COUNTING VISIBLE CIRCLES ON THE SPHERE AND KLEINIAN GROUPS

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Abstract. For a circle packing $\mathcal{P}$ on the sphere invariant under a nonelementary Kleinian group satisfying certain finiteness conditions, we describe the asymptotic distribution of circles in $\mathcal{P}$ of spherical curvature at most T as T tends to infinity.

1. Introduction

In the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ with the Riemannian metric induced from $\mathbb{R}^3$, the distance (or *the spherical distance*) between two points is simply the angle between the rays connecting them to the origin o.

Let $\mathcal{P}$ be a collection of circles on the sphere $\mathbb{S}^2$, also called a *circle packing* on $\mathbb{S}^2$. The *visual size* of a circle C in $\mathbb{S}^2$ can be measured by its spherical radius $0 < \theta(C) \le \pi/2$, that is, the half of the visual angle of C from the origin $o = (0, 0, 0)$. We label the circles by their spherical curvatures given by

$$\mathrm{Curv}_{\mathbb{S}^2}(C) := \cot\theta(C).$$

We suppose that $\mathcal{P}$ is *locally finite* in the sense that for any $T > 0$,

$$\#\{C \in \mathcal{P} : \mathrm{Curv}_{\mathbb{S}^2}(C) < T\} < \infty.$$

In the beautiful book *Indra's pearls*, Mumford, Series and Wright ask the following question: ([13, §5.4, pg.155])

How many visible circles are there?

To address this question, for any subset $E \subset \mathbb{S}^2$ and $T > 0$, we define

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset,\ \ \mathrm{Curv}_{\mathbb{S}^2}(C) < T\}.$$

The main goal of this article is to obtain an asymptotic formula for $N_T(\mathcal{P}, E)$ as $T \to \infty$ when $\mathcal{P}$ is invariant under a Kleinian group satisfying certain finiteness assumptions. Our formula involves notions from hyperbolic geometry. Consider the Poincare ball model $\mathbb{B} = \{x_1^2 + x_2^2 + x_3^2 < 1\}$ of the hyperbolic 3-space with the metric given by $\frac{2\cdot\sqrt{dx_1^2+dx_2^2+dx_3^2}}{1-(x_1^2+x_2^2+x_3^2)}$. The geometric boundary of $\mathbb{B}$ naturally identifies with $\mathbb{S}^2$.

In this article, let G denote the group of orientation preserving isometries of $\mathbb{B}$ and $\Gamma < G$ a non-elementary (=non virtually-abelian) Kleinian group. We denote by $\Lambda(\Gamma) \subset \mathbb{S}^2$ the limit set of Γ, and by $\delta = \delta_\Gamma$ the critical exponent of Γ. Let $\{\nu_x : x \in \mathbb{B}\}$ be a Patterson-Sullivan density, i.e., a Γ-invariant conformal density

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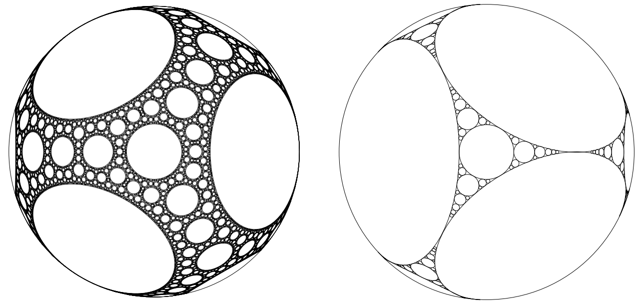

FIGURE 1. Sierpinski curve and Apollonian gasket (by C. McMullen)

of dimension δ on $\Lambda(\Gamma)$. We denote by $m^{\rm BMS}$ the Bowen-Margulis-Sullivan measure on the unit tangent bundle $\mathrm{T}^1(\Gamma\backslash\mathbb{B})$ associated to $\{\nu_x\}$, see §2.2.

For a vector $u\in \mathrm{T}^1(\mathbb{B})$, denote by $u^+\in\mathbb{S}^2$ the forward end point of the geodesic determined by u, and by $\pi(u)\in\mathbb{B}$ the base point of u. For $x_1,x_2\in\mathbb{B}$ and $\xi\in\mathbb{S}^2$, $\beta_\xi(x_1,x_2)$ denotes the signed distance between horospheres based at ξ and passing through x_1 and x_2.

Definition 1.1 (Skinning size of $\mathcal{P}$)**.** For a circle packing $\mathcal{P}$ on $\mathbb{S}^2$ invariant under Γ, we define $0\le \mathrm{sk}(\mathcal{P})\le\infty$ by

$$\mathrm{sk}(\mathcal{P}):=\sum_{i\in I}\int_{s\in \mathrm{Stab}_\Gamma(C_i^\dagger)\backslash C_i^\dagger} e^{\delta\beta_{s^+}(x,\pi(s))}d\nu_x(s^+)$$

where $x\in\mathbb{B}$, $\{C_i : i\in I\}$ is a set of representatives of Γ-orbits in $\mathcal{P}$ and $C_i^\dagger\subset \mathrm{T}^1(\mathbb{B})$ is the set of unit normal vectors to the convex hull of C_i.

By the conformal property of $\{\nu_x\}$, the definition of $\mathrm{sk}(\mathcal{P})$ is independent of the choice of $x\in\mathbb{B}$ and the choice of representatives $\{C_i\}$.

Theorem 1.2. *Let $\mathcal{P}$ be a locally finite Γ-invariant circle packing on the sphere $\mathbb{S}^2$ with finitely many Γ-orbits. Suppose that $|m^{\rm BMS}|<\infty$ and $\mathrm{sk}(\mathcal{P})<\infty$. Then for any Borel subset $E\subset\mathbb{S}^2$ with $\nu_o(\partial E)=0$,*

$$\lim_{T\to\infty}\frac{N_T(\mathcal{P},E)}{T^\delta}=\frac{2^\delta\cdot \mathrm{sk}(\mathcal{P})}{\delta\cdot|m^{\rm BMS}|}\cdot\nu_o(E).$$

where $o=(0,0,0)$. If $\mathcal{P}$ is infinite, $\mathrm{sk}(\mathcal{P})>0$.

