# TENT PROPERTY OF THE GROWTH INDICATOR FUNCTIONS AND APPLICATIONS

DONGRYUL M. KIM, YAIR N. MINSKY, AND HEE OH

Abstract. Let $\Gamma$ be a Zariski dense discrete subgroup of a connected semisimple real algebraic group $G$. Let $k = \operatorname{rank} G$. Let $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ be the growth indicator function of $\Gamma$, first introduced by Quint. In this paper, we obtain the following pointwise bound of $\psi_\Gamma$: for all $v \in \mathfrak{a}$,

$$\psi_\Gamma(v) \le \min_{1 \le i \le k} \delta_{\alpha_i} \alpha_i(v)$$

where $\Delta = \{\alpha_1, \cdots, \alpha_k\}$ is the set of all simple roots of $(\mathfrak{g}, \mathfrak{a})$ and $0 < \delta_{\alpha_i} \le \infty$ is the critical exponent of $\Gamma$ associated to $\alpha_i$. When $\Gamma$ is $\Delta$-Anosov, there are precisely $k$-number of directions where the equality is achieved, and the following strict inequality holds for $k \ge 2$: for all $v \in \mathfrak{a} - \{0\}$,

$$\psi_\Gamma(v) < \frac{1}{k} \sum_{i=1}^{k} \delta_{\alpha_i} \alpha_i(v).$$

We discuss applications for self-joinings of convex cocompact subgroups in $\prod_{i=1}^k \mathrm{SO}(n_i, 1)$ and Hitchin subgroups of $\mathrm{PSL}(d, \mathbb{R})$. In particular, for a Zariski dense Hitchin subgroup $\Gamma < \mathrm{PSL}(d, \mathbb{R})$, we obtain that for any $v = \operatorname{diag}(t_1, \cdots, t_d) \in \mathfrak{a}^+$,

$$\psi_\Gamma(v) \le \min_{1 \le i \le d-1} (t_i - t_{i+1}).$$

## Contents

1. Introduction 1
2. Tent property 8
3. Proof of Theorem 1.4 11
4. Applications to self-joinings 16
References 18

## 1. Introduction

Let $G$ be a connected semisimple real algebraic group. We let $P = MAN$ be a minimal parabolic subgroup of $G$ with a fixed Langlands decomposition, where $A$ is a maximal real split torus of $G$, $M$ is the maximal compact subgroup centralizing $A$ and $N$ is the unipotent radical of $P$. Let $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{a} = \operatorname{Lie} A$ and $\mathfrak{a}^+$ denote the positive Weyl chamber so that $\log N$ consists

Minsky and Oh are partially supported by the NSF.

of positive root subspaces. Let $K$ be a maximal compact subgroup so that the Cartan decomposition $G = K(\exp \mathfrak{a}^+)K$ holds. Let $\mu : G \to \mathfrak{a}^+$ denote the Cartan projection map defined by the condition $\exp \mu(g) \in KgK$ for all $g \in G$. Let $\Gamma < G$ be a Zariski dense discrete subgroup. We denote by $\mathcal{L} \subset \mathfrak{a}^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$. It is a convex cone with non-empty interior [1].

Following Quint [25], the growth indicator function $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ is defined as follows: choose any norm $\|\cdot\|$ on $\mathfrak{a}$. For an open cone $\mathcal{C}$ in $\mathfrak{a}$, let $\tau_{\mathcal{C}}$ denote the abscissa of convergence of $\sum_{\gamma\in\Gamma,\, \mu(\gamma)\in\mathcal{C}} e^{-s\|\mu(\gamma)\|}$ (that is, the infimum of the set of $s$ for which the series converges). Now for any non-zero $v \in \mathfrak{a}$, let

$$\psi_\Gamma(v) := \|v\| \inf_{v\in\mathcal{C}} \tau_{\mathcal{C}} \tag{1.1}$$

where the infimum is over all open cones $\mathcal{C}$ containing $v$, and let $\psi_\Gamma(0) = 0$. The definition of $\psi_\Gamma$ does not depend on the choice of a norm on $\mathfrak{a}$. Note that $\psi_\Gamma = -\infty$ outside $\mathcal{L}$. Quint showed that $\psi_\Gamma$ is a concave upper-semi continuous function satisfying $\mathcal{L} = \{\psi_\Gamma \geq 0\}$ and $\psi_\Gamma > 0$ on the interior $\operatorname{int}\mathcal{L}$.

The main aim of this paper is to present a pointwise bound for the growth indicator function together with some applications. Throughout the paper, for any non-negative function $f$ on $\mathfrak{a}^+$, we denote by

$$0 \leq \delta_{\Gamma,f} \leq \infty$$

or simply, $\delta_f$, the critical exponent of $\Gamma$ with respect to $f$, that is, the abscissa of convergence of the series $\sum_{\gamma\in\Gamma} e^{-sf(\mu(\gamma))}$.

Let

$$\Delta = \{\alpha_1, \cdots, \alpha_k\}$$

denote the set of simple roots for $(\mathfrak{g}, \mathfrak{a}^+)$.

**Definition 1.1** (Tent function)**.** Let $\Gamma < G$ be a Zariski dense discrete subgroup with $\delta_{\Gamma,\alpha_i} < \infty$ for some $1 \leq i \leq k$. We define a tent function $T_\Gamma : \mathfrak{a} \to [0, \infty)$ by

$$T_\Gamma(v) := \min_{1\leq i\leq k} \delta_{\Gamma,\alpha_i} \cdot \alpha_i(v).$$

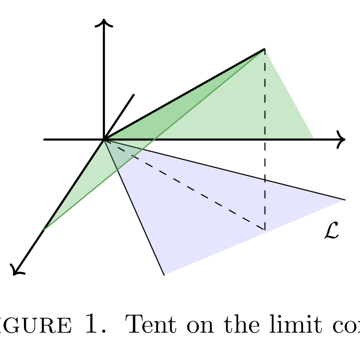


FIGURE 1. Tent on the limit cone

We obtain the following tent property of the growth indicator function:

**Theorem 1.2** (Tent property)**.** *For any Zariski dense discrete subgroup $\Gamma < G$ such that $\min_{1\le i\le k}\delta_{\Gamma,\alpha_i} < \infty$, we have*

$$\psi_\Gamma(v) \le T_\Gamma(v) \quad \text{for all } v \in \mathfrak{a}.$$

*Moreover, when $\delta_{\Gamma,\alpha_i} < \infty$, there exists $v_i \in \mathcal{L} - \{0\}$ such that $\psi_\Gamma(v_i) = T_\Gamma(v_i) = \delta_{\Gamma,\alpha_i}\alpha_i(v_i)$.*

**Remark 1.3.**
(1) Denote by $\pi_G$ the half-sum of all positive roots of $(\mathfrak{g},\mathfrak{a}^+)$ counted with multiplicity. Then for any discrete subgroup $\Gamma < G$, we have $\psi_\Gamma \le 2\pi_G$ [25, Thm. IV.2.2].
(2) If $G$ has property (T) and $\Gamma$ is of infinite co-volume, then $\psi_\Gamma \le 2\pi_G - \Theta$ where $\Theta$ is the half-sum of a maximal strongly orthogonal system ([26], [21], see also [19, Thm. 7.1]). Our bound in Theorem 1.2 provides a sharper bound for Hitchin subgroups; see Remark 3.5.

For a non-empty subset $\theta \subset \Delta$, a finitely generated subgroup $\Gamma < G$ is called a $\theta$-Anosov subgroup if there exist constants $C, C' > 0$ such that for all $\gamma \in \Gamma$ and all $\alpha_i \in \theta$,

$$\alpha_i(\mu(\gamma)) \ge C|\gamma| - C' \tag{1.2}$$

where $|\gamma|$ denotes the word length of $\gamma$ with respect to a fixed finite symmetric set of generators of $\Gamma$. The notion of Anosov subgroups was first introduced by Labourie for surface groups [17], and was extended to general word hyperbolic groups by Guichard-Wienhard [16]. Several equivalent characterizations have been established, one of which is the above definition (see [11], [12], [13], [14]). Anosov subgroups are regarded as natural generalizations of convex cocompact subgroups of rank one groups.

