

# PROPERNESS AND FINITENESS OF TOTALLY GEODESIC SUBMANIFOLDS IN THE CONVEX CORE

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ABSTRACT. We study totally geodesic submanifolds in the convex core of geometrically finite rank-one locally symmetric manifolds. Although the infinite-volume setting can exhibit highly complicated behavior, including geodesic planes with fractal closures, we show that a strong rigidity persists inside the convex core. This rigidity has striking consequences in the infinite volume setting: every maximal totally geodesic submanifold of dimension at least two contained in the convex core is properly immersed and has finite volume, and only finitely many such submanifolds can occur. These results stand in sharp contrast to the behavior in the finite-volume setting. Moreover, combining this finiteness result with the work of Bader-Fisher-Miller-Stover and of Gromov-Schoen, we deduce that any geometrically finite rank-one manifold with infinitely many maximal totally geodesic submanifolds of dimension at least two and of finite volume must be arithmetic.

## CONTENTS

|                                                                 |    |
|-----------------------------------------------------------------|----|
| 1. Introduction                                                 | 1  |
| 2. Closed subgroups of rank one Lie groups                      | 7  |
| 3. Totally geodesic submanifolds                                | 10 |
| 4. Totally geodesic submanifolds contained in the convex core   | 14 |
| 5. Orbit closure classification inside $\mathrm{RFM}$           | 17 |
| 6. Equidistribution in $\mathrm{RFM}$                           | 22 |
| 7. Rigidity of totally geodesic submanifolds in the convex core | 25 |
| References                                                      | 28 |

## 1. INTRODUCTION

The study of totally geodesic submanifolds occupies a central position in differential geometry, dynamics, and geometric topology. In locally symmetric spaces of finite volume, such manifolds arise naturally from algebraic subgroups and encode deep arithmetic and geometric structure. Their distribution reflects the intricate interplay between discrete subgroups, rigidity phenomena, and homogeneous dynamics.

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In the infinite volume setting, however, the picture is considerably more delicate. Although geometric finiteness provides a natural analogue of finite volume, many of the structural features familiar from the compact or finite volume world no longer hold. Even in real hyperbolic 3-manifolds, the qualitative behavior of geodesic planes depends sensitively on the geometry of the ambient space, and examples are known in which certain geodesic planes have fractal closures.

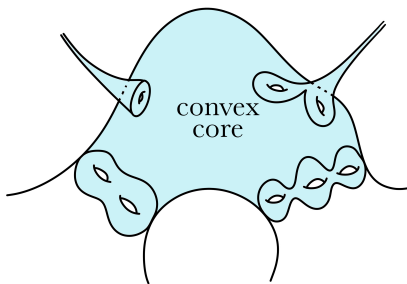
The main purpose of this paper is to establish that, despite these potential pathologies, a robust form of rigidity persists for totally geodesic submanifolds contained in the convex core of a geometrically finite rank-one locally symmetric manifold. This rigidity has striking implications for the geometry of infinite volume geometrically finite manifolds, in sharp contrast to what can occur in the finite volume case.

Let  $G$  be a connected simple real algebraic group of rank one, and let  $X = X_G$  denote its associated Riemannian symmetric space. Concretely,  $G$  is locally isomorphic to one of  $\mathrm{SO}(d, 1)$ ,  $\mathrm{SU}(d, 1)$ ,  $\mathrm{Sp}(d, 1)$  ( $d \geq 2$ ), or  $F_4^{-20}$  and  $X$  is the real  $\mathbb{H}_{\mathbb{R}}^d$ , complex  $\mathbb{H}_{\mathbb{C}}^d$ , quaternionic  $\mathbb{H}_{\mathbb{H}}^d$ , or octonionic hyperbolic space  $\mathbb{H}_{\mathbb{O}}^2$ , respectively.

Let  $\Gamma < G$  be a non-elementary discrete subgroup, and let  $\mathcal{M} = \Gamma \backslash X$  be the associated locally symmetric space. The convex core of  $\mathcal{M}$  is the smallest closed convex subset of  $\mathcal{M}$  whose inclusion into  $\mathcal{M}$  is a homotopy equivalence. Equivalently,

$$\mathrm{core}(\mathcal{M}) = \Gamma \backslash \mathrm{hull}(\Lambda)$$

where  $\Lambda$  denotes the limit set of  $\Gamma$  and  $\mathrm{hull}(\Lambda) \subset X$  its convex hull (see Section 4).



Throughout this paper, we assume that

$$\mathcal{M} \text{ is geometrically finite,}$$

that is, the unit neighborhood of  $\mathrm{core}(\mathcal{M})$  has finite volume. Clearly, every finite-volume rank-one manifold is geometrically finite, but the class of geometrically finite manifolds is much broader than that of finite-volume ones.

**Properness of totally geodesic submanifolds.** A totally geodesic subspace of  $X$  is a connected smooth submanifold  $Y$  that contains every complete geodesic in  $X$  tangent to  $Y$ . A totally geodesic submanifold  $\mathcal{N}$  of  $\mathcal{M}$  is the image of such a  $Y$  under the quotient map  $X \mapsto \Gamma \backslash X$ . Equivalently,  $\mathcal{N}$  is the image of the natural immersion

$$\iota : \text{Stab}_\Gamma(Y) \backslash Y \rightarrow \mathcal{M}$$

where  $\text{Stab}_\Gamma(Y) = \{\gamma \in \Gamma : \gamma Y = Y\}$ . The dimension of  $\mathcal{N}$  is defined to be the dimension of  $Y$ , and the volume of  $\mathcal{N}$  is defined as the volume of the quotient  $\text{Stab}_\Gamma(Y) \backslash Y$ . We say that  $\mathcal{N}$  is properly immersed if the immersion  $\iota$  is proper; equivalently, if  $\mathcal{N}$  is closed in  $\mathcal{M}$ , that is,  $\overline{\mathcal{N}} = \mathcal{N}$  (Lemma 3.5).

Throughout the paper, a totally geodesic submanifold of  $\mathcal{M}$  is assumed to have

### dimension at least two

so that we need not repeat this assumption. A totally geodesic submanifold of dimension one will simply be referred to as a geodesic.

We focus on totally geodesic submanifolds contained in  $\text{core}(\mathcal{M})$ . For instance, any bounded totally geodesic submanifold is automatically contained in  $\text{core}(\mathcal{M})$ . By a maximal totally geodesic submanifold of  $\text{core}(\mathcal{M})$ , we mean a totally geodesic submanifold  $\mathcal{N}$  contained in  $\text{core}(\mathcal{M})$  with  $\mathcal{N} \neq \mathcal{M}$  that is maximal with respect to inclusion: if  $\mathcal{N}'$  is a totally geodesic submanifold satisfying  $\mathcal{N} \subset \mathcal{N}' \subset \text{core}(\mathcal{M})$ , then  $\mathcal{N} = \mathcal{N}'$ .

**Theorem 1.1** (Properness). *Let  $\mathcal{M}$  be a geometrically finite manifold of infinite volume. Then every maximal totally geodesic immersed submanifold contained in  $\text{core}(\mathcal{M})$  is properly immersed and has finite volume.*

If  $\mathcal{M}$  has finite volume, then  $\mathcal{M} = \text{core}(\mathcal{M})$ , and the geodesic flow is ergodic with respect to the Liouville measure on the unit tangent bundle. In this case, almost every totally geodesic submanifold is dense in  $\mathcal{M}$ , so Theorem 1.1 fails completely in the finite volume setting.

Theorem 1.1 has several immediate consequences. First, it implies that if there exists a maximal totally geodesic submanifold  $\mathcal{N}$  contained in  $\text{core}(\mathcal{M})$  with  $\overline{\mathcal{N}} \neq \mathcal{N}$ , then  $\mathcal{M}$  must have finite volume. Second, it follows that  $\mathcal{M}$  contains a finite volume totally geodesic submanifold if and only if there exists a totally geodesic submanifold of dimension two contained in  $\text{core}(\mathcal{M})$ .

The minimal codimension of totally geodesic subspaces in  $X = \mathbb{H}_{\mathbb{K}}^d$  is equal to  $\dim_{\mathbb{R}}(\mathbb{K})$ , which is 1, 2, 4, 8 for  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ , respectively.

**Corollary 1.2.** *Every totally geodesic submanifold of  $\mathcal{M} = \Gamma \backslash \mathbb{H}_{\mathbb{K}}^d$  of codimension  $\dim_{\mathbb{R}}(\mathbb{K})$  contained in  $\text{core}(\mathcal{M})$  is properly immersed and has finite volume.*

**Finiteness of totally geodesic manifolds.** We next establish a finiteness property for totally geodesic submanifolds contained in the convex core.

**Theorem 1.3** (Finiteness I). *If  $\mathcal{M}$  is a geometrically finite manifold of infinite volume, then there exist only finitely many maximal totally geodesic submanifolds contained in  $\text{core}(\mathcal{M})$ .*

By Theorem 1.1, all of these submanifolds are properly immersed and have finite volume.

Although there may exist infinitely many maximal properly immersed totally geodesic submanifolds in  $\mathcal{M}$ , only finitely many of them have finite volume:

**Theorem 1.4** (Finiteness II). *If  $\mathcal{M}$  is a geometrically finite manifold of infinite volume, then there exist only finitely many maximal totally geodesic submanifolds of finite volume.*

Theorems 1.3 and 1.4 fail dramatically in the finite volume setting, since an arithmetic manifold containing one maximal totally geodesic submanifold of finite volume necessarily contains infinitely many such submanifolds. For example, the figure-eight knot complement contains infinitely many finite area immersed totally geodesic surfaces (see [32], [17]).

**Remark 1.5.** For real hyperbolic 3-manifolds, these theorems were proved by McMullen–Mohammadi–Oh [20] in the convex cocompact case and by Benoist–Oh [6] in full generality. In higher dimensions, they were established in [16] for convex cocompact real hyperbolic  $d$ -manifolds whose convex cores have totally geodesic boundary, for all  $d \geq 3$ .

**Arithmeticity from totally geodesic submanifolds.** Every quaternionic or octonionic hyperbolic manifold  $\mathcal{M} = \Gamma \backslash X$  of finite volume is arithmetic, that is,  $\Gamma$  is an arithmetic lattice of  $G$ , by the theorem of Gromov–Schoen [14]. Answering a question of McMullen and Reid, Bader–Fisher–Miller–Stover ([2], [3]) showed that if a real or complex hyperbolic manifold of finite volume contains infinitely many maximal totally geodesic submanifolds of finite volume, then it is arithmetic. See also Margulis–Mohammadi [18] for compact real hyperbolic 3-manifolds, Baldi–Ullmo [4] for related results in complex hyperbolic manifolds, and Filip–Fisher–Lowe [13] for analogous results in the setting of closed real-analytic manifolds of negative sectional curvature.

Combining these results with Theorem 1.4 gives the following:

**Theorem 1.6.** *If a geometrically finite manifold  $\mathcal{M}$  contains infinitely many<sup>1</sup> maximal totally geodesic submanifolds of finite volume, then  $\mathcal{M}$  is arithmetic.*

**Finiteness of totally geodesic boundaries.** We denote by  $\partial_\infty X$  the geometric boundary of  $X$ , that is, the set of equivalence classes of geodesic rays, where two rays are identified if they remain within bounded Hausdorff distance of each other. For a totally geodesic subspace  $Y \subset X$  of dimension at least two, we may regard its boundary  $\partial_\infty Y$  as a subset of  $\partial_\infty X$ . Let  $\mathcal{S}_X$

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<sup>1</sup>Throughout the paper, “infinitely many” means infinitely many pairwise distinct.

denote the space of all such boundaries  $\partial_\infty Y \subset \partial_\infty X$ , equipped with the Chabauty-Hausdorff topology.

For a totally geodesic submanifold  $\mathcal{N} = \Gamma \backslash \Gamma Y$ , we have  $\mathcal{N} \subset \text{core}(\mathcal{M})$  if and only if  $\partial_\infty Y \subset \Lambda$  where  $\Lambda \subset \partial_\infty X$  is the limit set of  $\Gamma$ . Using this criterion, we deduce the following consequence of Theorems 1.1 and 1.3:

**Theorem 1.7.** *Let  $\Gamma < G$  be a geometrically finite, non-lattice subgroup of  $G$ , and  $\Lambda \subset \partial_\infty X$  be its limit set. Then the collection*

$$\{S \in \mathcal{S}_X : S \subset \Lambda \text{ maximal}\}$$

*consists of finitely many closed  $\Gamma$ -orbits in  $\mathcal{S}_X$ .*

**Counting spheres in the limit set.** Specializing to  $G = \text{SO}(d+1, 1)^\circ$  ( $d \geq 2$ ), we have  $X = \mathbb{H}_{\mathbb{R}}^{d+1}$  and  $\partial_\infty X = \mathbb{R}^d \cup \{\infty\}$ .

