Abstract. (1) Let $\Gamma = (\rho_1 \times \rho_2)(\Delta)$ where $\rho_1, \rho_2 : \Delta \to \text{SO}^\circ(n, 1)$ are convex cocompact representations of a finitely generated group $\Delta$. We provide a sharp pointwise bound on the growth indicator function $\psi_\Gamma$ by a tent function: for any $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\psi_\Gamma(v) \leq \min(v_1 \text{dim}_H \Lambda_{\rho_1}, v_2 \text{dim}_H \Lambda_{\rho_2}). \quad (0.1)$$

We obtain several interesting consequences including the gap and rigidity property on the critical exponent.

(2) Generalizing this, we propose a conjecture that $\psi_\Gamma$ is at most the half sum of all positive roots for any Anosov subgroup of a semisimple real algebraic group of rank at least 2. We confirm this conjecture for Zariski dense Anosov subgroups of Hitchin groups.

(3) For each $v$ in the interior of the limit cone of $\Gamma$, we prove the following on the $v$-directional conical limit set $\Lambda_v \subset S^{n-1} \times S^{n-1}$:

$$\frac{\psi_\Gamma(v)}{\max(v_1, v_2)} \leq \text{dim}_H \Lambda_v \leq \frac{\psi_\Gamma(v)}{\min(v_1, v_2)}.$$

We also study the local behavior of the higher rank Patterson-Sullivan measures on each $\Lambda_v$.

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1. Introduction

Let $X$ denote the Riemannian product $(\mathbb{H}^n \times \mathbb{H}^n, d)$ of two hyperbolic $n$-spaces $(\mathbb{H}^n, d_{\mathbb{H}^n})$ of constant curvature $-1$. The Lie group $G = \text{SO}^\circ(n, 1) \times \text{SO}^\circ(n, 1)$ is the identity component of $\text{Isom}(X)$. The Riemannian product $\mathcal{F} = \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ is equal to the Furstenberg boundary of $G$. We consider a particular class of discrete subgroups of $G$, constructed as follows. Let $\rho_1, \rho_2 : \Delta \to \text{SO}^\circ(n, 1)$ be non-elementary convex cocompact faithful representations of a finitely generated group $\Delta$. Let

$$\Gamma = \{(\rho_1(\sigma), \rho_2(\sigma)) : \sigma \in \Delta\}.$$  

Such a subgroup $\Gamma$ is an Anosov subgroup of $G$ with respect to a minimal parabolic subgroup$^1$ in the sense of Guichard and Wienhard [16]. Moreover, any such Anosov subgroup of $G$ arises precisely in this way. The quotient $X := \Gamma \backslash (\mathbb{H}^n \times \mathbb{H}^n)$ is a locally symmetric rank-two Riemmanian manifold of infinite volume, called a self-joining of a hyperbolic manifold.

A well-known theorem of Sullivan [37] says that the critical exponent $\delta_\rho$ of $\rho(\Delta)$ is equal to the Hausdorff dimension of the limit set $\Lambda_\rho \subset \mathbb{S}^{n-1}$ of $\rho(\Delta)$:

$$\delta_\rho = \dim \Lambda_\rho \quad \text{for each } i = 1, 2.$$

The main aim of this paper is to investigate a higher rank analogue of this theorem. We begin by recalling the growth indicator function of $\Gamma$, which is a higher rank version of the critical exponent. Fix $o \in \mathbb{H}^n$, and by abuse of notation, we also set $o = (o, o) \in X$. For $g = (g_1, g_2) \in G$, the Cartan projection of $g$ is a vector-valued distance function:

$$\mu(g) = (d_{\mathbb{H}^n}(g_1 o, o), d_{\mathbb{H}^n}(g_2 o, o)).$$

Following Quint [31], the growth indicator function $\psi_\Gamma : \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty\}$ is defined as follows: If $\mathcal{C}$ is an open cone in $\mathbb{R}^2$, let $\tau_\mathcal{C}$ denote the abscissa of convergence of $\sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-sd(\gamma o, o)}$. Now for any non-zero $v \in \mathbb{R}^2$, let

$$\psi_\Gamma(v) := \|v\| \inf_{v \in \mathcal{C}} \tau_\mathcal{C}$$

where the infimum is over all open cones $\mathcal{C}$ containing $v$ and $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^2$, and let $\psi_\Gamma(0) = 0$. The supremum $\sup_{\|v\|=1} \psi_\Gamma(v)$ coincides with the critical exponent $\delta$, which is the abscissa of convergence of $\sum_{\gamma \in \Gamma} e^{-sd(\gamma o, o)}$. Moreover, $\psi_\Gamma = -\infty$ outside the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$. The limit cone $\mathcal{L}$ can also be defined as

$$\mathcal{L} := \{(v_1, v_2) \in \mathbb{R}^2_{\geq 0} : d_- v_1 \leq v_2 \leq d_+ v_1\}$$

where $d_+$ and $d_-$ are respectively the maximal and minimal geodesic stretching constants of $\rho_2$ relative to $\rho_1$: $d_+(\rho_1, \rho_2) = \sup_{\sigma \in \Delta^{-}\{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)}$ and $d_-(\rho_1, \rho_2) = \inf_{\sigma \in \Delta^{-}\{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)}$.

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$^1$From now on, all Anosov subgroups mentioned in this paper are with respect to a minimal parabolic subgroup.
where $\ell_i(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\rho_i(\Delta) \setminus \mathbb{H}^n$ corresponding to $\rho_i(\sigma)$. Note that $0 < d_- \leq d_+ < \infty$ and we have $\rho_1$ and $\rho_2$ are not conjugate of each other ($\rho_1 \not\sim \rho_2$) if and only if $d_- < d_+$ [2]. Quint showed that $\psi_T$ is a concave function which is positive on $\text{int} \, L$, and $L = \{ v \in \mathbb{R}^2 : \psi_T(v) \geq 0 \}$. For $\Gamma$ Anosov, $\psi_T$ is known to be analytic and strictly concave on $\text{int} \, L$ [28, Prop. 4.6 and 4.11].

**Tent property of $\psi_T$.** The limit set of $\Gamma$ is the set of all accumulation points of an orbit $\Gamma \circ \in F$, and is of the form

$$\Lambda = \{ (\xi, \zeta(\xi)) : \xi \in \Lambda_{\rho_1} \} \subset \Lambda_{\rho_1} \times \Lambda_{\rho_2}$$

for some Hölder isomorphism $\zeta : \Lambda_{\rho_1} \to \Lambda_{\rho_2}$. For a subset $S \subset F$, we denote by $\dim S$ the Hausdorff dimension of $S$ with respect to the Riemannian metric on $F = S^{n-1} \times S^{n-1}$.

The following theorem provides a tight tent over the graph of $\psi_T$: for $\rho_1 \not\sim \rho_2$, there exist unique vectors $u_{e_1}, u_{e_2} \in \text{int} \, L$ such that $\langle u_{e_i}, e_i \rangle = 1$ and $\nabla \psi_T(u_{e_i})$ is parallel to $e_i$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ (Lemma 3.1). See Figures 1 and 4.

**Theorem 1.1 (Tent property of $\psi_T$).** Let $\rho_1 \not\sim \rho_2$. For any $v = (v_1, v_2) \in \mathbb{R}^2$,

$$\psi_T(v) \leq \min(v_1 \cdot \dim \Lambda_{\rho_1}, v_2 \cdot \dim \Lambda_{\rho_2}),$$

(1.3)

where the equality holds if and only if $v = c \cdot u_{e_i}$ for some $c > 0$ and $i = 1, 2$, in which case $\psi_T(cu_{e_i}) = c \dim \Lambda_{\rho_i}$. In particular,

$$\psi_T(v) \leq \min(v_1, v_2) \dim \Lambda.$$

Although there have been studies on the bounds of the critical exponent of $\Gamma$ ([7], [28], [29], etc), our tent theorem seems the first sharp pointwise bound on the growth indicator function in this higher rank situation. See Theorem 4.1 and Corollary 1.10 where we present an analogous result in a more general setting. We remark that for a non-lattice discrete subgroup $\Gamma$ of a semisimple Lie group $G$ with property (T), some pointwise bound for $\psi_T$ was previously known ([10], [33]). This does not apply in our situation as $G = SO^o(n, 1) \times SO^o(n, 1)$ does not have property (T).

![Figure 1. Tent on the limit cone](image-url)
Similarly, in [13], it is shown that Corollary 1.4 implies that positive square-integrable harmonic function on $\rho$. Therefore if $v = \tilde{v}_1 u_{e_1} + \tilde{v}_2 u_{e_2}$ with $\tilde{v}_1, \tilde{v}_2 \geq 0$, then

$$\psi_T(v) \geq \tilde{v}_1 \cdot \dim \Lambda_{\rho_1} + \tilde{v}_2 \cdot \dim \Lambda_{\rho_2} \quad (1.4)$$

where the equality holds if and only if $\tilde{v}_1 \cdot \tilde{v}_2 = 0$.

**Remark 1.3.**

1. If $\rho_1 \sim \rho_2$, then $\psi_T = -\infty$ outside the ray $\mathbb{R}_{\geq 0}(1, 1)$, and $\psi_T(\frac{1}{\sqrt{2}}(1, 1)) = \frac{1}{\sqrt{2}} \dim \Lambda_{\rho_1} = \frac{1}{\sqrt{2}} \dim \Lambda_{\rho_2}$; hence Theorem 1.1 also holds in this case, except for the statement concerning the equality case.

2. Theorem 1.1 does not generalize to a general Zariski dense discrete subgroup of $G$; for instance, it is false for a product of two non-elementary convex cocompact subgroups of $SO^0(n, 1)$.

We now discuss some applications of the tent theorem.

**Corollary 1.4.** Let $\pi(v_1, v_2) = (n - 1)(v_1 + v_2)$, which is the sum of all positive roots of $G$. Then for all $v \in \mathbb{R}^2$,

$$\psi_T(v) \leq \frac{\pi(v)}{2}.$$ 

Moreover, if $n \geq 3$ and $\rho_1 \neq \rho_2$, we have $\psi_T(v) < \frac{\pi(v)}{2}$ for all non-zero $v \in \mathbb{R}^2$.

Observe that by Theorem 1.1,

$$\psi_T(v_1, v_2) \leq \min(v_1 \delta_{\rho_1}, v_2 \delta_{\rho_2}) \leq \min(v_1, v_2) \max(\delta_{\rho_1}, \delta_{\rho_2}) \leq \frac{\pi(v)}{2}.$$ 

Therefore if $\psi_T(v_1, v_2) = \pi(v)/2$, then $v_1 = v_2$ and $\delta_{\rho_1} = \delta_{\rho_2} = (n - 1)$; if $n \geq 3$, this can happen only when $\rho_1 \sim \rho_2$ by Mostow rigidity theorem.

The comparison of $\psi_T$ with the half-sum of positive roots is meaningful in view of Sullivan’s theorem that $\delta_{\rho_i} \sim \frac{n-1}{2}$ if and only if the bottom of the $L^2$-spectrum on $\rho_i(\Delta)\backslash \mathbb{H}^n$ is given by $\frac{(n-1)^2}{4}$ and there exists no positive square-integrable harmonic function on $\rho_i(\Delta)\backslash \mathbb{H}^n$ [38, Thm. 2.21]. Similarly, in [13], it is shown that Corollary 1.4 implies that $L^2(\Gamma\backslash G)$ is tempered, the bottom of the $L^2$-spectrum on $\Gamma\backslash X$ is given by $\frac{(n-1)^2}{2}$, and there exists no positive square-integrable harmonic function on $\Gamma\backslash X$.

There exist convex cocompact (non-lattice) subgroups of $SO^0(n, 1)$ whose critical exponents are arbitrarily close to $n - 1$, constructed by McMullen [20, Sec.6]. For Anosov subgroups of $G$, the trivial bound is given by $\delta \leq \max(\delta_{\rho_1}, \delta_{\rho_2}) \leq n - 1$, while the following corollary presents a strong gap for the value of $\delta$ from the trivial bound.

**Corollary 1.5 (Gap + Rigidity).** We have

$$\delta \leq \frac{1}{\sqrt{2}} \dim \Lambda \leq \frac{n - 1}{\sqrt{2}}$$

and the first equality holds if and only if $\rho_1 \sim \rho_2$. In particular, $\delta = \frac{1}{\sqrt{2}}(n-1)$ if and only if $\rho_1(\Delta), \rho_2(\Delta)$ are conjugate lattices of $SO^0(n, 1)$.
Some cases of Corollary 1.5 when \( \Delta \) is a surface group and \( n = 2 \) were also considered in [29, Thm. 1.8]. In Theorems 3.9, 4.1 and Corollary 4.4, a more general version of Theorem 1.1 and Corollary 1.5 are presented.

In view of special interests in low dimensional hyperbolic manifolds which come with huge deformation spaces, we also formulate the following corollary of Theorem 1.1, using the isomorphisms \( \text{PSL}_2(\mathbb{C}) \simeq \text{SO}^\circ(3,1) \) and \( \text{PSL}_2(\mathbb{R}) \simeq \text{SO}^\circ(2,1) \):

**Corollary 1.6.** For any non-elementary convex cocompact subgroup \( \Gamma_0 < \text{PSL}_2(\mathbb{R}) \) and any non-elementary convex cocompact representation \( \pi : \Gamma_0 \to \text{PSL}_2(\mathbb{C}) \), the critical exponent of the group \( \{(\gamma_0, \pi(\gamma_0)) : \gamma_0 \in \Gamma_0\} \) with respect to the Riemannian metric on \( \mathbb{H}^2 \times \mathbb{H}^3 \) is strictly less than \( \frac{2}{\sqrt{5}} \).