For a $\theta$-Anosov subgroup $\Gamma < G$, it follows from (1.2) that for some constant $C > 0$,

$$\max_{\alpha_i\in\theta}\delta_{\Gamma,\alpha_i} \le C\log \#\mathsf{S} < \infty$$

where $\mathsf{S}$ is a fixed finite generating set of $\Gamma$. Therefore Theorem 1.2 applies to any Zariski dense subgroup contained in some $\theta$-Anosov subgroup of $G$.

For $\Delta$-Anosov subgroups, we obtain the following sharper result:

**Theorem 1.4.** *Let $\Gamma$ be a Zariski dense $\Delta$-Anosov subgroup of $G$. The following hold:*

(1) *For each $1 \le i \le k$, there exists a unique $v_i \in \operatorname{int}\mathcal{L}$ such that $\alpha_i(v_i) = 1$ and $\psi_\Gamma(v_i) = \delta_{\Gamma,\alpha_i}$.*
(2) *For $v \in \mathfrak{a} - \{0\}$, we have $\psi_\Gamma(v) \le T_\Gamma(v)$ where equality holds if and only if $v = cv_i$ for some $1 \le i \le k$ and $c > 0$.*
(3) *If $k = \operatorname{rank} G \ge 2$, then*

$$\psi_\Gamma < \frac{1}{k}\sum_{i=1}^{k}\delta_{\Gamma,\alpha_i}\alpha_i.$$

When $\Gamma$ is $\Delta$-Anosov, $\psi_\Gamma$ is strictly concave[1] in $\operatorname{int}\mathcal{L}$ by ([28, Thm. A], [23, Prop. 4.11]). Therefore by the convexity of the unit norm ball $\{\|v\| \le 1\}$, there exists a unique unit vector $u_{\Gamma,\|\cdot\|} \in \mathfrak{a}^+$, called the direction of maximal growth, such that $\psi_\Gamma(u_{\Gamma,\|\cdot\|}) = \max_{\|v\|=1} \psi_\Gamma(v)$. By [25, Coro. III.1.4], we have

$$\delta_{\Gamma,\|\cdot\|} = \psi_\Gamma(u_{\Gamma,\|\cdot\|}). \tag{1.3}$$

**Corollary 1.5.** *Let $k = \operatorname{rank} G \ge 2$. Let $\Gamma$ be a Zariski dense $\Delta$-Anosov subgroup of $G$. For any norm $\|\cdot\|$ on $\mathfrak{a}$ induced from an inner product which is non-negative on $\mathfrak{a}^+$, we have*

$$\delta_{\Gamma,\|\cdot\|} < \min_{1\le i\le k} \delta_{\Gamma,\alpha_i} \cdot \alpha_i(u_{\Gamma,\|\cdot\|}).$$

In view of the above discussion, any upper bound on $\delta_{\Gamma,\alpha_i}$ for any $\alpha_i \in \Delta$ provides an explicit pointwise upper bound on $\psi_\Gamma$. We discuss some examples of $\Delta$-Anosov subgroups.

**Self-joinings of hyperbolic manifolds.** For $1 \le i \le k$, consider the hyperbolic space $(\mathbb{H}^{n_i}, d_i)$, $n_i \ge 2$, with constant sectional curvature $-1$, and let $G_i = \mathrm{SO}^\circ(n_i, 1) = \mathrm{Isom}^+(\mathbb{H}^{n_i})$. Let $G = \prod_{i=1}^k G_i$. Denote by $\alpha_i$ the simple root of $\mathfrak{g}_i = \operatorname{Lie} G_i$. Then $\Delta = \{\alpha_1, \cdots, \alpha_k\}$ is the set of simple roots of $\mathfrak{g}$. Via the map $v \mapsto (\alpha_1(v), \cdots, \alpha_k(v))$, we may identify $\mathfrak{a} = \mathbb{R}^k$ and $\mathfrak{a}^+ = \{(v_1, \cdots, v_k) \in \mathbb{R}^k : v_i \ge 0 \text{ for all } i\}$.

Let $\Sigma$ be a countable group and $\rho_i : \Sigma \to G_i$ be a faithful convex cocompact representation with Zariski dense image for each $1 \le i \le k$. Setting $\rho = (\rho_1, \cdots, \rho_k)$, the self-joining $\Gamma_\rho$ is defined as the following subgroup of $G$:

$$\Gamma_\rho = \left(\prod_{i=1}^k \rho_i\right)(\Sigma) = \{(\rho_1(\sigma), \cdots, \rho_k(\sigma)) \in G : \sigma \in \Sigma\}. \tag{1.4}$$

We also assume that no two of $\rho_i$'s are conjugate, so that $\Gamma_\rho$ is a Zariski dense discrete subgroup of $G$. The hypothesis on $\rho_i$'s implies that $\Gamma_\rho$ is a $\Delta$-Anosov subgroup of $G$ (cf. [16, Thm. 5.15]).

Fix $o_i \in \mathbb{H}^{n_i}$. For each $1 \le i \le k$, denote by $0 < \delta_{\rho_i} \le \infty$ the critical exponent of $\rho_i(\Sigma)$, that is, the abscissa of convergence of the series $\sum_{\sigma\in\Sigma} e^{-s d_i(\rho_i(\sigma)o_i, o_i)}$. We also denote by $\Lambda_{\rho_i} \subset \mathbb{S}^{n_i-1}$ the limit set of $\rho_i(\Sigma)$, which is the set of accumulation points of $\rho_i(\Sigma)o_i$ in the compactification $\mathbb{H}^{n_i} \cup \mathbb{S}^{n_i-1}$. These two notions are independent of the choice of $o_i \in \mathbb{H}^{n_i}$. By Patterson [22] and Sullivan [29], we have

$$\delta_{\rho_i} = \dim \Lambda_{\rho_i} \tag{1.5}$$

where $\dim \Lambda_{\rho_i}$ is the Hausdorff dimension of $\Lambda_{\rho_i}$ with respect to the spherical metric $d_{\mathbb{S}^{n_i-1}}$. We deduce from Theorem 1.4:

[1] Since $\psi_\Gamma$ is homogeneous, the strict concavity of $\psi_\Gamma$ is equivalent to saying that $\psi_\Gamma(v+w) > \psi_\Gamma(v) + \psi_\Gamma(w)$ for all $v, w \in \operatorname{int}\mathcal{L}$ in different directions

**Corollary 1.6.** *Let $\Gamma_\rho < G$ be a Zariski dense subgroup of $G = \prod_{i=1}^k \mathrm{SO}^\circ(n_i, 1)$, $n_i \geq 2$, as defined in* (1.4)*. Assume $k \geq 2$. For any $v = (v_1, \cdots, v_k) \in \mathbb{R}^k$, we have*

$$\psi_{\Gamma_\rho}(v) < \frac{1}{k} \sum_{i=1}^k \dim \Lambda_{\rho_i} \cdot v_i.$$

*In particular, we have*

$$\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}} < \frac{1}{k} \left( \sum_{i=1}^k (\dim \Lambda_{\rho_i})^2 \right)^{1/2}$$

*where $\|\cdot\|_{\mathrm{Euc}}$ denotes the standard Euclidean norm on $\mathbb{R}^k$.*

Let $\mathcal{F} = \prod_{i=1}^k \mathbb{S}^{n_i - 1}$, which is the Furstenberg boundary of $G$. The limit set of $\Gamma_\rho$ is the set of all accumulation points of an orbit $\Gamma_\rho(o_1, \cdots, o_k)$:

$$\Lambda_\rho = \left\{ (\xi_1, \cdots, \xi_k) \in \mathcal{F} : \begin{array}{c} \exists \text{ a sequence } \sigma_\ell \in \Delta \text{ s.t. } \forall\ 1 \leq i \leq k, \\ \xi_i = \lim_{\ell \to \infty} \rho_i(\sigma_\ell)(o_i) \end{array} \right\}. \quad (1.6)$$

In [15], we showed that

$$\dim \Lambda_\rho = \max_{1 \leq i \leq k} \dim \Lambda_{\rho_i} \quad (1.7)$$

where the Hausdorff dimension of $\Lambda_\rho$ is computed with respect to the Riemannian metric on $\mathcal{F}$ given by $\sqrt{\sum_{1 \leq i \leq k} {d_{\mathbb{S}^{n_i-1}}}^2}$. We deduce the following from Corollary 1.6 and (1.7):

**Corollary 1.7** (Gap theorem)**.** *For $k \geq 2$, we have*

$$\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}} < \frac{\dim \Lambda_\rho}{\sqrt{k}}.$$

The trivial bound for $\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}}$ is given by $\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}} \leq \min_i \delta_{\rho_i} \leq \dim \Lambda_\rho$. Hence Corollary (1.7) presents a strong gap for the value of $\delta_{\Gamma, \|\cdot\|_{\mathrm{Euc}}}$ from the trivial bound. This phenomenon is in contrast to the rank one case: there exist convex cocompact (non-lattice) subgroups $\Gamma$ of $\mathrm{SO}^\circ(n, 1)$ whose critical exponents $\delta_\Gamma$ are arbitrarily close to $n - 1$ (see e.g., [20, Sec.6] on the construction of McMullen).