Theorem 1.7 implies that if  $\Gamma < G$  is a geometrically finite non-lattice subgroup, then the collection of all  $(d-1)$ -round spheres contained in the limit set  $\Lambda$  forms a locally finite<sup>2</sup> sphere packing consisting of finitely many  $\Gamma$ -orbits. Spheres arising from the totally geodesic boundary components of  $\text{core}(\mathcal{M})$  may be regarded as *visible* spheres, but there may also be *invisible* spheres, namely those lying in the limit set but not corresponding to boundary components of  $\text{core}(\mathcal{M})$ . Given this local finiteness, the sphere counting results of ([26], [24, Theorem 7.5], see also [25]) apply and counts all such spheres, visible or invisible, contained in the limit set:

**Theorem 1.8.** *Let  $\Gamma < \text{SO}(d+1, 1)^\circ$  be a geometrically finite subgroup with  $d \geq 2$ . Assume that  $\Lambda \subset \mathbb{R}^d$  and that  $\Lambda$  contains a  $(d-1)$ -sphere. Then there exists  $c > 0$ , depending only on  $\Gamma$ , such that, as  $t \rightarrow \infty$ , the number  $N(\Lambda, t)$  of all  $(d-1)$ -spheres contained in  $\Lambda$  of Euclidean radius at least  $1/t$  satisfies*

$$N(\Lambda, t) \sim c t^{\dim(\Lambda)}$$

*where  $\dim(\Lambda)$  is the Hausdorff dimension of  $\Lambda$ .*

**Remark 1.9.** Al Assal and Lowe, building on the works [20] and [6], proved that a real-hyperbolic geometrically finite 3-manifold of infinite volume cannot contain infinitely many compact *asymptotically geodesic* surfaces [1]. In view of their methods, we expect an analogous statement in general rank one: any geometrically finite, infinite-volume rank-one manifold cannot contain infinitely many compact asymptotically geodesic maximal submanifolds.

**On the proof.** We emphasize that as far as we know, there are no direct geometric or topological arguments that establish the properness and finiteness theorems (Theorems 1.1 and 1.3). The key point is that these results arise as strict consequences of a much stronger rigidity phenomenon: the only available method for establishing properness and finiteness is to prove a rigidity-equidistribution theorem for totally geodesic submanifolds inside

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<sup>2</sup>This means that for any  $\varepsilon > 0$ , there are only finitely many spheres in a fixed bounded region of Euclidean radii larger than  $\varepsilon > 0$

the convex core of a geometrically finite manifold (not necessarily of infinite volume). This stronger statement is the foundation for the entire argument:

**Theorem 1.10** (Theorem 7.1, Theorem 7.4). *Let  $\mathcal{M}$  be geometrically finite.*

- (1) *The closure of any totally geodesic submanifold contained in  $\text{core}(\mathcal{M})$  is a totally geodesic submanifold of finite volume.*
- (2) *If there exist infinitely many maximal totally geodesic submanifolds  $\mathcal{N}_i$  contained in  $\text{core}(\mathcal{M})$ , then  $\mathcal{M}$  has finite volume and the sequence  $\mathcal{N}_i$  becomes equidistributed in  $\mathcal{M}$  as  $i \rightarrow \infty$ .*

Theorem 1.10 is proved using unipotent dynamics on the homogeneous space  $\Gamma \backslash G$ . Fix a one-parameter diagonal subgroup  $A < G$ . The relevant subspace of  $\Gamma \backslash G$  for studying totally geodesic submanifolds contained in  $\text{core}(\mathcal{M})$  is the smallest closed subset containing all bounded  $A$ -orbits, which we denote by  $\text{RFM}$ . Understanding the closures of totally geodesic submanifolds in  $\mathcal{M}$  amounts to describing the orbit closures of  $xH$  contained in  $\text{RFM}$  for all simple non-compact subgroups  $H < G$ ; this is given in Theorem 5.1 where we describe all orbit closures inside  $\text{RFM}$  for the action of any connected closed subgroup of  $G$  generated by unipotent elements. By Ratner's arguments [31], this is reduced to the study of the closure of  $xU \subset \text{RFM}$  for a one-parameter unipotent subgroup  $U = \{u_s\}$  of  $H$  (see the proof of Theorem 5.1). We use the geometric finiteness hypothesis together with the non-divergence of unipotent flows in the infinite volume rank one setting due to Benoist-Oh [6] and Buenger-Zheng [9] to deduce that any weak-\* limit  $\nu$  of the sequence

$$\nu_T = \frac{1}{T} \int_0^T \delta_{xu_s} ds,$$

is a probability measure on  $\Gamma \backslash G$ , where  $\delta_{xu_s}$  denotes the Dirac measure at  $xu_s \in \Gamma \backslash G$ . We then apply Ratner's measure classification theorem [29] and the avoidance theorem of Dani-Margulis [10] to the ergodic components of  $\nu$  to show that the closure of  $xU$  is a homogeneous space. This constitutes the main step of the proof of Theorem 1.10(1).

For the proof of Theorem 1.10(2), we first deduce from Theorem 1.10(1) that any maximal totally geodesic submanifold inside  $\text{core}(\mathcal{M})$  gives rise to an  $H$ -invariant *probability* measure in  $\Gamma \backslash G$  supported on a closed orbit  $xN_G(H)$ , where  $N_G(H)$  denotes the normalizer of  $H$ , and apply the Mozes-Shah theorem [23] on the limiting behavior of probability measures invariant under unipotent flows.

The earlier papers [20] and [6] on geometrically finite real hyperbolic 3-manifolds relied on topological arguments for unipotent dynamics. This was feasible because, in that specific setting, all totally geodesic submanifolds have codimension one in the ambient manifold, a feature unique to real hyperbolic 3-manifolds. In more general settings, where totally geodesic submanifolds have varying dimensions or maximal geodesic planes have codimension greater than one, extending topological methods becomes quite

complicated. In contrast, our current paper leverages the full power of the measure rigidity theorems of Ratner ([29], [30], [31]), Dani-Margulis [10], and Mozes-Shah [23] on unipotent flows, allowing us to obtain, in our view, rather surprising results in the geometry of geometrically finite manifolds.

Finally, we note that the geometric finiteness assumption on  $\mathcal{M}$  is essential: if  $\mathcal{M}$  is a  $\mathbb{Z}$ -cover of a compact hyperbolic 3-manifold, then  $\mathcal{M} = \text{core}(\mathcal{M})$ , and  $\mathcal{M}$  contains infinitely many dense geodesic surfaces.

### Organization.

- In Section 2, we review the structure of closed subgroups of rank-one Lie groups and introduce the collections  $\mathcal{H}$  and  $\mathcal{H}^*$  of closed subgroups of  $G$  stabilizing totally geodesic subspaces of  $X$ .
- In Section 3, we relate these to immersed totally geodesic submanifolds of  $\mathcal{M}$ .
- In Section 4, we characterize those contained in the convex core of  $\mathcal{M}$  via the renormalized frame bundle  $\text{RFM}$ .
- Sections 5-6 develop the orbit closure and equidistribution results for subgroups generated by unipotent one-parameter subgroups within  $\text{RFM}$ .
- Section 7 applies these results to prove the rigidity, properness and finiteness theorems announced in the introduction.

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## 2. CLOSED SUBGROUPS OF RANK ONE LIE GROUPS

Let  $G$  be a connected simple real algebraic group of rank one. In this section, we describe the structure of closed subgroups of  $G$ . In particular, we introduce two natural families of subgroups  $\mathcal{H}$  and  $\mathcal{H}^*$  which will play a central role throughout the paper.

Let  $\mathfrak{g}$  denote the Lie algebra of  $G$ . Let  $\theta$  be a Cartan involution of  $\mathfrak{g}$ , and  $\Theta$  the corresponding involution of  $G$  which induces  $\theta$ . Write  $\mathfrak{g}$  as the direct sum of  $\pm 1$  eigenspaces of  $\theta$ :

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{2.1}$$

where  $\mathfrak{k} = \{\theta(x) = x\}$  and  $\mathfrak{p} = \{\theta(x) = -x\}$ . Let  $K < G$  be the maximal compact subgroup with Lie algebra  $\mathfrak{k}$ ; equivalently,  $K$  is the fixed point subgroup of  $\Theta$ . Let  $B$  denote the Killing form on  $\mathfrak{g}$ , and define the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{g}$ :

$$\langle x, y \rangle := -B(x, \theta(y)), \quad x, y \in \mathfrak{g},$$

with associated norm  $\|\cdot\|$ . Let

$$X := G/K \text{ and } o = [K] \in X.$$

Since  $\langle \cdot, \cdot \rangle$  is  $\text{Ad}(K)$ -invariant, it induces a left  $G$ -invariant and right  $K$ -invariant metric on  $G$ , which descends to a Riemannian metric on  $X$ , which

we denote by  $\mathbf{d}$ . The Riemannian symmetric space  $(X, \mathbf{d})$  is isometric to one of the real, complex, quaternionic, and octonionic hyperbolic spaces.

Let  $\mathfrak{a} \subset \mathfrak{p}$  be a Cartan subalgebra, i.e., a maximal abelian subalgebra of  $\mathfrak{p}$ . Since  $G$  has real rank one,  $\mathfrak{a}$  is one-dimensional. Let  $A = \exp \mathfrak{a}$  and let  $M = C_K(A)$  denote the centralizer of  $A$  in  $K$ . We parametrize  $A = \{a_t : t \in \mathbb{R}\}$  so that  $\mathbf{d}(o, a_t o) = |t|$  for all  $t \in \mathbb{R}$ . Define the horospherical subgroups

$$N^\pm = \{g \in G : a_t g a_{-t} \rightarrow e \text{ as } t \rightarrow \pm\infty\}$$

and the corresponding minimal parabolic subgroups

$$P^\pm = MAN^\pm.$$

For simplicity, we write  $N = N^-$  and  $P = P^-$ .

Denote by  $\partial_\infty X$  the boundary of  $X$  at infinity; the set of equivalence classes of geodesic rays, where two rays are equivalent if they have finite Hausdorff distance. Every geodesic ray in  $X$  is of the form  $\{ga_t o : t \geq 0\}$  for some  $g \in G$ . The stabilizer of the equivalence class of  $\{a_t o : t \geq 0\}$  is equal to  $P$ ; hence we may identify

$$G/P \simeq \partial_\infty X,$$

where  $gP$  corresponds to the boundary point in  $\partial_\infty X$  represented by the ray  $\{ga_t o : t \geq 0\}$ .

**Families  $\mathcal{H}^*$  and  $\mathcal{H}$ .** A totally geodesic subspace  $Y$  is a connected smooth submanifold of  $X$  which contains all complete geodesics in  $X$  tangent to  $Y$ . For such  $Y \subset X$ , we define its stabilizer

$$G_Y := \{g \in G : gY = Y\}. \quad (2.2)$$

We introduce two families of subgroups of  $G$  that contain representatives corresponding to conjugacy classes of  $G_Y$ . For a subgroup  $H < G$ , denote by  $H^\circ$  its identity component, and by  $C_G(H)$  and  $N_G(H)$  the centralizer and normalizer of  $H$  in  $G$ , respectively.

We define

$$\mathcal{H}^* = \left\{ H < G : \begin{array}{l} \text{a connected simple } \Theta\text{-invariant} \\ \text{non-compact closed subgroup of } G \text{ containing } A \end{array} \right\}$$

and

$$\mathcal{H} = \{N_G(H)^\circ : H \in \mathcal{H}^*\}. \quad (2.3)$$

For a reductive subgroup  $L < G$ , let  $L^{\text{nc}}$  denote the maximal connected normal semisimple subgroup of  $L$  with no compact factors. Since  $G$  has rank one,  $L^{\text{nc}}$  is either trivial or a simple non-compact algebraic subgroup of rank one. We have that  $L = L^{\text{nc}} C_L(L^{\text{nc}})$  with  $C_L(L^{\text{nc}})$  compact.

The following lemma implies that the maps

$$H \mapsto H^{\text{nc}} \quad \text{and} \quad H \mapsto N_G(H)^\circ$$

define a bijection  $\mathcal{H} \rightarrow \mathcal{H}^*$  and its inverse, respectively.

