TENT PROPERTY OF THE GROWTH INDICATOR FUNCTIONS AND APPLICATIONS

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Abstract. Let $\Gamma$ be a Zariski dense discrete subgroup of a connected semisimple real algebraic group $G$. Let $k = \text{rank } G$. Let $\psi_T : \mathfrak{a} \rightarrow \mathbb{R} \cup \{-\infty\}$ be the growth indicator function of $\Gamma$, first introduced by Quint. In this paper, we obtain the following pointwise bound of $\psi_T$:

for all $v \in \mathfrak{a}$,

$$\psi_T(v) \leq \min_{1 \leq i \leq k} \delta_{\alpha_i}(v)$$ (0.1)

where $\Delta = \{\alpha_1, \cdots, \alpha_k\}$ is the set of all simple roots of $(g, \mathfrak{a})$ and $0 < \delta_{\alpha_i} \leq \infty$ is the critical exponent of $\Gamma$ associated to $\alpha_i$. When $\Gamma$ is $\Delta$-Anosov, there are precisely $k$-number of directions where the equality is achieved in (0.1), and the following strict inequality holds for $k \geq 2$:

for all $v \in \mathfrak{a} - \{0\}$,

$$\psi_T(v) < \frac{1}{k} \sum_{i=1}^{k} \delta_{\alpha_i}(v).$$

We discuss applications for self-joinings of convex cocompact subgroups in $\prod_{i=1}^{k} \text{SO}(n_i, 1)$ and Hitchin subgroups of $\text{PSL}(d, \mathbb{R})$.

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1. Introduction

Let $G$ be a connected semisimple real algebraic group. We let $P = MAN$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition, where $A$ is a maximal real split torus of $G$, $M$ is the maximal compact subgroup centralizing $A$ and $N$ is the unipotent radical of $P$. Let $\mathfrak{g} = \text{Lie } G$, $\mathfrak{a} = \text{Lie } A$ and $\mathfrak{a}^+$ denote the positive Weyl chamber so that $\log N$ consists of positive root subspaces. Let $K$ be a maximal compact subgroup so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Let $\mu : G \rightarrow \mathfrak{a}^+$ denote the Cartan projection map defined by the condition $\exp \mu(g) \in KgK$ for

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all \( g \in G \). Let \( \Gamma < G \) be a Zariski dense discrete subgroup. We denote by \( L \subset \mathfrak{a}^+ \) the limit cone of \( \Gamma \), which is the asymptotic cone of \( \mu(\Gamma) \). It is a convex cone with non-empty interior [1].

Following Quint [23], the growth indicator function \( \psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\} \) is defined as follows: choose any norm \( \| \cdot \| \) on \( \mathfrak{a} \). For an open cone \( \mathcal{C} \) in \( \mathfrak{a} \), let \( \tau_\mathcal{C} \) denote the abscissa of convergence of \( \sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}} e^{-s\|\mu(\gamma)\|} \) (that is, the infimum of the set of \( s \) for which the series converges). Now for any non-zero \( v \in \mathfrak{a} \), let

\[
\psi_\Gamma(v) := \|v\| \inf_{v \in \mathcal{C}} \tau_\mathcal{C}
\]

(1.1)

where the infimum is over all open cones \( \mathcal{C} \) containing \( v \), and let \( \psi_\Gamma(0) = 0 \). The definition of \( \psi_\Gamma \) does not depend on the choice of a norm on \( \mathfrak{a} \). Note that \( \psi_\Gamma = -\infty \) outside \( L \). Quint showed that \( \psi_\Gamma \) is a concave upper-semi continuous function satisfying \( L = \{ \psi_\Gamma \geq 0 \} \) and \( \psi_\Gamma > 0 \) on the interior \( \text{int} \ L \).

The main aim of this paper is to present a pointwise bound for the growth indicator function together with some applications. Throughout the paper, for any non-negative function \( f \) on \( \mathfrak{a}^+ \), we denote by

\[
0 \leq \delta_{\Gamma,f} \leq \infty
\]

or simply, \( \delta_f \), the critical exponent of \( \Gamma \) with respect to \( f \), that is, the abscissa of convergence of the series \( \sum_{\gamma \in \Gamma} e^{-sf(\mu(\gamma))} \).

Let

\[
\Delta = \{ \alpha_1, \cdots, \alpha_k \}
\]

denote the set of simple roots for \( (\mathfrak{g}, \mathfrak{a}^+) \).

**Definition 1.1 (Tent function).** Let \( \Gamma < G \) be a Zariski dense discrete subgroup with \( \delta_{\Gamma,\alpha_i} < \infty \) for some \( 1 \leq i \leq k \). We define a tent function \( T_\Gamma : \mathfrak{a} \to [0, \infty) \) by

\[
T_\Gamma(v) := \min_{1 \leq i \leq k} \delta_{\Gamma,\alpha_i} \cdot \alpha_i(v).
\]

**Figure 1. Tent on the limit cone**

We obtain the following tent property of the growth indicator function:
Theorem 1.2 (Tent property). For any Zariski dense discrete subgroup $\Gamma < G$ such that $\min_{1 \leq i \leq k} \delta_{\Gamma, \alpha_i} < \infty$, we have

$$\psi_T(v) \leq T_G(v) \quad \text{for all } v \in a.$$ 

Moreover, when $\delta_{\Gamma, \alpha_i} < \infty$, there exists $v_i \in L - \{0\}$ such that $\psi_T(v_i) = T_G(v_i)$.

Remark 1.3. (1) Denote by $\pi_G$ the half-sum of all positive roots of $(g, a^+)$ counted with multiplicity. Then for any discrete subgroup $\Gamma < G$, we have $\psi_T \leq 2\pi_G$ [23, Thm. IV.2.2].

(2) If $G$ has property (T) and $\Gamma$ is of infinite co-volume, then $\psi_T \leq 2\pi_G - \Theta$ where $\Theta$ is the sum of a maximal strongly orthogonal system ([24], [19], see also [17, Thm. 7.1]). Our bound in Theorem 1.2 provides a sharper bound for Hitchin subgroups; see Remark 3.5.

For a non-empty subset $\theta \subset \Delta$, a finitely generated subgroup $\Gamma < G$ is called a $\theta$-Anosov subgroup if there exist constants $C, C' > 0$ such that for all $\gamma \in \Gamma$ and all $\alpha_i \in \theta$,

$$\alpha_i(\mu(\gamma)) \geq C|\gamma| - C'$$

where $|\gamma|$ denotes the word length of $\gamma$ with respect to a fixed finite symmetric set of generators of $\Gamma$. The notion of Anosov subgroups was first introduced by Labourie for surface groups [15], and was extended to general word hyperbolic groups by Guichard-Wienhard [14]. Several equivalent characterizations have been established, one of which is the above definition (see [9], [10], [11], [12]). Anosov subgroups are regarded as natural generalizations of convex cocompact subgroups of rank one groups.

