EXPONENTIAL MIXING AND SHRINKING TARGETS FOR GEODESIC FLOW ON GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

DUBI KELMER AND HEE OH

Abstract. Let $\mathcal{M} = \Gamma \backslash \mathbb{H}^n$ be a geometrically finite hyperbolic manifold, which is either convex cocompact or of critical exponent $\delta$ strictly bigger than $(n-1)/2$. We present a very general theorem on the shrinking target problem for geodesic flow, using the exponential mixing for all bounded smooth functions on the unit tangent bundle $T^1(\mathcal{M})$. This includes a strengthening of Sullivan’s logarithm law for the excursion rate of the geodesic flow. More generally, we prove logarithm laws for the first hitting time for shrinking cusp neighborhoods, shrinking tubular neighborhoods of closed geodesics, and shrinking metric balls, as well as give quantitative estimates for the time a generic geodesic spends in such shrinking sets.

1. Introduction

Let $\mathcal{M}$ be a complete hyperbolic manifold of dimension $n \geq 2$. Denote by $\mathcal{G}^t$ the geodesic flow on the unit tangent bundle $T^1(\mathcal{M})$. If $\mathcal{M}$ is of finite volume, but non-compact, Sullivan [22] showed in 1982 the following logarithm law for the rate of the excursion of the geodesic flow: for any $o \in \mathcal{M}$, and for almost all $x \in T^1(\mathcal{M})$,

$$\limsup_{t \to \infty} \frac{d(\mathcal{G}^t(x), o)}{\log t} = \frac{1}{n-1} \quad (1.1)$$

where $d(\mathcal{G}^t(x), o)$ is the hyperbolic distance between the basepoint of $\mathcal{G}^t(x)$ and $o$.

This result can be viewed as a special case of the so-called shrinking target problem for the geodesic flow, which asks the behavior of a generic geodesic ray with respect to a given sequence of shrinking subsets. Indeed, if we consider the family of shrinking cuspidal neighborhoods $\mathfrak{h}_t := \{z \in \mathcal{M} : d(o, z) > t\}$, $t \gg 1$, then (1.1) is equivalent to the following logarithm law for the first hitting time: for almost all $x$,

$$\liminf_{t \to \infty} \frac{\log \tau_{\mathfrak{h}_t}(x)}{t} = n - 1 \quad (1.2)$$

where $\tau_{\mathfrak{h}_t}(x) := \inf\{s > 0 : \mathcal{G}^s(x) \in \mathfrak{h}_t\}$.

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In this paper, we investigate shrinking target problems for geodesic flow on a geometrically finite hyperbolic manifold $M$, and prove results which are far reaching strengthening and generalizations of (1.2), and hence of (1.1).

We present a hyperbolic manifold $M$ as the quotient manifold $\Gamma \backslash H^n$ where $\Gamma$ is a torsion-free discrete subgroup of $G = \text{SO}(n,1) = \text{Isom}^+(\mathbb{H}^n)$. We assume that $\Gamma$ is a Zariski dense geometrically finite subgroup. Denote by $\Lambda \subset \partial(\mathbb{H}^n)$ the limit set of $\Gamma$ and by $0 < \delta \leq n - 1$ the critical exponent of $\Gamma$. The maximal entropy of the geodesic flow on $T^1(M)$ is given by $\delta$, and there exists a unique probability ergodic measure of maximal entropy, called the Bowen-Margulis-Sullivan measure on $T^1(M)$, which we denote by $m$. The support of $m$ is precisely the non-wandering set for the geodesic flow and hence the shrinking target problem in this setting is interesting only for those shrinking subsets in the support of $m$ and for $m$-almost all geodesics.

Now since $G$ is ergodic for $m$, the Birkhoff ergodic theorem implies that for a given Borel set $B$,

$$\lim_{t \to \infty} \frac{\text{Leb}\{0 < s < t : G^s(x) \in B\}}{t} = m(B) \quad (1.3)$$

where $\text{Leb}$ denotes the Lebesgue measure on $\mathbb{R}$. The shrinking target problem asks a finer question on the set of times $\{0 < s : G^s(x) \in B_t\}$ for a given family $\{B_t\}$ of shrinking sets and for a $m$-generic point $x$. The three main questions we address in this paper for a $m$-generic point $x \in T^1(M)$:

1. (Logarithm laws) Is there a logarithm law for the first hitting time

$$\tau_{B_t}(x) := \inf\{s > 0 : G^s(x) \in B_t\}? \quad (1.4)$$

2. (Shrinking rate threshold) How fast can $B_t$ shrink so that

$$\{0 < s < t : G^s(x) \in B_t\} \neq \emptyset, \text{ or equivalently } \tau_{B_t}(x) < t$$

for an infinite sequence of time $t$, or for all sufficiently large $t \gg 1$?

3. (Quantitative estimates) How fast can $B_t$ shrink so that

$$\text{Leb}\{0 < s < t : G^s(x) \in B_t\} \asymp t \cdot m(B_t)$$

for an infinite sequence of times $t$, or for all sufficiently large $t \gg 1$?

Here the notation $f_t \asymp g_t$ means that $c_1 \leq \frac{f_t}{g_t} \leq c_2$ with some absolute constants $c_1, c_2 > 0$.

In order to address the above questions, we need to impose certain regularity conditions on the shrinking targets. Let $K < G$ be a maximal compact subgroup, $A = \{a_t\}$ a one parameter diagonalizable subgroup, and $M$ the centralizer of $A$ in $K$. We can then identify $M$ with $\Gamma \backslash G$ and the unit tangent bundle $T^1(M)$ with $\Gamma \backslash G/M$ in the way that the geodesic flow $\mathcal{G}^t$ on $T^1(M)$ is given by the right translation action of $a_t$ on $\Gamma \backslash G/M$. The space of functions on $T^1(M)$ can be regarded as the space of functions on $\Gamma \backslash G$ which are right $M$-invariant. We fix $\ell \gg \dim(M)$ and a Sobolev norm $S = S_{\infty, \ell}$ on $C_{\infty}(\Gamma \backslash G)$ given by

$$S(\Psi) = \sum \|D(\Psi)\|_\infty$$
where the sum is taken over all monomials in the basis of \( \text{Lie}(G) \) of order at most \( \ell \).

We say that a family \( \mathcal{B} = \{B_t : t \gg 1\} \) of subsets of \( T^1(M) \) is a family of shrinking targets if \( B_s \subset B_t \) for \( s > t \), \( m(B_t) > 0 \) and \( m(B_t) \to 0 \) as \( t \to \infty \). A family \( \{B_t\} \) of shrinking targets is said to be inner regular (resp. outer regular) if there exist \( c > 0, \alpha > 0 \) and a family of functions \( \Psi^- \in C^\infty(T^1(M)) \) (resp. \( \Psi^+ \in C^\infty(T^1(M)) \)) such that

\[
\begin{align*}
\bullet & \quad 0 \leq \Psi^- \leq \text{Id}_{B_t} \text{ (resp. } \text{Id}_{B_t} \leq \Psi^+ \leq c) ; \\
\bullet & \quad m(B_t) \leq c \cdot m(\Psi^-) \text{ (resp. } m(\Psi^+) \leq c \cdot m(B_t) ) ; \\
\bullet & \quad S(\Psi^\pm) \leq c \cdot m(B_t)^{-\alpha} .
\end{align*}
\]

A family \( \{B_t\} \) is said to be regular if it is both inner and outer regular.

We note that this regularity condition is rather mild and is satisfied by most families of naturally occurring shrinking targets. Such examples include shrinking cusp neighborhoods, shrinking tubular neighborhoods of a closed geodesic and shrinking metric balls.

In the rest of the introduction, we assume that

\[
\Gamma \text{ is either convex cocompact or } \delta > \frac{n-1}{2}
\]

and that \( \mathcal{B} = \{B_t : t \gg 1\} \) is a family of shrinking targets in \( T^1(M) \).

Remark 1.5. We note that any finitely generated Zariski dense group \( \Gamma \) of \( \text{SO}(2,1) \) satisfies this condition, since if \( \Gamma \) has cusp, then \( \delta > 1/2 \). For \( n = 3 \), the condition is also not very restrictive since any Zariski dense, finite generated discrete subgroup \( \Gamma < \text{SO}(3,1) \) whose limit set is not totally disconnected satisfies that \( \delta > 1 \) (c.f. [2]).

1.1. Logarithm laws. For discrete time dynamical systems it is expected that the first hitting time would be inversely proportional to the measure of the shrinking target. For continuous time flow we show it is inversely proportional to the measure of a thickened set \( \tilde{B}_t := \bigcup_{|s| < \epsilon_0} G^s(B_t) \), in the sense of the following logarithm law (see Theorem 4.10).

**Theorem 1.1.** Suppose that the thickened family \( \{\tilde{B}_t\} \) is inner regular. Then for \( m\)-a.e. \( x \in T^1(M) \), we have

\[
\lim \limits_{t \to \infty} \frac{\log(\tau_{B_t}(x))}{-\log(m(\tilde{B}_t))} = 1 .
\]

Remark 1.6. We note that such logarithm laws for the first hitting time were shown to hold for certain families of shrinking targets in many examples of discrete time dynamical systems with fast mixing, see e.g. [6, 7, 8].

1.2. Shrinking rate threshold. In order to ensure a generic orbit \( G^s(x) \) hits \( B_t \) before time \( t \) for an infinite sequence of \( t \)'s, it is not hard to see that the condition \( \lim \sup_{t \to \infty} \log^2(t) \cdot m(\tilde{B}_t) = \infty \) is necessary. In the first part of the following theorem we show that up to logarithmic factors it is also sufficient. In the second part, we obtain that a generic orbit \( G^s(x) \) hits \( B_t \)
for all sufficiently large time $t \gg 1$ under a slightly stronger assumption on the rate of shrinking (see Theorem 4.10).

**Theorem 1.2.** Suppose that $\{\tilde{B}_t\}$ is inner regular.

1. If $\limsup_{t \to \infty} \frac{t \log(m(\tilde{B}_t))}{\log(m(B_t))} = \infty$, then
   $$\liminf_{t \to \infty} \frac{\tau_{B_t}(x)}{t} \leq 1 \text{ for } m\text{-a.e. } x \in T^1(M).$$

2. If $\sum_{j=1}^{\infty} \frac{|\log(m(\tilde{B}_{t_j}))|}{t_j m(B_{t_j})} < \infty$ for some sequence $t_j \to \infty$, then
   $$\limsup_{t \to \infty} \frac{\tau_{B_t}(x)}{t} \leq 1 \text{ for } m\text{-a.e. } x \in T^1(M).$$

**1.3. Quantitative estimates.** In order to answer a more refined question regarding the amount of time that a geodesic flow spends in a shrinking target, we require our family of targets to be both inner and outer regular. In addition we also require that the measure of the shrinking targets does not change too fast in the sense that $m(B_t) \asymp m(B_{2t})$ for all $t \gg 1$.

With these additional regularity assumptions we have the following (see Theorem 4.7 below for a more general result).

**Theorem 1.3.** Suppose that $\{B_t\}$ is regular and that $m(B_{2t}) \asymp m(B_t)$ for all $t \gg 1$.

1. If $\lim inf \frac{\log(m(B_t))}{t m(B_t)} = 0$, then there exists a sequence $t_k \to \infty$ such that for $m$-a.e. $x \in T^1(M)$,
   $$\frac{\text{Leb}\{0 < s < t_k : G^s(x) \in B_{t_k}\}}{t_k} \asymp m(B_{t_k}).$$

2. If $\sum_{j=1}^{\infty} \frac{|\log(m(B_{t_j}))|}{2 t_j m(B_{2t_j})} < \infty$, then for $m$-a.e. $x$, and for all $t \gg 1$,
   $$\frac{\text{Leb}\{0 < s < t : G^s(x) \in B_t\}}{t} \asymp m(B_t).$$

We observe that unlike Theorems 1.1 and 1.2, the amount of time that the geodesic flow spends in the targets is governed by the measure of the original targets and not their thickenings.

**Remark 1.7.** We note that in many examples the measure of the shrinking targets decay regularly like $m(B_t) \asymp t^{-\eta}$ for some power $\eta > 0$. In such cases, we automatically have that $m(B_t) \asymp m(B_{2t})$ and the rest of the conditions of Theorems 1.2 and 1.3 are satisfied if $\eta < 1$.

**Remark 1.8.** We also obtain analogous statements for the shrinking target problem for the discrete time flow $\{G^n : n \in \mathbb{N}\}$. When the thinned sets $\tilde{B}_t$ have roughly the same measure as the original sets $B_t$, i.e., $m(\tilde{B}_t) \asymp m(B_t)$, the first hitting time for the discrete and continuous flow behave the same. This is indeed the case for shrinking cusp neighborhoods or tubular...
neighborhoods of closed geodesics. However, there are also cases when the measure of the thickened targets is much larger, such as the case of metric balls considered in Theorem 1.7, in which case the first hitting time for the continuous flow is much shorter.

Remark 1.9. All the results described above still hold as stated if we replace the unit tangent bundle $T^1(M)$ with the frame bundle $\Gamma \backslash G$. The only change is that when $n \geq 5$ and there are cusps we need to replace the condition $\delta > \frac{n-1}{2}$ by the stronger condition that $\delta > n - 2$. We note if $M$ contains a co-dimension one properly immersed totally geodesic submanifold of finite volume than $\delta > n - 2$, so this stronger condition still holds in many examples.

For some concrete applications of these results, we discuss three families of shrinking targets to which our theorems apply. In order to define these families we fix a left $G$-invariant and right $K$-invariant metric $d$ on $G$ which descends to the hyperbolic metric on $\mathbb{H}^n = G/K$. This metric then naturally defines a distance function, $\text{dist}(-,\cdot)$ on $T^1(M) = \Gamma \backslash G / M$.

1.4. Cusp excursion. The convex core of $M$ is defined by $\text{core}(M) = \Gamma \backslash \text{hull}(\Lambda)$. As $M$ is geometrically finite, there are finitely many disjoint cuspidal regions whose complement in $\text{core}(M)$ is a compact submanifold. We denote by $h_i$, $1 \leq i \leq k$ the pre-image in $T^1(M)$ of these cuspidal regions under the base point projection $\pi: T^1(M) \to M$ and we denote by $\kappa_i$ the rank of $h_i$, that is, the rank of a maximal free abelian subgroup of the stabilizer $\text{Stab}_\Gamma(h_i)$. It is known that $1 \leq \kappa_i < 2\delta$ for each $i$.

We show that the family of shrinking cusp neighborhoods $h_{i,t} := \{x \in h_i : \text{dist}(x,\partial(h_i)) > t\}$, (1.10)
is regular and that their measures decay like $m(h_{i,t}) \propto e^{-(2\delta - \kappa_i)t}$ (1.11) for $t \gg 1$ (see section 5.1). Applying our results to these cusp neighborhoods, we get the following.

**Theorem 1.4.** Fix $1 \leq i \leq k$.

1. For $m$-a.e. $x \in T^1(M)$,
$$\lim_{t \to \infty} \frac{\log \tau_{h_i,t}(x)}{t} = 2\delta - \kappa_i;$$

2. For any $0 < \eta < \frac{1}{2\delta - \kappa_i}$, and for $m$-a.e. $x \in T^1(M)$,
$$\text{Leb}\{0 < s < t : G^s(x) \in h_{i,\eta \log t}\} \propto t^{1-\eta(2\delta - \kappa_i)}.$$

Remark 1.12. As mentioned before, it is not hard to show that
$$\liminf_{t \to \infty} \frac{\log(\tau_{h_i}(x))}{t} = \left(\limsup_{t \to \infty} \frac{\text{dist}(G^t(x),o)}{\log t}\right)^{-1}$$ (1.13)
where \( h_i = \bigcup h_{i,t} \), and Stratmann and Velani showed that (1.13) is equal to \( 2\delta - \max \kappa_i \) [23]. Theorem 1.4(1) presents a strengthening of Sullivan’s logarithm law (1.1), as we consider excursion to individual cusps and we obtain an actual limit instead of \( \lim \inf \).

For the sake of a concrete application, we give a reformulation of Theorem 1.4(1) in the case of Apollonian manifolds. An Apollonian gasket \( \mathcal{P} = \bigcup \mathcal{C}_i \) is a countable union of circles obtained by repeatedly inscribing circles into the triangular interstices of four mutually tangent circles with disjoint interiors in the complex plane (where lines are considered as circles). The symmetry group \( \{ g \in \text{PSL}_2(\mathbb{C}) : g(\mathcal{P}) = \mathcal{P} \} \) is a discrete subgroup of \( \text{PSL}_2(\mathbb{C}) \) which acts on \( \hat{\mathbb{C}} \) by Möbius transformations and its torsion-free subgroup of finite index is called an Apollonian group, which we denote by \( \Gamma \). Via the Poincaré extension theorem, we can identify \( \text{PSL}_2(\mathbb{C}) \) with \( \text{Isom}^+(\mathbb{H}^3) \) for the upper-half space model \( \mathbb{H}^3 \) of the hyperbolic space. The quotient manifold \( \Gamma \backslash \mathbb{H}^3 \) is called an Apollonian manifold, which is known to be geometrically finite with all cusps having rank one. Its limit set is equal to the closure \( \overline{\mathcal{P}} \), and supports a locally finite Hausdorff measure \( \mathcal{H} \) of dimension \( \delta = 1.30568(8) \).