**Lemma 2.4.** *Let  $L < G$  be a connected reductive  $\Theta$ -invariant subgroup containing  $A$ . Suppose that  $L^{\text{nc}} \neq \{e\}$ , and let  $H := N_G(L^{\text{nc}})^\circ$ . Then we have*

$$H \in \mathcal{H}, \quad H^{\text{nc}} = L^{\text{nc}}, \quad \text{and} \quad H.o = L.o.$$

*In particular, for any  $H \in \mathcal{H}$ , we have*

$$H = N_G(H^{\text{nc}})^\circ \quad \text{and} \quad H.o = H^{\text{nc}}.o.$$

*Proof.* Since  $L \supset A$  and  $L^{\text{nc}}$  is normal in  $L$ , it follows that  $L^{\text{nc}} \supset A$ . Since  $L$  is  $\Theta$ -invariant, so is  $L^{\text{nc}}$ , and hence  $L^{\text{nc}} \in \mathcal{H}^*$ . Therefore  $H := N_G(L^{\text{nc}})^\circ \in \mathcal{H}$ . On the other hand, the normalizer of a reductive subgroup  $R < G$  is commensurable with  $RC_G(R)$ . Hence  $H$  is commensurable with  $L^{\text{nc}}C_G(L^{\text{nc}})$ . Since  $C_G(L^{\text{nc}})$  is compact, we have  $H^{\text{nc}} = L^{\text{nc}}$ . Since  $L$  and  $H$  are  $\Theta$ -invariant, the compact subgroup  $C_G(L^{\text{nc}})$  is also  $\Theta$ -invariant. This implies that  $C_G(L^{\text{nc}}) < K$ ; otherwise,  $\text{Lie}(C_G(L^{\text{nc}})) \cap \mathfrak{p}$  is non-zero and this is a contradiction since the image of any non-zero subspace of  $\mathfrak{p}$  is unbounded under the exponential map. It follows that  $H.o = L^{\text{nc}}.o = L.o$ .  $\square$

**Lemma 2.5.** *Let  $H < G$  be a  $\Theta$ -invariant closed subgroup such that  $N_G(H) \cap \exp \mathfrak{p} \subset H$ . If  $gHg^{-1}$  is  $\Theta$ -invariant for some  $g \in G$ , then there exists  $h \in H$  such that*

$$k := gh^{-1} \in K \quad \text{and} \quad gHg^{-1} = kHk^{-1}.$$

*Proof.* Let  $\mathfrak{h}$  be the Lie algebra of  $H$ . Since  $H$  is  $\Theta$ -invariant, we have

$$H = (K \cap H) \exp(\mathfrak{p} \cap \mathfrak{h}).$$

Since  $\Theta(g)H\Theta(g)^{-1} = gHg^{-1}$ , we have  $n_g := g^{-1}\Theta(g) \in N_G(H)$ . Since  $\Theta(n_g) = n_g^{-1}$ , we have  $n_g \in \exp \mathfrak{p}$ . By the hypothesis  $N_G(H) \cap \exp \mathfrak{p} \subset H$ , we have  $n_g \in H$ . Writing  $g = k \exp(y/2)$  for  $k \in K$  and  $y \in \mathfrak{p}$ , we have  $n_g = \exp(-y) \in H$ . It follows that  $y \in \mathfrak{h}$ , and hence  $h := \exp(y/2) \in H$ . Since  $k = gh^{-1}$ , this proves the claim.  $\square$

**Lemma 2.6.** *For any  $H \in \mathcal{H}^* \cup \mathcal{H}$ , we have  $N_G(H) \cap \exp \mathfrak{p} \subset H$ .*

*Proof.* Let  $H \in \mathcal{H}^* \cup \mathcal{H}$ . Set  $H' := N_G(H)$  and  $\mathfrak{h}' = \text{Lie}(H')$ . Since  $H$  is  $\Theta$ -invariant, so is  $H'$ . Hence  $H'$  admits the Cartan decomposition  $H' = (K \cap H') \exp(\mathfrak{p} \cap \mathfrak{h}')$ , and by the uniqueness of the Cartan decomposition,

$$H' \cap \exp(\mathfrak{p}) = \exp(\mathfrak{p} \cap \mathfrak{h}').$$

Now for  $H \in \mathcal{H}^* \cup \mathcal{H}$ ,  $C_G(H)$  is a subgroup of  $K$  and hence its Lie algebra is contained in  $\mathfrak{k}$ . Therefore

$$\mathfrak{p} \cap \mathfrak{h}' = \mathfrak{p} \cap (\mathfrak{h} \oplus \text{Lie}(C_G(H))) = \mathfrak{p} \cap \mathfrak{h}.$$

Thus

$$H' \cap \exp(\mathfrak{p}) = \exp(\mathfrak{p} \cap \mathfrak{h}) \subset H.$$

$\square$

**Lemma 2.7.** *Let  $L$  be a connected reductive subgroup of  $G$  containing some  $H \in \mathcal{H}^*$ . Then  $L$  is  $\Theta$ -invariant.*

*Proof.* By [22], there exists  $g \in G$  such that both  $gHg^{-1}$  and  $gLg^{-1}$  are  $\Theta$ -invariant. By applying Lemmas 2.5 and 2.6 to  $H$  and  $gHg^{-1}$ , we obtain  $h \in H$  such that  $k := gh^{-1} \in K$  and  $gHg^{-1} = kHk^{-1}$ . Since  $g = kh \in kL$ , we have  $gLg^{-1} = kLk^{-1}$ . Hence  $kLk^{-1}$  is  $\Theta$ -invariant. This implies that  $L$  is  $\Theta$ -invariant.  $\square$

**Unimodular connected subgroups of  $G$ .** A closed subgroup of  $G$  is unimodular if its left and right Haar measures coincide. A closed subgroup admitting a lattice is a unimodular subgroup [28]. We will describe all unimodular closed subgroups of  $G$  using the following lemma.

**Lemma 2.8.** *Let  $V < N$  be a non-trivial subgroup.*

- (1) *If  $g \in G$  satisfies  $gVg^{-1} \cap V \neq \{e\}$ , then  $g \in P$ .*
- (2) *The normalizer of  $V$  in  $G$  is contained in  $P$ .*

*Proof.* By the Bruhat decomposition in rank one, we have  $G = P \cup PwP$  where  $w$  is a Weyl element such that  $wPw^{-1} = P^+$ . Suppose that for some  $g = p_1wp_2 \in PwP$ , the intersection  $gVg^{-1} \cap V$  is non-trivial. Since  $p \in P$  and  $V < P$ , this implies that  $wPw^{-1} \cap P$  contains a nontrivial unipotent element. This is a contradiction, since  $wPw^{-1} \cap P$  is equal to the centralizer of  $A$  and has no unipotent element in the rank one setting. This proves (1). (2) follows from (1).  $\square$

**Lemma 2.9.** *Any connected closed unimodular subgroup  $L$  of  $G$  is either reductive or of the form  $L = QV$  where  $V$  is the unipotent radical of  $L$  and  $Q$  is a compact subgroup.*

*Proof.* Suppose  $L$  is not reductive, i.e., its unipotent radical  $V$  is not trivial. We have a Levi decomposition  $L = QV$  where  $Q$  is a reductive subgroup. Since  $Q$  normalizes  $V$ , it is contained in a parabolic subgroup by Lemma 2.8. This implies  $Q^{\text{nc}}$  should be trivial, since a parabolic subgroup in rank one cannot contain a noncompact simple Lie subgroup. Hence  $Q$  is an almost direct product of a torus and compact simple Lie group. Since  $L$  is unimodular,  $Q$  cannot have an  $\mathbb{R}$ -split torus, as its conjugation action on  $V$  would yield a non-trivial modular character of  $L$ . Therefore  $Q$  must be compact.  $\square$

### 3. TOTALLY GEODESIC SUBMANIFOLDS

In this section, we characterize the totally geodesic subspaces of the symmetric space  $X = G/K$  and describe their stabilizers in  $G$ . These subspaces correspond precisely to the orbits of subgroups belonging to the families  $\mathcal{H}$  and  $\mathcal{H}^*$  introduced in Section 2. We also explain how they descend to immersed totally geodesic submanifolds in the quotient manifold  $\mathcal{M} = \Gamma \backslash X$ . Recall the basepoint  $o = [K] \in X$ .

A totally geodesic subspace of  $X$  of dimension one is a complete geodesic, and it is of the form  $gA.o$  for some  $g \in G$ . Those of dimension at least two can be characterized as follows:

**Lemma 3.1.** *For a connected smooth submanifold  $Y$  of  $X$  of dimension at least two, the following are equivalent to each other:*

- (1)  $Y$  is totally geodesic;
- (2)  $Y = gH.o$  for some  $g \in G$  and  $H \in \mathcal{H}$ ;
- (3)  $Y = gH.o$  for some  $g \in G$  and  $H \in \mathcal{H}^*$ .

*Proof.* That a subspace  $Y$  as in (2) and (3) is totally geodesic follows from [15] (see also [11, 2.6]). Now suppose that  $Y$  is a totally geodesic subspace of  $X$  of dimension at least two. By translating by an element of  $G$ , we may assume that  $Y$  passes through  $o \in X$ . Then there exists a subspace  $\mathfrak{p}^*$  of  $\mathfrak{p}$  such that  $Y = \exp \mathfrak{p}^*.o$  and  $[[\mathfrak{p}^*, \mathfrak{p}^*], \mathfrak{p}^*] \subset \mathfrak{p}^*$  by [11, Proposition 2.6.1]. Moreover,  $\mathfrak{k}^* := [\mathfrak{p}^*, \mathfrak{p}^*]$  is a subalgebra of  $\mathfrak{k}$  and

$$\mathfrak{g}^* := \mathfrak{k}^* + \mathfrak{p}^*$$

is a  $\theta$ -invariant reductive subalgebra of  $\mathfrak{g}$ . Let  $L$  denote the connected reductive subgroup of  $G$  with Lie algebra  $\mathfrak{g}^*$ . Then  $L$  is  $\Theta$ -invariant and  $Y = L.o \simeq L/L \cap K$ . Since  $G$  has rank one and  $L = C_L(L)[L, L]$ , we must have  $L^{\text{nc}} \neq \{e\}$ ; otherwise  $[L, L] \subset K$  and hence  $Y = C_L(L).o$  would be at most one-dimensional, contradicting the hypothesis on  $Y$ . Moreover, because  $L$  is  $\Theta$ -invariant, there exists  $k \in K$  such that  $Q := k^{-1}Lk$  contains  $A$ . Then

$$Y = kQ.o \quad \text{and} \quad H := N_G(Q^{\text{nc}})^\circ \in \mathcal{H}.$$

By Lemma 2.4, we have  $Q.o = H.o = H^{\text{nc}}.o$ . This completes the proof.  $\square$

**Lemma 3.2.** (1) *If  $Y = gH.o$  for some  $H \in \mathcal{H} \cup \{A\}$ , then  $G_Y^\circ = g N_G(H)^\circ g^{-1}$ . In particular, if  $H \in \mathcal{H}$ , then  $G_Y^\circ = gHg^{-1}$ .*  
 (2) *If  $G_{Y_1}^\circ = G_{Y_2}^\circ$  for totally geodesic subspaces  $Y_1, Y_2$  of  $X$ , then  $Y_1 = Y_2$ .*

*Proof.* Without loss of generality, we may assume  $g = e$ . Since  $H$  is  $\Theta$ -invariant, we have the decomposition  $\mathfrak{h} = \mathfrak{k}^* \oplus \mathfrak{p}^*$  where  $\mathfrak{k}^* = \mathfrak{h} \cap \mathfrak{k}$  and  $\mathfrak{p}^* = \mathfrak{h} \cap \mathfrak{p}$ . Suppose  $g'Y = Y$  for some  $g' \in G$ . Then  $g'.o \in H.o$  and hence  $g' = hk$  for some  $h \in H$  and  $k \in K$ . So  $g'Hg'^{-1} = h(kHk^{-1})h^{-1}$ . Since  $kH.o = h^{-1}g'H.o = H.o$ , we have  $kHk^{-1}.o = H.o$ . Since  $k.o = o$ , this implies  $\text{Ad}(k)\mathfrak{p}^* = \mathfrak{p}^*$ . Since  $\mathfrak{k}^* = [\mathfrak{p}^*, \mathfrak{p}^*]$ , it follows that  $\text{Ad}(k)\mathfrak{h} = \mathfrak{h}$ , and hence  $k \in N_K(H)$ . Thus

$$G_Y = H N_K(H).$$

Hence  $G_Y^\circ \subset N_G(H)^\circ$ . Since  $N_G(H)$  is commensurable with  $H C_G(H)$  and  $C_G(H) = C_K(H)$  by the  $\Theta$ -invariance of  $H$  (see the proof of Lemma 2.4), we have  $G_Y^\circ \supset N_G(H)^\circ$ , proving (1).

To prove (2), suppose that  $G_{Y_1}^\circ = G_{Y_2}^\circ$ . Write  $Y_i = g_i H_i.o$  for some  $H_i \in \mathcal{H} \cup \{A\}$  and  $g_i \in G$ . By (1), we have  $g_1 H_1 g_1^{-1} = g_2 H_2 g_2^{-1}$ . If  $H_1 = A$ , then  $H_2 = A$ , and  $g_2^{-1} g_1 \in N_G(A) = A N_K(A)$ . Hence  $g_1 A.o = g_2 A.o$ . Now suppose  $H_1, H_2 \in \mathcal{H}$ . By Lemmas 2.5 and 2.6, we have  $g_1 = g_2 k h$  for some  $k \in K$  and  $h \in H_1$ , implying  $k H_1 k^{-1} = H_2$ . Hence  $Y_1 = g_2 k H_1 k^{-1}.o = g_2 H_2.o = Y_2$ .  $\square$

A totally geodesic subspace  $Y \subset X$  of dimension at least two equipped with the metric induced from  $(X, d)$ , is itself a rank-one symmetric space of non-compact type. Moreover, its boundary at infinity  $\partial_\infty Y$  can be naturally identified with a subset of  $\partial_\infty X$ . Indeed, if  $Y = gH.o$  for  $H \in \mathcal{H}$ , then, since  $A \subset H$ , we have

$$\partial_\infty Y = gHP/P \subset \partial_\infty X = G/P.$$

**Lemma 3.3.** *We have  $\text{hull}(\partial_\infty Y) = Y$ .*

*Proof.* We use the fact that between any two distinct points of  $\partial_\infty X$ , there is a unique geodesic with those endpoints. Let  $\ell \subset X$  be a geodesic with both end points in  $\partial_\infty Y$ . Since  $Y$  itself is a rank-one symmetric space, inside  $Y$ , there exists a unique geodesic  $\ell_Y$  with the same points. By the uniqueness,  $\ell = \ell_Y \subset Y$ , proving the inclusion  $\text{hull}(\partial_\infty Y) \subset Y$ . Conversely, for any  $y \in Y$ , there exists a complete geodesic through  $y$  with end points in  $\partial_\infty Y$ . Hence  $Y \subset \text{hull}(\partial_\infty Y)$ , and the equality follows.  $\square$

**Properly immersed totally geodesic submanifolds.** Let  $\Gamma < G$  be a discrete subgroup. Throughout the paper, we assume that all discrete subgroups are torsion-free. Consider the associated locally symmetric space

$$\mathcal{M} = \Gamma \backslash X.$$

Let  $p : X \rightarrow \Gamma \backslash X$  denote the quotient map. For a totally geodesic subspace  $Y$  of  $X$ , the restriction  $p|_Y : Y \rightarrow \Gamma \backslash X$  factors through the covering map  $Y \rightarrow (G_Y \cap \Gamma) \backslash Y$ , and hence induces an immersion

$$\iota : (G_Y \cap \Gamma) \backslash Y \rightarrow \mathcal{M}.$$

Its image  $\mathcal{N} = \Gamma \backslash \Gamma Y$  is a totally geodesic (immersed) submanifold of  $\mathcal{M}$ . If the map  $\iota$  is proper, then  $\mathcal{N}$  is a *properly immersed totally geodesic submanifold* of  $\mathcal{M}$ .