For a $\theta$-Anosov subgroup $\Gamma < G$, it follows from (1.2) that for some constant $C > 0$,

$$\max_{\alpha_i \in \theta} \delta_{\Gamma, \alpha_i} \leq C \log \#S < \infty$$

where $S$ is a fixed finite generating set of $\Gamma$. Therefore Theorem 1.2 applies to any Zariski dense subgroup contained in some $\theta$-Anosov subgroup of $G$.

For $\Delta$-Anosov subgroups, we obtain the following sharper result:

Theorem 1.4. Let $\Gamma$ be a Zariski dense $\Delta$-Anosov subgroup of $G$. The following hold:

1. For each $1 \leq i \leq k$, there exists a unique $v_i \in \text{int } L$ such that $\alpha_i(v_i) = 1$ and $\psi_T(v_i) = \delta_{\Gamma, \alpha_i}$.
2. For $v \in a - \{0\}$, we have $\psi_T(v) \leq T_G(v)$ where equality holds if and only if $v = cv_i$ for some $1 \leq i \leq k$ and $c > 0$.
3. If $k = \text{rank } G \geq 2$, then

$$\psi_T \leq \frac{1}{k} \sum_{i=1}^k \delta_{\Gamma, \alpha_i} \alpha_i.$$
When $\Gamma$ is $\Delta$-Anosov, $\psi_T$ is strictly concave in $\text{int} \mathcal{L}$ by [26 Thm. A, 21 Prop. 4.11]. Therefore by the convexity of the unit norm ball $\{\|v\| \leq 1\}$, there exists a unique unit vector $u_{\Gamma,\|\|} \in \mathfrak{a}^+$, called the direction of maximal growth, such that $\psi_T(u_{\Gamma,\|\|}) = \max_{\|v\|=1} \psi_T(v)$. By [23 Coro. III.1.4], we have

$$\delta_{\Gamma,\|\|} = \psi_T(u_{\Gamma,\|\|}). \quad (1.3)$$

**Corollary 1.5.** Let $k = \text{rank} G \geq 2$. Let $\Gamma$ be a Zariski dense $\Delta$-Anosov subgroup of $G$. For any norm $\|\cdot\|$ on $a$ induced from an inner product, we have

$$\delta_{\Gamma,\|\|} < \min_{1 \leq i \leq k} \delta_{\Gamma,\alpha_i} \cdot \alpha_i(u_{\Gamma,\|\|}).$$

In view of the above discussion, any upper bound on $\delta_{\Gamma,\alpha_i}$ for any $\alpha_i \in \Delta$ provides an explicit pointwise upper bound on $\psi_T$. We discuss some examples of $\Delta$-Anosov subgroups.

**Self-joinings of hyperbolic manifolds.** For $1 \leq i \leq k$, consider the hyperbolic space $(\mathbb{H}^{n_i}, d_i)$, $n_i \geq 2$, with constant sectional curvature $-1$, and let $G_i = \text{SO}^+(n_i, 1) = \text{Isom}^+(\mathbb{H}^{n_i})$. Let $G = \prod_{i=1}^k G_i$. Denote by $\alpha_i$ the simple root of $\mathfrak{g}_i = \text{Lie} G_i$. Then $\Delta = \{\alpha_1, \cdots, \alpha_k\}$ is the set of simple roots of $\mathfrak{g}$. Via the map $v \mapsto (\alpha_1(v), \cdots, \alpha_k(v))$, we may identify $a = \mathbb{R}^k$ and $a^+ = \{(v_1, \cdots, v_k) \in \mathbb{R}^k : v_i \geq 0 \text{ for all } i\}$.

Let $\Sigma$ be a countable group and $\rho_i : \Sigma \to G_i$ be a faithful convex cocompact representation with Zariski dense image for each $1 \leq i \leq k$. Setting $\rho = (\rho_1, \cdots, \rho_k)$, the self-joining $\Gamma_{\rho}$ is defined as the following subgroup of $G$:

$$\Gamma_{\rho} = \left( \prod_{i=1}^k \rho_i \right) (\Sigma) = \{(\rho_1(\sigma), \cdots, \rho_k(\sigma)) \in G : \sigma \in \Sigma\}. \quad (1.4)$$

We also assume that no two of $\rho_i$’s are conjugate, so that $\Gamma_{\rho}$ is a Zariski dense discrete subgroup of $G$. The hypothesis on $\rho_i$’s implies that $\Gamma_{\rho}$ is a $\Delta$-Anosov subgroup of $G$ (cf. [14 Thm. 5.15]).

Fix $\alpha_i \in \mathbb{H}^{n_i}$. For each $1 \leq i \leq k$, denote by $0 < \delta_{\rho_i} \leq \infty$ the critical exponent of $\rho_i(\Sigma)$, that is, the abscissa of convergence of the series $\sum_{\sigma \in \Sigma} e^{-sd_i(\rho_i(\sigma)\alpha_i)}$. We also denote by $\Lambda_{\rho_i} \subset \mathbb{S}^{n_i-1}$ the limit set of $\rho_i(\Sigma)$, which is the set of accumulation points of $\rho_i(\Sigma)\alpha_i$ in the compactification $\mathbb{H}^{n_i} \cup \mathbb{S}^{n_i-1}$. These two notions are independent of the choice of $\alpha_i \in \mathbb{H}^{n_i}$. By Patterson [20] and Sullivan [27], we have

$$\delta_{\rho_i} = \dim \Lambda_{\rho_i} \quad (1.5)$$

where $\dim \Lambda_{\rho_i}$ is the Hausdorff dimension of $\Lambda_{\rho_i}$ with respect to the spherical metric $d_{\mathbb{S}^{n_i-1}}$. We deduce from Theorem [1.4]

---

1. Since $\psi_T$ is homogeneous, the strict cocavity of $\psi_T$ is equivalent to saying that $\psi_T(v + w) > \psi_T(v) + \psi_T(w)$ for all $v, w \in \text{int} \mathcal{L}$ in different directions.
Corollary 1.6. Let \( \Gamma \rho < G \) be a Zariski dense subgroup of \( G = \prod_{i=1}^k \text{SO}^o(n_i, 1) \), \( n_i \geq 2 \), as defined in (1.4). Assume \( k \geq 2 \). For any \( v = (v_1, \cdots, v_k) \in \mathbb{R}^k \), we have
\[
\psi_{\Gamma \rho}(v) < \frac{1}{k} \sum_{i=1}^k \dim \Lambda_{\rho_i} \cdot v_i.
\]
In particular, we have
\[
\delta_{\Gamma \rho, \|\cdot\|_{\text{Euc}}} < \frac{1}{k} \left( \sum_{i=1}^k (\dim \Lambda_{\rho_i})^2 \right)^{1/2}
\]
where \( \|\cdot\|_{\text{Euc}} \) denotes the standard Euclidean norm on \( \mathbb{R}^k \).