Fix a tangent point \( \xi = C_i \cap C_j \) for \( i \neq j \) and consider a sufficiently small Euclidean ball \( B \) in \( \mathbb{H}^3 \) based at \( \xi \), so that \( \mathcal{B} = \Gamma(B) \) is a disjoint collection of Euclidean balls. Fix \( o \in \mathbb{H}^3 \) outside of \( B \), let \( B(t) \subset B \) be the Euclidean ball based at \( \xi \) and \( d_{\mathbb{H}^3}(o,B(t)) = d_{\mathbb{H}^3}(o,B) + t \). Set \( \mathcal{B}_t := \Gamma(B(t)) \).

The following is a consequence of Theorem 1.4:

**Corollary 1.5.** Let \( \mathcal{P} \) be an Apollonian gasket. For \( \mathcal{H} \)-almost all initial direction \( v \) toward \( \overline{\mathcal{P}} \),

\[
\lim_{t \to \infty} \frac{\log(\inf\{ s > 0 : v_s \in \mathcal{B}_t \})}{t} = 2\delta - 1(= 1.61137(6)) \tag{1.14}
\]

where \( v_s \) denotes the base point of the vector \( G^s(v) \) traveled by distance \( s \) from \( v \).

1.5. **Tubular neighborhoods.** Another natural family of shrinking targets is given by tubular neighborhoods of a closed geodesic.

For a closed geodesic \( \mathcal{C} \subset T^1(\mathcal{M}) \), we consider the \( \epsilon \)-neighborhood of \( \mathcal{C} \) given by:

\[
\mathcal{C}_\epsilon := \{ x \in T^1(\mathcal{M}) : \text{dist}(x, \mathcal{C}) \leq \epsilon \}.
\]

The family \( \{ \mathcal{C}_{1/t} : t \gg 1 \} \) forms a family of shrinking neighborhoods of \( \mathcal{C} \). We show that \( \{ \mathcal{C}_{1/t} : t \gg 1 \} \) is a regular family with \( m(\mathcal{C}_{1/t}) \asymp m(\tilde{\mathcal{C}}_{1/t}) \asymp t^{-2\delta} \) for \( t \gg 1 \). Moreover, the thickening, \( \tilde{\mathcal{C}}_{1/t} \), of \( \mathcal{C}_{1/t} \) is contained in a slightly larger tubular neighborhood, say \( \mathcal{C}_{1/(3t)} \) (see §5.3). Applying our results to this family of shrinking targets gives the following result on the amount of time a generic geodesic spirals near a fixed closed geodesic (cf. [10, Theorem 1.1] for a similar result in a negatively curved compact manifold).
Theorem 1.6. Let $C \subset T^1(M)$ be a closed geodesic. Then for $m$-a.e. $x \in T^1(M)$, we have the following:

(1) $$\lim_{t \to \infty} \frac{\log \tau_{C_1/t}(x)}{\log t} = 2\delta; \quad (1.15)$$

(2) For any $0 < \eta < \frac{1}{2\delta}$ and for all $t \gg 1$,

$$\text{Leb}\{0 < s < t : \text{dist}(G^s(x), C) < t^{-\eta}\} \asymp t^{1-2\delta \eta}.$$ 

Remark 1.16. Since for any point $x \in T^1(M)$ we have that

$$\liminf_{\epsilon \to 0} \frac{\log(\tau_{C_1}(x)(x))}{\log t} = \left( \limsup_{t \to \infty} -\frac{\log(\text{dist}(G^t(x), C))}{\log t} \right)^{-1},$$

Theorem 1.6 (1) implies that for $m$-a.e. starting points $x \in T^1(M)$

$$\limsup_{t \to \infty} -\frac{\log(\text{dist}(G^t(x), C))}{\log t} = \frac{1}{2\delta}, \quad (1.17)$$

which was previously shown in [5, Theorem 4] to hold for the special case of convex co-compact hyperbolic surfaces.

1.6. Shrinking balls. For any fixed $x_0 \in \text{supp}(m)$, we show that the family of shrinking metric balls $B_{1/t}(x_0) = \{x \in T^1(M) : \text{dist}(x, x_0) < 1/t\}$ is regular and satisfies that $m(B_{1/t}(x_0)) \asymp m(B_{2/t}(x_0))$. When $\Gamma$ is convex co-compact, $m(B_{1/t}(x_0)) \asymp t^{-(2\delta+1)}$ and $m(\tilde{B}_{1/t}(x_0)) \asymp t^{-2\delta}$ (see §5.2). In particular our results imply the following:

Theorem 1.7. Suppose $\mathcal{M}$ is convex cocompact. Fix $x_0 \in \text{supp}(m)$. Then for $m$-a.e. $x \in T^1(M)$,

(1) $$\lim_{t \to \infty} \frac{\log \tau_{B_1/t}(x_0)(x)}{\log t} = 2\delta; \quad (1.18)$$

(2) For $0 < \eta < \frac{1}{2\delta+1}$ and for all $t \gg 1$, we have

$$\text{Leb}\{0 < s < t : \text{dist}(G^s(x), x_0) \leq t^{\eta}\} \asymp t^{1-(2\delta+1)\eta}.$$ 

When $\mathcal{M}$ has cusps, the situation is more complicated as the measure of shrinking balls can fluctuate, with the fluctuation depending on $x_0$ (or more precisely on the cusp excursions of the geodesic emanating from $x_0 \in T^1(M)$). Combining our previous results on cusp excursions we can show the following

Theorem 1.8. Suppose that $\mathcal{M}$ has cusps.

(1) For $m$-a.e. center points $x_0 \in T^1(M)$, and for $m$-a.e. $x \in T^1(M)$

$$\lim_{t \to \infty} \frac{\log \tau_{B_1/t}(x_0)(x)}{\log t} = 2\delta.$$
(2) On the other hand, for any pair of cusps of ranks $\kappa_1, \kappa_2$, we can find a center point $x_0$ such that for $m$-a.e. $x \in T^1(M)$

$$\lim_{t \to \infty} \frac{\tau_{B_{1/t}(x_0)}(x)}{\log t} = 4\delta - \kappa_1 - \kappa_2.$$ 

Remark 1.19. We note that for hyperbolic manifold of finite volume, both compact and non-compact, we have that $\delta = n - 1$ and the measure of thickened metric balls in $T^1(M)$ satisfies $m(\tilde{B}_{1/t}(x_0)) \asymp t^{-(2n-2)}$. Hence in this case the same arguments imply that for $m$- a.e. $x \in T^1(M)$ we have

$$\lim_{t \to \infty} \frac{\log \tau_{B_{1/t}(x_0)}(x)}{\log t} = 2(n - 1).$$

Remark 1.20. We consider shrinking balls in $M$, in which case the limit is $n - 1$ (see also [15], for related result for the discrete time geodesic flow).

1.7. Strategy of proof. First we define an averaging operator, along the discrete time, acting on $L^2(T^1(M), m)$:

$$\lambda_T(\Psi)(x) = \frac{1}{T} \sum_{k=1}^{T} \Psi(G^k(x)).$$

If $\Psi$ is the characteristic function of $B$, we simply write $\lambda_T(B)$ instead of $\lambda_T(1_B)$. The Birkhoff ergodic theorem implies that for a.e. $x \in X$,

$$\lim_{T \to \infty} \lambda_T(\Psi)(x) = m(\Psi).$$

Denoting by $\tau_{B_t}^d(x) = \min\{k : G^k(x) \in B_t\}$ the first hitting discrete time, we note that if we had a rate control in this convergence such as

$$|\lambda_T(B_t)(x) - m(B_t)| \ll \frac{\sqrt{m(B_t)} \log(m(B_t))}{\sqrt{T}}$$

we would get

$$\log \tau_{B_t}^d(x) \leq - \log m(B_t) \quad (1.21)$$

just from the simple observation that $\lambda_{\tau_{B_t}^d(x)}(B_t) = 0$.

An estimate like (1.20) is too strong to be true for a.e. individual point $x \in X$. So, instead, we prove its mean-version for all smooth functions $\Psi$, that is,

$$\|\lambda_T(\Psi) - m(\Psi)\| \leq C \frac{\|\Psi\| \log(S(\Psi)\|\Psi\|)}{\sqrt{T}}$$

(1.22)

for some uniform constant $C > 0$. The regularity conditions imposed on the thickenings $\tilde{B}_t$ of our shrinking targets are precisely so that we could apply (1.22) to smooth functions which approximates $1_{\tilde{B}_t}$ and deduce

$$\|\lambda_T(\tilde{B}_t) - m(\tilde{B}_t)\| \ll \frac{\sqrt{m(\tilde{B}_t)} \log(m(\tilde{B}_t))}{\sqrt{T}}. \quad (1.23)$$
This effective mean ergodic theorem for $\tilde{B}_t$’s enables us to obtain that for a.e. $x \in X$,

$$\log \tau_{\tilde{B}_t}(x) \leq -\log m(\tilde{B}_t) \quad (1.24)$$

for all sufficiently large $t$. Using that $|\tau_{\tilde{B}_t}(x) - \tau_{B_t}(x)| \leq 1$, we deduce that

$$\limsup_{t \to \infty} \log \tau_{B_t}(x) - \log m(B_t) \leq 1.$$ 

This is the non-trivial direction of the logarithm law Theorem 1.1; the other direction follows from an abstract property of a shrinking family. Theorems 1.2 and 1.3 are also proved in a similar spirit using the effective mean ergodic theorem.

The use of quantitative mixing of geodesic flow in the shrinking target problem in the homogeneous setting goes back to the work of Kleinbock and Margulis [14], and the idea of using an effective mean ergodic theorem was first introduced in [9] and more explicitly in [12, 13], where this idea was used to prove the analogous results for finite volume hyperbolic manifold.

Here we will use the following exponential decay of matrix coefficients for geometrically finite hyperbolic manifolds:

**Theorem 1.9.** Let $\Gamma$ be either convex cocompact or $\delta > \frac{n-1}{2}$. Then there exists $\eta_0 > 0$ such that for any bounded $\Psi_1, \Psi_2 \in C^\infty(T^1(M))$ with support in one-neighborhood of $\text{supp}(m)$, for all $t \geq 1$,

$$\int_{T^1(M)} \Psi_1(G^t(x)) \Psi_2(x) \, dm(x) = m(\Psi_1)m(\Psi_2) + O(e^{-\eta_0 t}S(\Psi_1)S(\Psi_2)).$$

This theorem was obtained in [19] for compactly supported functions under the assumption $\delta > \frac{n-1}{2}$ and in [24] for any convex cocompact $\Gamma$. In order to study shrinking target problem for cusp neighborhoods as described in Theorem 1.4, removing the compact support condition is crucial as we need to study functions that are positive on cusps. We use the quantitative decay of the matrix coefficient of the functions $L^2(\Gamma \setminus G)$ with respect to the Haar measure $m^{\text{Haar}}$ in [19], and exploit the product structures of $m$ and $m^{\text{Haar}}$ to transfer the exponential rate information on the transversal intersections of $G^t(B_\epsilon(x))$ for the flow box $B_\epsilon(x)$ of size $\epsilon$, that we get from the behavior of the correlation function with respect to $m^{\text{Haar}}$, to the behavior of the correlation function with respect to $m$. Here $\epsilon$ depends on the injectivity radius of $x$, and as we need to control the exponential rate independent of the injectivity radius for Theorem 1.9, which is required to deal with functions which are not compactly supported, the whole procedure turns out to be technically quite subtle.

After some preliminaries given in section 2, we devote section 3 to prove Theorem 1.9. With this result in hand, we prove effective mean ergodic theorem in this setting (see Theorem 4.1), and use it in section 4 to establish results on shrinking target problems for both the discrete and continuous...
time flow. While the results we obtain for the discrete time flow are essentially optimal, this is not the case for some of the results for continuous time flow. Nevertheless, in section 4.5, we show how one can obtain optimal results for the continuous flow by translating it into a discrete time flow problem for a thickened target. In section 5, we deduce Theorems 1.4, 1.6, 1.7 and 1.8 by proving the regularity of the corresponding shrinking sets and by computing their volumes using Sullivan’s shadow lemma and the structure of cusps for geometrically finite manifolds.

2. Preliminaries and notation

2.1. Notations and conventions. Let $G \cong \text{SO}(n,1)^o$ be the group of orientation preserving isometries of $\mathbb{H}^n$, and $\Gamma \subset G$ a geometrically finite, Zariski dense discrete subgroup of $G$. We denote by $\Lambda$ its limit set, and by $0 < \delta \leq n - 1$ the Hausdorff dimension of $\Lambda$, which is equal to the critical exponent of $\Gamma$. Let $\mathcal{M} = \Gamma \backslash \mathbb{H}^n$. Let $K < G$ be a maximal compact subgroup, $A = \{a_t : t \in \mathbb{R}\}$ a one parameter diagonalizable subgroup, and $M$ the centralizer of $A$ in $K$. We can identify $\mathcal{M}$ with $G/K$ and the unit tangent bundle $T^1(\mathcal{M})$ with $G/K$ in the way that the geodesic flow $\mathcal{G}^t$ on $T^1(\mathcal{M})$ is given by the right translation action of $a_t$ on $G/K$. With this identification we can work in the homogeneous space $\Gamma \backslash G$ and think of subsets and functions on $T^1(\mathcal{M})$ and $\mathcal{M}$ respectively as $M$-invariant (resp. $K$ invariant) subsets and functions on $\Gamma \backslash G$.

We use the notation $A \ll B$ (as well as $A = O(B)$) to mean that there is some positive constant $c \geq 0$ such that $A \leq cB$. We will also use the notation $A \times B$ if $A \ll B \ll A$ and we will use subscripts to indicate that the implied constants depend on some parameters. All implied constants may depend on $\Gamma$ which we think of as fixed throughout.

We say that two families of shrinking sets, $\{B_t\}$ and $\{A_t\}$, are Lipschitz equivalent if there are some constants $c_1, c_2$ such that $B_{c_1t} \subseteq A_t \subseteq B_{c_2t}$ for all $t$ and we denote this by $B_t \asymp A_t$.

We fix a left $G$-invariant and right $K$-invariant metric $d$ on $G$ which descends to the hyperbolic metric on $\mathbb{H}^n = G/K$. This induces a unique metric on $G/M$ which we will also denote by $d$ by abuse of notation. The metric $d$ defines distance function on $T^1(\mathcal{M}) = \Gamma \backslash G/M$ given by $\text{dist}(\Gamma g, \Gamma h) = \inf_{\gamma \in \Gamma} d(\gamma g, h)$. For a subset $S \subseteq G$ and $\epsilon > 0$, $S_\epsilon$ denotes the $\epsilon$-neighborhood of $e$ in $S$, that is, $S_\epsilon = \{g \in S : d(g, e) \leq \epsilon\}$. Set $B_\epsilon := P_\epsilon N_\epsilon$; and note that $G_\epsilon \asymp B_\epsilon$ for all sufficiently small $\epsilon > 0$.

2.2. Invariant measures. For $\xi \in \partial \mathbb{H}^n$, let $\beta_\xi : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ denote the Busemann function for the geodesic flow. A family of measures $\mu_x : x \in \mathbb{H}^n$ is called a $\Gamma$-invariant conformal density of dimension $\delta_{\mu} > 0$ on $\partial \mathbb{H}^n$, if each $\mu_x$ is a non-zero finite Borel measure on $\partial \mathbb{H}^n$ satisfying for any $x, y \in \mathbb{H}^n$, $\xi \in \partial \mathbb{H}^n$ and $\gamma \in \Gamma$,

$$\gamma \ast \mu_x = \mu_{\gamma x} \quad \text{and} \quad \frac{d\mu_y}{d\mu_x}(\xi) = e^{-\delta_{\mu} \beta_\xi(y,x)},$$
where $\gamma_s \mu_x(F) = \mu_x(\gamma^{-1}(F))$ for any Borel subset $F$ of $\partial \mathbb{H}^n$.

In particular, the Patterson-Sullivan density, $\nu_x$, is a $\Gamma$ invariant conformal density supported on the limit set $\Lambda$ of dimension $\delta$ and the Lebesgue density $m_x$ is a $G$-invariant conformal density of dimension $(n-1)$ (both are unique up to scalar multiplication).