We will need the following general result<sup>3</sup>.

**Lemma 3.4.** *Let  $G$  be a locally compact, second countable topological group and  $H < G$  a closed subgroup. Let  $\Gamma < G$  a closed subgroup. Then the canonical projection  $(H \cap \Gamma) \backslash H \rightarrow \Gamma \backslash G$  is proper if and only if  $\Gamma H$  is closed in  $G$ .*

*Proof.* Suppose that  $\Gamma H$  is closed in  $G$ . Let  $x_0 = [e] \in \Gamma \backslash G$  and  $Z = x_0 H$ . Then  $Z$  is closed in  $\Gamma \backslash G$ ; in particular,  $Z$  is locally compact and second countable and  $H$  acts transitively on  $Z$ . We first claim, by Baire's category theorem, that the map  $h \mapsto x_0 h$  from  $H$  to  $Z$  is open. Indeed, let  $\Omega$  be a neighborhood of  $e$  in  $H$  and fix  $h \in H$ . We wish to show that  $x_0 h \Omega$  contains a neighborhood of  $xh$ . Choose an open neighborhood  $\Omega_1$  of  $e$  with compact closure and the closure of  $\Omega_1 \Omega_1^{-1}$  is contained in  $\Omega$ . There exists a sequence  $\{g_i\} \subset H$  such that  $H = h \cup (\cup_{i=1}^\infty \Omega_1) g_i$ . Hence  $Z = \cup_{i=1}^\infty x_0 h \overline{\Omega}_1$ . By Baire's

<sup>3</sup>The proof of the lemma appears in the arXiv version of [27], but was omitted from the published version; we reproduce it here for the reader's convenience.

category theorem, one of the sets  $x_0 h \overline{\Omega}_1 g_i$  has nonempty interior, say around  $x_0 h \omega g_i$  for some  $\omega \in \overline{\Omega}_1$ . Thus  $x_0 h \overline{\Omega}_1 g_i (g_i^{-1} \omega^{-1})$  contains a neighborhood of  $x_0 h$ , proving that the map  $h \mapsto x_0 h$  is open. Therefore the map

$$(H \cap \Gamma) \backslash H \rightarrow Z, \quad (H \cap \Gamma) h \mapsto x_0 h$$

is continuous, open, and one to one, hence a homeomorphism.

Now let  $C \subset \Gamma \backslash G$  be compact. Then its inverse image in  $(H \cap \Gamma) \backslash H$  coincides with the inverse image of  $C \cap Z$  in  $(H \cap \Gamma) \backslash H$ , which is compact by the above homeomorphism. This proves that the projection map is proper. The converse is straightforward.  $\square$

**Lemma 3.5.** *Let  $\Gamma < G$  be a discrete subgroup and let  $\mathcal{N} = xH.o$  be a totally geodesic submanifold for some  $x \in \Gamma \backslash G$  and  $H \in \mathcal{H}$ . Then the following are equivalent:*

- (1)  $\mathcal{N}$  is properly immersed in  $\mathcal{M}$ ;
- (2)  $\mathcal{N}$  is closed in  $\mathcal{M}$ ;
- (3)  $xH$  is closed in  $\Gamma \backslash G$ .

*Proof.* We may assume without loss of generality that  $x = [e]$ , so that  $\mathcal{N} = \Gamma \backslash \Gamma Y$  where  $Y = H.o$ . We first show that (3)  $\Rightarrow$  (1). By hypothesis, the image of  $(\Gamma \cap H) \backslash H \rightarrow \Gamma \backslash G$  is closed. Hence by Lemma 3.4, the inclusion map  $j : (\Gamma \cap H) \backslash H \rightarrow \Gamma \backslash G$  is proper. Since  $p : \Gamma \backslash G \rightarrow \mathcal{M} = \Gamma \backslash G.o$ ,  $x \mapsto x.o$  is a proper map, the composition  $p \circ j : (\Gamma \cap H) \backslash H \rightarrow \mathcal{M}$  is proper. Since  $p \circ j$  factors through the map  $j' : (\Gamma \cap H) \backslash Y \rightarrow \mathcal{M}$  induced by the inclusion  $Y \subset X$ ,  $j'$  is a proper map. Since  $j'$  factors through the map  $\iota : (\Gamma \cap G_Y) \backslash Y \rightarrow \mathcal{M}$ , this implies that  $\iota$  is proper;  $\mathcal{N}$  is properly immersed, as desired. The implication (1)  $\Rightarrow$  (2) is immediate.

We now show (2)  $\Rightarrow$  (3). Since  $[G_Y : H] < \infty$  by Lemma 3.2, it suffices to show that  $\Gamma G_Y$  is closed in  $G$ . Consider the subspace  $\mathcal{S} = \{gY : g \in G\} = GY$  with the topology given by the identification  $\mathcal{S} \simeq G/G_Y$ . Suppose that  $\mathcal{N}$  is closed in  $\mathcal{M}$ , that is,  $\Gamma Y$  is closed in  $X$ . This implies  $\Gamma Y$  is closed in  $\mathcal{S}$ ; if  $\gamma_i Y \rightarrow gY$ , then, since  $\Gamma Y$  is closed in  $X$ , we have  $gY \subset \Gamma Y = \cup_{\gamma \in \Gamma} \gamma Y$  as a subset of  $X$ . This implies that  $gY = \gamma Y$  for some  $\gamma \in \Gamma$ ; otherwise,  $gY \cap \gamma Y$  is a nowhere dense subset of  $gY$  for all  $\gamma \in \Gamma$ , contradicting the Baire Category theorem.

Now to show that  $\Gamma G_Y$  is closed in  $G$ , suppose that  $\gamma_i h_i$  converges to  $g \in G$  for some sequences  $\gamma_i \in \Gamma$  and  $h_i \in G_Y$ . This implies that  $\gamma_i Y$  converges to  $gY$  in  $\mathcal{S}$ . By the closedness of  $\Gamma Y$  in  $\mathcal{S}$ , we then have  $gY = \gamma Y$  for some  $\gamma \in \Gamma$ . Therefore  $g = \gamma h$  for some  $h \in G_Y$ . This implies that  $\Gamma G_Y$  is closed in  $G$ , completing the proof.  $\square$

In Lemma 3.5, it is important that  $H$  is chosen from the collection  $\mathcal{H}$  so that  $H$  is equal to the connected component of its normalizer, since  $xH^{\text{nc}}$  may not be closed even if  $xH^{\text{nc}}.o = xH.o$ .

#### 4. TOTALLY GEODESIC SUBMANIFOLDS CONTAINED IN THE CONVEX CORE

Let  $\Gamma < G$  be a torsion-free discrete subgroup, and consider the associated locally symmetric space  $\mathcal{M} = \Gamma \backslash X$ . In this section, we focus on totally geodesic submanifolds contained in the convex core of  $\mathcal{M}$ .

We begin by recalling several basic definitions. The *limit set* of  $\Gamma$ , denoted by  $\Lambda = \Lambda(\Gamma)$ , is the set of accumulation points of  $\Gamma(o)$  in  $\partial_\infty X$  within the compactification  $X \cup \partial_\infty X$ . We assume that  $\Gamma$  is non-elementary, that is,  $\#\Lambda \geq 3$ . Then  $\Lambda$  is the unique  $\Gamma$ -minimal subset of  $\partial_\infty X$ . The *convex hull* of  $\Lambda$ , denoted by  $\text{hull}(\Lambda)$ , is the smallest convex subset of  $X$  containing all geodesics<sup>4</sup> in  $X$  with endpoints in  $\Lambda$ . The *convex core* of  $\mathcal{M}$  is defined as the quotient manifold

$$\text{core}(\mathcal{M}) = \Gamma \backslash \text{hull}(\Lambda).$$

For  $p \in \mathcal{M}$ , the *injectivity radius* of  $p$  is the supremum of  $r > 0$  such that the ball  $B(\tilde{p}, r)$  injects into  $\mathcal{M}$ , where  $\tilde{p} \in X$  satisfies  $p = \Gamma \backslash \Gamma \tilde{p}$ . We denote it by  $\text{inj}(p)$ . For  $\varepsilon > 0$ , define the  $\varepsilon$ -thin and  $\varepsilon$ -thick parts of  $\mathcal{M}$  by

$$\mathcal{M}_\varepsilon = \{p \in \mathcal{M} : \text{inj}(p) < \varepsilon\} \quad \text{and} \quad \mathcal{M}_{\geq \varepsilon} = \mathcal{M} - \mathcal{M}_\varepsilon.$$

**Horospheres and horoballs.** If  $\xi = gP \in \partial_\infty X$  for  $g \in G$ , then a horosphere of  $X$  based at  $\xi$  is of the form  $gaN.o$  for some  $a \in A$ . Similarly, a horoball of  $X$  based at  $\xi$  is of the form  $gA_{[T, \infty)}N.o$  where  $A_{[T, \infty)} = \{a_t : t \geq T\}$ . A horosphere (resp. horoball) in  $\mathcal{M}$  is then the image of a horosphere (resp. horoball) in  $X$  under the quotient map  $X \rightarrow \Gamma \backslash X$ .

**Geometrically finite manifolds.** We now recall the notion of geometrically finite manifolds.

**Definition 4.1.** *We say that  $\mathcal{M}$ , or equivalently  $\Gamma$ , is geometrically finite, if the unit neighborhood of  $\text{core}(\mathcal{M})$  has finite volume.*

**Theorem 4.2.** [8] *The manifold  $\mathcal{M}$  is geometrically finite if and only if  $\mathcal{M}_{\geq \varepsilon} \cap \text{core}(\mathcal{M})$  is compact for all sufficiently small  $\varepsilon > 0$ .*

A limit point  $\xi \in \Lambda$  is *parabolic* if its stabilizer

$$\Gamma_\xi := \{g \in \Gamma : g\xi = \xi\}$$

contains a parabolic element. We denote by  $\Lambda_p = \Lambda_p(\Gamma)$  the set of all parabolic limit points of  $\Gamma$ .

For geometrically finite  $\mathcal{M}$ , there exists a finite set of  $\Gamma$ -representatives  $\xi_i \in \Lambda_p$  ( $1 \leq i \leq \ell$ ) such that

$$\Lambda_p = \Gamma\xi_1 \cup \cdots \cup \Gamma\xi_\ell. \quad (4.3)$$

Fix  $g_i \in G$  so that  $\xi_i = g_iP \in \Lambda_p$ . For  $T > 0$ , define the horoball based at  $\xi_i$  of depth  $T$  by

$$\tilde{\mathcal{H}}_{\xi_i, T} = g_iNA_{[T, \infty)}.o \subset X \quad \text{and} \quad \mathcal{H}_{\xi_i, T_i} = \Gamma \backslash \Gamma \tilde{\mathcal{H}}_{\xi_i, T_i} \subset \mathcal{M}. \quad (4.4)$$

---

<sup>4</sup>by geodesics, we mean complete geodesics

For all sufficiently small  $\varepsilon > 0$ , there exist  $T_1, \dots, T_\ell > 0$  such that for each  $1 \leq i \leq \ell$ , the collection  $\{\gamma \tilde{\mathcal{H}}_{\xi_i, T_i} : \gamma \in \Gamma\}$  consists of disjoint horoballs, and the following disjoint decomposition holds:

$$\mathcal{M} = \mathcal{M}_{\geq \varepsilon} \cup \mathcal{H}_{\xi_1, T_1} \cup \dots \cup \mathcal{H}_{\xi_\ell, T_\ell}. \quad (4.5)$$

We record a simple observation for later use.