Let \( F = \prod_{i=1}^k S^{n_i-1} \), which is the Furstenberg boundary of \( G \). The limit set of \( \Gamma \rho \) is the set of all accumulation points of an orbit \( \Gamma \rho(o_1, \cdots, o_k) \):
\[
\Lambda_{\rho} = \left\{ (\xi_1, \cdots, \xi_k) \in F : \exists \text{ a sequence } \sigma_\ell \in \Delta \text{ s.t. } \forall 1 \leq i \leq k, \right\}.
\]
In (1.6), we showed that
\[
\dim \Lambda_{\rho} = \max_{1 \leq i \leq k} \dim \Lambda_{\rho_i}
\]
where the Hausdorff dimension of \( \Lambda_{\rho} \) is computed with respect to the Riemannian metric on \( F \) given by \( \sqrt{\sum_{1 \leq i \leq k} d_{S^{n_i-1}}^2} \). We deduce the following from Corollary 1.6 and (1.7):

Corollary 1.7 (Gap theorem). For \( k \geq 2 \), we have
\[
\delta_{\Gamma \rho, \|\cdot\|_{\text{Euc}}} < \frac{\dim \Lambda_{\rho}}{\sqrt{k}}.
\]

The trivial bound for \( \delta_{\Gamma \rho, \|\cdot\|_{\text{Euc}}} \) is given by \( \delta_{\Gamma \rho, \|\cdot\|_{\text{Euc}}} \leq \min_i \delta_{\rho_i} \leq \dim \Lambda_{\rho} \). Hence Corollary (1.7) presents a strong gap for the value of \( \delta_{\Gamma \rho, \|\cdot\|_{\text{Euc}}} \) from the trivial bound. This phenomenon is in contrast to the rank one case: there exist convex cocompact (non-lattice) subgroups \( \Gamma \) of \( \text{SO}^o(n, 1) \) whose critical exponents \( \delta_{\Gamma} \) are arbitrarily close to \( n-1 \) (see e.g., [18, Sec.6] on the construction of McMullen).

Remark 1.8. Let \( \rho_1, \rho_2 \) be two convex cocompact faithful representations into \( \text{SO}^o(n, 1) = \text{Isom}^o(\mathbb{H}^n) \) and \( \rho = (\rho_1, \rho_2) \). Note that \( \Gamma \rho < \text{SO}^o(n, 1) \times \text{SO}^o(n, 1) \) is Zariski dense if and only if \( \rho_1 \) and \( \rho_2 \) are not conjugate by an element of \( \text{Isom}^o(\mathbb{H}^n) \). Hence Corollary 1.7 can be interpreted as the following rigidity statement: we have
\[
\delta_{\Gamma \rho, \|\cdot\|_{\text{Euc}}} \leq \frac{n-1}{\sqrt{2}}
\]
and the equality holds if and only if \( \rho_1(\Sigma) \) and \( \rho_2(\Sigma) \) are conjugate lattices of \( \text{SO}^o(n, 1) \). This particular rigidity statement is also presented in [3].
In view of special interests in low dimensional hyperbolic manifolds which come with huge deformation spaces, we also formulate the following consequence of Corollary \[\text{(1.5)}\] using the isomorphisms $\text{PSL}_2(\mathbb{C}) \cong \text{SO}^0(3,1)$ and $\text{PSL}_2(\mathbb{R}) \cong \text{SO}^0(2,1)$, the characterization of the critical exponent in \[\text{(1.3)}\], and the simple fact $\sup\{\min(v_1,2v_2) : v_1^2 + v_2^2 = 1\} = \frac{2}{\sqrt{5}}$. 

Corollary 1.9. Consider the metric on $\mathbb{H}^2 \times \mathbb{H}^3$ given by $d = \sqrt{d_{\mathbb{H}^2}^2 + d_{\mathbb{H}^3}^2}$. For any non-elementary faithful convex cocompact subgroup $\Gamma_0 < \text{PSL}_2(\mathbb{R})$ and any non-elementary faithful convex cocompact non-Fuchsian representation $\rho_0 : \Gamma_0 \to \text{PSL}_2(\mathbb{C})$, the critical exponent of the group $\{(\gamma_0, \rho_0(\gamma_0)) : \gamma_0 \in \Gamma_0\}$ with respect to $d$ is strictly less than $\frac{2}{\sqrt{5}}$.

Hitchin representations. We discuss applications to Hitchin representations. In $G = \text{PSL}(d,\mathbb{R})$, we have $a^+ = \{v = \text{diag}(t_1, \cdots, t_d) : t_1 \geq \cdots \geq t_d, \sum t_i = 0\}$ and $a_\alpha(v) = t_i - t_{i+1}$ for $1 \leq i \leq d - 1$. Let $\Sigma$ be a torsion-free uniform lattice of $\text{PSL}(2,\mathbb{R})$, and $\pi_d$ denote the $d$-dimensional irreducible representation $\text{PSL}(2,\mathbb{R}) \to \text{PSL}(d,\mathbb{R})$, which is unique up to conjugation. A Hitchin representation $\rho : \Sigma \to \text{PSL}(d,\mathbb{R})$ is a representation which belongs to the same connected component as $\pi_d|_{\Sigma}$ in the character variety $\text{Hom}(\Sigma, \text{PSL}(d,\mathbb{R}))/\sim$ where the equivalence is given by conjugations. We call the image of a Hitchin representation $\Gamma := \rho(\Sigma)$ a Hitchin subgroup of $G$. A Hitchin subgroup is known to be a $\Delta$-Anosov subgroup of $\text{PSL}(d,\mathbb{R})$ by Labourie \[\text{(15)}\]. By the work of Potrie-Sambarino \[\text{(21) Thm. B}\] (see also \[\text{(22) Coro. 9.4}\]), a Hitchin subgroup $\Gamma < \text{PSL}(d,\mathbb{R})$ satisfies:

$$\delta_{\Gamma, a_i} = 1 \quad \text{for all } 1 \leq i \leq d - 1.$$ \[\text{(1.9)}\]

Using this, Theorems \[\text{(1.2)}\] and \[\text{(1.4)}\] imply the following:

Corollary 1.10. Let $d \geq 3$ and $\Gamma < \text{PSL}(d,\mathbb{R})$ be a Zariski dense Hitchin subgroup of $\text{PSL}(d,\mathbb{R})$. Then for any $v = \text{diag}(t_1, \cdots, t_d) \in a^+$,

$$\psi_\Gamma(v) \leq \min_{1 \leq i \leq d-1} (t_i - t_{i+1}) \quad \text{and} \quad \psi_\Gamma(v) < (t_1 - t_d)/(d-1).$$ \[\text{(1.10)}\]

This pointwise bound for $\psi_\Gamma$ is sharper than the one from \[\text{(24, 19)}\], which for instance, for $d = 3$, gives the bound $\pi_G(v) = t_1 - t_3$ while the above corollary gives a bound $(t_1 - t_3)/2$.

