7.3, we can find \( w'_i \to w' \in H \) such that \( x_0 \ell_i w'_i \in RF M \), and \( x_0 w' \in \bigcap_{i=1}^{\ell} \mathcal{G}(U^{(i)} \pm, x_0 L) \); hence \( x_0 w' \subset x_0 L \cap RF_+ M \). Writing \( x'_i = x_0 \ell_i w'_i \) and \( r'_i = w''_i^{-1} r_i w'_i \), we have \( x'_i r'_i = xw_i w'_i \in X \), where \( x'_i \to x_0 w' \) in \( x_0 L \cap RF M \), and \( r'_i \to e \) in \( \exp L^+ \). Since \( F^* \) is \( H' \)-invariant, we have \( x_0 w' \in F^* \). Since \( F^* \) is open and \( x_0 w' \in F^* \), it follows that \( x'_i \in X \cap RF M \cap F^* \) for sufficiently large \( i \). Note that \( r'_i \not\in C(H) \), as \( r_i \not\in C(H) \). This proves the claim.

We may assume \( r_i \not\in N(U) \) for all \( i \), up to switching the roles of \( U \) and \( U^+ \), by Lemma 2.4. Note that \( x_i \to x_0 \) in \( RF M \cap x_0 L \) and \( x_0 \) satisfies (15.7). As we are assuming (2)_m, and (3)_m, we may now apply Proposition 14.1 to the sequence \( x_0 \ell_i r_i \to x_0 \) to obtain a non-trivial element \( v \in \hat{U}^\perp \) such that \( x_0 L v \cap RF_+ M \subset X \). Since \( x_0 \in F^* \cap RF M \), it follows from Lemma 7.12 that there exist \( x_2 \in F^* \cap RF M \) and a connected closed subgroup \( \hat{U} < N \) properly containing \( L \cap N \) such that \( x_2 \hat{U} A \subset X \). Since co-dim\( _N(\hat{U}) \leq m \), it remains to apply (2.a)_m to finish the proof of the proposition.

**Proof of (1)_m+1.** Combining Propositions 15.7 and 15.8, we now prove:

**Theorem 15.9.** If (2)_m and (3)_m are true, then (1)_m+1 is true.

**Proof.** Recall that we only need to consider the case \( X = \overline{xF} \) where \( x \in F^* \) and \( xH' \) is not closed in \( F^* \). By Proposition 15.7, there exists \( x_0 \in F^* \cap RF M \) and \( L \in \mathcal{L}_U \) such that \( x_0 L \) is closed and \( x_0 L \cap RF_+ M \subset X \). Since \( X \) is \( H' \)-invariant, it follows
\[ (x_0 L \cap RF_+ M) \cdot H' \subset X. \]

Note that \( (x_0 L \cap RF_+ M) \cdot H' = x_0 L \cdot C(H) \cap F \) is a closed set. We may assume the inclusion in (15.8) is proper, otherwise we have nothing further to prove. Then by Proposition 15.8, there exists \( \tilde{L} \in \mathcal{L}_{\hat{U}} \) for some \( \hat{U} < N \) properly containing \( L \cap N \), and a closed orbit \( x_1 \tilde{L} \) with \( x_1 \in F^* \cap RF M \) such that \( x_1 \tilde{L} \cap RF_+ M \subset X \). If \( (x_1 \tilde{L} \cap RF_+ M) \cdot C(H) \neq X \), then we can apply Proposition 15.8 on \( x_1 \tilde{L} \cap RF_+ M \) in \( X \), as \( \mathcal{L}_{\hat{U}} \subset \mathcal{L}_U \). Continuing in this fashion, the process terminates in a finite step for a dimension reason,
and hence $X = (x_1 \hat{L} \cap RF_+ M) \cdot H' = x_1 \hat{L} \cdot C(H) \cap F$ for some $\hat{L} \in \mathcal{L}_U$, completing the proof. □