**Remark 1.8.** Let $\rho_1, \rho_2$ be two convex cocompact faithful representations into $\mathrm{SO}^\circ(n, 1) = \mathrm{Isom}^\circ(\mathbb{H}^n)$ and $\rho = (\rho_1, \rho_2)$. Note that $\Gamma_\rho < \mathrm{SO}^\circ(n, 1) \times \mathrm{SO}^\circ(n, 1)$ is Zariski dense if and only if $\rho_1$ and $\rho_2$ are not conjugate by an element of $\mathrm{Isom}^\circ(\mathbb{H}^n)$. Hence Corollary 1.7 can be interpreted as the following rigidity statement: we have

$$\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}} \leq \frac{n-1}{\sqrt{2}} \quad (1.8)$$

and the equality holds if and only if $\rho_1(\Sigma)$ and $\rho_2(\Sigma)$ are conjugate *lattices* of $\mathrm{SO}^\circ(n, 1)$. This particular rigidity statement was recently extended in [3] even to geometrically finite representations.

In view of special interests in low dimensional hyperbolic manifolds which come with huge deformation spaces, we also formulate the following consequence of Corollary 1.5, using the isomorphisms $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{SO}^\circ(3,1)$ and $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}^\circ(2,1)$, the characterization of the critical exponent in (1.3), and the simple fact $\sup\{\min(v_1, 2v_2) : v_1^2 + v_2^2 = 1\} = \frac{2}{\sqrt{5}}$.

**Corollary 1.9.** *Consider the metric on $\mathbb{H}^2 \times \mathbb{H}^3$ given by $d = \sqrt{d_{\mathbb{H}^2}^2 + d_{\mathbb{H}^3}^2}$. For any non-elementary convex cocompact subgroup $\Gamma_0 < \mathrm{PSL}(2,\mathbb{R})$ and any non-elementary faithful convex cocompact Zariski dense representation $\rho_0 : \Gamma_0 \to \mathrm{PSL}(2,\mathbb{C})$, the critical exponent of the group $\{(\gamma_0, \rho_0(\gamma_0)) : \gamma_0 \in \Gamma_0\}$ with respect to $d$ is strictly less than $\frac{2}{\sqrt{5}}$.*

**Hitchin representations.** We discuss applications to Hitchin representations. In $G = \mathrm{PSL}(d,\mathbb{R})$, we have $\mathfrak{a}^+ = \{v = \mathrm{diag}(t_1, \cdots, t_d) : t_1 \geq \cdots \geq t_d, \sum t_i = 0\}$ and $\alpha_i(v) = t_i - t_{i+1}$ for $1 \leq i \leq d-1$. Let $\Sigma$ be a torsion-free uniform lattice of $\mathrm{PSL}(2,\mathbb{R})$, and $\pi_d$ denote the $d$-dimensional irreducible representation $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(d,\mathbb{R})$, which is unique up to conjugation. A Hitchin representation $\rho : \Sigma \to \mathrm{PSL}(d,\mathbb{R})$ is a representation which belongs to the same connected component as $\pi_d|_\Sigma$ in the character variety $\mathrm{Hom}(\Sigma, \mathrm{PSL}(d,\mathbb{R}))/\sim$ where the equivalence is given by conjugations.

We call the image of a Hitchin representation $\Gamma := \rho(\Sigma)$ a Hitchin subgroup of $G$.

A Hitchin subgroup is known to be a $\Delta$-Anosov subgroup of $\mathrm{PSL}(d,\mathbb{R})$ by Labourie [17]. By the work of Potrie-Sambarino [23, Thm. B] (see also [24, Coro. 9.4]), a Hitchin subgroup $\Gamma < \mathrm{PSL}(d,\mathbb{R})$ satisfies:

$$\delta_{\Gamma,\alpha_i} = 1 \quad \text{for all } 1 \leq i \leq d-1. \tag{1.9}$$

Together with this important result, Theorems 1.2 and 1.4 imply the following:

**Corollary 1.10.** *Let $d \geq 3$ and $\Gamma < \mathrm{PSL}(d,\mathbb{R})$ be a Zariski dense Hitchin subgroup of $\mathrm{PSL}(d,\mathbb{R})$. Then for any $v = \mathrm{diag}(t_1, \cdots, t_d) \in \mathfrak{a}^+$,*

$$\psi_\Gamma(v) \leq \min_{1 \leq i \leq d-1} (t_i - t_{i+1}); \tag{1.10}$$

$$\psi_\Gamma(v) < (t_1 - t_d)/(d-1). \tag{1.11}$$

This pointwise bound for $\psi_\Gamma$ is sharper than the one from ([26], [21], [19, Thm. 7.1]), which for instance, for $d = 3$, gives the upper bound $\frac{3}{2}(t_1 - t_3)$ while the above corollary gives a bound $\frac{1}{2}(t_1 - t_3)$.

**Remark 1.11.** Following [7], for any geometrically finite subgroup $\Sigma < \mathrm{PSL}(2,\mathbb{R})$, a representation $\rho : \Sigma \to \mathrm{PSL}(d,\mathbb{R})$ is called cusped Hitchin if there exists a *positive* $\rho$-equivariant map from the limit set of $\Sigma$ to the space $\mathcal{F}$ of complete $d$-dimensional flags. For a cusped Hithin subgroup $\Gamma < \mathrm{PSL}(d,\mathbb{R})$, i.e., the image of a cusped Hitchin representation of a geometrically finite $\Sigma < \mathrm{PSL}(2,\mathbb{R})$, the inequality

$$\max_{1 \leq i \leq d-1} \delta_{\Gamma,\alpha_i} \leq 1 \tag{1.12}$$

was obtained, with equality only when $\Sigma$ is a lattice, by Canary, Zhang and Zimmer [7, Thm. 1.1]. Although $\Gamma$ is not Anosov when $\Sigma$ is not convex cocompact, Theorem 1.2, using (1.12), implies that the pointwise bound (1.10) $\psi_\Gamma(v) \le \min_{1\le i\le d-1}(t_i - t_{i+1})$, and hence $\psi_\Gamma(v) \le (t_1 - t_d)/(d-1)$, also holds for any Zariski dense cusped Hitchin subgroup $\Gamma$ of $\mathrm{PSL}(d,\mathbb{R})$ as well.

**Remark 1.12.** The bound in Corollary 1.10 is stronger than [23, Coro. 1.4] (also [7, Thm. 1.1] for cusped Hitchin subgroups) in two aspects: first, the bound for $\psi_\Gamma$ given by [23, Coro. 1.4] is weaker than $\frac{t_1-t_d}{d-1}$ and stated only for vectors inside a strictly smaller cone than the limit cone (see Remark 3.5 for details).

**Remark 1.13.** The comparison of $\psi_\Gamma$ with the half sum $\pi_G$ of positive roots is meaningful in view of Sullivan's theorem that for a convex cocompact subgroup $\Gamma < \mathrm{SO}^\circ(n,1)$, the inequality $\delta_\Gamma \le \pi_G = \frac{n-1}{2}$ holds if and only if the bottom of the $L^2$-spectrum on $\Gamma\backslash\mathbb{H}^n$ is given by $(n-1)^2/4$ and there exists no positive square-integrable harmonic function on $\Gamma\backslash\mathbb{H}^n$ [30, Thm. 2.21].

Corollaries 1.6 and 1.10 imply that $\psi_\Gamma \le \pi_G$ in their respective settings (even with the strict inequality). In recent work [9], these results were used to show that the quasi-regular representation $L^2(\Gamma\backslash G)$ is tempered and there exists no positive square-integrable harmonic function on the associated locally symmetric manifold.