Remark 1.3. (1) If Γ is *geometrically finite*, that is, if Γ admits a finite sided fundamental domain in $\mathbb{B}$, then $|m^{\rm BMS}|<\infty$ [21].

By [16, Lemma 1.13 and Theorem 1.14(2)], if $\delta>1$ then $\mathrm{sk}(\mathcal{P})<\infty$; and $\delta\le 1$, then we have $\mathrm{sk}(\mathcal{P})<\infty$ if and only if $\mathcal{P}$ does not contains an *infinite bouquet of tangent circles glued at a parabolic fixed point* of Γ (see Fig. 2, or [15, Definition 1.3]). We note that by [15, Proposition 3.4] the nonexistence of such infinite bouquets corresponds to Γ-parabolic-coranks of $\hat{C}$ for each $C\in\mathcal{P}$ being equal to zero, and in this case [16, Theorem 1.14(2)] implies that $\mathrm{sk}(\mathcal{P})<\infty$.

(2) Under the assumption of $|m^{\rm BMS}|<\infty$, ν_o is atom-free by [18, Sec.1.5], and hence the above theorem works for any Borel subset E whose boundary intersects $\Lambda(\Gamma)$ in countably many points. If Γ is Zariski dense in G, then any proper real

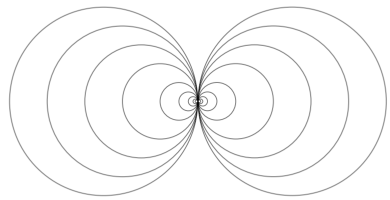

FIGURE 2. Infinite bouquet of tangent circles

subvariety of $\mathbb{S}^2$ has zero ν_o-measure [6, Cor. 1.4] and hence Theorem 1.2 applies to any Borel subset E of $\mathbb{S}^2$ whose boundary is contained in a countable union of real algebraic curves.

We combine the above remarks in the following:

Corollary 1.4. *Let Γ be a geometrically finite Zariski dense discrete subgroup of G. Let $\mathcal{P}$ be a locally finite Γ-invariant circle packing on $\mathbb{S}^2$ which is a union of finitely many Γ-orbits. Let E be a Borel subset of $\mathbb{S}^2$ such that $\partial E \cap \Lambda(\Gamma)$ is contained in a union of countably many proper real algebraic sub-varieties of $\mathbb{S}^2$. Then*

$$\lim_{T\to\infty} \frac{N_T(\mathcal{P}, E)}{T^\delta} = \frac{2^\delta \operatorname{sk}(\mathcal{P})}{\delta |m^{\mathrm{BMS}}|} \cdot \nu_o(E).$$

1.1. **Examples.** (1) If X is a finite volume hyperbolic 3 manifold with totally geodesic boundary, its fundamental group $\Gamma := \pi_1(X)$ is geometrically finite and X is homeomorphic to $\Gamma\backslash(B \cup \Omega(\Gamma))$ where $\Omega(\Gamma)$ is the domain of discontinuity for Γ [8]. The universal cover $\tilde{X}$ developed in $\mathbb{B}$ has geodesic boundary components which are Euclidean hemispheres normal to $\mathbb{S}^2$. Then $\Omega(\Gamma)$ is the union of a countably many disjoint open disks corresponding to the geodesic boundary components of $\tilde{X}$. The Ahlfors finiteness theorem [1] implies that the circle packing $\mathcal{P}$ on $\mathbb{S}^2$ consisting of the geodesic boundary components of $\tilde{X}$ is locally finite and has finitely many Γ-orbits. Moreover, $\operatorname{sk}(\mathcal{P}) < \infty$ as $\mathcal{P}$ contains no infinite bouquet of tangent circles.

In the case when $\pi_1(X)$ is convex co-compact, then no disks in $\Omega(\Gamma)$ are tangent to each other and $\Lambda(\Gamma)$ is known to be homeomorphic to a Sierpinski curve [4] (see Fig. 1).

(2) Starting with four mutually tangent circles on the sphere $\mathbb{S}^2$, one can inscribe into each of the curvilinear triangle a unique circle by an old theorem of Apollonius of Perga (c. BC 200). Continuing to inscribe the circles this way, one obtains an Apollonian circle packing on $\mathbb{S}^2$ (see Fig. 1). Apollonian circle packings are examples of circle packing obtained in the way described in (1) (cf. [5] and [10].).

(3) Take $k \geq 1$ pairs of mutually disjoint closed disks $\{(D_i, D_i') : 1 \leq i \leq k\}$ in $\mathbb{S}^2$. For each $1 \leq i \leq k$, choose $\gamma_i \in G$ which maps the interior of D_i to the exterior of D_i' and vice versa. The group $\Gamma := \langle \gamma_i : 1 \leq i \leq k \rangle$ is called a Schottky group of genus k (cf. [11, Sec. 2.7]). Let $\mathcal{P} := \cup_{1\leq i\leq k} \Gamma(C_i) \cup \Gamma(C_i')$, where C_i and C_i' are the boundaries of D_i and D_i', respectively. Then $\mathcal{P}$ is locally finite, as the Γ-orbit of the disks nest down onto the limit set $\Lambda(\Gamma)$, which is totally disconnected. Such a collection $\mathcal{P}$ is called a *Schottky dance* (see Fig. 3 or [13, Fig. 4.11]).

The common exterior of hemispheres above the initial disks D_i and D_i' is a fundamental domain for Γ in $\mathbb{B}$ and hence Γ is geometrically finite. Since $\mathcal{P}$ has no infinite bouquet of tangent circles, Theorem 1.2 applies by Remark 1.3 (1).

