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S-ARITHMETICITY OF DISCRETE SUBGROUPS CONTAINING LATTICES IN HOROSPHERICAL SUBGROUPS

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0. Introduction. Let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{Q}_\infty = \mathbb{R}$. Let $\mathbf{G}_p$ be a connected semisimple $\mathbb{Q}_p$-algebraic group. The unipotent radical of a proper parabolic $\mathbb{Q}_p$-subgroup of $\mathbf{G}_p$ is called a *horospherical* subgroup. Two horospherical subgroups are called *opposite* if they are the unipotent radicals of two opposite parabolic subgroups. In [5] and [6], we studied discrete subgroups generated by lattices in two opposite horospherical subgroups in a simple real algebraic group with real rank at least 2. This work was inspired by the following conjecture posed by G. Margulis.

CONJECTURE 0.1. *Let $\mathbf{G}$ be a connected semisimple $\mathbb{R}$-algebraic group such that $\mathbb{R}$-rank $(\mathbf{G}) \geq 2$, and let $\mathbf{U}_1$, $\mathbf{U}_2$ be a pair of opposite horospherical $\mathbb{R}$-subgroups of $\mathbf{G}$. For each $i = 1, 2$, let F_i be a lattice in $\mathbf{U}_i(\mathbb{R})$ such that $H \cap F_i$ is finite for any proper normal $\mathbb{R}$-subgroup H of G. If the subgroup generated by F_1 and F_2 is discrete, then it is an arithmetic lattice in $\mathbf{G}(\mathbb{R})$.*

We settled the conjecture in many cases, including the case when $\mathbf{G}$ is an absolutely simple real split group with $\mathbf{G}(\mathbb{R})$ not locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ (see [5]).

In this paper, we study a problem analogous to the conjecture in a product of real and p-adic algebraic groups. The following is a special case of the main theorem, Theorem 4.3.

THEOREM 0.2. *Let S be a finite set of valuations of $\mathbb{Q}$ including the archimedean valuation ∞. For each $p \in S$, let $\mathbf{G}_p$ be a connected semisimple algebraic $\mathbb{Q}_p$-group without any $\mathbb{Q}_p$-anisotropic factors, and let $\mathbf{U}_{1p}$, $\mathbf{U}_{2p}$ be a pair of opposite horospherical subgroups of $\mathbf{G}_p$. Set $G = \prod_{p \in S} \mathbf{G}_p(\mathbb{Q}_p)$, $U_1 = \prod_{p\in S} \mathbf{U}_{1p}(\mathbb{Q}_p)$, and $U_2 = \prod_{p\in S} \mathbf{U}_{2p}(\mathbb{Q}_p)$.*

Assume that $\mathbf{G}_\infty$ is absolutely simple $\mathbb{R}$-split with rank at least 2 and that if $\mathbf{G}_\infty(\mathbb{R})$ is locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$, then $\mathbf{U}_{1\infty}$ is not the unipotent radical of a Borel subgroup of $\mathbf{G}_\infty$. Let F_1 and F_2 be lattices in U_1 and U_2, respectively. If the subgroup generated by F_1 and F_2 is discrete, then it is a nonuniform S-arithmetic lattice in G.

If p is a nonarchimedean valuation of $\mathbb{Q}$, then no horospherical subgroup of $\mathbf{G}_p(\mathbb{Q}_p)$ admits a lattice. Moreover, there is no infinite unipotent discrete subgroup in a p-adic Lie group. Therefore it is necessary to assume in Theorem 0.2 that S contains the archimedean valuation ∞.

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In the group $G = \mathrm{SL}_m(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$, one can ask if it is possible to generate a discrete subgroup by taking lattices from two opposite horospherical subgroups of G. One interesting aspect of Theorem 0.2 says that, in general, the answer is *no*.

COROLLARY 0.3. *Keeping the same notation as in Theorem 0.2, set* $\mathbf{G} = \prod_{p \in S} \mathbf{G}_p$. *Suppose that there exist lattices* F_1 *and* F_2 *in* U_1 *and* U_2, *respectively, that generate a discrete subgroup of* G. *Then*

(1) $\mathbf{G}$ *is typewise homogeneous; that is, for each* $p \in S$, *there is an isogeny* $f_p : \mathbf{G}_p \to \mathbf{G}_\infty$; *in particular,* $\mathbf{G}_p$ *is absolutely almost simple;*
(2) *for each* $p \in S$, $\mathbf{U}_{1p}$ *is isomorphic to* $\mathbf{U}_{1\infty}$.

In particular, we have the following corollary.

COROLLARY 0.4. *With the same notation as in Corollary 0.3, suppose that* $\mathbf{G}$ *is not typewise homogeneous. Then any subgroup generated by lattices in a pair of opposite horospherical subgroups of* G *is not discrete.*

Corollary 0.3 follows from Theorem 0.2 simply by the definition of an S-arithmetic subgroup of G (see Section 1.6).

Examples. In the following groups there are no discrete subgroups containing lattices in opposite horospherical subgroups:

(1) $G = \mathrm{SL}_m(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$ for any $m \neq n$ such that $m \geq 4$ and $n \geq 2$;
(2) $G = \mathrm{SL}_m(\mathbb{R}) \times \mathrm{SL}_{n_1}(\mathbb{Q}_p) \times \mathrm{SL}_{n_2}(\mathbb{Q}_p)$ for any $n_1, n_2 \geq 2$ and $m \geq 4$;
(3) $G = SO(m, m)_{\mathbb{R}} \times \mathrm{SL}_n(\mathbb{Q}_p)$ for any $n \geq 2$ and $m \geq 2$.

As a corollary of Theorem 0.2, we obtain that as long as a discrete subgroup of G intersects a pair of opposite horospherical subgroups as lattices, then it is a lattice in the ambient group G as well. This is not always true in rank-1 simple groups, for instance, in $\mathrm{SL}_2(\mathbb{R})$ (see the remark after Theorem 0.2 in [5]).