Remark 1.11. The bound in Corollary \[\text{(1.10)}\] is stronger than \[\text{(21) Coro. 1.4}\] in two aspects: first, the bound for $\psi_\Gamma$ given by \[\text{(21) Coro. 1.4}\] is weaker than $\frac{t_1 - t_d}{d-1}$ and stated only for vectors inside a strictly smaller cone than the limit cone (see Remark \[\text{(3.3)}\] for details).

Remark 1.12. The comparison of $\psi_\Gamma$ with $\pi_G$ is meaningful in view of Sullivan’s theorem that for a convex cocompact subgroup $\Gamma < \text{SO}^0(n,1)$, the inequality $\delta^\Gamma \leq \delta^\Gamma < \frac{\pi_G}{2}$ holds if and only if the bottom of the $L^2$-spectrum on $\Gamma \backslash \mathbb{H}^n$ is given by $(n-1)^2/4$ and there exists no positive square-integrable harmonic function on $\Gamma \backslash \mathbb{H}^n$ \[\text{(28) Thm. 2.21}\].
Corollaries 1.6 and 1.10 imply that $\psi_{\Gamma} \leq \pi_{G}$ in their respective settings (even with the strict inequality). In recent work [7], these results were used to show that the quasi-regular representation $L^2(\Gamma \backslash G)$ is tempered and there exists no positive square-integrable harmonic function on the associated locally symmetric manifold.

For any discrete subgroup $\Gamma < G$, note that $\delta_{\Gamma, \pi_G} \leq 2$ as follows from Remark 1.3(1). We propose the following conjecture:

**Conjecture 1.13.** Let $k = \text{rank } G \geq 2$. If $\Gamma$ is a $\Delta$-Anosov subgroup of $G$, then $\delta_{\Gamma, \pi_G} \leq 1$, or equivalently $\psi_{\Gamma} \leq \pi_{G}$.

The equivalence is a consequence of [23, Lemma III.1.3].

**On the proofs.** The proof of Theorem 1.2 consists of two parts: first prove that each linear form $\delta_{\Gamma, \alpha_i}$ is tangent to $\psi_{\Gamma}$ whenever $\delta_{\Gamma, \alpha_i} < \infty$ and then take the minimum! Although taking the minimum seems a trivial step, the resulting tent function turns out to be quite useful, as discussed above. The proof of Theorem 1.4 is crucially based on special properties of $\psi_{\Gamma}$ for $\Delta$-Anosov subgroups (see Theorem 3.1).

**Organization.** In section 2, we prove Theorem 1.2. In section 3, we prove Theorem 1.4. In section 4, we discuss applications of tent property of $\psi_{\Gamma}$ to self-joining of hyperbolic manifolds.

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2. Tent property

Let $G$ be a connected, semisimple real algebraic group of rank $k \geq 1$. Let $\mathfrak{g}$ denote the Lie algebra of $G$, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the +1 and −1 eigenspaces of a fixed Cartan involution respectively. We denote by $K$ the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Let $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. Let $\Delta = \{\alpha_1, \cdots, \alpha_k\}$ be the set of simple roots $(\mathfrak{g}, \mathfrak{a}^+)$. As in the introduction, for $g \in G$, we denote by $\mu(g) \in \mathfrak{a}$ the unique element in $\mathfrak{a}^+$ such that $g \in K \exp(\mu(g))K$.

Let $\Gamma < G$ be a Zariski dense discrete subgroup. We denote by $\mathcal{L} \subset \mathfrak{a}^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$:

$$\mathcal{L} = \{\lim t_i \mu(\gamma_i) \in \mathfrak{a}^+ \text{ for some } t_i \to 0 \text{ and } \gamma_i \in \Gamma\}.$$ 

It is a convex cone with non-empty interior [11]. The growth indicator function $\psi_{\Gamma} : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ is defined as in (1.1). It follows easily from the definition that $\psi_{\Gamma}$ does not depend on the choice of a norm on $\mathfrak{a}$. 


Quint showed the following:

**Theorem 2.1.** [23 Thm. IV.2.2] The growth indicator function $\psi_T$ is concave, upper semi-continuous, and satisfies

$$\mathcal{L} = \{ u \in \mathfrak{a}^+ : \psi_T(u) > -\infty \}.$$  
Moreover, $\psi_T(u)$ is non-negative on $\mathcal{L}$ and positive on $\text{int} \mathcal{L}$.

**Lemma 2.2.** [23 Lem. III.1.3] Let $F$ be a non-negative continuous function on $\mathfrak{a}^+$ satisfying $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in \mathfrak{a}$. If $F(u) > \psi_T(u)$ for all $u \in \mathfrak{a} - \{0\}$, then

$$\sum_{\gamma \in \Gamma} e^{-F(\mu(\gamma))} < \infty.$$  
Moreover, we have $\delta_{\Gamma,F} < 1$.

**Proof.** Convergence of the series is shown in [23, Lem. III.1.3], and in particular $\delta_{\Gamma,F} \leq 1$. To obtain the strict inequality, we claim that there exists $0 < \varepsilon < 1$ such that

$$1 - \varepsilon F > \psi_T$$  
on $\mathfrak{a} - \{0\}$.

(2.1)

Since $\psi_T = -\infty$ outside $\mathcal{L}$ and both $F$ and $\psi_T$ are homogeneous functions, it suffices to prove (2.1) on $\{\|v\| = 1, v \in \mathcal{L}\}$. Since $\psi_T \geq 0$ on $\mathcal{L}$, we have $F > 0$ on $\mathcal{L} - \{0\}$. Hence the claim now follows because $\psi_T/F$ is upper semicontinuous and thus achieves its maximum on any compact set. \(\square\)

We denote by $\mathfrak{a}^*$ the set of all linear forms on $\mathfrak{a}$.

**Definition 2.3.** A linear form $\alpha \in \mathfrak{a}^*$ is called tangent to $\psi_T$ at $u \in \mathfrak{a} - \{0\}$ if $\alpha \geq \psi_T$ and $\alpha(u) = \psi_T(u)$.