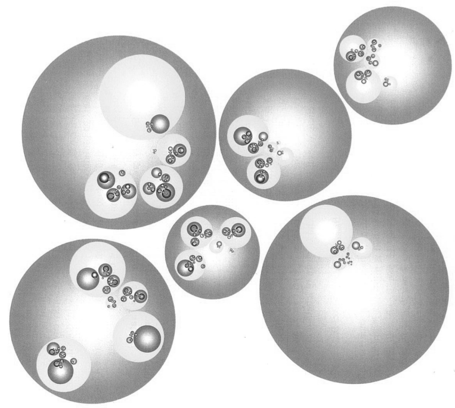

FIGURE 3. Schottky dance (from Indra's Pearls, by D.Mumford, C. Series and D. Wright, copyright Cambridge University Press 2002)

1.2. **Counting in terms of the visual distance.** Let $\hat{C} \subset \mathbb{B}$ denote the convex hull of C. Then (cf. [22, P.24]) since $o = (0,0,0)$, we have

$$\sin\theta(C) = 1/\cosh d(\hat{C}, o) \text{ and } \mathrm{Curv}_S(C) = \sinh d(\hat{C}, o). \tag{1.1}$$

Since $2\sinh(T) \sim e^T$ as $T \to \infty$, Theorem 1.2 follows from the following result for $x = o$.

Theorem 1.5. *Keeping the same assumption as in Theorem 1.2, we have, for any $x \in \mathbb{B}$,*

$$\lim_{T\to\infty} \frac{\#\{C \in \mathcal{P} : C \cap E \neq \emptyset,\ d(\hat{C}, x) < T\}}{e^{\delta \cdot T}} = \frac{\mathrm{sk}(\mathcal{P})}{\delta \cdot |m^{\mathrm{BMS}}|} \cdot \nu_x(E). \tag{1.2}$$

Using this result, in the last section we also obtain the counting estimate in terms of Euclidean curvatures of circles in $\mathcal{P}$ (see Theorem 4.1), and thus provide a shorter proof of [15, Theorem 1.4].

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2. Equistribution of normal translates of a hyperbolic surface

In this section, we set up notations as well as recall a result from [16] on limiting distribution of normal geodesic evolution of a hyperbolic surface.

2.1. **Group theoretic notations for hyperbolic space and hyperbolic surfaces.** Fix a circle $C_0 \subset \mathbb{S}^2$. Denote by $\hat{C}_0 \subset \mathbb{B}$ the convex hull of C_0, that is, $\hat{C}_0$ is the smallest convex set in $\mathbb{B}$ containing all geodesics whose end points are in C_0. Then $\hat{C}_0$ is a two dimensional hyperbolic disc isometrically imbedded in $\mathbb{B}$. Let $C_0^\dagger \subset \mathrm{T}^1(\mathbb{B})$ denote the set of unit normal vectors to $\hat{C}_0$. Let $p_0 \in \hat{C}_0$ and $X_0 \in C_0^\dagger$ based at p_0.

Let K be the stabilizer subgroup of p_0 in G and $M \subset K$ be the stabilizer of X_0 in G. Then under the maps $gK \mapsto gp_0$ and $gM \mapsto gX_0$ we identify G/K with $\mathbb{B}$ and G/M with $\mathrm{T}^1(\mathbb{B})$, respectively. Let

$$H = \{h \in G : hC_0 = C_0\} = \{h \in G : h\hat{C}_0 = \hat{C}_0\}.$$

Then $H \cdot p_0 = \hat{C}_0$ and $H \cdot X_0 = C_0^\dagger$. Here H has two connected components, one of which is the group of orientation preserving hyperbolic isometries of $\hat{C}_0$. Also $M \subset H$.

Proposition 2.1 ([15, Lemma 3.2])**.** *Let Γ be a discrete subgroup of G.*

(1) *If $\Gamma(C_0) = \{\gamma C_0 : \gamma \in \Gamma\}$ is infinite, then $[\Gamma : H \cap \Gamma] = \infty$.*
(2) *$\Gamma(C_0)$ is a locally finite packing*
$\Leftrightarrow$ *the natural projection map $(\Gamma \cap H)\backslash\hat{C}_0 \to \Gamma\backslash\mathbb{B}$ is proper*
$\Leftrightarrow$ *the natural inclusion $(\Gamma \cap H)\backslash H \to \Gamma\backslash G$ is proper.*

2.2. **Patterson-Sullivan conformal density and BMS and BR measures.** For $u \in \mathrm{T}^1(\mathbb{B})$, we define $u^+ \in \mathbb{S}^2 = \partial\mathbb{B}$ (resp. $u^- \in \mathbb{S}^2$) the forward (resp. backward) endpoint of the geodesic determined by u and $\pi(u) \in \mathbb{B}$ the basepoint.

Let Γ be a non-elementary Klenian group. Let $\{\nu_x : x \in \mathbb{B}\}$ be a Γ-invariant conformal density on $\mathbb{S}^2$ of dimension $\delta = \delta_\Gamma$; that is, each ν_x is a finite measure on $\mathbb{B}$ and

$$\gamma_*\nu_x = \nu_{\gamma x} \text{ for all } \gamma \in \Gamma \text{ and} \tag{2.1}$$

$$\frac{d\nu_y}{d\nu_x}(\xi) = e^{-\delta\beta_\xi(y,x)} \text{ for all } \xi \in \mathbb{S}^2, \tag{2.2}$$

where $\gamma_*\nu_x(R) = \nu_x(\gamma^{-1}R)$ and the *Busemann function* $\beta_\xi(y,x)$ is given by

$$\beta_\xi(y,x) = \lim_{t\to\infty} d(y,\xi_t) - d(x,\xi_t), \tag{2.3}$$

for any geodesic ray $\{\xi_t\}$ such that $\lim_{t\to\infty}\xi_t = \xi$.

Let $\{m_x : x \in \mathbb{B}\}$ be the G-invariant (Lebesgue) probability conformal density of dimension 2 on $\mathbb{S}^2$.