COROLLARY 0.5. *Let* S, $\mathbf{G}_p$, $p \in S$, *and* G *be as in Theorem 0.2. Let* Γ *be a discrete subgroup of* G. *Then* Γ *is a nonuniform* S*-arithmetic lattice in* G *if and only if for each* $p \in S$ *there exists a pair* $\mathbf{U}_{1p}$, $\mathbf{U}_{2p}$ *of opposite horospherical subgroups of* $\mathbf{G}_p$ *such that* $\Gamma \cap U_i$ *is a lattice in* U_i, *where* $U_i = \prod_{p \in S} \mathbf{U}_{ip}(\mathbb{Q}_p)$ *for each* $i = 1, 2$. *In that case,* $\mathbf{G}$ *is typewise homogeneous.*

For the proof of Theorem 0.2, denote by Γ_{F_1,F_2} the subgroup generated by F_1 and F_2, and denote by $\Gamma^{\infty}_{F_1,F_2}$ the image of the subgroup $\Gamma_{F_1,F_2} \cap \mathbf{G}_\infty(\mathbb{R}) \times \prod_{p \in S,\ p \neq \infty} \mathbf{G}_p(\mathbb{Z}_p)$ under the natural projection $G \to \mathbf{G}_\infty(\mathbb{R})$. Using the results from [5], we first obtain a $\mathbb{Q}$-form on $\mathbf{G}_\infty$ with respect to which the subgroup $\Gamma^{\infty}_{F_1,F_2}$ is an arithmetic lattice in $\mathbf{G}_\infty(\mathbb{R})$. Then applying a special case of Margulis's superrigidity (see Theorem 4.2), we show that this $\mathbb{Q}$-form of $\mathbf{G}_\infty$ endows a $\mathbb{Q}$-form on each $\mathbf{G}_p$, $p \in S$, so that Γ_{F_1,F_2} becomes an S-arithmetic subgroup in G. We also need some results on the classification of lattices in the product of real and p-adic nilpotent Lie groups (see Corollary 2.7). In fact our method shows that in Theorem 0.2 we can remove the assumption that $\mathbf{G}_\infty$ is $\mathbb{R}$-split as long as $\mathbf{G}_\infty$ is absolutely simple with real rank at least 2 and Conjecture 0.1 holds for $\mathbf{G}_\infty$.

In [5], we proved directly that any discrete subgroup of $\mathbf{G}_\infty(\mathbb{R})$ containing lattices in opposite horospherical subgroups is an arithmetic subgroup, rather than using Margulis's arithmeticity theorem or superrigidity theorem. Therefore, the methods used provide an alternative proof of the arithmeticity theorem in the case of nonuniform lattices for the groups considered in [5]. In the present paper, however, we use a special case of Margulis's superrigidity theorem in order to extend the arithmetic structure of $\Gamma^\infty_{F_1,F_2}$, obtained in [5], to an S-arithmetic structure of Γ_{F_1,F_2}.

Before we close the introduction, we describe the following open case of Margulis's conjecture, which is believed to be a challenging case.

Open problem. Consider the following two subgroups of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$:

$$U_1=\begin{pmatrix}1 & \mathbb{R}^2\\ 0 & 1\end{pmatrix},\qquad U_2=\begin{pmatrix}1 & 0\\ \mathbb{R}^2 & 1\end{pmatrix}.$$

For $i=1,2$, choose two linearly independent vectors u_i and v_i in $\mathbb{R}^2$ such that $\{nu_i+mv_i \mid n,m\in\mathbb{Z}\}$ does not contain any element of the form $(x,0)$ or $(0,x)$ for any $x\neq 0$. By the natural isomorphism of U_i with $\mathbb{R}^2$, we consider u_i and v_i as elements of U_i. Then the question dealt with by Conjecture 0.1 can be regarded as the following discreteness criterion problem:

Which four elements u_1, u_2, v_1, and v_2 generate a discrete subgroup?

From the classification of the $\mathbb{Q}$-forms of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ (cf. [11]), it is not hard to see that Conjecture 0.1 implies that u_1, v_1, u_2, and v_2 can generate a discrete subgroup only in the case when the elements u_1, v_1, u_2, and v_2 are from some Hilbert modular group of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$. It then follows from the results of [12] that the discrete subgroup generated by those four elements is in fact a Hilbert modular group. Here we say that Γ is a Hilbert modular group of $\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ if there is a real quadratic extension field k of $\mathbb{Q}$ such that Γ is conjugate to a subgroup of finite index in

$$\left\{\left(g,{}^\sigma g\right)\mid g\in\mathrm{PSL}_2(J)\right\},$$

where J is the ring of integers of k and $\sigma: k\to k$ is the nontrivial Galois automorphism of k.

It seems plausible that an analogous conjecture in the setting of Theorem 0.2 holds under the assumption that the S-rank of G, that is, the $\sum_{p\in S}\mathbb{Q}_p$-rank of G, is at least 2 (without any assumption on G_∞). The first question in this regard would be to ask whether the conjecture is true for $G=\mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{Q}_p)$.

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1. Notation

1.1. For a set S of valuations of $\mathbb{Q}$, denote by S_f the subset of S consisting of nonarchimedean valuations (i.e., $S_f = S - \{\infty\}$), where ∞ denotes the archimedean valuation of $\mathbb{Q}$. Denote by $\mathbb{Q}_p$ the field of p-adic numbers with the normalized absolute value $| \, |_p$ and set $\mathbb{Q}_\infty = \mathbb{R}$. Denote by $\mathbb{Z}_p$ the ring of p-adic integers, that is, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$. For a valuation p on $\mathbb{Q}$ and a connected $\mathbb{Q}_p$-algebraic group $\mathbf{G}_p$, we denote by G_p the $\mathbb{Q}_p$-rational points of $\mathbf{G}_p$ and by G_p^+ the subgroup of G_p generated by all of its unipotent 1-parameter subgroups.

1.2. By Lie G_p we denote the Lie algebra of the group G_p considered as a Lie group over $\mathbb{Q}_p$, which is naturally identified with the $\mathbb{Q}_p$-points of the Lie algebra Lie $\mathbf{G}_p$.