**Dimension of the limit set.** The following theorem can be proved by an easy adaptation of Sullivan’s proof of \( \dim \Lambda_{\rho_i} = \delta_{\rho_i} \) [37], based on the fact that \( \Lambda \) consists of conical limit points for Anosov subgroups (2.2):

**Theorem 1.7.** We have

\[
\dim \Lambda = \delta_{\min} = \max(\dim \Lambda_{\rho_1}, \dim \Lambda_{\rho_2}) \quad (1.5)
\]

where \( \delta_{\min} \) denotes the critical exponent of \( \Gamma \) with respect to the minimum of the metrics on \( \mathbb{H}^n \times \mathbb{H}^n \).

It is easy to observe that \( \sqrt{2}\delta \leq \delta_{\min} \) in general, and \( \sqrt{2}\delta = \delta_{\min} \) when \( \rho_1 \sim \rho_2 \). See Theorem 5.1 for a more general version. We refer to [23] and [27] for some interesting discussion on some aspects of the \( \dim \Lambda \) as we vary \( \rho_2 \) while fixing \( \rho_1 \).

**Dimension of the directional limit sets.** We also obtain estimates on the Hausdorff dimension of directional conical limit sets of \( \Gamma \). Given \( v = (v_1, v_2) \in \mathbb{R}^2 \) with \( v_1, v_2 > 0 \), a point \((\xi_1, \xi_2) \in \mathcal{F} = S^{n-1} \times S^{n-1}\) is called a \( v \)-directional conical limit point of \( \Gamma \) if the geodesic ray

\[
\{(\xi_1(v_1t), \xi_2(v_2t)) : t \geq 0\}
\]

accumulates on \( \Gamma \setminus X \), where \( \{\xi(t) : t \geq 0\} \) denotes a unit speed geodesic in \( \mathbb{H}^n \) toward \( \xi \in S^{n-1} \). We denote by

\[
\Lambda_v \subset \Lambda
\]

the set of all \( v \)-directional conical limit points of \( \Gamma \); note that \( \Lambda_v \) depends only on the slope \( v_2/v_1 \) of the line \( \mathbb{R}v \). It is not hard to observe that \( \Lambda_v = \emptyset \) if \( v_2/v_1 \notin [d_-, d_+] \). On the other hand, the non-triviality of \( \Lambda_v \) is much more subtle to decide.

**Theorem 1.8.** Assume that \( \rho_1 \not\sim \rho_2 \). If \( v_2/v_1 \in (d_-, d_+) \), i.e., \( v \in \text{int} \mathcal{L} \), then

\[
\frac{\psi_T(v)}{\max(v_1, v_2)} \leq \dim \Lambda_v \leq \frac{\psi_T(v)}{\min(v_1, v_2)}.
\]
See Theorem 7.1 and Corollary 7.7 for a more general result.

If \( \rho_1 \sim \rho_2 \), the growth indicator function \( \psi \Gamma \) is symmetric for an obvious reason. We note that there are many geometric examples where \( \rho_1 \) and \( \rho_2 \) are non-conjugate and \( \psi \Gamma \) is symmetric. For instance, when \( \Delta = \pi_1(S) \) is a surface group, there always exists an involution \( \iota \) in the outer automorphism group \( \text{Out} \Delta \) and a Fuchsian representation \( \rho \) of \( \Delta \) which is not a fixed point of \( \iota \) in the Teichmüller space \( \mathcal{T}(S) \), i.e., \( \rho \circ \iota \neq \rho \). See Figure 2. See section 8 for more examples including 3-manifold groups.

![Figure 2. Example of an involution](image)

**On the proofs.** The proof of Theorem 1.1 is based on the observation that the value \( \psi \Gamma(v) \) coincides with the critical exponent \( D_\alpha \) of \( \Gamma \) with respect to the Manhattan metric \( d_\alpha \) on \( \Gamma_0 \) for the dual vector \( \alpha \in \mathbb{R}^2 \) and the observation that \( T_i(v) = v_i \cdot \dim \Lambda_{\rho_i} \) can be interpreted as a linear form tangent to the growth indicator function \( \psi \Gamma \) which is known to be strictly concave [28].

Indeed, we propose the following conjecture generalizing Corollary 1.4. See Section 4 for the definition of an Anosov subgroup (with respect to a minimal parabolic subgroup).

**Conjecture 1.9.** If \( \Gamma \) is an Anosov subgroup of a connected semisimple real algebraic group \( G \) with rank \( G \geq 2 \), then

\[
\psi \Gamma \leq \rho
\]

where \( \rho \) is the half-sum of all positive roots.

This conjecture, if true, has applications in the study of the \( L^2 \)-spectrum and positive eigenfunctions of the associated locally symmetric manifold by the recent work [13]. The following theorem provides a partial confirmation of this conjecture:

**Theorem 1.10.** Let \( \Gamma \) be a Zariski dense Anosov subgroup of a Hitchin subgroup of \( \text{PSL}(d, \mathbb{R}) \). Then for any \( v = \text{diag}(t_1, \cdots, t_d) \) with \( \sum_{i=1}^d t_i = 0 \),

\[
\psi \Gamma(v) \leq \frac{t_1 - t_d}{d-1} \leq \rho(v).
\]

We mention that this pointwise bound is sharper than the one from ([33], [25]), which for instance, for \( d = 3 \), gives the bound \( t_1 - t_3 \) while the above corollary gives a bound \( (t_1 - t_3)/2 \).
The proof of Theorem 1.8 is based on the reparameterization theorem ([6], [9], [35]) which provides us with a trivial vector bundle associated to the dynamics of one-dimensional diagonal flow exp\(tv\). We combine this with the non-triviality of the Patterson-Sullivan measure \(\nu_u\) on \(\Lambda_v\) [8].

**Organization.** In section 2, we review basic notions and state known results about Anosov subgroups of \(\prod_{i=1}^k \text{SO}^o(n,1)\). In section 3, we prove Theorem 1.1 and related corollaries. In section 4, we discuss Theorem 4.1 and Corollary 1.10. In section 5, we prove Theorem 1.7. In section 6, we discuss the trivial vector bundle mentioned above, and prove a result that the associated vector-valued coordinate map grows with speed \(o(t)\) under the time \(t\)-flow (Theorem 6.4). In section 7, we prove Theorem 1.8. We also obtain the local behavior of the measures \(\nu_u\) in Theorems 7.5 and 7.8. In the last section 8, we discuss some geometric examples with symmetric growth indicator functions.

As mentioned before, our approach works for any Anosov subgroup of a semisimple real algebraic group of rank at most 3, provided the Hausdorff dimension of the limit set is computed with respect to a well-chosen metric on the Furstenberg boundary. The reason we have chosen to write this paper mainly for the product of \(\text{SO}^o(n,1)\)'s is because \(\mathcal{F}\) in this case is simply the Riemannian product of the spheres \(\mathbb{S}^{n-1}\) and hence is equipped with a natural Riemannian metric.

**Acknowledgements.** Our work has been largely inspired by a pioneering paper of Marc Burger [7] on a higher rank Patterson-Sullivan theory. In particular, the upper bound of Theorem 1.8 was already noted in [7, Thm. 2]. We would like to dedicate this paper to him on the occasion of his sixtieth birthday with affection and admiration. Marc also informed us that Corollary 1.5 can be deduced by combining Theorems 1(a) and 3(a) of [7]. We are grateful to Dick Canary for many helpful remarks on an earlier version of this paper, and in particular for pointing out how to strengthen our original version of Theorem 1.7. He also explained to us a different argument proving Corollary 1.5 following [5]. We would like to thank Minju Lee for helpful discussions. We also thank Andres Sambarino for pointing out some redundant rank restriction in our earlier version.

2. Preliminaries

Fixing \(k \geq 1\), let \(G = \prod_{i=1}^k \text{SO}^o(n,1)\), and consider the Riemannian symmetric space \((X = \prod_{i=1}^k \mathbb{H}^n, d)\) where

\[
d((x_i), (y_i)) = \sqrt{\sum_{i=1}^k d_{\mathbb{H}^n}(x_i, y_i)^2}.
\]

Consider the Furstenberg boundary \(\mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n-1}\) equipped with the Riemannian metric. Let \(\Delta\) be a finitely generated group and \(\rho_i : \Delta \rightarrow \text{SO}^o(n,1)\)
a non-elementary convex cocompact representation with finite kernel for each $1 \leq i \leq k$. In the whole paper, let $\Gamma$ be the subgroup of $G$ defined as

$$\Gamma = \{(\rho_1(\sigma), \cdots, \rho_k(\sigma)) \in G : \sigma \in \Delta \}.$$  

We remark that the class of these groups is precisely the class of Anosov subgroups of $G$ with respect to a minimal parabolic subgroup in the sense of Guichard and Wienhard [16].

We will assume that no two $\rho_i$ are conjugate to each other. This assumption implies that if $H$ denotes the identity component of the Zariski closure of $\Gamma$, then $H$ is isomorphic to $\prod_{i=1}^k SO(n_i, 1)$ for some $2 \leq n_i \leq n$. Hence $\Gamma$ is a Zariski dense Anosov subgroup of a semisimple real algebraic group $H$ of rank equal to $k = \text{rank } G$, which enables us to use the general theory developed for such groups. Fix a basepoint $o \in \mathbb{H}^n$. By abuse of notation, we write $H = \mathbb{R}^k$ and $a^+ = \{(u_1, \cdots, u_k) \in \mathbb{R}^k : u_i \geq 0 \text{ for all } i \}$. We denote by $\| \cdot \|$ the Euclidean norm on $a$.

The limit set of $\Gamma$, which we denote by $\Lambda = \Lambda_\Gamma$, is defined as the set of all accumulation points of $\Gamma$ in the Furstenberg boundary $\mathcal{F}$. It is the unique $\Gamma$-minimal subset of $\mathcal{F}$ ([2], [18, Lem. 2.3]).

For each $\xi = (\xi_1, \cdots, \xi_k) \in \mathcal{F}$ and $(t_1, \cdots, t_k) \in a$, we write

$$\xi(t_1, \cdots, t_k) = (\xi_1(t_1), \cdots, \xi_k(t_k))$$

where $\{\xi_i(t) : t \geq 0\}$ denotes the unit speed geodesic from $o$ to $\xi_i$ in $\mathbb{H}^n$. Set

$$\xi(a^+) := \{\xi(t_1, \cdots, t_k) \in X : t_i \geq 0 \text{ for all } i\}. \quad (2.1)$$

Recall that $\xi \in \mathcal{F}$ is called a conical limit point if there exists a sequence $\gamma_j \in \Gamma$ such that

$$\sup_j d(\xi(a^+), \gamma_j o) < \infty.$$ 

If $\Lambda_\mathcal{C}$ denotes the set of all conical limit points, then it is a well-known property of an Anosov subgroup (cf. [18, Prop. 7.4]) that

$$\Lambda = \Lambda_\mathcal{C}. \quad (2.2)$$

The Cartan projection of $g = (g_i)_{i=1}^k \in G$ is given by

$$\mu(g) = (d_{\mathbb{H}^n}(g_1o, o), \cdots, d_{\mathbb{H}^n}(g_ko, o)) \in a^+.$$ 

In particular, $d(go, o) = \|\mu(g)\|$. We denote by $\mathcal{L} \subset a^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$. It is a convex cone with non-empty interior [2]. For $k = 2$, $\mathcal{L}$ coincides with $\{(u_1, u_2) \in \mathbb{R}^2 : u_2 \in u_1[d_{-}, d_{+}]\}$ where $d_{\pm}$ is defined as in (1.2). Let $\delta$ denote the critical exponent of $\Gamma$, which is the abscissa of convergence of the Poincaré series $P(\Gamma)(s) = \sum_{\gamma \in \Gamma} e^{-s\|\mu(\gamma)\|}$. It follows from the non-elementary assumption on $\rho_i(\Delta)$ that $\delta > 0$.

Let $\psi_\Gamma : a^+ \to \mathbb{R} \cup \{-\infty\}$ denote the growth indicator function of $\Gamma$ following Quint [31]: for any non-zero $u \in a^+$,

$$\psi_\Gamma(u) := \|u\| \inf_{\text{open cones } \mathcal{C} \subset a^+, \gamma \in \mathcal{C}} \tau_\mathcal{C} \quad (2.3)$$
where $\tau_\gamma$ is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma} e^{-s(\mu(\gamma))}$, and $\psi_\Gamma(0) = 0$. Quint also showed that $\psi_\Gamma$ is a concave function such that $\psi_\Gamma > 0$ on int $\mathcal{L}$,

$$\delta = \sup_{\|u\|=1} \psi_\Gamma(u)$$

and that there exists a unique unit vector

$$u_\Gamma \in \mathcal{L} \quad (2.4)$$

in the maximal growth direction, that is, $\delta = \psi_\Gamma(u_\Gamma)$ [31].