For any discrete subgroup $\Gamma < G$, note that $\delta_{\Gamma,\pi_G} \le 2$ as follows from Remark 1.3(1). We propose the following conjecture:

**Conjecture 1.14.** *Let $k = \operatorname{rank} G \ge 2$. If $\Gamma$ is a $\Delta$-Anosov subgroup of $G$, then*
$$\delta_{\Gamma,\pi_G} \le 1,$$
*or equivalently $\psi_\Gamma \le \pi_G$.*

The equivalence is a consequence of [25, Lemma III.1.3].

**On the proofs.** The proof of Theorem 1.2 consists of two parts: first prove that each linear form $\delta_{\Gamma,\alpha_i}\alpha_i$ is tangent to $\psi_\Gamma$ whenever $\delta_{\Gamma,\alpha_i} < \infty$ and then take the *minimum*! Although taking the minimum seems a trivial step, the resulting tent function turns out to be quite useful, as discussed above. The proof of Theorem 1.4 is crucially based on special properties of $\psi_\Gamma$ for $\Delta$-Anosov subgroups (see Theorem 3.1).

**Organization.** In section 2, we prove Theorem 1.2. In section 3, we prove Theorem 1.4. In section 4, we discuss applications of tent property of $\psi_\Gamma$ to self-joining of hyperbolic manifolds.

**Acknowledgements.** We would like to thank Marc Burger and Dick Canary for useful comments and Andres Sambarino for pointing out some redundant rank restriction in our earlier version.

## 2. Tent property

Let $G$ be a connected, semisimple real algebraic group of rank $k \geq 1$. Let $\mathfrak{g}$ denote the Lie algebra of $G$, and decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and $-1$ eigenspaces of a fixed Cartan involution respectively. We denote by $K$ the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$. We also choose a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Let $A := \exp \mathfrak{a}$. Choosing a closed positive Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. Let

$$\Delta = \{\alpha_1, \cdots, \alpha_k\}$$

be the set of simple roots $(\mathfrak{g}, \mathfrak{a}^+)$.

As in the introduction, for $g \in G$, we denote by $\mu(g) \in$ the unique element in $\mathfrak{a}^+$ such that

$$g \in K \exp(\mu(g)) K.$$

Let $\Gamma < G$ be a Zariski dense discrete subgroup. We denote by $\mathcal{L} \subset \mathfrak{a}^+$ the limit cone of $\Gamma$, which is the asymptotic cone of $\mu(\Gamma)$:

$$\mathcal{L} = \{\lim t_i \mu(\gamma_i) \in \mathfrak{a}^+ \text{ for some } t_i \to 0 \text{ and } \gamma_i \in \Gamma\}.$$

It is a convex cone with non-empty interior [1]. The growth indicator function $\psi_\Gamma : \mathfrak{a} \to \mathbb{R} \cup \{-\infty\}$ is defined as in (1.1). It follows easily from the definition that $\psi_\Gamma$ does not depend on the choice of a norm on $\mathfrak{a}$.

Quint showed the following:

**Theorem 2.1.** [25, Thm. IV.2.2] *The growth indicator function $\psi_\Gamma$ is concave, upper semi-continuous, and satisfies*

$$\mathcal{L} = \{u \in \mathfrak{a}^+ : \psi_\Gamma(u) > -\infty\}.$$

*Moreover, $\psi_\Gamma(u)$ is non-negative on $\mathcal{L}$ and positive on* $\operatorname{int} \mathcal{L}$.

**Lemma 2.2.** [25, Lem. III.1.3] *Let $F$ be a continuous function on $\mathfrak{a}^+$ satisfying $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in \mathfrak{a}$. If $F(u) > \psi_\Gamma(u)$ for all $u \in \mathfrak{a} - \{0\}$, then*

$$\sum_{\gamma \in \Gamma} e^{-F(\mu(\gamma))} < \infty.$$

*Moreover, we have $\delta_{\Gamma, F} < 1$.*

*Proof.* Convergence of the series is shown in [25, Lem. III.1.3], and in particular $\delta_{\Gamma,F} \leq 1$. To obtain the strict inequality, we claim that there exists $0 < \varepsilon < 1$ such that

$$(1 - \varepsilon) F > \psi_\Gamma \quad \text{on } \mathfrak{a} - \{0\}. \tag{2.1}$$

Since $\psi_\Gamma = -\infty$ outside $\mathcal{L}$ and both $F$ and $\psi_\Gamma$ are homogeneous functions, it suffices to prove (2.1) on $\{\|v\| = 1, v \in \mathcal{L}\}$. Since $\psi_\Gamma \geq 0$ on $\mathcal{L}$, we have $F > 0$ on $\mathcal{L} - \{0\}$. Hence the claim now follows because $\frac{\psi_\Gamma}{F}$ is upper semi-continuous and thus achieves its maximum on any compact set. $\square$

We denote by $\mathfrak{a}^*$ the set of all linear forms on $\mathfrak{a}$.

**Definition 2.3.** A linear form $\alpha \in \mathfrak{a}^*$ is called *tangent* to $\psi_\Gamma$ at $u \in \mathfrak{a} - \{0\}$ if $\alpha \geq \psi_\Gamma$ and $\alpha(u) = \psi_\Gamma(u)$.

Consider the following dual cone of the limit cone $\mathcal{L}$:

$$\mathcal{L}^\star := \{\alpha \in \mathfrak{a}^* : \alpha(v) \geq 0 \text{ for all } v \in \mathcal{L}\}. \tag{2.2}$$

Observe that the set of all positive roots is contained in $\mathcal{L}^\star$.

Note that the interior of $\mathcal{L}^\star$ is given as

$$\operatorname{int} \mathcal{L}^\star = \{\alpha \in \mathfrak{a}^* : \alpha(v) > 0 \text{ for all } v \in \mathcal{L} - \{0\}\}.$$

For any $\alpha \in \mathcal{L}^\star$, we set

$$\delta_\alpha = \delta_{\Gamma,\alpha}.$$

**Lemma 2.4.** *If $\alpha \in \operatorname{int} \mathcal{L}^\star$, then*

$$\delta_\alpha \leq \sup_{v \in \mathcal{L} - \{0\}} \frac{\psi_\Gamma(v)}{\alpha(v)} < \infty.$$

*Proof.* Let $\kappa := \sup_{v \in \mathcal{L} - \{0\}} \frac{\psi_\Gamma(v)}{\alpha(v)}$. Since $\alpha > 0$ on $\mathcal{L} - \{0\}$, $0 \leq \kappa = \sup_{v \in \mathcal{L} - \{0\}} \frac{\psi_\Gamma(v)}{\alpha(v)} < \infty$ is well-defined. Since $\psi_\Gamma < (\kappa + \varepsilon)\alpha$ on $\mathfrak{a} - \{0\}$ for any $\varepsilon > 0$, we have, by Lemma 2.2, that $\delta_{(\kappa+\varepsilon)\alpha} < 1$. Hence $\delta_\alpha < \kappa + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\delta_\alpha \leq \kappa$. $\square$

**Theorem 2.5.** *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any non-zero $\alpha \in \mathcal{L}^\star$ with $\delta_\alpha < \infty$, the linear form*

$$T_\alpha := \delta_\alpha \alpha$$

*is tangent to $\psi_\Gamma$ and $\delta_\alpha > 0$. In particular, for any subset $S \subset \operatorname{int} \mathcal{L}^\star$,*

$$\psi_\Gamma \leq \inf_{\alpha \in S} T_\alpha.$$

*Proof.* Fix any norm $\|\cdot\|$ on $\mathfrak{a}$ and we use this norm in the definition of $\psi_\Gamma$. We first claim

$$\psi_\Gamma(v) \leq \delta_\alpha \alpha(v) \quad \text{for all } v \in \operatorname{int} \mathcal{L}. \tag{2.3}$$

Fix $v \in \operatorname{int} \mathcal{L}$ and $\varepsilon > 0$. We then consider

$$\mathcal{C}_\varepsilon(v) = \left\{ w \in \mathfrak{a} : \alpha(w) > 0 \text{ and } \left| \frac{\|w\|}{\alpha(w)} - \frac{\|v\|}{\alpha(v)} \right| < \varepsilon \right\};$$

since $\alpha(v) > 0$, this is a well-defined open cone containing $v$. Therefore by the definition of $\psi_\Gamma$, we have

$$\psi_\Gamma(v) \leq \|v\| \tau_{\mathcal{C}_\varepsilon(v)}. \tag{2.4}$$

Observe that for any $s \geq 0$,

$$\begin{aligned} \sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}_\varepsilon(v)} e^{-s\|\mu(\gamma)\|} &\leq \sum_{\gamma \in \Gamma, \mu(\gamma) \in \mathcal{C}_\varepsilon(v)} e^{-s\alpha(\mu(\gamma))\left(\frac{\|v\|}{\alpha(v)} - \varepsilon\right)} \\ &\leq \sum_{\gamma \in \Gamma} e^{-s\alpha(\mu(\gamma))\left(\frac{\|v\|}{\alpha(v)} - \varepsilon\right)}. \end{aligned}$$