1.3. For each $p \in S$, we denote by pr_p the natural projection map from $\prod_{p \in S} \mathbf{G}_p$ to $\mathbf{G}_p$. The notation pr^∞ denotes the projection of $G_\infty \times \prod_{p \in S_f} \mathbf{G}(\mathbb{Z}_p)$ to G_∞. For a subgroup H of $\prod_{p \in S} G_p$, the notation H^∞ denotes the image of $H \cap (G_\infty \times \prod_{p \in S_f} \mathbf{G}_p(\mathbb{Z}_p))$ under the projection pr^∞.

1.4. The notation $\mathbb{Z}_S$ denotes the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$ and $\{(1/p) \mid p \in S_f\}$.

1.5. For $\mathbf{G} = \prod_{p \in S} \mathbf{G}_p$, we say that $\mathbf{G}$ has a $\mathbb{Q}$-form if there exist a connected algebraic $\mathbb{Q}$-group $\mathbf{H}$ and a $\mathbb{Q}_p$-isomorphism $\phi_p : \mathbf{H} \to \mathbf{G}_p$ for each $p \in S$. A subgroup $\mathbf{M}$ of $\mathbf{G}$ is said to be defined over $\mathbb{Q}$ if there is a $\mathbb{Q}$-subgroup $\mathbf{M}'$ of $\mathbf{H}$ such that $\mathbf{M} = \prod_{p \in S} \phi_p(\mathbf{M}')$. For a subring J of $\mathbb{Q}$ and a subgroup $\mathbf{M}$ of $\mathbf{G}$ defined over $\mathbb{Q}$, the notation $\mathbf{M}(J)$ denotes the set $\{\prod_{p \in S} \phi_p(x) \in \mathbf{G} \mid x \in \mathbf{M}'(J)\}$, where $\mathbf{M} = \prod_{p \in S} \phi_p(\mathbf{M}')$.

1.6. Let $\mathbf{G}_p$ be a connected algebraic $\mathbb{Q}_p$-group for each $p \in S$. A subgroup Γ of $G = \prod_{p \in S} \mathbf{G}_p(\mathbb{Q}_p)$ is called an S-arithmetic (or simply arithmetic if $S = \{\infty\}$) subgroup of G if there exist a connected algebraic $\mathbb{Q}_p$-group $\mathbf{G}'_p$, a $\mathbb{Q}_p$-isogeny $f_p : \mathbf{G}_p \to \mathbf{G}'_p$ for each $p \in S$, and a $\mathbb{Q}$-form on $\mathbf{G}' = \prod_{p \in S} \mathbf{G}'_p$ such that $(\prod_{p \in S} f_p)(\Gamma)$ is commensurable to $\mathbf{G}'(\mathbb{Z}_S)$. An epimorphism with finite kernel is called an isogeny.

1.7. A discrete subgroup Γ of a locally compact group G is called a lattice if G/Γ has a finite G-invariant Borel measure. A lattice Γ in G is called uniform if G/Γ is compact, and it is nonuniform otherwise.

2. Discrete unipotent subgroups

2.1. Let S be a finite set of valuations of $\mathbb{Q}$ including ∞. For each $p \in S$, let $\mathbf{G}_p$ be a connected algebraic $\mathbb{Q}_p$-group. Let $G = \prod_{p \in S} G_p$.

PROPOSITION 2.1. *If Γ is a discrete subgroup (resp., lattice) in G, then Γ^∞ is a discrete subgroup (resp., lattice) in G_∞.*

Proof. Since the kernel of pr^{∞} is compact, the subgroup Γ^{∞} is discrete if Γ is. Let Γ be a lattice in G. Since $G_{\infty} \times \prod_{p \in S_f} \mathbf{G}_p(\mathbb{Z}_p)$ is an open subgroup of G, the intersection $\Gamma \cap (G_{\infty} \times \prod_{p \in S_f} \mathbf{G}_p(\mathbb{Z}_p))$ is also a lattice in $G_{\infty} \times \prod_{p \in S_f} \mathbf{G}_p(\mathbb{Z}_p)$. Hence, Γ^{∞} is a lattice in G_{∞} by the compactness of the kernel of pr^{∞}. □

It also follows from the above proof that if Γ is a uniform lattice in G, then Γ^{∞} is a uniform lattice in G_{∞} as well.

LEMMA 2.2. *For $p \in S_f$, let G_p be unipotent. If L is a closed subgroup of G_p such that G_p/L carries a finite G_p-invariant Borel measure, then $L = G_p$.*

Proof. Suppose this is not so. The general case is easily reduced to the case when G_p is abelian. Then there is a 1-parameter unipotent subgroup $U = \mathbb{Q}_p x$ of G_p that is not contained in L. If $\{p^{-n}x \mid n > n_0\} \cap L = \emptyset$ for some positive integer n_0, then it contradicts the assumption that G_p/L carries a finite G_p-invariant Borel measure (since $\mathbb{Q}_p/p^{-n_0}\mathbb{Z}_p$ is an infinite countable set). Hence there exists a sequence $x_i \in L \cap U$ for all $i \geq 1$ such that $x_i \to \infty$ as $i \to \infty$. Since L is closed and $\mathbb{Z}(x_i) \subset L$, we have $\mathbb{Z}_p(x_i) \subset L$ for all $i \geq 1$. Note that $U = \cup_{i \geq 1} \mathbb{Z}_p(x_i)$ since $x_i \to \infty$ as $i \to \infty$. Therefore $U \subset L$, contradicting the assumption. □

LEMMA 2.3. *For each $p \in S$, let G_p be unipotent. If Γ is a lattice in G, then*
(1) *$pr_p(\Gamma)$ is dense in G_p for each $p \in S_f$;*
(2) *$pr^{\infty}(\Gamma)$ is Zariski-dense in $\mathbf{G}_{\infty}$.*

Proof. Let $p \in S_f$. Denote by L the closure of $pr_p(\Gamma)$ in G_p. Since G/Γ has a finite G-invariant measure, then so does G_p/L (see, e.g., [4, Chap. II, Lemma 6.1]). Therefore by Lemma 2.2, $L = G_p$. This implies (1).