We define the Bowen-Margulis-Sullivan measure m^{BMS} ([2], [12], [21]) and the Burger-Roblin measure m^{BR} ([3], [18]) associated to $\{\nu_x\}$ and $\{m_x\}$ to be the measures on $\mathrm{T}^1(\Gamma\backslash\mathbb{B})$ induced by the following Γ-invariant measures on $\mathrm{T}^1(\mathbb{B})$ respectively: for $x \in \mathbb{B}$,

$$d\tilde{m}^{\mathrm{BMS}}(u) = e^{\delta\beta_{u^+}(x,\pi(u))}\; e^{\delta\beta_{u^-}(x,\pi(u))}\; d\nu_x(u^+)d\nu_x(u^-)dt;$$

$$d\tilde{m}^{\mathrm{BR}}(u) = e^{2\beta_{u^+}(x,\pi(u))}\; e^{\delta\beta_{u^-}(x,\pi(u))}\; dm_x(u^+)d\nu_x(u^-)dt.$$

By the conformal properties (2.2) of $\{\nu_x\}$ and $\{m_x\}$, these definitions are independent of the choice of $x \in \mathbb{B}$. Moreover both of these measures are invariant under the left action of Γ on $\mathrm{T}^1(\mathbb{B})$. Let m^{BMS} and m^{BR} denote the corresponding measures on $\Gamma\backslash\,\mathrm{T}^1(\mathbb{B}) = \mathrm{T}^1(\Gamma\backslash\mathbb{B})$.

2.3. **Comparison of visual density of circles corresponding to different base points.** Let $\mathcal{P}$ and Γ be as in the statement of Theorem 1.5. For any $y \in \mathbb{B}$ and any $E \subset \mathbb{S}$, set

$$\mathcal{N}_T^y(\mathcal{P}, E) = \#\{C \in \mathcal{P} : C \cap E \neq \emptyset,\ d(y,\hat{C}) < T\}. \tag{2.4}$$

Proposition 2.2. *Choose $x \in \mathbb{B}$. Suppose that there exists a constant $D \geq 0$ such that for any $F \subset \mathbb{S}^2$ with $\nu_o(\partial F) = 0$, we have*

$$\lim_{T\to\infty} \frac{\mathcal{N}_T^x(\mathcal{P}, F)}{e^{\delta T}} = D \cdot \nu_x(F). \tag{2.5}$$

Then for any $y \in \mathbb{B}$ and any $E \subset \mathbb{S}^2$ with $\nu_o(\partial E) = 0$, we get

$$\lim_{T\to\infty} \frac{\mathcal{N}_T^y(\mathcal{P}, E)}{e^{\delta T}} = D \cdot \nu_y(E).$$

Remark 2.3. (1) By the conformality property (2.2), ν_o and ν_y are absolutely continuous with respect to each other, so $\nu_o(\partial F) = 0$ if and only if $\nu_y(\partial F) = 0$.

(2) Due to Proposition 2.2, in order to prove Theorem 1.5, it is enough to derive its conclusion (1.2) for a particular choice $x = p_0$.

Proof. Let $\epsilon > 0$ be given. For any $\xi \in \mathbb{S}^2$ there exists an open disc D in $\mathbb{S}^2$ centered at ξ such that

$$|\beta_{\xi_1}(x, y) - \beta_{\xi_2}(x, y)| \leq \epsilon, \text{ for all } \xi_1, \xi_2 \in D, \tag{2.6}$$

and $\nu_o(\partial D) = 0$; such a disc D exists because we can let the radius of D tend to 0 over an uncountable set or radii, the boundaries of concentric discs of distinct radii are disjoint, and ν_o is finite. By (2.2) and (2.6), for any $\xi \in D$ we have

$$\nu_y(F)e^{-\delta\epsilon} \leq \nu_x(F)e^{\delta\beta_\xi(x,y)} \leq \nu_y(F)e^{\delta\epsilon}, \text{ for any measurable } F \subset D. \tag{2.7}$$

We cover $\mathbb{S}^2$ by finitely many such discs D_i, $1 \leq i \leq k$, and let

$$E_i = E \cap D_i \setminus (\cup_{j<i} D_j), \text{ for } 1 \leq i \leq k.$$

For subsets A and B of $\mathbb{S}^2$, $\partial(A \cap B) \subset \partial(A) \cup \partial(B)$. Therefore

$$\partial(E_i) \ \subset\ \partial(E) \cup \cup_{j=1}^k \partial(D_j).$$

Therefore $\nu_o(\partial(E_i)) = 0$, and hence $\nu_y(\partial E_i) = 0$ by Remark 2.3(1), and in particular $\nu_y(\mathrm{int}(E_i)) = \nu_y(E_i)$.

For each i, choose a closed set $F_i = (E_i)_\eta^- := \{\xi \in E_i : d(\xi, \partial(E_i)) > \eta\}$ such that $\nu_y(F_i) \geq e^{-\epsilon}\nu_y(E_i)$ and $\nu_o(\partial F_i) = 0$. Such F_i exists because $(E_i)_\eta^- \uparrow \mathrm{int}(E)$ and hence $\nu_y((E_i)_\eta^-) \to \nu_y(\mathrm{int}(E_i)) = \nu_y(E_i)$ as $\eta \to 0$, and $\partial((E_i)_\eta^-)$ are disjoint for distinct η's, so we can find arbitrarily small $\eta > 0$ such that $\nu_o(\partial((E_i)_\eta^-)) = 0$.

By (2.3), there exists $T_\epsilon > 0$ such that if $C \in \mathcal{P}$ with $d(x, \hat{C}) \geq T_\epsilon$ and $C \cap E_i \neq \emptyset$ for some i, then for any $\xi \in E_i$,

$$|d(y, \hat{C}) - (d(x, \hat{C}) + \beta_\xi(x, y))| \leq \epsilon, \tag{2.8}$$

and further if

$$C \cap F_i \neq \emptyset \Rightarrow C \cap F_j = \emptyset, \text{ for all } j \neq i, \tag{2.9}$$

because for sufficiently large $T_\epsilon > 0$, $d(x, \hat{C}) > T_\epsilon$ implies that the spherical diameter of C is less than the minimum of spherical distances between distinct nonempty F_i and F_j.