Since $\tau_{\mathcal{C}_\varepsilon(v)}$ is the abscissa of convergence of the series

$$\sum_{\gamma\in\Gamma,\mu(\gamma)\in\mathcal{C}_\varepsilon(v)} e^{-s\|\mu(\gamma)\|},$$

it follows from the definition of $\delta_\alpha$ that

$$\tau_{\mathcal{C}_\varepsilon(v)} \le \frac{\delta_\alpha}{\|v\|\alpha(v)^{-1}-\varepsilon} = \frac{\delta_\alpha\alpha(v)}{\|v\|-\varepsilon\alpha(v)}.$$

Together with (2.4), we have

$$\psi_\Gamma(v) \le \|v\|\frac{\delta_\alpha\alpha(v)}{\|v\|-\varepsilon\alpha(v)}.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\psi_\Gamma(v) \le \delta_\alpha\alpha(v).$$

This proves the claim (2.3).

We now claim that the inequality (2.3) also holds for any $v$ in the boundary $\partial\mathcal{L}$. Choose any $v_0 \in \operatorname{int}\mathcal{L}$. From the concavity of $\psi_\Gamma$, we have

$$t\psi_\Gamma(v_0) + (1-t)\psi_\Gamma(v) \le \psi_\Gamma(tv_0 + (1-t)v) \quad \text{for all } 0 < t < 1.$$

Since $\mathcal{L}$ is convex, $tv_0 + (1-t)v \in \operatorname{int}\mathcal{L}$ for all $0 < t < 1$. As we have already shown $\psi_\Gamma \le T_\alpha$ on $\operatorname{int}\mathcal{L}$, we get

$$t\psi_\Gamma(v_0) + (1-t)\psi_\Gamma(v) \le T_\alpha(tv_0 + (1-t)v) \quad \text{for all } 0 < t < 1.$$

By sending $t \to 0^+$, we get

$$\psi_\Gamma(v) \le T_\alpha(v).$$

Since $\psi_\Gamma = -\infty$ outside $\mathcal{L}$, we have established $\psi_\Gamma \le T_\alpha$. It remains to show that $\psi_\Gamma(v) = T_\alpha(v)$ for some $v \in \mathfrak{a} - \{0\}$. Suppose not, i.e., $\psi_\Gamma < \delta_\alpha\alpha$ on $\mathfrak{a} - \{0\}$. By Lemma 2.2, the abscissa of convergence of the series

$$\sum_{\gamma\in\Gamma} e^{-s\delta_\alpha\alpha(\mu(\gamma))} \tag{2.5}$$

is strictly less than 1. However the abscissa of convergence of the series (2.5) is equal to 1 by the definition of $\delta_\alpha$. Therefore we have obtained a contradiction.

Note that this implies $\delta_\alpha > 0$ since $\psi_\Gamma > 0$ on $\operatorname{int}\mathcal{L}$, which is non-empty by Zariski density hypothesis by Theorem 2.1. The last part of the theorem follows from Lemma 2.4. □

**Remark 2.6.** We also note the following lower bound for $\psi_\Gamma$: let $T_\ell \in \mathcal{L}^\star$, $\ell \in I$, be a finite collection of linear forms which are tangent to $\psi_\Gamma$ at some $v_\ell \in \mathcal{L} - \{0\}$. Then the concavity property of $\psi_\Gamma$ implies that for any $v = \sum_{\ell\in I} c_\ell v_\ell$ with $c_\ell \ge 0$,

$$\sum_{\ell\in I} c_\ell T_\ell(v_\ell) \le \psi_\Gamma(v).$$

**Proof of Theorem 1.2** Note that $\Delta \subset \mathcal{L}^\star$. Hence this follows from Theorem 2.5 by taking the minimum over all simple roots $\alpha_i \in \Delta$ with $\delta_{\alpha_i} < \infty$.

We also note the following corollary of Theorem 2.5:

**Corollary 2.7.** *Let $\Gamma < G$ be a Zariski dense discrete subgroup. For any $\alpha \in \operatorname{int} \mathcal{L}^\star$, we have*

$$0 < \delta_\alpha = \max_{v \in \mathcal{L} - \{0\}} \frac{\psi_\Gamma(v)}{\alpha(v)} < \infty.$$

*Proof.* By Lemma 2.4, $\delta_\alpha < \infty$. Hence Theorem 2.5 implies $\psi_\Gamma \leq \delta_\alpha \alpha$ and $\psi_\Gamma(v) = \delta_\alpha \alpha(v)$ for some $v \neq 0$. This implies the claim. $\square$

By the following theorem, the above corollary applies to $\alpha \in \theta$ for $\theta$-Anosov subgroups.

**Theorem 2.8** ([11], [13])**.** *If $\Gamma$ is $\theta$-Anosov, then*

$$\theta \subset \operatorname{int} \mathcal{L}^\star.$$

*In particular, if $\Gamma$ is $\Delta$-Anosov, then*

$$\mathcal{L} \subset \operatorname{int} \mathfrak{a}^+ \cup \{0\}. \tag{2.6}$$

## 3. Proof of Theorem 1.4

In this section, let

$$\Gamma < G \text{ be a Zariski dense } \Delta\text{-Anosov subgroup,}$$

as defined in the introduction (1.2).

By Quint's duality lemma [27, Lem. 4.3] and the works of Quint [27], Sambarino [28, Lem. 4.8] and Potrie-Sambarino [23, Prop. 4.6 and 4.11], which is based on the work [4], we have the following fundamental properties of $\Gamma$:

**Theorem 3.1.** *On $\operatorname{int} \mathcal{L}$, $\psi_\Gamma$ is analytic, strictly concave, and vertically tangent on $\partial \mathcal{L}$.*

The vertical tangency of $\psi_\Gamma$ on $\partial \mathcal{L}$ means that there are no linear forms which are tangent to $\psi_\Gamma$ at a point of $\partial \mathcal{L}$.

In the following, we fix a norm on $\mathfrak{a}$ induced from an inner product $\langle \cdot, \cdot \rangle$ which is non-negative on $\mathfrak{a}^+$, i.e., $\langle u, v \rangle \geq 0$ for all $u, v \in \mathfrak{a}^+$. We denote by $\nabla \psi_\Gamma(u) \in \mathfrak{a}$ the gradient of $\psi_\Gamma$ at $u$ so that $d(\psi_\Gamma)_u(v) = \langle \nabla \psi_\Gamma(u), v \rangle$ for all $v \in \mathfrak{a} - \{0\}$.

The following theorem was first observed by Quint for Schottky groups [27] and is deduced from Theorem 3.1 in general:

**Theorem 3.2** ([8, Coro. 7.8] [18, Prop. 4.4])**.** *Let $u \in \operatorname{int} \mathcal{L}$.*

1. *There exists a unique $\psi_u \in \mathfrak{a}^*$ which is tangent to $\psi_\Gamma$ at $u$.*
2. *We have $\psi_u \in \operatorname{int} \mathcal{L}^\star$ and*

$$\psi_u(\cdot) = \langle \nabla \psi_\Gamma(u), \cdot \rangle = d(\psi_\Gamma)_u. \tag{3.1}$$

(3) *The map $u \mapsto \psi_u$ induces a bijection between directions in* $\operatorname{int}\mathcal{L}$ *and directions in* $\operatorname{int}\mathcal{L}^\star$.
(4) *We have $\delta_{\psi_u} = 1$.*

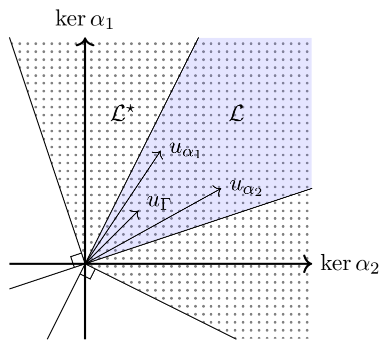


FIGURE 2. Limit cone and its dual cone.