The following proposition follows from the fact that $\Gamma$ is a Zariski dense Anosov subgroup of $H \simeq \prod_{i=1}^k SO^o(n_i,1)$ for $2 \leq n_i \leq k$ by the works of Sambarino [35, Lem. 4.8] and Potrie-Sambarino [28, Prop. 4.6 and 4.11].

**Theorem 2.1.** We have the following:

1. $\mathcal{L} \subset \text{int } a^+ \cup \{0\}$;
2. $\psi_\Gamma$ is strictly concave and analytic on int $\mathcal{L}$;
3. $u_\Gamma \in \mathcal{L}$.

For $x = (x_1, \cdots, x_k), y = (y_1, \cdots, y_k) \in X$, and $\xi = (\xi_1, \cdots, \xi_k) \in \mathcal{F}$, the $a$-valued Busemann function is given as

$$\beta_\xi(x,y) = (\beta_{\xi_1}(x_1,y_1), \cdots, \beta_{\xi_k}(x_k,y_k)) \in a$$

where $\beta_{\xi}(x_i,y_i) = \lim_{t \to +\infty} d_{\mathbb{H}^n}(\xi_i(t),x_i) - d_{\mathbb{H}^n}(\xi_i(t),y_i)$ is the Busemann function on $\mathbb{H}^{n-1} \times \mathbb{H}^n \times \mathbb{H}^n$.

**Definition 2.2.** For a linear form $\psi \in a^*$, a Borel probability measure $\nu$ on $\Lambda$ is called a $(\Gamma, \psi)$-Patterson-Sullivan measure if the following holds: for any $\xi \in \Lambda$ and $\gamma \in \Gamma$,

$$\frac{d\gamma_*\nu}{d\nu}(\xi) = e^{-\psi(\beta_{\xi}(\gamma,o,o))}$$

where $\gamma_*\nu(W) = \nu(\gamma^{-1}W)$ for any Borel subset $W \subset \Lambda$.

**Theorem 2.3** ([12, Thm. 7.7 and Cor. 7.8], [18, Cor. 7.12]). Let $u \in \text{int } \mathcal{L}$.

1. There exists a unique $\psi_u \in a^*$ such that $\psi_u \geq \psi_\Gamma$ and $\psi_u(u) = \psi_\Gamma(u)$. Moreover, $\psi_u(\cdot) = \frac{1}{\|u\|}(\nabla \psi_\Gamma(u),\cdot)$.
2. There exists a unique $(\Gamma, \psi_u)$-Patterson-Sullivan measure, say, $\nu_u$.
3. The abscissa of convergence of the series $\mathcal{P}_u(s) := \sum_{\gamma \in \Gamma} e^{-s(\psi_u(\mu(\gamma)))}$ is equal to 1 and $\mathcal{P}_u(1) = \infty$.

**Construction of $\nu_u$.** Fix $u \in \text{int } \mathcal{L}$. It follows from Theorem 2.1(1) that $\Gamma o \cup \Lambda$ is a compact space. For $s > 1$, consider the probability measure on $\Gamma o \cup \Lambda$ given by

$$\nu_{u,s} := \frac{1}{\mathcal{P}_u(s)} \sum_{\gamma \in \Gamma} e^{-s(\psi_u(\mu(\gamma)))} D_{\gamma_0} \quad (2.5)$$

where $D_{\gamma_0}$ denotes the Dirac measure on $\gamma_0$; this is a probability measure by Theorem 2.3(3). Note that the space of probability measures on $\Gamma o \cup \Lambda$ is a weak* compact space. Therefore, by passing to a subsequence, it weakly
converges to a probability measure, say $\nu_u$, on $\Gamma \cup \Lambda$. Since $\mathcal{P}_u(1) = \infty$ by Theorem 2.3, $\nu_u$ is supported on $\Lambda$. Now the uniqueness of $(\Gamma, \psi_u)$-Patterson-Sullivan measure (Theorem 2.3) implies that as $s \to 1^+$, $\nu_{u,s}$ weakly converges to $\nu_u$.

Hausdorff dimension. We consider the Riemannian metric on $F = S^{n-1} \times S^{n-1}$. For a subset $S \subset F$ and $s > 0$, the $s$-dimensional Hausdorff measure of $S$ is defined by

$$H^S(S) = \lim_{\varepsilon \to 0} \inf \{ \sum_{j=1}^k r_j^s : S \subset \bigcup_{j \in J} B(x_j, r_j) : 0 < r_j \leq \varepsilon \}$$

where the infimum is taken over all countable covers of $S$ by balls of radius at most $\varepsilon$. The Hausdorff dimension of $S$, which we denote by $\dim S$, is the infimum $s \geq 0$ such that $H^S(S) = 0$, or equivalently the supremum $s$ such that $H^S(S) = +\infty$. We refer to [3] for general facts on Hausdorff dimension.

3. Tent property of the growth indicator function

The main goal of this section is to prove Theorem 3.3 which provides the pointwise bound of $\psi_\Gamma$ by a tent function. This theorem implies Theorem 1.1 of the introduction.

Let $G = \prod_{i=1}^k \text{SO}^\circ(n,1)$ and $\Gamma = (\prod_{i=1}^k \rho_i)\Delta < G$ be an Anosov subgroup as in the last section. In particular, we assume that no $\rho_i$ and $\rho_j$ are conjugate for all $i \neq j$. Write $\rho = \prod \rho_j$ so that an element of $\Gamma = \rho(\Delta)$ is of the form $\rho(\sigma) = (\rho_1(\sigma), \ldots, \rho_k(\sigma))$ for $\sigma \in \Delta$.

Consider the following dual cone of the limit cone $\mathcal{L}$:

$$\mathcal{L}^* := \{ \alpha \in \mathbb{R}^k : \langle \mathcal{L}, \alpha \rangle \geq 0 \}.$$ (3.1)

Note that $\text{int} \mathcal{L}^* = \{ \alpha \in \mathbb{R}^k : \langle \mathcal{L} - \{\alpha\}, \alpha \rangle > 0 \}$. Since $\mathcal{L} \subset \text{int} \mathfrak{a}^+$, $\text{int} \mathcal{L}^*$ contains $\mathfrak{a}^+$ properly. In particular, it contains the standard unit vectors $e_1, \ldots, e_k \in \mathbb{R}^k$. See Figure 3.

![Figure 3. Limit cone and its dual cone.](image)

For $u \in \mathcal{L}$, we write

$$\delta_u = \psi_\Gamma(u).$$

The main ingredients of the following lemma are Quint’s duality lemma [34] and the strict concavity of $\psi_\Gamma$ proved by Potrie-Sambarino [28].
Lemma 3.1. The map \( u \mapsto \alpha_u := \frac{1}{\delta_u} \nabla \psi_T(u) \) defines a bijection \( \text{int} \ L \rightarrow \text{int} \ L^* \). If we denote its inverse map \( \text{int} \ L^* \rightarrow \text{int} \ L \) by \( \alpha \mapsto u_\alpha \), then for \( \alpha \in \text{int} \ L^* \),

\[
\frac{1}{\delta_u} \frac{\nabla \psi_T(u)}{\|u\|} \Gamma(\alpha) \text{ is the unique vector such that } \alpha \text{ is parallel to } \nabla \psi_T(\alpha) \text{ and } \langle \alpha, u_\alpha \rangle = 1.
\]

Moreover, \( \psi_{u_\alpha}(\cdot) = \langle \delta_u \alpha, \cdot \rangle \), \( \delta_u \alpha = \max_{v \in L, \langle v, \alpha \rangle = 1} \psi_T(v) \), and \( \nabla \psi_{u_\Gamma} \in \mathbb{R} u_\Gamma \).

Proof. The first claim is given in [12, Lem 2.24], which gives an alternative argument of Quint’s duality lemma [34] and uses Theorem 2.1. Since \( \alpha \in \text{int} \ L^* \), the hyperplane \( \ker \alpha = \{ w \in \mathbb{R}^k : \langle w, \alpha \rangle = 0 \} \) intersects \( L \) only at the origin, and hence the set \( S_\alpha = \{ u \in L : \langle \alpha, u \rangle = 1 \} \), parallel to \( \ker \alpha \), is a compact convex subset. Since \( \psi_T \) is concave, and strictly concave in \( \text{int} \ L \), there exists a unique vector \( v_\alpha \in S_\alpha \) such that \( \psi_T(v_\alpha) = \max_{v \in S_\alpha} \psi_T(v) \). Hence \( \alpha \) is parallel to \( \nabla \psi_T(v_\alpha) \). Since \( \alpha = \frac{1}{\delta_u} \frac{\nabla \psi_T(u_\alpha)}{\|u_\alpha\|} \) and \( \langle \alpha, u_\alpha \rangle = 1 \), it follows that \( \nabla \psi_T(v_\alpha) = \nabla \psi_T(u_\alpha) \). By the uniqueness of \( v_\alpha \), this implies \( u_\alpha = v_\alpha \). Since \( \langle \alpha, u_\alpha \rangle = 1 \) and \( \delta_u = \psi_{u_\alpha}(u_\alpha) \), this implies that \( \psi_{u_\alpha}(\cdot) = \langle \delta_u \alpha, \cdot \rangle \) by Theorem 2.3. See Figure 4. See [12, Lem. 2.24] for the claim \( \nabla \psi_{u_\Gamma} \in \mathbb{R} u_\Gamma \). \( \square \)

![Figure 4. From \( \alpha \) to \( u_\alpha \)](image)

Definition 3.2 (Tent function). We define the tent function \( T : \mathbb{R}^k \rightarrow \mathbb{R} \) of \( \Gamma \) by

\[
T(v) = \min_{1 \leq i \leq k} v_i \delta_{\rho_i}.
\]

Theorem 3.3 (Tent property). For any \( v = (v_1, \cdots, v_k) \in \mathbb{R}^k \),

\[
\psi_T(v) \leq T(v). \tag{3.2}
\]

Moreover the equality holds if and only if \( v = cu_{e_i} \) for some \( c > 0 \) and \( 1 \leq i \leq k \), in which case we have \( \psi_T(cu_{e_i}) = c\delta_{\rho_i} = T(cu_{e_i}) \).

The key part of the argument is Theorem 3.9, which we state after discussing some of the corollaries.
Denote by \( L^\dagger = \{ \sum_{i=1}^k \tilde{v}_i u_{e_i} : \tilde{v}_i \geq 0 \} \) the closed convex cone spanned by \( u_{e_1}, \cdots, u_{e_k} \); then \( L^\dagger \subset \text{int} \mathcal{L} \). The last claim of Theorem 3.3 together with the strict concavity of \( \psi_\Gamma \) on \( \text{int} \mathcal{L} \) implies the following lower bound of \( \psi_\Gamma \) on \( L^\dagger \):

**Corollary 3.4.** If \( v = \sum_{i=1}^k \tilde{v}_i u_{e_i} \) with \( \tilde{v}_i \geq 0 \), then

\[
\psi_\Gamma(v) \geq \sum_{i=1}^k \tilde{v}_i \delta_{\rho_i},
\]

where the inequality is strict if and only if \( \tilde{v}_1 \cdot \tilde{v}_2 + \tilde{v}_2 \cdot \tilde{v}_3 + \tilde{v}_3 \cdot \tilde{v}_1 > 0 \).

Note that Theorem 3.3 implies that for any \( v \in \text{int} \mathcal{L} \),

\[
\psi_\Gamma(v) \leq \min_{1 \leq i \leq k} v_i \dim \Lambda_{\rho_i} \leq \left( \min_{1 \leq i \leq k} v_i \right) \dim \Lambda
\]

since \( \dim \Lambda_{\rho_i} = \delta_{\rho_i} \), and \( \max_i \dim \Lambda_{\rho_i} \leq \dim \Lambda \), as the projection \( \Lambda \to \Lambda_{\rho_i} \) is Lipschitz. Therefore Theorem 1.1 follows.

**Corollary 3.5.** Let \( \pi(v_1, \cdots, v_k) = (n-1)(\sum_{i=1}^k v_i) \), which is the sum of all positive roots of \( G \). Then for all \( v \in \mathbb{R}^k \),

\[
\psi_\Gamma(v) \leq \frac{\pi(v)}{k} \leq \frac{\pi(v)}{2}.
\]

If \( \psi_\Gamma(v) = \frac{\pi(v)}{k} \) for \( v \neq 0 \), then \( v_1 = \cdots = v_k \) and each \( \rho_i(\Delta) \) is a lattice of \( \text{SO}_\circ^\infty(n, 1) \).

**Proof.** Since \( \delta_{\rho_i} \leq (n-1) \), we get \( \psi_\Gamma(v) \leq (\min_i v_i)(n-1) \leq \frac{1}{k}(\sum_{i=1}^k v_i)(n-1) \); hence the claim follows. \( \square \)

We also get the following gap+rigidity result for the critical exponent \( \delta \):

**Corollary 3.6** (Gap + Rigidity property). We have

\[
\delta < \min_{1 \leq i \leq k} u_i \delta_{\rho_i} = \min_{1 \leq i \leq k} u_i \dim \Lambda_{\rho_i}
\]

where \( u_\Gamma = (u_1, \cdots, u_k) \) is the unit vector of the maximal growth direction.

