SHRINKING TARGETS FOR THE GEODESIC FLOW ON GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

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ABSTRACT. Let $\mathcal{M}$ be a geometrically finite hyperbolic manifold. We present a very general theorem on the shrinking target problem for the geodesic flow, using its exponential mixing. This includes a strengthening of Sullivan's logarithm law for the excursion rate of the geodesic flow. More generally, we prove logarithm laws for the first hitting time for shrinking cusp neighborhoods, shrinking tubular neighborhoods of a closed geodesic, and shrinking metric balls, as well as give quantitative estimates for the time a generic geodesic spends in such shrinking targets.

1. INTRODUCTION

Let $\mathcal{M}$ be a complete hyperbolic manifold of dimension $n \geq 2$. Denote by $\mathcal{G}^t$ the geodesic flow on the unit tangent bundle $\mathrm{T}^1(\mathcal{M})$. If $\mathcal{M}$ is of finite volume, but non-compact, Sullivan [27] showed in 1982 the following logarithm law for the rate of the excursion of the geodesic flow: for any $o \in \mathcal{M}$, and for almost all $x \in \mathrm{T}^1(\mathcal{M})$,

$$\limsup_{t\to\infty} \frac{d(\mathcal{G}^t(x), o)}{\log t} = \frac{1}{n-1} \tag{1.1}$$

where $d(\mathcal{G}^t(x), o)$ is the hyperbolic distance between the basepoint of $\mathcal{G}^t(x)$ and o.

This result can be viewed as a special case of the so-called shrinking target problem for the geodesic flow, which asks the behavior of a generic geodesic ray with respect to a given sequence of shrinking subsets. Indeed, if we consider the family of shrinking cuspidal neighborhoods $\mathfrak{h}_t := \{z \in \mathcal{M} : d(o, z) > t\}$, $t > 1$, then (1.1) is equivalent to the following logarithm law for the first hitting time: for almost all x,

$$\liminf_{t\to\infty} \frac{\log \tau_{\mathfrak{h}_t}(x)}{t} = n - 1 \tag{1.2}$$

where $\tau_{\mathfrak{h}_t}(x) := \inf\{s > 0 : \mathcal{G}^s(x) \in \mathfrak{h}_t\}$.

In this paper, we investigate shrinking target problems for the geodesic flow on a geometrically finite hyperbolic manifold $\mathcal{M}$, and prove results

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which are far reaching strengthening and generalizations of (1.2), and hence of (1.1).

Let $\mathbb{H}^n$ denote the n-dimensional hyperbolic space and let $G := \mathrm{Isom}^+(\mathbb{H}^n)$ be the group of all orientation preserving isometries. We may present a complete hyperbolic manifold $\mathcal{M}$ as the quotient $\Gamma\backslash\mathbb{H}^n$ where Γ is a torsion-free discrete subgroup of G. We assume that Γ is Zariski dense and geometrically finite in the whole paper. Denote by $\Lambda \subset \partial\mathbb{H}^n$ the limit set of Γ and by $0 < \delta \leq n-1$ the critical exponent of Γ. The maximal entropy of the geodesic flow on $\mathrm{T}^1(\mathcal{M})$ is given by δ, and there exists a unique ergodic probability measure of maximal entropy, called the Bowen-Margulis-Sullivan measure, which we denote by m. The support of m is precisely the non-wandering set for the geodesic flow, and hence the shrinking target problem in this setting is interesting only for those shrinking subsets in the support of m and for m-almost all points. Now since $\mathcal{G}^t$ is ergodic for m, the Birkhoff ergodic theorem says that for a given Borel subset $B \subset \mathrm{T}^1(\mathcal{M})$, we have the following: for m-almost all $x \in \mathrm{T}^1(\mathcal{M})$,

$$\lim_{t\to\infty} \frac{\ell\{0 < s < t : \mathcal{G}^s(x) \in B\}}{t} = \mathsf{m}(B), \tag{1.3}$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$. The shrinking target problem asks a finer question on the set of times $\{s > 0 : \mathcal{G}^s(x) \in B_t\}$ for a given family $\{B_t\}$ of shrinking sets and for m-a.e. x. The three main questions we address in this paper for m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$ are as follows:

(1) (Logarithm laws) Is there a logarithm law for the first hitting time

$$\tau_{B_t}(x) := \inf\{s > 0 : \mathcal{G}^s(x) \in B_t\}? \tag{1.4}$$

(2) (Shrinking rate threshold) How fast can B_t shrink so that

$$\tau_{B_t}(x) < t$$

for an infinite sequence of times t tending to ∞ or for all sufficiently large $t \gg 1$?

(3) (Quantitative estimates) How fast can B_t shrink so that[1]

$$\ell\{0 < s < t : \mathcal{G}^s(x) \in B_t\} \asymp t \cdot \mathsf{m}(B_t)$$

for an infinite sequence of times t tending to ∞, or for all sufficiently large $t \gg 1$?

In order to address the above questions, we need to impose certain regularity conditions on the shrinking targets. Let $K < G$ be a maximal compact subgroup and identify $\mathcal{M}$ with $\Gamma\backslash G/K$. There exists a one parameter diagonalizable subgroup $A = \{a_t\}$ so that if M denotes the centralizer of A in K, then the unit tangent bundle $\mathrm{T}^1(\mathcal{M})$ can be identified with $\Gamma\backslash G/M$ in the way that the geodesic flow $\mathcal{G}^t$ on $\mathrm{T}^1(\mathcal{M})$ corresponds to the right

[1]The notation $f_t \ll g_t$ means that for all $t > 1$, $f_t \leq c\, g_t$ for some absolute constant $c > 0$, and we write $f_t \asymp g_t$ if $f_t \ll g_t$ and $g_t \ll f_t$. We sometimes indicate the dependence of the implied constant in subscripts.

translation action of a_t on $\Gamma\backslash G/M$. We fix $\ell \gg \dim(\mathcal{M})$ and the Sobolev norm $\mathcal{S} = \mathcal{S}_{\infty,\ell}$ on $C^\infty(\Gamma\backslash G)$ given by

$$\mathcal{S}(\Psi) = \sum \|D(\Psi)\|_\infty$$

where the sum is taken over all monomials in a fixed basis of $\mathrm{Lie}(G)$ of order at most ℓ.

A family of shrinking targets in $\mathrm{T}^1(\mathcal{M})$ means a collection $\mathcal{B} = \{B_t \subset \mathrm{T}^1(\mathcal{M}) : t > 1\}$ such that $\mathsf{m}(B_t) > 0$, $B_t \supset B_s$ for $s > t$, and $\lim_{t\to\infty} \mathsf{m}(B_t) = 0$. A family $\{B_t\}$ of shrinking targets is said to be *inner regular* (resp. *outer regular*) if there exist $\alpha > 0$ and a family of functions $\Psi_t^- \in C^\infty(\mathrm{T}^1(\mathcal{M}))$ (resp. $\Psi_t^+ \in C^\infty(\mathrm{T}^1(\mathcal{M}))$) such that

- $0 \le \Psi_t^- \le \mathrm{Id}_{B_t}$ (resp. $\mathrm{Id}_{B_t} \le \Psi_t^+ \ll 1$);
- $\mathsf{m}(B_t) \ll \mathsf{m}(\Psi_t^-)$ (resp. $\mathsf{m}(\Psi_t^+) \ll \mathsf{m}(B_t)$);
- $\mathcal{S}(\Psi_t^\pm) \ll \mathsf{m}(B_t)^{-\alpha}$

where the implied constants are independent of t. A family $\{B_t\}$ is said to be *regular* if it is both inner and outer regular.

We note that this regularity condition is rather mild, and is satisfied by most families of naturally occurring shrinking targets. Such examples include shrinking cusp neighborhoods, shrinking tubular neighborhoods of a closed geodesic and shrinking metric balls, as will be shown later.

In the rest of the introduction, we assume that $\mathcal{B} = \{B_t : t \gg 1\}$ is a family of shrinking targets in $\mathrm{T}^1(\mathcal{M})$.

1.1. **Logarithm laws.** For discrete time dynamical systems, it is expected that the first hitting time would be inversely proportional to the measure of the shrinking target; it is indeed the case for the discretized geodesic flow. For the continuous geodesic flow, it turns out that it is inversely proportional to the measure of a thickened set $\tilde{B}_t := \cup_{|s|<1/2}\mathcal{G}^s(B_t)$:

Theorem 1.1. (1) *If $\{B_t\}$ is inner regular, then*

$$\lim_{t\to\infty} \frac{\log(\tau^d_{B_t}(x))}{-\log(\mathsf{m}(B_t))} = 1 \quad \text{for } \mathsf{m}\text{-a.e. } x \in \mathrm{T}^1(\mathcal{M}).$$

where $\tau^d_B(x) = \min\{n \in \mathbb{N} : \mathcal{G}^n x \in B\}$.

(2) *If $\{\tilde{B}_t\}$ is inner regular, then*

$$\lim_{t\to\infty} \frac{\log(\tau_{B_t}(x))}{-\log(\mathsf{m}(\tilde{B}_t))} = 1 \quad \text{for } \mathsf{m}\text{-a.e. } x \in \mathrm{T}^1(\mathcal{M}).$$

Remark 1.5. When $|\log \mathsf{m}(\tilde{B}_t)| \asymp |\log \mathsf{m}(B_t)|$, the first hitting time for the discrete flow $\{\mathcal{G}^n : n \in \mathbb{N}\}$ behaves in the same way for the continuous flow. This is indeed the case for shrinking cusp neighborhoods or tubular neighborhoods of a closed geodesic. However, there are also cases when $|\log \mathsf{m}(\tilde{B}_t)|$ is much larger than $|\log \mathsf{m}(B_t)|$, such as the case of shrinking metric balls.

We note that logarithm laws for the first hitting time were studied for certain families of shrinking targets in many examples of discrete time dynamical systems with fast mixing, see e.g. [6, 7, 8].

1.2. **Shrinking rate threshold.** In order to ensure that a generic orbit $\mathcal{G}^s(x)$ hits B_t before time t for an infinite sequence of t tending to ∞, the easy half of the Borel-Canteli lemma implies that it is necessary to have $\sum_k \mathsf{m}(\tilde{B}_k) = \infty$, from which $\limsup_{t\to\infty} \log^2(t) t \cdot \mathsf{m}(\tilde{B}_t) = \infty$ follows. The first part of the following theorem says that this condition is also sufficient, up to logarithmic factors. The second part says that a generic orbit $\mathcal{G}^s(x)$ hits B_t before time t, for all sufficiently large t, under a slightly stronger assumption on the rate of shrinking (see Theorem 4.10).

Theorem 1.2. *Suppose that $\{\tilde{B}_t\}$ is inner regular.*

(1) *If $\limsup_{t\to\infty} \frac{t\mathsf{m}(\tilde{B}_t)}{|\log(\mathsf{m}(\tilde{B}_t))|} = \infty$, then*

$$\liminf_{t\to\infty} \frac{\tau_{B_t}(x)}{t} \le 1 \quad \text{for } \mathsf{m}\text{-a.e. } x \in \mathrm{T}^1(\mathcal{M}).$$

(2) *If $\sum_{j=1}^{\infty} \frac{|\log(\mathsf{m}(\tilde{B}_{t_j}))|}{t_j \mathsf{m}(\tilde{B}_{t_j})} < \infty$ for some sequence $t_j \to \infty$, then*

$$\limsup_{t\to\infty} \frac{\tau_{B_t}(x)}{t} \le 1 \quad \text{for } \mathsf{m}\text{-a.e. } x \in \mathrm{T}^1(\mathcal{M}).$$

1.3. **Quantitative estimates.** In order to answer a more refined question regarding the amount of time that a generic geodesic ray spends in a shrinking target, we require our family of targets to be regular and their measures do not change too fast in the sense that $\mathsf{m}(B_t) \asymp \mathsf{m}(B_{2t})$.

With these additional regularity assumptions, we have the following (see Theorem 4.7 below for a more general result).

Theorem 1.3. *Suppose that $\{B_t\}$ is regular and that $\mathsf{m}(B_{2t}) \asymp \mathsf{m}(B_t)$.*

(1) *If $\limsup_{t\to\infty} \frac{t\mathsf{m}(B_t)}{|\log(\mathsf{m}(B_t))|} = \infty$, then there exists a sequence $t_k \to \infty$ such that for m-a.e. x,*

$$\frac{\ell\{0 < s < t_k : \mathcal{G}^s(x) \in B_{t_k}\}}{t_k} \asymp \mathsf{m}(B_{t_k}).$$

(2) *If $\sum_{j=1}^{\infty} \frac{|\log(\mathsf{m}(B_{2^j}))|}{2^j \mathsf{m}(B_{2^j})} < \infty$, then for m-a.e. x,*

$$\frac{\ell\{0 < s < t : \mathcal{G}^s(x) \in B_t\}}{t} \asymp \mathsf{m}(B_t).$$

We observe that unlike Theorems 1.1 and 1.2, the amount of time that the geodesic flow spends in the targets is governed by the measure of the original targets rather than by their thickenings.

Remark 1.6. (1) We note that in many examples the measure of the shrinking targets decays like $\mathsf{m}(B_t) \asymp t^{-\eta}$ for some $\eta > 0$. In such

cases, we have $\mathsf{m}(B_t) \asymp \mathsf{m}(B_{2t})$ and the rest of the conditions of Theorems 1.2 and 1.3 are satisfied if $\eta < 1$.

(2) As mentioned before, the extra conditions on the rate of decay we have in Theorems 1.2 and 1.3 are sharp, but up to logarithmic factors. While it would be very interesting to have sharp conditions on the nose, we note that such a result is notoriously hard. Even when $\mathcal{M}$ has finite volume, sharp results regarding Theorem 1.3(1) are known only in some very special cases when the shrinking targets are cusp neighborhoods [27], or spherical balls [21] (or general spherical targets if one considers discrete time dynamics [13]). There are no known sharp results regarding Theorem 1.3(2). We refer to [15] where this kind of problem is studied for systems with almost perfect mixing.

(3) All the results described above still hold as stated if we replace the unit tangent bundle $\mathrm{T}^1(\mathcal{M})$ with the frame bundle $\Gamma\backslash G$, provided $\delta > n-2$. We note if $\mathcal{M}$ contains a co-dimension one properly immersed totally geodesic sub-manifold of finite volume, then $\delta > n-2$, so this stronger condition still holds in many examples.

For some concrete applications of these results, we discuss three families of shrinking targets to which our theorems apply. In order to define these families, we fix a left G-invariant and right K-invariant metric d on G which descends to the hyperbolic metric on $\mathbb{H}^n = G/K$. This metric then naturally defines a distance function, $\mathrm{dist}(\cdot,\cdot)$ on $\mathrm{T}^1(\mathcal{M}) = \Gamma\backslash G/M$.

1.4. **Cusp excursion.** The convex core of $\mathcal{M}$ is defined by $\mathrm{core}(\mathcal{M}) = \Gamma\backslash \mathrm{hull}(\Lambda)$, where $\mathrm{hull}(\Lambda)$ defines the convex hull of the limit set Λ. As $\mathcal{M}$ is geometrically finite, there are finitely many disjoint cuspidal regions whose complement in $\mathrm{core}(\mathcal{M})$ is a compact submanifold. Let $\mathfrak{h}_i$, $1 \le i \le k$, denote the pre-images in $\mathrm{T}^1(\mathcal{M})$ of these cuspidal regions under the base point projection $\pi : \mathrm{T}^1(\mathcal{M}) \to \mathcal{M}$. For each i, we denote by κ_i the rank of $\mathfrak{h}_i$, that is, the rank of the maximal free abelian subgroup of the stabilizer $\mathrm{Stab}_\Gamma(\mathfrak{h}_i)$. It is known that $\kappa_i < 2\delta$.

For each i and $t > 1$, consider the following cusp neighborhood

$$\mathfrak{h}_{i,t} := \{x \in \mathfrak{h}_i : \mathrm{dist}(x, \partial\mathfrak{h}_i) > t\}. \tag{1.7}$$

For each i, we show that the shrinking family $\{\mathfrak{h}_{i,t} : t > 1\}$ is regular and that

$$\mathsf{m}(\mathfrak{h}_{i,t}) \asymp e^{-(2\delta-\kappa_i)t}. \tag{1.8}$$

(see section 5.1). Applying our results to this family, we get the following:

Theorem 1.4. *Fix $1 \le i \le k$.*

(1) *For m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$,*

$$\lim_{t\to\infty} \frac{\log \tau_{\mathfrak{h}_{i,t}}(x)}{t} = 2\delta - \kappa_i.$$

(2) *For any* $0 < \eta < \frac{1}{2\delta - \kappa_i}$, *and for* m*-a.e.* $x \in \mathrm{T}^1(\mathcal{M})$,

$$\ell\{0 < s < t : \mathcal{G}^s(x) \in \mathfrak{h}_{i,\eta \log t}\} \asymp t^{1-\eta(2\delta-\kappa_i)}.$$

Remark 1.9. As mentioned before, it is not hard to show that

$$\liminf_{t\to\infty} \frac{\log(\tau_{\mathfrak{h}_t}(x))}{t} = \left(\limsup_{t\to\infty} \frac{\mathrm{dist}(\mathcal{G}^t(x), o)}{\log t}\right)^{-1} \tag{1.10}$$

where $\mathfrak{h}_t = \bigcup_{1\le i\le k} \mathfrak{h}_{i,t}$. Stratmann and Velani showed that (1.10) is equal to $2\delta - \max_i \kappa_i$ [28], and hence extended Sullivan's logarithm law (1.1) to geometrically finite manifolds. Theorem 1.4(1) presents a stronger version, as we consider excursion to individual cusps as well as obtain an actual limit rather than lim inf.