We deduce the following from the above two theorems:

**Proposition 3.3.** *Consider the map* $\operatorname{int}\mathcal{L} \to \operatorname{int}\mathcal{L}^\star$ *given by* $u \mapsto \alpha_u$ *where*

$$\alpha_u := \frac{\psi_u}{\psi_\Gamma(u)}.$$

(1) *The map $u \mapsto \alpha_u$ is a bijection.*
(2) *Its inverse map* $\operatorname{int}\mathcal{L}^\star \to \operatorname{int}\mathcal{L}$ *is given by* $\alpha \to u_\alpha$ *where* $u_\alpha \in \operatorname{int}\mathcal{L}$ *is the unique vector such that* $\nabla\psi_\Gamma(u_\alpha)$ *is perpendicular to* $\ker\alpha$ *and*

$$\alpha(u_\alpha) = 1.$$

*We also have*

$$\psi_\Gamma(u_\alpha) = \max_{v\in\mathcal{L},\alpha(v)=1} \psi_\Gamma(v). \tag{3.2}$$

*Proof.* For $t > 0$, $\psi_{tu} = \psi_u$ and $\psi_\Gamma(tu) = t\psi_\Gamma(u)$; hence $\alpha_{tu} = t^{-1}\alpha_u$. Therefore (1) follows from Theorem 3.2.

Let $\alpha \in \operatorname{int}\mathcal{L}^\star$. Let $u_\alpha \in \operatorname{int}\mathcal{L}$ be the vector given by the relation $\alpha_{u_\alpha} = \alpha$, that is, $\alpha = \frac{\psi_{u_\alpha}}{\psi_\Gamma(u_\alpha)}$. By the definition of $\psi_{u_\alpha}$ given in (3.1), $\nabla\psi_\Gamma(u_\alpha)$ is perpendicular to $\ker\alpha$, and

$$\alpha(u_\alpha) = \frac{\psi_{u_\alpha}(u_\alpha)}{\psi_\Gamma(u_\alpha)} = \frac{\psi_\Gamma(u_\alpha)}{\psi_\Gamma(u_\alpha)} = 1.$$

To show the uniqueness, suppose that $v \in \operatorname{int}\mathcal{L}$ is a vector such that $\nabla\psi_\Gamma(v)$ is parallel to $\nabla\psi_\Gamma(u_\alpha)$ and $\alpha(v) = 1$. The strict concavity of $\psi_\Gamma$ on $\operatorname{int}\mathcal{L}$ as in Theorem 3.1 implies that $v$ must be parallel to $u_\alpha$. Since $\alpha(v) = \alpha(u_\alpha) = 1$, it follows that $v = u_\alpha$.

Observe that for any $v \in \mathcal{L}$ with $\alpha(v) = 1$, we have

$$\psi_\Gamma(v) \le \psi_{u_\alpha}(v) = \psi_\Gamma(u_\alpha)\alpha(v) = \psi_\Gamma(u_\alpha) = \psi_\Gamma(u_\alpha).$$

Since $\alpha(u_\alpha) = 1$, this implies (3.2). $\square$

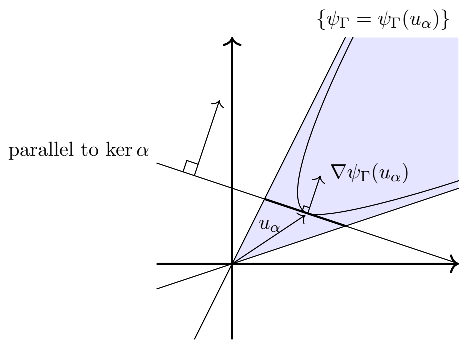


Figure 3. From $\alpha$ to $u_\alpha$

**Theorem 3.4.** *For any $\alpha \in \operatorname{int}\mathcal{L}^\star$, we have*

$$\delta_\alpha = \psi_\Gamma(u_\alpha) \quad \text{and} \quad \psi_{u_\alpha} = \delta_\alpha \alpha.$$

*Proof.* The first claim follows from (3.2) and Corollary 2.7. Since $\psi_{u_\alpha} = \psi_\Gamma(u_\alpha)\alpha$ by Proposition 3.3, the first claim implies the second. □

**Proof of Theorem 1.4.** For (1), we claim that $v_i := u_{\alpha_i}$ satisfies the claim. By Proposition 3.3, we have $u_{\alpha_i} \in \operatorname{int}\mathcal{L}$ and it satisfies $\alpha_i(u_{\alpha_i}) = 1$. By Lemma 3.4, $\psi_\Gamma(u_{\alpha_i}) = \delta_{\alpha_i}$. The uniqueness follows easily from the strict concavity of $\psi_\Gamma$ (Theorem 3.1).

For (2), suppose that for some $v \in \mathfrak{a}$ and $1 \le i \le k$, we have $\psi_\Gamma(v) = \delta_{\alpha_i}\alpha_i(v)$. Since $\psi_{u_{\alpha_i}} = \delta_{\alpha_i}\alpha_i$ is a tangent form to $\psi_\Gamma$ at $u_{\alpha_i}$, it follows again from the strict concavity of $\psi_\Gamma$ and the vertical tangency property (Theorem 3.1 that $v$ is parallel to $u_{\alpha_i}$.

By Theorem 1.2, we have

$$\psi_\Gamma(v) \le \min_{1\le i\le k} \delta_{\alpha_i}\alpha_i(v) \le \frac{1}{k}\sum_{1\le i\le k} \delta_{\alpha_i}\alpha_i(v). \tag{3.3}$$

Suppose that $\psi_\Gamma(v) = \frac{1}{k}\sum_{1\le i\le k}\delta_{\alpha_i}\alpha_i(v)$ for some $v \ne 0$. It then follows from (3.3) that

$$\psi_\Gamma(v) = \min_{1\le i\le k} \delta_{\alpha_i}\alpha_i(v) = \frac{1}{k}\sum_{1\le i\le k} \delta_{\alpha_i}\alpha_i(v).$$

It implies that for all $1 \le i \le k$,

$$\psi_\Gamma(v) = \delta_{\alpha_i}\alpha_i(v).$$

Then, as we just have seen, this implies that $v$ is parallel to all $u_{\alpha_i}$, $1 \le i \le k$. When $k \ge 2$, this contradicts Theorem 3.2. This proves (3).

**Proof of Corollary 1.5.** For simplicity, we omit $\|\cdot\|$ in the subscript in this proof, e.g., $u_\Gamma = u_{\Gamma,\|\cdot\|}$. Recall that $\delta_\Gamma = \psi_\Gamma(u_\Gamma)$. Since $\delta_\Gamma = \max_{\|v\|=1}\psi_\Gamma(v)$, [25, Lem. III.3.4], applied to $\psi_\Gamma$, implies that there exists a tangent form, $\psi_{u_\Gamma}$, to $\psi_\Gamma$ at $u_\Gamma$. By the vertical tangent condition in Theorem 3.1, it follows that $u_\Gamma \in \operatorname{int}\mathcal{L}$. Moreover, we have $\nabla\psi_\Gamma(u_\Gamma) \in \mathbb{R}_{>0}u_\Gamma$ [8, Lem. 2.24]. Therefore, by Theorem 3.2(2), there exists $c_0 > 0$ such that

$$\psi_{u_\Gamma}(\cdot) = \langle c_0 u_\Gamma, \cdot\rangle. \tag{3.4}$$

We now claim that

$$\psi_\Gamma(u_\Gamma) < T_\Gamma(u_\Gamma).$$

Suppose not. Then, by Theorem 1.4, there exist $c > 0$ and $1 \le i \le k$ such that $u_\Gamma = cu_{\alpha_i}$ and hence $\psi_{u_\Gamma} = \psi_{u_{\alpha_i}} = \delta_{\alpha_i}\alpha_i$. By (3.4), it follows that $\alpha_i(\cdot) = \langle c_1 u_\Gamma, \cdot\rangle$ for some $c_1 > 0$.

Since $u_\Gamma \in \operatorname{int}\mathcal{L}$, and $\langle,\rangle$ is non-negative on $\mathfrak{a}^+$ by the hypothesis, the linear form $\langle c_1 u_\Gamma, \cdot\rangle$ is positive on $\mathfrak{a}^+ - \{0\}$. On the other hand, the simple root $\alpha_i$ is zero on a wall of $\mathfrak{a}^+$. Therefore we obtained a contradiction. This finishes the proof.