16. $U$ and $AU$-orbit closures: proof of $(2)_{m+1}$

In this section, we fix a closed orbit $x_0 \hat{L}$ for $x_0 \in F^*$ and $\hat{L} \in \mathcal{L}_U$. Let $U < \hat{L} \cap N$ be a connected closed subgroup with $\text{co-dim} \hat{L} \cap N \leq m + 1$. By replacing $U$ and $\hat{L}$ by their conjugates using an element $m \in M$, we may assume that $U \subseteq \hat{L} \cap H \cap N$. We keep the same notation $H, F, \partial F, F^*$ etc from section 15. If $x \in RF_+ M \cap \partial F$ (resp. if $x \in RF M \cap \partial F$), then $(2.a)$ (resp. $(2.b)$) follows from Theorem 10.1.

We fix $x \in RF M \cap x_0 \hat{L} \cap F^*$, and set

$$ (16.1) \quad X := \overline{xU} \quad \text{and assume that } X \neq x_0 \hat{L} \cap RF_+ M. $$

This assumption implies that $U$ is a proper connected closed subgroup of $\hat{L} \cap N$ and hence $\text{dim}(\hat{L} \cap N) > \text{dim} U \geq 1$.

By Proposition 4.16, either $x_0 \hat{L}$ is compact or $\mathcal{S}(U, x_0 \hat{L})$ contains a compact orbit $zL_0$ with $L_0 \in \mathcal{L}_U$. If $x_0 \hat{L}$ is compact, then $(2)_{m+1}$ follows from Theorem 10.1. Therefore we assume in the rest of the section that

$$ (16.2) \quad \mathcal{S}(U, x_0 \hat{L}) \text{ contains a compact orbit } zL_0 \text{ with } L_0 \in \mathcal{L}_U. $$

**Lemma 16.1.** If $(1)_{m+1}$ and $(2)_{m}$ hold, then $\overline{xAU} \cap \mathcal{S}(U, x_0 \hat{L}) \neq \emptyset$.

**Proof.** Since $(1)_{m+1}$ is true, we have $\overline{xH} = xQ \cap F$ for some $Q \in \mathcal{L}_U$ such that $xQ$ is closed. By Lemma 15.1, $Q < \hat{L}$. It follows from Lemma 4.11 that either $Q = \hat{L}$ or $\text{dim}(Q \cap N) < \text{dim}(\hat{L} \cap N)$. Suppose that $Q = \hat{L}$. By (16.2), there exists a compact orbit $zL_0 \subseteq \mathcal{S}(U, x_0 \hat{L})$ for some $L_0 \in \mathcal{L}_U$. On the other hand, $x_0 \hat{L} \cap F = \overline{xH} = \overline{xAU}(K \cap H)$. Hence for some $k \in K \cap H$, $zk \in \overline{xAU}$. Since $H \subseteq L_0$, $zk \in zL_0$. So $\overline{xAU}$ intersects $zL_0$, proving the claim. If $\text{dim}(Q \cap N) < \text{dim}(\hat{L} \cap N)$, then $\overline{xAU} \subset xQ \subset \mathcal{S}(U, x_0 \hat{L})$. □

**Lemma 16.2.** If $(1)_{m+1}$ and $(2)_{m}$ hold, then $\overline{xU} \cap \mathcal{S}(U, x_0 \hat{L}) \neq \emptyset$.

**Proof.** Since

$$ (16.3) \quad (x_0 \hat{L} \cap RF_+ M) \cdot F^* \subset \mathcal{S}(U, x_0 \hat{L}), $$

it suffices to consider the case when $X := \overline{xU} \subset F^*$. Let $Y \subset X$ be a $U$-minimal set with respect to $RF M$. Since $Y \subset F^*$, by Proposition 9.6, there exists an unbounded one-parameter subsemigroup $S$ inside $AU^+ C_2(U) \cap \hat{L}$ such that $YS \subset Y$. In view of Lemma 2.2, we could remove $C_2(U)$-component of $S$ so that $S$ is either of the following