For the sake of a concrete application, we give a reformulation of Theorem 1.4(1) in the case of Apollonian manifolds. An Apollonian gasket $\mathcal{P} = \bigcup C_i$ is a countable union of circles obtained by repeatedly inscribing circles into the triangular interstices of four mutually tangent circles with disjoint interiors in the complex plane (where lines are considered as circles). The symmetry group $\{g \in \mathrm{PSL}_2(\mathbb{C}) : g(\mathcal{P}) = \mathcal{P}\}$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ which acts on $\widehat{\mathbb{C}}$ by Möbius transformations and its torsion-free subgroup of finite index is called an Apollonian group, which we denote by Γ. Via the Poincaré extension theorem, we can identify $\mathrm{PSL}_2(\mathbb{C})$ with $\mathrm{Isom}^+(\mathbb{H}^3)$ for the upper-half space model $\mathbb{H}^3$ of the hyperbolic space. The quotient manifold $\Gamma\backslash\mathbb{H}^3$ is called an Apollonian manifold, which is known to be geometrically finite with all cusps having rank one. Its limit set is equal to the closure $\overline{\mathcal{P}}$, and supports a locally finite Hausdorff measure $\mathcal{H}$ of dimension $\delta = 1.30568(8)$ [11].

Fix a tangent point $\xi = C_i \cap C_j$ for $i \neq j$ and consider a sufficiently small Euclidean ball B in $\mathbb{H}^3$ based at ξ, so that $\mathcal{B} = \Gamma(B)$ is a disjoint collection of Euclidean balls.

Fix $o \in \mathbb{H}^3$ outside of B, let $B(t) \subset B$ be the Euclidean ball based at ξ and $d_{\mathbb{H}^3}(o, B(t)) = d_{\mathbb{H}^3}(o, B) + t$. Set $\mathcal{B}_t := \Gamma(B(t))$.

The following is a consequence of Theorem 1.4:

Corollary 1.5. *Let* $\mathcal{P}$ *be an Apollonian gasket. For* $\mathcal{H}$*-almost all initial direction* v *toward* $\overline{\mathcal{P}}$,

$$\lim_{t\to\infty} \frac{\log(\inf\{s > 0 : v_s \in \mathcal{B}_t\})}{t} = 2\delta - 1 (= 1.6113...) \tag{1.11}$$

where v_s *denotes the base point of* $\mathcal{G}^s(v)$.

1.5. **Tubular neighborhoods.** Another natural family of shrinking targets is given by tubular neighborhoods of a closed geodesic. For a closed geodesic $\mathcal{C} \subset \mathrm{T}^1(\mathcal{M})$ and $\epsilon > 0$, we consider the ϵ-tubular neighborhood of $\mathcal{C}$:

$$\mathcal{C}_\epsilon := \left\{x \in \mathrm{T}^1(\mathcal{M}) : \mathrm{dist}(x, \mathcal{C}) \le \epsilon\right\}.$$

The family $\{\mathcal{C}_{1/t} : t > 1\}$ forms a family of shrinking neighborhoods of $\mathcal{C}$. We show that $\{\mathcal{C}_{1/t} : t > 1\}$ is a regular family with $\mathsf{m}(\mathcal{C}_{1/t}) \asymp \mathsf{m}(\tilde{\mathcal{C}}_{1/t}) \asymp t^{-2\delta}$. Applying our results to this family of shrinking targets gives the following result on the amount of time a generic geodesic spirals near a fixed closed geodesic (cf. [10, Theorem 1.1] for a similar result in a negatively curved compact manifold).

Theorem 1.6. *Let $\mathcal{C} \subset \mathrm{T}^1(\mathcal{M})$ be a closed geodesic. Then for m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$, we have the following:*

(1)
$$\lim_{t\to\infty} \frac{\log \tau_{\mathcal{C}_{1/t}}(x)}{\log t} = 2\delta;$$

(2) *For any $0 < \eta < \frac{1}{2\delta}$ and for all $t > 1$,*
$$\ell\{0 < s < t : \mathrm{dist}(\mathcal{G}^s(x), \mathcal{C}) < t^{-\eta}\} \asymp t^{1-2\delta\eta}.$$

Remark 1.12. Since for any $x \in \mathrm{T}^1(\mathcal{M})$ we have that
$$\liminf_{t\to\infty} \frac{\log(\tau_{\mathcal{C}_{1/t}}(x))}{\log t} = \left(\limsup_{t\to\infty} \frac{-\log(\mathrm{dist}(\mathcal{G}^t(x), \mathcal{C}))}{\log t}\right)^{-1},$$
Theorem 1.6 (1) implies that for m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$,
$$\limsup_{t\to\infty} \frac{-\log(\mathrm{dist}(\mathcal{G}^t(x), \mathcal{C}))}{\log t} = \frac{1}{2\delta}, \tag{1.13}$$
which was previously shown in [4, Theorem 4] to hold for the special case of convex co-compact hyperbolic *surfaces*.

1.6. **Shrinking balls.** For any fixed $x_0 \in \mathrm{supp}(\mathsf{m})$, we show that the family of shrinking metric balls $B_t(x_0) := \{x \in \mathrm{T}^1(\mathcal{M}) : \mathrm{dist}(x, x_0) < 1/t\}$ is regular and satisfies $\mathsf{m}(B_t(x_0)) \asymp \mathsf{m}(B_{2t}(x_0))$. When Γ is convex co-compact, $\mathsf{m}(B_t(x_0)) \asymp t^{-(2\delta+1)}$ and $\mathsf{m}(\tilde{B}_t(x_0)) \asymp t^{-2\delta}$ (see §5.2). In particular our results imply the following:

Theorem 1.7. *Let $\mathcal{M}$ be convex cocompact. Fix $x_0 \in \mathrm{supp}(\mathsf{m})$. Then for m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$,*

(1)
$$\lim_{t\to\infty} \frac{\log \tau_{B_t(x_0)}(x)}{\log t} = 2\delta. \tag{1.14}$$

(2) *For $0 < \eta < \frac{1}{2\delta+1}$, we have*
$$\ell\{0 < s < t : \mathrm{dist}(\mathcal{G}^s(x), x_0) \le t^{\eta}\} \asymp t^{1-(2\delta+1)\eta}.$$

When $\mathcal{M}$ has cusps, the situation is more complicated as $\mathsf{m}(B_t(x_0))$ can fluctuate, with the fluctuation depending on x_0 (or more precisely on the cusp excursions of the geodesic emanating from $x_0 \in \mathrm{T}^1(\mathcal{M})$). Combining our previous results on cusp excursions, we can show the following

Theorem 1.8. *Suppose that $\mathcal{M}$ has cusps.*

(1) *For* m*-a.e.* $x_0 \in \mathrm{T}^1(\mathcal{M})$*, and for* m*-a.e.* $x \in \mathrm{T}^1(\mathcal{M})$*,*

$$\lim_{t\to\infty} \frac{\log \tau_{B_t(x_0)}(x)}{\log t} = 2\delta.$$

(2) *For any pair of distinct cusps of ranks* κ_1, κ_2*, we can find* $x_0 \in \mathrm{T}^1(\mathcal{M})$ *such that for* m*-a.e.* $x \in \mathrm{T}^1(\mathcal{M})$*,*

$$\lim_{t\to\infty} \frac{\log(\tau_{B_t(x_0)}(x))}{\log t} = 4\delta - \kappa_1 - \kappa_2.$$

Remark 1.15. We note that if $\mathcal{M}$ has finite volume, then $\delta = n-1$ and $\mathsf{m}(\tilde{B}_t(x_0)) \asymp t^{-(2n-2)}$. Hence, in this case, the same arguments imply that for m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$, we have $\lim_{t\to\infty} \frac{\log \tau_{B_t(x_0)}(x)}{\log t} = 2(n-1)$. We note that here the shrinking targets are in $\mathrm{T}^1(\mathcal{M})$, unlike the results of [21] which considered shrinking balls inside $\mathcal{M}$, in which case the limit is $n-1$ (see also [17], for related result for the discrete time geodesic flow).

1.7. **Strategy of proof.** First we define an averaging operator, along the discrete time, acting on $L^2(\mathrm{T}^1(\mathcal{M}), \mathsf{m})$:

$$\lambda_T(\Psi)(x) = \frac{1}{T}\sum_{k=1}^{T} \Psi(\mathcal{G}^k(x)).$$

If Ψ is the characteristic function of B, we simply write $\lambda_T(B)$ instead of $\lambda_T(1_B)$. The Birkhoff ergodic theorem implies that for a.e. $x \in X$,

$$\lim_{T\to\infty} \lambda_T(\Psi)(x) = \int_{\mathrm{T}^1(\mathcal{M})} \Psi d\mathsf{m}.$$

We note that if we had a rate control in this convergence such as

$$|\lambda_T(B_t)(x) - \mathsf{m}(B_t)| \ll \frac{\sqrt{\mathsf{m}(B_t)}|\log(\mathsf{m}(B_t))|}{\sqrt{T}}, \tag{1.16}$$

we would get

$$\log \tau^d_{B_t}(x) \le |\log \mathsf{m}(B_t)| + 2\log|\log \mathsf{m}(B_t)| \tag{1.17}$$

just from the simple observation that $\lambda_{\tau^d_{B_t}(x)}(B_t) = 0$.

An estimate like (1.16) is too strong to be true for a.e. individual points x. So, instead, we prove its mean-version for all smooth functions $\Psi \in L^2(\mathrm{T}^1(\mathcal{M}), \mathsf{m})$, that is,

$$\|\lambda_T(\Psi) - \mathsf{m}(\Psi)\|_2 \le C\frac{\|\Psi\|\log(\frac{\mathcal{S}(\Psi)}{\|\Psi\|_2})}{\sqrt{T}} \tag{1.18}$$

for some uniform constant $C > 0$. The regularity conditions imposed on the thickenings $\tilde{B}_t$ of our shrinking targets are precisely so that we could apply

(1.18) to smooth functions which approximates $1_{\tilde{B}_t}$ and deduce

$$\|\lambda_T(\tilde{B}_t) - \mathsf{m}(\tilde{B}_t)\|_2 \ll \frac{\sqrt{\mathsf{m}(\tilde{B}_t)}\log|(\mathsf{m}(\tilde{B}_t))|}{\sqrt{T}}. \tag{1.19}$$

This effective mean ergodic theorem for $\tilde{B}_t$'s enables us to obtain that for a.e. x,

$$\log \tau^d_{\tilde{B}_t}(x) \le |\log \mathsf{m}(\tilde{B}_t)| + O(\log|\log \mathsf{m}(\tilde{B}_t)|), \tag{1.20}$$

for all sufficiently large t. Using that $|\tau^d_{\tilde{B}_t}(x) - \tau_{B_t}(x)| \le 1$, we deduce that

$$\limsup_{t\to\infty} \frac{\log \tau_{B_t}(x)}{-\log \mathsf{m}(\tilde{B}_t)} \le 1.$$

This is the non-trivial direction of the logarithm law Theorem 1.1; the other direction holds for general shrinking targets in any dynamical system (see e.g. [14, Lemma 2.2]). Theorems 1.2 and 1.3 are also proved in a similar spirit using the effective mean ergodic theorem.

The use of quantitative mixing of geodesic flow in the shrinking target problem in the homogeneous setting goes back to the work of Kleinbock and Margulis [16], and the idea of using an effective mean ergodic theorem was first introduced in [9] and more explicitly in [13, 14], where these ideas were used to prove the analogous results for finite volume hyperbolic manifold. In this paper we follow closely the arguments of [13], in particular, for the discretized geodesic flows; once we establish the relevant result on exponential decay of matrix coefficients as in Theorem 1.9, the arguments are essentially identical, although establishing the needed regularity conditions for explicit examples is technically more difficult in the infinite volume case (see Section 5). On the other hand, our results for continuous flows are new even for the finite volume case. It is worthwhile mentioning that, for spherical shrinking targets, the results of [13] also dealt with unipotent flows having only polynomial decay of matrix coefficients, and established a dynamical Borel-Cantelli Lemma, which is stronger than the logarithm law. While it would be very interesting to extend such results to the infinite volume setting, our current methods do not seem to apply.

Here we will use the following exponential decay of matrix coefficients for geometrically finite hyperbolic manifolds:

Theorem 1.9. *There exists $\eta_0 > 0$ such that for any $\Psi_1, \Psi_2 \in C^\infty(\mathrm{T}^1(\mathcal{M}))$ with support in one-neighborhood of* $\mathrm{supp}(\mathsf{m})$*, for all $t \ge 1$,*

$$\int_{\mathrm{T}^1(\mathcal{M})} \Psi_1(\mathcal{G}^t(x))\Psi_2(x)\, d\mathsf{m}(x) = \mathsf{m}(\Psi_1)\mathsf{m}(\Psi_2) + O(e^{-\eta_0 t}\mathcal{S}(\Psi_1)\mathcal{S}(\Psi_2)). \tag{1.21}$$

Moreover, η_0 is explicitly computable when $\delta > \frac{n-1}{2}$, depending only on the spectral gap for the Laplacian on $L^2(\mathcal{M})$. If Γ is convex cocompact or $\delta > n-2$, (1.21) with m replaced by its M-invariant lift on $\Gamma\backslash G$ holds for any $\Psi_1, \Psi_2 \in C^\infty(\Gamma\backslash G)$.

This theorem was obtained in ([22], [5]) for compactly supported functions under the assumption $\delta > \frac{n-1}{2}$ and in [25] for any convex cocompact Γ (see also [24] for the same result for the frame flow). In order to study shrinking target problem for cusp neighborhoods as described in Theorem 1.4, removing the compact support condition is crucial as we need to study functions that are positive on cusps. We use the quantitative decay of the matrix coefficient of the functions $L^2(\Gamma\backslash G)$ with respect to the Haar measure $\mathsf{m}^{\mathrm{Haar}}$ in [22], and exploit the product structures of m and $\mathsf{m}^{\mathrm{Haar}}$ to transfer the exponential rate information on the transversal intersections of $\mathcal{G}^t(B_\epsilon(x))$ for the flow box $B_\epsilon(x)$, that we get from the behavior of the correlation function with respect to $\mathsf{m}^{\mathrm{Haar}}$, to the behavior of the correlation function with respect to m. Here ϵ depends on the injectivity radius of x, and as we need to control the exponential rate independent of the injectivity radius for Theorem 1.9, which is required to deal with functions which are not compactly supported, the whole procedure turns out to be technically quite subtle. The remaining cases of geometrically finite manifolds with cusps are proved in a recent work of Li-Pan [20].

After some preliminaries given in section 2, we devote section 3 to the proof of Theorem 1.9. With this result in hand, we prove effective mean ergodic theorem in this setting (see Theorem 4.1), and use it in section 4 to establish results on shrinking target problems for both the discrete and continuous time flow. While the results we obtain for the discrete time flow are essentially optimal, this is not the case for some of the results for continuous time flow. Nevertheless, in section 4.5, we show how one can obtain optimal results for the continuous flow by translating it into a discrete time flow problem for a thickened target. In section 5, we deduce Theorems 1.4, 1.6, 1.7 and 1.8 by proving the regularity of the corresponding shrinking sets and by computing their volumes using Sullivan's shadow lemma and the structure of cusps for geometrically finite manifolds.

2. Preliminaries and notation

2.1. **Notations and conventions.** Let $G \cong \mathrm{SO}(n,1)^o$ be the group of orientation preserving isometries of $\mathbb{H}^n$, and $\Gamma < G$ a geometrically finite, torsion-free, Zariski dense, discrete subgroup of G. We denote by Λ the limit set of Γ, and by $0 < \delta \leq n-1$ the Hausdorff dimension of Λ, which is equal to the critical exponent of Γ. Let $\mathcal{M} = \Gamma\backslash\mathbb{H}^n$. Let $K < G$ be a maximal compact subgroup and identify $\mathcal{M}$ with $\Gamma\backslash G/K$. There exists a one parameter diagonalizable subgroup $A = \{a_t\}$ so that if M denotes the centralizer of A in K, then the unit tangent bundle $\mathrm{T}^1(\mathcal{M})$ can be identified with $\Gamma\backslash G/M$ in the way that the geodesic flow $\mathcal{G}^t$ on $\mathrm{T}^1(\mathcal{M})$ corresponds to the right translation action of a_t on $\Gamma\backslash G/M$. With this identification we can work in the homogeneous space $\Gamma\backslash G$ and think of subsets and functions on $\mathrm{T}^1(\mathcal{M})$ and $\mathcal{M}$ respectively as M-invariant (resp. K invariant) subsets and functions on $\Gamma\backslash G$.

We say that two families $\{B_t\}$ and $\{A_t\}$ of shrinking sets are Lipschitz equivalent and write $B_t \asymp A_t$, if there are some positive constants c_1, c_2 such that $B_{c_1 t} \subseteq A_t \subseteq B_{c_2 t}$ for all $t > 1$.

We fix a left G-invariant and right K-invariant metric d on G which descends to the hyperbolic metric on $\mathbb{H}^n = G/K$. This induces a unique metric on G/M which we will also denote by d by abuse of notation. The metric d defines a distance function on $\mathrm{T}^1(\mathcal{M}) = \Gamma\backslash G/M$ given by $\mathrm{dist}(\Gamma g, \Gamma h) = \inf_{\gamma\in\Gamma} d(\gamma g, h)$.