Then by (2.8) and (2.9), for any $\xi \in E_i$,

$$\mathcal{N}_T^y(\mathcal{P}, E_i) - \mathcal{N}_{T_\epsilon}^y(\mathcal{P}, E_i) \leq \mathcal{N}_{T+\beta_\xi(x,y)+\epsilon}^x(\mathcal{P}, E_i) \text{ and} \tag{2.10}$$

$$\mathcal{N}_T^y(\mathcal{P}, F_i) \geq N_{T+\beta_\xi(x,y)-\epsilon}^x(\mathcal{P}, F_i) - N_{T_\epsilon+d(x,y)+1}^x(\mathcal{P}, F_i). \tag{2.11}$$

By local finiteness of $\mathcal{P}$, $\mathcal{N}^y_{T_\epsilon}(\mathcal{P}, E_i) < \infty$. Hence by (2.10), (2.5) and (2.7),

$$\limsup_{T\to\infty} \frac{\mathcal{N}^y_T(\mathcal{P}, E_i)}{e^{\delta T}} \le D \cdot \nu_x(E_i) e^{\delta(\beta_\xi(x,y)+\epsilon)} \le D \cdot \nu_y(E_i) e^{\delta(2\epsilon)},$$

and by (2.11), (2.5) and (2.7),

$$\liminf_{T\to\infty} \frac{\mathcal{N}^y_T(\mathcal{P}, F_i)}{e^{\delta T}} \ge D \cdot \nu_x(F_i) e^{\delta(\beta_\xi(x,y)-\epsilon)} \ge D \cdot \nu_y(E_i) e^{-(1+2\delta)\epsilon}.$$

By summing over $1 \le i \le k$, we get

$$\limsup_{T\to\infty} \frac{\mathcal{N}^y_T(\mathcal{P}, E)}{e^{\delta T}} - \liminf_{T\to\infty} \frac{\mathcal{N}^y_T(\mathcal{P}, E)}{e^{\delta T}} \le D \cdot \nu_y(E)(e^{2\delta\epsilon} - e^{-(1+2\delta)\epsilon}),$$

and the conclusion follows by taking the limit as $\epsilon \to 0$. $\square$

2.4. **Equidistribution for orthogonal translates of a hyperbolic surface.** We consider the following two measures on $H/M \cong H \cdot X_0 = C_0^\dagger$: Choose any $x \in \mathbb{B}$, and define

$$d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s) = e^{2\beta_{s^+}(x,\pi(s))} dm_x(s^+) \text{ and } d\mu^{\mathrm{PS}}_{C_0^\dagger}(s) := e^{\delta\beta_{s^+}(x,\pi(s))} d\nu_x(s^+), \tag{2.12}$$

we note that the map $s \mapsto s^+$ from $C_0^\dagger \to \mathbb{S}^2 \setminus C_0$ is a differomorphism [16, Lemma 2.1]. These definitions are in fact independent of the choice of x. The measures $\mu^{\mathrm{Leb}}_{C_0^\dagger}$ and $\mu^{\mathrm{PS}}_{C_0^\dagger}$ are invariant under the action of H and $H \cap \Gamma$, respectively. We will denote the corresponding measures on the quotient space $(H\cap\Gamma)\backslash C_0^\dagger$ by μ^{Leb} and μ^{PS}, respectively.

Let $\mathcal{G}^t$ denote the geodesic flow on $\mathrm{T}^1(\mathbb{B})$. Then for any $v \in C_0^\dagger$, $t \mapsto \mathcal{G}^t(v)$ is the geodesic orthogonal to C_0 with tangent v. And the image of $\mathcal{G}^t(C_0^\dagger)$ in $\mathbb{B}$ is a union of two connected codimension one submanifolds in $\mathbb{B}$ consisting of points at distance t from C_0 on each side of C_0. In the next result we describe the limiting distribution of the geodesic evolution of $C_0^\dagger$ modulo Γ in $\Gamma\backslash \mathrm{T}^1(\mathbb{B}) = \mathrm{T}^1(\Gamma\backslash\mathbb{B})$.

Let $A = \{a_t : t \in \mathbb{R}\}$ be the one-parameter subgroup of G such that $a_t X_0 = \mathcal{G}^t(X_0)$ for all $t \in \mathbb{R}$. Then M is the centralizer of A in K. Now if we write $v \in C_0^\dagger = H\cdot X_0 \cong H/M$ as $v = sX_0 = sM = [s]$ for $s \in H$, then $\mathcal{G}^t(v) = sa_tX_0 = [sa_t]$.

Theorem 2.4 ([16, Theorem 1.2])**.** *Suppose that the natural projection map $(\Gamma \cap H)\backslash\hat{C}_0 \to \Gamma\backslash\mathbb{B}$ is proper. If $|m^{\mathrm{BMS}}| < \infty$ and $|\mu^{\mathrm{PS}}| := \mu^{\mathrm{PS}}(H\cap\Gamma\backslash C_0^\dagger) < \infty$, then for any $\psi \in C_c(\Gamma\backslash G/M)$, we have*

$$\lim_{t\to\infty} e^{(2-\delta)t} \int_{[s]\in(\Gamma\cap H)\backslash C_0^\dagger} \psi([sa_t]) d\mu^{\mathrm{Leb}}([s]) = \frac{|\mu^{\mathrm{PS}}|}{|m^{\mathrm{BMS}}|} m^{\mathrm{BR}}(\psi) \quad \text{as } t\to\infty.$$

Moreover if $[\Gamma : H\cap\Gamma] = \infty$ then $|\mu^{\mathrm{PS}}| > 0$.

Note that due to Proposition 2.1, the properness condition in Theorem 2.4 is satisfied if ΓC_0 is a locally finite circle packing.

2.5. **Haar measure on G in terms of $\mu^{\mathrm{Leb}}_{C_0^\dagger}$.** Let $A^+ = \{a_t : t \ge 0\}$. We have the following generalized Cartan decomposition ([19, Prop. 7.1.3]): $G = HA^+K$, in the sense that every element of $g \in G$ can be written as $g = ha_tk$, where $t \ge 0$, $h \in H$ and $k \in K$. Also if $ha_tk = h'a_{t'}k'$, with $t, t' > 0$, then that $t = t'$, $h = h'm_2$, and $k = m^{-1}k'$ for some $m \in M$.