For (2), the subgroup $pr^{\infty}(\Gamma)$ is a lattice in G_{∞} by Proposition 2.1. Note that G_{∞} is a connected and simply connected nilpotent Lie group as is any real unipotent algebraic group. It is well known that any lattice in a connected and simply connected nilpotent Lie group is Zariski-dense (cf. [8]). This completes the proof. □

2.2. Let $\mathbf{V}$ be a connected unipotent algebraic $\mathbb{Q}$-group, $V_p = \mathbf{V}(\mathbb{Q}_p)$ and $V = \prod_{p \in S} V_p$. For a subring J of $\mathbb{Q}$, we identify $V(J)$ with its image under the diagonal embedding of $\mathbf{V}(\mathbb{Q})$ into V. It is well known that $V(\mathbb{Z}_S)$ is a uniform lattice in V (cf. [7, Thm. 5.7]).

LEMMA 2.4. *Let F be a discrete subgroup of V. Then the restriction $pr_{\infty}|_F$ is injective.*

Proof. Without loss of generality, we may assume that $\mathbf{V} \subset \mathrm{GL}_N$. Suppose that there is a nontrivial element $x \in F$ such that $pr_{\infty}(x) = e$. Since x is unipotent, $(x - e)^n = 0$ for some $n \in \mathbb{N}$. Then, by the binomial formula, we have that for each $p \in S_f$, $pr_p(x^{s^m})$ tends to e as $m \to \infty$, where $s = \prod_{p \in S_f} p$. Hence, $x^{s^m} \to e$ as $m \to \infty$. This contradicts the assumption that F is discrete and thus proves our claim. □

LEMMA 2.5. *For any nonzero integers m and d, there exists a nonzero integer k such that $V(k\mathbb{Z}_S) \subset mV(d\mathbb{Z}_S)$, where $mV(\mathbb{Z}_S) = \{x^m \mid x \in V(\mathbb{Z}_S)\}$.*

Proof. Since V is unipotent, there exists an integer n such that $(x-e)^n = e$ for any $x \in V$. Without loss of generality, we may assume that $\mathbf{V} \subset \mathrm{GL}_N$ and

$$V(d\mathbb{Z}_S) = \{x \in V \mid (x-e) \text{ is a matrix whose entries are in } d\mathbb{Z}_S\}.$$

For any $x = e+u \in V$, we have $\log x^{1/m} = (1/m)(\sum_{j=1}^{n-1}((-1)^{j+1}/j)u^j)$. Therefore we can find k such that if $x \in V(k\mathbb{Z}_S)$, then $\log x^{1/m} \in \log V(d\mathbb{Z}_S)$. Hence $V(k\mathbb{Z}_S) \subset mV(d\mathbb{Z}_S)$. □

2.3. It is well known that any lattice in a real algebraic unipotent group is an arithmetic subgroup (cf. [3]). Analogously, we now prove that any lattice in $V = \prod_{p\in S} V_p$ is an S-arithmetic subgroup.

We denote by $\mathcal{V}$ the product $\prod_{p\in S} \mathrm{Lie}\, V_p$. There exists an integer b such that for any subgroup U in V, $b\langle \log U\rangle \subset \log U$, where $\langle \log U\rangle$ denotes the subring of $\mathcal{V}$ generated by $\log U$ (cf. [3, Lemma 5.2]). For a discrete subgroup F in V, we set $\Delta_F = b\langle \log F\rangle$. It is then clear that Δ_F is a discrete subgroup in $\mathcal{V}$ such that $b \log F \subset \Delta_F \subset \log F$.

PROPOSITION 2.6. *Let F be a discrete subgroup of V such that $pr^\infty(F)$ is Zariski-dense in V_∞ and $pr_p(F)$ is dense in V_p for each $p \in S_f$. Then F is an S-arithmetic subgroup of V.*

Proof. Since $\log : V_p \to \mathcal{V}_p$ is both a rational map and a homeomorphism for each $p \in S$, we have that $pr^\infty(\Delta_F)$ is Zariski-dense in $\mathcal{V}_\infty$ and $pr_p(\Delta_F)$ is dense in $\mathcal{V}_p$ for each $p \in S_f$.

We first show that there exists a $\mathbb{Q}$-form on $\mathcal{V}$ such that $\Delta_F \subset \mathcal{V}(\mathbb{Q})$. Since Δ_{F^∞} is a Zariski-dense discrete subgroup in $\mathcal{V}_\infty$, which is a connected and simply connected nilpotent Lie group, then Δ_{F^∞} is a lattice in $\mathcal{V}_\infty$ (see, e.g., [8]). Therefore there exists a $\mathbb{Q}$-form on $\mathcal{V}_\infty$ such that $\Delta_{F^\infty} = \mathcal{V}_\infty(\mathbb{Z})$. (Note that this $\mathbb{Q}$-form on $\mathcal{V}_\infty$ does not necessarily coincide with the $\mathbb{Q}$-form on $\mathcal{V}_\infty$ given by the original $\mathbb{Q}$-form on $\mathbf{V}$ with which we started.) We denote by $\mathcal{V}_\infty(\mathbb{Q}_p)$ the completion of $\mathcal{V}(\mathbb{Q})$ with respect to the p-adic norm. Note that Δ_{F^∞} is a basis of the vector space $\mathcal{V}_\infty(\mathbb{Q}_p)$ over $\mathbb{Q}_p$. Therefore, in order to define a $\mathbb{Q}_p$-linear map $\phi_p : \mathcal{V}_\infty(\mathbb{Q}_p) \to \mathcal{V}_p$, it is enough to define it on Δ_{F^∞}.