2.2. **Invariant measures.** For $\xi \in \partial\mathbb{H}^n$, let $\beta_\xi : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ denote the Busemann function for the geodesic flow, defined by

$$\beta_\xi(x, y) = \lim_{t\to\infty} d(x, \xi(t)) - d(y, \xi(t)),$$

with $\xi(t)$ a unit speed geodesic ray toward ξ. A family of measures $\{\mu_x : x \in \mathbb{H}^n\}$ is called a Γ-*invariant conformal density* of dimension $\delta_\mu > 0$ on $\partial\mathbb{H}^n$, if each μ_x is a non-zero finite Borel measure on $\partial\mathbb{H}^n$ satisfying for any $x, y \in \mathbb{H}^n$, $\xi \in \partial\mathbb{H}^n$ and $\gamma \in \Gamma$,

$$\gamma_*\mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_\mu \beta_\xi(y,x)},$$

where $\gamma_*\mu_x(F) = \mu_x(\gamma^{-1}(F))$ for any Borel subset F of $\partial\mathbb{H}^n$.

In particular, the *Patterson-Sullivan density* $\{\nu_x\}$ is a Γ-invariant conformal density supported on the limit set Λ of dimension δ and the *Lebesgue density* $\{m_x\}$ is a G-invariant conformal density of dimension $(n-1)$ (both are unique up to scalar multiplications).

Let $\pi : \mathrm{T}^1(\mathbb{H}^n) \to \mathbb{H}^n$ be the basepoint projection. For $u \in \mathrm{T}^1(\mathbb{H}^n)$, we denote by $u^\pm \in \partial\mathbb{H}^n$ the forward and the backward endpoints of the geodesic determined by u. Fix $o \in \mathbb{H}^n$ so that K fixes o. The map

$$u \mapsto (u^+, u^-, s = \beta_{u^-}(o, \pi(u)))$$

is a homeomorphism between $\mathrm{T}^1(\mathbb{H}^n)$ and $(\partial\mathbb{H}^n \times \partial\mathbb{H}^n - \{(\xi,\xi) : \xi \in \partial\mathbb{H}^n\}) \times \mathbb{R}$. In these coordinates, the BMS measure $\mathsf{m} = \mathsf{m}^{\mathrm{BMS}}$, the Haar measure $\mathsf{m}^{\mathrm{Haar}}$, and the Burger-Roblin measure m^{BR} on $\mathrm{T}^1(\mathbb{H}^n)$ are given by

(1) $d\mathsf{m}(u) = e^{\delta\beta_{u^+}(o,\pi(u))}\, e^{\delta\beta_{u^-}(o,\pi(u))}\, d\nu_o(u^+) d\nu_o(u^-) ds.$
(2) $d\mathsf{m}^{\mathrm{Haar}}(u) = e^{(n-1)\beta_{u^+}(o,\pi(u))}\, e^{(n-1)\beta_{u^-}(o,\pi(u))}\, dm_o(u^+) dm_o(u^-) ds.$
(3) $d\mathsf{m}^{\mathrm{BR}}(u) = e^{(n-1)\beta_{u^+}(o,\pi(u))}\, e^{\delta\beta_{u^-}(o,\pi(u))}\, dm_o(u^+) d\nu_o(u^-) ds.$

These measures are all left Γ-invariant, and hence descend to corresponding measures on $\mathrm{T}^1(\mathcal{M})$. Using $\mathrm{T}^1(\mathbb{H}^n) = G/M$, we can lift the above measures to right M-invariant measures on $\Gamma\backslash G$, which we still denote by m, $\mathsf{m}^{\mathrm{Haar}}$ and m^{BR} by abuse of notation. The measure m is finite and ergodic with respect to the geodesic flow [27]. We will normalize the Patterson-Sullivan density $\{\nu_x\}$ so that $\mathsf{m}(\mathrm{T}^1(\mathcal{M})) = \mathsf{m}(\Gamma\backslash G) = 1$.

Let $N = N^+$ and N^- denote the expanding and the contracting horospherical subgroups respectively, i.e.,

$$N^\pm = \{g \in G : a_s g a_{-s} \to e \text{ as } s \to \pm\infty\}.$$

Note that

$$\Omega := \mathrm{supp}(\mathsf{m}) = \{[g] \in \Gamma\backslash G : g^+, g^- \in \Lambda(\Gamma)\},$$

where $g^\pm := [gM]^\pm \in \partial\mathbb{H}^n$.

The BMS measure m has a natural foliation corresponding to the decomposition $PN = G$ (modulo a Zariski closed subset) with $P = N^-AM$. Explicitly, for any $g \in G$, we define the PS-measure and the Lebesgue measure on the coset gN, by

$$d\tilde{\mu}^{\mathrm{PS}}_{gN}(gn) = e^{\delta\beta_{(gn)^+}(o,gn)} d\nu_o(gn)^+, \tag{2.1}$$

and

$$d\tilde{\mu}^{\ell}_{gN}(gn) = e^{(n-1)\beta_{(gn)^+}(o,gn)} dm_o(gn)^+, \tag{2.2}$$

respectively. We also define the measure $\tilde{\nu}_{gP}$ on the coset gP by

$$d\tilde{\nu}_{gP}(gp) = e^{\delta t} d\nu_o(gp)^- dt \tag{2.3}$$

for $t = \beta_{(gp)^-}(o, gp)$. Using the decomposition $G = gPN$ and noting that $(gpn)^- = (gp)^-$, we have that for any $\Psi \in C_c(G)$,

$$\mathsf{m}(\Psi) = \int_{gP}\int_N \Psi(gpn) d\tilde{\mu}^{\mathrm{PS}}_{gpN}(gpn) d\nu_{gP}(gp). \tag{2.4}$$

Finally, for $x = [g] \in \Gamma\backslash G$ and $\epsilon > 0$ smaller than the injectivity radius at x, we denote by $d\mu^{\mathrm{PS}}_{xN_\epsilon}$ and $d\nu_{xP_\epsilon}$ the measures induced by $d\tilde{\mu}^{\mathrm{PS}}_{gN}$ and $d\tilde{\nu}_{gP}$ on xN_ϵ and xP_ϵ respectively.

2.3. **Cusp decomposition.** Let X_0 be the pre-image of the convex core of $\mathcal{M}$ under the base point projection map $\pi : \Gamma\backslash G \to \Gamma\backslash G/K = \mathcal{M}$ and let X be the unit neighborhood of X_0. Then $\Omega \subseteq X_0 \subseteq X$ and since $\mathcal{M}$ is geometrically finite, X has finite Haar-measure. When $\mathcal{M}$ is convex cocompact, X is compact, and otherwise it can be decomposed into a compact part and finitely many cusp neighborhoods, as we describe below.

Let $\Lambda_p \subset \Lambda$ denote the set of parabolic fixed points (i.e. points fixed by some parabolic element of Γ). Since Γ is geometrically finite, Λ_p consists of finitely many Γ-orbits represented by $\{\xi_1, \dots, \xi_k\}$ which are called cusps of $\mathcal{M}$. A cuspidal neighborhood of $\xi_i \in \Lambda_p$ is a set of the form

$$\mathfrak{h}_i = \pi^{-1}(\Gamma\backslash\Gamma H_{\xi_i}) \tag{2.5}$$

where $H_\xi \subseteq \mathbb{H}^n$ is some fixed horoball tangent to ξ such that $\gamma H_\xi \cap H_\xi \neq \emptyset$ if and only if γ fixes ξ. For each i, the stabilizer $\mathrm{Stab}_\Gamma(\xi_i)$ is a free abelian subgroup and we denote its rank by κ_i. We set $\kappa_{\max} := \max \kappa_i$ and $\kappa_{\min} := \min \kappa_i$. Note that $2\delta > k_{\max}$(see [3, Lem. 3.5]).

For $x \in \Gamma\backslash G$, we denote by r_x the injectivity radius at x. For all sufficiently small $\epsilon > 0$, let $X(\epsilon) = \{x \in X : r_x < \epsilon\}$, so that

$$Y(\epsilon) := X \setminus X(\epsilon)$$

is compact, and the family $X(\epsilon)$ with $\epsilon < \epsilon_0$ forms a shrinking family of cusp neighborhoods.

More explicitly, we show in section 5.1 that for all sufficiently small $\epsilon > 0$,

$$X(\epsilon) \cap \mathfrak{h}_i \asymp X \cap \mathfrak{h}_{i,\log(\epsilon^{-1})}, \tag{2.6}$$

and using the measure estimate $\mathsf{m}(\mathfrak{h}_{i,\log(\epsilon^{-1})}) \asymp \epsilon^{2\delta-\kappa_i}$ (see Proposition 5.5), we get that

$$\mathsf{m}(X(\epsilon)) \asymp \epsilon^{2\delta-\kappa_{\max}}. \tag{2.7}$$

2.4. **Sobolev norms.** The mixing rate of the geodesic flow depends on the smoothness of the test functions which can be captured by appropriate Sobolev norms we now define. Given a fixed basis of $\operatorname{Lie} G$, $l \in \mathbb{N}$, and $1 \le p \le \infty$, the Sobolev norm $\mathcal{S}_{p,l}(\Psi)$ of $\Psi \in C^\infty(\Gamma\backslash G)$ is defined by

$$\mathcal{S}_{p,l}(\Psi) = \sum \|D(\Psi)\|_p^{\mathrm{Haar}} \tag{2.8}$$

where the sum is taken over all monomials D of order at most l in the basis elements, and $\|\Psi\|_p^{\mathrm{Haar}}$ denotes the $L^p(\Gamma\backslash G, \mathsf{m}^{\mathrm{Haar}})$-norm of Ψ. While this norm depends on the choice of basis, changing the basis will only change the norm by some bounded factor.

We will mostly use the norms $\mathcal{S}_{\infty,l}$, which we will denote by $\mathcal{S}_l$ to simplify notation. Since $\operatorname{supp}(\mathsf{m}) \subset X$, it is sufficient for our purpose to consider functions supported on X, and since X has finite Haar measure we can, and will, use the bound

$$\mathcal{S}_{p,l}(\Psi) \le \mathcal{S}_l(\Psi)\mathsf{m}^{\mathrm{Haar}}(X)^{1/p} \ll \mathcal{S}_l(\Psi),$$

where the implied constant is independent of $\Psi \in C^\infty(X)$.

3. Decay of matrix coefficients

A crucial ingredient in our proof is the exponential mixing of the geodesic flow with respect to the BMS-measure. We use the inner product notation:

$$\langle a_t\Psi, \Phi\rangle = \int_{\Gamma\backslash G} \Psi(xa_t)\Phi(x)\, d\mathsf{m}(x).$$

By the remarks following Theorem 1.9, this following theorem is the only missing part of it, given the works [24] and [20].

Theorem 3.1. *Suppose that $\delta > \max\{\frac{n-1}{2}, n-2\}$ (resp. $\delta > \frac{n-1}{2}$). Then there exist an explicit $\eta_0 > 0$ (depending only on the spectral gap of $L^2(\mathcal{M})$) and $l \in \mathbb{N}$, such that for any bounded $\Psi, \Phi \in C^\infty(X)$ (resp. $\Psi, \Phi \in C^\infty(X)^M$)*

$$\langle a_t\Psi, \Phi\rangle = \mathsf{m}(\Psi)\cdot\mathsf{m}(\Phi) + O(e^{-\eta_0 t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi)).$$

In the rest of this section, we assume

$$\delta > (n-1)/2.$$

Theorem 3.1 with an explicit η_0 depending only on the spectral gap of $L^2(\mathcal{M})$ is then proved in ([22, Theorem 6.16], [5]) under the assumption that the test functions are compactly supported. In order to complete the

proof of the theorem we need to remove the assumption on the support of the test functions.

To do this, we will approximate Ψ as the sum $\Psi_\epsilon + (\Psi - \Psi_\epsilon)$ where Ψ_ϵ is a smooth function supported on $Y(\epsilon)$, and similarly for Φ. In view of (2.7), the main term will be reduced to $\langle a_t\Psi_\epsilon, \Phi_\epsilon\rangle$, for which the result follows from [22, Theorem 6.16]. However, since the dependence on the supports of Ψ_ϵ and Φ_ϵ was not made explicit in terms of ϵ in [22], we need to redo their arguments while keeping track of the dependence on ϵ as well as on all implied constants along the proof.

3.1. **Control of BR measures.** Since $\mathsf{m}^{\mathrm{BR}}(\Gamma\backslash G) = \infty$ when $\Gamma < G$ is not a lattice, and some of the implied constants in [22, Thm. 6.16] depend on $\mathsf{m}^{\mathrm{BR}}(\mathrm{supp}(\Psi))$, we need the following result to control the dependence on these measures.

Lemma 3.2. *Assume that $\delta > \frac{n-1}{2}$. Then there exists $c > 0$ such that for any K-invariant subset $Y \subset \Gamma\backslash G$ with $\mathsf{m}^{\mathrm{Haar}}(Y) < \infty$, we have*

$$\mathsf{m}^{\mathrm{BR}}(Y) \le c \cdot \sqrt{\mathsf{m}^{\mathrm{Haar}}(Y)}.$$

Proof. Recall that by [26] and [19], there exists a positive eigenfunction $\phi_0 \in C^\infty(\Gamma\backslash G)^K$ for the Laplace operator such that

$$-\Delta\phi_0 = \delta(n-1-\delta)\phi_0.$$

Under the assumption $\delta > \frac{n-1}{2}$, we have $\|\phi_0\|_2^{\mathrm{Haar}} < \infty$. If Ψ denotes the indicator function of Y, then Ψ is K-invariant and hence by [18, Lem. 6.7]

$$\mathsf{m}^{\mathrm{BR}}(\Psi) = \int_X \Psi(x)\phi_0(x)d\mathsf{m}^{\mathrm{Haar}}(x),$$

and in particular $\mathsf{m}^{\mathrm{BR}}(Y) \le \|\phi_0\|_2^{\mathrm{Haar}}\sqrt{\mathsf{m}^{\mathrm{Haar}}(Y)}$, as claimed. □

Since X is K-invariant with $\mathsf{m}^{\mathrm{Haar}}(X) < \infty$, the following is a consequence of Lemma 3.2:

Corollary 3.3. *If $\delta > \frac{(n-1)}{2}$, then $\mathsf{m}^{\mathrm{BR}}(X) < \infty$.*

3.2. **Test function supported on small balls.** For a subset $S \subseteq G$ and $\epsilon > 0$, S_ϵ denotes the ϵ-neighborhood of e in S, that is, $S_\epsilon = \{g \in S : d(g,e) \le \epsilon\}$. Set $B_\epsilon := P_\epsilon N_\epsilon$; and note that $G_\epsilon \asymp B_\epsilon$ for all sufficiently small $\epsilon > 0$. In this subsection, we will prove the following.

Proposition 3.4. *Suppose $\delta > \max\{\frac{n-1}{2}, n-2\}$ (resp. $\delta > \frac{n-1}{2}$). There exist $l \in \mathbb{N}$ depending only on $\dim(G)$ and $\eta > 0$ (depending only on the spectral gap of Γ) such that for any $\epsilon \in (0,1)$ small and any $x \in Y(\epsilon) \cap \Omega$, for all $\Phi, \Psi \in C^\infty(xB_\epsilon)$ (resp. $\Phi, \Psi \in C^\infty(xB_\epsilon M)^M$), we have that*

$$\langle a_t\Psi, \Phi\rangle = \mathsf{m}(\Psi)\mathsf{m}(\Phi) + O(e^{-\eta t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi))$$

where the implied constant is absolute.

Proof. Fix $\Phi, \Psi \in C^\infty(xB_\epsilon)$. In the case when $\frac{n-1}{2} < \delta \leq n-2$, we assume that $\Phi, \Psi \in C^\infty(xB_\epsilon M)$ are M-invariant. We have

$$\langle a_t \Psi, \Phi \rangle = \int_{xp \in xP_\epsilon} \int_{xpN_\epsilon} \Psi(xpna_t)\Phi(xpn) d\mu^{\mathrm{PS}}_{xpN}(xpn) d\nu_{xP}(xp).$$

Now, for fixed $p \in P_\epsilon$, letting $\phi = \Phi|_{xpN_\epsilon} \in C^\infty_c(xpN_\epsilon)$, we estimate the inner integral

$$\int_{xpN_\epsilon} \Psi(xpna_t)\Phi(xpn) d\mu^{\mathrm{PS}}_{xpN}(xpn) = \int_{xpN_\epsilon} \Psi(xpna_t)\phi(xpn) d\mu^{\mathrm{PS}}_{xpN}(xpn) \tag{3.1}$$

as follows.

Fix a small $0 < \epsilon_0 < \epsilon^2$ and consider the functions $\Psi^\pm_{\epsilon_0}$ on $\Gamma\backslash G$ defined by

$$\Psi^+_{\epsilon_0}(y) = \sup_{g \in G_{\epsilon_0}} \Psi(yg), \quad \Psi^-_{\epsilon_0}(y) = \inf_{g \in G_{\epsilon_0}} \Psi(yg)$$

and let

$$\psi^\pm_{\epsilon_0}(xp) = \int_{xpN} \Psi^\pm_{\epsilon_0}(xpn) d\mu^{\mathrm{PS}}_{xpN}(xpn).$$

We then have that

$$\nu_{xP}(\psi^\pm_{\epsilon_0}) = \mathsf{m}(\Psi^\pm_{\epsilon_0}) \quad \text{and} \quad \int_{xP_\epsilon} \mu^{\mathrm{PS}}_{xpN}(\phi) d\nu_{xP}(xp) = \mathsf{m}(\Phi).$$

Moroever, since $\Psi(x) = \Psi^\pm_{\epsilon_0}(x) + O(\epsilon_0 \mathcal{S}_{\infty,1}(\Psi))$, we get that

$$\mathsf{m}(\Psi^\pm_{\epsilon_0}) = \mathsf{m}(\Psi) + O(\epsilon_0 \mathcal{S}_{\infty,1}(\Psi)),$$

where we used that $\mathsf{m}(X) < \infty$. We will also use the notation

$$\phi^+_{\epsilon_1}(y) := \sup_{n \in N_{\epsilon_1}} \phi(yn),$$

and similarly get that $\mu^{\mathrm{PS}}_{yN}(\phi^+_{\epsilon_1}) = \mu^{\mathrm{PS}}_{yN}(\phi) + O(\epsilon_1 \mathcal{S}_{\infty,1}(\phi))$.