We note that in the above proof, the hypothesis that the norm $\|\cdot\|$ is induced from an inner product was used to deduce that $\psi_{u_{\Gamma,\|\cdot\|}}$ is strictly positive on $\mathfrak{a}^+ - \{0\}$.

**Remark 3.5.** We explain how Theorem 1.4 can be compared with [23, Coro. 1.4]. Let $(\mathfrak{a}^+)^\star = \{\alpha \in \mathfrak{a}^* : \alpha(v) \ge 0 \text{ for all } v \in \mathfrak{a}^+\}$ so that

$$\operatorname{int}(\mathfrak{a}^+)^\star = \{\alpha \in \mathfrak{a}^* : \alpha(v) > 0 \text{ for all } v \in \mathfrak{a}^+ - \{0\}\}.$$

Recall that [23, Coro. 1.4] concerns the Hitchin representations, but their argument applied to our Zariski dense Anosov subgroups yields the following: For any $\alpha \in \operatorname{int}(\mathfrak{a}^+)^\star$, the quantity $\delta_\alpha$ satisfies

$$\delta_\alpha \le \frac{1}{\sum_{i=1}^k a_i} \tag{3.5}$$

where $\alpha = \sum_{i=1}^k (a_i\delta_{\alpha_i})\alpha_i$; the hypothesis $\alpha \in \operatorname{int}(\mathfrak{a}^+)^\star$ is equivalent to $\alpha \neq 0$ and $a_i > 0$ for all $1 \le i \le k$.

On the other hand, our Theorem 1.4 says that for all $\alpha \in \operatorname{int}\mathcal{L}^\star$,

$$\delta_\alpha = \psi_\Gamma(u_\alpha) \le \min_{1\le i\le k} \delta_{\alpha_i}\alpha_i(u_\alpha); \tag{3.6}$$

this is equivalent to saying that for all $v \in \mathfrak{a}$, $\psi_\Gamma(v) \le \min_{1\le i\le k}\delta_{\alpha_i}\alpha_i(v)$.

Since

$$1 = \alpha(u_\alpha) = \sum_{i=1}^k a_i\delta_{\alpha_i}\alpha_i(u_\alpha) \ge \left(\sum_{i=1}^k a_i\right)\min_{1\le i\le k}\delta_{\alpha_i}\alpha_i(u_\alpha) \tag{3.7}$$

we have

$$\min_{1\le i\le k}\delta_{\alpha_i}\alpha_i(u_\alpha) \le \frac{1}{\sum_{i=1}^k a_i}$$

where the equality is strict except for one direction of $u_\alpha$ satisfying

$$\delta_{\alpha_i}\alpha_i(u_\alpha) = \delta_{\alpha_j}\alpha_j(u_\alpha) \quad \text{for all } i, j = 1, \cdots, k.$$

Therefore our bound (3.6) is sharper than the bound (3.5) in addition to the point that it applies to the optimal cone $\operatorname{int}\mathcal{L}^\star$, while [23, Coro. 1.4] applies only for $\alpha \in \operatorname{int}(\mathfrak{a}^+)^\star$, which is strictly smaller than $\operatorname{int}\mathcal{L}^\star$.

Both approaches are based on the observation that the linear forms $\delta_\alpha\alpha$'s are tangent to $\psi_\Gamma$ for $\alpha \in \Delta$, but [23, Coro. 1.4] considers these tangent forms as points on the boundary of the subset $\mathcal{D} = \{\varphi \in \operatorname{int}(\mathfrak{a}^+)^\star : \delta_\varphi \le 1\}$ and deduce (3.5) from the convexity of $\mathcal{D}$, whereas we think of the tangent forms as functions on $\mathfrak{a}$ and obtain a stronger bound of (3.6) simply by taking *minimum* of these tangent forms over $\alpha \in \Delta$.

**Alternate proof of Theorem 1.4**(2)**.** For Anosov subgroups, we present an alternate proof of

$$\psi_\Gamma \le T_\Gamma \tag{3.8}$$

for $k \le 3$, using the following "strip theorem":

**Theorem 3.6** (Strip theorem)**.** [6, Thm. 6.3] *Let $\Gamma$ be a Zariski dense $\Delta$-Anosov subgroup of $G$. Let $k = \#\Delta \le 3$ and $v \in \operatorname{int}\mathcal{L}$. For all sufficiently large $R > 0$, the abscissa of convergence of the series*

$$\sum_{\gamma\in\Gamma, \|\mu(\gamma)-\mathbb{R}v\|\le R} e^{-s\psi_v(\mu(\gamma))}$$

*is equal to* 1*.*

To show the inequality (3.8), we fix $v \in \operatorname{int}\mathcal{L}$ and $1 \le i \le k$. For $R > 0$, we write $S_R := \{g \in G : \|\mu(g) - \mathbb{R}v\| < R\}$. By Theorem 3.6, there exists $R > 0$ such that the series $\mathcal{D}_R(s) = \sum_{\gamma\in\Gamma\cap S_R} e^{-s\psi_v(\mu(\gamma))}$ has the abscissa of convergence 1. Recalling that $\alpha_i > 0$ on $\operatorname{int}\mathcal{L}_\Gamma$, there exists $C > 0$ so that for any $\gamma \in S_R$, we have

$$\left\|\mu(\gamma) - \frac{\alpha_i(\mu(\gamma))}{\alpha_i(v)}v\right\| \le C.$$

It then follows that

$$\mathcal{D}_R(s) = \sum_{\gamma\in\Gamma\cap S_R} e^{-s\psi_v(\mu(\gamma))} \ll \sum_{\gamma\in\Gamma\cap S_R} e^{-s\frac{\alpha_i(\mu(\gamma))}{\alpha_i(v)}\psi_\Gamma(v)} \le \sum_{\gamma\in\Gamma} e^{-s\frac{\alpha_i(\mu(\gamma))}{\alpha_i(v)}\psi_\Gamma(v)}.$$

Since the series $\sum_{\gamma\in\Gamma} e^{-s\frac{\alpha_i(\mu(\gamma))}{\alpha_i(v)}\psi_\Gamma(v)}$ is finite whenever $s > \frac{\alpha_i(v)}{\psi_\Gamma(v)}\delta_{\alpha_i}$, we have $1 \le \frac{\alpha_i(v)}{\psi_\Gamma(v)}\delta_{\alpha_i}$. Hence

$$\psi_\Gamma(v) \le \delta_{\alpha_i}\alpha_i(v).$$

Since $v \in \operatorname{int}\mathcal{L}$ and $1 \le i \le k$ are arbitrary, we get

$$\psi_\Gamma \le T_\Gamma \quad \text{on } \operatorname{int}\mathcal{L}.$$

By the concavity of $\psi_\Gamma$, this implies $\psi_\Gamma \le T_\Gamma$ on $\mathcal{L}$ as well (see the proof of Theorem 2.5). Since $\psi_\Gamma = -\infty$ outside $\mathcal{L}$, (3.8) follows.

## 4. Applications to self-joinings

We consider the case when $G = \prod_{i=1}^k \mathrm{SO}^\circ(n_i,1)$, $n_i \geq 2$, and $\rho_i : \Sigma \to \mathrm{SO}^\circ(n_i,1)$ is a faithful convex cocompact representation with Zariski dense image. We let $\Gamma_\rho < G$ be the subgroup defined as in (1.4). The hypothesis on $\rho_i$'s implies that $\Gamma_\rho$ is $\Delta$-Anosov. We assume

$$k \geq 2 \text{ and } \Gamma_\rho \text{ is Zariski dense in } G$$

in the entire section.