For each $x \in \Delta_{F^\infty}$, there exists an element $y_x \in \Delta_F$ such that $pr_\infty(y_x) = x$. By Lemma 2.4, such an element y_x is unique. We set $\phi_p(x) = pr_p(y_x)$. We show that the map ϕ_p is a $\mathbb{Q}_p$-isomorphism. Since $\dim \mathcal{V}_\infty(\mathbb{Q}_p) = \dim \mathcal{V}_p$, it suffices to show that ϕ_p is onto. To show this, it is again enough to show that $pr_p(\Delta_F) \subset \mathrm{Im}\,\phi_p$, since $pr_p(\Delta_F)$ is dense in $\mathcal{V}_p$ by assumption. For $x = \log y \in \Delta_F$, there is an $n \in \mathbb{N}$ such that $pr_\infty(y^{s^n}) \in F^\infty$ where $s = \prod_{p\in S_f} p$. Then $pr_\infty(s^n x) \in \Delta_{F^\infty}$, and hence $pr_p(x) = \phi_p(s^{-n} pr_\infty(s^n x))$. Therefore $pr_p(\Delta_F) \subset \mathrm{Im}\,\phi_p$, proving that ϕ_p is an isomorphism over $\mathbb{Q}_p$ for each $p \in S_f$. Set ϕ_∞ to be the identity map of $\mathcal{V}_\infty$. Hence, $(\mathcal{V}_\infty, (\phi_p, p \in S))$ provides a $\mathbb{Q}$-form on $\mathcal{V}$ such that $\Delta_F \subset \mathcal{V}(\mathbb{Q})$. Using the exponential map, we obtain a $\mathbb{Q}$-form on $\mathbf{V}$ such that $F \subset V(\mathbb{Q})$.

We now show that $k\Delta_{V(\mathbb{Z}_S)} \subset \Delta_F$ for some nonzero integer k. It is easy to see that $\Delta_{V(m\mathbb{Z})} \subset \Delta_{V(\mathbb{Z}_S)} \cap \Delta_F$ for some nonzero integer m.

Now let B be a basis of $\Delta_{V(m\mathbb{Z})}$ over $\mathbb{Z}$. To show that Δ_F contains the $\mathbb{Z}_S$-module generated by B, it is enough to show that for any $x \in B$, we have $p^{-n}x \in \Delta_F$ for all $n \geq 1$ and for all $p \in S_f$, since the $\mathbb{Z}$-span of $\{p^{-n} \mid p \in S_f,\ n \geq 1\}$ is equal to $\mathbb{Z}_S$. For $p \in S_f$, since $pr_p(\Delta_F)$ is dense in $\mathcal{V}_p$, there exists $n_i \in \mathbb{N}$, going to infinity as $i \to \infty$, such that $p^{-n_i}x \in \Delta_F$. Let $n \geq 1$, and take any integer i such that $n_i \geq n$. Since $p^n = p^{n_i - n}p^{-n_i}$ and $p^{-n_i}x \in \Delta_F$, we have $p^{-n}x \in \Delta_F$. Therefore Δ_F contains the $\mathbb{Z}_S$-module generated by $\Delta_{V(m\mathbb{Z})}$ as well as by B.

Since $V(m\mathbb{Z})$ has finite index in $V(\mathbb{Z})$, we can find a nonzero integer k such that $k\Delta_{V(\mathbb{Z})} \subset \Delta_{V(m\mathbb{Z})}$. For any $x \in \Delta_{V(\mathbb{Z}_S)}$, there exists n such that $s^n x \in \Delta_{V(\mathbb{Z})}$ for $s = \prod_{p \in S_f} p$, and hence $ks^n x \in \Delta_{V(m\mathbb{Z})}$. Therefore $kx \in \Delta_F$ since Δ_F contains the $\mathbb{Z}_S$-module generated by $\Delta_{V(m\mathbb{Z})}$. This proves that $k\Delta_{V(\mathbb{Z}_S)} \subset \Delta_F$.

Since $kb \log V(\mathbb{Z}_S) \subset k\Delta_{V(\mathbb{Z}_S)}$, $\Delta_F \subset \log F$, we have $kbV(\mathbb{Z}_S) \subset F$. By Lemma 2.5, there exists a nonzero integer j such that $V(j\mathbb{Z}_S) \subset kbV(\mathbb{Z}_S)$. Therefore $V(j\mathbb{Z}_S) \subset F$ and F is commensurable with $V(\mathbb{Z}_S)$. This shows that F is an S-arithmetic subgroup of V. □

2.4. By Lemma 2.3 and the remark in Section 2.2 that any S-arithmetic subgroup of V is a uniform lattice in V, we obtain the following two corollaries of Proposition 2.6.

COROLLARY 2.7. *Any lattice in V is an S-arithmetic subgroup of V.*

COROLLARY 2.8. *Let F be a discrete subgroup of V. Then the following are equivalent:*

(1) *F is a lattice in V;*
(2) *$pr^\infty(F)$ is Zariski-dense in V_∞ and $pr_p(F)$ is dense in V_p for each $p \in S_f$;*
(3) *V/F is compact.*

PROPOSITION 2.9. *Let F be a lattice in V. If $F \subset V(\mathbb{Q})$, then F is commensurable with $V(\mathbb{Z}_S)$.*

Proof. By Proposition 2.6, there is a $\mathbb{Q}$-form on $\mathbf{V}$ with respect to which F is an S-arithmetic subgroup. Since $F \subset V(\mathbb{Q})$, this $\mathbb{Q}$-form must coincide with the original $\mathbb{Q}$-form of $\mathbf{V}$. Therefore F is commensurable with $V(\mathbb{Z}_S)$. □

3. Discrete subgroups in semisimple groups

3.1. Throughout this section, let S be a finite set of valuations of $\mathbb{Q}$ including ∞. For each $p \in S$, let $\mathbf{G}_p$ be a connected adjoint semisimple $\mathbb{Q}_p$-algebraic group without any $\mathbb{Q}_p$-anisotropic factors, and let $\mathbf{U}_{1p}$, $\mathbf{U}_{2p}$ be a pair of opposite horospherical subgroups of $\mathbf{G}_p$. Set $\mathbf{U}_1 = \prod_{p\in S}\mathbf{U}_{1p}$, $\mathbf{U}_2 = \prod_{p\in S}\mathbf{U}_{2p}$, $U_1 = \prod_{p\in S}\mathbf{U}_{1p}(\mathbb{Q}_p)$, and $U_2 = \prod_{p\in S}\mathbf{U}_{2p}(\mathbb{Q}_p)$. For lattices F_1 and F_2 in U_1 and U_2, respectively, we denote by Γ_{F_1,F_2} the subgroup generated by F_1 and F_2.