In particular,

\[
\delta < \frac{\dim \Lambda}{\sqrt{k}}.
\]

**Proof.** Note that \( \psi_\Gamma(u_\Gamma) = \delta \), and \( \nabla \psi_\Gamma(u_\Gamma) \) is a positive multiple of \( u_\Gamma \). In particular, \( \nabla \psi_\Gamma(u_\Gamma) \in \text{int} \mathfrak{a}^\dagger \) and hence not parallel to any \( e_i, 1 \leq i \leq k \). Therefore the first claim follows from Theorem 3.3. The second claim follows since \( \min_i u_i \leq 1/\sqrt{k} \) for \( \sum_{i=1}^k u_i^2 = 1 \). \( \square \)
Critical exponents for the Manhattan metrics. The proof of Theorem 3.3 uses the realization of $\psi_T(u)$ as the critical exponent of $\Gamma$ with respect to the Manhattan metric $d_{\alpha_o}$ which we now recall.

**Definition 3.7** (Manhattan metric $d_{\alpha}$ on $\Gamma_o$). For $\alpha = (\alpha_1, \cdots, \alpha_k) \in \text{int} \mathcal{L}^*$, define the following function $d_{\alpha}$ on $\Gamma_o \times \Gamma_o$ by

$$d_{\alpha}(\rho(\sigma_1)o, \rho(\sigma_2)o) := \sum_{i=1}^k \alpha_i \cdot d_{\mathbb{Z}^k}(\rho_i(\sigma_1)o, \rho_i(\sigma_2)o)$$

for any $\sigma_1, \sigma_2 \in \Delta$. In other words, for $\gamma_1, \gamma_2 \in \Gamma$,

$$d_{\alpha}(\gamma_1o, \gamma_2o) = \langle \alpha, \mu(\gamma_1^{-1}\gamma_2) \rangle. \quad (3.3)$$

This defines a metric on $\{ \gamma o : \mu(\gamma) \in \mathcal{L} \}$ which is a co-finite subset of $\Gamma$, since $\mathcal{L}$ is the asymptotic cone of $\mu(\Gamma)$. We call $d_{\alpha}$ the Manhattan metric on $\Gamma_o$ for the parameter $\alpha$.

Let $D_{\alpha}$ be the critical exponent of $\Gamma$ with respect to $d_{\alpha}$, i.e., it is the abscissa of convergence of the series $\sum_{\gamma \in \Gamma} e^{-s d_{\alpha}(\gamma o, o)}$.

We remark that by [31, Lemma 3.1.1], we have for each $\alpha \in \text{int} \mathcal{L}^*$,

$$D_{\alpha} = \lim_{T \to \infty} \frac{\log \#\{ \gamma \in \Gamma : \langle \alpha, \mu(\gamma) \rangle \leq T \}}{T}. \quad (3.3)$$

For each $1 \leq i \leq k$, it follows from the definition that $D_{e_i} = \delta_{\rho_i}$.

**Lemma 3.8.** For any $\alpha \in \text{int} \mathcal{L}^*$,

$$\psi_\Gamma(u_\alpha) = D_{\alpha}.$$

In particular, $\psi_T(u_{e_i}) = \delta_{\rho_i}$.

**Proof.** By Theorem 2.3(3), the series $\sum_{\gamma \in \Gamma} e^{-s \psi_\alpha(\mu(\gamma))}$ has the abscissa of convergence 1. Since $\psi_\alpha(\cdot) = \langle \delta_{u_\alpha} \alpha, \cdot \rangle$ by Lemma 3.1, we have

$$\sum_{\gamma \in \Gamma} e^{-s \psi_\alpha(\mu(\gamma))} = \sum_{\gamma \in \Gamma} e^{-s d_{u_\alpha} \alpha \mu(\gamma)}.$$

Therefore $D_{\alpha} = \delta_{u_\alpha}$.

**Proof of Theorem 3.3:** For each $1 \leq i \leq k$, define linear forms $\beta_i$ and $T_i$ on $\mathbb{R}^k$ by setting $\beta_i(v) = \langle v, e_i \rangle$ and $T_i(v) = \delta_{\rho_i} \beta_i(v)$ for all $v \in \mathbb{R}^k$. Since $T = \min_i T_i$, the claim (3.2) follows if we show $\psi_T \leq T_i$ for each $1 \leq i \leq k$. Fix $i$. By Theorem 2.1(1), $e_i \in \text{int} \mathcal{L}^*$. Since $\delta_{\rho_i} = \psi_T(u_{e_i})$ by Lemma 3.8, it follows from Lemma 3.1 that $T_i = \psi_{u_{e_i}}$.

On the other hand, recall that $\psi_{u_{e_i}}$ is tangent to $\psi_T$ at $u_{e_i}$, i.e., $\psi_T \leq \psi_{u_{e_i}}$ and $\psi_T(u_{e_i}) = \psi_{u_{e_i}}(u_{e_i})$. Moreover, the strict concavity of $\psi_T$ implies that $\psi_T < \psi_{u_{e_i}}$ on $\mathbb{R}^k - \mathbb{R} u_{e_i}$. Therefore $\psi_T \leq T_i$, and $\psi_T(v) = T_i(v)$ if and only if $v$ is parallel to $u_{e_i}$. This finishes the proof.

**Different proof of Theorem 3.3 for $k \leq 3$.**
We also give a more direct proof of Theorem 3.3 which does not use the fact that \( \psi_{u_i} \) is a tangent linear form to \( \psi_T \) in the case of \( k \leq 3 \), and which may be of independent interest.

We begin by the following slightly weaker form:

**Theorem 3.9.** Let \( \alpha \in \text{int} \mathcal{L}^* \). Then
\[
\psi_T(u_\alpha) = D_\alpha \leq T(u_\alpha). \tag{3.4}
\]
Moreover, the equality holds if and only if \( \alpha = ce_i \) for some \( 1 \leq i \leq k \) and \( c > 0 \), in which case, \( D_{ce_i} = \delta_{\rho_i}/c = T(u_{ce_i}) \).

We will use the following strip theorem.

**Theorem 3.10** (Strip theorem). [8, Thm. 6.3] Let \( k \leq 3 \) and \( u \in \text{int} \mathcal{L} \). For all sufficiently large \( R > 0 \), the abscissa of convergence of the series
\[
\sum_{\gamma \in \Gamma, \|\mu(\gamma) - R\| \leq R} e^{-s\psi_\alpha(\mu(\gamma))}
\]
is equal to 1.

**Proof of Theorem 3.9 for \( k \leq 3 \):** Fix \( \alpha \in \text{int} \mathcal{L}^* \). Let \( S_R := \{ g \in G : \|\mu(g) - R\| < R \} \) for \( R > 0 \). By Theorem 3.10, there exists \( R > 0 \) such that the series \( D_R(s) = \sum_{\gamma \in \Gamma \cap S_R} e^{-s\psi_\alpha(\mu(\gamma))} \) has the abscissa of convergence 1.

Write \( u_\alpha = (u_1, \cdots, u_k) \in \text{int} \mathcal{L} \). Fix \( 1 \leq i \leq k \). Recalling that \( u_i > 0 \), there exists \( C > 0 \) so that for any \( \rho(\sigma) \in S_R \), we have
\[
\left\| \mu(\rho(\sigma)) - \frac{\mu(\rho_i(\sigma))}{u_i} u_\alpha \right\| \leq C. \tag{3.5}
\]
It then follows that
\[
D_R(s) = \sum_{\rho(\sigma) \in \Gamma \cap S_R} e^{-s\delta_{\rho\alpha}(\mu(\rho(\sigma)), \alpha)} \ll \sum_{\gamma \in \Gamma} e^{-s\delta_{\rho\alpha}(\mu(\rho(\sigma))/u_i)}
\]
which is finite whenever \( s > \frac{u_i}{\delta_{\rho\alpha}} \delta_{\rho_i} \). Therefore we have
\[
1 \leq \frac{u_i}{\delta_{\rho\alpha}} \delta_{\rho_i}.
\]
Hence by Lemma 3.8,
\[
D_\alpha \leq u_\alpha \delta_{\rho_i} = u_i D_{e_i}.
\]
Since \( 1 \leq i \leq k \) is arbitrary, we get \( D_\alpha \leq T(u_\alpha) \). It completes the proof of (3.4).

Now we show the last assertion: If \( \alpha = ce_i \) for some \( c > 0 \) and \( 1 \leq i \leq k \), then the \( i \)-th component of \( u_\alpha \) must be \( 1/c \), as \( \langle \alpha, u_\alpha \rangle = 1 \); hence
\[
D_\alpha = D_{e_i}/c \leq T(u_\alpha) = \min_j (u_j D_{e_j}) \leq D_{e_i}/c.
\]
Therefore we have the equality \( D_{ce_i} = D_{e_i}/c = T(u_{ce_i}) \) in (3.4). Now suppose that \( \alpha \) is not parallel to any coordinate axis and that \( D_\alpha = \min_j (u_j D_{e_j}) \), that is, \( \psi_T(u_\alpha) = T(u_\alpha) \). Let \( S_\alpha := \{ v \in \text{int} \mathcal{L} : \langle \alpha, v \rangle = 1 \} \). Since \( \psi_T|S_\alpha \) is maximized at \( u_\alpha \) by Lemma 3.1 and \( \psi_T \) is analytic at \( u_\alpha \), \( \psi_T|S_\alpha \)
is constant to first order at \( u_\alpha \); more precisely, for some neighborhood \( U \subset S_\alpha \) of \( u_\alpha \), \( \psi_T(v) = \psi_T(u_\alpha) + O(\|v - u_\alpha\|^2) \) for all \( v \in U \). On the other hand, if we set \( T_i(v) = v_i \delta_{p_i} \), then by the hypothesis, \( \nabla T_i \), which is in the direction of the \( i \)-th coordinate axis, is not normal to \( S_\alpha \) for each \( 1 \leq i \leq k \). Hence there exists a tangent direction, say, \( w \), to \( S_\alpha \) at \( u_\alpha \) where \( T|_{S_\alpha} \) strictly decreases to first order, that is, there exists \( c > 0 \) such that \( T(u_\alpha + sw) = T(u_\alpha) - cs + O(\|sw\|^2) \) for all sufficiently small \( s \geq 0 \). Since \( \psi_T(u_\alpha + sw) \leq T(u_\alpha + sw) \) for all sufficiently small \( s \geq 0 \) by (3.4), we get \( 0 \leq -c \), yielding a contradiction. This finishes the proof.

We now explain how to finish the proof of Theorem 3.3 using Theorem 3.9. Observe that for \( c > 0 \) and \( \alpha \in \text{int} \mathcal{L}^* \), we have \( cu_\alpha = u_\alpha/c \). Therefore by Lemmas 3.1 and 3.8, Theorem 3.9 implies Theorem 3.3 for any \( v \in \text{int} \mathcal{L} \). Now this inequality extends to \( \mathcal{L} \) by the following lemma, and this proves the claim since \( \psi_T = -\infty \) outside \( \mathcal{L} \).

**Lemma 3.11.** For any sequence \( v_j \in \text{int} \mathcal{L} \) converging to a non-zero vector \( w \in \partial \mathcal{L} \) along a fixed line \( L \), we have

\[
\psi_T(w) = \lim_{j \to \infty} \psi_T(v_j).
\]

**Proof.** Let \( L \subset \mathbb{R}^k \) be a line and \( v_j \in L \cap \text{int} \mathcal{L} \) be a sequence converging to \( w \in \partial \mathcal{L} \). By the upper semi-continuity of \( \psi_T \), we have \( \psi_T(w) \geq \limsup_j \psi_T(v_j) \). Since \( \psi_T \) is concave, the restriction \( \psi_T|_L \) is a concave function on the line \( L \) in particular. It follows that \( \psi_T(w) \leq \liminf_j \psi_T(v_j) \). This implies the claim. \( \square \)

We now discuss some consequences of the Tent theorem.

**Corollary 3.12.** For each \( 1 \leq i, j \leq k \), the \( j \)-th component of \( u_{e_i} \in \text{int} \mathcal{L} \) is at least \( \delta_{p_i} \delta_{p_j} \), and the \( i \)-th component is equal to 1.

**Proof.** We just consider the case when \( k = 3 \). Note that \( u_{e_1} \) is of the form \((1, c_2, c_3)\) for some \( c_2, c_3 > 0 \). Since \( \psi_T(u_{e_1}) = \min(\delta_{p_1}, c_2\delta_{p_2}, c_3\delta_{p_3}) = \delta_{p_1} \) by Theorem 3.9, it follows \( c_j\delta_{p_j} \geq \delta_{p_1} \) from which the claim on \( u_{e_1} \) follows. The argument is now symmetric for other \( u_{e_i} \)'s. \( \square \)

The case of \( k = 2 \) in the above corollary implies the following theorem of Burger [7, Thm. 1 and its Coro.], which generalizes a theorem of Thurston on the stretch constant of a finite area hyperbolic surface (see also [14]):

**Corollary 3.13.** If \( \rho_1 \neq \rho_2 \), then

\[
d_+(\rho_1, \rho_2) > \frac{\dim \Lambda_{\rho_1}}{\dim \Lambda_{\rho_2}}.
\]

**Proof.** Corollary 3.12 for \( k = 2 \) implies that the slope of the vector \( u_{e_1} \) is at least \( \delta_{p_1}/\delta_{p_2} \). Since \( u_{e_1} \in \text{int} \mathcal{L} \) and \( \mathcal{L} \) is the convex cone whose largest slope is \( d_+(\rho_1, \rho_2) \), this implies that

\[
d_+(\rho_1, \rho_2) > \delta_{p_1}/\delta_{p_2}.
\]
Example 3.14. Recalling that $D_{(1,\ldots,1)}$ denotes the critical exponent of $\Gamma$ for the $L^1$-metric on $\mathbb{H}^n \times \cdots \times \mathbb{H}^n$, a particular case of the above theorem implies that

$$D_{(1,\ldots,1)} < \dim \Lambda,$$

(3.6)

since $\min u_i \leq 1/k$ when $\sum_{1 \leq i \leq k} u_i = 1$.