Let $\pi : T^1(\mathbb{H}^n) \rightarrow \mathbb{H}^n$ be the natural projection. For $u \in T^1(\mathbb{H}^n)$, we denote by $u^\pm \in \partial \mathbb{H}^n$ the forward and the backward endpoints of the geodesic determined by $u$. Fix $o \in \mathbb{H}^n$ once and for all. The map

$$u \mapsto (u^+, u^-, s = \beta_u(o, \pi(u)))$$

is a homeomorphism between $T^1(\mathbb{H}^n)$ and

$$(\partial \mathbb{H}^n \times \partial \mathbb{H}^n - \{(\xi, \xi) : \xi \in \partial \mathbb{H}^n\}) \times \mathbb{R}.$$  

This homeomorphism gives us coordinates $(u^+, u^-, s)$ on $T^1(\mathbb{H}^n)$. In these coordinates, the BMS measure $m = m^{\text{BMS}}$, the Haar measure $m^{\text{Haar}}$, and the Burger-Roblin measure $m^{\text{BR}}$ on $T^1(\mathbb{H}^n)$ are given by

1. $dm(u) = \epsilon^{\delta_{u^+}(o, \pi(u))} \delta_x^{\delta_{u^-}(o, \pi(u))} dv_o(u^+) dv_o(u^-) ds.$
2. $dm^{\text{Haar}}(u) = \epsilon^{(n-1)\delta_{u^+}(o, \pi(u))} \delta_x^{(n-1)\delta_{u^-}(o, \pi(u))} dm_o(u^+) dm_o(u^-) ds.$
3. $dm^{\text{BR}}(u) = \epsilon^{(n-1)\delta_{u^+}(o, \pi(u))} \delta_x^{(n-1)\delta_{u^-}(o, \pi(u))} dm_o(u^+) dm_o(u^-) ds.$

Using the identification of $T^1(\mathbb{H}^n)$ with $G/M$, we can lift the above measures to right $M$ invariant measures on $G$. These measures are all left $\Gamma$-invariant, and hence descend to corresponding measures on $T^1(\mathcal{M}) = \Gamma \backslash G/M$ and $\Gamma \backslash G$, which we still denote by $m$, $m^{\text{Haar}}$ and $m^{\text{BR}}$ by abuse of notation. The measure $m$ is finite and ergodic with respect to the geodesic flow [22]. We recall that the conformal density $\nu_x$ and hence also the measure $m$ was defined up to a scalar multiplication and we will choose our normalization so that $m(T^1(\mathcal{M})) = m(\Gamma \backslash G) = 1$.

Let $N = N^+$ and $N^-$ denote the expanding and the contracting horospherical subgroups respectively, i.e.,

$$N^\pm = \{ g \in G : a_s g a_{-s} \rightarrow e \text{ as } s \rightarrow \pm \infty \}.$$  

Note that

$$\Omega := \text{supp}(m) = \{ [g] \in \Gamma \backslash G : g^+, g^- \in \Lambda(\Gamma) \},$$

where $g^\pm \in \partial \mathbb{H}^n$ means the end points of the geodesic defined by the coset $gM \in G/M \cong T^1(\mathbb{H}^n)$.

The BMS measure $m$ has a natural foliation corresponding to the decomposition $PN = G$ (modulo a Zariski closed subset) with $P = N^+ AM$. Explicitly, for any $g \in G$ we define the PS-measure and the Lebesgue measure on the coset $gN$, by

$${d\mu}_g^{\text{PS}}(gn) = e^{\delta_{(gn)^+}(o, gn)} dv_o(gn)^+$$

and

$${d\mu}_g^{\text{Leb}}(gn) = e^{(n-1)\delta_{(gn)^+}(o, gn)} dm_o(gn)^+.$$

(2.1)
respectively, and we define the measure $\tilde{\nu}_{gP}$ on the coset $gP$ by.

$$d\tilde{\nu}_{gP}(gp) = e^{\delta t}d\nu_o(gp)^{-}dt$$  \hspace{1cm} (2.3)

for $t = \beta_{(gp)} -(o, gp)$. Using the decomposition $G = gPN$ and noting that $(gp)^{-} = (gp)^{-}$, we have that for any $\Psi \in C_c(G),$

$$m(\Psi) = \int_{gP} \int_{N} \Psi(gpn)d\tilde{\mu}_{PS}^{gP}N(gpm)d\nu_{gP}(gp).$$  \hspace{1cm} (2.4)

Finally, for $x = [g] \in \Gamma \backslash G$ and $\epsilon > 0$ smaller than the injectivity radius at $x$, we denote by $d\mu_{xN}$ and $d\nu_{xP}$ the measures induced by $d\tilde{\mu}_{gN}$ and $d\tilde{\nu}_{gP}$ on the orbits $xN$ and $xP$ respectively.

2.3. Cusp decomposition. Let $X_0$ be the pre-image of the convex core of $M$ under the base point projection map $\pi: \Gamma \backslash G \rightarrow \Gamma \backslash G/K = M$ and let $X$ be the unit neighborhood of $X_0$. Then $\Omega \subseteq X_0 \subseteq X$ and since $M$ is geometrically finite, $X$ has finite Haar-measure. When $M$ is convex cocompact, $X$ is compact, and otherwise it can be decomposed into a compact part and finitely many cusp neighborhoods, as we describe below.

Let $\Lambda_p \subset \Lambda$ denote the set of parabolic fixed points (i.e. points fixed by some parabolic element of $\Gamma$). Since $\Gamma$ is geometrically finite, $\Gamma_p$ consists of finitely many $\Gamma$-orbits represented by $\{\xi_1, \ldots, \xi_k\}$ which are the cusps of $\Gamma$.

A cusp neighborhood of $\xi_i \in \Lambda_p$ is a set of the form

$$h_i = \pi^{-1}(\Gamma \backslash \Gamma H_{\xi_i})$$  \hspace{1cm} (2.5)

where $H_{\xi} \subseteq \mathbb{H}^n$ is some fixed horoball tangent to $\xi$ such that $\gamma H_{\xi} \cap H_{\xi} \neq \emptyset$ if and only if $\gamma$ fixes $\xi$, and $\pi: \Gamma \backslash G \rightarrow \Gamma \backslash \mathbb{H}$ is again the natural projection. For each cusp $\xi_i$, its rank is defined to be the rank of the maximal abelian subgroup contained in the stabilizer $\Gamma_{\xi_i}$ of $\xi_i$ in $\Gamma$; we will denote it by $\kappa_i$. We denote by $\kappa_{\text{max}}$ and $\kappa_{\text{min}}$ the maximal and minimal ranks of cusps of $\Gamma$ respectively, and note that

$$2\delta > k_{\text{max}}$$

(see [4, Lem. 3.5]).

For $x \in \Gamma \backslash G$, we denote by $r_x$ the injectivity radius at $x$. For all sufficiently small $\epsilon > 0$, let

$$X(\epsilon) = \{x \in X : r_x < \epsilon\},$$

so that

$$Y(\epsilon) := X \setminus X(\epsilon)$$

is compact, and the family $X(\epsilon)$ with $\epsilon < \epsilon_0$ forms a shrinking family of cusp neighborhoods.

More explicitly, we show in section 5.1 that for $\epsilon > 0$ sufficiently small

$$X(\epsilon) \cap h_i \asymp X \cap h_{i,\log(\epsilon^{-1})},$$  \hspace{1cm} (2.6)

and using the measure estimate $m(h_{i,\log(\epsilon^{-1})}) \asymp \epsilon^{2\delta - \kappa_i}$ (see Proposition 5.5) we get that

$$m(X(\epsilon)) \asymp \epsilon^{2\delta - \kappa_{\text{max}}}.$$  \hspace{1cm} (2.7)
2.4. Sobolev norms. The mixing rate of the geodesic flow depends on the smoothness of the test functions which can be captured by appropriate Sobolev norms we now define. Given some fixed basis for the Lie algebra of $G$, $l \in \mathbb{N}$, and $1 \leq p \leq \infty$ the Sobolev norm $S_{p,l}$ is defined on $\Psi \in C^\infty(\Gamma \setminus G)$ by

$$S_{p,l}(\Psi) = \sum \|D(\Psi)\|_p^{\text{Haar}}$$

where the sum is taken over all monomials $D$ of order at most $l$ in the basis elements, and $\|\Psi\|_p^{\text{Haar}}$ denotes the $L^p(\Gamma \setminus G, m^{\text{Haar}})$-norm of $\Psi$. While this norm does depend on the choice of basis, changing the basis will only change the norm by some bounded factor.

We will mostly use the norms $S_{\infty,l}$, which we will denote by $S_l$ to simplify notation. Since supp$(m) \subset X$ it is sufficient for our purpose to consider functions supported on $X$, and since $X$ has finite Haar measure we can, and will use the bound

$$S_{p,l}(\Psi) \leq S_{\infty,l}(\Psi) m^{\text{Haar}}(X)^{1/p} \ll S_l(\Psi),$$

uniformly for all $\Psi \in C^\infty(X)$.

3. Decay of matrix coefficients

A crucial ingredient in our proof is the exponential mixing of the geodesic flow with respect to the BMS-measure. We use the inner product notation:

$$\langle a_t \Psi, \Phi \rangle = \int_{\Gamma \setminus G} \Psi(xa_t)\Phi(x) \, dm(x).$$

**Theorem 3.1.** Suppose that either $\Gamma$ is convex cocompact or that $\delta > \max\{\frac{n-1}{2}, n-2\}$ (resp. $\delta > \frac{n-1}{2}$). Then, there exist $\eta_0 > 0$ and $l \in \mathbb{N}$, such that for any bounded $\Psi, \Phi \in C^\infty(X)$ (resp. $\Psi, \Phi \in C^\infty(X)^M$)

$$\langle a_t \Psi, \Phi \rangle = m(\Psi) \cdot m(\Phi) + O(e^{-\eta_0 t} S_l(\Psi) S_l(\Phi)).$$

When $\Gamma$ is convex co-compact this result is given in [24, Theorem 1.1]. In the rest of this section, we assume

$$\delta > (n-1)/2.$$

Theorem 3.1 is then proved in [19, Theorem 6.16] under the assumption that the test functions are compactly supported. In order to complete the proof of the theorem we need to remove the assumption on the support of the test functions.

To do this, we will approximate $\Psi$ as the sum $\Psi_\epsilon + (\Psi - \Psi_\epsilon)$ where $\Psi_\epsilon$ is a smooth function supported on $Y(\epsilon)$, and similarly for $\Phi$. In view of (2.7), the main term will be reduced to $\langle a_t \Psi_\epsilon, \Phi_\epsilon \rangle$, for which the result follows from [19, Theorem 6.16]. However, since the dependence on the supports of $\Psi_\epsilon, \Phi_\epsilon$ was not made explicit in terms of $\epsilon$ in [19], we need to redo their arguments while keeping track of the dependence on $\epsilon$ as well as on all implied constants along the proof.
3.1. Control of BR measures. Since \( m^{BR}(\Gamma \backslash G) = \infty \) when \( \Gamma < G \) is not a lattice, and some of the implied constants in [19, Thm. 6.16] depend on \( m^{BR}(\text{supp}(\Psi)) \) we need the following result to control the dependence on these measures.

**Lemma 3.2.** Assume that \( \delta > \frac{n-1}{2} \). Then there exists \( c > 0 \) such that for any \( K \)-invariant subset \( Y \subset \Gamma \backslash G \) with \( m^{\text{Haar}}(Y) < \infty \), we have

\[
m^{BR}(Y) \leq c \cdot \sqrt{m^{\text{Haar}}(Y)}.
\]

**Proof.** Recall that by [21] and [17], there exists a positive eigenfunction \( \phi_0 \in C^\infty(\Gamma \backslash G)^K \) for the Laplace operator such that

\[
-\Delta \phi_0 = \delta (n-1-\delta) \phi_0
\]

Under the assumption \( \delta > \frac{n-1}{2} \), we have

\[
\|\phi_0\|^2_{\text{Haar}} < \infty.
\]

If \( \Psi \) denotes the indicator function of \( Y \), then \( \Psi \) is \( K \)-invariant and hence by [16, Lem. 6.7]

\[
m^{BR}(\Psi) = \int_X \Psi(x) \phi_0(x) d\text{m}^{\text{Haar}}(x),
\]

and in particular \( m^{BR}(Y) \leq \|\phi_0\|^2_{\text{Haar}} \sqrt{m^{\text{Haar}}(Y)} \), as claimed. \( \square \)

Since \( X \) is \( K \)-invariant with \( m^{\text{Haar}}(X) < \infty \), the following follows from Lemma 3.2:

**Corollary 3.3.** If \( \delta > \frac{(n-1)}{2} \), then

\[
m^{BR}(X) < \infty.
\]

3.2. Test function supported on small balls. Next we prove the result for the special case when the function \( \Psi, \Phi \) are supported on a small neighborhood of a point \( x \in \Omega := \text{supp}(m) \). For that, we consider a small number \( \epsilon > 0 \) and a point \( x \in Y(\epsilon) \cap \Omega \), and consider \( \Phi, \Psi \in C^\infty(xB_\epsilon) \) supported in an \( \epsilon \)-neighborhood of \( x \), where we recall the notation \( B_\epsilon := P_\epsilon N_\epsilon \) where \( P = N^{-AM} \). In this section, we will prove the following.

**Proposition 3.4.** Suppose \( \delta > \max\{\frac{n-1}{2}, n-2\} \) (resp. \( \delta > \frac{n-1}{2} \)). For \( \epsilon \in (0,1) \) small and any \( x \in Y(\epsilon) \cap \Omega \), there exist \( l \in \mathbb{N} \) depending only on \( \text{dim}(G) \) and \( \eta > 0 \) (depending only on the spectral gap of \( \Gamma \)) such that for all \( \Phi, \Psi \in C^\infty(xB_\epsilon) \) (resp. \( \Phi, \Psi \in C^\infty(xB_\epsilon M)^M \)) we have that

\[
\langle a_t \Psi, \Phi \rangle = m(\Psi)m(\Phi) + O(e^{-\eta t} S_l(\Psi) S_l(\Phi))
\]

where the implied constant is absolute.

**Proof.** Fix \( \Phi, \Psi \in C^\infty(xB_\epsilon) \). In the case when \( \frac{n-1}{2} < \delta \leq \max\{\frac{n-1}{2}, n-2\} \), we assume that \( \Phi, \Psi \in C^\infty(xB_\epsilon M) \) are \( M \)-invariant. We have

\[
\langle a_t \Psi, \Phi \rangle = \int_{xP} \int_{xP N_\epsilon} \Psi(xpm \alpha_t) \Phi(xpm) d\mu_{xP N}(xpm) d\nu_P(xP).
\]
Now, for fixed \( p \in P \), letting \( \phi = \Phi|_{xpNa} \in C_c^\infty(xpNa) \), we estimate the inner integral
\[
\int_{xpNa} \Psi(xpna_t)\phi(xp) d\mu_{xpN}(xp) = \int_{xpNa} \Psi(xpma_t)\phi(xp) d\mu_{xpN}(xp)
\]
as follows.

Fix a small \( 0 < \epsilon_0 < \epsilon^2 \) and consider the functions \( \Psi_{\epsilon_0}^\pm \) on \( \Gamma \setminus G \) defined by
\[
\Psi_{\epsilon_0}^+(y) = \sup_{g \in G_{\epsilon_0}} \Psi(yg), \quad \Psi_{\epsilon_0}^-(y) = \inf_{g \in G_{\epsilon_0}} \Psi(yg)
\]
and let
\[
\psi_{\epsilon_0}^\pm(xp) = \int_{xpN} \Psi_{\epsilon_0}^\pm(xp) d\mu_{xpN}(xp).
\]
We then have that
\[
\nu_{xp}(\psi_{\epsilon_0}^\pm) = m(\Psi^\pm) \quad \text{and} \quad \int_{xpN} \mu_{xpN}(\phi) d\nu_{xp}(xp) = m(\Phi).
\]
Moreover, since \( \Psi(x) = \Psi_{\epsilon_0}^\pm(x) + O(\epsilon_0 S_{\infty,1}(\Psi)) \), we get that
\[
m(\Psi_{\epsilon_0}^\pm) = m(\Psi) + O(\epsilon_0 S_{\infty,1}(\Psi)),
\]
where we used that \( m(X) < \infty \) is finite. We will also use the notation
\[
\phi_{\epsilon_1}^+(y) := \sup_{n \in N_{\epsilon_1}} \phi(yn),
\]
and similarly get that \( \mu_{ypN}^\pm(\phi_{\epsilon_1}^+) = \mu_{ypN}^\pm(\phi) + O(\epsilon_1 S_{\infty,1}(\phi)). \)