LEMMA 3.1. (1) *The subgroups* $\mathbf{U}_{1p}(\mathbb{Q}_p)$ *and* $\mathbf{U}_{2p}(\mathbb{Q}_p)$ *generate the subgroup* G_p^+ *(see [2]).*

(2) *Any subgroup of* G_p *normalized by* G_p^+ *is either trivial or contains* H_p^+ *for some nontrivial normal simple* $\mathbb{Q}_p$*-subgroup* H_p *of* G_p *(see [10]).*

If G_p is $\mathbb{Q}_p$-simple, it is well known [10] that any subgroup of G_p normalized by G_p^+ is either central (and hence trivial in our case since $\mathbf{G}_p$ is adjoint) or contains G_p^+. It is not difficult to see that this implies (2) of the above lemma, since a connected adjoint semisimple $\mathbb{Q}_p$-algebraic group is a direct product of adjoint $\mathbb{Q}_p$-simple groups.

LEMMA 3.2. *Let* F_1 *and* F_2 *be lattices in* U_1 *and* U_2*, respectively. Then for each* $p \in S_f$, $pr_p(\Gamma_{F_1,F_2})$ *is dense in* G_p^+.

Proof. By Lemma 2.3, the closure of $pr_p(F_i)$ contains $\mathbf{U}_{ip}(\mathbb{Q}_p)$. Therefore the closure of $pr_p(\Gamma_{F_1,F_2})$ contains the subgroup generated by $\mathbf{U}_{1p}(\mathbb{Q}_p)$ and $\mathbf{U}_{2p}(\mathbb{Q}_p)$, which is G_p^+ by Lemma 3.1. □

PROPOSITION 3.3. *If* Γ *is a discrete subgroup of* G *containing* F_1 *and* F_2*, then the restriction* $pr_\infty|_\Gamma$ *of* pr_∞ *is injective.*

Proof. We show that the subgroup $\Gamma_0 = \{\gamma \in \Gamma \mid pr_\infty(\gamma) = e\}$ is trivial. Without loss of generality, we may assume that $\Gamma_0 \subset G_{S_f} = \prod_{p\in S_f} G_p$. Note that Γ_0 is normalized by $pr_{S_f}(\Gamma)$ as well as by Γ. We claim that Γ_0 is normalized by $G_{S_f}^+ = \prod_{p\in S_f} G_p^+$. For each $g \in G_{S_f}^+$, there is a sequence $\{g_i \mid i = 1, 2, \ldots\}$ in $pr_{S_f}(\Gamma)$ converging to g as $i \to \infty$, since $pr_{S_f}(\Gamma)$ is dense in $G_{S_f}^+$ by Lemma 3.2. Note that $g_i x g_i^{-1} \in \Gamma_0$ for any $x \in \Gamma_0$ and any $i \geq 1$. But Γ_0 is discrete, and in particular, it is closed. Therefore $gxg^{-1} \in \Gamma_0$, proving that Γ_0 is normalized by $G_{S_f}^+$. Let $p \in S_f$. Since $pr_p(\Gamma_0)$ is normalized by G_p^+ and $pr_p(\Gamma_0)$ is countable, it follows from Lemma 3.1 that $pr_p(\Gamma_0)$ is trivial. Therefore Γ_0 is trivial, yielding that $pr_\infty|_\Gamma$ is injective. □

THEOREM 3.4 (See [1] and [4, Chap. I, Thm. 3.2.4]). *Let* $\mathbf{G}$ *be a connected semisimple* $\mathbb{Q}$*-algebraic group, and let* $G = \prod_{p\in S}\mathbf{G}(\mathbb{Q}_p)$*. Then the* S*-arithmetic subgroup* $\mathbf{G}(\mathbb{Z}_S)$ *is a lattice in* G.

3.2. Let $\mathbf{G}$ be a connected $\mathbb{Q}$-simple algebraic group with $\mathbb{Q}$-rank at least 1, S-rank at least 2 (S-rank of $\mathbf{G} = \sum_{p\in S}\mathbb{Q}_p$-rank of $\mathbf{G}$), and $\mathbf{U}_1$, $\mathbf{U}_2$ a pair of opposite horospherical $\mathbb{Q}$-subgroups of $\mathbf{G}$. It was proved by Raghunathan [9] for $\mathbb{Q}$-rank at least 2 and by Venkataramana [12] for $\mathbb{Q}$-rank 1 that for any ideal A of $\mathbb{Z}_S$, the subgroup generated by $\mathbf{U}_1(A)$ and $\mathbf{U}_2(A)$ is of finite index in $\mathbf{G}(A)$. It is not hard to see that the following theorem is a consequence of the above result.

THEOREM 3.5. *Let F_1 and F_2 be lattices in U_1 and U_2 commensurable to $U_1(\mathbb{Z}_S)$ and $U_2(\mathbb{Z}_S)$, respectively. If the subgroup Γ_{F_1,F_2} is discrete, then it is commensurable with the S-arithmetic subgroup $G(\mathbb{Z}_S)$.*

4. Main theorem

4.1. As before, let S be a finite set of valuations of $\mathbb{Q}$ including ∞, and for each $p \in S$, let $\mathbf{G}_p$ be a connected semisimple $\mathbb{Q}_p$-algebraic group without any $\mathbb{Q}_p$-anisotropic factors and let $\mathbf{U}_{1p}$, $\mathbf{U}_{2p}$ be a pair of opposite horospherical subgroups of $\mathbf{G}_p$. We set $\mathbf{G} = \prod_{p\in S} \mathbf{G}_p$, $G = \prod_{p\in S} G_p$, $\mathbf{U}_1 = \prod_{p\in S} \mathbf{U}_{1p}$, $\mathbf{U}_2 = \prod_{p\in S} \mathbf{U}_{2p}$, $U_1 = \prod_{p\in S} \mathbf{U}_{1p}(\mathbb{Q}_p)$, and $U_2 = \prod_{p\in S} \mathbf{U}_{2p}(\mathbb{Q}_p)$.