For $k = 2$ and $\dim \Lambda_{\rho_i} = 1$, this was proved by Bishop and Steger [4].

4. General Anosov subgroups and Hitchin representations

Let $G$ be a connected semisimple real algebraic group of rank $k$, and let $P = MAN$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition. So $A$ is a maximal real split torus of $G$. Let $\mathfrak{a} = \text{Lie } A$ and $\mathfrak{g} = \text{Lie } G$. Let $G = K(\exp \mathfrak{a}^+)K$ be a Cartan decomposition of $G$ where $K$ is a maximal compact subgroup and $\mathfrak{a}^+ \subset \mathfrak{a}$ is a closed positive Weyl chamber. For $g \in G$, denote the Cartan projection of $g$ by $\mu(g)$ the unique element of $\mathfrak{a}^+$ such that $g \in K \exp \mu(g)K$. The quotient $\mathcal{F} = G/P$ is the Furstenberg boundary of $G$. We fix a basepoint $o = [K] \in G/K$. Let $\Phi$ (resp. $\Phi^+$) denote the set of all (resp. positive) roots for $(\mathfrak{g}, \mathfrak{a}^+)$. Consider the inner product on $\mathfrak{g}$ given by the killing form:

$$B(v, w) = \text{Tr}(\text{ad } v \circ \text{ad } w) = \sum_{\beta \in \Phi} \beta(v)\beta(w).$$

Let $\Gamma < G$ be a Zariski dense subgroup. Suppose that $\Gamma$ is Anosov $G$ with respect to $P$; this means that $\Gamma$ arises as the image $\rho(\Delta)$ where $\rho : \Delta \to G$ is a representation of a Gromov hyperbolic group $\Delta$ which induces a $\Delta$-equivariant continuous embedding $\partial \Delta \to \mathcal{F}$ of the Gromov boundary $\partial \Delta$ mapping two distinct points to two points in general position. Two points $\xi, \eta$ of $\mathcal{F}$ are said to be in general position if $(\xi, \eta)$ belongs to a unique open $G$-orbit of $\mathcal{F} \times \mathcal{F}$.

The limit cone $\mathcal{L} \subset \mathfrak{a}^+$ of $\Gamma$ is the asymptotic cone of $\mu(\Gamma)$, and the growth indicator function $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ is defined as in (2.3) using the norm induced by the killing form. Then Theorems 2.1, 2.3 and 3.10 are all known to hold for $\Gamma$ in this generality (see the same references quoted there). In particular, $\psi_\Gamma$ is analytic in int $\mathcal{L}$. For $u \in \text{int } \mathcal{L}$, we denote by $\nabla \psi_\Gamma(u)$ the gradient of $\psi_\Gamma$ at $u$ so that the directional derivative $\partial_v \psi_\Gamma(u)$ satisfies $\partial_v \psi_\Gamma(u) = \langle \nabla \psi_\Gamma(u), v \rangle$ for all non-zero $v \in \mathfrak{a}$. Lemma 3.1 also holds giving the bijection $u \mapsto u_{\alpha}$ between int $\mathcal{L}^*$ and int $\mathcal{L}$. Let $\{\beta_1, \cdots, \beta_k\} \subset \Phi^+$ be the set of simple roots, and let $\delta_{\beta_i}$ denote the abscissa of convergence of the series $\sum_{\gamma \in \Gamma} e^{-s_{\beta_i}(\mu(\gamma))}$ for each $1 \leq i \leq k$. Let $e_1, \cdots, e_k \in \text{int } \mathcal{L}^*$ be the unique unit vectors such that

$$\langle e_i, v \rangle = \beta_i(v)$$
for all $v \in \mathfrak{a}$. For each $\alpha \in \text{int } \mathcal{L}^*$, we denote by $D_\alpha$ the critical exponent of $\Gamma$ with respect to the Manhattan metric $d_\alpha(\alpha, \gamma \alpha) = \langle \alpha, \mu(\gamma) \rangle$ on $\Gamma$. Then $D_\alpha$ coincides with $\delta_\beta$. The proof of Theorem 4.3 works precisely in the same way for the following theorem:

**Theorem 4.1 (Tent property).** Let $\Gamma$ be a Zariski dense Anosov subgroup of $G$ as above. Then for any $v \in \mathfrak{a}$,

$$\psi_T(v) \leq \min_{1 \leq i \leq k} (\beta_i(v)\delta_\beta_i).$$

Moreover, the equality holds if and only if $v = cu_i$ for some $c > 0$ and $1 \leq i \leq k$. In particular, if $u_T \in \mathfrak{a}^+$ is the unit vector of the maximal growth,

$$\delta = \psi_T(u_T) < \min_{1 \leq i \leq k} (\beta_i(u_T)\delta_\beta_i). \quad (4.1)$$

**Remark 4.2.** It will be very interesting to compute the value $\delta_\beta$, or give a good upper bound, which is a key to the conjecture 1.9. For Hitchin representations, $\delta_\beta = 1$; see (4.3) below, as computed in [28]. See also [30] for some partial results for $(1, 1, 2)$ hyperconvex representations.

**Hitchin representations.** We discuss the implication of Theorem 4.1 for Hitchin representations. Let $\Delta$ be a surface group, i.e., $\Delta = \pi_1(\Sigma)$ for a closed orientable surface $\Sigma$ of genus at least 2. Fix a realization of $\Delta$ as a uniform lattice of $\text{PSL}(2, \mathbb{R})$, and let $\pi_d$ denote the $d$-dimensional irreducible representation $\text{PSL}(2, \mathbb{R}) \to \text{PSL}(d, \mathbb{R})$ which is unique up to conjugation. Let $\rho : \Delta \to \text{PSL}(d, \mathbb{R})$ be a Hitchin representation, that is, $\rho$ belongs to the same connected component as $\pi_d$ in the character variety $\text{Hom}(\Delta, \text{PSL}(d, \mathbb{R}))/\sim$ where the equivalence is given by conjugations. Then

$$\Gamma := \rho(\Sigma)$$

is an Anosov subgroup of $\text{PSL}(d, \mathbb{R})$ with respect to a minimal parabolic subgroup [17]. We call $\Gamma$ Fuchsian if $\rho$ is conjugate to $\pi_d$.

In this case, $\mathfrak{a}^+ = \{v = \text{diag}(t_1, \ldots, t_d) : t_1 \geq \cdots \geq t_d, \sum t_i = 0\}$ and $\beta_i(v) = t_i - t_{i+1}$ for $1 \leq i \leq d - 1$.

To ease the notation, we write $(t_1, \ldots, t_d)$ for $\text{diag}(t_1, \ldots, t_d)$. We set $v_d := \pi_d(1/2, -1/2) \in \mathfrak{a}^+$ for $d \geq 2$; hence $v_3 = (1, 0, -1)$ and $v_4 = (3/2, 1/2, -1/2, -3/2)$. Since the killing form of $\text{PSL}(d, \mathbb{R})$ is given by

$$\langle (t_1, \ldots, t_d), (s_1, \ldots, s_d) \rangle = 2d \sum t_is_i,$$

we get $B(v_3, v_3) = 12$ and $B(v_4, v_4) = 40$. We consider the following normalized inner product: for $v = (t_1, \ldots, t_d), w = (s_1, \ldots, s_d) \in \mathfrak{a}$,

$$\langle v, w \rangle = \frac{1}{B(v_d, v_d)} B(v, w) = \frac{2d}{B(v_d, v_d)} \sum_{i=1}^d t_is_i \quad (4.2)$$
so that \(|v_d| = 1\). We then consider \(\psi_\Gamma\) and \(\delta\) defined with respect to this inner product. Then for \(\Gamma \subset \text{PSL}(d, \mathbb{R})\) Fuchsian, it is easy to check that \(\delta = 1\).

By the work of Potrie-Sambarino [28, Thm. B] (see also [30, Cor. 9.4]), we have for any Hitchin \(\Gamma < \text{PSL}(d, \mathbb{R})\),

\[
\delta_{\beta_i} = 1 \quad \text{for all } 1 \leq i \leq d - 1. \tag{4.3}
\]

Theorem 4.1 implies The following:

**Corollary 4.3.** Let \(\Gamma < \text{PSL}(d, \mathbb{R})\) be a Zariski dense Anosov subgroup of the image of a Hitchin representation \(\rho : \Delta \to \text{PSL}(d, \mathbb{R})\). Then for any \(v = (t_1, \ldots, t_d) \in \mathfrak{a}^+\),

\[
\psi_\Gamma(v) \leq \min_{1 \leq i \leq d-1} (t_i - t_{i+1}) \leq (t_1 - t_d)/(d - 1). \tag{4.4}
\]

In particular, we have

\[
\delta < 1
\]

for \(\Gamma\) non-Fuchsian.

**Anosov groups of a product of rank one groups.** For \(1 \leq i \leq k\), let \((X_i, d_i)\) be a rank one Riemannian symmetric space whose maximum sectional curvature is \(-1\), and \(G_i\) denote the identity component of the isometry group of \(X_i\). Then \(G_i\) is isomorphic to one of

\[
\text{SO}^o(n, 1), \text{SU}(n, 1), \text{Sp}(n, 1), F_4^{-20}
\]

and the volume entropy \(D(X_i)\) is respectively given by

\[
n - 1, 2n, 4n + 2, 22.
\]

Let \((X = \prod_{i=1}^k X_i, d)\) denote the Riemannian product, and let \(G = \prod_{i=1}^k G_i\). Let \(\Delta\) be a finitely generated group and \(\rho_i : \Delta \to G_i\) be a faithful convex cocompact representation. Let \(\Gamma = (\prod_i \rho_i)(\Delta) < G\). The notation \(\delta_{\rho_i(\Delta)}\) denotes the critical exponent of \(\rho_i(\Delta)\) with respect to \(d_i\). Let \(\pi(v) = \sum_{i=1}^k v_i D(X_i)\) denote the sum of all positive roots of \(G\).

**Corollary 4.4.** Let \(k \geq 2\). Suppose that \(\Gamma < G\) is Zariski dense. Then for any \(v = (v_1, \ldots, v_k) \in \mathfrak{a}^+ = \mathbb{R}^k_{\geq 0}\),

\[
\psi_\Gamma(v) \leq \min_i v_i \delta_{\rho_i(\Sigma)} \leq \frac{1}{k} \sum_{i=1}^k v_i D(X_i) \leq \frac{\pi(v)}{2}. \tag{4.5}
\]

Moreover if \(\psi_\Gamma(v) = \pi(v)/2\) for some non-zero \(v \in \mathfrak{a}^+\), then \(k = 2\), \(G_1 \simeq G_2 \simeq \text{SO}^o(2, 1)\), and \(\rho_i(\Sigma)\) is a lattice in \(G_i\) for each \(i = 1, 2\).

The first part of the corollary is an immediate consequence of Theorem 4.1 since \(\delta_{\rho_i(\Gamma)} \leq D(X_i)\). To see the second part, suppose that \(\psi_\Gamma(v) = \pi(v)/2\) for \(v \neq 0\). Then \(v_i > 0\) for all \(i\), since \(\{u : \psi_\Gamma(u) \geq 0\}\) is equal to the limit cone of \(\Gamma\), which is contained in the interior of \(\mathfrak{a}^+\). It follows from (4.5) that \(k = 2\), \(v_1 = v_2\) and \(\delta_{\rho_i(\Gamma)} = D(X_i)\) for each \(i = 1, 2\). This implies
that \( \rho_1(\Gamma) \) and \( \rho_2(\Gamma) \) are co-compact lattices in \( G_1 \) and \( G_2 \) respectively by (37), [38], [10], [11]). If at least one of \( G_1 \) and \( G_2 \) is not isomorphic to \( \text{SO}^\circ(2,1) \), then \( \rho_1 \sim \rho_2 \) by Mostow rigidity theorem (see [24, Thm. 24.1]), and hence that cannot happen by the Zariski density hypothesis on \( \Gamma \). Hence \( G_1 \simeq G_2 \simeq \text{SO}^\circ(2,1) \).