**Lemma 4.6.** *For each  $1 \leq i \leq \ell$ , the horoball  $\mathcal{H}_{\xi_i, T_i}$  contains no complete geodesic.*

*Proof.* Suppose that  $\mathcal{H}_{\xi_i, T_i}$  contains a complete geodesic. By the disjointness property of the horoballs  $\{\gamma \tilde{\mathcal{H}}_{\xi_i, T_i} : \gamma \in \Gamma\}$  some translate  $\gamma \tilde{\mathcal{H}}_{\xi_i, T_i}$  would then contain a complete geodesic of  $X$ . However, every complete geodesic in  $X$  connects two distinct points of  $\partial_\infty X$ , a contradiction.  $\square$

**Totally geodesic submanifolds contained in the convex core.** For each closed subgroup  $H$  of  $G$  with Lie algebra  $\mathfrak{h}$ , the restriction  $\langle \cdot, \cdot \rangle|_{\mathfrak{h}}$  defines an inner product on  $\mathfrak{h}$ . Left translation within  $H$  produces a left-invariant sub-Riemannian metric on  $H$ , which in turn determines a unique left-invariant volume form  $\text{vol}_H$ , or equivalently a left-invariant Haar measure  $\mu_H$ , on  $H$ . If  $H$  is unimodular,  $\mu_H$  is also right  $H$ -invariant.

For an orbit  $xH \subset \Gamma \backslash G$  with  $H \in \mathcal{H}$ , the measure  $\mu_H$  induces a unique right  $H$ -invariant measure on  $\text{Stab}_H(x) \backslash H$ , which we again denote by  $\mu_H$  by abuse of notation. We write

$$\text{vol}(xH) = \mu_H(\text{Stab}_H(x) \backslash H)$$

and set

$$\text{vol}(xH.o) = \frac{\text{vol}(xH)}{\mu_{H \cap K}(H \cap K)}.$$

We define the set<sup>5</sup>

$$\text{RFM} = \{[g] \in \Gamma \backslash G : gP^+, gP^- \in \Lambda\}. \quad (4.7)$$

For  $g \in G$ , the points  $gP^+$  and  $gP^-$  are precisely the two endpoints of the geodesic  $gA.o$ . Note that if one of the end points of  $gA.o$  lies outside the limit set,  $gA.o$  is unbounded. Therefore, any bounded  $A$ -orbit is contained in  $\text{RFM}$ ;

Note that  $\text{RFM}$  is right-invariant under the subgroup  $P^+ \cap P^- = C_G(A)$  and the centralizer  $C_G(A)$  is equal to  $AC_K(A)$ . It was shown in [19] that the periodic  $AC_K(A)$  orbits are equidistributed in  $\text{RFM}$  with respect to the Bowen-Margulis-Sullivan measure; in particular, they are dense in  $\text{RFM}$ . Since such periodic orbits are bounded, their closure is contained in the smallest closed subset of  $\Gamma \backslash G$  containing all bounded  $A$ -orbits, while the

<sup>5</sup>The notation  $\text{RFM}$  was introduced in [20] for the case  $G = \text{SO}(3, 1)^\circ$ , where it denotes the *renormalized frame bundle*. When  $G = \text{SO}(n, 1)^\circ$ , the quotient  $\Gamma \backslash G$  can be identified with the frame bundle of  $\mathcal{M} = \Gamma \backslash X$ . This identification does not extend to general rank-one spaces, so the term "frame bundle" is slightly abusive. Nevertheless, we retain the notation for consistency.

previous paragraph shows that any bounded  $A$ -orbit lies in  $\text{RF}\mathcal{M}$ . Therefore we obtain:

**Proposition 4.8.** *For geometrically finite  $\mathcal{M}$ , the set  $\text{RF}\mathcal{M}$  coincides with the smallest closed subset of  $\Gamma \backslash G$  containing all bounded  $A$ -orbits in  $\Gamma \backslash G$ .*

The image of  $\text{RF}\mathcal{M}$  under the basepoint projection  $\Gamma \backslash G \rightarrow \mathcal{M}$  is contained in  $\text{core}(\mathcal{M})$ , though the two sets need not coincide in general.

The next lemma characterizes when a totally geodesic submanifold of  $\mathcal{M}$  lies in the convex core in terms of its associated  $H$ -orbit in  $\text{RF}\mathcal{M}$ .

**Lemma 4.9.** *Let  $\mathcal{N}$  be a totally geodesic submanifold of  $\mathcal{M}$  of dimension at least two. Let  $\mathcal{N} = xH.o$  for some  $H \in \mathcal{H}$ . Then*

$$(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$$

where

- (1)  $\mu_H(xH) < \infty$ , equivalently,  $\text{vol}(\mathcal{N}) < \infty$ .
- (2)  $\mathcal{N} \subset \text{core}(\mathcal{M})$ .
- (3)  $xH \subset \text{RF}\mathcal{M}$ .
- (4)  $xH^{\text{nc}} \subset \text{RF}\mathcal{M}$ .

*Proof.* Let  $g \in G$  represent  $x = \Gamma g$  and let  $\pi : \Gamma \backslash G \rightarrow \Gamma \backslash X$  denote the natural projection  $x \mapsto x.o$ . The implication (3)  $\Rightarrow$  (2) holds since  $\pi(\text{RF}\mathcal{M}) \subset \text{core}(\mathcal{M})$ . To prove (2)  $\Rightarrow$  (3), suppose  $xH.o \subset \text{core}(\mathcal{M})$ . Then  $gH.o \subset \text{hull}(\Lambda)$ .

For  $h \in H$ , consider

$$(gh)^\pm := ghP^\pm \in \partial_\infty X.$$

The points  $(gh)^\pm$  are represented by the geodesic rays  $\{gha_t.o : t \geq 0\}$  and  $\{gha_t.o : t \leq 0\}$  respectively. Recall the fact that  $\overline{\text{hull}(\Lambda)} \cap \partial_\infty X$  coincides with  $\Lambda$ , where  $\overline{\text{hull}(\Lambda)}$  denotes the closure in the compactification  $X \cup \partial_\infty X$ . Since  $ghA.o \subset \text{hull}(\Lambda)$ , it follows that  $(gh)^\pm \in \Lambda$ , that is,  $gh \in \text{RF}\mathcal{M}$ , proving (2)  $\Leftrightarrow$  (3). Since  $xH^{\text{nc}}.o = \mathcal{N}$ , this also implies that (2)  $\Leftrightarrow$  (4).

We now prove (1)  $\Rightarrow$  (3). Suppose that the orbit  $xH$  is not contained in  $\text{RF}\mathcal{M}$ . By replacing  $x$  by  $xh$  for some  $h \in H$ , we may assume that one of  $gP^+$  or  $gP^-$  does not belong to the limit set. Suppose  $\xi = gP^- \notin \Lambda$ . Since  $\Gamma$  acts properly discontinuously on  $\partial_\infty X - \Lambda$ , there exists a neighborhood  $\mathcal{O}$  of  $\xi$  in the compactification  $X \cup \partial_\infty X$  such that  $\gamma\mathcal{O} \cap \mathcal{O} = \emptyset$  for all  $\gamma \in \Gamma - \{e\}$ . Consider the geodesic ray  $t \mapsto ga_t.o$ ,  $t \geq 0$ . For all sufficiently large  $t$ , we have  $ga_t.o \in \mathcal{O}$ , and the injectivity radius  $r_t := \text{inj}_{\Gamma \backslash G}(xa_t)$  tends to  $\infty$  as  $t \rightarrow \infty$ . Since  $A \subset H$  and the ball  $B_{r_t}^H = \{h \in H : d(e, h) < r_t\}$  injects to  $\text{Stab}_H(x) \backslash H$ , we have

$$\mu_H(xH) \geq \limsup_{t \rightarrow \infty} \mu_H(B_{r_t}^H) = \mu(H) = \infty.$$

Hence  $\mu_H(xH) = \infty$ . The case  $\xi = gP^+ \notin \Lambda$  can be proved similarly by sending  $t \rightarrow -\infty$ .  $\square$

We also observe:

**Lemma 4.10.** *Let  $\mathcal{N}$  be as in Lemma 4.9. If  $\mathcal{N}$  is bounded, then  $\mathcal{N} \subset \text{core}(\mathcal{M})$ .*

*Proof.* Suppose that  $\mathcal{N} = xH.o$  is bounded, or equivalently,  $xH$  is bounded. For any  $h \in H$ , since  $xhA \subset xH$  is bounded, we have  $xh \in \text{RFM}$ . Hence  $xH \subset \text{RFM}$ , which implies that  $\mathcal{N} \subset \text{core}(\mathcal{M})$ .  $\square$

**Remark 4.11.** We remark that an analogue of Lemma 4.9 does not hold for horospheres. There are maximal horospheres  $xN.o$  contained in the convex core of  $\mathcal{M}$  but  $xNk$  does not need to be contained in  $\text{RFM}$  for any  $k \in K$ . For example, consider a Kleinian group  $\Gamma < \text{PSL}_2(\mathbb{C})$  such that  $\infty$ , the fixed point of  $N$ , is a rank two parabolic limit point of  $\Gamma$  and  $\Lambda \neq \hat{\mathbb{C}}$ . Then  $[e]a_tNk \subset \Gamma \backslash G$  is not contained in  $\text{RFM}$  for any diagonal  $a_t \in A$  and  $k \in K$ . But for all sufficiently large  $t$ ,  $[e]a_tN.o$  is contained in the convex core of  $\Gamma \backslash \mathbb{H}^3$ .

## 5. ORBIT CLOSURE CLASSIFICATION INSIDE $\text{RFM}$

In this section, we classify the closures of orbits of connected closed subgroups generated by unipotent elements contained in the renormalized frame bundle  $\text{RFM}$ . This orbit-closure theorem forms the dynamical backbone of the rigidity and finiteness results proved later. Its proof relies on Ratner's measure classification theorem and the avoidance theorem of Dani-Margulis, adapted to the geometrically finite, possibly infinite-volume, setting.

**Theorem 5.1.** *Let  $\Gamma < G$  be geometrically finite and  $W < G$  a connected closed subgroup generated by one-parameter unipotent subgroups. Suppose that  $xW \subset \text{RFM}$  for  $x \in \Gamma \backslash G$ . Then there exists a connected Lie subgroup  $L$  containing  $W$  such that*

$$\overline{xW} = xL$$

*and  $\text{Stab}_L(x)$  is a lattice in  $L$ . Moreover,  $L$  is either reductive or a compact extension of a connected unipotent subgroup (as given by Lemma 2.9).*

**Three theorems for general  $\Gamma$  and  $G$ .** The rest of the section is devoted to proving Theorem 5.1. We begin by recalling several foundational results on unipotent dynamics:

**Theorem 5.2** (Ratner [29]). *Let  $G$  be a connected linear Lie group and  $\Gamma < G$  be a discrete subgroup. Let  $U$  be a one-parameter unipotent subgroup of  $G$ . Any  $U$ -invariant, ergodic probability measure  $\nu$  on  $\Gamma \backslash G$  is an  $L$ -invariant measure supported on a closed orbit  $xL \subset \Gamma \backslash G$  for some  $x \in \Gamma \backslash G$  and some connected closed subgroup  $L < G$  containing  $U$ .*

This fundamental result describes all ergodic invariant probability measures for one-parameter unipotent flows. The next theorem, due to Ratner as well, asserts that for a fixed base point, only countably many homogeneous subspaces can arise as supports of such invariant measures.

**Theorem 5.3** (Ratner [31, Theorem 5]). *Let  $G$  be a connected linear Lie group and  $\Gamma < G$  be a discrete subgroup. Let  $x \in \Gamma \backslash G$ . Let  $\mathcal{A}_x$  denote the set of all closed connected subgroups  $L < G$  such that  $xL$  is closed and has an  $L$ -invariant probability measure  $\mu_L$  and there is a one-parameter unipotent subgroup  $U < L$  acting ergodically on  $(xL, \mu_L)$ . Then  $\mathcal{A}_x$  is countable.*

To formulate the avoidance theorem of Dani–Margulis, we introduce the notion of the singular set associated to a subgroup.

**Definition 5.4.** *For a connected closed subgroup  $W < G$ , define the singular set with respect to  $W$ :*

$$\mathcal{S}(W) = \left\{ x \in \Gamma \backslash G : \begin{array}{l} \text{there exists a closed connected subgroup } L \subsetneq G \text{ s.t.} \\ W < L \text{ and } xL \text{ admits a finite } L\text{-invariant measure} \end{array} \right\}.$$

When an orbit  $xL \subset \Gamma \backslash G$  admits a finite  $L$ -invariant measure,  $xL$  is automatically a closed subset of  $\Gamma \backslash G$  [28, Theorem 1.13].

The avoidance theorem of Dani–Margulis asserts that the orbit of a generic point under a one-parameter unipotent subgroup spends a uniformly small proportion of time near the singular set. This quantitative form of avoidance will later ensure that the measures obtained from orbit averages do not concentrate on lower-dimensional homogeneous subsets.

**Theorem 5.5** (Dani–Margulis [10, Theorem 1]). *Let  $G$  be a connected linear Lie group and  $\Gamma < G$  be a discrete subgroup. Let  $W$  be a connected closed subgroup of  $G$  which is generated by unipotent elements in it. Let  $\mathcal{K} \subset \Gamma \backslash G$  be a compact subset disjoint from  $\mathcal{S}(W)$ . For any  $\varepsilon > 0$ , there exists a neighborhood  $\Omega$  of  $\mathcal{S}(W)$  such that for any one-parameter unipotent subgroup  $U = \{u_s : s \in \mathbb{R}\}$  of  $G$ , any  $x \in \mathcal{K}$ , and any  $T > 0$ ,*

$$\text{Leb}\{s \in [0, T] : xu_s \in \Omega\} \leq \varepsilon T.$$

where  $\text{Leb}$  denotes the Lebesgue measure of  $\mathbb{R}$ .