THEOREM 4.1 (See [5] and [6]). *Let $S = \{\infty\}$ and let $\mathbf{G}$ be an absolutely simple real algebraic group with $\mathbb{R}$-rank at least 2. Denote by $Z(\mathbf{U}_i)$ the center of $\mathbf{U}_i$ for each $i = 1, 2$. Let the pair $(\mathbf{G}, \mathbf{U}_1)$ be as follows:*

(1) *for commutative $\mathbf{U}_1$, assume that $\mathbf{G} \neq E_6^2$;*
(2) *for Heisenberg $\mathbf{U}_1$, assume that $\mathbf{G} \neq A_2^2, B_n^2, D_n^2$;*
(3) *for $\mathbf{U}_1$ such that $Z(\mathbf{U}_1)$ is not the root group of a highest real root, assume that $\mathbf{G}_0 \neq E_6^2$, where $\mathbf{G}_0$ is the algebraic subgroup generated by $Z(\mathbf{U}_1)$ and $Z(\mathbf{U}_2)$;*
(4) *for $\mathbf{U}_1$ such that $Z(\mathbf{U}_1)$ is the root group of a highest real root, assume that $[\mathbf{U}_1, \mathbf{U}_1] \neq Z(\mathbf{U}_1)$ and $\mathbf{G}_0' \neq E_6^2$, where $\mathbf{G}_0'$ is the algebraic subgroup generated by $Z(\mathbf{U}_1')$ and $Z(\mathbf{U}_2')$ and where $\mathbf{U}_i'$ is the centralizer of the subgroup $\{g \in \mathbf{U}_i \mid gug^{-1}u^{-1} \in Z(\mathbf{U}_i)$ for all $u \in \mathbf{U}_i\}$ in $\mathbf{U}_i$.*

For any lattices F_1 and F_2 in U_1 and U_2, respectively, the subgroup Γ_{F_1,F_2} is discrete if and only if there exists a $\mathbb{Q}$-form on $\mathbf{G}$ such that Γ_{F_1,F_2} is a subgroup of finite index in $\mathbf{G}(\mathbb{Z})$ and hence a nonuniform arithmetic lattice in $G = \mathbf{G}(\mathbb{R})$.

Remark. As for the hypothesis on the pair $(\mathbf{G}, \mathbf{U}_1)$, if $\mathbf{G}$ is split over $\mathbb{R}$ and G is not locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$, then $\mathbf{U}_1$ can be any horospherical subgroup. If G is locally isomorphic to $\mathrm{SL}_3(\mathbb{R})$ (i.e., is of type A_2^2), then the above hypothesis excludes only the case when $\mathbf{U}_1$ is Heisenberg. If $\mathbb{R}$-rank $(G) \geq 3$, then $\mathbf{U}_1$ can be any commutative or Heisenberg horospherical subgroup.

4.2. The following is a special case of Margulis's superrigidity theorem (see [4, Chap. VIII, Thm. 3.6]).

THEOREM 4.2. *Let $\mathbf{G}$ be a connected almost $\mathbb{Q}$-simple algebraic group without any $\mathbb{R}$-anisotropic factors. Assume that $\mathbb{R}$-rank $G \geq 2$ and that $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is an arithmetic subgroup of G. Let l be any field of char 0, $\mathbf{H}$ a connected adjoint semisimple l-group, and $j : \Gamma \to \mathbf{H}(l)$ a homomorphism with the image being Zariski-dense in $\mathbf{H}$.*

Then there exists a rational l-epimorphism $\phi : \mathbf{G} \to \mathbf{H}$ such that $\phi(x) = j(x)$ for all $x \in \Gamma$.

4.3. We now prove the main theorem of this paper. The notation continues from Section 4.1.

THEOREM 4.3. *Let $\mathbf{G}_p$ be a connected semisimple adjoint $\mathbb{Q}_p$-algebraic group without any $\mathbb{Q}_p$-anisotropic factors for each $p \in S$. Let F_1 and F_2 be lattices in U_1 and U_2, respectively, such that Γ_{F_1,F_2} is discrete. Assume that $(\mathbf{G}_\infty, \mathbf{U}_{1\infty})$ satisfies the conditions in Theorem 4.1. Then there exists a $\mathbb{Q}$-form on $\mathbf{G}$ (in the sense of Section 1.5) such that Γ_{F_1,F_2} is a subgroup of finite index in the S-arithmetic subgroup $\mathbf{G}(\mathbb{Z}_S)$. Hence Γ_{F_1,F_2} is a nonuniform S-arithmetic lattice in G.*

Proof. Since $\Gamma^\infty_{F_1,F_2}$ is a discrete subgroup of G_∞ (by Proposition 2.1) containing the lattices F_1^∞ and F_2^∞ in $\mathbf{U}_{1\infty}(\mathbb{R})$ and $\mathbf{U}_{2\infty}(\mathbb{R})$, respectively, Theorem 4.1 implies that there exists a $\mathbb{Q}$-form on $\mathbf{G}_\infty$ such that $\Gamma^\infty_{F_1,F_2}$ is a subgroup of finite index in $\mathbf{G}_\infty(\mathbb{Z})$. By Proposition 3.3, the map $pr_\infty|_{\Gamma_{F_1,F_2}}$ is injective. Therefore we can define a map $j_p : \Gamma^\infty_{F_1,F_2} \to G_p$ as follows: For $x \in \Gamma^\infty_{F_1,F_2}$, set $j_p(x) = pr_p \circ (pr^\infty)^{-1}(x)$. It is clear from the definition of $\Gamma^\infty_{F_1,F_2}$ that $j_p(\Gamma^\infty_{F_1,F_2}) \subset \mathbf{G}_p(\mathbb{Z}_p)$.