5. Hausdorff Dimension of \( \Lambda \)

In this section, we prove Theorem 1.7. It is convenient to use the upper half-space model of \( \mathbb{H}^n \) so that \( \partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{ \infty \} \). Let \( o = (0, \cdots , 0, 1) \in \mathbb{H}^n \) and \( v_o = (0, \cdots, 0, -1) \in T^1_o \mathbb{H}^n \) denote the downward normal vector based at \( o \). By abuse of notation, we set \( o = (o, \cdots, o) \in \prod_{i=1}^k \mathbb{H}^n \) and \( v_o = (v_o, \cdots, v_o) \in \prod_{i=1}^k T^1_o \mathbb{H}^n \). For \( \xi \in \mathcal{F} \cap \prod_{i=1}^k \mathbb{R}^{n-1} \) and \( r > 0 \), let \( B(\xi, r) \) denote the ball in \( \prod_{i=1}^k \mathbb{R}^{n-1} \) centered at \( \xi \) of radius \( r \).

Let \( K_0 \simeq \text{SO}(n) \) be the maximal compact subgroup of \( \text{SO}^\circ(n,1) \) given as the stabilizer of \( o \in \mathbb{H}^n \). Let \( M_0 := \text{Stab}(v_o) \). We then have the following identification: \( \text{SO}^\circ(n,1)/K_0 = \mathbb{H}^n \) and \( \text{SO}^\circ(n,1)/M_0 = T^1 \mathbb{H}^n \). Let \( A_0 = \{ a_t : t \in \mathbb{R} \} < \text{SO}^\circ(n,1) \) denote the one-parameter subgroup of semisimple elements whose right translation action on \( \text{SO}^\circ(n,1)/M_0 \) corresponds to the geodesic flow on \( T^1 \mathbb{H}^n \). Set \( K = \prod K_0 < G, \ M = \prod M_0 < G, \) and \( A = \prod A_0 \). We also set \( A^+ = \prod A^+_0 \) where \( A^+_0 = \{ a_t : t \geq 0 \} \). Then \( X = G/K \).

Let \( \delta_{\min} \) denote the abscissa of convergence of the series \( \sum_{\lambda \in \Gamma} e^{-sd_{\min}(o, \gamma o)} \) where \( d_{\min} \) denotes the minimum of the Riemannian metrics on \( \mathbb{H}^n \times \cdots \times \mathbb{H}^n \).

**Theorem 5.1.** For any \( k \geq 1 \), we have

\[
\dim \Lambda = \delta_{\min} = \max_{1 \leq i \leq k} \dim \Lambda_{\rho_i}.
\]

In particular,

\[
\delta \leq \frac{n-1}{\sqrt{k}}.
\]

**Proof.** Since \( \sqrt{k} \cdot d_{\min} \leq d \), we have \( \sqrt{k} \cdot \delta \leq \delta_{\min} \). Therefore the second assertion follows from the first one.

In order to show (5.1), we use shadows. For \( R > 0 \) and \( x \in X \), the shadow \( O_R(o, x) \) is defined as

\[
O_R(o, x) = \{ \eta \in \mathcal{F} : \exists k \in K \text{ s.t. } k^+ = \eta, a \in A^+, \text{ and } d(kao, x) \leq R \}.
\]

For each \( N \in \mathbb{N} \), let \( \Lambda_N := \Lambda \cap \limsup_{\gamma \in \Gamma} O_N(o, \gamma o) \), that is,

\[
\Lambda_N = \{ \xi \in \Lambda : \exists \gamma_i \to \infty \text{ in } \Gamma \text{ such that } \xi \in O_N(o, \gamma_i o) \text{ for all } i \geq 1 \}.
\]

There exists a constant \( c_N > 0 \) such that for any \( \gamma \in \Gamma \), the shadow \( O_N(o, \gamma o) \) is contained in a ball \( B(\xi, c_N e^{-\min \mu(\gamma)}) \) for some \( \xi, \gamma \in \mathcal{F} \).
From Theorem 2.3 (1), $\gamma_i \to \infty$ if and only if $\min \mu(\gamma_i) \to \infty$. Hence, for any fixed $t > 0$, we have $$\Lambda_N \subseteq \bigcup_{\gamma \in \Gamma, \min \mu(\gamma) > t} O_N(\alpha, \gamma \alpha) \subseteq \bigcup_{\gamma \in \Gamma, \min \mu(\gamma) > t} B(\xi, \epsilon_N e^{-\min \mu(\gamma)}).$$

Let $s > \delta_{\min}$. Since $\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)} < \infty$,
$$\lim_{t \to \infty} \sum_{\gamma \in \Gamma, \min \mu(\gamma) > t} e^{-s \min \mu(\gamma)} = 0.$$ 

This implies that the $s$-dimensional Hausdorff measure of $\Lambda_N$ is equal to zero; so $\dim \Lambda_N \leq \delta_{\min}$. Since $\Lambda$ is equal to the conical limit set $\Lambda_c$ by (2.2), we have $\Lambda = \bigcup_{N \in \mathbb{N}} \Lambda_N$. Consequently,
$$\dim \Lambda \leq \sup_{N \in \mathbb{N}} \dim \Lambda_N \leq \delta_{\min}.$$ 

Since, for any $s > 0$,
$$\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)} \leq \sum_{i=1}^{k} \sum_{\sigma \in \Delta} e^{-s \mu(\rho_i(\sigma))},$$
the series $\sum_{\gamma \in \Gamma} e^{-s \min \mu(\gamma)}$ converges when $s > \max \delta_{\rho_i}$. It implies that $\delta_{\min} \leq \max_i \delta_{\rho_i} = \max_i \dim \Lambda_{\rho_i}$. Since the projection map $\Lambda \to \Lambda_{\rho_i}$ is Lipschitz, we have $\dim \Lambda \geq \max_i \dim \Lambda_{\rho_i}$. As a result, we have
$$\max_i \dim \Lambda_{\rho_i} \leq \dim \Lambda \leq \delta_{\min} \leq \max_i \dim \Lambda_{\rho_i},$$
which completes the proof. \hfill \Box

6. **FIBERED DYNAMICAL SYSTEMS AND $\ker \psi_u$-COORDINATE MAP**

We continue to use notation $M_0, K_0, M, K, A$ from the last section. For $[g] \in SO^0(n, 1)/M_0$, we denote by $g^+ \in S^{n-1}$ and $g^- \in S^{n-1}$ respectively the forward and backward endpoints of the geodesic determined by the tangent vector $[g] \in T^1 \mathbb{H}^n$. Now the map $[g] \to (g^+, g^-, \beta_{g^+}(o, go))$ gives an $SO^0(n, 1)$-equivariant homeomorphism between the space $SO^0(n, 1)/M_0$ and \{$(\xi, \eta) \in \mathbb{H}^n \times \mathbb{H}^n : \xi \neq \eta$\} \times $\mathbb{R}$, where the $SO^0(n, 1)$-action on the latter space is given by $g.(\xi, \eta, s) = (g\xi, g\eta, s + \beta_{\xi}(g^{-1}o, o))$. This homeomorphism is called the Hopf parametrization of $SO^0(n, 1)/M_0$ under which the right $A$-action on $SO^0(n, 1)/M_0$ corresponds to the translation flow on $\mathbb{R}$.

For $\xi \in \mathcal{F} = \prod_{i=1}^{k} S^{n-1}$, we write $\xi_i$ for its $i$-th component. We set $\mathcal{F}^{(2)} = \{(\xi, \eta) \in \mathcal{F} \times \mathcal{F} : \xi_i \neq \eta_i \text{ for all } i\}$. Then the Hopf parametrization of $SO^0(n, 1)/M_0$ extends to the Hopf-parametrization of $G/M$ componentwise, and gives the $G$-equivariant homeomorphism $G/M \simeq \mathcal{F}^{(2)} \times a$ given by
$$[g] \to (g^+, g^-, \beta_{g^+}(o, go)) \text{ where } g^\pm = (g_i^\pm).$$

Set $\Lambda^{(2)} = \mathcal{F}^{(2)} \cap (\Lambda \times \Lambda)$. Then $\Omega := \Gamma \setminus (\Lambda^{(2)} \times a)$ is identified with the closed subspace $\{[g] \in \Gamma \setminus G/M : g^\pm \in \Lambda\}$ of $\Gamma \setminus G/M$ via the Hopf parameterization.
Trivial ker $\psi_u$-vector bundle. We fix a unit vector $u \in \text{int} \mathcal{L}$ in the rest of this section. Consider the $\Gamma$-action on the space $\Lambda^{(2)} \times \mathbb{R}$ by

$$\gamma.(\xi, \eta, s) = (\gamma \xi, \gamma \eta, s + \psi_u(\beta_\xi(\gamma^{-1}o, o))).$$

The reparametrization theorems for Anosov groups ([6, Prop. 4.1], [9, Thm. 4.15]) imply that $\Gamma$ acts properly discontinuously and cocompactly on $\Lambda^{(2)} \times \mathbb{R}$. Hence $Z := \Gamma \setminus (\Lambda^{(2)} \times \mathbb{R})$ is a compact space. Now the $\Gamma$-equivariant projection $\Lambda^{(2)} \times \mathfrak{a} \rightarrow \Lambda^{(2)} \times \mathbb{R}$ given by $(\xi, \eta, v) \mapsto (\xi, \eta, \psi_u(v))$ induces an affine bundle with fiber ker($\psi_u$):

$$\pi: \Omega = \Gamma \setminus (\Lambda^{(2)} \times \mathfrak{a}) \rightarrow Z = \Gamma \setminus (\Lambda^{(2)} \times \mathbb{R}).$$

It is well known that such a bundle is indeed a trivial vector bundle, and hence we can choose a continuous global section $s: Z \rightarrow \Omega$ so that $\pi \circ s = \text{id}_Z$. Denote by $\{\tau_t : t \in \mathbb{R}\}$ the flow on $Z$ given by translations on $\mathbb{R}$. For $v = (v_1, \ldots, v_k) \in \mathfrak{a}$, we write $a_v = (a_{v_1}, \ldots, a_{v_k}) \in \mathcal{A}$.

**Definition 6.1** (ker $\psi_u$-coordinate map). We define the following continuous ker $\psi_u$-valued map:

$$\hat{K}_u: Z \times \mathbb{R} \rightarrow \ker \psi_u$$

defined as follows: for $z \in Z$ and $t \in \mathbb{R}$,

$$s(z) a_{tu} = s(z) \gamma_{z} a_{K_u(z,t)},$$

(6.1)

Fix a compact subset $D \subset G$ such that $s(Z) = \Gamma \setminus \Gamma D$, and for each $z \in Z$, write $s(z) = \Gamma \hat{s}(z)$ for some $\hat{s}(z) \in D$. Hence

$$\Lambda^{(2)} \times \mathfrak{a} = \Gamma \setminus D a_{\ker \psi_u}.$$  

(6.2)

**Lemma 6.2.** For any $g \in G$ with $g^\perp \in \Lambda$, there exist $z_g \in Z$ and $w_g \in \ker \psi_u$ such that for all $t \in \mathbb{R}$, there exists $\gamma_{g,t} \in \Gamma$ satisfying

$$\gamma_{g,t} a_{tu} = \hat{s}(z_g) a_{K_u(z_g,t)+w_g}.$$  

(6.3)

**Proof.** By (6.2), there exist $\gamma \in \Gamma$, $z \in Z$ and $w \in \ker \psi_u$ such that $\gamma g = \hat{s}(z) a_w$, and hence

$$\gamma a_{tu} = \hat{s}(z) a_{tu+w}.$$  

On the other hand, by (6.1), there exists $\gamma_{z,t} \in \Gamma$ such that

$$\gamma_{z,t} a_{tu} = \hat{s}(z) a_{K_u(z,t)}.$$  

Therefore,

$$\gamma_{z,t} \gamma a_{tu} = \hat{s}(z) a_{K_u(z,t)+w}.$$  

(6.4)

We also set $K_u^+(g, t) := K_u(z_g, t) + w_g \in \ker \psi_u$, so that for all $t \in \mathbb{R}$,

$$\gamma_{g,t} a_{tu} \in D a_{K_u^+(g, t)}.$$  

(6.4)
Lemma 6.3. For any \( g \in G \) with \( g^+ \in \Lambda \), we have
\[
Q(g) := \sup_{t > 0} \inf_{\gamma \in \Gamma} \psi_u(\mu(\gamma g a tu)) < \infty.
\]
Moreover, \( \sup_{g \in G, g^± \in \Lambda} Q(g) < \infty \).

Proof. If \( g, h \in G \) satisfy \( g^+ = h^+ \), then \( \sup_{t > 0} d(ga tu o, ha tu o) < \infty \). It follows that \( Q(g) < \infty \) if and only if \( Q(h) < \infty \). Therefore, by replacing \( g \) with \( h \in G \) satisfying \( g^+ = h^+ \) and \( h^± \in \Lambda \), we may assume without loss of generality that \( g^± \in \Lambda \). By (6.4), we have that for all \( t > 0 \),
\[
\gamma g, t ga tu \in D \cdot a \ker \psi_u,
\]
and hence
\[
\mu(\gamma g, t ga tu) \in \mu(D \cdot a \ker \psi_u) \subset C + \ker \psi_u,
\]
where \( C \) is a fixed compact subset of \( a \) depending only on \( D \). Therefore
\[
Q(g) = \sup_{t > 0} \psi_u(\mu(\gamma g, t ga tu)) \leq \max_{c \in C} \psi_u(c) < \infty.
\]
It is clear from the above proof that \( \sup_{g \in G, g^± \in \Lambda} Q(g) \leq \max_{c \in C} \psi_u(c) \). □

Theorem 6.4. For \( \nu_u \)-a.e. \( \xi \in \Lambda \), we have
\[
\lim_{t \to \infty} \frac{1}{t} \hat{K}_u(z, t) = 0
\]
whenever \( z = [\xi, \eta, s] \) for some \( \eta \in \Lambda \) and \( s \in \mathbb{R} \).