Now by ([22, Lem. 6.2], [5]), there exists some absolute constant $c > 0$, such that the integral

$$\int_{xpN_\epsilon} \Psi(xpna_t)\phi(xpn) d\mu^{\mathrm{PS}}_{xpN}(xpn) \tag{3.2}$$

is bounded from above and below, respectively, by

$$(1 \pm c\epsilon_0) e^{-\delta t} \sum_{p \in \mathcal{P}_x(t)} \psi^\pm_{c\epsilon_0}(xp) \phi^\pm_{ce^{-t}\epsilon_0}(xpa_{-t}),$$

where $\mathcal{P}_x(t)$ is the finite set defined by

$$\mathcal{P}_x(t) = \{p \in P_\epsilon : xpN_\epsilon a_t \cap xpN_\epsilon \neq \emptyset\}.$$

Moreover, by the proof of [22, Thm. 6.7], there are positive constants $\eta > 0$ (depending only on the spectral gap of Γ) and $\alpha > 0$ such that

$$e^{-\delta t}\sum_{p\in\mathcal{P}_x(t)}\psi^{\pm}_{\epsilon_0}(xp)\phi^{+}_{e^{-t}\epsilon_0}(xpa_{-t}) = \nu_{xP}(\psi^{\pm}_{\epsilon_0})\mu^{\mathrm{PS}}_{xpN}(\phi^{\pm}_{e^{-t}\epsilon_0}) + O(e^{-\eta t}+\epsilon_0^{\alpha})A^{\mathrm{BR}}_{\Psi}A^{\mathrm{PS}}_{\phi} + O(e^{-\eta t}\mathcal{S}_{2,l}(\Psi)\mathcal{S}_{2,l}(\phi))$$

where

$$A^{\mathrm{BR}}_{\Psi} := \mathcal{S}_{\infty,1}(\Psi)\mathsf{m}^{\mathrm{BR}}(\mathrm{supp}(\Psi)) \ll \mathcal{S}_{\infty,1}(\Psi),$$

(by Lemma 3.2), and

$$A^{\mathrm{PS}}_{\phi} := \mathcal{S}_{\infty,1}(\phi)\mu^{\mathrm{PS}}_{xpN}(\mathrm{supp}(\phi)) \le \mathcal{S}_{\infty,1}(\phi)\mu^{\mathrm{PS}}_{xpN}(xpN_\epsilon).$$

Notice that the injectivity radii of the supports of ϕ and ψ are at least ϵ and since we chose $\epsilon_0 \ll \epsilon^2$ much smaller, all the implied constants are absolute and independent of ϵ and ϵ_0.

Combining these results and estimating

$$\nu_{xP}(\psi^{\pm}_{\epsilon_0}) = \mathsf{m}(\Psi^{\pm}_{\epsilon_0}) = \mathsf{m}(\Psi) + O(\epsilon_0\mathcal{S}_{\infty,1}(\Psi)),$$

and

$$\mu^{\mathrm{PS}}_{xpN}(\phi^{\pm}_{e^{-t}\epsilon_0}) = \mu^{\mathrm{PS}}_{xpN}(\phi) + O(\epsilon_0\mathcal{S}_{\infty,1}(\phi)),$$

we get that

$$\int_{xpN_\epsilon}\Psi(xpna_t)\phi(xpn)d\mu^{\mathrm{PS}}_{xpN}(xpn) = \mathsf{m}(\Psi)\mu^{\mathrm{PS}}_{xpN}(\phi)(1+O(\epsilon_0)) + O(\epsilon_0\mathcal{S}_{\infty,1}(\Psi)\mathcal{S}_{\infty,1}(\Psi)) + O(e^{-\eta t}+\epsilon_0^{\alpha})\mathcal{S}_{\infty,1}(\Psi)\mathcal{S}_{\infty,1}(\phi)\mu^{\mathrm{PS}}_{xpN}(xpN_\epsilon) + O(e^{-\eta t}\mathcal{S}_{2,l}(\Psi)\mathcal{S}_{2,l}(\phi)).$$

Since all implied constants are independent of ϵ_0, taking the limit as $\epsilon_0 \to 0$ gives

$$\int_{xpN_\epsilon}\Psi(xpna_t)\phi(xpn)d\mu^{\mathrm{PS}}_{xpN}(xpn) = \mathsf{m}(\Psi)\mu^{\mathrm{PS}}_{xpN}(\phi) + O(e^{-\eta t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi)\mu^{\mathrm{PS}}_{xpN}(xpN_\epsilon))) + O(e^{-\eta t}\mathcal{S}_{2,l}(\Psi)\mathcal{S}_{2,l}(\phi))$$

where we used that $\mathcal{S}_l(\phi) \le \mathcal{S}_l(\Phi)$.

Now, integrating over xP_ϵ, and noting that $\int_{xP_\epsilon}\mu^{\mathrm{PS}}_{xpN}(\phi)d\nu_{xP}(xp) = \mathsf{m}(\Phi)$, the main term is indeed $\mathsf{m}(\Psi)\mathsf{m}(\Phi)$. Next, since

$$\int_{xP_\epsilon}\mu^{\mathrm{PS}}_{xpN}(xpN_\epsilon))d\nu_{xP}(xp) = \int_{xP_\epsilon}\int_{xpN_\epsilon}d\mu^{\mathrm{PS}}_{xpN}d\nu_{xP}(xp) = \mathsf{m}(B_\epsilon) \le 1,$$

the integral of the first remainder term is bounded by $O(e^{-\eta t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi))$. For the second remainder term, we bound $\mathcal{S}_{2,l}(\phi) \le \mathcal{S}_l(\Phi)\sqrt{\mu^{\ell}_{xpN}(xpN_\epsilon)}$ to get that

$$\begin{aligned}\int_{xP_\epsilon} \mathcal{S}_{2,l}(\Phi_{|_{xpN_\epsilon}})d\nu_{xP}(xp) &\ll \mathcal{S}_l(\Phi)(\mu^\ell_{xpN}(xpN_\epsilon))^{-1/2}\int_{xP_\epsilon}\int_{xpN_\epsilon} d\mu^\ell_{xpN}(xpn)d\nu_{xP}(xp)\\ &= \mathcal{S}_l(\Phi)(\mu^\ell_{xpN}(xpN_\epsilon))^{-1/2}\mathsf{m}^{\mathrm{BR}}(xP_\epsilon N_\epsilon).\end{aligned}$$

We now use Lemma 3.2 to bound

$$\mathsf{m}^{\mathrm{BR}}(xP_\epsilon N_\epsilon) \le \mathsf{m}^{\mathrm{BR}}(xP_\epsilon N_\epsilon K) \ll \sqrt{\mathsf{m}^{\mathrm{Haar}}(xP_\epsilon N_\epsilon K)},$$

and since there is a uniform constant $c > 0$ such that $P_\epsilon N_\epsilon K \subseteq P_{c\epsilon}K$, noting that $P_\epsilon K = N^-_\epsilon A_\epsilon K$, we can bound

$$\mathsf{m}^{\mathrm{Haar}}(xP_\epsilon N_\epsilon K) \ll \mathsf{m}^{\mathrm{Haar}}(xP_{c\epsilon}K) \ll \mu^\ell_{xpN}(xpN_\epsilon)$$

to get that

$$\int_{xP_\epsilon} \mathcal{S}_{2,l}(\Phi_{|_{xpN_\epsilon}})d\nu_{xP}(xp) \ll \mathcal{S}_l(\Phi).$$

Combining the two remainder terms, and bounding all norms by $\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi)$ we get that

$$\langle a_t\Psi, \Phi\rangle = \mathsf{m}(\Psi)\mathsf{m}(\Phi) + O(e^{-\eta t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi))$$

where the implied constant is absolute. □

3.3. **General test functions.** We now use a partition of unity to reduce the case of a general test function to the case of functions with small support.

For $\epsilon \in (0,1)$ sufficiently small, let Q_ϵ be a maximal family of points in $X \cap Y_\epsilon$ such that the sets yB_{ϵ^3}, $y \in Q_\epsilon$, are disjoint and meet $Y_{2\epsilon}$, and let $Q'_\epsilon := \{y \in Q_\epsilon : yB_{\epsilon^2} \cap Y_{4\epsilon} \neq \emptyset\}$. Note that the collection $\{yB_{\epsilon^2} : y \in Q_\epsilon\}$ covers $X \cap Y_{2\epsilon}$ and that $\{yB_{\epsilon^3}B_{\epsilon^3} : y \in Q'_\epsilon\}$ covers $X \cap Y_{4\epsilon}$. Since $\mathsf{m}^{\mathrm{Haar}}(X) < \infty$, we have $\#Q_\epsilon = O(\epsilon^{-3\dim(G)})$.

Fix a non-negative function $\beta_\epsilon \in C^\infty(B_\epsilon)$ taking values in $[0,1]$ which is 1 on $B_{\epsilon^3}B_{\epsilon^3}$ and 0 outside B_{ϵ^2} (note that $B_{\epsilon^3}B_{\epsilon^3} \subseteq B_{2\epsilon^3} \subset B_{\epsilon^2}$). We can choose β_ϵ so that $\mathcal{S}_l(\beta_\epsilon) \ll \epsilon^{-3l}$. For each $y \in Q_\epsilon$, define a function $\beta_{y,\epsilon}(yb) := \beta_\epsilon(b)$ on yB_ϵ.

Lemma 3.5. *For any $x \in \bigcup_{y\in Q'_\epsilon} yB_{\epsilon^2}$, we have*

$$\sum_{z\in Q_\epsilon} \beta_{z,\epsilon}(x) \ge 1.$$

Proof. Let $y \in Q'_\epsilon$. If $x \in yB_{\epsilon^3}B_{\epsilon^3}$, then $\beta_{y,\epsilon}(x) = 1$ and hence $\sum_{z\in Q_\epsilon}\beta_{z,\epsilon}(x) \ge 1$. Now suppose that $x \in yB_{\epsilon^2} \setminus yB_{\epsilon^3}B_{\epsilon^3}$, in which case $xB_{\epsilon^3} \cap yB_{\epsilon^3} = \emptyset$. Since $y \in Y_{3\epsilon}$ and $x \in yB_{\epsilon^2}$ we have that $x \in Y_{2\epsilon} \cap X$. By the maximality of Q_ϵ, there exists $z \in Q_\epsilon$ such that $xB_{\epsilon^3} \cap zB_{\epsilon^3} \neq \emptyset$. This implies that $x \in zB_{\epsilon^3}B_{\epsilon^3}$ and hence $\beta_{z,\epsilon}(x) \ge 1$. □

Now consider the normalized function (supported on yB_ϵ) given by

$$\alpha_{y,\epsilon} := \frac{\beta_{y,\epsilon}}{\sum_{z\in Q_\epsilon}\beta_{z,\epsilon}}.$$

Lemma 3.6. *For any $y \in Q'_\epsilon$, we have $\mathcal{S}_l(\alpha_{y,\epsilon}) \ll \epsilon^{-p}$, where p and the implied constant depend only on l and $\dim(G)$.*

Proof. Let $s_\epsilon(x) = \sum_{z\in Q_\epsilon} \beta_{z,\epsilon}(x)$ so that $\alpha_{y,\epsilon}(x) = \frac{\beta_{y,\epsilon}(x)}{s_\epsilon(x)}$. Since $\alpha_{y,\epsilon}$ is supported on yB_{ϵ^2}, we only need to bound its derivatives there in which case we have that $s_\epsilon(x) = \sum_{z\in Q_\epsilon} \beta_{z,\epsilon}(x) \geq 1$. Taking derivatives of the quotient $\alpha_{y,\epsilon} = \frac{\beta_{y,\epsilon}}{s_\epsilon}$ and using the bound $s_\epsilon(x) \geq 1$ together with the bound $\mathcal{S}_{\infty,l}(s_\epsilon) \ll \epsilon^{-3l}\#Q_\epsilon \ll \epsilon^{-3(l+\dim(G))}$ proves the lemma. □

Lemma 3.7. *The function $\tau_\epsilon := \sum_{y\in Q'_\epsilon} \alpha_{y,\epsilon}$ belongs to $C^\infty(X)$ and satisfies that $0 \leq \tau_\epsilon \leq 1$, $\tau_\epsilon = 1$ on $X \cap Y_{4\epsilon}$, and $\tau_\epsilon = 0$ outside Y_ϵ.*

Proof. Since $\tau_\epsilon = \frac{\sum_{y\in Q'_\epsilon} \beta_{y,\epsilon}}{\sum_{y\in Q_\epsilon} \beta_{y,\epsilon}}$, it is clear that $0 \leq \tau_\epsilon \leq 1$. Note that if $y \in Q_\epsilon \setminus Q'_\epsilon$, then $yB_{\epsilon^2} \cap Y_{4\epsilon} = \emptyset$. Hence if $x \in X \cap Y_{4\epsilon}$ satisfies $x \notin yB_{\epsilon^2}$, then $\beta_{y,\epsilon}(x) = 0$. This shows that $\sum_{y\in Q_\epsilon} \beta_{y,\epsilon}(x) = \sum_{y\in Q'_\epsilon} \beta_{y,\epsilon}(x)$. Moroever, since $X \cap Y_{4\epsilon}$ is covered by $\{yB_{\epsilon^3}B_{\epsilon^3} : y \in Q'_\epsilon\}$, we have that $\sum_{y\in Q'_\epsilon} \beta_{y,\epsilon}(x) \neq 0$ on $X \cap Y_{4\epsilon}$ and hence indeed $\tau_\epsilon = 1$ there. Next, since for any $y \in Q'_\epsilon$, we have that $yB_{\epsilon^2} \subseteq Y_\epsilon$; so $\tau_\epsilon(x) = 0$ outside of Y_ϵ. Finally we can bound $\mathcal{S}_l(\tau_\epsilon) \leq \sum_{y\in Q'_\epsilon} \mathcal{S}_l\alpha_{y,\epsilon} \ll \epsilon^{-p+3\dim(G)}$. □

Proof of Theorem 3.1. Suppose first that $\delta > \max\{\frac{n-1}{2}, n-2\}$. Now, for given $\Psi, \Phi \in C^\infty(X)$, consider

$$\Psi_\epsilon := \Psi \cdot \tau_\epsilon = \sum_{y\in Q'_\epsilon} \Psi \cdot \alpha_{y,\epsilon} \quad \text{and} \quad \Phi_\epsilon := \Phi \cdot \tau_\epsilon = \sum_{y\in Q'_\epsilon} \Phi \cdot \alpha_{y,\epsilon}.$$

Note that $\mathcal{S}_l(\Psi \cdot \alpha_{y,\epsilon}) \ll \mathcal{S}_l(\alpha_{y,\epsilon})\mathcal{S}_l(\Psi) \ll \epsilon^{-p}\mathcal{S}_l(\Psi)$, with p as in Lemma 3.6. Now applying Proposition 3.4 to each $\Psi \cdot \alpha_{y,\epsilon}$ and $\Phi \cdot \alpha_{y',\epsilon}$ for $y, y' \in Q'_\epsilon$, and recalling that $\#Q_\epsilon = O(\epsilon^{-3\dim G})$, we get that

$$\langle a_t\Psi_\epsilon, \Phi_\epsilon\rangle \quad = \quad \mathsf{m}(\Psi_\epsilon)\mathsf{m}(\Phi_\epsilon) + O(\epsilon^{-p_0}e^{-\eta t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi)) \tag{3.3}$$

with $p_0 = 2p + 6\dim(G)$.

It follows from (2.7) that for $\delta_0 := 2\delta - \kappa_{\max} > 0$,

$$\mathsf{m}(\Psi - \Psi_\epsilon) \leq \|\Psi\|_\infty \mathsf{m}(X_{4\epsilon}) \ll \epsilon^{\delta_0}\|\Psi\|_\infty,$$

and similarly $\mathsf{m}(\Phi - \Phi_\epsilon) \ll \epsilon^{\delta_0}\|\Phi\|_\infty$. Hence

$$|\langle a_t\Psi, \Phi\rangle - \langle a_t\Psi_\epsilon, \Phi_\epsilon\rangle| \ll \epsilon^{\delta_0}\|\Psi\|_\infty\|\Phi\|_\infty.$$

We then deduce

$$\langle a_t\Psi, \Phi\rangle \quad = \quad \mathsf{m}(\Psi)\mathsf{m}(\Phi) + O(\epsilon^{\delta_0}\|\Psi\|_\infty\|\Phi\|_\infty) + O(\epsilon^{-p_0}e^{-\eta t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi)).$$

Taking $\epsilon = e^{-\frac{\eta t}{\delta_0+p_0}}$ and recalling that $\mathcal{S}_{\infty,0} \ll \mathcal{S}_l$, we get that

$$\langle a_t\Psi, \Phi\rangle \quad = \quad \mathsf{m}(\Psi)\mathsf{m}(\Phi) + O(e^{-\eta_0 t}\mathcal{S}_l(\Psi)\mathcal{S}_l(\Phi))$$

with $\eta_0 = \frac{\eta\delta_0}{\delta_0+p_1}$. This concluds the proof when $\delta > \max\{\frac{n-1}{2}, n-2\}$.