Consider the following dual cone of the limit cone $\mathcal{L}$:

$$\mathcal{L}^* := \{ \alpha \in \mathfrak{a}^* : \alpha(v) \geq 0 \text{ for all } v \in \mathcal{L} \}.  \quad (2.2)$$

Observe that the set of all positive roots is contained in $\mathcal{L}^*$. Note that the interior of $\mathcal{L}^*$ is given as

$$\text{int} \mathcal{L}^* = \{ \alpha \in \mathfrak{a}^* : \alpha(v) > 0 \text{ for all } v \in \mathcal{L} - \{0\} \}.$$  

For any $\alpha \in \mathcal{L}^*$, we set

$$\delta_{\alpha} = \delta_{\Gamma,\alpha}.$$  

**Lemma 2.4.** If $\alpha \in \text{int} \mathcal{L}^*$, then

$$\delta_{\alpha} \leq \sup_{v \in \mathcal{L} - \{0\}} \frac{\psi_T(v)}{\alpha(v)} < \infty.$$  

**Proof.** Let $\kappa := \sup_{v \in \mathcal{L} - \{0\}} \frac{\psi_T(v)}{\alpha(v)}$. Since $\alpha > 0$ on $\mathcal{L} - \{0\}$, $0 \leq \kappa = \sup_{v \in \mathcal{L} - \{0\}} \frac{\psi_T(v)}{\alpha(v)} < \infty$ is well-defined. Since $\psi_T < (\kappa + \varepsilon)\alpha$ on $\mathfrak{a} - \{0\}$ for any $\varepsilon > 0$, we have, by Lemma 2.2, that $\delta_{(\kappa + \varepsilon)\alpha} < 1$. Hence $\delta_{\alpha} \leq (\kappa + \varepsilon)\alpha$. Since $\varepsilon > 0$ is arbitrary, we get $\delta_{\alpha} \leq \kappa$. \(\square\)
Theorem 2.5. Let \( \Gamma < G \) be a Zariski dense discrete subgroup. For any non-zero \( \alpha \in \mathcal{L}^* \) with \( \delta_\alpha < \infty \), the linear form

\[
T_\alpha := \delta_\alpha \alpha
\]

is tangent to \( \psi_\Gamma \) and \( \delta_\alpha > 0 \). In particular, for any subset \( S \subset \text{int} \mathcal{L}^* \),

\[
\psi_\Gamma \leq \inf_{\alpha \in S} T_\alpha.
\]

Proof. Fix any norm \( \| \cdot \| \) on \( a \) and we use this norm in the definition of \( \psi_\Gamma \).

We first claim

\[
\psi_\Gamma(v) \leq \delta_\alpha \alpha(v) \quad \text{for all } v \in \text{int} \mathcal{L}. \quad (2.3)
\]

Fix \( v \in \text{int} \mathcal{L} \) and \( \varepsilon > 0 \). We then consider

\[
C_\varepsilon(v) = \left\{ w \in a : \alpha(w) > 0 \text{ and } \left| \frac{\|w\|}{\alpha(w)} - \frac{\|v\|}{\alpha(v)} \right| < \varepsilon \right\};
\]

since \( \alpha(v) > 0 \), this is a well-defined open cone containing \( v \). Therefore by the definition of \( \psi_\Gamma \), we have

\[
\psi_\Gamma(v) \leq \|v\| \tau_{C_\varepsilon(v)}. \quad (2.4)
\]

Observe that for any \( s \geq 0 \),

\[
\sum_{\gamma \in \Gamma : \mu(\gamma) \in C_\varepsilon(v)} e^{-s\|\mu(\gamma)\|} \leq \sum_{\gamma \in \Gamma : \mu(\gamma) \in C_\varepsilon(v)} e^{-s\alpha(\mu(\gamma))\left(\frac{\|v\|}{\alpha(v)} - \varepsilon\right)} \leq \sum_{\gamma \in \Gamma} e^{-s\alpha(\mu(\gamma))\left(\frac{\|v\|}{\alpha(v)} - \varepsilon\right)}.
\]

Since \( \tau_{C_\varepsilon(v)} \) is the abscissa of convergence of the series

\[
\sum_{\gamma \in \Gamma : \mu(\gamma) \in C_\varepsilon(v)} e^{-s\|\mu(\gamma)\|},
\]

it follows from the definition of \( \delta_\alpha \) that

\[
\tau_{C_\varepsilon(v)} \leq \frac{\delta_\alpha}{\|v\|\alpha(v)^{-1} - \varepsilon} = \frac{\delta_\alpha \alpha(v)}{\|v\| - \varepsilon \alpha(v)}.
\]

Together with (2.4), we have

\[
\psi_\Gamma(v) \leq \|v\| \frac{\delta_\alpha \alpha(v)}{\|v\| - \varepsilon \alpha(v)}.
\]

Since \( \varepsilon > 0 \) is arbitrary, we get

\[
\psi_\Gamma(v) \leq \delta_\alpha \alpha(v).
\]

This proves the claim (2.3).

We now claim that the inequality (2.3) also holds for any \( v \) in the boundary \( \partial \mathcal{L} \). Choose any \( v_0 \in \text{int} \mathcal{L} \). From the concavity of \( \psi_\Gamma \), we have

\[
t\psi_\Gamma(v_0) + (1-t)\psi_\Gamma(v) \leq \psi_\Gamma(tv_0 + (1-t)v) \quad \text{for all } 0 < t < 1.
\]
Since $L$ is convex, $tv_0 + (1 - t)v \in \text{int } L$ for all $0 < t < 1$. As we have already shown $\psi_T \leq T_\alpha$ on int $L$, we get

$$t\psi_T(v_0) + (1 - t)\psi_T(v) \leq T_\alpha(tv_0 + (1 - t)v) \quad \text{for all } 0 < t < 1.$$ 

By sending $t \to 0^+$, we get

$$\psi_T(v) \leq T_\alpha(v).$$

Since $\psi_T = -\infty$ outside $L$, we have established $\psi_T \leq T_\alpha$.

It remains to show that $\psi_T(v) = T_\alpha(v)$ for some $v \in a - \{0\}$. Suppose not, i.e., $\psi_T < \delta_\alpha \alpha$ on $a - \{0\}$. By Lemma 2.2, the abscissa of convergence of the series

$$\sum_{\gamma \in \Gamma} e^{-\delta_\alpha \alpha(\mu(\gamma))}$$

is strictly less than 1. However the abscissa of convergence of the series (2.5) is equal to 1 by the definition of $\delta_\alpha$. Therefore we have obtained a contradiction.

Note that this implies $\delta_\alpha > 0$ since $\psi_T > 0$ on int $L$, which is non-empty by Zariski density hypothesis by Theorem 2.1. The last part of the theorem follows from Lemma 2.4.

**Remark 2.6.** We also note the following lower bound for $\psi_T$: let $T_\ell \in L^*$, $\ell \in I$, be a finite collection of linear forms which are tangent to $\psi_T$ at some $v_\ell \in L - \{0\}$. Then the concavity property of $\psi_T$ implies that for any $v = \sum_{\ell \in I} c_\ell v_\ell$ with $c_\ell \geq 0$,

$$\sum_{\ell \in I} c_\ell T_\ell(v_\ell) \leq \psi_T(v).$$

**Proof of Theorem 1.2** Note that $\Delta \subset L^*$. Hence this follows from Theorem 2.5 by taking the minimum over all simple roots $\alpha_i \in \Delta$ with $\delta_{\alpha_i} < \infty$.