**Orbit closures inside RFM for geometrically finite  $\mathcal{M}$ .** We now return to the setting of a geometrically finite manifold  $\mathcal{M} = \Gamma \backslash X$ . Denote by

$$\pi : \Gamma \backslash G \rightarrow \mathcal{M}$$

the projection  $x \mapsto x.o$ . To apply the general results recalled above, we need a quantitative non-divergence estimate for unipotent flows: unipotent orbits spend most of its time in the thick part of  $\mathcal{M}$ .

**Theorem 5.6.** ([6, Corollary 5.5], [9, Theorem 1.1]) *Let  $\Gamma < G$  be a discrete subgroup. For any  $\varepsilon > 0$  and a compact set  $\mathcal{K} \subset \Gamma \backslash G$ , there exists a constant  $\eta > 0$  depending on  $\mathcal{K}$  such that for any one-parameter unipotent subgroup  $U = \{u_s : s \in \mathbb{R}\}$ , any  $x \in \mathcal{K}$  and any  $T > 0$ ,*

$$\text{Leb}\{s \in [0, T] : \pi(xu_s) \in \mathcal{M}_\eta\} \leq \varepsilon T$$

where  $\mathcal{M}_\eta$  denotes the  $\eta$ -thin part of  $\mathcal{M} = \Gamma \backslash X$  as in Section 4.

The following proposition is the key step where the geometric finiteness assumption plays a decisive role. It guarantees that, within the renormalized frame bundle  $\text{RFM}$ , any limiting measure arising from time-averages of unipotent orbits remains supported inside  $\text{RFM}$ .

Denote by  $\mathcal{P}(\Gamma \backslash G \cup \{\infty\})$  the space of probability measures on the one-point compactification of  $\Gamma \backslash G$ , equipped with the weak-\* topology. This is a compact metrizable space.

**Proposition 5.7** (Non-divergence within  $\text{RFM}$ ). *Let  $\Gamma < G$  be a geometrically finite subgroup and  $\mathcal{M} = \Gamma \backslash X$ . Let  $\mathcal{K} \subset \Gamma \backslash G$  be a compact subset, and let  $U_i = \{u_{i,s} : s \in \mathbb{R}\}$  be a sequence of one-parameter unipotent subgroups of  $G$ . Let  $x_i \in \mathcal{K}$  be a sequence such that  $x_i U_i \subset \text{RFM}$  for each  $i \geq 1$ . Then for any sequence  $T_i \rightarrow \infty$ , any weak-\* limit of the sequence*

$$\nu_{T_i} := \frac{1}{T_i} \int_0^{T_i} \delta_{x_i u_{i,s}} ds$$

*in  $\mathcal{P}(\Gamma \backslash G \cup \{\infty\})$ , as  $i \rightarrow \infty$ , is a probability measure supported on  $\text{RFM}$ .*

*Proof.* Fix  $\varepsilon > 0$  and a compact subset  $\mathcal{K} \subset \Gamma \backslash G$ . Let  $\eta > 0$  be as in Theorem 5.6. Then for any  $x_i \in \mathcal{K}$  such that  $x_i U_i \subset \text{RFM}$ , we have  $\pi(x_i U_i) \subset \text{core}(\mathcal{M})$ . Hence Theorem 5.6 implies that for all  $i \geq 1$  and  $T > 0$ ,

$$\text{Leb}\{s \in [0, T] : \pi(x_i u_{i,s}) \in \text{core}(\mathcal{M}) - \mathcal{M}_\eta\} \geq (1 - \varepsilon)T. \quad (5.8)$$

Since  $\mathcal{M}$  is geometrically finite, the set  $\text{core}(\mathcal{M}) - \mathcal{M}_\eta$  is compact. Hence  $\mathcal{C} := \{z \in \Gamma \backslash G : \pi(z) \in \text{core}(\mathcal{M}) - \mathcal{M}_\eta\}$  is a compact subset of  $\Gamma \backslash G$ . By (5.8),

$$\nu_{T_i}(\mathcal{C}) \geq (1 - \varepsilon) \quad \text{for all } i.$$

Therefore, for any weak-\* limit  $\nu$  of the sequence  $\nu_{T_i}$ , we have  $\nu(\mathcal{C}) \geq 1 - \varepsilon$ . As  $\varepsilon > 0$  is arbitrary, it follows that  $\nu(\Gamma \backslash G) = 1$ . Since  $\text{supp}(\nu_{T_i}) \subset \text{RFM}$ , we conclude that  $\nu$  is supported on  $\text{RFM}$ .  $\square$

The next proposition is a key application of the avoidance and measure-classification theorems. It shows that, apart from points in the singular set, a unipotent orbit contained in  $\text{RFM}$  is already dense in  $\Gamma \backslash G$  which in turn forces  $\Gamma$  to be a lattice.

**Proposition 5.9.** *Let  $\Gamma < G$  be a geometrically finite subgroup. Let  $U$  be a one-parameter unipotent subgroup of  $G$ . Suppose  $xU \subset \text{RFM}$  for  $x \in \Gamma \backslash G$ . Then*

$$\text{either } x \in \mathcal{S}(U) \quad \text{or} \quad \overline{xU} = \Gamma \backslash G.$$

*In the second case,  $\Gamma$  is a lattice in  $G$ .*

*Proof.* Suppose that  $x \notin \mathcal{S}(U)$ . Let  $U = \{u_s : s \in \mathbb{R}\}$  and assume  $xU \subset \text{RFM}$ . For  $T > 0$ , let

$$\nu_T = \frac{1}{T} \int_0^T \delta_{xu_s} ds$$

be the probability measure on  $\Gamma \backslash G$ . Let  $\nu$  be a weak-\* limit of a sequence  $\nu_i := \nu_{T_i}$  for some  $T_i \rightarrow \infty$  in the one-point compactification of  $\Gamma \backslash G$ . Since  $xU \subset \text{RFM}$ , Proposition 5.7 implies that

$$\nu(\Gamma \backslash G) = 1.$$

On the other hand, since  $x \notin \mathcal{S}(U)$ , Theorem 5.5 implies that for any  $\varepsilon > 0$ , there exists a neighborhood  $\Omega$  of  $\mathcal{S}(U)$  such that for all  $i$ ,

$$\nu_i(\Omega) \leq \varepsilon.$$

As  $\varepsilon > 0$  is arbitrary, it follows that

$$\nu(\mathcal{S}(U)) = 0. \tag{5.10}$$

Since  $\nu$  is a  $U$ -invariant probability measure on  $\Gamma \backslash G$ , it admits a  $U$ -ergodic decomposition

$$\nu = \int_{\alpha \in \Gamma \backslash G} \nu_\alpha d\nu(\alpha)$$

where  $\nu_\alpha$  is a  $U$ -invariant ergodic probability measure (cf. [12]). By Ratner's Theorem 5.2, for  $\nu$ -a.e.  $\alpha$ , the measure  $\nu_\alpha$  is an  $L_\alpha$ -invariant probability measure supported on a closed orbit  $x_\alpha L_\alpha$  for some  $x_\alpha \in \Gamma \backslash G$  and some connected closed subgroup  $L_\alpha < G$  containing  $U$ .

Let  $E$  be the set of all  $\alpha$  such that  $L_\alpha \neq G$ . Then for all  $\alpha \in E$ , we have

$$x_\alpha L_\alpha \subset \mathcal{S}(U).$$

Since

$$\nu(E) = \int_{\alpha \in \Gamma \backslash G} \nu_\alpha(E) d\nu(\alpha) \leq \nu(\mathcal{S}(U)),$$

it follows from (5.10) that  $\nu(E) = 0$ . Therefore

$$L_\alpha = G \quad \text{for } \nu\text{-a.e. } \alpha.$$

Hence  $\nu$  is the  $G$ -invariant measure. Since the support of  $\nu$  is contained in  $\overline{xU}$ , it follows that  $\overline{xU} = \Gamma \backslash G$ . As  $\nu(\Gamma \backslash G) = 1$ , this forces  $\Gamma$  to be a lattice in  $G$ .  $\square$

To complete the argument, we recall the following version of Ratner's orbit-closure theorem for finite-volume homogeneous spaces, which will be used to analyze the structure of orbit closures arising in the previous proposition.

**Theorem 5.11** (Ratner [31, Theorem 4]). *Let  $L$  be a connected linear Lie group and  $\Delta < L$  be a lattice. For any connected closed subgroup  $W < L$  generated by one-parameter unipotent subgroups in it and  $x \in \Delta \backslash L$ , there exists a connected closed subgroup  $L_0$  containing  $W$  such that  $\overline{xW} = xL_0$  and  $\text{Stab}_{L_0}(x)$  is a lattice in  $L_0$ . Moreover there is a one-parameter unipotent subgroup  $V < W$  which acts ergodically on  $(xL_0, \mu_{L_0})$  where  $\mu_{L_0}$  is the  $L_0$ -invariant probability measure on  $xL_0$ .*

With all the necessary dynamical ingredients in place—namely, the avoidance theorem, the non-divergence result within RFM and Ratner’s orbit-closure theorem—we can now prove Theorem 5.1.

*Proof of Theorem 5.1.* Suppose that  $xW \subset \text{RFM}$ . First, consider the case when  $W$  is a one-parameter unipotent subgroup. By Proposition 5.9, either  $\overline{xW} = \Gamma \backslash G$  and  $\text{Vol}(\Gamma \backslash G) < \infty$ , or  $x \in \mathcal{S}(W)$ . Assume now that  $x \in \mathcal{S}(W)$ . Then there exists a minimal connected closed subgroup  $L < G$  containing  $W$  such that  $xL$  is closed and carries an  $L$ -invariant probability measure. We claim that  $\overline{xW} = xL$ . Let  $g \in G$  with  $x = \Gamma g$ . Then  $\text{Stab}_L(x) = L \cap g^{-1}\Gamma g$  is a lattice in  $L$ . The map  $x\ell \mapsto [e]\ell$  defines an  $L$ -equivariant homeomorphism  $xL \simeq (L \cap g^{-1}\Gamma g) \backslash L$  for  $[e]$  denotes the identity coset in  $L \cap g^{-1}\Gamma g \backslash L$ . By Theorem 5.11,

$$\overline{[e]W} = [e]L_0$$

for some connected closed subgroup  $L_0$  containing  $W$  and  $W$  acts ergodically on  $([e]L_0, \mu_{L_0})$ . By the minimality of  $L$ , it follows that  $L = L_0$ , proving that  $\overline{xW} = xL$ .

The general case can be deduced from this following Ratner [31]. For the reader’s convenience, we recall Ratner’s argument, which is slightly simpler in the present rank-one setting. Suppose that  $W$  is a connected unipotent subgroup. Let  $\mathcal{U}$  denote the set of all one-parameter subgroups of  $W$ . By the previous case, for each  $V \in \mathcal{U}$ , there exists a closed connected subgroup  $L(V) < G$  such that  $V \subset L(V)$  and  $\overline{xV} = xL(V)$  and  $V$  acts ergodically on  $(xL(V), \mu_{L(V)})$ . We then have

$$W = \bigcup \{W \cap L(V) : V \in \mathcal{U}\},$$

which is a countable union by Theorem 5.3. By the Baire category theorem, it follows that  $W \subset L(V)$  for some  $V \in \mathcal{U}$ . Hence  $\overline{xW} = xL(V)$ , proving the claim for unipotent  $W$ . Now consider a general case. Since  $G$  has rank one,  $W$  is either unipotent or a simple non-compact closed subgroup. As the former case has already been treated, we may assume that  $W$  is a connected simple non-compact subgroup. Then  $W$  is generated by a pair of opposite maximal unipotent subgroups in it, say,  $V^+$  and  $V^-$ . By applying the previous case to  $xwV^+w^{-1} \subset \text{RFM}$  for each  $w \in V^-$ , and using the countability result (Theorem 5.3), we obtain a closed connected subgroup  $L < G$  and a subset  $S \subset V^-$  of positive Haar measure such that, for all  $w \in S$ ,  $wV^+w^{-1} \subset L$  and

$$\overline{xwV^+w^{-1}} = xL$$

and  $\text{Stab}_x L$  is a lattice in  $L$ . The set of elements  $w \in V^-$  for which the closed group generated by  $V^+$  and  $wV^+w^{-1}$  is a proper algebraic subgroup of  $W$  is contained in a countable union of proper algebraic subsets of  $V^-$ . Since  $S$  has positive Haar measure in  $V^-$ , it follows that  $W \subset L$ , and hence  $\overline{xW} = xL$ . Since  $L$  admits a lattice, it is unimodular. Hence the “moreover” part follows from Lemma 2.9. This completes the proof.  $\square$

**Remark 5.12.** In the proof of Theorem 5.1, the geometric finiteness assumption and the rank one setting were needed only to guarantee that the measures  $\nu_T$  in the proof of Proposition 5.9 admit weak-\* limits that are probability measures in  $\Gamma \backslash G$ . Since Theorems 5.2, 5.5, and 5.11 remain valid for any connected Lie group  $G$ , the same argument applies verbatim and yields:

**Theorem 5.13.** *Let  $G$  be a connected linear Lie group and  $\Gamma < G$  a discrete subgroup. Let  $W < G$  be a connected closed subgroup generated by one-parameter unipotent subgroups. If  $xW$  is bounded for  $x \in \Gamma \backslash G$ , then there exists a connected Lie subgroup  $L$  containing  $W$  such that*

$$\overline{xW} = xL$$

and  $\text{Stab}_L(x)$  is a lattice in  $L$ .