Now by [19, Lem. 6.2] there exists some absolute constant \( c > 0 \), such that the integral
\[
\int_{xpNa} \Psi(xpna_t)\phi(xp) d\mu_{xpN}(xp)
\]
is bounded from above and below (respectively) by
\[
(1 \pm ce\epsilon_0)e^{-\delta t} \sum_{p \in P_x(t)} \psi_{\epsilon_0}^\pm(xp)\phi_{\epsilon_0}^\pm(xpna_{-t}),
\]
where \( P_x(t) \) is the finite set defined by
\[
P_x(t) = \{ p \in P : xpNa_t \cap xpN = \emptyset \}.
\]
Moreover, by the proof of [19, Thm. 6.7], there are positive constants \( \eta > 0 \) (depending only on the spectral gap of \( \Gamma \)) and \( \alpha > 0 \) such that
\[
e^{-\delta t} \sum_{p \in P_x(t)} \psi_{\epsilon_0}^\pm(xp)\phi_{\epsilon_0}^\pm(xpna_{-t}) = \nu_{xp}(\psi_{\epsilon_0}^\pm)\mu_{xpN}(\phi_{\epsilon_0}^\pm)
\]
\[
+ O(e^{-\eta t} + \epsilon_0^\alpha A_{\Psi}^{BR} A_{\phi}^{PS} + O(e^{-\eta t} S_{2,1}(\Psi) S_{2,1}(\phi))
\]
where
\[
A_{\Psi}^{BR} := S_{\infty,1}(\Psi)m^{BR}(\text{supp}(\Psi)) \ll S_{\infty,1}(\Psi),
\]
We now use Lemma 3.2 to bound
\[
A^\mathrm{PS}_\phi := S_{\infty,1}(\phi)\mu^\mathrm{PS}_{xpN}(\text{supp}(\phi)) \leq S_{\infty,1}(\phi)\mu^\mathrm{PS}_{xpN}(xpN_\epsilon).
\]
Combining these results and estimating
\[
\nu_{xP}(\psi_{t_0}^\pm) = m(\Psi_{t_0}^\pm) = m(\Psi) + O(\epsilon_0S_{\infty,1}(\Psi)),
\]
and
\[
\mu^\mathrm{PS}_{xpN}(\phi_{t_0}^\pm) = \mu^\mathrm{PS}_{xpN}(\phi) + O(\epsilon_0S_{\infty,1}(\phi)),
\]
we get that
\[
\int_{xpN_\epsilon} \Psi(xpna_t)\phi(xp) d\mu^\mathrm{PS}_{xpN}(xpn) = m(\Psi)\mu^\mathrm{PS}_{xpN}(\phi)(1 + O(\epsilon_0))
\]
\[
+ O(\epsilon_0S_{\infty,1}(\Psi)S_{\infty,1}(\Psi)) + O(e^{-\eta t} + \epsilon_0^2)S_{\infty,1}(\Psi)S_{\infty,1}(\phi)\mu^\mathrm{PS}_{xpN}(xpN_\epsilon)
\]
\[
+ O(e^{-\eta t}S_{2,l}(\Psi)S_{2,l}(\phi)).
\]
Since all implied constants are independent of \(\epsilon_0\), taking the limit as \(\epsilon_0 \to 0\) gives
\[
\int_{xpN_\epsilon} \Psi(xpna_t)\phi(xp) d\mu^\mathrm{PS}_{xpN}(xpn) = m(\Psi)\mu^\mathrm{PS}_{xpN}(\phi)
\]
\[
+ O(e^{-\eta t}S_{\infty,l}(\Psi)S_{\infty,l}(\phi)\mu^\mathrm{PS}_{xpN}(xpN_\epsilon)) + O(e^{-\eta t}S_{2,l}(\Psi)S_{2,l}(\phi))
\]
where we used that \(S_{\infty,l}(\phi) \leq S_{\infty,l}(\Phi)\).

Now, integrating over \(xP_\epsilon\), and noting that \(\int_{xP_\epsilon} \mu^\mathrm{PS}_{xpN}(\phi) d\nu_{xP}(xp) = m(\Phi)\), the main term is indeed \(m(\Psi)m(\Phi)\). Next, since
\[
\int_{xP_\epsilon} \mu^\mathrm{PS}_{xpN}(xpN_\epsilon) d\nu_{xP}(xp) = \int_{xP_\epsilon} \int_{xpN_\epsilon} d\mu^\mathrm{PS}_{xpN} d\nu_{xP}(xp) = m(B_\epsilon) \leq 1,
\]
the integral of the first remainder term is bounded by \(O(e^{-\eta t}S_{\infty,l}(\Psi)S_{\infty,l}(\Phi))\).

For the second remainder term we bound \(S_{2,l}(\phi) \leq S_{\infty,l}(\Phi)\sqrt{\mu^\mathrm{Leb}_{xpN}(xpN_\epsilon)}\)
to get that
\[
\int_{xP_\epsilon} S_{2,l}(\Phi_{xP_\epsilon N_\epsilon}) d\nu_{xP}(xp) \ll S_{\infty,l}(\Phi)(\sqrt{\mu^\mathrm{Leb}_{xpN}(xpN_\epsilon)})^{-1/2} \int_{xP_\epsilon} \int_{xpN_\epsilon} d\mu^\mathrm{Leb}_{xpN}(xpn) d\nu_{xP}(xp)
\]
\[
= S_{\infty,l}(\Phi)(\sqrt{\mu^\mathrm{Leb}_{xpN}(xpN_\epsilon)})^{-1/2} m^\mathrm{BR}(xP_\epsilon N_\epsilon).\]

We now use Lemma 3.2 to bound
\[
m^\mathrm{BR}(xP_\epsilon N_\epsilon) \leq m^\mathrm{BR}(xP_\epsilon N_\epsilon K) \ll \sqrt{m^\mathrm{Haar}(xP_\epsilon N_\epsilon K)},
\]
and since there is a uniform constant \(c > 0\) such that \(P_\epsilon N_\epsilon K \subseteq P_\epsilon K\), noting that \(P_\epsilon K = N_\epsilon A_\epsilon K\), we can bound
\[
m^\mathrm{Haar}(xP_\epsilon N_\epsilon K) \ll m^\mathrm{Haar}(xP_\epsilon K) \ll \mu^\mathrm{Leb}_{xpN}(xpN_\epsilon)
\]
to get that
\[
\int_{xP_\epsilon} S_{2,l}(\Phi_{xP_\epsilon N_\epsilon}) d\nu_{xP}(xp) \ll S_{\infty,l}(\Phi),
\]
Lemma 3.5. For any $\beta \in B_{2\epsilon}$, we get that
\[ \langle a_1(\Psi, \Phi) \rangle = m(\Psi)m(\Phi) + O(\epsilon^{-\eta}S_l(\Psi)S_l(\Phi)) \]
where the implied constant is absolute.

□

3.3. General test functions. We now use a partition of unity to reduce the case of a general test function to the case of functions with small support.

For $\epsilon \in (0, 1)$ sufficiently small, let $Q_\epsilon$ be a maximal family of points in $X \cap Y_\epsilon$ such that the sets $yB_{3\epsilon}$, $y \in Q_\epsilon$, are disjoint and meet $Y_{2\epsilon}$, and let $Q_\epsilon := \{ y \in Q_\epsilon : yB_{3\epsilon} \cap Y_{3\epsilon} \neq \emptyset \}$. Note that the collection $\{ yB_{2\epsilon} : y \in Q_\epsilon \}$ covers $X \cap Y_{2\epsilon}$ and the collection $\{ yB_{3\epsilon}B_{\epsilon^3} : y \in Q_\epsilon \}$ covers $X \cap Y_{4\epsilon}$. Since $m_{\text{Haar}}(X) < \infty$, we have $\#Q_\epsilon = O(\epsilon^{-3\dim(G)})$.

Fix a non-negative function $\beta_\epsilon \in C^\infty(B_{\epsilon})$ taking values in $[0, 1]$ which is 1 on $B_{3\epsilon}B_{\epsilon^3}$ and 0 outside $B_{2\epsilon}$ (note that $B_{3\epsilon}B_{\epsilon^3} \subseteq B_{2\epsilon} \subseteq B_{\epsilon}$). We can choose $\beta_\epsilon$ so that $S_{\infty, l}(\beta_\epsilon) \ll \epsilon^{-3l}$. For each $y \in Q_\epsilon$ define a function on $yB_\epsilon$ by $\beta_{y_\epsilon}(y) := \beta_\epsilon(b)$.

Lemma 3.5. For any $y \in Q_\epsilon$ and $x \in yB_{2\epsilon}$, we have
\[ \sum_{z \in Q_\epsilon} \beta_{z_\epsilon}(x) \geq 1. \]

Proof. First note that if $y \in Q_\epsilon$ then $yB_{2\epsilon} \cap Y_{3\epsilon} \neq \emptyset$ so there is some $b \in B_{2\epsilon}$ and $y' \in Y_{3\epsilon}$ so that $yb \in Y_{3\epsilon}$ implying that $y \in Y_{3\epsilon}$ (since $yG_{3\epsilon} = y'bG_{3\epsilon} \subseteq y'G_{4\epsilon}$). Now there are two possibilities, either $x \in yB_{3\epsilon}B_{\epsilon^3}$ in which case $\beta_{y_\epsilon}(x) = 1$ so $\sum_{z \in Q_\epsilon} \beta_{z_\epsilon}(x) \geq 1$. Otherwise, $x \in yB_{2\epsilon} \setminus yB_{3\epsilon}B_{\epsilon^3}$, in which case $xB_{3\epsilon} \cap yB_{3\epsilon} = \emptyset$. Since $y \in Y_{3\epsilon}$ and $x \in yB_{2\epsilon}$ we have that $x \in Y_{2\epsilon} \cap X$. Hence, from maximality of the set $Q_\epsilon$ there is some $z \in Q_\epsilon$ such that $xB_{3\epsilon} \cap zB_{\epsilon^3} \neq \emptyset$ (otherwise we could have added $x$ to $Q_\epsilon$). This implies that $x \in zB_{3\epsilon}B_{\epsilon^3}$ so $\beta_{z_\epsilon}(x) \geq 1$, implying the result in this case.

Now consider the normalized function (supported on $yB_\epsilon$) given by:
\[ \alpha_{y_\epsilon} := \frac{\beta_{y_\epsilon}}{\sum_{z \in Q_\epsilon} \beta_{z_\epsilon}}. \]

Lemma 3.6. For any $y \in Q_\epsilon$, we can bound
\[ S_{\infty, l}(\alpha_{y_\epsilon}) \ll \epsilon^{-p}, \]
where the exponent $p$, and the implied constant depend only on $l$ and $\dim(G)$.

Proof. Let $s_\epsilon(x) = \sum_{z \in Q_\epsilon} \beta_{z_\epsilon}(x)$ so that $\alpha_{y_\epsilon}(x) = \frac{\beta_{y_\epsilon}(x)}{s_\epsilon(x)}$. Since $\alpha_{y_\epsilon}$ is supported on $yB_{2\epsilon}$ we only need to bound its derivatives there in which case we have that $s_\epsilon(x) = \sum_{z \in Q_\epsilon} \beta_{z_\epsilon}(x) \geq 1$. Taking derivatives of the quotient $\alpha_{y_\epsilon} = \frac{\beta_{y_\epsilon}}{s_\epsilon}$ and using the bound $s_\epsilon(x) \geq 1$ together with the bound $S_{\infty, l}(s_\epsilon) \ll \epsilon^{-3l} \#Q_\epsilon \ll \epsilon^{-3(l+\dim(G))}$, gives our result.

□

Lemma 3.7. The function $\tau_\epsilon = \sum_{y \in Q_\epsilon} \alpha_{y_\epsilon}$ is in $C^\infty(X)$ and satisfies that
• $0 \leq \tau_\epsilon \leq 1$
• $\tau_\epsilon = 1$ on $X \cap Y_{4\epsilon}$, and
• $\tau_\epsilon = 0$ outside $Y_\epsilon$.

Proof. Since $\tau_\epsilon = \frac{\sum_{y \in Q_\epsilon} \beta_{y,\epsilon}}{\sum_{y \in Q_\epsilon} \beta_{y,\epsilon}}$, it is clear that $0 \leq \tau_\epsilon \leq 1$. Next for $x \in X \cap Y_{4\epsilon}$ if $y \in Q_\epsilon \setminus Q'_\epsilon$ then $y B_{\epsilon} \cap Y_{4\epsilon} = \emptyset$, and hence $x \notin y B_{\epsilon}$ so $\beta_{y,\epsilon}(x) = 0$. This shows that $\sum_{y \in Q_\epsilon} \beta_{y,\epsilon}(x) = \sum_{y \in Q_\epsilon} \beta_{y,\epsilon}(x)$. Moreover, since $X \cap Y_{4\epsilon}$ is covered by $\{y B_{3} B_{3} : y \in Q'_\epsilon\}$, we have that $\sum_{y \in Q'_\epsilon} \beta_{y,\epsilon}(x) \neq 0$ on $X \cap T_{4\epsilon}$ and hence indeed $\tau_\epsilon = 1$ there. Next, since for any $y \in Q'_\epsilon$, we have that $y B_{\epsilon} \subseteq Y_\epsilon$, indeed $\tau_\epsilon(x) = 0$ outside of $Y_\epsilon$. Finally we can bound

$$S_{\infty, l}(\tau_\epsilon) \leq \sum_{y \in Q'_\epsilon} S_{\infty, l}(\alpha_{y,\epsilon}) \ll \epsilon^{-p+3 \dim(G)}.$$  

Proof of Theorem 3.1. Suppose first that $\delta > \max\{\frac{n-1}{2}, n-2\}$. Now, for given bounded $\Psi, \Phi$ in $C^\infty(X)$, consider

$$\Psi_\epsilon := \Psi \cdot \tau_\epsilon = \sum_{y \in Q'_\epsilon} \Psi \cdot \alpha_{y,\epsilon} \text{ and } \Phi_\epsilon := \Phi \cdot \tau_\epsilon = \sum_{y \in Q'_\epsilon} \Phi \cdot \alpha_{y,\epsilon}.$$  

Note that $S_{l}(\Psi \cdot \alpha_{y,\epsilon}) \ll S_{l}(\alpha_{y,\epsilon}) S_{l}(\Psi) \ll \epsilon^{-p} S_{l}(\Psi)$, with $p$ as in Lemma 3.6. Now applying Proposition 3.4 to each $\Psi \cdot \alpha_{y,\epsilon}$ and $\Phi \cdot \alpha_{y,\epsilon}$ for $y, y' \in Q'_\epsilon$, and recalling that $\#Q_{\epsilon} = O(\epsilon^{-3 \dim(G)})$ we get that

$$\langle a_t \Psi_\epsilon, \Phi_\epsilon \rangle = m(\Psi_\epsilon)m(\Phi_\epsilon) + O(\epsilon^{-p_0} e^{-\eta t} S_{l}(\Psi) S_{l}(\Phi))$$  

(3.3)

with $p_0 = 2p + 6 \dim(G)$.

Now, by (2.7), for $\delta_0 := 2\delta - \kappa_{\max} > 0$,

$$m(\Psi - \Psi_\epsilon) \ll \|\Psi\|_\infty m(X_{4\epsilon}) \ll \epsilon^{\delta_0}\|\Psi\|_\infty,$$

and similarly $m(\Phi - \Phi_\epsilon) \ll \epsilon^{\delta_0}\|\Phi\|_\infty$. Hence

$$|\langle a_t \Psi, \Phi \rangle - \langle a_t \Psi_\epsilon, \Phi_\epsilon \rangle| \ll \epsilon^{\delta_0}\|\Psi\|_\infty\|\Phi\|_\infty,$$

and therefore we deduce

$$\langle a_t \Psi, \Phi \rangle = m(\Psi)m(\Phi) + O(\epsilon^{\delta_0}\|\Psi\|_\infty\|\Phi\|_\infty) + O(\epsilon^{-p_0} e^{-\eta t} S_{l}(\Psi) S_{l}(\Phi)).$$

Taking $\epsilon = e^{-\frac{\eta t}{p_0 + 1}}$ and recalling that $S_{\infty, 0} \ll S_{l}$, we get that

$$\langle a_t \Psi, \Phi \rangle = m(\Psi)m(\Phi) + O(\epsilon^{-\eta t} S_{l}(\Psi) S_{l}(\Phi)).$$

with $\gamma_0 = \frac{\eta t}{p_0 + 1}$, thus concluding the proof when $\delta > \max\{\frac{n-1}{2}, n-2\}$.

Finally, for $n > 3$, if $\frac{n-1}{2} < \delta \leq n-2$, and $\Psi$ and $\Phi$ are $M$-invariant, we can replace $\alpha_{y,\epsilon}$ with an $M$-invariant function $\alpha_{y,\epsilon}^{M}(x) = \int_{M} \alpha_{y,\epsilon}(x m)dm$ and run the same argument to get (3.3). Then the rest of the proof is identical.
4. Shrinking target problems

We now use the results on the exponential decay of matrix coefficients to prove an effective mean ergodic theorem and apply it to various shrinking target problems. As before, we assume $\Gamma$ is a geometrically finite Zariski dense subgroup of $G = \text{SO}(n,1)^\circ$. We assume either $\delta > \frac{n-1}{2}$ or $\Gamma$ is convex cocompact. For $n \geq 5$ in the case where $\Gamma$ has a cusp and $\delta \leq n-2$, all functions and shrinking targets on $X$ we consider below are assumed to be $M$-invariant so that Theorem 3.1 applies to them. All functions below are also assumed to be real-valued functions.

4.1. Effective mean ergodic theorem. For $\lambda$ a measure on $\mathbb{R}$ and $T \geq 1$, consider the averaging operator $\lambda_T$ on $L^2(X, m)$ given by

$$\lambda_T(\Psi)(x) = \frac{1}{T} \int_0^T \Psi(xa_t) d\lambda(t).$$

We will take $\lambda$ to be either the Lebesgue measure on $\mathbb{R}$ (when considering continuous time flow) or the counting measure on $\mathbb{Z}$ (for discrete time) and note that in both cases the corresponding averaging operator is unitary.