We also note the following corollary of Theorem 2.5.

**Corollary 2.7.** Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $\alpha \in \text{int } L^*$, we have

$$0 < \delta_\alpha = \max_{v \in L - \{0\}} \frac{\psi_T(v)}{\alpha(v)} < \infty.$$ 

**Proof.** By Lemma 2.4, $\delta_\alpha < \infty$. Hence Theorem 2.5 implies $\psi_T \leq \delta_\alpha \alpha$ and $\psi_T(v) = \delta_\alpha \alpha(v)$ for some $v \neq 0$. This implies the claim.

By the following theorem, the above corollary applies to $\alpha \in \theta$ for $\theta$-Anosov subgroups.

**Theorem 2.8** ([9], [11]). If $\Gamma$ is $\theta$-Anosov, then

$$\theta \subset \text{int } L^*.$$ 

In particular, if $\Gamma$ is $\Delta$-Anosov, then

$$L \subset \text{int } a^+ \cup \{0\}. \quad (2.6)$$
3. Proof of Theorem 1.4

In this section, let 
\[ \Gamma < G \] be a Zariski dense ∆-Anosov subgroup, as defined in the introduction (1.2).

By Quint’s duality lemma [25, Lem. 4.3] and the works of Quint [25], Sambarino [26, Lem. 4.8] and Potrie-Sambarino [21, Prop. 4.6 and 4.11], which is based on the work [4], we have the following fundamental properties of \( \Gamma \):

**Theorem 3.1.** On \( \text{int} \, L \), \( \psi_\Gamma \) is analytic, strictly concave, and vertically tangent on \( \partial L \).

The vertical tangency of \( \psi_\Gamma \) on \( \partial L \) means that there are no linear forms which are tangent to \( \psi_\Gamma \) at a point of \( \partial L \).

In the following, we fix a norm on \( a \) induced from an inner product \( \langle \cdot, \cdot \rangle \). We denote by \( \nabla \psi_\Gamma(u) \in a \) the gradient of \( \psi_\Gamma \) at \( u \) so that \( d(\psi_\Gamma)_u(v) = \langle \nabla \psi_\Gamma(u), v \rangle \) for all \( v \in a - \{0\} \).

The following theorem was first observed by Quint for Schottky groups [25] and is deduced from Theorem 3.1 in general:

**Theorem 3.2** ([6, Coro. 7.8] [16, Prop. 4.4]). Let \( u \in \text{int} \, L \).

1. There exists a unique \( \psi_u \in a^* \) which is tangent to \( \psi_\Gamma \) at \( u \).
2. We have \( \psi_u \in \text{int} \, L^* \) and

\[
\psi_u(\cdot) = \langle \nabla \psi_\Gamma(u), \cdot \rangle = d(\psi_\Gamma)_{u}. \tag{3.1}
\]

3. The map \( u \mapsto \psi_u \) induces a bijection between directions in \( \text{int} \, L \) and directions in \( \text{int} \, L^* \).
4. We have \( \delta_{\psi_u} = 1 \).

![Figure 2. Limit cone and its dual cone.](image)

We deduce the following from the above two theorems:

**Proposition 3.3.** Consider the map \( \text{int} \, L \to \text{int} \, L^* \) given by \( u \mapsto \alpha_u \) where

\[
\alpha_u := \frac{\psi_u}{\psi_\Gamma(u)}.
\]
(1) The map $u \mapsto \alpha_u$ is a bijection.

(2) Its inverse map $\text{int} \mathcal{L}^* \rightarrow \text{int} \mathcal{L}$ is given by $\alpha \mapsto u_\alpha$ where $u_\alpha \in \text{int} \mathcal{L}$ is the unique vector such that $\nabla \psi_\Gamma(u_\alpha)$ is perpendicular to $\ker \alpha$ and
\[
\alpha(u_\alpha) = 1.
\]

We also have
\[
\psi_T(u_\alpha) = \max_{v \in \mathcal{L}, \alpha(v) = 1} \psi_T(v). \tag{3.2}
\]

Proof. For $t > 0$, $\psi_{tu} = \psi_u$ and $\psi_T(tu) = t\psi_T(u)$; hence $\alpha_{tu} = t^{-1}\alpha_u$. Therefore (1) follows from Theorem 3.2.

Let $\alpha \in \text{int} \mathcal{L}^*$. Let $u_\alpha \in \text{int} \mathcal{L}$ be the vector given by the relation $\alpha_{u_\alpha} = \alpha$, that is, $\alpha = \frac{\psi_{u_\alpha}}{\psi_T(u_\alpha)}$. By the definition of $\psi_{u_\alpha}$ given in (3.1), $\nabla \psi_T(u_\alpha)$ is perpendicular to $\ker \alpha$, and
\[
\alpha(u_\alpha) = \frac{\psi_{u_\alpha}(u_\alpha)}{\psi_T(u_\alpha)} = \frac{\psi_T(u_\alpha)}{\psi_T(u_\alpha)} = 1.
\]

To show the uniqueness, suppose that $v \in \text{int} \mathcal{L}$ is a vector such that $\nabla \psi_T(v)$ is parallel to $\nabla \psi_T(u_\alpha)$ and $\alpha(v) = 1$. The strict concavity of $\psi_T$ on $\text{int} \mathcal{L}$ as in Theorem 3.1 implies that $v$ must be parallel to $u_\alpha$. Since $\alpha(v) = \alpha(u_\alpha) = 1$, it follows that $v = u_\alpha$.

Observe that for any $v \in \mathcal{L}$ with $\alpha(v) = 1$, we have
\[
\psi_T(v) \leq \psi_{u_\alpha}(v) = \psi_T(u_\alpha)\alpha(v) = \psi_T(u_\alpha) = \psi_T(u_\alpha).
\]

Since $\alpha(u_\alpha) = 1$, this implies (3.2). \qed

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{From $\alpha$ to $u_\alpha$}
\end{figure}

**Theorem 3.4.** For any $\alpha \in \text{int} \mathcal{L}^*$, we have
\[
\delta_\alpha = \psi_T(u_\alpha) \quad \text{and} \quad \psi_{u_\alpha} = \delta_\alpha \alpha.
\]

Proof. The first claim follows from (3.2) and Corollary 2.7. Since $\psi_{u_\alpha} = \psi_T(u_\alpha)\alpha$ by Proposition 3.3, the first claim implies the second. \qed
Proof of Theorem 1.4. For (1), we claim that \( v_i := u_{\alpha_i} \) satisfies the claim. By Proposition 3.3, we have \( u_{\alpha_i} \in \text{int} \mathcal{L} \) and it satisfies \( \alpha_i(u_{\alpha_i}) = 1 \). By Lemma 3.4, \( \psi_T(u_{\alpha_i}) = \delta_{\alpha_i} \). The uniqueness follows easily from the strict concavity of \( \psi_T \) (Theorem 3.1).