- $v^{-1}A^+ v$ for a one-parameter semigroup $A^+ \subset A$ and $v \in U^+ \cap \hat{L}$;
- $V^+$ for a one-parameter semigroup $V^+ \subset U^+ \cap \hat{L}$,

and $YS \subset X(C_2(U) \cap \hat{L})$. Since $\mathcal{S}(U, x_0 \hat{L})$ is invariant by $NC_2(U) \cap \hat{L}$, it suffices to show that $X(NC_2(U) \cap \hat{L}) \cap \mathcal{S}(U, x_0 \hat{L}) \neq \emptyset$. If $S = v^{-1}A^+ v$, then $Yv^{-1}A^+ \subset Xv^{-1}(C_2(U) \cap \hat{L})$. Choose $y \in Y$. We may assume that
$yv^{-1} \in F^*$ by (16.3). Then, replacing $y$ with an element in $yU$ if necessary, we may assume $yv^{-1} \in RF M \cap F^*$. Choose a sequence $a_n \to \infty$ in $A^+$. Then $yv^{-1}a_n$ converges to some $y_0 \in RF M$ by passing to a subsequence. Since $\liminf a_n^{-1}A^+ = A$, and $(yv^{-1}a_n)(a_n^{-1}A^+) \subset Xv^{-1}(C_2(U) \cap \hat{L})$, we obtain that $yvA \subset Xv^{-1}(C_2(U) \cap \hat{L})$. Since $yv\hat{A}U \subset Xv^{-1}(C_2(U) \cap \hat{L})$ and $\overline{y\hat{A}U}$ meets $\mathcal{S}(U, x_0 \hat{L})$ by Lemma 16.1, the claim follows. Next, assume that $S = V^+$, so that $VV^+ \subset X C_2(U) \cap \hat{L}$. Let $v_n \to \infty$ be a sequence in $V^+$. We have $Yv_n \subset X \subset F^*$. Together with the fact $Yv_n$ is $U$-invariant, this implies $Yv_n \cap M \subset RF M$. Note that $Yv_n(v_n^{-1}V^+) \subset X(C_2(U) \cap \hat{L})$. Choose $y_n \in Yv_n \cap RF M$. As $RF M$ is compact, $y_n$ converges to some $y_0 \in RF M$, by passing to a subsequence, and hence $y_0\hat{U} \subset X(C_2(U) \cap \hat{L})$. Since co-dim$_N(UV) \leq m$, the conclusion follows from (2)$_m$.

**Lemma 16.3.** If (1)$_{m+1}$ and (2)$_m$ hold, then $\overline{xU} \cap \mathcal{S}(U, x_0 \hat{L}) \cap F^* \neq \emptyset$.

**Proof.** By Lemma 16.2, there exists $y \in \overline{xU} \cap \mathcal{S}(U, x_0 \hat{L})$. Hence by (2)$_m$, $\overline{yU} = yL \cap RF_+ M \subset \overline{xU}$ for some $L \in \mathcal{Q}_U$ properly contained in $\hat{L}$. Consider the collection of all subgroups $L \in \mathcal{Q}_U$ such that $yL \subset \overline{xU}$ for some $y \in RF_+ M$. Choose $L$ from this collection so that $L \cap N$ has maximal dimension. If $yL \cap F^* \neq \emptyset$, then the claim follows.