Let dm correspond the Haar probability measure on M. Writing $h = sm \in C_0^\dagger \times M$, let $dh = d\mu^{\mathrm{Leb}}_{C_0^\dagger}(s)dm$ and let $dk = dm_{p_0}(k.X_0^-)dm$, then dh and dk correspond to Haar measures on H and K, respectively. Then the following defines a Haar measure on G [19, Prop. 8.1.1]: for any $\psi \in C_c(G)$,

$$\int_G \psi(g)dg = \int_{HA^+K} \psi(ha_tk)\, 4\sinh t \cdot \cosh t \; dhdtdk. \tag{2.13}$$

We denote by $d\lambda$ the unique right G-invariant measure on $H\backslash G$ which is compatible with the choice of dg and dh: for $\psi \in C_c(G)$ and $\bar\psi \in C_c(H\backslash G)$ given by $\bar\psi[g] := \int_{h\in H} \psi(hg)dh$,

$$\int_G \psi\; dg = \int_{[g]\in H\backslash G} \overline{\psi}([g])d\lambda[g].$$

Hence $d\lambda([a_tk]) = (4\sinh t \cdot \cosh t)\; dtdk$.

2.6. **An asymptotic property relating m^{BR} to ν_{p_0}.** Fixing a left-invariant metric on G, we denote by U_ϵ an ϵ-ball around e, and for $S \subset G$, we set $S_\epsilon = S \cap U_\epsilon$. For each small $\epsilon > 0$, we choose a non-negative function $\psi_\epsilon \in C_c(G)$ supported inside U_ϵ and $\int_G \psi_\epsilon dg = 1$ and define $\Psi_\epsilon \in C_c(\Gamma\backslash G)$ by

$$\Psi_\epsilon(g) = \sum_{\gamma\in\Gamma} \psi_\epsilon(\gamma g). \tag{2.14}$$

For a Borel subset $E \subset \mathbb{S}^2$, let

$$E_{X_0} := \{k \in K : kX_0^- \in E\} \subset K, \tag{2.15}$$

and define functions ψ_ϵ^E on G/M and Ψ_ϵ^E on $\Gamma\backslash G/M$ by

$$\psi_\epsilon^E(g) = \int_{k\in(E_{X_0})^{-1}} \psi_\epsilon(gk)dk \text{ and } \Psi_\epsilon^E(g) = \int_{k\in(E_{X_0})^{-1}} \Psi_\epsilon(gk)dk. \tag{2.16}$$

Proposition 2.5. *If $\nu_{p_0}(\partial E) = 0$, then*

$$\lim_{\epsilon\to 0} m^{\mathrm{BR}}(\Psi_\epsilon^E) = \nu_{p_0}(E).$$

Proof. We have $m^{\mathrm{BR}}(\Psi_\epsilon^E) = \tilde m^{\mathrm{BR}}(\psi_\epsilon^E)$. Let $\Omega = (E_{X_0})^{-1}$. Then

$$\nu_{p_0}(\partial(\Omega^{-1}X_0^-)) = \nu_{p_0}(\partial E) = 0.$$

Set $f = \chi_K$, the characteristic function of K. Then for any $g \in G$,

$$\psi_\epsilon^E(g) = \int_{k\in\Omega} f(k)\psi_\epsilon(gk)dk =: f *_\Omega \psi_\epsilon(g),$$

as per the notation of [16, eq.(7.4)]. By [16, Prop. 7.5],

$$\lim_{\epsilon\to 0} \tilde m^{\mathrm{BR}}(f *_\Omega \psi_\epsilon) = \int_{k\in\Omega^{-1}} f(k^{-1})d\nu_{p_0}(kX_0^-) = \nu_{p_0}(E),$$

here we note that the choice of the Haar measure dg considered in (2.13) is same as the one considered for [16, Prop. 7.5] due to [16, Section 8]. ∎

3. Proof of Theorem 1.5

It is enough to prove the theorem under the assumption that $\mathcal{P} = \Gamma C_0$.

Let $t_0 = d(o, p_0)$ and $\epsilon_0 > 0$ be such that $\sin(\epsilon_0) = 1/\cosh(t_0)$ (see (1.1)). Given $0 < \epsilon < \epsilon_0$, let $\mathcal{P}_\epsilon = \{C \in \mathcal{P} : \theta(C) \leq \epsilon/2\}$, and let $T_\epsilon > 0$ be such that $\sin(\epsilon/2) = 1/\cosh(T_\epsilon)$. Note that $T_\epsilon > t_0$.

Given $E \subset \mathbb{S}^2$, define

$$E_\epsilon^+ = \{x \in \mathbb{S}^2 : \mathrm{dist}(x, \overline{E}) < \epsilon\} \text{ and } E_\epsilon^- = \mathbb{S}^2 \setminus (\mathbb{S}^2 \setminus E)_\epsilon^+.$$

Then E_ϵ^+ is open and E_ϵ^- is compact, and as $\epsilon \to 0$,

$$E_\epsilon^+ \downarrow \overline{E} \text{ and } E_\epsilon^- \uparrow \mathrm{int}(E). \tag{3.1}$$

For $T > 0$, define (see (2.15))

$$B_T(E) = HA_T^+(E_{X_0})^{-1}, \text{ where } A_T^+ = \{a_t : 0 \leq t < T\}.$$

Let $c_\epsilon^+ = \#(\mathcal{P} \setminus \mathcal{P}_\epsilon) < \infty$ and $c_\epsilon^- = \#([e]\Gamma \cap [e]A^+_{t_0+T_\epsilon}K) < \infty$, where $[e]$ represents the coset of identity in $(\Gamma \cap H)\backslash G$, and note that $[e]\Gamma$ is discrete and $A^+_{t_0+T_\epsilon}K$ is relatively compact.