For (2), suppose that for some \( v \in a \) and \( 1 \leq i \leq k \), we have \( \psi_T(v) = \delta_{\alpha_i} \alpha_i(v) \). Since \( \psi_{u_{\alpha_i}} = \delta_{\alpha_i} \alpha_i \) is a tangent form to \( \psi_T \) at \( u_{\alpha_i} \), it follows again from the strict concavity of \( \psi_T \) and the vertical tangency property (Theorem 3.1) that \( v \) is parallel to \( u_{\alpha_i} \).

By Theorem 1.2, we have

\[
\psi_T(v) \leq \min_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(v) \leq \frac{1}{k} \sum_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(v). \tag{3.3}
\]

Suppose that \( \psi_T(v) = \frac{1}{k} \sum_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(v) \) for some \( v \neq 0 \). It then follows from (3.3) that

\[
\psi_T(v) = \min_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(v) = \frac{1}{k} \sum_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(v).
\]

It implies that for all \( 1 \leq i \leq k \),

\[
\psi_T(v) = \delta_{\alpha_i} \alpha_i(v).
\]

Then, as we just have seen, this implies that \( v \) is parallel to all \( u_{\alpha_i}, 1 \leq i \leq k \).

When \( k \geq 2 \), this contradicts Theorem 3.2. This proves (3).

Proof of Corollary 1.5. For simplicity, we omit \( \| \cdot \| \) in the subscript in this proof, e.g., \( u_T = u_T_{\| \cdot \|} \). Recall that \( \delta_T = \psi_T(u_T) \). Since \( \delta_T = \max_{\|v\|=1} \psi_T(v) \), [23] Lem. III.3.4], applied to \( \psi_T \), implies that there exists a tangent form, \( \psi_{u_T} \), to \( \psi_T \) at \( u_T \). By the vertical tangent condition in Theorem 3.1, it follows that \( u_T \in \text{int} \mathcal{L} \). Moreover, we have \( \nabla \psi_T(u_T) \in \mathbb{R}_{>0} u_T \) [3] Lem. 2.24]. Therefore, by Theorem 3.2(2), there exists \( c_0 > 0 \) such that

\[
\psi_{u_T}(\cdot) = \langle c_0 u_T, \cdot \rangle. \tag{3.4}
\]

We now claim that

\[
\psi_T(u_T) < T_T(u_T).
\]

Suppose not. Then, by Theorem 1.4, there exist \( c > 0 \) and \( 1 \leq i \leq k \) such that \( u_T = cu_{\alpha_i} \) and hence \( \psi_{u_T} = \psi_{u_{\alpha_i}} = \delta_{\alpha_i} \alpha_i \).

By (3.4), it follows that \( \alpha_i(\cdot) = \langle c_1 u_T, \cdot \rangle \) for some \( c_1 > 0 \).

Since \( u_T \in \text{int} \mathcal{L} \), the linear form \( \langle c_1 u_T, \cdot \rangle \) is strictly positive on \( a^+ \). Therefore, we obtained a contradiction. This finishes the proof.

We note that in the above proof, the hypothesis that the norm \( \| \cdot \| \) is induced from an inner product was used to deduce that \( \psi_{u_T_{\| \cdot \|}} \) is strictly positive on \( a^+ \).
Remark 3.5. We explain how Theorem 1.4 can be compared with [21, Coro. 1.4]. Let \((a^+)\) so that \(\text{int}(a^+) = \{\alpha \in a^* : \alpha(v) > 0 \text{ for all } v \in a^+\}\).

Recall that [21, Coro. 1.4] concerns the Hitchin representations, but their argument applied to our Zariski dense Anosov subgroups yields the following: For any \(\alpha \in \text{int}(a^+)\), the quantity \(\delta_\alpha\) satisfies
\[
\delta_\alpha \leq \frac{1}{\sum_{i=1}^{k} a_i} \tag{3.5}
\]
where \(\alpha = \sum_{i=1}^{k} (a_i, \alpha_i)\); the hypothesis \(\alpha \in \text{int}(a^+)\) is equivalent to \(\alpha \neq 0\) and \(a_i > 0\) for all \(1 \leq i \leq k\).

On the other hand, our Theorem 1.4 says that for all \(\alpha \in \text{int}L^\star\),
\[
\delta_\alpha = \psi_T(u_\alpha) \leq \min_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(u_\alpha); \tag{3.6}
\]
this is equivalent to saying that for all \(v \in a\), \(\psi_T(v) \leq \min_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(v)\).

Since
\[
1 = \alpha(u_\alpha) = \sum_{i=1}^{k} a_i \delta_{\alpha_i} \alpha_i(u_\alpha) \geq \left(\sum_{i=1}^{k} a_i\right) \min_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(u_\alpha) \tag{3.7}
\]
we have
\[
\min_{1 \leq i \leq k} \delta_{\alpha_i} \alpha_i(u_\alpha) \leq \frac{1}{\sum_{i=1}^{k} a_i}
\]
where the equality is strict except for one direction of \(u_\alpha\) satisfying
\[
\delta_{\alpha_i} \alpha_i(u_\alpha) = \delta_{\alpha_j} \alpha_j(u_\alpha) \text{ for all } i, j = 1, \ldots, k.
\]
Therefore our bound (3.6) is sharper than the bound (3.5) in addition to the point that it applies to the optimal cone \(\text{int}L^\star\), while [21, Coro. 1.4] applies only for \(\alpha \in \text{int}(a^+)\), which is strictly smaller than \(\text{int}L^\star\).

Both approaches are based on the observation that the linear forms \(\delta_\alpha\alpha_i\)’s are tangent to \(\psi_T\) for \(\alpha \in \Delta\), but [21, Coro. 1.4] considers these tangent forms as points on the boundary of the subset \(D = \{\varphi \in \text{int}(a^+) : \delta_\varphi \leq 1\}\) and deduce (3.5) from the convexity of \(D\), whereas we think of the tangent forms as functions on \(a\) and obtain a stronger bound of (3.6) simply by taking minimum of these tangent forms over \(\alpha \in \Delta\).