Finally, for $n > 3$, if $\frac{n-1}{2} < \delta \leq n-2$, and Ψ and Φ are M-invariant, we can replace $\alpha_{y,\epsilon}$ with an M-invariant function $\alpha^M_{y,\epsilon}(x) = \int_M \alpha_{y,\epsilon}(xm)dm$ and run the same argument to get (3.3). Then the rest of the proof is identical. □

4. Shrinking target problems

We now use the results on the exponential decay of matrix coefficients to prove an effective mean ergodic theorem and apply it to various shrinking target problems. As before, we assume that Γ is a geometrically finite, Zariski dense subgroup of $G = \mathrm{SO}(n,1)^\circ$. For $n \geq 5$, in the case where Γ has a cusp and $\delta \leq n-2$, all functions and shrinking targets on X we consider below are assumed to be M-invariant so that Theorem 1.9 applies to them. All functions below are also assumed to be real-valued functions.

Remark 4.1. While we state our results for the geodesic flow on geometrically finite hyperbolic manifolds, we note that the results in this section are quite general and hold for any dynamical system on a measure space (X, m) for which one has exponential decay of correlation in the sense of Theorem 3.1

4.1. **Effective mean ergodic theorem.** Fix ℓ as given in Theorem 3.1. For notational convenience, we introduce the norm

$$\mathcal{S}^*(\Psi) := \frac{\mathcal{S}_l(\Psi)}{\|\Psi\|} \quad \text{for any non-zero } \psi \in C^\infty(X) \cap L^2(X, \mathsf{m}),$$

where $\|\Psi\|$ denotes the L^2-norm of Ψ. In the entire section, we will take λ to be either the Lebesgue measure on $\mathbb{R}$ (when considering a continuous time flow) or the counting measure on $\mathbb{Z}$ (for a discrete time flow). For $T \geq 1$, consider the averaging operator λ_T on $L^2(X, \mathsf{m})$ given by

$$\lambda_T(\Psi)(x) = \frac{1}{T}\int_0^T \Psi(xa_t)d\lambda(t).$$

Theorem 4.1. *For any non-zero $\Psi \in C^\infty(X)$, and for all $T \gg 1$,*

$$\|\lambda_T(\Psi) - \mathsf{m}(\Psi)\|^2 \ll \frac{(1+\log(\mathcal{S}^*(\Psi))) \cdot \|\Psi\|^2}{T}.$$

Proof. Since we have $\|\lambda_T(\Psi) - \mathsf{m}(\Psi)\|^2 = \|\lambda_T(\Psi)\|^2 - \mathsf{m}(\Psi)^2$, it is enough to estimate $\|\lambda_T(\Psi)\|^2$. Now, expand

$$\begin{aligned}\|\lambda_T\Psi\|^2 &= \frac{1}{T^2}\int_0^T\int_0^T\int_X \Psi(xa_{t_1-t_2})\Psi(x)d\mathsf{m}(x)d\lambda(t_1)d\lambda(t_2)\\ &= \frac{1}{T^2}\int_{-T}^T\int_X \Psi(xa_t)\Psi(x)d\mathsf{m}(x)(T-|t|)d\lambda(t)\end{aligned}$$

where we used that λ is translation invariant and $\lambda([0,T)\cap[t,t+T)) = T-|t|$ (where in the discrete case we may and will assume that t and T are integers).

Now fix a large parameter M to be determined later. For $|t| \geq M$ large we use Theorem 1.9 to get that

$$\int_X \Psi(xa_t)\Psi(x)d\mathsf{m}(x) = \mathsf{m}(\Psi)^2 + O(\mathcal{S}(\Psi)^2 e^{-\eta_0|t|}),$$

for some $\eta_0 \in (0,1)$. On the other hand, for $|t| < M$ small, we bound $\langle a_t\Psi, \Psi\rangle \leq \|\Psi\|^2$, to get that

$$\|\lambda_T\Psi\|^2 \quad = \quad \mathsf{m}(\Psi)^2 + O(\|\Psi\|^2 \tfrac{M}{T}) + O(\tfrac{\mathcal{S}(\Psi)^2 e^{-\eta_0 M}}{T}),$$

using $\mathsf{m}(\Psi) \leq \|\Psi\|$. Using these estimates, we get that

$$\|\lambda_T(\Psi) - \mathsf{m}(\Psi)\|^2 \ll \frac{M\|\Psi\|^2 + \mathcal{S}(\Psi)^2 e^{-\eta_0 M}}{T}.$$

It remains to set $M = 2\log(\mathcal{S}^*(\psi))\eta_0{}^{-1}$ to finish the proof. □

Following [13], for a non-negative function Ψ on X, we define

$$\mathcal{C}_{T,\Psi} := \{x \in X : |\lambda_T(\Psi)(x) - \mathsf{m}(\Psi)| \geq \tfrac{\mathsf{m}(\Psi)}{2}\}; \tag{4.2}$$

$$\mathcal{C}^o_{T,\Psi} := \{x \in X : \lambda_T(\Psi)(x) = 0\}. \tag{4.3}$$

Note that $\mathcal{C}^o_{T,\Psi} \subseteq \mathcal{C}_{T,\Psi}$.

As a direct consequence of the effective mean ergodic theorem, we get the following bounds:

Proposition 4.2. *For a non-negative $\Psi \in C^\infty(X)$ and $T \geq 1$, we have*

$$\mathsf{m}(\mathcal{C}_{T,\Psi}) \ll \frac{\log(\mathcal{S}^*(\Psi))\|\Psi\|^2}{T \cdot \mathsf{m}(\Psi)^2}.$$

Proof. On one hand,

$$\|\lambda_T(\Psi) - \mathsf{m}(\Psi)\|^2 \geq \int_{\mathcal{C}_{T,\Psi}} |\lambda_T(\Psi)(x) - \mathsf{m}(\Psi)|^2 d\mathsf{m} \geq \frac{(\mathsf{m}(\Psi))^2 \mathsf{m}(\mathcal{C}_{T,\Psi})}{4}.$$

On the other hand, by Theorem 4.1,

$$\|\lambda_T(\Psi) - \mathsf{m}(\Psi)\|^2 \ll \frac{\log(\mathcal{S}^*(\Psi))\|\Psi\|^2}{T}.$$

Putting these two together gives the result. □

Having control on the measures of these subsets has immediate consequences to several shrinking target problems. Indeed, a simple adaptation of [13, Lemmas 13 and 14] gives the following result.

Lemma 4.3. *Let $\{\Psi_t\}_{t\geq 1} \subseteq L^2(X,\mathsf{m})$ be a decreasing family of bounded non-negative functions.*

(1) *If $\sum_j \mathsf{m}(\mathcal{C}^o_{t_{j-1},\Psi_{t_j}}) < \infty$ for some subsequence $t_j \to \infty$, then for m-a.e. $x \in X$,*

$$\lambda_t\Psi_t(x) \neq 0 \quad \text{for all } t \gg_x 1.$$

(2) *If there exists* $C > 1$ *such that* $\mathsf{m}(\Psi_{2^j}) \leq C \cdot \mathsf{m}(\Psi_{2^{j+1}})$ *for all* $j \gg 1$ *and* $\sum_j \mathsf{m}(\mathcal{C}_{2^{j-1},\Psi_{2^j}}) < \infty$, *then for* m*-a.e.* $x \in X$,

$$\lambda_t(\Psi_t)(x) \geq \frac{\mathsf{m}(\Psi_t)}{4C} \quad \text{for all } t \gg_x 1.$$

(3) *If there exists* $C > 1$ *such that* $\mathsf{m}(\Psi_{2^j}) \leq C \cdot \mathsf{m}(\Psi_{2^{j+1}})$ *for all* $j \gg 1$ *and* $\sum_j \mathsf{m}(\mathcal{C}_{2^{j+1},\Psi_{2^j}}) < \infty$, *then for* m*-a.e.* $x \in X$,

$$\lambda_t(\Psi_t)(x) \leq (4C) \cdot \mathsf{m}(\Psi_t) \quad \text{for all } t \gg_x 1.$$

4.2. **Hitting along a subsequence.** In the rest of this section, let $\mathcal{B} = \{B_t\}$ be a family of shrinking targets in X. Recall that a family $\mathcal{B}$ is inner regular (resp. outer regular) if there exist $c > 0, \alpha > 0$ and smooth positive functions $0 \leq \Psi_t^- \leq \mathrm{Id}_{B_t}$ (resp. $\mathrm{Id}_{B_t} \leq \Psi_t^+ \leq c$) such that

- $\mathsf{m}(B_t) \leq c \cdot \mathsf{m}(\Psi_t^-)$ (resp. $\mathsf{m}(\Psi_t^+) \leq c \cdot \mathsf{m}(B_t)$);
- $\mathcal{S}(\Psi_t^{\pm}) \leq c \cdot \mathsf{m}(B_t)^{-\alpha}$.

A family $\mathcal{B}$ is regular if it is inner and outer regular. When we want to emphasize the parameters c and α, we say that a family is (c, α)-regular. Our first application of the effective mean ergodic theorem is the following.

Proposition 4.4. *Assume that* $\mathcal{B}$ *is inner regular and satisfies*

$$\liminf_{t\to\infty} \frac{|\log(\mathsf{m}(B_t))|}{t\mathsf{m}(B_t)} = 0.$$

Then there is a subsequence $t_j \to \infty$ *such that for* m*-a.e.* $x \in X$,

$$\lambda(\{t \leq t_j : xa_t \in B_{t_j}\}) \gg t_j \mathsf{m}(B_{t_j}).$$

If $\mathcal{B}$ *is also outer regular, then for* m*-a.e.* $x \in X$,

$$\lambda(\{t \leq t_j : xa_t \in B_{t_j}\}) \asymp t_j \mathsf{m}(B_{t_j}).$$

Proof. Since $\mathcal{B}$ is inner regular, there are functions $\Psi_t \in C^\infty(X)$ with $0 \leq \Psi_t \leq \mathrm{Id}_{B_t}$ such that $\log(\mathcal{S}^*(\Psi_t)) \ll \log(\mathsf{m}(B_t))$ and $\mathsf{m}(\Psi_t) \gg \mathsf{m}(B_t)$. The mean ergodic theorem (Theorem 4.1) applied to Ψ_t implies that

$$\|\lambda_t(\Psi_t) - \mathsf{m}(\Psi_t)\|^2 \ll \frac{(1 + \log(\mathcal{S}^*(\Psi_t)))\|\Psi_t\|^2}{t}.$$

Set $\tilde{\Psi}_t := \frac{\Psi_t}{\mathsf{m}(\Psi_t)}$ to get that

$$\|\lambda_t(\tilde{\Psi}_t) - 1\|^2 \ll \frac{(1 + \log(\mathcal{S}^*(\Psi_t)))\|\Psi_t\|^2}{\mathsf{m}(\Psi_t)^2 \cdot t} \ll \frac{|\log(\mathsf{m}(B_t))|}{\mathsf{m}(B_t) \cdot t},$$

where we used that $\|\Psi_t\|^2 \leq \mathsf{m}(B_t)$. From our assumption, there is some subsequence t_j such that $\frac{\log(\mathsf{m}(B_{t_j}))}{\mathsf{m}(B_{t_j}) \cdot t_j} \to 0$. Hence $\lambda_{t_j}(\tilde{\Psi}_{t_j}) \to 1$ in $L^2(\Gamma\backslash G, \mathsf{m})$ and, after perhaps passing to another subsequence, we get $\lambda_{t_j}(\tilde{\Psi}_{t_j})(x) \to 1$ for m-a.e $x \in X$. For any x in this full measure subset, the inequality $\lambda_{t_j}(\tilde{\Psi}_{t_j})(x) \leq \frac{\lambda(\{t \leq t_j : xa_t \in B_{t_j}\})}{t_j \mathsf{m}(\Psi_{t_j})}$ implies that $\lambda(\{t \leq t_j : xa_t \in B_{t_j}\}) \gg t_j \mathsf{m}(B_{t_j})$ as claimed. Assuming that$\{B_t\}$ is also outer regular, repeating

the same argument for functions approximating Id_{B_t} from above gives the other inequality. □

In particular, taking λ to be the Lebesgue measure gives the first part of Theorem 1.3. Moreover, by taking λ to be the counting measure, we get the following consequence implying a discrete version of Theorem 1.2(1).

Corollary 4.5. *If $\mathcal{B}$ is inner regular and $\liminf_{t\to\infty} \frac{|\log(\mathsf{m}(B_t))|}{\mathsf{m}(B_t)t} = 0$, then $\{k \in \mathbb{N} : xa_k \in B_k\}$ is unbounded for m-a.e. $x \in X$.*

Proof. Applying the above result with λ the counting measure shows that for m-a.e. $x \in X$,

$$\#\{k \le t_j : xa_k \in B_{t_j}\} \gg t_j \mathsf{m}(B_{t_j}) \to \infty,$$

along some subsequence t_j. Since $B_{t_j} \subseteq B_k$ for any $k \le t_j$, it follows that the subset $\{k : xa_k \in B_k\}$ is unbounded as well. □

4.3. **Orbits eventually always hitting.** The results of the previous section allow us to control how orbits hit the shrinking targets along a subsequence of times, however, under the same hypothesis we could also have different subsequences for which this asymptotic fails, and for which the set $\{k \le k_j : xa_k \in B_{k_j}\}$ may even be empty (see e.g. [13, Proposition 12]). A more subtle question is to ask what conditions on the shrinking sets guarantee that the truncated orbit $\{xa_j : j \le k\}$ is eventually always hitting the targets B_k, and moreover, how large is their intersection? This is the content of the following Theorem 4.6, which is a discrete version of Theorem 1.2(2).

Theorem 4.6. *Assume that $\mathcal{B}$ is inner regular and that $\sum_{j=1}^{\infty} \frac{|\log(\mathsf{m}(B_{t_j}))|}{t_{j-1}\mathsf{m}(B_{t_j})} < \infty$ for some sequence $t_j \to \infty$. Then for m-a.e. $x \in X$ and for all $t \gg_x 1$, we have $\{k \in \mathbb{N} : k \le t,\ xa_k \in B_t\} \neq \emptyset$.*

Proof. From the inner regularity, we can find smooth functions $0 \le \Psi_t \le \mathrm{Id}_{B_t}$ satisfying $\log(\mathcal{S}_l(\Psi_t)) \ll \log(\mathsf{m}(B_t))$ and $\mathsf{m}(B_t) \ll \mathsf{m}(\Psi_t)$. By Proposition 4.2, we can estimate thatfor any $s, t > 1$

$$\mathsf{m}(\mathcal{C}_{s,\Psi_t}) \ll \frac{|\log(\mathcal{S}^*(\Psi_t))| \cdot \|\Psi_t\|^2}{s\mathsf{m}(\Psi_t)^2} \ll \frac{|\log(\mathsf{m}(B_k))|}{s\mathsf{m}(B_t)}.$$

Since $\mathsf{m}(\mathcal{C}_{t_{j-1},\Psi_{t_j}}) \ll \frac{|\log(\mathsf{m}(B_{t_j}))|}{t_{j-1}\mathsf{m}(B_{t_j})}$, we obtain $\sum_j \mathsf{m}(\mathcal{C}_{t_{j-1},\Psi_{t_j}}) < \infty$. Hence by the first part of Lemma 4.3, we have that for m-a.e. $x \in X$, $\lambda_t \Psi_t(x) \neq 0$ for all sufficiently large t. Taking λ to be the counting measure on $\mathbb{N}$, this implies that $\{k \in \mathbb{N} : k \le t,\ xa_k \in B_t\} \neq \emptyset$ for all sufficiently large t. □

Theorem 4.7 implies Theorem 1.3(2).