Now suppose that $yL \subset \partial F$. As $y \in RF_+ M \cap \partial F$, we have $y = zw_0c_0$ for some $z \in BF M$, $w_0 \in \hat{V}^+$ and $c_0 \in C(H)$. Since $y \in \overline{xU}$, there exists $u \in U$ such that $zu_i$ converges to $y$ as $n \to \infty$. Set $z_i := xu_i^{-1}v^{-1} \in \overline{xU}c_0^{-1}v^{-1}$ so, $z_i \to z$. As $w_0 \in \hat{V}^+$ and hence $v^{-1} \in \hat{V}^-$ and $xu_i \in F^*$, we have $z_i \in F^* \cap RF_+ M \subset RF M \cdot U$. By Lemma 7.6, we may modify $z_i$ by elements of $U$ so that $z_i \in RF M$ and $z_i$ converges to some $z_0 \in \hat{H}$. Write $z_i = z_0\ell_i r_i$ for some $\ell_i \in \hat{H}$ and $r_i \in \exp \hat{h}^\perp$ converging to $e$. Since $z_i \in F^*$ and $z_0\ell_i \in \partial F$, we have $r_i \neq e$. By Theorem 10.1, we have $\overline{z_0\ell_i U} = z_0\ell_i L_i$ for some $L_i \in \mathcal{Q}_U$ contained in $\hat{H}$.

**Case 1:** $r_i \in N(U)$ for some $i$. Then $\overline{xU} = z_0\ell_i r_i v_0 c_0 U = z_0\ell_i U(r_i v_0 c_0) = z_0\ell_i L_i(r_i v_0 c_0)$. As $\overline{xU} \neq x_0 \hat{L}$ by the hypothesis, it follows that $x \in \mathcal{S}(U, x_0 \hat{L}) \cap F^*$, proving the claim.

**Case 2:** $r_i \notin N(U)$ for all $i$. Then there exists a one-parameter subgroup $U_0 \subset U$ such that $r_i \notin N(U_0)$ for all $i$, by passing to a subsequence.

By Lemma 8.3, we can find $u_{t_i} \to \infty$ in $U_0$ so that $z_0\ell_i u_{t_i} \in RF M$ and $u_{t_i}^{-1}r_i u_{t_i}$ converges to a non-trivial element $v \in \hat{V}$, whose size is strictly bigger than $\|v_0\|$. As $z_0\ell_i u_{t_i}$ is contained in the compact subset $z_0\hat{H}$, we may assume that $z_0\ell_i u_{t_i}$ converges to some $z' \in z_0\hat{H}$. Hence $z_i u_{t_i} = z_0\ell_i u_{t_i}(u_{t_i}^{-1}r_i u_{t_i}) \to z'v \in RF M \cap \overline{xU}c_0^{-1}v^{-1}$. Since $z' \in BF M$ and $z'v \in RF M$, we have $v \in \hat{V}^-$.

By Theorem 10.1, $\overline{xU} = z'Q_1$ for some $Q_1 \in \mathcal{Q}_U$. Since $z'v v_0 c_0 \in \overline{xU}$, we get $\overline{xU} \supset z'Q_1(v_0 c_0)$. Since the size of $v$ is larger than the size of $v_0$, then $v v_0$ is a non-trivial element of $\hat{V}^-$.

Since $z'Q_1 \subset BF M$, the closed orbit $z'Q_1(vv_0)c_0$ meets $F^*$. Hence the claim follows. □
Theorem 16.4. If $(1)_{m+1}, (2)_m,$ and $(3)_m$ are true, then $(2)_{m+1}$ is true.

Proof. We first show $(2.a)_{m+1}$ holds for $X = \overline{xU}$. By Lemma 16.3 and $(2)_m$, there exists a closed orbit $yL$ with $y \in F^*$ and $L \in Q_U$ such that $\overline{xL} \supset yL \cap RF^+_M$ and $L \cap N \neq \emptyset \cap N$. We choose $L \in Q_U$ so that $\dim(L \cap N)$ is maximal. Note that $\operatorname{co-dim}_{L \cap N} U \leq m$. By Theorem 12.1, we can assume that

$$(16.4) \quad y \in \bigcap_{i=1}^\ell \mathcal{G}(U^{(i)}) \cap F^* \cap RF^+_M$$

where $U^{(1)}, \ldots, U^{(\ell)}$ are one-parameter subgroups generating $U$. As $y \in \overline{xU}$, there exists $u_i \in U$ such that $xu_i \to y$ as $i \to \infty$. Since $y \in F^*$, we can assume $xu_i \in RF^+_M$ after possibly modifying $u_i$ by Lemma 7.6. We will write $xu_i = y\ell_i r_i$ where $\ell_i \in L$ and $r_i \in \exp \ell_i \cap \hat{L}$.