We claim that $j_p(\Gamma^\infty_{F_1,F_2})$ is Zariski-dense in $\mathbf{G}_p$. Since the subgroup generated by $\mathbf{U}_{1p}$ and $\mathbf{U}_{2p}$ is Zariski-dense in $\mathbf{G}_p$, it suffices to show that the subgroup $j_p(F_i^\infty)$ is Zariski-dense in $\mathbf{U}_{ip}$ for each $i = 1, 2$. It is clear for $p = \infty$ since $j_\infty(F_i^\infty) = F_i^\infty$ is a lattice in $\mathbf{U}_{i\infty}(\mathbb{R})$. For $p \in S_f$, note that $j_p(F_i^\infty) = pr_p(F_i) \cap \mathbf{U}_{ip}(\mathbb{Z}_p)$. Since $pr_p(F_i)$ is dense in U_{ip} by Lemma 2.3 and since $\mathbf{U}_{ip}(\mathbb{Z}_p)$ is open in U_{ip}, $j_p(F_i^\infty)$ is dense in $\mathbf{U}_{ip}(\mathbb{Z}_p)$. Therefore the Zariski closure of $j_p(F_i^\infty)$ contains $\mathbf{U}_{ip}(\mathbb{Z}_p)$ and hence $\mathbf{U}_{ip}$, since it is well known that $\mathbf{U}_{ip}(\mathbb{Z}_p)$ is Zariski-dense in $\mathbf{U}_{ip}$.

By Theorem 4.2, for each $p \in S$, there exists a $\mathbb{Q}_p$-epimorphism $\phi_p : \mathbf{G}_\infty \to \mathbf{G}_p$ such that $\phi_p(x) = j_p(x)$ for all $x \in \Gamma^\infty_{F_1,F_2}$. Since $\mathbf{G}_\infty$ is absolutely simple in our case and hence has no nontrivial normal subgroup, ϕ_p is in fact an isomorphism. Therefore $(\mathbf{G}_\infty, (\phi_p, p \in S))$ endows a $\mathbb{Q}$-form on $\mathbf{G}$ with respect to which $\mathbf{U}_1$ and $\mathbf{U}_2$ are defined over $\mathbb{Q}$.

Since $F_i \subset \mathbf{U}_i(\mathbb{Q})$, F_i is commensurable with $\mathbf{U}_i(\mathbb{Z}_S)$ by Proposition 2.9. Since Γ_{F_1,F_2} is discrete, it follows from Theorem 3.5 that the subgroup Γ_{F_1,F_2} is commensurable with the S-arithmetic subgroup $\mathbf{G}(\mathbb{Z}_S)$. Since each $\mathbf{G}_p$ is adjoint, we can assume that $\mathbf{G}_p \subset \mathrm{SL}_N$ by considering the adjoint representation of $\mathbf{G}_p$. Moreover we may assume $\mathbf{G}(\mathbb{Q}) \subset \{\prod_{p\in S} g \mid g \in \mathrm{SL}_N(\mathbb{Q})\} = \mathrm{SL}_N(\mathbb{Q})$ by considering the isomorphisms ϕ_p. Since Γ_{F_1,F_2} is an S-arithmetic subgroup contained in $\mathbf{G}(\mathbb{Q})$, there exists a $\mathbb{Z}_S$-module L in $\mathbb{Q}^N$ of rank N that is invariant by Γ_{F_1,F_2} (cf. [7, Prop. 4.2]); hence $\Gamma_{F_1,F_2} \subset \mathbf{G}^L = \{g \in \mathbf{G}(\mathbb{Q}) \mid g(L) \subset L\}$. Now, by applying the automorphism of $\mathrm{SL}_N(\mathbb{C})$ that changes the standard basis to a basis of L, we may assume $\mathbf{G}(\mathbb{Z}_S) = \mathbf{G}^L$ so that $\Gamma_{F_1,F_2} \subset \mathbf{G}(\mathbb{Z}_S)$.

By Theorem 3.4, Γ_{F_1,F_2} is a lattice in G. Since the lattice $\Gamma^\infty_{F_1,F_2}$ in $\mathbf{G}_\infty$ contains a nontrivial unipotent element, $\Gamma^\infty_{F_1,F_2}$ is a nonuniform lattice by Godement's criterion (cf. [8]). Therefore, by the remark following Proposition 2.1, the lattice Γ_{F_1,F_2} is nonuniform. □

Proof of Theorem 0.2. The hypothesis on $(G_\infty, U_{1\infty})$ in Theorem 4.2 is satisfied for the groups considered in Theorem 0.2 by the remark following Theorem 4.1. To go from an adjoint group to its finite covers, we now give a standard argument. For each $p \in S$, there exists a connected semisimple adjoint $\mathbb{Q}_p$-group $\mathbf{G}'_p$ and a $\mathbb{Q}_p$-isogeny $f_p : \mathbf{G}_p \to \mathbf{G}'_p$ (cf. [4, Chap. I, Prop. 1.4.11]). Set $f = \prod_{p\in S} f_p$, the direct product of the f_p's. Set $F'_i = f(F_i)$ for each $i = 1, 2$, and let Γ'_{F_1,F_2} be the subgroup generated by F'_1 and F'_2. Since the kernel of f is finite, it follows that F'_i is a lattice in $f(U_i)$ and Γ'_{F_1,F_2} is discrete since $\Gamma'_{F_1,F_2} \subset f(\Gamma_{F_1,F_2})$. Hence by Theorem 4.3, there exists a $\mathbb{Q}$-form on $\mathbf{G}' = \prod_{p\in S} \mathbf{G}'_p$ such that Γ'_{F_1,F_2} is a subgroup of finite index in $\mathbf{G}'(\mathbb{Z}_S)$. Since $f(\Gamma_{F_1,F_2})$ is a discrete subgroup containing the S-arithmetic subgroup Γ'_{F_1,F_2}, the subgroup $f(\Gamma_{F_1,F_2})$ is commensurable with $\mathbf{G}'(\mathbb{Z}_S)$. Hence Γ_{F_1,F_2} is an S-arithmetic subgroup of G by the definition in Section 1.6. Hence Theorem 0.2 is proved. □

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