Theorem 4.7. *Assume that $\mathcal{B}$ is regular and that $\mathsf{m}(B_{2t}) \asymp \mathsf{m}(B_t)$. If $\sum_{j=1}^{\infty} \frac{|\log(\mathsf{m}(B_{2^j}))|}{2^j \mathsf{m}(B_{2^j})} < \infty$, then, for m-a.e. x, and for all $t \gg_x 1$,*

$$\frac{\#\{j \le t : xa_j \in B_t\}}{t} \asymp \frac{\ell\{s \le t : xa_s \in B_t\}}{t} \asymp \mathsf{m}(B_t).$$

Proof. Let $\Psi_t^{\pm}$ be functions which approximate Id_{B_t} from above and below such that $0 \le \Psi_t^- \le \mathrm{Id}_{B_t} \le \Psi_t^+ \le c$, $\log(\mathcal{S}_l(\Psi_t^{\pm})) \ll \log(\mathsf{m}(B_t))$ and $\mathsf{m}(\Psi_t^+) \asymp \mathsf{m}(\Psi_t^-) \asymp \mathsf{m}(B_t)$. For each of these functions we can use Proposition 4.2 as before to estimate $\mathsf{m}(\mathcal{C}_{s,\Psi_t^{\pm}}) \ll \frac{|\log(\mathsf{m}(B_t))|}{s\mathsf{m}(B_t)}$. Taking $s = 2^{j\pm 1}$ and $t = 2^j$, we get that $\sum_j \mathsf{m}(\mathcal{C}_{2^{j\pm1},\Psi_{2^j}^{\pm}}) < \infty$. So by the second and third part of Lemma 4.3 we get that for m-a.e. $x \in X$ and for all sufficiently large t, we have

$$\mathsf{m}(B_t) \ll \mathsf{m}(\Psi_t^-) \ll \lambda_t \Psi_t^- \le \lambda_t(\mathrm{Id}_{B_t}) \le \lambda_t \Psi_t^+ \ll \mathsf{m}(\Psi_t^+) \ll \mathsf{m}(B_t).$$

This implies that $\lambda_t(\mathrm{Id}_{B_t}) \asymp \mathsf{m}(B_t)$. Finally, taking λ to be the counting measure on $\mathbb{N}$ (resp. the Lebesgue measure) gives the result for discrete (resp. continuous) time flow. □

4.4. **Logarithm law for the first hitting time.** Using similar arguments utilizing the effective mean ergodic theorem, we can prove the logarithm law for the first hitting time for the discrete flow. Recall the discrete first hitting time function

$$\tau_B^d(x) = \min\{k \in \mathbb{N} : xa_k \in B\}. \tag{4.4}$$

Theorem 4.8. *If $\mathcal{B}$ is inner regular, then*

$$\lim_{t\to\infty} \frac{\log(\tau_{B_t}^d(x))}{-\log(\mathsf{m}(B_t))} = 1 \quad \text{for } \mathsf{m}\text{-a.e. } x \in X.$$

Proof. We first note that the bound

$$\liminf_{t\to\infty} \frac{\log(\tau_{B_t}^d(x))}{-\log(\mathsf{m}(B_t))} \ge 1,$$

holds for m-a.e.x; indeed, this holds in general for any monotone sequence of shrinking targets in a measure preserving dynamical system (see [14, Lemma 2.2]). It is thus sufficient to show that for m-a.e. x,

$$\limsup_{t\to\infty} \frac{\log(\tau_{B_t}^d(x))}{-\log(\mathsf{m}(B_t))} \le 1.$$

Fix a small $\epsilon > 0$ and set

$$\mathcal{A}_\epsilon^+ := \{x \in X : \limsup_{t\to\infty} \frac{\log(\tau_{B_t}^d(x))}{-\log \mathsf{m}(B_t)} > 1 + 2\epsilon\}.$$

Note that if $x \in \mathcal{A}_\epsilon^+$, then there are arbitrarily large values of t for which $\tau_{B_t}^d(x) \ge \frac{1}{\mathsf{m}(B_t)^{1+2\epsilon}}$, and hence $x \in \mathcal{C}^o_{k_\epsilon(t),\Psi_t}$ where $\Psi_t = \mathrm{Id}_{B_t}$ and

$$k_\epsilon(t) = \lfloor \frac{1}{(\mathsf{m}(B_t))^{1+2\epsilon}} \rfloor.$$

Now for any $j \in \mathbb{N}$, we choose $y_j \in (\frac{1}{2^{j+1}}, \frac{1}{2^j}]$ such that either $t_j = \sup\{t : \mathsf{m}(B_t) \geq y_j\}$ satisfies $\mathsf{m}(B_{t_j}) = y_j$ or there is no t with $\mathsf{m}(B_t) \in [y_j, y_{j-1})$ (if the function $t \mapsto \mathsf{m}(B_t)$ is continuous, we may simply take $y_j = 2^{-j}$. In general, since the function $t \mapsto \mathsf{m}(B_t)$ is monotone decreasing, it has at most countably many points of discontinuity and hence we can always find such points). We partition $[0, \infty)$ into intervals $I_j = \{t : \mathsf{m}(B_t) \in [y_{j+1}, y_j)\}$ and write

$$\mathcal{A}_\epsilon^+ \subseteq \bigcap_{k\in\mathbb{N}} \bigcup_{j>k} \bigcup_{t\in I_j} \mathcal{C}^o_{k_\epsilon(t),\Psi_t}.$$

For all sufficiently large j and any t with $\mathsf{m}(B_t) \in [y_{j+1}, y_j)$, we have that $k_\epsilon(t) \in [2^{(1+\epsilon)(j)}, 2^{(1+2\epsilon)(j+2)}]$ so that $\mathcal{C}^o_{k_\epsilon(t),\Psi_t} \subseteq \mathcal{C}^o_{2^{(1+\epsilon)j},\Psi_t}$. Since $B_{t_j} \subseteq B_t$ for all $t < t_j$, we get $\mathcal{C}^o_{2^{(1+\epsilon)j},\Psi_t} \subseteq \mathcal{C}^o_{2^{(1+\epsilon)j},\Psi_{t_j}}$. We can thus further bound

$$\mathcal{A}_\epsilon^+ \subseteq \bigcap_{k\in\mathbb{N}} \bigcup_{j>k, I_j\neq\emptyset} \mathcal{C}^o_{2^{(1+\epsilon)j},\Psi_{t_j}}.$$

From our choice of y_j and t_j, we have that $\mathsf{m}(\Psi_{t_j}) = y_j \in (\frac{1}{2^{j+1}}, \frac{1}{2^j}]$. Since $\{B_t\}$ is inner regular, we have $0 \leq \Psi^-_{t_j} \leq \Psi_{t_j}$ with $\mathsf{m}(\Psi^-_{t_j}) \asymp \mathsf{m}(\Psi_{t_j})$ and $\log(\mathcal{S}^*(\Psi^-_{t_j})) \ll |\log(\mathsf{m}(\Psi_{t_j}))| \ll j$. Using Proposition 4.2 for the smooth functions as before, we bound

$$\mathsf{m}(\mathcal{C}_{2^{j(1+\epsilon)},\Psi_{t_j}}) \leq \mathsf{m}(\mathcal{C}_{2^{j(1+\epsilon)},\Psi^-_{t_j}}) \ll \frac{j}{2^{j(1+\epsilon)}2^{-j}} \ll j2^{-\epsilon j}.$$

Hence $\mathsf{m}(\mathcal{A}_\epsilon^+) \leq \sum_{j>k} j2^{-\epsilon j} \ll k2^{-\epsilon k}$ for all $k \in \mathbb{N}$. Therefore $\mathsf{m}(\mathcal{A}_\epsilon^+) = 0$ and

$$\limsup_{t\to\infty} \frac{\log(\tau^d_{B_t}(x))}{-\log \mathsf{m}(B_t)} \leq 1 + 2\epsilon \quad \text{for } \mathsf{m}\text{-a.e. } x \in X.$$

This holds for any $\epsilon > 0$. Hence, by taking a sequence of $\epsilon_j \to 0$, we finish the proof. □

4.5. **Thickening along the flow.** We note that if $\{k \in \mathbb{N} : xa_k \in B_k\}$ is unbounded (resp. $\{j \leq k : xa_j \in B_k\} \neq \emptyset$), then $\{t \in \mathbb{R} : xa_t \in B_t\}$ is unbounded (resp. $\{t \leq k : xa_t \in B_k\} \neq \emptyset$). Hence the same assumptions on the shrinking rate of $\mathsf{m}(B_t)$ as in Proposition 4.4 give the same conclusions also for the continuous flow. However, it is possible for the set $\{t \in (0, \infty) : xa_t \in B_t\}$ to be unbounded even when it is bounded for the discrete time flow. In order to get the correct thresholds for the continuous flow, one needs to consider the thickened targets.

For any set $B \subseteq X$ we define its thickening $\tilde{B}$ to be

$$\tilde{B} = \bigcup_{|s|<1/2} Ba_s. \tag{4.5}$$

In the following lemma we observe that the shrinking target problems for the continuous flow can be translated to similar problems for the discrete flow hitting the thickened targets.

Lemma 4.9. *For any $B \subseteq X$ and $x \in X$, we have:*

(1) *If $xa_t \in B$ for some $t \in \mathbb{R}$, then $xa_k \in \tilde{B}$ for $k \in \mathbb{Z}$ with $|t-k| \leq 1/2$.*
(2) *If $xa_k \in \tilde{B}$ with $k \in \mathbb{Z}$, then $xa_t \in B$ for some t with $|t-k| \leq 1/2$.*
(3) $|\tau_B(x) - \tau^d_{\tilde{B}}(x)| \leq 1/2$.

The proof of these observations is easy once stated and we omit the details. Using this, we get the following sharper results for the continuous time flow, which imply Theorems 1.1 and 1.2.

Theorem 4.10. *Suppose that the family $\{\tilde{B}_t\}_{t\geq 1}$ of thickened targets is inner regular.*

(1) *If $\liminf_{k\to\infty} \frac{|\log(\mathsf{m}(\tilde{B}_k)|}{\mathsf{m}(\tilde{B}_k)k} = 0$ then for m-a.e. $x \in X$, $\{t \in \mathbb{R} : xa_t \in B_t\}$ is unbounded.*
(2) *If $\sum_{j=1}^{\infty} \frac{|\log(\mathsf{m}(\tilde{B}_{2^j}))|}{2^j \mathsf{m}(\tilde{B}_{2^j})} < \infty$, then for m-a.e. $x \in X$,*

$$\{0 < s < t : xa_s \in B_t\} \neq \emptyset \quad \text{for all } t \gg_x 1.$$

(3) *For m-a.e. $x \in X$,*

$$\lim_{t\to\infty} \frac{\log \tau_{B_t}(x)}{-\log \mathsf{m}(\tilde{B}_t)} = 1.$$

Proof. The first condition (with k replaced by $k+1$) implies that the set $\{k \in \mathbb{N} : xa_k \in \tilde{B}_{k+1}\}$ is unbounded. For each k in this set, there is some $t_k \in [k-1/2, k+1/2]$ with $xa_{t_k} \in B_{k+1} \subseteq B_{t_k}$, proving the first part.

For the second part, the summability condition implies that for m-a.e. x, we have that $\{xa_j : j \leq k)\} \cap \tilde{B}_k \neq \emptyset$ for all sufficiently large $k > k_0$. Now for $t \geq k_0 + 1$ and $k := \lfloor t \rfloor$, there is some $j \leq k$ with $xa_j \in \tilde{B}_k$; hence there is $s \leq t$ with $xa_s \in B_k \subseteq B_t$.

Finally for the last part, since $|\tau_B(x) - \tau^d_{\tilde{B}}(x)| \leq 1/2$, we get that

$$\lim_{t\to\infty} \frac{\log \tau_{B_t}(x)}{-\log \mathsf{m}(\tilde{B}_t)} = \lim_{t\to\infty} \frac{\log \tau^d_{\tilde{B}_t}(x)}{-\log \mathsf{m}(\tilde{B}_t)}.$$

□

Remark 4.6. The problem of estimating $\ell\{t \leq k : xa_t \in B_k\}$, for the continuous time flow, does not easily reduce to the discrete time problem for the thickened targets. Here, knowing that $xa_k \in \tilde{B}_k$ only tells us that $xa_t \in B_k$ for some t close to k but not on the amount of time spent there. Hence, to get asymptotics we need the stronger condition that $\sum_{j=1}^{\infty} \frac{|\log(\mathsf{m}(B_{2^j}))|}{2^j \mathsf{m}(B_{2^j})} < \infty$ for the original sets and not the thickened sets. In particular, if $\mathsf{m}(B_k) \asymp k^{-a}$ for some $a \geq 1$ and $\mathsf{m}(\tilde{B}_k) \asymp k^{-b}$ for some $b < 1$, then by reducing to the thickened case, we know that for all sufficiently large k, $\{t \leq k : xa_t \in B_k\} \neq \emptyset$, but we do not get an asymptotic estimate for the size of these sets.

5. Explicit examples

In this section, we consider explicit examples of shrinking targets given by shrinking cusp neighborhoods, shrinking metric balls and shrinking tubular neighborhoods, and show that they are regular and approximate their measure.

5.1. **Cusp neighborhoods.** Let $\mathfrak{h}_1, \cdots, \mathfrak{h}_k$ and $\mathfrak{h}_{i,t}$ be the cusp neighborhoods defined in (2.5). In order to apply our results for these sets we need to verify that the family $\{\mathfrak{h}_{i,t}\}_{t\geq 1}$ is regular and satisfies $\mathsf{m}(\mathfrak{h}_{i,t}) \asymp e^{-t(2\delta-\kappa_i)}$ where κ_i is the rank of the parabolic fixed points associated to $\mathfrak{h}_i$. While the upper bound $\mathsf{m}(\mathfrak{h}_{i,t}) \ll e^{-t(2\delta-\kappa_i)}$ is proved in [2] and [23], we could not find a reference where the lower bound is established; so we include a proof for the convenience of readers.

The important feature of a geometrically finite group is that all of its parabolic fixed points are bounded, i.e., the stabilizer of ξ in Γ acts cocompactly on $\Lambda - \{\xi\}$ for each parabolic fixed point ξ. This is the main ingredient of the argument below. We refer to [1] for the description of horoballs in geometrically finite manifolds that will be used below.

We will work here with the upper half space model

$$\mathbb{H}^n = \{z = (x, y) : x \in \mathbb{R}^{n-1},\ y > 0\},$$

and fix our base point to be $o = (0, 1)$. Since we will work with one fixed cusp, we may assume without loss of generality that it is the infiniy ∞. Set $\Gamma_\infty := \mathrm{Stab}_\Gamma \infty$ and κ to be the rank of ∞. Without loss of generality, we assume that $\Gamma_\infty = \mathbb{Z}^\kappa$. Fix a horoball $\tilde{H}(0) \subset \mathbb{H}^n$ such that

$$\Gamma_\infty = \{\gamma \in \Gamma : \tilde{H}(0) \cap \gamma\tilde{H}(0) \neq \emptyset\} = \{\gamma \in \Gamma : \tilde{H}(0) = \gamma\tilde{H}(0)\}.$$

In fact, $\tilde{H}(0)$ is of the form $\{(x, y) : y = y_0\}$ for some $y_0 > 0$. For the notational simplicity, we assume $y_0 = 1$. Set $\tilde{H}(t) = \{z \in \tilde{H}(0) : d(z, \partial\tilde{H}(0)) \geq t\} = \{(x, y) : y \geq e^t\}$. Without loss of generality, we may assume $\pi(\mathfrak{h}_t) = \Gamma_\infty \backslash \tilde{H}(t)$ where $\mathfrak{h}_{\infty,t} = \mathfrak{h}_t$.

Choose a fundamental domain $\mathcal{F}_\infty \subseteq \mathbb{R}^{n-1}$ for the action of Γ_∞ on $\mathbb{R}^{n-1}$ containing the origin so that the sets, $\mathrm{int}(\gamma\mathcal{F}_\infty)$, are mutually disjoint for $\gamma \in \Gamma_\infty$. Note that $H'(t) = \{z = (x, y) : x \in \mathcal{F}_\infty : y \geq e^t\}$ is a fundamental domain for $\pi(\mathfrak{h}_t) = \Gamma_\infty \backslash \tilde{H}(t)$. We can choose a compact fundamental parallelepiped $\mathcal{P}$ containing $\mathcal{F}_\infty \cap \Lambda$ such that $\Gamma_\infty\mathcal{P}$ covers $\Lambda \setminus \{\infty\}$ and $\mathrm{int}(\gamma\mathcal{P})$s are mutually disjoint for all $\gamma \in \Gamma_\infty$. We may choose $\mathcal{P}$ to contain the origin so that if $H(t) := H'(t) \cap \mathrm{hull}(\Lambda)$, then

$$(\mathcal{F}_\infty \cap \Lambda) \times [e^t, \infty) \subset H(t) \subset \mathcal{P} \times [e^t, \infty). \tag{5.1}$$

As $\mathcal{P}$ is compact, we have for any $z \in H(t)$, we have $d(\Gamma o, z) = d(o, z)$, and for $z \in \partial H(t)$,

$$d(\Gamma o, z) = d(o, z) = t + O(e^{-t}).$$

The following is also clear from (5.1):

Proposition 5.1. *The injectivity radius r_z at any point $z \in \partial H(t)$ satisfies $r_z \asymp e^{-t}$, where the implied constants are uniform for all $t \gg 1$.*

We will use the following well-known fact:

Proposition 5.2. *There exists $c > 0$ such that for all $t \geq 0$,*

$$H(t+c) \subset \{z \in \mathrm{hull}(\Lambda) \cap H(0) : d(z, \Gamma o) \geq t\} \subseteq H(t-c).$$

Next we want to estimate the measure $\mathsf{m}(\mathfrak{h}_t)$ for large t. For any $\xi^- \neq \xi^+ \in \partial\mathbb{H}^n - \{\infty\}$ and $s \in \mathbb{R}$, we denote by ξ_s the unit speed geodesic from ξ^- to ξ^+ (where s is the signed distance from the highest point of the geodesic), and recall that this gives us the coordinates (ξ^-, ξ^+, s) parametrizing $\mathrm{T}^1(\mathcal{M})$. Let $\Lambda' = \Lambda \setminus \{\infty\}$ and let $\mathcal{P}_0 = \mathcal{F}_\infty \cap \Lambda$.