Fix $\ell$ as given in Theorem 3.1. For $\Psi \in L^2(X, m)$ let $\|\Psi\|$ denote the $L^2(X, m)$. For notational convience we introduce the norm

$$S^*(\Psi) := \frac{S(\Psi)}{\|\Psi\|}.$$

for non-zero $\psi \in C^\infty(X) \cap L^2(X, m)$.

**Theorem 4.1.** Let $\lambda$ denote Lebesgue measure on $\mathbb{R}$ or the counting measure on $\mathbb{Z}$. Then for any non-zero $\Psi \in C^\infty(X)$, and for all $T \gg 1$,

$$\|\lambda_T(\Psi) - m(\Psi)\|^2 \ll \frac{(1 + \log(S^*(\Psi)) \cdot \|\Psi\|^2}{T}.$$

**Proof.** Since we have

$$\|\lambda_T(\Psi) - m(\Psi)\|^2 = \|\lambda_T(\Psi)\|^2 - m(\Psi)^2,$$

it is enough to estimate $\|\lambda_T(\Psi)\|^2$. Now, expand

$$\|\lambda_T \Psi\|^2 = \frac{1}{T^2} \int_0^T \int_0^T \int_X \Psi(xa_{t_1}a_{t_2}) \Psi(g) dm(g) d\lambda(t_1) d\lambda(t_2)$$

$$= \frac{1}{T^2} \int_{-T}^T \int_X \Psi(xa_t) \Psi(x) dm(x) (T - |t|) d\lambda(t)$$

where we used that $\lambda$ is translation invariant and $\lambda((0, T) \cap (t, t + T)) = T - |t|.$

Now fix a large parameter $M$ to be determined later. For $|t| \geq M$ large we use Theorem 1.9 to get that

$$\int_X \Psi(xa_t) \Psi(x) dm(x) = m(\Psi)^2 + O(S(\Psi)^2 e^{-M|t|}).$$
for some $\eta_0 \in (0, 1)$, while for $|t| < M$ small we bound $\langle a_t \Psi, \Psi \rangle \leq \|\Psi\|^2$, to get that
\[
\|\lambda_T \Psi\|^2 = m(\Psi)^2 + O(\|\Psi\|^2 \frac{M}{T}) + O(S(\Psi)^2 e^{-\eta_0 M} T),
\]
where we used that $m(\Psi) \leq \|\Psi\|$. Using this estimate we conclude that
\[
\|\lambda_T \Psi - m(\Psi)\|^2 \ll \frac{M \|\Psi\|^2 + S(\Psi)^2 e^{-\eta_0 M} T}{T},
\]
and taking $M = \frac{2 \log(S^*(\psi))}{\eta_0}$ concludes the proof.

Following [12], for a non-negative function $\Psi$ on $X$, we define the following two sets:
\[
C_{T,\Psi} = \{ x \in X : |\lambda_T(\Psi)(x) - m(\Psi)| \geq \frac{m(\Psi)}{2} \}, \quad C_{T,\Psi}^0 = \{ x \in X : \lambda_T(\Psi)(x) = 0 \},
\]
and note that $C_{T,\Psi}^0 \subseteq C_{T,\Psi}$.

As a direct consequence of the effective mean ergodic theorem, we get the following bounds for the measures of $C_{T,\Psi}$ for smooth non-negative functions.

**Proposition 4.2.** For a non-negative $\Psi \in C^\infty(X)$ and $T \geq 1$, we have
\[
m(C_{T,\Psi}) \ll \frac{\log(S^*(\Psi))\|\Psi\|^2}{T \cdot m(\Psi)^2}.
\]

**Proof.** On one hand
\[
\|\lambda_T(\Psi) - m(\Psi)\|^2 \geq \int_{C_{T,\Psi}} |\lambda_T(\Psi)(x) - m(\Psi)|^2 dm \geq \frac{(m(\Psi))^2 m(C_{T,\Psi})}{4},
\]
and on the other hand by Theorem 4.1
\[
\|\lambda_T(\Psi) - m(\Psi)\|^2 \ll \frac{\log(S^*(\Psi))\|\Psi\|^2}{T}.
\]

Having control on the measures of these sets has immediate consequence to several shrinking target problems. Indeed, a simple adaptation of [12, Lemmas 13 and 14] gives the following result.

**Lemma 4.3.** Let $\{\Psi_t\}_{t \geq 1} \subseteq L^2(X, m)$ be a decreasing family of bounded non-negative functions.

1. If $\sum_j m(C_{t_j-1,\Psi_{t_j}}) < \infty$ for some subsequence $t_j \to \infty$, then for $m$-a.e. $x \in X$,
\[
\lambda_{t_j} \Psi_{t_j}(x) \neq 0 \quad \text{for all } t \gg x 1.
\]
2. If there exists $C > 1$ such that $m(\Psi_{2j}) \leq C \cdot m(\Psi_{2j+1})$ for all $j \gg 1$ and $\sum_j m(C_{2j-1,\Psi_{2j}}) < \infty$, then for $m$-a.e. $x \in X$,
\[
\lambda_{t} (\Psi_{t})(x) \geq \frac{m(\Psi_{t})}{4C} \quad \text{for all } t \gg x 1.
\]
(3) If there exists $C > 1$ such that $m(\Psi_{2j}) \leq C \cdot m(\Psi_{2j+1})$ for all $j \gg 1$ and $\sum_j m(C_{2j+1}, \Psi_{2j}) < \infty$, then for $m$-a.e. $x \in X$,

$$
\lambda_t(\Psi_t)(x) \leq (4C) \cdot m(\Psi_t) \quad \text{for all } t \gg 1.
$$

\vspace{1em}

4.2. Hitting along a subsequence. In the rest of this section, let $B = \{B_t\}$ be a family of shrinking targets in $X$. Recall that a family $B$ is inner regular (resp. outer regular) if there exist $c > 0, \alpha > 0$ and smooth positive functions $0 \leq \Psi_t^- \leq \operatorname{Id}_{B_t}$ (resp. $\operatorname{Id}_{B_t} \leq \Psi_t^+ \leq c$) such that

- $m(B_t) \leq c \cdot m(\Psi_t^-)$ (resp. $m(\Psi_t^+) \leq c \cdot m(B_t)$);
- $S(\Psi_t^+) \leq c \cdot m(B_t)^{-\alpha}$.

A family $B$ is regular if it is inner and outer regular. When we want to emphasize the parameters $c$ and $\alpha$ we say a family is $(c, \alpha)$ regular.

Our first application of the effective mean ergodic theorem is the following

**Proposition 4.4.** Assume that $B$ is inner regular and satisfies that

$$
\liminf_{t \to \infty} \frac{\log(m(B_t))}{tm(B_t)} = 0.
$$

Then then there is a subsequence $t_j$ such that for $m$-a.e. $x \in X$

$$
\lambda(\{t \leq t_j : xa_t \in B_{t_j}\}) \gg t_j m(B_{t_j}).
$$

If $B$ is also outer regular, then for $m$-a.e. $x \in X$

$$
\lambda(\{t \leq t_j : xa_t \in B_{t_j}\}) \asymp t_j m(B_{t_j}).
$$

where $\lambda$ is either Lebesgue measure on $\mathbb{R}$ or the counting measure on $\mathbb{Z}$.

**Proof.** Since $B$ is inner regular, there are $\Psi_t \in C^\infty(X)$ with $0 \leq \Psi_t \leq \operatorname{Id}_{B_t}$ such that $\log(S^*(\Psi_t)) \ll \log(\mu(B_t))$ and $m(\Psi_t) \gg m(B_t)$. The mean ergodic theorem 4.1 applied to $f_t$ implies that

$$
\|\lambda_t(\Psi_t) - m(\Psi_t)\|^2 \ll \frac{(1 + \log(S^*(\Psi_t)))\|\Psi_t\|^2}{t}.
$$

Let $\Psi_t = \frac{\Psi_t}{m(\Psi_t)}$ to get that

$$
\|\lambda_t(\Psi_t) - 1\|^2 \ll \frac{(1 + \log(S^*(\Psi_t)))\|\Psi_t\|^2}{m(\Psi_t)^2 \cdot t} \ll \frac{|\log(m(B_t))|}{m(B_t) \cdot t},
$$

where we used that $\|\Psi_t\|^2 \leq m(B_t)$. From our assumption, there is some subsequence for which $\frac{\log(m(B_{t_j}))}{m(B_{t_j})} \cdot t_j \to 0$, so $\lambda_{t_j}(\Psi_{t_j}) \to 1$ in $L^2(\Gamma \setminus G, m)$ and, after perhaps passing to another subsequence $\lambda_{t_j}(\Psi_{t_j})(x) \to 1$ for $m$-a.e $x \in X$. For any $x$ in this full measure set, estimating $\lambda_{t_j}(\Psi_{t_j})(x) \leq \frac{\lambda(\{t \leq t_j : xa_t \in B_{t_j}\})}{t_j m(\Psi_{t_j})}$ implies that $\lambda(\{t \leq t_j : xa_t \in B_{t_j}\}) \gg t_j m(B_{t_j})$ as claimed. Assuming $\{B_t\}$ is also outer regular, repeating the same argument for functions approximating $\operatorname{Id}_{B_t}$ from above gives the other inequality. $\square$
In particular, taking \( \lambda \) to be the Lebesgue measure gives the first part of Theorem 1.3. Moreover, by taking \( \lambda \) to be the counting measure we get the following consequence implying a discrete version of Theorem 1.2(1).

**Corollary 4.5.** If \( \{ B_t \} \) is inner regular and \( \liminf_{t \to \infty} \frac{\log(m(B_t))}{m(B_t)t} = 0 \), then \( \{ k \in \mathbb{N} : x_{a_k} \in B_k \} \) is unbounded for \( m \)-a.e. \( x \in X \).

**Proof.** Applying the above result with \( \lambda \) the counting measure shows that for \( m \)-a.e. \( x \in X \),

\[
\#\{ k \leq t_j : x_{a_k} \in B_{t_j} \} \gg t_j m(B_{t_j}) \to \infty,
\]

along some subsequence \( t_j \). Since for any \( k \leq t_j \) we have that \( B_{t_j} \subseteq B_k \) and this implies that the set \( \{ k : x_{a_k} \in B_k \} \) is unbounded as well. \( \square \)

### 4.3. Orbits eventually always hitting.

Again we let \( \mathcal{B} = \{ B_t \} \) denote a family of shrinking targets in \( X \). The results of the previous section allows us to control how orbits hit the shrinking targets along a subsequence of times, however, under the same hypothesis we could also have different subsequences for which this asymptotic fails, and for which the set \( \{ k \leq k_j : x_{a_k} \in B_{k_j} \} \) may even be empty (see e.g. [12, Proposition 12]). A more subtle question is to ask what conditions on the shrinking sets guarantees that the truncated orbits \( \{ x_{a_j} : j \leq k \} \) is eventually always hitting the targets \( B_k \), and moreover, how large is their intersection? This is the content of the following Theorem 4.6, which is a discrete version of Theorems 1.2(2), and Theorem 4.7 which implies Theorem 1.3(2).

**Theorem 4.6.** Assuming \( \mathcal{B} \) is inner regular and \( \sum_{j=1}^{\infty} \frac{|\log(m(B_{t_j}))|}{m(B_{t_j})t_j} < \infty \) for some sequence \( t_j \to \infty \), then for \( m \)-a.e. \( x \in X \) and for all \( t \gg_1 \) we have that \( \{ k \in \mathbb{N} : k \leq t, x_{a_k} \in B_t \} \neq \emptyset \).

**Proof.** From the inner regularity we can find smooth \( 0 \leq \Psi_t \leq \mathbf{1}_{B_t} \) satisfying that \( \log(S_t(\Psi_t)) \ll \log(m(B_t)) \) and \( m(B_t) \ll \Psi_t \). By Proposition 4.2 we can estimate for any \( s, t \)

\[
m(C_{s, \Psi_t}) \ll \frac{|\log(S^*(\Psi_t))||\Psi_t|^2}{sm(\Psi_t)^2} \ll \frac{|\log(m(B_t))|}{sm(B_t)}.
\]

In particular, \( m(C_{t_j-1, \Psi_{t_j}}) \ll \frac{|\log(m(B_{t_j}))|}{m(B_{t_j})t_j} \) implying that \( \sum_j m(C_{t_j-1, \Psi_{t_j}}) < \infty \) so by the first part of Lemma 4.3 we have that for \( m \)-a.e. \( x \in X \), \( \lambda_t \Psi_t(x) \neq 0 \) for all sufficiently large \( t \). Taking \( \lambda \) to be the counting measure on \( \mathbb{N} \), this implies that the sets \( \{ k \in \mathbb{N} : k \leq t, x_{a_k} \in B_t \} \) are not empty for all sufficiently large \( t \). \( \square \)

**Theorem 4.7.** Assume that \( \mathcal{B} \) is regular and that \( m(B_{2t}) \asymp m(B_t) \) for \( t \gg 1 \). Assuming the summability condition

\[
\sum_{j=1}^{\infty} \frac{|\log(m(B_{2^{j+1}}))|}{2^j m(B_{2^{j+1}})} < \infty,
\]

we have that for m-a.e. \( x \in X \) the \( \{ x_{a_k} : k \leq t \} \) is eventually always hitting the shrinking targets \( B_k \) along some subsequence \( t_j \).

**Proof.** From the summability we can find smooth \( 0 \leq \Psi_t \leq \mathbf{1}_{B_t} \) satisfying that \( \log(S_t(\Psi_t)) \ll \log(m(B_t)) \) and \( m(B_t) \ll \Psi_t \). By Proposition 4.2 we can estimate for any \( s, t \)

\[
m(C_{s, \Psi_t}) \ll \frac{|\log(S^*(\Psi_t))||\Psi_t|^2}{sm(\Psi_t)^2} \ll \frac{|\log(m(B_t))|}{sm(B_t)}.
\]

In particular, \( m(C_{t_j-1, \Psi_{t_j}}) \ll \frac{|\log(m(B_{t_j}))|}{m(B_{t_j})t_j} \) implying that \( \sum_j m(C_{t_j-1, \Psi_{t_j}}) < \infty \) so by the first part of Lemma 4.3 we have that for m-a.e. \( x \in X \), \( \lambda_t \Psi_t(x) \neq 0 \) for all sufficiently large \( t \). Taking \( \lambda \) to be the counting measure on \( \mathbb{N} \), this implies that the sets \( \{ k \in \mathbb{N} : k \leq t, x_{a_k} \in B_t \} \) are not empty for all sufficiently large \( t \). \( \square \)
we have that for \( m \)-a.e. \( x \), and for all \( t \gg x \),
\[
\#\{j \leq t : x_{a_j} \in B\} \gg \frac{|s \leq t : x_s \in B|}{t} \gg m(B_t).
\]

**Proof.** Let \( \Psi_t^{\pm} \) to approximate \( \text{Id}_{B_t} \) from above and below with \( 0 \leq \Psi_t^- \leq \Psi_t^+ \leq c \) such that \( \log(S_t(f^{\pm})) \ll \log(m(B_t)) \) and \( m(f^+) \asymp m(f^-) \asymp m(B_t) \). For each of these functions we can use Proposition 4.2 as before to estimate \( m(C_{s,f^\pm}) \ll \frac{|\log(m(B_t))|}{\log m(B_t)} \). Taking \( s = 2^{j+1} \) and \( t = 2^j \) we get that \( \sum_j m(C_{2^j+1,f^{\pm}}) < \infty \) so by the second and third part of Lemma 4.3 we get that for \( m \)-a.e. \( x \in X \) for all sufficiently large \( t \),
\[
m(B_t) \ll m(\Psi_t^-) \ll \lambda_t \Psi_t^- \ll \lambda_t (\text{Id}_{B_t}) \ll \lambda_t \Psi_t^+ \ll m(\Psi_t^+) \ll m(B_t)
\]
implies that \( \lambda_t (\text{Id}_{B_t}) \asymp m(B_t) \). Finally, taking \( \lambda \) to be the counting measure on \( \mathbb{N} \) gives the result for discrete time and taking \( \lambda \) to be Lebesgue measure gives the result for continuous time.