Proof. Let \( m_u \) denote the \( \psi_u \)-Bowen-Margulis-Sullivan measure on \( Z \); that is, \( m_u \) is the unique probability measure on \( Z \) which is locally equivalent to \( \nu_u \otimes \nu_u \otimes ds \). It follows from [6] that \( m_u \) is the measure of maximal entropy and in particular ergodic for the \( \tau_t \)-flow. Combining the reparametrization theorem [6, Prop. 4.1], and [36, Prop. 3.5], we deduce that there exists a Hölder continuous function \( F : Z \to \ker \psi_u \) with \( \int_Z F dm_u = 0 \) such that for all \( z \in Z \) and \( t \in \mathbb{R} \),
\[
\hat{K}_u(z, t) = \int_0^t F(z \tau_s) \, ds + E(z) - E(z \tau_t)
\]
for some bounded measurable function \( E : Z \to \ker \psi_u \). The Birkhoff ergodic theorem for the \( \tau_s \) flow on \((Z, m_u)\) implies that for \( m_u \)-almost all \( z \in Z \), we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t F(z \tau_s) \, ds = \int_Z F dm_u = 0;
\]
hence
\[
\lim_{t \to \infty} \frac{1}{t} \hat{K}_u(z, t) = 0
\]
since \( E \) is bounded. This proves the claim. □

Corollary 6.5. For \( \nu_u \)-a.e. \( \xi \in \Lambda \), we have the following: for any \( g \in G \) with \( g^+ = \xi \) and \( g^- \in \Lambda \),
\[
\lim_{t \to \infty} \frac{1}{t} \hat{K}_u^1(g, t) = 0.
\]
Proof. Let $z = [(\xi_1, \eta_1, s)] \in Z$ be such that $\gamma_{g,0}g = \tilde{s}(z)aK^1_{(g,0)}$ as given by (6.4). It follows that $g^+ = \xi \in \Gamma_\xi_1$. Hence if $\xi_1 \in \Lambda$ satisfies Theorem 6.4, so does $g^+ = \xi$. Since $\sup_{t>0} \|K^+_{(g,t)} - \hat{K}_u(z,t)\| < \infty$ by their definitions, the claim now follows from Theorem 6.4.

Let $m_{u}^{BMS}$ denote the Bowen-Margulis-Sullivan measure on $\Omega \subset \Gamma \setminus G$ given by $m_{u}^{BMS} = m_u \otimes \text{Leb}|_{\ker \psi_u}$; this is an $A$-invariant ergodic (infinite) Radon measure, as shown in [19]. We also remark that by [8], $m_{u}^{BMS}$ is $\{a_{tu}: t \in \mathbb{R}\}$-ergodic if and only if $k \leq 3$. In terms of this measure, Corollary 6.5 can be formulated as the following which may be regarded as an analogue of Sullivan’s result [37, Coro. 19], which predates his logarithm law.

**Theorem 6.6.** For $m_{u}^{BMS}$-a.e. $x \in \Gamma \setminus G$, we have

$$\lim_{t \to \infty} \frac{d(xa_{tu}o, o)}{t} = 0.$$  

Since $D$ is compact, this theorem follows from Corollary 6.5 in view of (6.4).

## 7. Hausdorff Dimension of $\Lambda_u$ and Local Behavior of $\nu_u$

### The directional conical limit sets $\Lambda_u$.

For each $u = (u_1, \cdots, u_k) \in a^+$, the $u$-directional conical limit set $\Lambda_u \subset \Lambda_\xi$ is defined as

$$\Lambda_u := \{\xi \in F : \liminf_{t \to +\infty} d(\xi(u_1t, \cdots, u_kt), \Gamma o) < \infty\}.$$  

In this section, we obtain estimates on $\dim \Lambda_u$ and local estimates on $\nu_u$ for $u \in \text{int} \mathcal{L}$. As in section 5, we use the upper half-space model of $\mathbb{H}^n$ so that $\partial \mathbb{H}^n = \mathbb{R}^{n-1} \cup \{\infty\}$. For $r > 0$, let $B(r)$ denote the ball in $\mathbb{R}^{n-1}$ centered at 0 of radius $r$. Note that for $w = (w_1, \cdots, w_k) \in a$, the set $\{a_{tw}v_o = (a_{w_1tv_o}, \cdots, a_{w_ktv_o}) : t > 0\}$ represents the geodesic ray emanating from $(0, \cdots, o)$ pointing toward $(0, \cdots, 0) \in \prod_{i=1}^k \mathbb{R}^{n-1}$, with the speed controlled by $w$. By abuse of notation, we write $d(o, a_{tw}v_o)$ for the distance between $o$ and the basepoint of $a_{tw}v_o$ in $X$.

In the whole section, we fix a unit vector with respect to the Euclidean norm

$$u = (u_1, \cdots, u_k) \in \text{int} \mathcal{L}.$$  

However, note that statements below still hold for an arbitrary vector in $\text{int} \mathcal{L}$ since all quantities are homogeneous. We also set $M_u = \max_{1 \leq i \leq k} u_i$, $m_u = \min_{1 \leq i \leq k} u_i$ and $\delta_u := \psi_u(u) = \psi_T(u) > 0$.

### Upper bound for dimension.

**Theorem 7.1.** For any $k \geq 1$, we have

$$\dim \Lambda_u \leq \frac{\delta_u}{m_u}. \quad (7.1)$$
Proof. For any $N \in \mathbb{N}$, set

$$\Gamma_N(u) := \{ \gamma \in \Gamma : \| \mu(\gamma) - t_\gamma u \| \leq N \text{ for some } t_\gamma > 0 \}$$

and

$$\Lambda^*_N(u) := \limsup_{t \to \infty} \{ O_N(o, \gamma) : \gamma \in \Gamma_N(u), \| \mu(\gamma) \| \geq t \},$$

where $O_N(o, \gamma)$ is defined as in (5.2).

There exists $d_N > 0$ such that for any $\gamma \in \Gamma_N(u)$, the shadow $O_N(o, \gamma)$ is contained in a ball $B(\xi_\gamma, d_N e^{-t_\gamma m_u})$ for some $\xi_\gamma \in F$. Since $\| \mu(\gamma) - t_\gamma u \| \leq N$, by applying $\psi_u$, we get

$$|\psi_u(\mu(\gamma)) \delta_u^{-1} - t_\gamma| \leq N \delta_u^{-1} \| \psi_u \|_{op}$$

where $\| \cdot \|_{op}$ denotes the operator norm of $\psi_u$. Fix $N \in \mathbb{N}$. For any $t > 1$,

$$\Lambda^*_N(u) \subset \bigcup_{\gamma \in \Gamma_N(u), \| \mu(\gamma) \| \geq t} O_N(o, \gamma) \subset \bigcup_{\gamma \in \Gamma_N(u), \| \mu(\gamma) \| \geq t} B(\xi_\gamma, d_N e^{-t_\gamma m_u})$$

$$\subset \bigcup_{\gamma \in \Gamma, \| \mu(\gamma) \| \geq t} B\left(\xi_\gamma, d'_N e^{-m_u \delta_u^{-1} \psi_u(\mu(\gamma))} \right)$$

for some constant $d'_N \geq 1$. On the other hand, by Theorem 2.3(3), for any $s > \delta_u/m_u$, we have

$$\lim_{t \to \infty} \sum_{\gamma \in \Gamma, \| \mu(\gamma) \| \geq t} e^{-sm_u \delta_u^{-1} \psi_u(\mu(\gamma))} = 0.$$ 

It follows that the $s$-dimensional Hausdorff measure of $\Lambda^*_N(u)$ is zero. Hence

$$\dim \Lambda^*_N(u) \leq \delta_u/m_u.$$ 

Since

$$\Lambda_u \subset \bigcup_{N \in \mathbb{N}} \Lambda^*_N(u),$$

it follows that $\dim \Lambda_u \leq \delta_u/m_u$. \qed

Remark 7.2. In the case when $m_u(n-1)(k-1) < \delta_u$, the upper bound in (7.1) can be improved to

$$\delta_u + (M_u - m_u)(n-1)(k-1)$$

by replacing $B(\xi_\gamma, d_N e^{-t_\gamma m_u})$ with

$$e^{(M_u - m_u)t_\gamma} \text{ number of balls of radius } d_N e^{-t_\gamma M_u}.$$ 

By Theorems 7.1 and 3.3, we deduce:

Corollary 7.3. 

$$\dim \Lambda_u \leq \frac{\min_{1 \leq i \leq k} (u_i \dim \Lambda_{p_i})}{\min_{1 \leq i \leq k} u_i}$$

Lower bound for dimension.

Lemma 7.4. [18, Lem. 10.6] There exists a compact subset $S \subset G$ such that for any $\eta \in \Lambda$, there exists $g \in S$ such that $g^+ = \eta$ and $g^- = \Lambda$. 

Theorem 7.5. For any $k \geq 1$, there exists $C > 0$, depending only on $\psi_u$, such that for any $\xi = (\xi_1, \ldots, \xi_k) \in \Lambda_u$ and for all $t > 0$,

$$\nu_u \left( \prod_{i=1}^{k} B(\xi_i, e^{-u_i t}) \right) \leq Ce^{-\delta_u t}.$$ 

Proof. Without loss of generality, we may assume $\xi = (0, \ldots, 0) \in \Lambda_u$ by translating $\xi$ to 0. By Lemma 7.4, we can choose $g \in S$ so that $g^+ = \xi$ and $g^- \in \Lambda$. By Lemma 6.3, there exists $c > 0$ (depending only on $\Gamma$) such that for any $t > 0$, there exists $\gamma_{g,t} \in \Gamma$ such that $|\psi_u(\mu(\gamma_{g,t}ga_{tu}))| \leq c$.

Let $R_t$ denote the semiball centered at 0 of radius $e^{-t}$. Using the explicit construction of $\nu_u = \lim_{s \to 1} \nu_{u,s}$ with $\nu_{u,s}$ given in (2.5), we get

$$\nu_u \left( \prod_{i=1}^{k} B(e^{-u_i t}) \right) = \lim_{s \to 1} \frac{\sum_{\gamma \in \Gamma, \gamma_0 \in \prod R_{u_i t}} e^{-s \psi_u(\mu(\gamma))}}{\sum_{\gamma \in \Gamma} e^{-s \psi_u(\mu(\gamma))}}. \quad (7.2)$$

Let $\gamma_0 \in \prod R_{u_i t}$ (such $\gamma$ exists since $\xi \in \Lambda_u$) and write $\gamma = (\rho_1(\sigma), \ldots, \rho_k(\sigma))$ for $\sigma \in \Delta$. By the hyperbolic geometry on $\mathbb{H}^n$, there exists a constant $C > 0$ (depending only on $\Gamma$) such that for all $1 \leq i \leq k$ and $t > 1$,

$$u_i t + d(ga_{tu_i}, v_0, \rho_i(\sigma) o) - C \leq d(\rho_i(\sigma) o, o) \leq u_i t + d(ga_{tu_i}, v_0, \rho_i(\sigma) o). \quad (7.3)$$

This implies that $ut + \mu(a_{tu_i}^{-1} g^{-1} \gamma) - \mu(\gamma) \in Q_0$ for some compact subset $Q_0 \subset a$. Therefore

$$c_0 := \sup_{t > 0} ||tu + \mu(\gamma_{g,t} \gamma) - \mu(\gamma_{g,t}ga_{tu}) - \mu(\gamma)|| < \infty.$$ 

By applying $\psi_u$ and using $|\psi_u(\gamma_{g,t}ga_{tu})| \leq c$, we then get

$$c_1 := \sup_{t > 0} |\delta_u t + \psi_u(\mu(\gamma_{g,t} \gamma)) - \psi_u(\mu(\gamma))| < \infty,$$

where $c_1$ depends only on $S$ and $\psi_u$.

Hence for all $t > 0$ and $s > 1$,

$$\sum_{\gamma \in \Gamma, \gamma_0 \in \prod R_{u_i t}} e^{-s \psi_u(\mu(\gamma))} \leq Ce^{c_1} e^{-s \delta_u t} \sum_{\gamma \in \Gamma} e^{-s \psi_u(\mu(\gamma))}.$$ 

By (7.2), it follows that for all $t > 0$,

$$\nu_u \left( \prod_{i=1}^{k} B(e^{-u_i t}) \right) \leq e^{c_1} e^{-\delta_u t}.$$ 

This finishes the proof. \qed

To deduce a lower bound on $\dim \Lambda_u$ from the above estimate, we use:

Theorem 7.6. [8, Thm. 1.6] Let $k \leq 3$. For any $u \in \text{int} \mathcal{L}$, we have $\nu_u(\Lambda_u) = 1$.