We first show the following:

Lemma 5.3.

$$\mathsf{m}(\mathfrak{h}_t) = \sum_{\gamma \in \Gamma_\infty} \int_{\mathcal{P}_0} \int_{\gamma\mathcal{P}_0} \int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s)) d\mathsf{m}(\xi^-, \xi^+, s).$$

Proof. Let $\mathcal{F}_\Gamma \subseteq \mathbb{H}^n$ be a fundamental domain for $\Gamma\backslash\mathbb{H}^n$ containing o such that for $t \geq 0$ sufficiently large, we have that $\mathcal{F}_\Gamma \cap \tilde{H}(t) = H'(t)$, so that $\mathsf{m}(\mathfrak{h}_t) = \int_{\mathrm{T}^1(\mathcal{M})} \mathrm{Id}_{H'(t)}\, d\mathsf{m}$. Since $\{(\xi^-, \xi^+, s) : \{\xi^\pm\} \cap \{\infty\} \neq \emptyset\}$ has m-measure zero, we can rewrite this in the (ξ^-, ξ^+, s) coordinates as

$$\int_{\mathrm{T}^1(\mathcal{M})} \mathrm{Id}_{H'(t)}\, d\mathsf{m} = \int_{\Lambda'} \int_{\Lambda'} \int_{\mathbb{R}} \mathrm{Id}_{H'(t)}(\pi(\xi_s)) d\mathsf{m}(\xi^-, \xi^+, s).$$

Now decomposing Λ' as a union over translates $\gamma\mathcal{P}_0$ with $\gamma \in \Gamma_\infty$, we can rewrite

$$\begin{aligned}
\mathsf{m}(\mathfrak{h}_t) &= \sum_{\gamma,\gamma' \in \Gamma_\infty} \int_{\gamma\mathcal{P}_0} \int_{\gamma'\mathcal{P}_0} \int_{\mathbb{R}} \mathrm{Id}_{H'(t)}(\pi(\xi_s)) d\mathsf{m}(\xi^-, \xi^+, s) \\
&= \sum_{\gamma,\gamma' \in \Gamma_\infty} \int_{\mathcal{P}_0} \int_{\gamma'\mathcal{P}_0} \int_{\mathbb{R}} \mathrm{Id}_{\gamma^{-1}H'(t)}(\pi(\xi_s)) d\mathsf{m}(\xi^-, \xi^+, s) \\
&= \sum_{\gamma \in \Gamma_\infty} \int_{\mathcal{P}_0} \int_{\gamma\mathcal{P}_0} \int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s)) d\mathsf{m}(\xi^-, \xi^+, s)
\end{aligned}$$

where for the second line we made a change of variables $\xi \mapsto \gamma\xi$ and in the last line we used that $\tilde{H}(t) = \bigcup_{\gamma \in \Gamma_\infty} \gamma H'(t)$. □

In order to evaluate this, we need the following geometric estimate.

Lemma 5.4. *Let $\xi^- \in \mathcal{P}_0$ and $\xi^+ \in \gamma\mathcal{P}_0$ with $\gamma \in \Gamma_\infty$. Then there exists $c > 0$ such that*

$$\int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s)) ds = \begin{cases} d(o, \gamma o) - 2t + O(1) & \text{if } d(o, \gamma o) > 2t - c \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Proof. Recall that $o = (0, 1)$ and note that $\gamma o = (v, 1)$ for some $v \in \mathbb{R}^{n-1}$. Since $\mathcal{P}_0$ is a compact set containing the origin, then $\gamma\mathcal{P}_0$ is a compact set (of the same diameter) containing v and hence $\|\xi^- - \xi^+\| = \|v\| + O(1)$ where $\|\cdot\|$ is the Euclidian norm on $\mathbb{R}^{n-1}$. Note that $\sup\{t : \xi \cap H(t) \neq \emptyset\} = \log(\frac{\|\xi^- - \xi^+\|}{2})$ and $d(o, \gamma o) = \log(\|v\|) + O(1)$. Hence if $d(o, \gamma o) < 2t - c$, then $\xi \cap H(t) = \emptyset$.

Now assume that $\xi \cap H(t) \neq \emptyset$ and let $z_1, z_4 \in \mathbb{H}^n$ be the first and second intersections of the geodesic ξ_s with $\partial\tilde{H}(0)$ and z_2, z_3 the first and second intersections with $\partial H(t)$. Writing $z_i = (x_i, y_i)$, we have that $\|x_1\|$ and $\|x_4 - v\|$ are uniformly bounded and that $\|x_2\|$ and $\|x_3 - v\|$ are bounded by $O(e^t)$; this implies that $d(z_1, o)$, $d(z_4, \gamma o)$, $d(z_2, a_t o)$ and $d(z_3, \gamma a_t o)$ are all uniformly bounded. Now on one hand, $d(z_1, z_4) = d(o, \gamma o) + O(1)$, and on the other hand, since z_1, z_2, z_3, z_4 all lie on the same geodesic, we have $d(z_1, z_4) = d(z_1, z_2) + d(z_2, z_3) + d(z_3, z_4)$. The middle term is precisely $\int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s))ds$ and $d(z_1, z_2) = d(o, a_t o) + O(1) = t + O(1)$ and similarly $d(z_3, z_4) = t + O(1)$, concluding the proof. □

Recall the notation $\tilde{\mathfrak{h}}_t = \cup_{|s|<1/2}\mathcal{G}^s\mathfrak{h}_t$.

Proposition 5.5. *We have* $\mathsf{m}(\mathfrak{h}_t) \asymp \mathsf{m}(\tilde{\mathfrak{h}}_t) \asymp e^{-t(2\delta-\kappa)}$.

Proof. From Lemma 5.3, we have

$$\mathsf{m}(\mathfrak{h}_t) = \sum_{\gamma\in\Gamma_\infty} \int_{\mathcal{F}_\infty} \int_{\gamma\mathcal{F}_\infty} \int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s)) d\mathsf{m}(\xi^-, \xi^+, s) =$$
$$\sum_{\gamma\in\Gamma_\infty} \int_{\mathcal{P}_0} \int_{\gamma\mathcal{P}_0} \int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s)) e^{\delta(\beta_{\xi^+}(o,\pi(\xi_s))+\beta_{\xi^-}(o,\pi(\xi_s)))} d\nu_o(\xi^-) d\nu_o(\xi^+) ds$$

Next note that for any $\xi^- \in \mathcal{P}_0$ and $\xi^+ \in \gamma\mathcal{P}_0$ and $\gamma \in \Gamma_\infty$, the sum $\beta_{\xi^+}(o, \pi(\xi_s)) + \beta_{\xi^-}(o, \pi(\xi_s))$ is independent of s and is uniformly bounded. Indeed, let s_1 be the least time such that $z_1 := \pi(\xi_{s_i}) \in H(0)$ and note that $d(z_1, o) = O(1)$ is uniformly bounded. Now, for $z = \pi(\xi_s)$, on one hand $\beta_{\xi^+}(z_1, z) + \beta_{\xi^-}(z_1, z) = s - s_1 + s_1 - s = 0$, and on the other hand $|\beta_{\xi^\pm}(z_1, z) - \beta_{\xi^\pm}(o, z)| \leq d(z_1, o)$ which is uniformly bounded.

With this observation together with Lemma 5.4, we get that

$$\begin{aligned} \mathsf{m}(\mathfrak{h}_t) &\asymp \sum_{\gamma\in\Gamma_\infty} \nu_o(\mathcal{P}_0)\nu_o(\gamma\mathcal{P}_0) \int_{\mathbb{R}} \mathrm{Id}_{\tilde{H}(t)}(\pi(\xi_s)) ds \\ &\asymp \sum_{\substack{\gamma\in\Gamma_\infty \\ d(o,\gamma o)\geq 2t-c}} \nu_o(\mathcal{P}_0)\nu_o(\gamma\mathcal{P}_0)(d(o, \gamma o) - 2t + O(1)) \end{aligned}$$

Next, to estimate $\nu_o(\gamma\mathcal{P}_0) = \nu_{\gamma o}(\mathcal{P}_0)$, we use the Γ-conformality to get that

$$\nu_o(\gamma\mathcal{P}_0) = \int_{\mathcal{P}_0} e^{-\delta\beta_\xi(\gamma o, o)} d\nu_o(\xi).$$

To estimate $\beta_\xi(\gamma o, o)$, let z_1, z_2 be the two points in the intersection of $\partial H(0)$ and the geodesic connecting ξ to $\gamma\xi$. Then $d(z_1, o)$ and $d(z_2, \gamma o)$ are uniformly bounded and $\beta_\xi(z_1, z_2) = d(z_1, z_2) = d(\gamma o, o) + O(1)$ implying that $\beta_\xi(\gamma o, o) = d(\gamma o, o) + O(1)$. Plugging in this estimate gives

$$\mathsf{m}(\mathfrak{h}_t) \asymp \sum_{\substack{\gamma\in\Gamma_\infty \\ d(o,\gamma o)\geq 2t-c}} e^{-\delta d(o,\gamma o)}(d(o,\gamma o) - 2t + O(1)).$$

We may write Γ_∞ as $\{\gamma_v : v \in \mathbb{Z}^\kappa\}$ where γ_v is the translation by v. Note that $d(o, \gamma_v(o)) = 2\log\|v\| + O(1)$. Hence

$$\begin{aligned}
\mathsf{m}(\mathfrak{h}_t) &\asymp \sum_{\substack{v\in\mathbb{Z}^\kappa \\ \|v\|\geq Ce^t}} e^{-\delta(2\log\|v\|+O(1))}(2\log\|v\| - 2t + O(1)) \\
&\asymp \sum_{\substack{v\in\mathbb{Z}^\kappa \\ \|v\|\geq Ce^t}} \|v\|^{-2\delta}\log(\|v\|e^{-t}) \\
&\asymp \int_{x\in\mathbb{R}^k, |x|\geq e^t} \|x\|^{-2\delta}\log(\|x\|e^{-t})dx \asymp e^{-t(2\delta-\kappa)}
\end{aligned}$$

as claimed.

For the thickened target, for any $x \in \mathfrak{h}_t$ and $|s| \leq 1/2$, if $xa_t \in \mathfrak{h}_{t-1}$, then $\mathfrak{h}_t \subseteq \tilde{\mathfrak{h}}_t \subseteq \mathfrak{h}_{t-1}$. Hence $\mathsf{m}(\tilde{\mathfrak{h}}_t) \asymp e^{-t(2\delta-\kappa)}$ as well. □

Next we show regularity.

Proposition 5.6. *Both families $\{\mathfrak{h}_t : t \geq 1\}$ and $\{\tilde{\mathfrak{h}}_t : t \geq 1\}$ are regular.*

Proof. Since $\mathfrak{h}_t \subseteq \tilde{\mathfrak{h}}_t \subseteq \mathfrak{h}_{t-1}$ it is enough to show that $\{\mathfrak{h}_t\}$ is regular. Let $H'(t)$ denote the fundamental domain for $\Gamma_\infty\backslash\tilde{H}(t)$ defined above, and $\mathcal{F}_\Gamma$ a fundamental domain for $\Gamma\backslash\mathbb{H}^n$ such that $\mathcal{F}_\Gamma \cap \tilde{H}(t) = H'(t)$. For any $t \geq 1$ let $\psi_t^\pm$ be smooth functions on $\Gamma_\infty\backslash\tilde{H}(t)$ taking values in $[0, 1]$ satisfying

$$\mathrm{Id}_{H'(t+1)} \leq \psi_t^- \leq \mathrm{Id}_{H'(t)} \leq \psi_t^+ \leq \mathrm{Id}_{H'(t-1)},$$

and we can choose them so that $\mathcal{S}(\psi_t^\pm) = O(1)$, independent of t.

Since $\mathcal{F}_\Gamma \cap \tilde{H}(t) = H'(t)$, we can lift the functions $\psi_t^\pm$ to right K-invariant, and left Γ-invariant functions $\Psi_t^\pm$ on G. As such, by looking at their values on a fixed fundamental domain, we see that

$$0 \leq \mathrm{Id}_{\mathfrak{h}_{t+1}} \leq \Psi_t^- \leq \mathrm{Id}_{\mathfrak{h}_t} \leq \Psi_t^+ \leq \mathrm{Id}_{\mathfrak{h}_{t-1}} \leq 1.$$

Since $\mathsf{m}(\mathfrak{h}_t) \asymp \mathsf{m}(\mathfrak{h}_{t\pm1})$, we also get that $\mathsf{m}(\Psi_t^\pm) \asymp \mathsf{m}(\mathfrak{h}_t)$, implying that $\{\mathfrak{h}_t\}$ is regular. □

Proof of Theorem 1.4. Applying Theorem 1.1 to the shrinking targets $B_t = \mathfrak{h}_{i,t}$ gives (1). For (2), fix some $\eta < \frac{1}{2\delta-\kappa_i}$ and let $c := 1 - \eta(2\delta - \kappa_j) > 0$.

Consider the shrinking family $\{B_t = \mathfrak{h}_{i,\eta\log(t)}\}$, which is regular and satisfies $\mathsf{m}(B_{2t}) \asymp \mathsf{m}(B_t) \asymp t^{-(2\delta-\kappa_i)\eta}$. In particular we have that

$$\sum_j \frac{\log(\mathsf{m}(B_{2^j}))}{2^j \mathsf{m}(B_{2^j})} \asymp \sum_j \frac{\log(j)}{2^{cj}} < \infty,$$

so Theorem 1.3(2) implies Theorem 1.4(2). □

5.2. **Shrinking balls in $\Gamma\backslash G$.** In this subsection, our goal is to show that for $x \in \mathrm{supp}(\mathsf{m})$, the family $\{xG_\epsilon : 0 < \epsilon < 1\}$ is regular as stated in Proposition 5.9. We may assume that $o \in \mathrm{hull}\,\Lambda$ and fix $v_o \in \mathrm{T}_o(\mathbb{H}^n)$ so that $M = \mathrm{Stab}(v_o)$. For $\xi \in \partial\mathbb{H}^n$ and $\epsilon > 0$, let $B_\xi(\epsilon)$ denote the Euclidian ball of radius ϵ around ξ. When Γ is convex co-compact, Sullivan's shadow lemma implies that $\nu_o(B_\xi(\epsilon)) \asymp \epsilon^\delta$, but when Γ has cusps, the measure $\nu_o(B_\xi(\epsilon))$ fluctuates as $\epsilon \to 0$. Nevertheless, we have the following:

Lemma 5.7. *For any $\xi \in \Lambda$, the following holds for all sufficiently small $\epsilon > 0$:*

(1) $\min\{\epsilon^\delta, \epsilon^{2\delta-\kappa_{\min}}\} \ll \nu_o(B_\xi(\epsilon)) \ll \max\{\epsilon^\delta, \epsilon^{2\delta-\kappa_{\max}}\}$.
(2) $\nu_o(B_\xi(2\epsilon)) \ll \nu_o(B_\xi(\epsilon))$.

Proof. For $\xi \in \Lambda$ let $\xi_t \subset \mathrm{hull}(\Lambda)$ denote the unit speed geodesic ray connecting o to ξ. Let $b(\xi_t) \subset \partial(\mathbb{H}^n)$ denote the shadow at infinity of the hyperbolic hyperplane meeting ξ_t orthogonally. Then

$$b(\xi_t) = B_\xi(\epsilon) \quad \text{for } \epsilon \asymp e^{-t}.$$

If $\{\xi_1, \cdots, \xi_k\}$ denotes the set of all representatives of Γ-orbits in the set of parabolic limit points, and $H_{\xi_i} \subset \mathbb{H}^n$ is a sufficiently deep horoball based at ξ_i, then $H := \bigcup_{i=1}^k \Gamma(H_{\xi_i})$ forms a family of disjoint horoballs.

Now by [28, Theorem 2], we have

$$\nu_o(b(\xi_t)) \asymp e^{-\delta t + d(\xi_t, \Gamma o)(\kappa(\xi_t)-\delta)}, \tag{5.3}$$

where $\kappa(\xi_t)$ is the rank of ξ_i if $\xi_t \in \Gamma(H_{\xi_i})$ for some i, and $\kappa(\xi_t) = \delta$ otherwise. Now, to prove (1), let $\epsilon = e^{-t}$. First, if ξ_t is not in H, the claim follows easily. Next, if $\xi_t \in \Gamma(H_{\xi_i})$, then $\kappa(\xi_t) = \kappa_i$ and $\nu_o(b(\xi_t)) \asymp e^{-\delta t + d(\xi_t,\Gamma o)(\delta-\kappa_i))}$. If $\kappa_i \le \delta$, then

$$\delta t \le \delta t + d(\xi_t, \Gamma o)(\delta - \kappa_i) \le t(2\delta - \kappa_{\min})$$

and hence $e^{-t(2\delta-\kappa_{\min})} \ll \nu_o(b(\xi_t)) \ll e^{-\delta t}$. Now if $\kappa_i > \delta$, then

$$-\delta t \le -\delta t + d(\xi_t, \Gamma o)(\kappa_i - \delta) \le t(\kappa_{\max} - 2\delta),$$

so that $e^{-\delta t} \ll \nu_o(b(\xi_t)) \ll e^{-(2\delta-\kappa_{\max})t}$. This proves (1).

For (2), we claim that $\nu_o(b(\xi_{t+1})) \asymp \nu_o(b(\xi_t))$. In the case when $\xi_t, \xi_{t+1} \in \Gamma(H_{\xi_i})$ for some i, we have that $|d(\xi_t, \Gamma o) - d(\xi_{t+1}, \Gamma o)| \ll 1$. Now, using (5.3) we get

$$\frac{\nu_o(b(\xi_{t+1}))}{\nu_o(b(\xi_t))} \asymp e^{(d(\xi_{t+1},\Gamma o)-d(\xi_t,\Gamma o))(\delta-k_i)} \asymp 1.$$

If this case does not happen, there must be some $t' \in [t, t+1]$ such that the projection of $\xi_{t'}$ in $\mathrm{core}(\mathcal{M})$ lies in the compact part $\mathrm{core}(\mathcal{M}) - \cup_i \Gamma\backslash H_{\xi_i}$, and hence $d(\xi_{t'}, \Gamma o) = O(1)$. But then also $d(\xi_t, \Gamma o)$ and $d(\xi_{t+1}, \Gamma o)$ are bounded and $\nu_o(b(\xi_{t+1})) \asymp \nu_o(b(\xi_t)) \asymp e^{-\delta t}$ as well. □

Proposition 5.8. *Let $\mathcal{K} \subseteq X$ be a compact subset. Let $\delta_- = \min\{\delta, 2\delta - k_{\max}\}$ and $\delta_+ = \max\{\delta, 2\delta - k_{\min}\}$. For any $x \in \mathcal{K} \cap \Omega$, we have that for all $0 < \epsilon < r_x$,*

1. $\epsilon^{1+\dim M+2\delta_+} \ll \mathsf{m}(xG_\epsilon) \ll \epsilon^{1+\dim M+2\delta_-}$,
2. $\mathsf{m}(xG_{2\epsilon}) \asymp \mathsf{m}(xG_\epsilon)$,
3. $\mathsf{m}(xG_\epsilon A_1) \asymp \epsilon^{-1}\mathsf{m}(xG_\epsilon)$,
4. $\mathsf{m}(xG_\epsilon M) \asymp \epsilon^{-\dim(M)}\mathsf{m}(xG_\epsilon)$,

where all the implied constants above are uniform over all $x \in \mathcal{K}$.