\( \square \)

**4.4. Logarithm law for the first hitting time.** Using similar arguments utilizing the effective mean ergodic theorem we can prove logarithm law for the first hitting time for the discrete flow. Define the discrete first hitting time function
\[
\tau^d_B(x) = \min \{k \in \mathbb{N} : x_{a_k} \in B\}
\]

**Theorem 4.8.** If \( \{B_t\} \) is inner regular, then
\[
\lim_{t \to \infty} \frac{\log(\tau^d_{B_t}(x))}{-\log(m(B_t))} = 1 \quad \text{for } m \text{-a.e. } x \in X.
\]

**Proof.** We first note that the bound
\[
\liminf_{t \to \infty} \frac{\log(\tau^d_{B_t}(x))}{-\log(m(B_t))} \geq 1,
\]
holds for \( m \)-a.e. \( x \) in general for any monotone sequence of shrinking targets in a measure preserving dynamical system (see [13, Lemma 2.2]). It is thus sufficient to show that for \( m \)-a.e. \( x \) we have
\[
\limsup_{t \to \infty} \frac{\log(\tau^d_{B_t}(x))}{-\log(m(B_t))} \leq 1.
\]

Fix a small \( \epsilon > 0 \) and let
\[
\mathcal{A}^+_t = \{x \in X : \limsup_{t \to \infty} \frac{\log(\tau^d_{B_t}(x))}{-\log m(B_t)} \geq 1 \pm 2\epsilon\}.
\]
Note that if \( x \in \mathcal{A}^+_t \) then there are arbitrarily large values of \( t \) for which
\[
\tau^d_{B_t}(x) \geq \frac{1}{m(B_t)^{1+\epsilon}},
\]
and hence \( x \in C^0_{k_t(t),\Psi_t} \) where \( \Psi_t = \text{Id}_{B_t} \) and
\[
k_t(t) = \left[\frac{1}{(m(B_t))^{1+\epsilon}}\right].
\]

Now for any \( j \in \mathbb{N} \) we choose \( y_j \in \left(\frac{1}{2^j}, \frac{1}{2^{j-1}}\right] \) such that either \( t_j = \sup\{t : m(B_t) \geq y_j\} \) satisfies \( m(B_{t_j}) = y_j \) or there is no \( t \) with \( m(B_t) \in [y_j, y_{j-1}) \).
(if the function \( t \mapsto m(B_t) \) is continuous we many simply take \( y_j = 2^{1-j} \), in general, since the function \( t \mapsto m(B_t) \) is monotone decreasing it has at most countably many points of discontinuity so we can always find such points).

We partition \([0, \infty)\) into intervals \( I_j = \{ t : m(B_t) \in [y_j, y_{j-1}) \} \) and bound 
\[
A_+^{\epsilon} \subseteq \bigcap_{k \in \mathbb{N}} \bigcup_{j > k} C_{k,t}^{0},
\]
for any \( t \) with \( m(B_t) \in [y_{j-1}, y_j) \), we have that \( k_\epsilon(t) \in [2^{(1+\epsilon)j}, 2^{(1+\epsilon)(j+1)}] \) so that \( C_{k_\epsilon(t), \Psi_t} \subseteq C_{2^{(1+\epsilon)j}, \Psi_t} \) and moreover \( B_{t_j} \subseteq B_t \) for all \( t < t_j \) so \( C_{2^{(1+\epsilon)j}, \Psi_t} \subseteq C_{2^{(1+\epsilon)j}, \Psi_{t_j}} \). We can thus further bound
\[
A_+^{\epsilon} \subseteq \bigcap_{k \in \mathbb{N}} \bigcup_{j > k} C_{2^{(1+\epsilon)j}, \Psi_{t_j}},
\]
where the notation \( \bigcup' \) means that we only take values of \( j \) for which \( I_j \) is nonempty.

From our choice of \( y_j \) and \( t_j \), we have that \( m(\Psi_{t_j}) = y_j = \left(\frac{1}{2^j}, \frac{1}{2^{j+1}}\right] \). Since the family \( \{ B_t \} \) is inner regular, we have \( 0 \leq \Psi_{t_j}^{-} \leq \Psi_{t_j} \) with \( m(\Psi_{t_j}^{-}) \asymp m(\Psi_{t_j}) \) and \( \log(S^*(\Psi_{t_j}^{-})) \ll |\log(m(\Psi_{t_j}))| \ll j \). Using Proposition 4.2 for the smooth functions as before we bound
\[
m(C_{2^{(1+\epsilon)j}, \Psi_{t_j}}) \leq m(C_{2^{(1+\epsilon)j}, \Psi_{t_j}^{-}}) \ll \frac{j}{2^{j(1+\epsilon)2^{-j}}} \ll j^{2-\epsilon j}.
\]
Hence \( m(A_+^{\epsilon}) \leq \sum_{j > k} j^{2-\epsilon j} \ll k^{2-\epsilon k} \) for all \( k \in \mathbb{N} \), so \( m(A_+^{\epsilon}) = 0 \) and
\[
\limsup_{t \to \infty} \frac{\log(\tau_{B_t}^0(x))}{-\log m(B_t)} \leq 1 + \epsilon
\]
for \( m \)-a.e. \( x \in X \). This holds for any \( \epsilon > 0 \) and taking a sequence of \( \epsilon_j \to 0 \) concludes the proof. \( \square \)

4.5. Thickening along the flow. We note that if \( \{ k \in \mathbb{N} : x a_k \in B_k \} \) is unbounded (resp. \( \{ j \leq k : x a_j \in B_k \} \) is not empty) for the discrete time flow, then \( \{ t \in \mathbb{R} : x a_t \in B_t \} \) is unbounded (resp. \( \{ t \leq k : x a_t \in B_k \} \) is non-empty) also for the continuous flow. Hence the same assumptions on the shrinking rate of \( m(B_t) \) as in Proposition 4.4 give the same conclusions also for the continuous flow. However, it is possible that for the continuous flow \( \{ t : x a_t \in B_t \} \) to be unbounded even when it is bounded for the discrete time flow. In order to get the correct thresholds for the continuous flow one needs to consider the thickened targets.

For any set \( B \subseteq X \) we consider its thickening \( \tilde{B} \) to be
\[
\tilde{B} = BA_{1/2} = \bigcup_{|s| < 1/2} Ba_s. \tag{4.3}
\]

In the following lemma we observe that the shrinking target problems for the continuous flow can be translated to similar problems for the discrete flow hitting the thickened targets.
Lemma 4.9. For $B \subseteq \Gamma \setminus G$ and $\hat{B}$ its thickening, the following holds for any $x \in X$:

1. If $xa_t \in B$ for some $t \in \mathbb{R}$, then $xa_k \in \hat{B}$ for $k \in \mathbb{Z}$ with $|x-k| \leq 1/2$.
2. If $xa_k \in \hat{B}$ with $k \in \mathbb{Z}$, then $xa_t \in B$ for some $t \in \mathbb{R}$ with $|t-k| \leq 1/2$.
3. $|\tau_B(x) - \tau_{\hat{B}}^d(x)| \leq 1/2$.

The proof of these observations is obvious once stated and we omit the details. Using this, we get the following sharper results for continuous time flow, which imply Theorems 1.1 and 1.2.

Theorem 4.10. Let $\{B_t\}_{t \geq 1}$ denote a family of shrinking targets and assume that the family of thickened targets $\{\hat{B}_t\}_{t \geq 1}$ is inner regular.

1. If $\lim \inf_{k \to \infty} \frac{|\log(m(B_k))|}{m(B_k)k} = 0$ then for $m$-a.e. $x \in X$, the set $\{t \in \mathbb{R} : xa_t \in B_t\}$ is unbounded.
2. If $\sum_{j=1}^{\infty} \frac{|\log(m(B_{j}))|}{\log(m(B_{j}))} < \infty$, then for $m$-a.e. $x$,
   \[
   \{0 < s < t : xa_s \in B_t \neq \emptyset \text{ for all } t \gg x 1. \}
   \]
3. For $m$-a.e. $x \in X$,
   \[
   \lim_{t \to \infty} \frac{\log \tau_{B_t}(x)}{\log m(B_t)} = 1.
   \]

Proof. The first condition (with $k$ replaced by $k + 1$) implies that the set $\{k \in \mathbb{N} : xa_k \in \hat{B}_{k+1}\}$ is unbounded. For each $k$ in this set there is some $t_k \in [k - 1/2, k + 1/2]$ with $xa_{t_k} \in \hat{B}_{k+1} \subseteq B_{t_k}$ proving the first part.

For the second part, the summability condition implies that for $m$-a.e. $x$, we have that $\{xa_j : j \leq k\} \cap \hat{B}_k \neq \emptyset$ for all sufficiently large $k > k_0$. Now for $t \geq k_0 + 1$ let $k = \lfloor t \rfloor$ then there is some $j \leq k$ with $xa_j \in \hat{B}_k$ and hence there is $s \leq t$ with $xa_s \in B_k \subseteq B_t$.

Finally for the last part since $|\tau_B(x) - \tau_{\hat{B}}^d(x)| \leq 1/2$, we get that
\[
\lim_{t \to \infty} \frac{\log \tau_{B_t}(x)}{\log m(B_t)} = \lim_{t \to \infty} \frac{\log \tau_{\hat{B}_t}^d(x)}{\log m(B_t)}.
\]

Remark 4.4. The problem of estimating Leb$\{t \leq k : xa_t \in B_k\}$, for the continuous time flow, does not easily reduce to the discrete time problem for the thickened target. Here, knowing that $xa_k \in \hat{B}_k$ only tells us that $xa_t \in B_k$ for some $t$ close to $k$ but not on the amount of time spent there. Hence, to get asymptotics we need the stronger condition that $\sum_{j=1}^{\infty} \frac{|\log(m(B_{j}))|}{\log(m(B_{j}))} < \infty$ for the original sets and not the thickened sets. In particular, if say $m(B_k) \propto k^{-a}$ for some $a \geq 1$ and $m(\hat{B}_k) \propto k^{-b}$ for some $b < 1$ then by reducing to the thickened case we know that for all sufficiently large $k$ the sets $\{t \leq k : xa_t \in B_k\}$ are not empty, but we do not get an asymptotic estimate for the size of these sets.
5. Explicit examples

In this section, we consider the explicit examples of shrinking targets given by shrinking cusp neighborhoods, shrinking metric balls and shrinking tubular neighborhoods, and show that they are regular and approximate their measure. We note that the results on the regularity and measure estimates for these sets obtained below are valid for any geometrically finite non-elementary subgroup $\Gamma$ of $G$ with no hypothesis on $\delta$.

5.1. Cusp neighborhoods. Let $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$ and $\mathfrak{h}_{i,t}$ be the cusp neighborhoods defined in (2.5). In order to apply our results for these sets we need to verify that the family $\{\mathfrak{h}_{i,t}\}_{t \geq 1}$ is regular and satisfies $m(\mathfrak{h}_{i,t}) \asymp e^{-t(2\delta - \kappa_i)}$ where $\kappa_i$ is the rank of the parabolic fixed points associated to $\mathfrak{h}_i$. While the upper bound is proved in [3] and [20], we could not find a reference where the lower bound is established, so we include a proof for the readers convenience.

The important feature of a geometrically finite group is that all of its parabolic fixed points are bounded, i.e., the stabilizer of $\xi$ in $\Gamma$ acts cocompactly on $\Lambda - \{\xi\}$ for each parabolic fixed point $\xi$. This is the main ingredient of the argument below. We refer to [1] for the description of horoballs in geometrically finite manifolds that will be used below.

We will work here with the upper half space model

$$\mathbb{H}^n = \{z = (x,y) : x \in \mathbb{R}^{n-1}, y > 0\},$$

for hyperbolic space with the metric $ds^2 = \sum_i dx_i^2 + dy_i^2$ and hyperbolic measure $dz = dx dy$, and we fix our base point to be $o = (0,1)$. Since we will work with one fixed cusp we may assume without loss of generality that it is the cusp at infinity. Set $\Gamma_\infty$ to be the stabilizer of $\infty$ in $\Gamma$. Let $\kappa$ be the rank of $\infty$. Then $\Gamma_\infty$ contains a subgroup $\mathbb{Z}^\kappa$ of finite index, which acts cocompactly as translations on an affine subspace $L$ of $\mathbb{R}^{n-1}$, unique to parallel translation. Without loss of generality, we assume $\Gamma_\infty = \mathbb{Z}^\kappa$.

Fix a horoball $\tilde{H}(0) \subset \mathbb{H}^n$ at $\infty$ such that $\Gamma_\infty(\tilde{H}(0)) = \tilde{H}(0)$ and if $\gamma \in \Gamma$ is such that $\tilde{H}(0) \cap \gamma \tilde{H}(0) \neq \emptyset$, then $\gamma \in \Gamma_\infty$. In fact, in the upper half space model, $\tilde{H}(0)$ is of the form $\{(x,y) : y = y_0\}$. For the notational simplicity, we assume $y_0 = 1$. Set $\tilde{H}(t) = \{z \in \tilde{H}(0) : d(z, \partial \tilde{H}(0)) \geq t\} = \{(x,y) : y \geq e^t\}$. Without loss of generality, we may assume $\pi(\mathfrak{h}_t) = \Gamma_\infty \setminus \tilde{H}(t)$ where $\mathfrak{h}_{\infty,t} = \mathfrak{h}_t$.

Choose a fundamental domain $F_\infty \subseteq \mathbb{R}^{n-1}$ for the action of $\Gamma_\infty$ on $\mathbb{R}^{n-1}$ containing the origin so that $\text{int}(\gamma F_\infty)$s are mutually disjoint for $\gamma \in \Gamma_\infty$. Note that $H'(t) = \{z = (x,y) : x \in F_\infty : y \geq e^t\}$ is a fundamental domain for $\pi(\mathfrak{h}_t) = \Gamma_\infty \setminus \tilde{H}(t)$. We can choose a compact fundamental parallelepiped $P$ containing $F_\infty \cap \Lambda$ such that $\Gamma_\infty P$ covers $\Lambda \setminus \{\infty\}$ and $\text{int}(\gamma P)$s are mutually disjoint for all $\gamma \in \Gamma_\infty$. We may choose $P$ to contain the origin so that if $H(t) := H'(t) \cap \text{hull}(\Lambda)$, then

$$\left(F_\infty \cap \Lambda\right) \times [e^t, \infty) \subset H(t) \subset P \times [e^t, \infty).$$

(5.1)
As \( \mathcal{P} \) is compact, we have for any \( z \in H(t) \), we have \( d(\Gamma o, z) = d(o, z) \), and for \( z \in \partial H(t) \),

\[
d(\Gamma o, z) = d(o, z) = t + O(e^{-t}).
\]

The following is also clear from (5.1):

**Proposition 5.1.** The injectivity radius \( r_z \) at any point \( z \in \partial H(t) \) satisfies \( r_z \asymp e^{-t} \), where the implied constants are uniform for all \( t \gg 1 \).

We will use the following observation:

**Proposition 5.2.** There is \( c > 0 \) such that for all \( t \geq 0 \),

\[
H(t + c) \subset \{ z \in \text{hull}(\Lambda) \cap H(0) : d(z, \Gamma o) \geq t \} \subset H(t - c).
\]

**Proof.** For \( t \geq t_0 \) we have that \( d(z, \Gamma o) = d(z, o) \) and that \( d(\Gamma o, \Gamma o) \leq t \) for all \( z \in H(t) \). Now, for \( zd H(t + c) \) we have that \( d(z, \Gamma o) \geq t \) and by definition \( z \in \text{hull}(\Lambda) \). On the other hand if \( z \in H(0) \) with \( d(z, \Gamma o) \geq t \) then \( z \in H(t - c) \) and the result follows. \( \square \)

Next we want to estimate the measure \( m(\mathcal{P}) \) for large \( t \). For any \( \xi^-, \xi^+ \in \partial \mathbb{H}^n \) with \( \xi^- \neq \xi^+ \) and \( s \in \mathbb{R} \) we denote by \( \xi_s \) the unit speed geodesic connecting \( \xi^- \) to \( \xi^+ \) (where \( s \) is the signed distance from the highest point of the geodesic), and recall that this gives us the coordinates \( (\xi^-, \xi^+, s) \) parametrizing \( T^1(M) \). Let \( \Lambda' = \Lambda \setminus \infty \) and let \( \mathcal{P}_0 = \mathcal{F}_\infty \cap \Lambda \).

We first show the following:

**Lemma 5.3.**

\[
m(\mathcal{P}) = \sum_{\gamma \in \Gamma} \int_{\mathcal{P}_0} \int_{\gamma \mathcal{P}_0} \int_{\mathbb{R}} \text{Id}_{H(t)}(\pi(\xi_s))dm(\xi^-, \xi^+, s).
\]

**Proof.** Let \( \mathcal{F}_T \subset \mathbb{H}^n \) be a fundamental domain for \( \Gamma \setminus \mathbb{H}^n \) containing \( o \) such that for \( t \geq 0 \) sufficiently large we have that \( \mathcal{F}_T \cap \tilde{H}(t) = H'(t) \), so that \( m(\mathcal{P}) = \int_{T^1(M)} \text{Id}_{H'(t)}dm \). Since the set \( \{ (\xi^-, \xi^+, s) : \xi^\pm = \infty \} \) has m-measure zero we can rewrite this in the \( (\xi^-, \xi^+, s) \) coordinates as

\[
\int_{T^1(M)} \text{Id}_{H'(t)}dm = \int_{\Lambda'} \int_{\mathcal{P}_0} \int_{\mathbb{R}} \text{Id}_{H'(t)}(\pi(\xi_s))dm(\xi^-, \xi^+, s).
\]

Now decomposing \( \Lambda' \) as a union over translates \( \gamma \mathcal{P}_0 \) with \( \gamma \in \Gamma \), we can rewrite

\[
m(\mathcal{P}) = \sum_{\gamma, \gamma' \in \Gamma} \int_{\gamma \mathcal{P}_0} \int_{\gamma' \mathcal{P}_0} \int_{\mathbb{R}} \text{Id}_{H'(t)}(\pi(\xi_s))dm(\xi^-, \xi^+, s)
\]

\[
= \sum_{\gamma, \gamma' \in \Gamma} \int_{\mathcal{P}_0} \int_{\gamma' \mathcal{P}_0} \int_{\mathbb{R}} \text{Id}_{\gamma^{-1}H'(t)}(\pi(\xi_s))dm(\xi^-, \xi^+, s)
\]

\[
= \sum_{\gamma \in \Gamma} \int_{\mathcal{P}_0} \int_{\mathcal{P}_0} \int_{\mathbb{R}} \text{Id}_{H'(t)}(\pi(\xi_s))dm(\xi^-, \xi^+, s)
\]
where for the second line we made a change of variables \( \xi \mapsto \gamma \xi \) and in the last line we used that \( \tilde{H}(t) = \bigcup_{\gamma \in \Gamma_\infty} \gamma H'(t) \).