**Case 1:** $r_i \in N(U)$ for some $i$. Then $y\ell_i \in RF^+_M$ and $X = \overline{xu_i U} = y\ell_i U r_i$. Since $y\ell_i U \subset yL$, and $\operatorname{co-dim}_{L \cap N} (U) \leq m$, we have $X = y\ell_i U r_i = y\ell_i L \cap RF^+_M$ for some $L \in Q_U$, proving the claim.

**Case 2:** $r_i \notin N(U)$ for all $i$. By (16.4), we can apply Proposition 14.2 to the sequence $xu_i \to y$ and obtain a sequence $v_j \to \infty$ in $(L \cap N)^\perp$ such that $yL v_j \cap RF^+_M \subset X$. Since $y \in F^*$, by Lemma 7.10, there exists a one-parameter subgroup $V \subset (L \cap N)^\perp$ such that $y_1 (L \cap N) V \subset X$ for some $y_1 \in F^* \cap RF^+_M$. Hence, by $(2)_m$, we get a contradiction to the maximality of $L \cap N$; this proves $(2.a)_{m+1}$.

Now we show $(2.b)_{m+1}$ for the closure $\overline{xAU}$. By $(1)_{m+1}$, we have $\overline{xH} = xL \cap F$ for some $L \in L_U$ contained in $\hat{L}$. Hence $\overline{xAU} \subset xL \cap RF^+_M$. It suffices to show that

$$(16.5) \quad \overline{xAU} = xL \cap RF^+_M.$$ 

If $U = L \cap N$, then $\overline{xU} = xL \cap RF^+_M$ by Theorem 12.1, which implies (16.5). So, suppose that $U$ is a proper closed subgroup of $L \cap N$. Since $\overline{xAU} (K \cap H) = \overline{xH} = xL \cap F$, it follows from Lemma 4.15 that we can choose $y \in \overline{xAU} \cap \mathcal{G}(U, xL)$. By $(2.a)_{m+1}$ and Lemma 13.4, we have $yU = xL \cap RF^+_M$, finishing the proof. \hfill \Box

17. Topological equidistribution: proof of $(3)_{m+1}$

In this section, we prove $(3)_{m+1}$. Let $U < N$ be a non-trivial connected closed subgroup. Let $x_0 \hat{L}$ be a closed orbit for $x_0 \in F^* \cap RF^+_M$ and $\hat{L} \in L_U$ such that $\operatorname{co-dim}_{L \cap N} (U) = m + 1$. As before we may assume that $U \subset \hat{L} \cap \hat{H} \cap N$. Let $x_i L_i \subset x_0 \hat{L}$ be a sequence of closed orbits intersecting $RF^+_M$ where $x_i \in RF^+_M$, $\hat{L}_i \in Q_U$. We write $x_i L_i$ as $y_i L_i v_i$ where $y_i \in RF^+_M$, $L_i \in L_U$, and $v_i \in (L_i \cap N)^\perp \cap \hat{L}$. Assume that no infinite subsequence of $y_i L_i v_i$ is contained in a subset of the form $y_0 L_0 D \subset \mathcal{G}(U, x_0 \hat{L})$ where $y_0 L_0$ is a closed orbit for some $L_0 \in L_U$ and $D \subset N(U)$ is a compact subset. Let $E = \limsup_{i \to \infty} (y_i L_i v_i \cap RF^+_M)$. Note that $\liminf_{i \to \infty} (y_i L_i v_i \cap RF^+_M)$
coincides with the intersection of the subsets \( \limsup(y_{i_k}L_{i_k}v_{i_k} \cap RF_+ M) \) for all infinite subsequences \( \{i_k : k \in \mathbb{N}\} \) of \( \mathbb{N} \). If the hypothesis of (3)_{m+1} holds for a given sequence \( y_iL_i v_i \), then it also holds for all subsequences. Hence to prove (3)_m holds, it suffices to show that \( E = RF_+ M \cap x_0 \hat{L} \). We note that by (3)_m, we may assume that \( L_i \cap N = U \) for all \( i \). This in particular implies that each \( y_iL_i v_i \cap RF_+ M \) is \( U \)-minimal by Theorem 12.1.