Lemma 3.1 (Basic counting)**.** *Given $T > T_\epsilon$,*

$$\#([e](\Gamma \cap B_T(E_\epsilon^-))) - c_\epsilon^- \leq \mathcal{N}_T^{p_0}(\mathcal{P}, E) \leq \#([e](\Gamma \cap B_T(E_\epsilon^+))) + c_\epsilon^+. \tag{3.2}$$

Proof. Let $C \in \mathcal{P}$. Assume that $\theta(C) < \epsilon$, or equivalently $d(\hat{C}, o) > T_\epsilon > t_0$. Let $\mathrm{int}(C)$ denote the smaller of the two open discs in $\mathbb{S}^2$ bounded by C. Then the following statements are equivalent:

1. $d(\hat{C}, p_0) = t$;
2. the distance between the orthogonal projection x of p_0 onto $\hat{C}$ is t;
3. there exists $\xi \in \mathrm{int}(C)$ such that the directed geodesic from p_0 to ξ intersects $\hat{C}$ perpendicularly at a distance t from p_0;
4. there exists $k \in K$ such that $\xi = kX_0^- \in \mathrm{int}(C)$ and $ka_{-t}X_0 \in C^\dagger$;
5. there exists $\gamma \in \Gamma$ such that $\gamma C_0 = C \in \mathcal{P}_\epsilon$ and there exists $k \in K$ such that $kX_0^- \in \mathrm{int}(\gamma C_0)$, and $ka_{-t}X_0 \in \gamma C_0^\dagger = \gamma H X_0$;
6. $t > T_\epsilon$, $k \in (\mathrm{int}(\gamma C_0))_{X_0}$, and $\gamma \in \Gamma \cap ka_{-t}H$.

Therefore if $C = \gamma C_0 \in \mathcal{P}_\epsilon$ for some $\gamma \in \Gamma$, $d(\hat{C}, p_0) < T$ and $C \cap E \neq \emptyset$ then $\mathrm{int}(C) \subset E_\epsilon^+$, and

$$\gamma \in \Gamma \cap (E_\epsilon^+)_{X_0}(A_T^+)^{-1}H = \Gamma \cap B_T(E_\epsilon^+)^{-1}.$$

Also $\gamma C_0 = \gamma' C_0$ for some $\gamma' \in \Gamma$ if and only if $\gamma^{-1}\gamma' \in H \cap \Gamma$. Therefore

$$\begin{aligned} &\#\{\gamma C_0 \in \mathcal{P}_\epsilon : \gamma \in \Gamma, d(\gamma\hat{C}_0, p_0) < T, \gamma C_0 \cap E \neq \emptyset\} \\ &\leq \#\{(\Gamma \cap B_T(E_\epsilon^+)^{-1})/(H \cap \Gamma)\} = \#\{(\Gamma \cap H)\backslash(\Gamma \cap B_T(E_\epsilon^+))\}. \end{aligned}$$

This gives the second inequality in (3.2).

Conversely, if $\gamma \in \Gamma \cap (E_\epsilon^-)_{X_0}\big(A_T^+ \setminus \overline{A^+_{t_0+T_\epsilon}}\big)^{-1}H$ then $\mathrm{int}(\gamma C_0) \cap E_\epsilon^- \neq \emptyset$, $d(\gamma\hat{C}_0, p_0) < T$ and $d(\gamma\hat{C}_0, o) > T_\epsilon$. Therefore $\theta(\gamma C_0) < \epsilon/2$, and hence $\gamma C_0 \subset E$. Therefore $\gamma C_0 \in \mathcal{P}$. This leads to the first inequality in (3.2). $\square$

Strong wavefront lemma. By [7, Thm. 1.6], there exists $\epsilon_1 > 0$, such that given $0 < \epsilon < \epsilon_1$, there exists $T'_\epsilon > 0$ such that

$$Ha_t kU_\epsilon \subset Ha_t A_{2\epsilon} k K_{2\epsilon}, \text{ for all } t \geq T'_\epsilon, \text{ and } k \in K. \tag{3.3}$$

There exists $0 < \alpha < 1$ (depending only on $d(o, p_0)$) such that for any $\epsilon > 0$ and $k \in K_{2\alpha\epsilon}$, we have $\mathrm{dist}(k\xi, \xi) < \epsilon$ for all $\xi \in \mathbb{S}^2$. Therefore if $T > T'_{\alpha\epsilon}$ then by (3.3)

$$B_T(E_\epsilon^+) \subset B_{T+2\epsilon}(E_{2\epsilon}^+)U_{\alpha\epsilon} \text{ and } B_{T-2\epsilon}(E_{2\epsilon}^-)U_{\alpha\epsilon} \subset B_T(E_\epsilon^-).$$

Fix $0 < \epsilon < \min(\epsilon_0, \epsilon_1)$. Define the counting functions $F_T^\pm$ on $\Gamma\backslash G$ by

$$F_T^\pm(g) := \sum_{[\gamma]\in(\Gamma\cap H)\backslash\Gamma} \chi_{B_{T\pm2\epsilon}(E_{2\epsilon}^\pm)}(\gamma g).$$

Then for any $g \in U_{\alpha\epsilon}$ and $T > T'_{\alpha\epsilon}$,

$$F_T^-(g) - d_1 \leq \#[e](\Gamma \cap B_T(E_\epsilon^-)), \quad F_T^+(g) \geq \#[e](\Gamma \cap B_T(E_\epsilon^+)) - d_1,$$

where $d_1 = \#(\Gamma \cap A^+_{T'_{\alpha\epsilon}} KU_{\epsilon_1}) < \infty$. Put $m^\pm = d_1 + c_\epsilon^\pm$. By Lemma 3.1, for all $T > \max(T_\epsilon, T'_{\alpha\epsilon})$ we have

$$F_T^-(g) - m^- \leq \mathcal{N}_T^{p_0}(\mathcal{P}, E) \leq F_T^+(g) + m^+.$$

Integrating against Ψ_ϵ (see (2.14)), we obtain

$$\langle F_T^-, \Psi_\epsilon\rangle - m^- \leq \mathcal{N}_T^{p_0}(\mathcal{P}, E) \leq \langle F_T^+, \Psi_\epsilon\rangle + m^+,$$

where the inner product is taken with respect dg.