In order to evaluate this we need the following geometric estimate.

**Lemma 5.4.** Let \( \xi^- \in \mathcal{P}_0 \) and \( \xi^+ \in \gamma \mathcal{P}_0 \) with \( \gamma \in \Gamma_\infty \). Then there is a constant \( c > 0 \) such that

\[
\int_{\mathbb{R}} \text{Id} \tilde{H}(t)(\pi(\xi_s)) ds = d(o, \gamma o) - 2t + O(1),
\]

when \( d(o, \gamma o) > 2t - c \) and it is zero otherwise.

**Proof.** Recall that \( o = (0, 1) \) and note that \( \gamma o = (v, 1) \) for some \( v \in \mathbb{R}^{n-1} \). Since \( \mathcal{P}_0 \) is a compact set containing the origin then \( \gamma \mathcal{P}_0 \) is a compact set (of the same diameter) containing \( v \) and hence \( \|\xi^- - \xi^+\| = \|v\| + O(1) \) where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^{n-1} \). Since \( \text{sup}\{t : \xi \cap H(t) \neq \emptyset\} = \log(\|\xi^- - \xi^+\|) \) and \( d(o, \gamma o) = \log(\|v\|) + O(1) \) indeed \( d(o, \gamma o) < 2t - c \) implies that \( \xi \cap H(t) = \emptyset \).

Now assume \( \xi \cap H(t) \neq \emptyset \) and let \( \zeta_1, \zeta_4 \in \mathbb{H}^n \) be the first and second intersections of the geodesic \( \xi_s \) with \( \partial H(0) \) and \( \zeta_2, \zeta_3 \) the first and second intersections with \( \partial \mathbb{H}(t) \). Writing \( \zeta_i = (x_i, y_i) \) we have that \( \|x_1\| \) and \( \|x_4 - v\| \) are uniformly bounded and that \( \|x_2\| \) and \( \|x_3 - v\| \) are bounded by \( O(e^t) \) implying that \( d(z_1, o), d(z_4, \gamma o), d(z_3, a_t o) \) and \( d(\zeta_4, a_t o) \) are all uniformly bounded. Now on one hand, \( d(z_1, \zeta_4) = d(o, \gamma o) + O(1) \), and on the other hand, since \( z_1, z_2, z_3, \zeta_4 \) all lie on the same geodesic we have \( d(z_1, z_4) = d(z_1, z_2) + d(z_2, z_3) + d(z_3, z_4) \). The middle term is precisely \( \int_{\mathbb{R}} \text{Id} \tilde{H}(t)(\pi(\xi_s)) ds \) and \( d(z_1, z_2) = d(o, a_t o) + O(1) = t + O(1) \) and similarly \( d(z_3, z_4) = t + O(1) \) concluding the proof. \( \square \)

**Proposition 5.5.** We have \( m(\mathfrak{h}_t) \asymp m(\tilde{\mathfrak{h}}_t) \asymp e^{-t(2d - \kappa)} \), where \( \tilde{\mathfrak{h}}_t = \bigcup_{|s| < 1/2} \mathfrak{h}^s \vartheta_0 \) is the thickening of \( \mathfrak{h}_t \) by the geodesic flow.

**Proof.** From Lemma 5.3 we have

\[
m(\mathfrak{h}_t) = \sum_{\gamma \in \Gamma_\infty} \int_{\mathcal{F}_\infty} \int_{\mathcal{F}_\infty} \int_{\mathbb{R}} \text{Id} \tilde{H}(t)(\pi(\xi_s)) dm(\xi^-, \xi^+, s)
\]

\[
= \sum_{\gamma \in \Gamma_\infty} \int_{\mathcal{P}_0} \int_{\mathcal{P}_0} \int_{\mathbb{R}} \text{Id} \tilde{H}(t)(\pi(\xi_s)) e^{\delta(\beta_{\xi^+} o, (s) + \beta_{\xi^-} (0, \pi(\xi_s)))} \nu_o(\xi^-) \nu_o(\xi^+) ds
\]

Next note that for any \( \xi^- \in \mathcal{P}_0 \) and \( \xi^+ \in \gamma \mathcal{P}_0 \) and \( \gamma \in \Gamma_\infty \) we have that \( \beta_{\xi^+} (0, \pi(\xi_s)) + \beta_{\xi^-} (0, \pi(\xi_s)) \) is independent of \( s \) and is uniformly bounded. Indeed, let \( s_1 \) be the least time such that \( z_1 = \pi(\xi_{s_1}) \in H(0) \) and note that \( d(z_1, o) = O(1) \) is uniformly bounded. Now, for \( z = \pi(\xi_s) \) on one hand \( \beta_{\xi^+} (z_1, z) + \beta_{\xi^-} (z_1, z) = s - s_1 + s_1 - s = 0 \), and on the other hand \( |\beta_{\xi^+} (z_1, z) - \beta_{\xi^-} (0, z)| \leq d(z_1, o) \) which is uniformly bounded.
With this observation together with Lemma 5.4 we get that
\[
m(h_t) \asymp \sum_{\gamma \in \Gamma_t} \nu_{0}(P_0)\nu_{0}(\gamma P_0) \int_{\mathbb{R}} \text{Id}_{\bar{H}(t)}(\pi(\xi_s))ds
\]
\[
\asymp \sum_{\gamma \in \Gamma_t \atop d(o,\gamma o) \geq 2t-c} \nu_{0}(P_0)\nu_{0}(\gamma P_0)(d(o,\gamma o) - 2t + O(1))
\]
Next, to estimate \( \nu_{0}(\gamma P_0) = \nu_{\gamma o}(P_0) \), we use conformity to get that
\[
\nu_{0}(\gamma P_0) = \int_{P_0} e^{-\delta \beta_{\xi}(\gamma o, o)} d\nu_{0}(\xi).
\]
To estimate \( \beta_{\xi}(\gamma o, o) \) let \( z_1, z_2 \) be the two points in \( H(0) \) connecting \( \xi \) to \( \gamma \xi \) then \( d(z_1, o) \) and \( d(z_2, \gamma o) \) are uniformly bounded and \( \beta_{\xi}(z_1, z_2) = d(z_1, z_2) = d(\gamma o, o) + O(1) \) implying that \( \beta_{\xi}(\gamma o, o) = d(\gamma o, o) + O(1) \). Plugging in this estimate gives
\[
m(h_t) \asymp \sum_{\gamma \in \Gamma_t \atop d(o,\gamma o) \geq 2t-c} e^{-\delta(d(o,\gamma o))}(d(o,\gamma o) - 2t + O(1)).
\]
We parametrize the elements in \( \Gamma_t \) as \( \{\gamma_v : v \in \mathbb{Z}^n\} \) and note that \( d(o, \gamma_v(o)) = 2 \log \|v\| + O(1) \). Hence
\[
m(h_t) \asymp \sum_{\gamma \in \Gamma_t \atop \|v\| \geq C e^t} e^{-\delta(2 \log \|v\| + O(1))}(2 \log \|v\| - 2t + O(1))
\]
\[
\asymp \sum_{\gamma \in \Gamma_t \atop \|v\| \geq C e^t} \|v\|^{-2\delta} \log(\|v\|e^{-t})
\]
\[
\asymp \int_{x \in \mathbb{R}^k, |x| \geq e^t} \|x\|^{-2\delta} \log(\|x\|e^{-t})dx \asymp e^{-t(2\delta - \kappa)}
\]
as claimed.

For the thickened target, since for any \( x \in h_t \), and \( |s| \leq 1/2 \) we have that \( xa_t \in h_{t-1} \), then \( h_t \subseteq h_{t} \subseteq h_{t-1} \) and hence \( m(h_t) \asymp e^{-t(2\delta - \kappa)} \) as well. \( \Box \)

Next we show regularity.

**Proposition 5.6.** The family \( \{h_t\}_{t \geq t_0} \) of shrinking cusp neighborhood, and their thickening \( \{\hat{h}_t\}_{t \geq t_0} \) are both regular.

**Proof.** To ease the notation, we assume \( t_0 = 0 \), as in the discussion above.
Since \( h_1 \subseteq h_t \subseteq h_{t-1} \) it is enough to show that \( \{h_t\}_{t \geq t_0} \) is regular. Let \( H'(t) \) denote the fundamental domain for \( \Gamma_t \setminus \bar{H}(t) \) defined above, and \( F_T \) a fundamental domain for \( \Gamma \setminus \mathbb{H}^n \) such that \( F_T \cap \bar{H}(t) = H'(t) \). For any \( t \geq 1 \) let \( \psi_t^\pm \) be smooth functions on \( \Gamma_t \setminus \bar{H}(t) \) taking values in \([0,1]\) satisfying that
\[
\text{Id}_{H'(t+1)} \leq \psi_t^- \leq \text{Id}_{H'(t)} \leq \psi_t^+ \leq \text{Id}_{H'(t-1)},
\]
and we can choose them so that $S(\psi^\pm) = O(1)$, independent of $t$.

Since $\mathcal{F}_t \cap \mathbb{H}(t) = H'(t)$, we can lift the functions $f^\pm_t$ to right $K$-invariant left $\Gamma$-invariant functions $\Psi^\pm_t$ on $G$ that we can think of as $K$-invariant functions on $\Gamma \backslash G/M = T^1(M)$. As such, by looking at their values on a fixed fundamental domain, we see that

$$0 \leq \text{Id}_{\mathcal{H}_{t+1}} \leq \Psi^+_t \leq \text{Id}_{\mathcal{H}_t} \leq \Psi^-_t \leq \text{Id}_{\mathcal{H}_{t-1}} \leq 1.$$ 

Since $m(\mathcal{H}_t) \times m(\mathcal{H}_{t+1})$ we also get that $m(\Psi^+_t) \times m(\mathcal{H}_t)$ implying that the family of cusp neighborhoods are $(c,0)$-regular for some $c \geq 1$. \hfill \Box

**Proof of Theorem 1.4.** Applying Theorem 1.1 to the shrinking targets $B_t = \mathcal{H}_{i,t}$ gives the first part.

For the second part, fix some $\eta < \frac{1}{2^\delta - \kappa}$ and let $c := 1 - \eta(2\delta - \kappa) > 0$. Consider the shrinking targets $B_t = \mathcal{H}_{i,t} \log(t)$, which are regular and satisfy that $m(B_{2t}) \times m(B_t) \asymp t^{-(2\delta - \kappa)\eta}$. In particular we have that

$$\sum_j \log(m(B_{2^j})) \asymp \sum_j \log(j) \asymp \infty,$$

so the second part of Theorem 1.3 implies the second part. \hfill \Box

### 5.2. Shrinking balls in $\Gamma \backslash G$. In this subsection, our goal is to show that for $x \in \text{supp}(m)$, the balls $\{xG_t\}$ forms a regular family as stated in Proposition 5.9.

For any $\xi \in \partial \mathbb{H}^n$ and $\epsilon > 0$, let $B_\xi(\epsilon)$ denote the Euclidian ball of radius $\epsilon$ around $\xi$. When $\Gamma$ is convex co-compact Sullivan’s Shadow lemma implies that $\nu_0(B_\xi(\epsilon)) \asymp \epsilon^\delta$, but when $\Gamma$ has cusps the measure does not decay as regularly and fluctuate as $\epsilon \to 0$. Nevertheless, we can use the results of Sullivan to show the following. Fix $o$ in the convex hull of $\Lambda$ and we may assume $K = \text{Stab}(o)$ and fix $\nu_o \in T_o(\mathbb{H}^n)$ so that $M = \text{Stab}(\nu_o)$.

**Lemma 5.7.** For any $\xi \in \Lambda$, the following holds for all $\epsilon > 0$ sufficiently small:

1. $\min \{\epsilon^\delta, \epsilon^{2\delta - \kappa_{\min}}\} \ll \nu_0(B_\xi(\epsilon)) \ll \max \{\epsilon^\delta, \epsilon^{2\delta - \kappa_{\max}}\}.$
2. $\nu_0(B_\xi(2\epsilon)) \ll \nu_0(B_\xi(\epsilon)).$

**Proof.** For $\xi \in \Lambda$ let $\xi_t, 0 \leq t < \infty$ denote the unit speed parametrization of the the geodesic connecting $o$ to $\xi$. Note that $\xi_t \subset \text{hull}(\Lambda)$. Let $b(\xi_t) \subset \partial(\mathbb{H}^n)$ denote the shadow at infinity of the hyperbolic hyperplane meeting $\xi_t$ orthogonally. Then $b(\xi_t) = B_\xi(\epsilon)$ for $\epsilon \asymp e^{-t}$.

As before, the set of parabolic limit points has finitely many $\Gamma$-orbits. If $\xi_1, \ldots , \xi_k$ is the set of representatives, and $H_{\xi_t} \subset \mathbb{H}^n$ is a sufficiently deep horoball based at $\xi_t$, then $H := \bigcup_{t=1}^k \Gamma(H_{\xi_t})$ forms a family of disjoint horoballs.

Now by [23, Theorem 2] the measure of $b(\xi_t)$ satisfies

$$\nu_0(b(\xi_t)) \asymp e^{-\delta t + d(\xi_t, \text{Tr}(o)(\xi_t) - \delta)}, \quad (5.2)$$
where \( \kappa(\xi_t) \) is the rank of \( \xi_t \) if \( \xi_t \in \Gamma(H_{\xi_t}) \) for some \( i \), and \( \kappa(\xi_t) = \delta \) otherwise. Now, to prove the first estimate, let \( \epsilon = e^{-t} \). First, if \( \xi_t \) is not in \( H \), the claim follows easily. Next, if \( \xi_t \in \Gamma(H_{\xi_t}) \) then \( \kappa(\xi_t) = \kappa_i \) and \( \nu_0(b(\xi_t)) \ll e^{-\delta t + d(\xi_t, \Gamma_0)(\delta - \kappa_i)} \). There are two possibilities, either \( \kappa_i < \delta \) in which case

\[
\delta t \leq \delta t + d(\xi_t, \Gamma_0)(\delta - \kappa_i) \leq t(2\delta - \kappa_{\text{min}})
\]

and \( e^{-t(2\delta - \kappa_{\text{min}})} \ll \nu_0(b(\xi_t)) \ll e^{-\delta t} \), or \( \kappa_i > \delta \) in which case

\[
-\delta t \leq -\delta t + d(\xi_t, \Gamma_0)(\kappa_i - \delta) \leq t(\kappa_{\text{max}} - 2\delta)
\]

so that \( e^{-\delta t} \ll \nu_0(b(\xi_t)) \ll e^{-(2\delta - \kappa_{\text{max}})(t)} \).

For the second part, we need to show that \( \nu_0(b(\xi_{t+1})) \asymp \nu_0(b(\xi_t)) \) (with the implied constants independent on \( t \)). Here there are two cases, either \( \xi_t, \xi_{t+1} \in \Gamma(H_{\xi_t}) \) for some \( i \), or not. In the first case, we have that \( |d(\xi_t, \Gamma_0) - d(\xi_{t+1}, \Gamma_0)| \ll 1 \). Now, using (5.2) we get that in this case

\[
\frac{\nu_0(b(\xi_{t+1}))}{\nu_0(b(\xi_t))} \asymp e^{(d(\xi_{t+1}, \Gamma_0) - d(\xi_t, \Gamma_0))(\delta - \kappa_i)} \gg 1.
\]

If not, there must be some \( t' \in [t, t + 1] \) such that the projection of \( \xi_{t'} \) in \( \text{core}(\mathcal{M}) \) lies in the compact part \( \text{core}(\mathcal{M}) - \bigcup_{\delta} H_{\xi_t} \), and hence \( d(\xi_{t'}, \Gamma_0) = O(1) \). But then also \( d(\xi_t, \Gamma_0) \) and \( d(\xi_{t+1}, \Gamma_0) \) are bounded and \( \nu_0(b(\xi_{t+1})) \asymp \nu_0(b(\xi_t)) \asymp e^{-\delta t} \) as well.

**Proposition 5.8.** Let \( \mathcal{K} \subseteq X \) be a compact subset. Let \( \delta_- = \min\{\delta, 2\delta - k_{\text{max}}\} \) and \( \delta_+ = \max\{\delta, 2\delta - k_{\text{min}}\} \). For any \( x \in \mathcal{K} \cap \Omega \) we have that for all \( \epsilon < r_x \)

1. \( e^{1+\dim M + 2\delta_+} \ll m(xG_\epsilon) \ll e^{1+\dim M + 2\delta_-}, \)
2. \( m(xG_{2\epsilon}) \asymp m(xG_\epsilon), \)
3. \( m(xG_{\epsilon A_1}) \asymp \epsilon^{-1} m(xG_\epsilon), \)
4. \( m(xG_\epsilon M) \asymp e^{-\dim(M)} m(xG_\epsilon), \)

where all the implied constants above are uniform for all \( x \in \mathcal{K} \).