**Lemma 17.1.** Assume that (1)_{m+1}, (2)_{m+1} and (3)_m are true. Then there exist \( y \in F^* \cap RF M \) and \( L \in \mathcal{U} \) with \( \dim(L \cap N) > \dim U \) such that \( yL \) is closed and \( E \supset yL \cap RF_+ M \).

**Proof.** By (2)_m, it suffices to show that there exist \( y_0 \in F^* \cap RF M \) and \( \hat{U} < N \) properly containing \( U \) such that \( E \supset y_0 \hat{U} \). Suppose that \( y_iL_i v_i \subset \partial F \) for infinitely many \( i \). Since \( y_iL_i v_i \cap RF M \neq \emptyset \), we may assume \( y_iL_i v_i \subset z_i \hat{H} C(H) \) for some \( z_i \in BF M \) by (3.3). Since \( L_i \cap N = U \), we get \( y_iL_i v_i = \hat{y}_i U \subset z_i \hat{H} C(H) \) by Theorem 10.1. This contradicts the hypothesis on \( y_iL_i v_i \). Therefore by passing to a subsequence, for all \( i \), \( y_iL_i v_i \cap RF_+ M \cap F^* \neq \emptyset \). Since \( AU \) is open and for some \( r \), it follows that \( E = \limsup(y_iL_i v_i \cap RF_+ M) \cap F^* \). Hence, after passing to a subsequence,

\[
y_0 \limsup_{i \to \infty} (v_i^{-1}AU v_i) \subset E. \tag{17.1}
\]

If \( v_i \to \infty \), then \( \limsup_{i}(v_i^{-1}AU v_i) \) contains \( \hat{A} \) for some \( \hat{U} \) properly containing \( U \) by Lemma 2.3. Therefore, we get the conclusion \( y_0 \hat{U} \subset E \) from (17.1). Now suppose that, by passing to a subsequence, \( v_i \) converges to some \( v \in N \cap \hat{L} \). Then (17.1) gives \( y_0 v^{-1}AU v \subset E \). Then by (2)_{m+1}, \( y_0 v^{-1}AU \) is of the form \( y_0 v^{-1}L_0 \cap RF_+ M \) for some \( L_0 \in \mathcal{L} \). Hence \( E \supset y_0 L \cap RF_+ M \) where \( L := v^{-1}L_0v \). If \( L \cap N \) contains \( U \) properly, this proves the claim. So we suppose that \( L \cap N = U \). By Theorem 12.1, we can assume that \( y_0 \in \bigcap_{i=1}^\infty G(U^{(i)},y_0 L) \cap F^* \cap RF M \), where \( U^{(1)}, \ldots, U^{(\ell)} \) are one-parameter subgroups generating \( U \). By replacing \( y_i \) by an element of \( y_i L \cap RF_+ M \), we may assume that \( y_i v_i \to y_0 \). Furthermore, as \( y_0 \in F^* \cap RF M \), for all \( i \) sufficiently large, \( y_i v_i \in F^* \cap RF_+ M \subset RF M \cdot U \) (as \( F^* \) is open). Hence we can also assume \( y_i v_i \in RF M \) by Lemma 7.7. Therefore we may write \( y_i v_i = y_0 \ell_i r_i \) for some \( \ell_i \to e \) in \( L \) and non-trivial \( r_i \to e \) in \( \exp L \).