Proof. Fix a compact subset $\mathcal{F}_0 \subseteq G$ such that $\mathcal{K} = \Gamma\backslash\Gamma\mathcal{F}_0$. First, since we assume $\epsilon \leq r_x$, we have that $\mathsf{m}(xG_\epsilon) = \mathsf{m}(gG_\epsilon)$ for $x = [g]$. We will use the flow boxes

$$\mathcal{B}(g, \epsilon) := g\mathcal{B}(\epsilon) = g(N_\epsilon^+ N^- \cap N_\epsilon^- N^+ AM)M_\epsilon A_\epsilon. \tag{5.4}$$

It is shown in [12, Lemma 4.7] that $\mathcal{B}(g, \epsilon) \asymp gG_\epsilon$ and that

$$\mathsf{m}(\mathcal{B}(g, \epsilon)) = (1 + O(\epsilon))2\epsilon\nu_{g(o)}(gN_\epsilon^+ v_o^+)\nu_{g(o)}(gN_\epsilon^- v_o^-)\,\mathrm{vol}_M(M_\epsilon), \tag{5.5}$$

where $\mathrm{vol}_M(M_\epsilon) \asymp \epsilon^{\dim(M)}$ and all implied constants are absolute.

We can estimate $\nu_{g(o)}(gN_\epsilon^\pm v_o^\pm) \asymp \nu_o(B_{g^\pm}(\epsilon))$, with the implied constants uniform for $g \in \mathcal{F}_0$. Hence by Lemma 5.7, we have

$$\epsilon^{\delta_+} \ll \nu_{g(o)}(gN_\epsilon^\pm v_o^\pm) \ll \epsilon^{\delta_-}.$$

Since $\mathrm{vol}_M(M_\epsilon) \asymp \epsilon^{\dim M}$, we get that

$$\epsilon^{2\delta_+ + 1 + \dim M} \ll \mathsf{m}(g\mathcal{B}(\epsilon)) \ll \epsilon^{2\delta_- + 1 + \dim M}$$

proving (1). (2) follows similarly from Lemma 5.7(2). (3) and (4) follow easily from the above description of $g\mathcal{B}(\epsilon)$. □

Proposition 5.9. *Fix a compact set $\mathcal{K} \subseteq X$. There exist some $c > 1$ and $\alpha > 1$ (depending on ℓ, and $\mathcal{K}$) such that the family $\{xG_\epsilon : x \in \mathcal{K} \cap \Omega,\ \epsilon < r_x\}$ and the family of their thickenings are regular for $\mathcal{S}_l$.*

Proof. We can find smooth functions $\Psi_\epsilon^\pm : G \to [0, 1)$ such that

$$\Psi_\epsilon^-(g) = \left\{ \begin{array}{cc} 1 & g \in G_{\epsilon/2} \\ 0 & g \notin G_\epsilon \end{array} \right\} \quad \Psi_\epsilon^+(g) = \left\{ \begin{array}{cc} 1 & g \in G_\epsilon \\ 0 & g \notin G_{2\epsilon} \end{array} \right\}$$

satisfying $\mathcal{S}_l(\Psi_\epsilon^\pm) \ll \epsilon^{-l}$. For $x \in \mathcal{K} \cap \Omega$, let $\Psi_{x,\epsilon}^\pm(xg) := \Psi_\epsilon^\pm(g)$. Then

$$0 \leq \Psi_{x,\epsilon}^- \leq \mathrm{Id}_{xG_\epsilon} \leq \Psi_{x,\epsilon}^+.$$

We then have that

$$\mathcal{S}_l(\Psi_{x,\epsilon}^\pm) \ll \epsilon^{-l} \ll \mathsf{m}(G_\epsilon)^{-\alpha},$$

for $\alpha = \frac{l}{1+\dim(M)+\delta_-}$ and that $\mathsf{m}(xG_{\epsilon/2}) \le \mathsf{m}(\Psi^-_{x,\epsilon}) \le \mathsf{m}(xG_\epsilon)$ so that $\mathsf{m}(xG_\epsilon) \ll \mathsf{m}(xG_{\epsilon/2}) \le \mathsf{m}(\Psi^-_{x,\epsilon})$, and similarly $\mathsf{m}(\Psi^+_{x,\epsilon}) \ll \mathsf{m}(xG_\epsilon)$.

The same argument shows that the thickened sets $xG_\epsilon A_1$ are (c,α)-regular for some constant $c>1$ and $\alpha = \frac{l}{\dim(M)+\delta_-}$. □

The proofs of Propositions 5.8 and 5.9 can easily be adapted for the following:

Proposition 5.10. *Let $\mathcal{M}$ be convex cocompact. Fix $x_0 \in \mathrm{supp}(\mathsf{m})$. Then the families $\{x_0G_\epsilon M\}$ and $\{x_0G_\epsilon MA_{1/2}\}$ are regular and $\mathsf{m}(x_0G_\epsilon M) \asymp \epsilon^{2\delta+1}$ and $\mathsf{m}(x_0G_\epsilon MA_{1/2}) \asymp \epsilon^{2\delta}$ with the implied constants uniform over all x_0.*

When $\mathcal{M}$ has cusps, we do not have such asymptotics for $\mathsf{m}(x_0G_\epsilon M)$ and $\mathsf{m}(x_0G_\epsilon MA_{1/2})$ uniformly for all $x_0 \in \mathrm{supp}(\mathsf{m})$. Nevertheless, we have the following estimates:

Proposition 5.11. *Let $\mathcal{K} \subseteq X$ be a compact subset of X, and let $x_0 = [g_0M] \in \mathcal{K} \cap \Omega$.*

(1) *If both $g_0^+, g_0^- \in \partial\mathbb{H}^n$ are parabolic fixed points corresponding to cusps of ranks κ_1 and κ_2 respectively, then*
$$\mathsf{m}(x_0G_\epsilon M) \asymp \epsilon^{4\delta+1-\kappa_1-\kappa_2}.$$
(2) *If x_0A is bounded, then*
$$\mathsf{m}(x_0G_\epsilon M) \asymp \epsilon^{2\delta+1}.$$
(3) *If $\sup_{t\in\mathbb{R}} \frac{d(x_0a_t,\Gamma o)}{\log|t|} < \infty$, then*
$$\lim_{\epsilon\to 0} \frac{\log(\mathsf{m}(x_0G_\epsilon M))}{\log\epsilon} = 2\delta+1.$$

Proof. Without loss of generality, we may assume that $g_0 = e$. Set $\xi_t := a_t$ and let $\xi^\pm := \lim_{t\to\pm\infty}\xi_t$. Recall that by [12, Lemma 4.7] and (5.5), we have
$$\mathsf{m}(x_0G_\epsilon M) \asymp \mathsf{m}(\mathcal{B}(g_0,\epsilon)M) \asymp \epsilon\cdot\nu_o(B_{\xi^+}(\epsilon))\nu_o(B_{\xi^-}(\epsilon)).$$
It thus remains to estimate $\nu_o(B_{\xi^\pm}(\epsilon))$ in each of the above cases. When $\xi^\pm$ are parabolic limit points, there exists t_0 such that for all $t \ge t_0$ (resp, $t < -t_0$), we have that $\xi_{\pm t} \in H_{\xi^\pm}$ is in the horoball centered at $\xi^\pm$. Since $\xi^\pm$ are parabolic limit points, this implies that for $t \ge t_0$, we have that $d(\xi_t,\Gamma o) = |t| + O(1)$, and hence, setting $\epsilon = e^{-|t|}$ by (5.3), we can estimate $\nu_o(B_{\xi^+}(\epsilon)) \asymp \epsilon^{2\delta-\kappa_1}$ and similarly $\nu_o(B_{\xi^-}(\epsilon)) \asymp \epsilon^{2\delta-\kappa_2}$, proving (1).

Next, the boundedness of x_0A means that $\sup_t d(\xi_t,\Gamma o) < \infty$. In this case, (5.3) implies that $\nu_o(B_{\xi^\pm}(\epsilon)) \asymp \epsilon^\delta$, proving (2).

Finally, assuming that $\sup \frac{d(x_0a_t,\Gamma o)}{\log|t|} < \infty$, again taking $\epsilon = e^{-t}$, (5.3) now implies that
$$\epsilon^\delta |\log(\epsilon)|^{-c_1} \ll \nu_o(B_{\xi^\pm}(\epsilon)) \ll \epsilon^\delta |\log(\epsilon)|^{c_1},$$

and hence

$$\log(\mathsf{m}(x_0 G_\epsilon M)) = (2\delta+1)\log\epsilon + O(\log|\log\epsilon|).$$

This proves (3). □

We finish this section with the proofs of Theorems 1.7 and 1.8.

Proof of Theorem 1.7. (1) follows by applying Theorem 1.1 to the shrinking targets $B_t = x_0 G_{1/t} M$ with thickening $\tilde{B}_t = x_0 G_{1/t} M A_{1/2}$ which is inner regular with $\log(\mathsf{m}(\tilde{B}_t)) = -2\delta t + O(1)$ by Proposition 5.10.

For (2) we consider the shrinking targets $B_t := x_0 G_{t^{-\eta}} M$. Note that for any $x \in \mathcal{M}$, we have that $d(\mathcal{G}^s(x), x_0) < t^{-\eta}$ exactly when $\mathcal{G}^s(x) \in B_t$. Since $\mathsf{m}(B_t) \asymp t^{-\eta(2\delta+1)}$, we have $\sum_j \frac{\log(\mathsf{m}(B_{2^j}))}{2^j \mathsf{m}(B_{2^j})} < \infty$ when $(2\delta+1)\eta < 1$. So (2) follows from Theorem 1.3(2). □

Proof of Theorem 1.8. For (1), we note that Theorem 1.4(1) implies that for m-a.e. $x_0 \in \mathrm{T}^1(\mathcal{M})$,

$$\limsup_{t\to\infty} \frac{d(\mathcal{G}^s(x_0), o)}{\log(t)} \le \frac{1}{2\delta - \kappa_{\max}}.$$

For any such x_0, the families $\{B_t = x_0 G_{1/t} M\}$ and $\{\tilde{B}_t\}$ are regular.

By Proposition 5.11(3), we have $\lim_{t\to\infty} \frac{-\log(\mathsf{m}(B_t))}{\log t} = 2\delta + 1$, and hence $\lim_{t\to\infty} \frac{\log(\mathsf{m}(\tilde{B}_t))}{-\log t} = 2\delta$. Now, using this limit together with Theorem 1.1, we get that for m-a.e. $x \in \mathrm{T}^1(\mathcal{M})$

$$\lim_{t\to\infty} \frac{\log(\tau_{B_t}(x))}{\log(t)} = 2\delta \quad \text{and} \quad \lim_{t\to\infty} \frac{\log(\tau_{B_t}(x))}{-\log(\mathsf{m}(\tilde{B}_t))} = 1.$$

For (2), given two cusps ξ_1, ξ_2 with ranks κ_1, κ_2, consider a geodesic connecting ξ_1 to ξ_2 and let $g_0 \in \mathrm{T}^1(\mathbb{H}^n)$ be any point on this geodesic, and set $x_0 = [g_0] \in \mathrm{T}^1(\mathcal{M})$. Consider the shrinking targets $B_t = x_0 G_{1/t} M$. By Proposition 5.11(1), we have $\log(\mathsf{m}(\tilde{B}_t)) = -(4\delta - \kappa_1 - \kappa_2)\log(t) + O(1)$ and hence (2) follows from Theorem 1.1.

□

5.3. **Shrinking tubular neighborhoods.** For a fixed closed geodesic $\mathcal{C} \subset \mathrm{T}^1(\mathcal{M})$ and $\epsilon > 0$, we set $\mathcal{C}_\epsilon = \{x \in \mathrm{T}^1(\mathcal{M}) : d(\mathcal{C}, x) < \epsilon\}$. The proof of Theorem 1.6 follows as above from the following.

Proposition 5.12. *The families* $\{\mathcal{C}_\epsilon : \epsilon < \epsilon_0\}$ *and* $\tilde{\mathcal{C}}_\epsilon = \{x a_s : x \in \mathcal{C}_\epsilon,\ |s| \le 1/2\}$ *are both regular and satisfy* $\mathsf{m}(\mathcal{C}_\epsilon) \asymp \mathsf{m}(\tilde{\mathcal{C}}_\epsilon) \asymp \epsilon^{2\delta}$.

Proof. Recall the notations $X, X(\epsilon)$ and $Y(\epsilon) = X - X(\epsilon)$ from section 2.3. They are all M-invariant subsets of $\Gamma\backslash G$, and in the following proof, we will regard them as subsets in $\Gamma\backslash G/M$. We can present $\mathcal{C} = [g_0]AM/M$ and an element of $\mathcal{C}$ is represented by $[g_0]a_t M$ for a unique $0 \le t < L$ where L is the length of $\mathcal{C}$. Let ϵ_0 be sufficiently small so that $\mathcal{C} \subseteq Y(\epsilon_0)$ and let $0 < \epsilon \le \epsilon_0$. Let Q_ϵ denote a maximal set of points $x_i \in \mathcal{C}$ such that

the sets $x_iG_\epsilon M$ are pairwise disjoint. Writing $x_i = x_0a_{t_i}$ the condition that $x_iG_\epsilon \cap x_jG_\epsilon = \emptyset$ imply that $|t_i - t_j| \geq \epsilon$ and the maximality condition implies that $|t_i - t_{i+1}| \leq 3\epsilon$. Hence $\#Q_\epsilon \asymp L\epsilon^{-1}$. Since

$$\bigcup_{x_i\in Q_\epsilon} x_iG_\epsilon M \subseteq \mathcal{C}_\epsilon \subseteq \bigcup_{x_i\in Q_\epsilon} x_iG_{3\epsilon}M,$$

we can estimate

$$\sum_{x_i\in Q_\epsilon} \mathsf{m}(x_iG_\epsilon M) \leq \mathsf{m}(\mathcal{C}_\epsilon) \leq \sum_{x_i\in Q_\epsilon} \mathsf{m}(x_iG_{3\epsilon}M).$$

Now Proposition 5.11(2) implies that $\mathsf{m}(g_iG_\epsilon M) \asymp \epsilon^{2\delta+1}$ where the implied constant does not depend on i. Summing over all $x_i \in Q_\epsilon$, we get that indeed $\mathsf{m}(\mathcal{C}_\epsilon) \asymp \epsilon^{2\delta}$.

Next, to show regularity, for each point $x_i \in Q_\epsilon$, let $\Psi^\pm_{\epsilon,i}$ be smooth non-negative functions approximating $x_iG_\epsilon M$ from below and $x_iG_{3\epsilon}M$ from above respectively, with $\mathcal{S}_l(\Psi^\pm_{\epsilon,i}) \ll \epsilon^{-l}$, and define $\Psi^\pm_\epsilon = \sum_i \Psi^\pm_{\epsilon,i}$. Since the sets $x_iG_\epsilon M$ are pairwise disjoint, we have that $\Psi^-_\epsilon \leq \mathrm{Id}_{\mathcal{C}_\epsilon} \leq \Psi^+_\epsilon$ and moreover

$$\mathsf{m}(\Psi^+_\epsilon) \leq \sum_{Q_\epsilon} \mathsf{m}(x_iG_{3\epsilon}M) \ll \sum_{Q_\epsilon} \mathsf{m}(x_iG_\epsilon M) \leq \mathsf{m}(\mathcal{C}_\epsilon),$$

and similarly that $\mathsf{m}(\mathcal{C}_\epsilon) \ll \mathsf{m}(\Psi^-_\epsilon)$. Since $\#Q_\epsilon \ll \epsilon^{-1}$, we can bound $\mathcal{S}_l(\Psi^\pm_\epsilon) \ll \epsilon^{-(l+1)} \ll \mathsf{m}(\mathcal{C}_\epsilon)^{-\alpha}$ with $\alpha = \frac{l+1}{2\delta}$, showing that the family $\{\mathcal{C}_\epsilon\}$ is (c,α) regular for some $c > 1$, and $\alpha = \frac{l+1}{2\delta}$.

Finally, note that there is $c \geq 1$ such that $a_{-s}G_\epsilon a_s \subseteq G_{c\epsilon}$ for all $|s| \leq 1/2$. Then any point $x \in \tilde{\mathcal{C}}_\epsilon$ is of the form $x = x_0a_tga_sM$ with $0 \leq t \leq L, g \in G_\epsilon$ and $|s| \leq 1/2$. We can write $ga_s = a_sa_{-s}ga_s \in a_sG_{c\epsilon}$, to get that $x \in x_0a_{t+s}G_{c\epsilon} \in \mathcal{C}_{c\epsilon}$. Therefore $\mathcal{C}_\epsilon \subseteq \tilde{\mathcal{C}}_\epsilon \subseteq \mathcal{C}_{c\epsilon}$, implying that $\{\tilde{\mathcal{C}}_\epsilon\}$ is also regular with $\mathsf{m}(\tilde{\mathcal{C}}_\epsilon) \asymp \mathsf{m}(C_\epsilon)$. □

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