Setting $\Xi_t = 4\sinh t \cdot \cosh t$, we have

$$\begin{aligned}
\langle F_T^\pm, \Psi_\epsilon\rangle &= \int_{g\in\Gamma\cap H\backslash G} \chi_{B_{T\pm2\epsilon}}(g)\Psi_\epsilon(g)\, dg \\
&= \int_{k\in((E_{2\epsilon}^\pm)_{X_0})^{-1}} \int_0^{T\pm2\epsilon} \int_{s\in\Gamma\cap H\backslash C_0^\dagger} \left(\int_{m\in M} \Psi_\epsilon(sa_t mk)\, dm\right) \Xi_t\, d\mu^{\mathrm{Leb}}(s)dtdk \\
&= \int_{k\in((E_{2\epsilon}^\pm)_{X_0})^{-1}} \left(\int_0^{T\pm2\epsilon} \Xi_t \int_{s\in\Gamma\cap H\backslash C_0^\dagger} \Psi_k^\epsilon(sa_t)\, d\mu^{\mathrm{Leb}}(s)dt\right)dk,
\end{aligned}$$

where $\Psi_{g_1}^\epsilon \in C_c(\Gamma\backslash G)^M$ is given by $\Psi_{g_1}^\epsilon(g) = \int_{m\in M} \Psi_\epsilon(gmg_1)\, dm$.

Hence by Theorem 2.4, and using $\Xi_t \sim e^{2t}$, and $\delta > 0$, we deduce that

$$\begin{aligned}
\lim_{T\to\infty} e^{-\delta(T\pm2\epsilon)}\langle F_{T\pm2\epsilon}^\pm, \Psi_\epsilon\rangle &= \frac{|\mu_{C_0^\dagger}^{\mathrm{PS}}|}{\delta\cdot|m^{\mathrm{BMS}}|} \int_{k\in((E_{2\epsilon}^\pm)_{X_0})^{-1}} m^{\mathrm{BR}}(\Psi_k^\epsilon)\, dk \\
&= \frac{|\mu_{C_0^\dagger}^{\mathrm{PS}}|}{\delta\cdot|m^{\mathrm{BMS}}|} m^{\mathrm{BR}}(\Psi_\epsilon^{E_{2\epsilon}^\pm}), \text{ by (2.16)}.
\end{aligned}$$

Hence

$$\frac{|\mu_{C_0^\dagger}^{\mathrm{PS}}|}{\delta\cdot|m^{\mathrm{BMS}}|} m^{\mathrm{BR}}(\Psi_\epsilon^{E_{2\epsilon}^-})e^{\delta(-2\epsilon)} \leq \liminf_{T\to\infty} \frac{N_T^{p_0}(\mathcal{P}, E)}{e^{\delta T}} \tag{3.4}$$

$$\frac{|\mu_{C_0^\dagger}^{\mathrm{PS}}|}{\delta\cdot|m^{\mathrm{BMS}}|} m^{\mathrm{BR}}(\Psi_\epsilon^{E_{2\epsilon}^+})e^{\delta(2\epsilon)} \geq \limsup_{T\to\infty} \frac{N_T^{p_0}(\mathcal{P}, E)}{e^{\delta T}}. \tag{3.5}$$

Fix any $\eta > 0$. By Proposition 2.5,

$$\limsup_{\epsilon\to 0} m^{\mathrm{BR}}(\Psi_\epsilon^{E_{2\epsilon}^+}) \le \lim_{\epsilon\to 0} m^{\mathrm{BR}}(\Psi_\epsilon^{E_\eta^+}) = \nu_{p_0}(E_\eta^+)$$

$$\liminf_{\epsilon\to 0} m^{\mathrm{BR}}(\Psi_\epsilon^{E_{2\epsilon}^-}) \ge \lim_{\epsilon\to 0} m^{\mathrm{BR}}(\Psi_\epsilon^{E_\eta^-}) = \nu_{p_0}(E_\eta^-).$$

By (3.1)

$$\lim_{\eta\to 0} \nu_{p_0}(E_\eta^+ \setminus E_\eta^-) = \nu_{p_0}(\partial E).$$

Therefore if we assume that $\nu_{p_0}(\partial E) = 0$, then

$$\lim_{T\to\infty} \frac{N_T^{p_0}(\mathcal{P}, E)}{e^{\delta T}} = \frac{|\mu^{\mathrm{PS}}|}{\delta \cdot |m^{\mathrm{BMS}}|} \nu_{p_0}(E).$$

□

4. Counting with respect to Euclidean curvature

Let $o = (0,1) \in \mathbb{R}^2 \times \mathbb{R}_{>0} \cong \mathbb{H}^2$. Let $\xi \in \mathbb{R}^2$ and consider unit speed hyperbolic geodesic $[o,\xi) := \{\xi_t : t \ge 0\}$ joining o to ξ. For any $t > 0$, let $C_\xi(t)$ be the circle in $\mathbb{R}^2$ such that the hyperbolic geodesic joining $\xi(t)$ with any point of $C_\xi(t)$ is perpendicular to the geodesic $[o,\xi)$. Then $d(\hat{C}_\xi(t), o) = t$. Let $\mathrm{Curv}(C_\xi(t))$ denote the Euclidean curvature of $C_\xi(t)$. Then

$$\lim_{t\to\infty} \mathrm{Curv}(C_\xi(t))/e^t = 1/(1+|\xi|^2),$$

and the converge is uniform for ξ in a compact set. Now using the arguments as in the Proof of Proposition 2.2, it is straightforward to deduce the following result from Theorem 1.5.

Theorem 4.1 ([15, Theorem 1.4])**.** *Let $\mathcal{P}$ be a locally finite packing of circles in $\mathbb{R}^2$ invariant under a non-elementary Klenian group Γ with finitely many Γ-orbits. Suppose that $|m^{\mathrm{BMS}}| < \infty$ and $\mathrm{sk}(\mathcal{P}) < \infty$. Then for any bounded set $E \subset \mathbb{R}^2$ with $\nu_o(\partial E) = 0$, we have*

$$\lim_{T\to\infty} \frac{\#\{C \in \mathcal{P} : \mathrm{Curv}(C) < T,\ C \cap E \ne \emptyset\}}{T^\delta} = \frac{\mathrm{sk}(\mathcal{P})}{\delta \cdot |m^{\mathrm{BMS}}|} \int_{\xi\in E} (1+|\xi|^2)^\delta \, d\nu_o(\xi).$$

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