4. Applications to self-joinings

We consider the case when \(G = \prod_{i=1}^{k} SO^0(n_i, 1)\), \(n_i \geq 2\), and \(\rho_i : \Sigma \to SO^0(n_i, 1)\) is a faithful convex cocompact representation with Zariski dense image. We let \(\Gamma_\rho < G\) be the subgroup defined as in (1.4). The hypothesis on \(\rho_i\)’s implies that \(\Gamma_\rho\) is \(\Delta\)-Anosov. We assume
\[
k \geq 2 \text{ and } \Gamma_\rho \text{ is Zariski dense in } G
\]
in the entire section.
Proof of Corollaries 1.6 and 1.7. Corollary 1.6 follows since \( \delta_{\alpha_i} = \delta_{\rho_i} = \dim \Lambda_{\rho_i} \). For Corollary 1.7, note that we have

\[
\delta_{\Gamma, \| \cdot \|_{\text{Euc}}} < \frac{1}{k} \left( \sum_{i=1}^{k} \left( \dim \Lambda_{\rho_i} \right)^2 \right)^{1/2} \leq \frac{1}{k} \left( k \max_{1 \leq i \leq k} \left( \dim \Lambda_{\rho_i} \right)^2 \right)^{1/2} = \frac{1}{\sqrt{k}} \max_{1 \leq i \leq k} \dim \Lambda_{\rho_i}.
\]

On the other hand, we showed in [13],

\[
\dim \Lambda_{\rho} = \max_{i} \dim \Lambda_{\rho_i}.
\]

Hence

\[
\delta_{\Gamma, \| \cdot \|_{\text{Euc}}} < \frac{1}{\sqrt{k}} \dim \Lambda_{\rho}.
\]

Critical exponent with respect to the \( L^1 \)-metric. Set \( \delta_{L^1} := \delta_{\sum_{i=1}^{k} \alpha_i} \), which is the critical exponent of \( \Gamma_{\rho} \) for the \( L^1 \)-metric \( \sum_{i=1}^{k} d_i \) on \( X = \prod_{i=1}^{k} \mathbb{H}^n \). We deduce the following from Corollary 1.6, whose special case when \( k = 2 \) and \( \dim \Lambda_{\rho_i} = 1 \) was proved by Bishop and Steger [2]:

Corollary 4.1. We have

\[
\delta_{L^1} < \frac{\dim \Lambda_{\rho}}{k}.
\]

Proof. Noting \( \alpha := \sum_{i=1}^{k} \alpha_i \in \text{int} \mathcal{L}^* \), write \( u_{\alpha} = (u_1, \cdots, u_k) \in \text{int} \mathcal{L} \). Lemma 3.4 and Corollary 1.6 imply

\[
\delta_{L^1} = \psi_{\Gamma}(u_{\alpha}) < \frac{1}{k} \sum_{i=1}^{k} \dim \Lambda_{\rho_i} u_i \leq \max_{i} \dim \Lambda_{\rho_i} \sum_{i=1}^{k} u_i.
\]

Since \( \alpha(u_{\alpha}) = \sum_{1 \leq i \leq k} u_i = 1 \) by Lemma 3.3(2) and \( \max_{i} \dim \Lambda_{\rho_i} = \dim \Lambda_{\rho} \) by [13], we get the desired inequality.

Geodesic stretching between two hyperbolic manifolds. When \( k = 2 \), the limit cone \( \mathcal{L} \) of \( \Gamma_{\rho} \) can also be described as

\[
\mathcal{L} := \{ (v_1, v_2) \in \mathbb{R}^2_{\geq 0} : d_- v_1 \leq v_2 \leq d_+ v_1 \}
\]

where \( d_+ \) and \( d_- \) are respectively the maximal and minimal geodesic stretching constants of \( \rho_2 \) relative to \( \rho_1 \):

\[
d_+(\rho_1, \rho_2) = \sup_{\sigma \in \Sigma - \{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)} \quad \text{and} \quad d_-(\rho_1, \rho_2) = \inf_{\sigma \in \Sigma - \{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)}
\]

where \( \ell_i(\sigma) \) denotes the length of the closed geodesic in the hyperbolic manifold \( \rho_i(\Delta) \backslash \mathbb{H}^n \) corresponding to \( \rho_i(\sigma) \) (cf. [3], [1]).
Thurston [29] showed that the maximal geodesic stretching constant is always strictly bigger than 1 for finite-area hyperbolic surfaces. (See also [8].) Theorem 1.4 implies the following corollary; this was already observed by Burger [5, Thm. 1 and its Coro.] and generalizes a theorem of Thurston [29, Thm. 3.1]:

**Corollary 4.2.** We have

\[
\frac{d_-(\rho_1, \rho_2)}{\dim \Lambda_{\rho_2}} < d_+(\rho_1, \rho_2) < \frac{\dim \Lambda_{\rho_1}}{\dim \Lambda_{\rho_2}}.
\]

**Proof.** By Theorem 1.4,

\[
\psi_{\Gamma} \leq \min(\delta_1 \alpha_1, \delta_2 \alpha_2).
\]

(4.2)

By Theorem 3.4 we have \(\psi_{\Gamma}(u_{\alpha_1}) = \delta_1 \alpha_1(u_{\alpha_1})\). Hence

\[
\delta_1 \alpha_1(u_{\alpha_1}) \leq \min(\delta_1 \alpha_1(u_{\alpha_1}), \delta_2 \alpha_2(u_{\alpha_1}))
\]

which implies \(\delta_1 \alpha_1(u_{\alpha_1}) \leq \delta_2 \alpha_2(u_{\alpha_1})\). Therefore,

\[
\frac{\delta_1}{\delta_2} \leq \frac{\alpha_2(u_{\alpha_1})}{\alpha_1(u_{\alpha_1})}.
\]

Similarly, we have \(\delta_2 \alpha_2(u_{\alpha_2}) \leq \min(\delta_1 \alpha_1(u_{\alpha_2}), \delta_2 \alpha_2(u_{\alpha_2}))\), and hence

\[
\frac{\alpha_2(u_{\alpha_2})}{\alpha_1(u_{\alpha_2})} \leq \frac{\delta_1}{\delta_2}.
\]

Since \(\dim \Lambda_{\rho_i} = \delta_i\) for \(i = 1, 2\) by Patterson [20] and Sullivan [27], we now have

\[
\frac{\alpha_2(u_{\alpha_2})}{\alpha_1(u_{\alpha_2})} \leq \frac{\dim \Lambda_{\rho_1}}{\dim \Lambda_{\rho_2}} \leq \frac{\alpha_2(u_{\alpha_1})}{\alpha_1(u_{\alpha_1})}.
\]

Since \(u_{\alpha_1}, u_{\alpha_2} \in \text{int} \mathcal{L}\), \(d_-(\rho_1, \rho_2) < \frac{\alpha_2(u_{\alpha_2})}{\alpha_1(u_{\alpha_2})}\) and \(\frac{\alpha_2(u_{\alpha_1})}{\alpha_1(u_{\alpha_1})} < d_+(\rho_1, \rho_2)\). It completes the proof. \(\square\)

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