## 6. EQUIDISTRIBUTION IN RFM

In this section, we establish an equidistribution theorem for a sequence maximal  $H_i$ -orbits, with  $H_i \in \mathcal{H}$ , contained in the renormalized frame bundle RFM. Combined with the orbit-closure classification from Section 5, this result provides the key dynamical ingredient in the proof of the finiteness theorem (Theorem 1.3) in Section 7.

We will use the following equidistribution theorem of Mozes-Shah:

**Theorem 6.1** (Mozes-Shah [23, Theorem 1.1]). *Let  $G$  be a connected Lie group and  $\Gamma < G$  be a discrete subgroup. Let  $U_i$  be a sequence of one-parameter unipotent subgroups of  $G$ . Assume that there exists an infinite sequence of  $U_i$ -invariant ergodic probability measures  $\nu_i$  on  $\Gamma \backslash G$  converging to a probability measure  $\nu$  on  $\Gamma \backslash G$  as  $i \rightarrow \infty$  in the weak-\* topology. Let  $x \in \text{supp } \nu$ . Then the following holds:*

- (1)  $\text{supp } \nu = xL$  where  $L = \{g \in G : \nu.g = \nu\}$ <sup>6</sup>;
- (2) Let  $g'_i \rightarrow e$  be a sequence in  $G$  as  $i \rightarrow \infty$  such that for all  $i \in \mathbb{N}$ ,  $xg'_i \in \text{supp } \nu_i$  and  $\{xg'_i U_i\}$  is uniformly distributed<sup>7</sup> with respect to  $\nu_i$ . Then there exists  $i_0 \geq 1$  such that

$$\text{supp } \nu_i \subset (\text{supp } \nu).g'_i \quad \text{for all } i \geq i_0.$$

- (3)  $\nu$  is invariant and ergodic for the action of the subgroup generated by  $\{g'_i U_i g_i'^{-1} : i \geq i_0\}$ .

The Mozes–Shah theorem provides the fundamental tool for analyzing weak-\* limits of invariant probability measures arising from unipotent orbits. To apply it effectively to our setting, we first record a structural lemma describing the closure of an  $H^{\text{nc}}$ -orbit inside a closed  $H$ -orbit.

<sup>6</sup> $\nu.g(E) = \nu(Eg)$  for a Borel subset  $E \subset \Gamma \backslash G$

<sup>7</sup>We say that  $xU$  is uniformly distributed with respect to  $\nu$  if the sequence of measures  $\frac{1}{T} \int_0^T \delta_{xus} ds$  converges to  $\nu$  as  $T \rightarrow \infty$

**Lemma 6.2.** *Let  $\Gamma < G$  be a discrete subgroup. Suppose that  $xH$  is closed for some  $x \in \Gamma \backslash G$  and  $H \in \mathcal{H}$ . Then there exists a connected closed subgroup  $F < G$  such that  $H^{\text{nc}} < F < H$  and*

$$\overline{xH^{\text{nc}}} = xF.$$

*Proof.* Choose  $g \in G$  so that  $x = [g]$ . By replacing  $\Gamma$  by  $g\Gamma g^{-1}$ , we may assume that  $g = e$ . Note that  $H = N_G(H^{\text{nc}})^\circ$  is an almost direct product  $H^{\text{nc}} \cdot C_G(H^{\text{nc}})^\circ$ . Let  $S := H^{\text{nc}} \times C_G(H^{\text{nc}})^\circ$  be the corresponding direct product. The multiplication map  $j : S \rightarrow H$  is surjective with finite kernel. Let  $\pi_1$  and  $\pi_2$  denote the projections from  $S$  to  $H^{\text{nc}}$  and  $C_G(H^{\text{nc}})^\circ$ , respectively. Set  $\Gamma_0 := j^{-1}(\Gamma \cap H) < S$ . A direct computation shows that

$$\overline{\Gamma_0(e, e)(H^{\text{nc}} \times \{e\})} = H^{\text{nc}} \times \overline{\pi_2(\Gamma_0)}. \quad (6.3)$$

Define  $F := j(H^{\text{nc}} \times \overline{\pi_2(\Gamma_0)})$ . Then clearly  $H^{\text{nc}} \subset F \subset H$ . Let  $\iota : \Gamma_0 \cap S \backslash S \rightarrow \Gamma \backslash G$  be the map induced by  $s \mapsto j(s)$  for  $s \in S$ . This map is clearly injective. Since  $[e]H$  is closed, it follows from Lemma 3.5 that  $\iota$  is proper. Hence its image of the closed set  $[e](H^{\text{nc}} \times \overline{\pi_2(\Gamma_0)})$  is closed in  $\Gamma \backslash G$ . This implies that  $\overline{[e]H^{\text{nc}}} = [e]F$ , completing the proof.  $\square$

We are now ready to combine the preceding results with the Mozes–Shah theorem to obtain the desired equidistribution statement:

**Theorem 6.4.** *Let  $\Gamma < G$  be a geometrically finite subgroup. If there exists infinitely many maximal  $x_i H_i \subset \text{RFM}$  with  $x_i \in \Gamma \backslash G$  and  $H_i \in \mathcal{H}$ , then  $\Gamma$  is a lattice. Moreover,  $\overline{x_i H_i^{\text{nc}}} = x_i F_i$  where  $H_i^{\text{nc}} < F_i < H_i$  and the  $F_i$ -invariant probability measure  $\mu_{F_i}$  converges to the  $G$ -invariant probability measure on  $\Gamma \backslash G$  as  $i \rightarrow \infty$ .*

*Proof.* Since  $H_i^{\text{nc}}$  is generated by unipotent one-parameter subgroups, Theorem 5.1 implies that  $\overline{x_i H_i^{\text{nc}}} = x_i F_i$  where  $H_i^{\text{nc}} < F_i$  and  $\text{Stab}_{F_i}(x_i)$  is a lattice in  $F_i$ . Since  $F_i \supset H_i^{\text{nc}}$ , it cannot be a compact extension of a connected unipotent subgroup. Therefore, by Lemma 2.9,  $F_i$  is reductive. By Lemma 2.7,  $F_i$  is  $\Theta$ -invariant, and hence  $F_i^{\text{nc}} \in \mathcal{H}^*$ . Since  $x_i F_i \subset \text{RFM}$ , Lemma 4.9 implies that  $x_i N_G(F_i^{\text{nc}}) \subset \text{RFM}$  and  $N_G(F_i^{\text{nc}}) \in \mathcal{H}$ . By maximality, we must have  $N_G(F_i^{\text{nc}}) = H_i$ , and therefore

$$F_i < H_i.$$

Let  $\mu_i = \mu_{F_i}$ , and let

$$U_i = \{u_{i,s} : s \in \mathbb{R}\} < H_i$$

be a one-parameter unipotent subgroup. By [23, Proposition 2.1], the subgroup  $H_i^{\text{nc}}$  acts ergodically on  $x_i F_i$  with respect to  $\mu_i$ . Since  $H_i^{\text{nc}}$  is simple in our setting, any unipotent subgroup of  $H_i^{\text{nc}}$  acts ergodically on  $(x_i F_i, \mu_i)$  by the Mautner phenomenon. Hence  $U_i$  acts ergodically on  $(x_i F_i, \mu_i)$ .

We claim that any weak-\* limit  $\mu$  of the sequence  $\mu_i$  on the one-point compactification of  $\Gamma \backslash G$  is supported on  $\Gamma \backslash G$ . Without loss of generality, assume that  $\mu_i \rightarrow \mu$  as  $i \rightarrow \infty$ .

Since  $\mu_i$  is  $U_i$ -ergodic, the Birkhoff ergodic theorem yields a  $\mu_i$ -conull set  $E_i \subset x_i F_i$  such that for all  $y_i \in E_i$  and for all  $\varphi \in C_c(\Gamma \backslash G)$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varphi(y_i u_{i,s}) ds = \int_{\Gamma \backslash G} \varphi(x) d\mu_i(x).$$

Choose  $\eta > 0$  so that (4.5) holds. By Lemma 4.6,  $x_i F_i.o$  intersects  $\mathcal{M}_{\geq \eta}$  non-trivially for each  $i \geq 1$ . Since  $E_i$  is conull in  $x_i F_i$  and  $x_i F_i.o \cap \mathcal{M}_{\geq \eta} \subset \text{core}(\mathcal{M}) \cap \mathcal{M}_{\geq \eta}$  is compact, we may, after replacing  $x_i$  if necessary, assume that  $x_i \in E_i$  is contained in some fixed compact subset  $\mathcal{K} \subset \Gamma \backslash G$ .

Observe that  $\mu$  is the weak-\* limit of  $\mu_{T_i} := \frac{1}{T} \int_0^T \delta_{x_i u_{i,s}} ds$  for some sequence  $T_i \rightarrow \infty$ . Hence by Proposition 5.7, we obtain that

$$\mu(\text{RFM}) = 1.$$

Since  $\mu$  is a probability measure on  $\Gamma \backslash G$ , we can apply Theorem 6.1 to the sequence  $\mu_i \rightarrow \mu$ . Thus

$$\text{supp } \mu = xL$$

for some  $x \in \text{supp } \mu$  and  $L = \{g \in G : \mu g = \mu\}$ . Let  $g_i \in G$  be such that  $x_i = \Gamma g_i$ , and let  $p_0 \in G$  be such that  $x = \Gamma p_0$ . Since  $x \in \text{supp } \mu$ , there exist  $\gamma_i \in \Gamma$  and  $u_i \in U_i$  such that

$$q_i := \gamma_i g_i u_i \rightarrow p_0 \quad \text{as } i \rightarrow \infty.$$

Set  $g'_i := p_0^{-1} q_i$ ; then  $g'_i \rightarrow e$  as  $i \rightarrow \infty$ . Since  $x_i U_i = x g'_i U_i$  is uniformly distributed with respect to  $\mu_i$ , Theorem 6.1(2) implies that for all sufficiently large  $i$ ,

$$\text{supp } \mu_i \subset x L g_i \subset \text{RFM}. \quad (6.5)$$

In particular,

$$\Gamma q_i F_i = \Gamma g_i F_i \subset \Gamma p_0 L g'_i = \Gamma p_0 L p_0^{-1} q_i,$$

and hence

$$\Gamma q_i F_i q_i^{-1} \subset \Gamma p_0 L p_0^{-1}.$$

Therefore for all large  $i$ , we have

$$q_i F_i q_i^{-1} \subset p_0 L p_0^{-1}.$$

We claim that  $L$  is reductive. Since  $xL$  supports a finite  $L$ -invariant measure,  $L$  is unimodular. Moreover,  $L$  contains a conjugate of  $F_i$  and hence cannot be a compact extension of a unipotent subgroup. By Lemma 2.9,  $L$  is reductive. Set

$$L_i := q_i^{-1} p_0 L p_0^{-1} q_i.$$

Since  $L_i$  contains  $F_i$ , Lemma 2.7 implies that  $L_i$  is  $\Theta$ -invariant. Hence  $[q_i] L_i.o$  is a totally geodesic submanifold. Since  $[q_i] L_i.o$  has finite volume, Lemma 4.9 gives

$$[q_i] L_i.o \subset \text{core}(\mathcal{M}).$$

Since  $x_i F_i.o = [q_i] F_i.o \subset [q_i] L_i.o$ , by the maximality of  $x_i F_i.o$  implies that

$$\text{either } L_i = G \quad \text{or} \quad [q_i] F_i.o = [q_i] L_i.o.$$

We claim that  $L_i = G$  for all sufficiently large  $i$ . Suppose not. Then, for infinitely many  $i$ ,  $F_i \subset L_i \subset H_i$ , and hence

$$q_i H_i^{\text{nc}} q_i^{-1} \subset p_0 L p_0^{-1} \subset q_i H_i q_i^{-1}.$$

Since  $H_i = N(H_i^{\text{nc}})^\circ$ , it follows that all subgroups  $q_i H_i q_i^{-1}$ s and hence all  $\gamma_i g_i H_i g_i^{-1} \gamma_i^{-1}$  are equal to one another.

On the other hand, for all  $i \neq j$ , we have  $x_i H_i.o \neq x_j H_j.o$  and thus  $\gamma g_i H_i.o \neq g_j H_j.o$  for all  $\gamma \in \Gamma$ . By Lemma 3.2,

$$\gamma g_i H_i g_i^{-1} \gamma^{-1} \neq g_j H_j g_j^{-1}.$$

This contradiction shows that  $L_i = G$  for all sufficiently large  $i$ . In particular, this implies that  $L = G$ . Hence  $\Gamma \backslash G$  is a lattice and  $\mu$  is the  $G$ -invariant probability measure on  $\Gamma \backslash G$ . This completes the proof.  $\square$

## 7. RIGIDITY OF TOTALLY GEODESIC SUBMANIFOLDS IN THE CONVEX CORE

In this section, we deduce the rigidity, properness, and finiteness results for totally geodesic submanifolds contained in the convex core of  $\mathcal{M}$  from the dynamical statements proved in Sections 5 and 6.