Suppose that \( r_i \) belongs to \( N(U) \) for infinitely many \( i \). Then

\[
y_i L_i v_i \cap RF_+ M = y_i v_i U = y_0 \ell_i U r_i = y_0 L r_i \cap RF_+ M.
\]

Hence \( y_i L_i v_i r_i^{-1} \cap RF_+ M = y_0 L \cap RF_+ M \). In particular, \( y_i L_i v_i r_i^{-1} \cap RF M \) is non-empty (as it contains \( y_0 \)) and is contained in \( y_0 L \). By Lemma 15.1, this implies that \( y_i L_i v_i \subset y_0 L r_i \). As \( r_i \to e \), this contradicts the hypothesis on \( y_i L_i v_i 's \). Therefore \( r_i \notin N(U) \) for all \( i \) but finitely many. We may now apply Proposition 14.2 and Lemma 7.10 to deduce that \( E \) contains an orbit \( z_0 \hat{U} \) for some \( \hat{U} < L \cap N \) containing \( U \) properly and for some \( z_0 \in RF_+ M \cap F^* \). This proves the claim. \( \square \)
Theorem 17.2. If \((1)_{m+1}\), \((2)_{m+1}\), and \((3)_m\) are true, then \((3)_{m+1}\) is true.

Proof. We claim that \(x_0\hat{L} \cap RF_+ M = E\). By Lemmas 17.1, we can take a maximal \(\hat{U}\) such that \(E \supset y\hat{U}\) for some \(y \in F^* \cap RF M\). By \((2)_m\), we get a closed orbit \(yL\) for some \(L \in Q\hat{U}\) such that \(yL \cap RF_+ M \subset E\). If \(L = \hat{L}\), then the claim is clear. Now suppose that \(L\) is a proper subgroup of \(\hat{L}\). This implies that \(L \cap N\) is a proper subgroup of \(\hat{L} \cap N\), since \(\hat{L} \cap N\) acts minimally on \(x_0\hat{L} \cap RF_+ M\) as \(\hat{L} \in L_U\). By Theorem 12.1, we can assume that \(y \in \bigcap_{i=1}^l \mathcal{G}(U^{(i)}, yL) \cap F^* \cap RF M\), where \(U^{(1)}, \ldots, U^{(l)}\) are one-parameter subgroups generating \(U\). As \(y \in E\), there exists a sequence \(x_i = y_iL_i v_i \cap RF_+ M\) converging to \(y\), by passing to a subsequence. Since \(U = v_i^{-1}L_i v_i \cap N\), we have \(x_i \in RF M \cdot U\). By Lemma 7.7, by replacing \(x_i\) with \(x_i u_i\) for some \(u_i \to e\) in \(U\), we may assume \(x_i \in RF M\).

We claim that \(x_i \notin yL N(U)\). Suppose not, i.e., \(x_i = y\ell_i r_i\) for some \(\ell_i \in L\) and \(r_i \in N(U)\). Then \(y_i L_i v_i \cap RF_+ M = x_i U = y L_i U r_i \subset y L r_i\). By the assumption on \(y_i L_i v_i\)’s, this cannot happen as \(r_i\)’s are bounded. On the other hand, \(\dim(L_i \cap N)\) is strictly smaller than \(\dim(L \cap N)\), since \(L_i \cap N = U\) and \(\hat{U} < L \cap N\), yielding a contradiction. Hence \(x_i \notin yL N(U)\).

We can now apply Proposition 14.2 and Lemma 7.10 and deduce that \(E\) contains \(y_1 \hat{U} V\) for some \(y_1 \in F^* \cap RF M\). This is a contradiction to the maximality assumption on \(\dim \hat{U}\).

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