Corollary 7.7. For $k \leq 3$, we have

$$\dim \Lambda_u \geq \frac{\delta_u}{M_u}.$$
Proof. By Theorem 7.5, there exists $C > 0$ such that for all $0 < r < 1$,
\[ \nu_u(B_{\max}(\xi, r)) \leq C \delta_u/M_u \]
where $B_{\max}(\xi, r)$ is a radius $r > 0$ ball centered at $\xi$ in $F = \prod_{i=1}^k S^{n-1}$ with respect to the maximum metric. Since $k \leq 3$, we have $\nu_u(\Lambda_u) = 1$ by Theorem 7.6. Since the Riemannian metric and the maximum metric on $F$ are Lipschitz equivalent to each other, the well-known mass distribution principle (cf. [3, Lemma 1.2.8]) now implies that
\[ \dim \Lambda_u \geq \frac{\delta_u}{M_u}. \]

Lower bound on the local size of $\nu_u$. Let $\Lambda_u^*$ be a subset of $\Lambda_u$ defined as follows: $\xi \in \Lambda_u^*$ if and only if for any $g \in G$ with $g^+ = \xi$ and $g^- \in \Lambda$,
\[ \limsup_{t \to \infty} \frac{1}{t} K_{u}^*(g, t) = 0. \tag{7.4} \]

By Corollary 6.5, $\nu_u(\Lambda_u^*) = 1$.

**Theorem 7.8.** Let $k \geq 1$. There exists $C > 0$ such that for any $\xi = (\xi_1, \cdots, \xi_k) \in \Lambda_u^*$, and for any sufficiently small $\epsilon > 0$, there exists $t_0 = t_{\epsilon, \xi} > 0$ such that for all $t \geq t_0$,
\[ C \cdot e^{-\delta_u(1+\epsilon)t} \leq \nu_u \left( \prod_{i=1}^k B(\xi_i, e^{-u_i t}) \right). \]

Proof. Without loss of generality, we may assume $\xi = 0$ and choose $g \in S$ such that $g^+ = \xi = 0$ and $g^- \in \Lambda$ where $S$ is a compact subset of $G$ given in Lemma 7.4. Let $\epsilon > 0$. Since $0 = g^+ \in \Lambda_u^*$, there exists $t_0 = t_{\epsilon, g} > 0$ such that for each $1 \leq i \leq k$, absolute of the $i$-th component of $K_{u}^*(g, t) \in a = \mathbb{R}^k$ is at most $\frac{\delta_u}{t}$ for all $t > t_0$.

Recall the definition of $\gamma_{g,t}$ from (6.4): $\gamma_{g,t} \in \Delta_{a_K^*(g,t)}$ where $d_i \in D$. Therefore $\gamma_{g,t}^{-1} = g a_{t_-^{-1}} K_{u}^*(g, t) d_i^{-1}$.

Let $q$ be the diameter of $D^{-1}o$.

Note that there exists $c_0 > 0$ such that for all $t > t_0$,
\[ O_1(o, \gamma_{g,t}^{-1} o) \subset O_{q+1}(o, g a_{t_-^{-1}} K_{u}^*(g, t) v_o) \subset \prod_{i=1}^k B(c_0 e^{-u_i(1-\epsilon/4)t}). \]

On the other hand, the shadow lemma (cf. [18, Lem. 7.8]) says that for any $R > 0$, there exists $c = c(\psi_u, R) \geq 1$ such that for any $\gamma \in \Gamma$,
\[ c^{-1} e^{-\psi_u(\mu(\gamma))} \leq \nu_u(O_R(o, \gamma o)) \leq c e^{-\psi_u(\mu(\gamma))}. \]

Hence we deduce that for all $t > \max(t_0, 2 \log c_0/(u_i \epsilon))$,
\[ \beta \cdot e^{-\delta_u t} \leq \nu_u \left( \prod_{i=1}^k B(\xi_i, c_0 e^{-u_i(1-\epsilon/4)t}) \right) \leq \nu_u \left( \prod_{i=1}^k B(\xi_i, e^{-u_i(1-\epsilon/2)t}) \right). \]
for some constant $\beta = \beta(\psi_u, D) > 0$. By reparameterizing $(1 - \varepsilon/2)t = s$, this implies the claim. \qed

**Critical exponents $\delta_{\text{max}}$ and $\delta_{\text{min}}$.** Let $\delta_{\text{max}}$ denote the abscissa of convergence of the series $\sum_{\gamma \in \Gamma} e^{-s d_{\text{max}}(o, \gamma o)}$ where $d_{\text{max}}$ denotes the maximum of the Riemannian metrics on $\mathbb{H}^n \times \cdots \times \mathbb{H}^n$. Since $d \geq d_{\text{max}}$ and $\sqrt{k} d_{\text{min}} \leq d \leq \sqrt{k} d_{\text{max}}$, we have $\delta \leq \delta_{\text{max}} \leq \sqrt{k} \delta \leq \delta_{\text{min}}$. Then by [31, Lemma 3.1.1], we have

$$\delta_{\text{\spadesuit}} = \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \mu(\gamma) < T \}}{T}$$

(7.5)

where $\spadesuit = \text{max}$ or $\min$ and $\text{max} \mu(\gamma)$ and $\text{min} \mu(\gamma)$ denote respectively the maximum and minimum of the entries of the vector $\mu(\gamma) \in \mathbb{a}^+$. Although [31, Lemma 3.1.1] is stated only when $\spadesuit$ is a norm, the same proof works for the min function.

**Lemma 7.9.** We have

$$\delta_{\text{max}} = \sup_{u \in \mathcal{L}} \frac{\delta_u}{M_u} \quad \text{and} \quad \delta_{\text{min}} \geq \sup_{u \in \mathcal{L}} \frac{\delta_u}{m_u}.$$  

Moreover, there exists a unique vector $w_T \in \mathcal{L}$ such that $\|w_T\|_{\text{max}} = 1$ and $\delta_{\text{max}} = \psi_T(w_T)$, and thus $\delta_{\text{max}} > \delta$.

**Proof.** Note that

$$\sup_{u \in \mathcal{L}} \frac{\delta_u}{m_u} = \sup_{u \in \mathbb{a}^+, \|u\|_{\text{max}} = 1} \psi_T(u),$$

and this is equal to $\delta_{\text{max}}$ by [31, Cor. 3.14]. By the strict concavity of $\psi_T$ on int $\mathcal{L}$ (see Theorem 2.1), there exists a unique $w_T \in \mathcal{L}$ as claimed in the statement. Since $\mathcal{L} \subset \text{int} \mathbb{a}^+$ by Theorem 2.1(1), for any $v \in \mathcal{L}$, we have $\|v\| < \|v\|_{\text{max}}$ from which $\delta_{\text{max}} > \delta$ follows.

To show the second inequality, let $u \in \mathcal{L} \subset \text{int} \mathbb{a}^+$. For $\varepsilon > 0$, set $C_\varepsilon = \{v = (v_i) \in \text{int} \mathbb{a}^+ : \max_{1 \leq i \leq k} \|v - u\|_i < \varepsilon\}$. For all sufficiently small $\varepsilon > 0$, if $v = (v_i) \in C_\varepsilon$ and $\min_i v_i = v_j$, then $m_u = u_j$. It follows that if $\mu(\gamma) \in C_\varepsilon$ and $\|\mu(\gamma)\| < T/m_u$, then $\min \mu(\gamma) \leq T/(1 - \varepsilon m_u)$. Hence we deduce from the definition of $\psi_T(u)$ that for all small $\varepsilon > 0$,

$$\delta_u/m_u \leq \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \mu(\gamma) \in C_\varepsilon, \|\mu(\gamma)\| < T/m_u \}}{T} \leq \limsup_{T \to \infty} \frac{\log \# \{ \gamma \in \Gamma : \min \mu(\gamma) < T/(1 - \varepsilon m_u) \}}{T} = \delta_{\text{min}}/(1 - \varepsilon m_u)$$

by (7.5). As $\varepsilon > 0$ is arbitrary, we get $\delta_{\text{min}} \geq \delta_u/m_u$ for all $u \in \mathcal{L}$. \qed
8. Examples of symmetric growth indicator functions

Let $\Delta$ be a Gromov hyperbolic group, and $\text{Out} \Delta$ denote its outer automorphism group, i.e., the group of automorphisms of $\Delta$ modulo the inner automorphisms. Note that, for a representation $\rho : \Delta \to \text{SO}^\circ(n,1)$ and $\iota \in \text{Out} \Delta$, $\rho \circ \iota$ is well-defined up to conjugation in $\text{SO}^\circ(n,1)$.

**Lemma 8.1.** Let $k \geq 2$. Let $\rho_1 : \Delta \to \text{SO}^\circ(n,1)$ be a non-elementary convex cocompact representation and $\iota \in \text{Out} \Delta$ of order $k$. Let $\rho_i = \rho_1 \circ \iota^{i-1}$ for $2 \leq i \leq k$ and let $\Gamma_i := (\prod_{i=1}^k \rho_i)(\Delta)$. Then

$$\psi_{\Gamma_i} = \psi_{\Gamma_i} \circ \theta$$

and

$$u_{\Gamma_i} = \frac{1}{\sqrt{k}}(1, \ldots, 1)$$

where $\theta$ denotes the cyclic permutation $(x_1, \ldots, x_k) \mapsto (x_2, \ldots, x_k, x_1)$.

**Proof.** For each $1 \leq n \leq k$, let $\Gamma^{(n)} = (\prod_{i=1}^k \rho_i \circ \iota^n)(\Delta)$. Since $\iota^k = 1$ in $\text{Out} \Delta$, $\Gamma^{(n)}$ can be regarded as a group obtained by permuting coordinates in a cyclic way. Hence,

$$\mathcal{L}_{\Gamma^{(n)}} = \theta(\mathcal{L}_{\Gamma^{(n-1)}}) \quad \text{and} \quad \psi_{\Gamma^{(n)}} = \psi_{\Gamma^{(n-1)}} \circ \theta^{-1}. \quad (8.1)$$

However, $\Gamma^{(n)} = \Gamma_i$ for all $n$; since applying an automorphism to all coordinates does not change the group. Hence, (8.1) implies that $\mathcal{L}_{\Gamma_i}$ and $\psi_{\Gamma_i}$ are invariant under the cyclic permutation $\theta$ of coordinates. $\square$

**Examples in $\mathbb{H}^2 \times \mathbb{H}^2$.** Let us describe some examples to which Lemma 8.1 can be applied. We begin in dimension 2. For a closed surface $S$ of genus $g \geq 2$, one can obtain homeomorphisms $\iota : S \to S$ of order 2 in a number of ways. Figure 5 indicates how this can be done: Arrange the surface in $\mathbb{R}^3$ so that it is symmetric by a 180° rotation. There are several possibilities distinguished by the number of intersection points of the surface with the rotation axis, which yield fixed points of $\iota$.

![Figure 5. Examples of involutions $\iota \in \text{Out} \pi_1(S_3)$. Indicated curves are mapped to each other by $\iota$.](image)

In order for the example $(\rho, \rho \circ \iota)$ not to be trivial, we need the groups not to be conjugate in $\text{SO}^\circ(2,1)$. That is, $\rho$ should not represent a point of Teichmüller space $\mathcal{T}(S)$ which is fixed by $\iota$. This is always possible when $g \geq 3$; to see this, note that there are disjoint, non-homotopic simple closed
curves exchanged by $\iota$ in each case. They can be assigned different lengths by a hyperbolic structure, which would then not be fixed by $\iota$. In genus 2, one just needs to avoid the hyperelliptic involution – the one with 6 fixed points – which fixes every point in $\mathcal{T}(S)$. All other rotations will do.

Examples in $\mathbb{H}^3 \times \mathbb{H}^3$. Examples involving 3-manifolds are also plentiful. Consider for example a “book of $I$-bundles” constructed as follows (see Anderson-Canary [1]). Let $S_1, \ldots, S_\ell$ be $\ell$ copies of a surface of genus $g \geq 1$ with one boundary component and let $Y$ be the 2-complex obtained by identifying all the boundary circles to one. A choice of cyclic order $c$ on the $\ell$ surfaces determines a thickening of $Y$ to a 3-manifold $N_c$: form $S_i \times [-1,1]$ for each $i$, and identify the annulus $\partial S_i \times [0,1]$ with $\partial S_j \times [-1,0]$ whenever $j$ follows $i$ in the order $c$ (the identification should take $[0,1] \to [-1,0]$ by an orientation-reversing homeomorphism, and should respect the original identification of the boundary circles). See Figure 6.

![Figure 6. Book of $I$-bundles with three surfaces and patterns indicating the identification.](image)

The result $N_c$ is homotopy equivalent to $Y$, and has $\ell$ boundary components of genus $2g$. It admits many convex cocompact hyperbolic structures: it is easy to construct one “by hand” by attaching Fuchsian structures along the common boundary using the Klein-Maskit combination theorem [22]. The Ahlfors-Bers theory parametrizes all convex cocompact representations as the Teichmüller space of $\partial N_c$ (cf. [21]). A permutation of $(1, \ldots, \ell)$ induces a homeomorphism of $Y$ which extends to a homotopy equivalence of $N_c$ which, if the permutation does not preserve or reverse the cyclic order, will not correspond to a homeomorphism. Selecting such a permutation of order 2, we have an automorphism that cannot be an isometry for any hyperbolic structure on $N_c$. (Even if it does correspond to a homeomorphism one can choose the hyperbolic structure on $N_c$ using a point in $\mathcal{T}(\partial N_c)$ that is not symmetric with respect to the involution).

References


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