**Proof.** Fix a compact subset \( \mathcal{F}_0 \subseteq G \) such that \( \mathcal{K} = \Gamma \setminus \mathcal{F}_0 \). First, since we assume \( \epsilon \leq r_x \) we have that \( m(xG_\epsilon) = m(gG_\epsilon) \) for \( x = [g] \).

We will use the flow boxes

\[
\mathcal{B}(g, \epsilon) = g\mathcal{B}(\epsilon) = g(N_\epsilon^+ N^- \cap N_\epsilon^+ N^+ \mathcal{M}) M A_\epsilon.
\]

(5.3)

It is shown in [11, Lemma 4.7] that \( \mathcal{B}(g, \epsilon) \asymp gG_\epsilon \), and that

\[
m(B(g, \epsilon)) = (1 + O(\epsilon))2\epsilon \nu_{g(o)}(gN_\epsilon^+ v_\omega^+) \nu_{g_0}(g_0N_\epsilon^- v_\omega^-) \text{vol}_M(M_\epsilon),
\]

(5.4)

where \( \text{vol}_M(M_\epsilon) \asymp \epsilon^{\dim(M)} \) and all implied constants are absolute.

We can estimate \( \nu_{g(o)}(gN_\epsilon^+ v_\omega^+) \asymp \nu_0(B_{g\pm}(\epsilon)) \), with the implied constants uniform for \( g \in \mathcal{F}_0 \), by Lemma 5.7 we have

\[
\epsilon^{\delta_-} \ll \nu_{g(o)}(gN_\epsilon^+ v_\omega^+) \ll \epsilon^{\delta_-}.
\]

Since \( \text{vol}_M(M_\epsilon) \asymp \epsilon^{\dim(M)} \) we get that

\[
\epsilon^{2\delta_- + 1 + \dim M} \ll m(g\mathcal{B}(\epsilon)) \ll \epsilon^{2\delta_- + 1 + \dim M}.
\]
proving (1). The second claim follows similarly from the second part of Lemma 5.7. The third and fourth claims follow easily from the above description of \( gB(\epsilon) \).

\[ \square \]

Proposition 5.9. Fix a compact set \( K \subseteq X \). There exist some \( c > 1 \) and \( \alpha > 1 \) (depending on \( \ell \), and \( K \)) such that the family of shrinking balls \( \{ xG_\epsilon : x \in K \cap \Omega, \epsilon < r_x \} \) and the family of their thickenings are regular for \( S_1 \).

Proof. We can find smooth functions \( \Psi^\pm_\epsilon : G \to [0,1) \) such that

\[
\Psi^-_\epsilon(g) = \begin{cases} 1 & g \in G_{\epsilon/2} \\ 0 & g \notin G_\epsilon \end{cases}, \quad \Psi^+_\epsilon(g) = \begin{cases} 1 & g \in G_\epsilon \\ 0 & g \notin G_{2\epsilon} \end{cases},
\]

satisfying \( S_1(\Psi^\pm_\epsilon) \ll \epsilon^{-\ell} \). For \( x \in K \cap \Omega \), let \( \Psi^\pm_\epsilon(xg) := \Psi^\pm_\epsilon(xg) \). Then

\[
0 \leq \Psi^-_\epsilon \leq \text{Id}_{xG_\epsilon} \leq \Psi^+_\epsilon.
\]

We then have that \( S_1(\Psi_\epsilon^x) \ll \epsilon^{-\ell} \ll m(xG_\epsilon)^{-\alpha} \),

for \( \alpha = \frac{\ell}{\dim(M)+\delta} \) and that \( m(xG_{\epsilon/2}) \leq m(\Psi^-_\epsilon) \leq m(xG_\epsilon) \) so that \( m(xG_\epsilon) \ll m(xG_{\epsilon/2}) \leq m(\Psi^-_\epsilon) \), and similarly \( m(\Psi^+_\epsilon) \ll m(xG_\epsilon) \).

The same argument shows that the thickened sets \( xG_\epsilon A_1 \) are \((c,\alpha)\)-regular for some constant \( c > 1 \) and \( \alpha = \frac{\ell}{\dim(M)+\delta} \).

\[ \square \]

The proofs of Propositions 5.8 and 5.9 can easily be adapted for the following:

Proposition 5.10. Let \( M \) be convex cocompact. Fix \( x_0 \in \text{supp}(m) \). Then the families \( \{ x_0G_\epsilon M \} \) and \( \{ x_0G_\epsilon MA_1/2 \} \) are regular and \( m(x_0G_\epsilon M) \asymp \epsilon^{2\delta+1} \) and \( m(x_0G_\epsilon MA_1/2) \asymp \epsilon^{2\delta} \).

When \( M \) has cusps we don’t have such asymptotics for \( m(x_0G_\epsilon M) \) and \( m(x_0G_\epsilon MA_1/2) \) in general. Nevertheless, under certain condition on the center point \( x_0 \) (and the geodesic emanating from it) we get the following.

Proposition 5.11. Let \( K \subseteq X \) be a compact subset of \( X \), and let \( x_0 \in K \cap \Omega \).

1. If the two limit points in \( \partial H^n \) of the (lift of the) geodesic emanating from \( x_0 \) are both parabolic fixed points corresponding to cusps of ranks \( \kappa_1, \kappa_2 \), then

\[
m(x_0G_\epsilon M) \asymp e^{4\delta+1-\kappa_1-\kappa_2}.
\]

2. If the geodesic emanating from \( x_0 \) is contained a compact set in \( T^1(M) \) then

\[
m(x_0G_\epsilon M) \asymp e^{2\delta+1}.
\]

3. If the ratio \( \frac{d(x_0a_t, \Gamma_0)}{\log |t|} \) remains bounded as \( t \to \pm \infty \) then

\[
\lim_{\epsilon \to 0} \frac{\log(m(x_0G_\epsilon M))}{\log \epsilon} = 2\delta + 1.
\]
Proof. Let $g_0 \in G$ with $x_0 = [g_0]$. Since this point is fixed we may assume with out loss of generality that $g_0 = e$. We denote by $\xi_t = g_0a_t$ and let $\xi_{\pm} \in \partial \mathbb{H}^n$ be the limit points $\operatorname{lim}_{t \to \pm \infty} \xi_t$. Recall that by [11, Lemma 4.7] and (5.4) we have

$$m(x_0G_xM) \propto m(B(g_0, \epsilon)M) \propto \nu_0(B_{\xi_+}(\epsilon))\nu_0(B_{\xi_-}(\epsilon)).$$

It thus remains to estimate $\nu_0(B_{\xi_{\pm}}(\epsilon))$ in each of the above cases.

When $\xi_{\pm}$ are parabolic fixed points, there is some $t_0$ such that for all $t \geq t_0$ (resp, $t < -t_0$) we have that $\xi_{\pm} t \in H_{\xi_{\pm}}$ is in the Horroball centered at $\xi_{\pm}$. Since $\xi_{\pm}$ are parabolic cusp points this implies that for $t \geq t_0$, we have that $d(\xi_t, \Gamma \alpha) = |t| + O(1)$, and hence, setting $\epsilon = e^{-|t|}$ by 5.2 we can estimate $\nu_0(B_{\xi_+}(\epsilon)) \propto \epsilon^{2\delta - \kappa_1}$ and similarly $\nu_0(B_{\xi_-}(\epsilon)) \propto \epsilon^{2\delta - \kappa_2}$ concluding the proof in the first case.

Next, the condition that the geodesic emanating from $x_0$ is contained in a compact set is equivalent to the condition that $d(\xi_t, \Gamma \alpha) \leq c$ for some constant $c_0 > 0$. In this case 5.2 implies that $\nu_0(B_{\xi_{\pm}}(\epsilon)) \propto \epsilon^b$ completing the proof of the second case.

Finally, assuming that the ratio $\frac{d(x_0G_x, \Gamma \alpha)}{\log |t|}$ is bounded, again taking $\epsilon = e^{-t}$ 5.2 now implies that

$$\epsilon^b \log(\epsilon)^{-c_1} \nu_0(B_{\xi_{\pm}}(\epsilon)) \ll \epsilon^b \log(\epsilon)^{c_1}.$$  

$\log(m(x_0G_xM)) = (2\delta + 1)\log \epsilon + O(\log \log \epsilon)$ concluding the proof. \hfill \Box

Remark 5.5. We note that the first two cases are quite rare, but do happen (explicitly for cuspidal geodesic and periodic closed geodesics respectively). The third case on the other hand is quite common, as Theorem 1.4 implies that it holds for $m$-a.e. $x_0 \in T^1(M)$.

We finish this section with the proofs of Theorems 1.7 and 1.8.

Proof of Theorem 1.7. Applying Theorem 1.1 to the shrinking targets $B_t = x_0G_{t/1}M$ with thickening $\tilde{B}_t = x_0G_{t/1}MA_{1/2}$ which is inner regular with

$$\log(m(\tilde{B}_t)) = -2\delta t + O(1)$$

by Proposition 5.10.

For the second part, we consider the shrinking targets $B_t = x_0G_{t-\eta}M$. Note that for any $x \in M$ we have that $d(G^s(x), x_0) < t^{-\eta}$ exactly when $G^s(x) \in B_t$, and since $m(B_t) \propto t^{-\eta(2\delta + 1)}$ the series $\sum_j \frac{\log(m(B_{2j}))}{2m(B_{2j})}$ converges when $(2\delta + 1)\eta < 1$ so the result follows by by the second part of Theorem 1.3. \hfill \Box

Proof of Theorem 1.8. For the first part we note that by the first part of Theorem 1.4 we have that for $m$- a.e. $x_0 \in T^1(M)$,

$$\limsup_{t \to \infty} \frac{d(G^s(x_0), o)}{\log(t)} \leq \frac{1}{2\delta - \kappa_{\max}}.$$  

For any such center point, the shrinking targets $B_t = x_0G_{t/1}M$ and their thickening $\tilde{B}_t = x_0G_{1/1}MA_{1/2}$ are regular. By the third part of Proposition
5.11 we have \( \lim_{t \to \infty} \frac{-\log(m(B_t))}{\log t} = 2\delta + 1 \), and hence for the thickened targets, \( \lim_{t \to \infty} \frac{\log(m(B_t))}{-\log t} = 2\delta \). Now, using this limit together with Theorem 1.1 implies that for \( m \text{-a.e. } x \in T^1(M) \)

\[
\lim_{t \to \infty} \frac{\log(\tau_{B_t}(x))}{\log(t)} = 2\delta \quad \text{and} \quad \lim_{t \to \infty} \frac{\log(\tau_{B_{t,i}}(x))}{-\log(m(B_{t,i}))} = 2\delta.
\]

For the second part, given two cusps \( \xi_1, \xi_2 \) with ranks \( \kappa_1, \kappa_2 \) consider a geodesic connecting \( \xi_1 \) to \( \xi_2 \) and let \( x_0 \in T^1(M) \) be any point on this geodesic, and consider the shrinking targets \( B_t = x_0 G_{1/t} M \). By the first part of Proposition 5.11 we have that \( \log(m(B_t)) = -(4\delta - \kappa_1 - \kappa_2) \log(t) + O(1) \) and the result follows by Theorem 1.1.

\[\square\]

5.3. Shrinking tubular neighborhoods. For a fixed closed geodesic \( C \subset T^1(M) \), we recall that an \( \epsilon \) tubular neighborhoods of \( C \) in \( T^1(M) \cong \Gamma \backslash G/M \) is given by

\[ C_\epsilon = \{ x \in T^1(M) : d(C, x) < \epsilon \}. \]

The proof of Theorem 1.6 follows as above from the following.

**Proposition 5.12.** For any closed geodesic \( C \subset T^1(M) \), the family of shrinking tubular neighborhoods \( C_\epsilon \) with \( \epsilon < \epsilon_0 \), and their thickening \( \tilde{C}_\epsilon = \{ x_\epsilon : x \in C_\epsilon, |s| \leq 1/2 \} \) are regular and satisfy \( m(C_\epsilon) \asymp m(\tilde{C}_\epsilon) \asymp \epsilon^{2\delta} \).

**Proof.** Recall the notations \( X, X(\epsilon) \) and \( Y(\epsilon) = X - X(\epsilon) \) from section 2.3. They are all \( M \)-invariant subsets of \( \Gamma \backslash G \), and in the following proof, we will regard them as subsets in \( \Gamma \backslash G/M \). As \( C \subset \text{supp}(m) \), it is contained in \( X \) in particular. We can present \( C = [g_0]AM/M \) and an element of \( C \) is represented by \([g_0]a_1 \epsilon \) for a unique \( 0 \leq t < L \) where \( L \) is the length of \( C \).

Let \( \epsilon_0 \) be sufficiently small so that \( C \subseteq Y(\epsilon_0) \) and let \( 0 < \epsilon < \epsilon_0 \). Let \( Q_\epsilon \) denote a maximal set of points \( x_i \in C \) such that the sets \( x_i G_\epsilon M \) are pairwise disjoint. Writing \( x_i = x_0 a_i \epsilon \) the condition that \( x_i G_\epsilon M \cap x_j G_\epsilon M = \emptyset \) imply that \( |t_i - t_j| \geq \epsilon \) and the maximality condition implies that \( |t_i - t_{i+1}| \leq 3\epsilon \). Hence \( \#Q_\epsilon \asymp L \epsilon^{-1} \). Since

\[
\bigcup_{x_i \in Q_\epsilon} x_i G_\epsilon M \subseteq C_\epsilon \subseteq \bigcup_{x_i \in Q_\epsilon} x_i G_{3\epsilon} M,
\]

we can estimate

\[
\sum_{x_i \in Q_\epsilon} m(x_i G_\epsilon M) \leq m(C_\epsilon) \leq \sum_{x_i \in Q_\epsilon} m(x_i G_{3\epsilon} M).
\]

Now the second part of Proposition 5.11 implies that \( m(g_i G_\epsilon M) \asymp \epsilon^{2\delta + 1} \) where the implied constant does not depend on \( i \). Summing over all \( x_i \in Q_\epsilon \) we get that indeed \( m(C_\epsilon) \asymp \epsilon^{2\delta} \).

Next, to show regularity, for each point \( x_i \in Q_\epsilon \), let \( \Psi_{\epsilon,i}^\pm \) be smooth non-negative functions approximating \( x_i G_\epsilon M \) from below and \( x_i G_{3\epsilon} M \) from above respectively, with \( \mathcal{S}_t(\Psi_{\epsilon,i}^\pm) \ll \epsilon^{-1} \), and define \( \Psi_\epsilon^\pm = \sum_i \Psi_{\epsilon,i}^\pm \). Since
the sets \(x_iG_{\epsilon}M\) are pairwise disjoint we have that \(\Psi_{\epsilon}^- \leq \text{Id}_{C_{\epsilon}} \leq \Psi_{\epsilon}^+\) and moreover

\[
m(\Psi_{\epsilon}^+) \leq \sum_{Q_{\epsilon}} m(x_iG_{3\epsilon}M) \ll \sum_{Q_{\epsilon}} m(x_iG_{\epsilon}M) \leq m(C_{\epsilon}),
\]

and similarly that \(m(C_{\epsilon}) \ll m(\Psi_{\epsilon}^-)\). Since \(\#Q_{\epsilon} \ll \epsilon^{-1}\) we can bound

\[
S_l(\Psi_{\epsilon}^+) \ll \epsilon^{-(l+1)} \ll m(C_{\epsilon})^{-\alpha} \quad \text{with} \quad \alpha = \frac{l+1}{2}\delta,
\]

showing that the shrinking tubular neighborhoods are all \((c,\alpha)\) regular for some \(c > 1\), and \(\alpha = \frac{l+1}{2}\delta\).

Finally, for the thickening, note that there is some \(c \geq 1\) such that

\[
a_{-s}G_{\epsilon}a_s \subseteq G_{c\epsilon}\quad \text{for all} \quad |s| \leq 1/2.
\]

Then any point in \(x \in \tilde{C}_{\epsilon}\) is of the form \(x = x_0a_{t}ga_{s}M\) with \(0 \leq t \leq L, g \in G_{\epsilon}\) and \(|s| \leq 1/2\). We can write

\[
a_{s}a_{-s}ga_{s} \in G_{c\epsilon},
\]

to get that \(x \in x_0a_{t+s}G_{c\epsilon} \subseteq \tilde{C}_{c\epsilon}\). We thus get that \(\tilde{C}_{\epsilon} \subseteq \tilde{C} \subseteq C_{c\epsilon}\) implying that \(\tilde{C}_{\epsilon}\) is also regular with \(m(\tilde{C}_{\epsilon}) \asymp m(C_{\epsilon})\).

\[\square\]

References


[15] D. Kleinbock and X. Zhao \textit{An application of lattice points counting to shrinking target problems} Discrete & Continuous Dynamical Systems - A, 38, 38: 155–168, 2018

[16] A. Kontorovich and H. Oh \textit{Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds} JAMS, 24:603-648, 2011