**Rigidity.** We begin with the topological rigidity of totally geodesic submanifolds:

**Theorem 7.1.** *Let  $\mathcal{M}$  be geometrically finite. If  $\mathcal{N}$  is a totally geodesic immersed submanifold of dimension at least two contained in  $\text{core}(\mathcal{M})$ , then the closure of  $\mathcal{N}$  is a totally geodesic, properly immersed submanifold of finite volume.*

*Proof.* Let  $\mathcal{N}$  be a totally geodesic immersed submanifold of dimension at least two. By Lemma 3.1,

$$\mathcal{N} = xH.o \quad \text{for some } H \in \mathcal{H}^* \text{ and } x \in \Gamma \backslash G.$$

Suppose that  $\mathcal{N} \subset \text{core}(\mathcal{M})$ . By Lemma 4.9,  $xH \subset \text{RFM}$ . By the definition of  $\mathcal{H}^*$ , we have  $H = H^{\text{nc}}$  and hence  $H$  is generated by one-parameter unipotent subgroups. Therefore, by Theorem 5.1,  $\overline{xH} = xL$  for some connected closed subgroup  $L$  containing  $H$  such that  $\text{Stab}_L(x)$  is a lattice in  $L$ . By Lemma 2.7,  $L$  is  $\Theta$ -invariant and hence  $L^{\text{nc}} \in \mathcal{H}^*$ . Consequently,

$$xL.o = xL^{\text{nc}}.o$$

is a properly immersed totally geodesic submanifold. Since  $\overline{xH} = xL$ , we conclude that  $\overline{\mathcal{N}} = xL.o$ , and that  $xL.o$  has finite volume.  $\square$

We remark that this type of rigidity fails for a general totally geodesic submanifold not contained in the core; see [20] for an example of quasifuchsian manifolds which contain geodesic planes whose closures are not submanifolds.

**Properness.** When the ambient manifold  $\mathcal{M}$  has infinite volume, Theorem 7.1 immediately yields the following consequence for maximal totally geodesic submanifolds contained in the convex core.

**Corollary 7.2.** *Let  $\mathcal{M}$  be geometrically finite and  $\text{Vol}(\mathcal{M}) = \infty$ . Every maximal totally geodesic immersed submanifold  $\mathcal{N}$  of dimension at least two contained in  $\text{core}(\mathcal{M})$  is necessarily properly immersed, and has finite volume.*

*Proof.* By Theorem 7.1,  $\overline{\mathcal{N}}$  is a totally geodesic properly immersed submanifold of finite volume. Since  $\overline{\mathcal{N}}$  is contained in  $\text{core}(\mathcal{M})$ , the maximality of  $\mathcal{N}$  implies  $\mathcal{N} = \overline{\mathcal{N}}$ , completing the proof.  $\square$

For geometrically finite real hyperbolic manifolds, any properly immersed totally geodesic submanifold of  $\mathcal{M}$  is itself geometrically finite [27]. In particular, any properly immersed  $\mathcal{N}$  contained in  $\text{core}(\mathcal{M})$  must have finite volume. A direct consequence of Theorem 7.1 shows that this phenomenon extends to all geometrically finite rank-one manifolds:

**Corollary 7.3.** *Let  $\mathcal{M}$  be geometrically finite. Then any properly immersed totally geodesic submanifold contained in  $\text{core}(\mathcal{M})$  has finite volume.*

**Equidistribution.** Building on the rigidity results above, we now examine whether infinitely many maximal totally geodesic submanifolds can exist inside the convex core. The following theorem shows that this phenomenon occurs only in the finite-volume case.

For a finite-volume geodesic submanifold  $\mathcal{N} = xH.o$  for  $H \in \mathcal{H}$ , the normalized volume measure  $\mu_{\mathcal{N}}$  is the probability measure supported on  $\mathcal{N} \subset \mathcal{M}$ , defined as the push-forward  $\pi_*(\mu_H)$  of the  $H$ -invariant probability measure  $\mu_H$  under the projection  $\pi : (\Gamma \cap H) \backslash H \rightarrow \mathcal{M} = \Gamma \backslash G.o$ . Similarly, when  $\mathcal{M}$  has finite volume, we denote by  $\mu_{\mathcal{M}}$  the probability measure on  $\mathcal{M}$  which is the push-forward of the  $G$ -invariant probability measure of  $\Gamma \backslash G$ .

**Theorem 7.4.** *Let  $\mathcal{M}$  be a geometrically finite manifold. If there exist infinitely many maximal totally geodesic submanifolds  $\mathcal{N}_i$  contained in  $\text{core}(\mathcal{M})$  of dimension at least two, then*

$$\text{Vol}(\mathcal{M}) < \infty$$

*and the normalized volume measures  $\mu_{\mathcal{N}_i}$  become equidistributed in  $\mathcal{M}$  as  $i \rightarrow \infty$ : for any  $f \in C_c(\mathcal{M})$ ,*

$$\lim_{i \rightarrow \infty} \int_{\mathcal{N}_i} f(x) d\mu_{\mathcal{N}_i}(x) = \int_{\mathcal{M}} f(x) d\mu_{\mathcal{M}}(x).$$

*Proof.* By Lemma 3.1, we have  $\mathcal{N}_i = x_i H_i.o$  for some  $H_i \in \mathcal{H}$  and  $x_i \in \Gamma \backslash G$ . By Lemma 3.5, each  $x_i H_i$  is closed. Since  $\mathcal{N}_i$  is a maximal totally geodesic submanifold contained in  $\text{core}(\mathcal{M})$ , the orbit  $x_i H_i$  is maximal among those contained in RFM. By Theorem 6.4,  $\Gamma$  is a lattice and  $\mu_{F_i}$  converges to the  $G$ -invariant probability measure on  $\Gamma \backslash G$ , where  $\mu_{F_i}$  is the  $F_i$ -invariant

probability measure supported on  $\overline{x_i H_i^{\text{nc}}} = x_i F_i$ . Since  $\pi_*(\mu_i) = \pi_*(\mu_{H_i})$ , it follows that  $\mu_{N_i} = \pi_*(\mu_i)$  converges to  $\mu_{\mathcal{M}} = \pi_*(\mu)$ . This completes the proof.  $\square$

**Finiteness.** Combining Theorem 7.1 and Theorem 7.4, we obtain the following finiteness result for infinite-volume manifolds.

**Corollary 7.5.** *Let  $\mathcal{M}$  be geometrically finite with  $\text{Vol}(\mathcal{M}) = \infty$ . Then there exist only finitely many maximal totally geodesic submanifolds in  $\text{core}(\mathcal{M})$  of dimension at least two.*

Having established the finiteness of totally geodesic submanifolds in the convex core, we now prove Theorem 1.7, which reformulates this statement on the level of their ideal boundaries.

**Proof of Theorem 1.7.** Let  $S = \partial_\infty Y \in \mathcal{S}_X$  be a maximal element contained in  $\Lambda$ . Since  $Y = \text{hull}(\partial_\infty Y)$  by Lemma 3.3, the subspace  $Y$  is a maximal totally geodesic subspace inside  $\text{hull}(\Lambda)$ . We claim that  $\Gamma S$  is closed in  $\mathcal{S}_X$  with respect to the Chabauty-Hausdorff topology. To see this, suppose that  $\gamma_i S = \gamma_i \partial_\infty Y$  converges to  $S' = \partial_\infty Y' \in \mathcal{S}_X$ . We claim that as  $i \rightarrow \infty$ ,

$$\gamma_i Y = \text{hull}(\gamma_i S) \rightarrow Y' = \text{hull}(S').$$

To verify this, let  $x_i \in \gamma_i Y$  converge to some  $x \in X$ , and let  $\ell_i$  be a geodesic passing through  $x_i$ . Since  $x_i \rightarrow x$  and hence all  $\ell_i$  pass through a fixed compact subset of  $X$ , and hence, after passing to a subsequence,  $\ell_i$  converges to some geodesic  $\ell$  in  $X$ . Since  $x \in \ell$  and the endpoints of  $\ell$  lie in  $S'$ , we have  $x \in \text{hull}(S') = Y'$ . Therefore any Hausdorff limit of  $\gamma_i Y$  is contained in  $Y'$ . Conversely, let  $x \in Y'$ . Then  $x$  lies on a geodesic  $\ell$  with endpoints  $\xi, \xi'$  in  $S'$ . Since  $\gamma_i S \rightarrow S'$ , we can choose sequences  $\xi_i \neq \xi'_i$  in  $\gamma_i S$  with  $\xi_i \rightarrow \xi$  and  $\xi'_i \rightarrow \xi'$ . Let  $\ell_i$  denote the geodesic connecting  $\xi_i$  and  $\xi'_i$ . Then  $\ell_i$  converges to  $\ell$ , and we may choose a sequence  $x_i \in \ell_i$  converging to  $x$ . Hence  $\gamma_i Y \rightarrow Y'$  in the Chabauty-Hausdorff topology.

Since  $\Gamma \backslash \Gamma Y$  is closed in  $\mathcal{M}$ , it follows that  $\text{hull}(S') \subset \Gamma Y$ , and hence  $S' \in \Gamma \partial_\infty Y$ . This proves that  $\Gamma S$  is closed in  $\mathcal{S}_X$ . The finiteness then follows from Corollary 7.5.  $\square$

**Rigidity of bounded geodesic planes.** In the proof of Corollary 7.5, the geometric finiteness was used only in Proposition 5.7 to ensure that weak-\* limits of the measures  $\mu_i$  were probability measures in  $\Gamma \backslash G$ . When the corresponding totally geodesic submanifolds are contained in a fixed compact subset of  $\Gamma \backslash G$ , the same reasoning applies without assuming geometric finiteness:

**Theorem 7.6.** *Let  $\mathcal{M} = \Gamma \backslash X$  be a rank-one locally symmetric space which is non-compact. Then the closure of any bounded totally geodesic submanifold of dimension at least two is a totally geodesic submanifold. Moreover, any compact subset of  $\mathcal{M}$  contains only finitely many maximal totally geodesic submanifolds of dimension at least two.*

We record the following characterization of  $\Gamma$ , up to finite index, in terms of totally geodesic subspaces contained in  $\text{hull}(\Lambda)$ .

**Theorem 7.7.** *Let  $\Gamma < G$  be a geometrically finite Zariski dense non-lattice subgroup. Let  $\mathsf{T}$  be any non-empty  $\Gamma$ -invariant collection of maximal totally geodesic subspaces of dimension at least two contained in  $\text{hull}(\Lambda)$ . Then*

$$\Gamma \text{ has finite index in the subgroup } \{g \in G : g(\mathsf{T}) = \mathsf{T}\}.$$

*Proof.* We may assume without loss of generality that  $Y = H.o \in \mathsf{T}$  for some  $H \in \mathcal{H}$ . Let  $\mathsf{T}'$  denote the subcollection of  $\mathsf{T}$  consisting of all subspaces of the form  $gH.o$  for  $g \in G$ . If  $g(\mathsf{T}) = \mathsf{T}$ , then  $g(\mathsf{T}') = \mathsf{T}'$ . Hence we may assume  $\mathsf{T} = \mathsf{T}'$  without loss of generality.

Let

$$\Delta = \{g \in G : g(\mathsf{T}) = \mathsf{T}\}.$$

By hypothesis,  $\Gamma$  is Zariski dense and not a lattice, so  $\Lambda \neq \partial_\infty X$  and hence  $\Delta \neq G$ . Since  $\Delta$  contains the Zariski dense subgroup  $\Gamma$ , it follows that  $\Delta$  is a discrete subgroup of  $G$ . Indeed, if  $L$  denotes the identity component of the closure of  $\Delta$ , then  $L$  is normalized by  $\Gamma$ , and hence by  $G$ . Since  $G$  is simple, this forces  $L = \{e\}$ , proving that  $\Delta$  is discrete.

Set

$$\Delta^* := \{g \in G : g\Delta H = \Delta H\}.$$

Clearly,  $\Delta < \Delta^*$ . Hence it suffices to show that

$$[\Delta^* : \Gamma] < \infty. \tag{7.8}$$

By the same argument as above,  $\Delta^*$  is discrete. By Corollary 7.2 and Lemma 4.9, the orbit  $\Gamma \backslash \Gamma H$  is closed and  $\text{Vol}(\Gamma \backslash \Gamma H) < \infty$ . Hence the intersection  $\Gamma \cap H$  is a lattice in  $H$ . Since  $\Gamma \cap H < \Delta^* \cap H$ , it follows that  $\Delta^* \cap H$  is also a lattice in  $H$ . Therefore

$$[\Delta^* \cap H : \Gamma \cap H] < \infty. \tag{7.9}$$

To prove (7.8), suppose on the contrary that there exists a sequence  $\delta'_i \in \Delta^*$  tending to infinity mod  $\Gamma$ . Since  $\delta'_i \in \Delta^*$ , by definition of  $\Delta^*$ , there exist  $\delta_i \in \Gamma$  such that  $\delta'_i H = \delta_i H$ , and hence  $\delta_i^{-1} \delta'_i \in H$ . Since  $\Gamma \backslash \Gamma H$  is closed, Lemma 3.5 implies that the projection map

$$(H \cap \Gamma) \backslash H \rightarrow \Gamma \backslash G$$

is proper. It follows that the sequence  $\delta_i^{-1} \delta'_i$  tends to infinity modulo  $H \cap \Gamma$ , contradicting (7.9). This proves (7.8).  $\square$

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