**Proof of Corollaries 1.6 and 1.7.** Corollary 1.6 follows since $\delta_{\alpha_i} = \delta_{\rho_i} = \dim \Lambda_{\rho_i}$. For Corollary 1.7, note that we have

$$\begin{aligned}\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}} &< \frac{1}{k}\left(\sum_{i=1}^k (\dim \Lambda_{\rho_i})^2\right)^{1/2} \\ &\leq \frac{1}{k}\left(k \max_{1\leq i \leq k} (\dim \Lambda_{\rho_i})^2\right)^{1/2} \\ &= \frac{1}{\sqrt{k}} \max_{1 \leq i \leq k} \dim \Lambda_{\rho_i}.\end{aligned}$$

On the other hand, we showed in [15],

$$\dim \Lambda_\rho = \max_i \Lambda_{\rho_i}.$$

Hence

$$\delta_{\Gamma_\rho, \|\cdot\|_{\mathrm{Euc}}} < \frac{1}{\sqrt{k}} \dim \Lambda_\rho.$$

**Critical exponent with respect to the $L^1$-metric.** Set $\delta_{L^1} := \delta_{\sum_{i=1}^k \alpha_i}$, which is the critical exponent of $\Gamma_\rho$ for the $L^1$-metric $\sum_{i=1}^k d_i$ on $X = \prod_{i=1}^k \mathbb{H}^{n_i}$. We deduce the following from Corollary 1.6, whose special case when $k=2$ and $\dim \Lambda_{\rho_i} = 1$ was proved by Bishop and Steger [2]:

**Corollary 4.1.** *We have*

$$\delta_{L^1} < \frac{\dim \Lambda_\rho}{k}. \tag{4.1}$$

*Proof.* Noting $\alpha := \sum_{i=1}^k \alpha_i \in \operatorname{int} \mathcal{L}^\star$, write $u_\alpha = (u_1, \cdots, u_k) \in \operatorname{int} \mathcal{L}$. Lemma 3.4 and Corollary 1.6 imply

$$\delta_{L^1} = \psi_\Gamma(u_\alpha) < \frac{1}{k}\sum_{i=1}^k \dim \Lambda_{\rho_i} u_i \leq \frac{\max_i \dim \Lambda_{\rho_i}}{k} \sum_{i=1}^k u_i.$$

Since $\alpha(u_\alpha) = \sum_{1 \leq i \leq k} u_i = 1$ by Lemma 3.3(2) and $\max_i \dim \Lambda_{\rho_i} = \dim \Lambda_\rho$ by [15], we get the desired inequality. $\square$

**Geodesic stretching between two hyperbolic manifolds.** When $k = 2$, the limit cone $\mathcal{L}$ of $\Gamma_\rho$ can also be described as

$$\mathcal{L} := \{(v_1, v_2) \in \mathbb{R}^2_{\geq 0} : d_- v_1 \leq v_2 \leq d_+ v_1\}$$

where $d_+$ and $d_-$ are respectively the maximal and minimal geodesic stretching constants of $\rho_2$ relative to $\rho_1$:

$$d_+(\rho_1, \rho_2) = \sup_{\sigma \in \Sigma - \{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)} \quad \text{and} \quad d_-(\rho_1, \rho_2) = \inf_{\sigma \in \Sigma - \{e\}} \frac{\ell_2(\sigma)}{\ell_1(\sigma)}$$

where $\ell_i(\sigma)$ denotes the length of the closed geodesic in the hyperbolic manifold $\rho_i(\Delta)\backslash\mathbb{H}^{n_i}$ corresponding to $\rho_i(\sigma)$ (cf. [5], [1]).

Thurston [31] showed that the maximal geodesic stretching constant is always strictly bigger than 1 for finite-area hyperbolic surfaces. (See also [10]). Theorem 1.4 implies the following corollary; this was already observed by Burger [5, Thm. 1 and its Coro.] and generalizes a theorem of Thurston [31, Thm. 3.1]:

**Corollary 4.2.** *We have*

$$d_-(\rho_1, \rho_2) < \frac{\dim \Lambda_{\rho_1}}{\dim \Lambda_{\rho_2}} < d_+(\rho_1, \rho_2).$$

*Proof.* By Theorem 1.4,

$$\psi_\Gamma \leq \min(\delta_1 \alpha_1, \delta_2 \alpha_2). \tag{4.2}$$

By Theorem 3.4, we have $\psi_\Gamma(u_{\alpha_1}) = \delta_1 \alpha_1(u_{\alpha_1})$. Hence

$$\delta_1 \alpha_1(u_{\alpha_1}) \leq \min(\delta_1 \alpha_1(u_{\alpha_1}), \delta_2 \alpha_2(u_{\alpha_1})),$$

which implies $\delta_1 \alpha_1(u_{\alpha_1}) \leq \delta_2 \alpha_2(u_{\alpha_1})$. Therefore,

$$\frac{\delta_1}{\delta_2} \leq \frac{\alpha_2(u_{\alpha_1})}{\alpha_1(u_{\alpha_1})}.$$

Similarly, we have $\delta_2 \alpha_2(u_{\alpha_2}) \leq \min(\delta_1 \alpha_1(u_{\alpha_2}), \delta_2 \alpha_2(u_{\alpha_2}))$, and hence

$$\frac{\alpha_2(u_{\alpha_2})}{\alpha_1(u_{\alpha_2})} \leq \frac{\delta_1}{\delta_2}.$$

Since $\dim \Lambda_{\rho_i} = \delta_i$ for $i = 1, 2$ by Patterson [22] and Sullivan [29], we now have

$$\frac{\alpha_2(u_{\alpha_2})}{\alpha_1(u_{\alpha_2})} \leq \frac{\dim \Lambda_{\rho_1}}{\dim \Lambda_{\rho_2}} \leq \frac{\alpha_2(u_{\alpha_1})}{\alpha_1(u_{\alpha_1})}.$$

Since $u_{\alpha_1}, u_{\alpha_2} \in \operatorname{int} \mathcal{L}$, $d_-(\rho_1, \rho_2) < \frac{\alpha_2(u_{\alpha_2})}{\alpha_1(u_{\alpha_2})}$ and $\frac{\alpha_2(u_{\alpha_1})}{\alpha_1(u_{\alpha_1})} < d_+(\rho_1, \rho_2)$. It completes the proof. $\square$

## References


[1] Y. Benoist. *Propriétés asymptotiques des groupes linéaires.* Geom. Funct. Anal. (1997), 1-47.

[2] C. Bishop and T. Steger. *Representation theoretic rigidity in* $\mathrm{PSL}(2,\mathbb{R})$. Acta Math., 170: 121-149, 1993

[3] H. Bray, R. Canary, L. Kao, and G. Martone. *Counting, equidistribution and entropy gaps at infinity with applications to cusped Hitchin representations.* Preprint, arXiv:2102.08552

[4] M. Bridgeman, R. Canary, F. Labourie and A. Sambarino. *The pressure metric for Anosov representations.* Geom. Funct. Anal. 25 (2015), 1089–1179.

[5] M. Burger. *Intersection, the Manhattan curve, and Patterson-Sullivan theory in rank 2.* Int. Math. Res. Not. 1993, no. 7, 217-225.

[6] M. Burger, O. Landesberg, M. Lee and H. Oh. *The Hopf-Tsuji-Sullivan dichotomy in higher rank and applications to Anosov subgroups.* J. Mod. Dyn., 19:301-330, 2023.

[7] R. Canary, T. Zhang and A. Zimmer *Entropy rigidity for cusped Hitchin representations.* Preprint, arXiv 2201.04859.

[8] S. Edwards, M. Lee and H. Oh. *Anosov groups: local mixing, counting, and equidistribution.* Geom. Topol., 27(2):513–573, 2023.

[9] S. Edwards and H. Oh. *Temperedness of* $L^2(\Gamma\backslash G)$ *and positive eigenfunctions in higher rank.* Preprint, arXiv:2202.06203. To appear in Comm. Amer. Math. Soc.

[10] F. Guéritaud and F. Kassel. *Maximally stretched laminations on geometrically finite hyperbolic manifolds.* Geom. Topol. 21 (2017), p. 693-840.

[11] F. Guéritaud, O. Guichard, F. Kassel and A. Wienhard. *Anosov representations and proper actions.* Geom. Topol. 21 (2017), no. 1, 485-584.

[12] M. Kapovich and B. Leeb. *Discrete isometry groups of symmetric spaces.* Handbook of group actions. Vol. IV, 191-290, Adv. Lect. Math. (ALM), 41, Int. Press, Somerville, MA, 2018.

[13] M. Kapovich, B. Leeb, and J. Porti. *Anosov subgroups: dynamical and geometric characterizations.* Eur. J. Math. 3 (2017), no. 4, 808-898.

[14] M. Kapovich, B. Leeb and J. Porti. *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings.* Geom. Topol. 22 (2018), no. 7, 3827-3923.

[15] D. Kim, Y. Minsky and H. Oh. *Hausdorff dimension of directional limit sets of self-joinings of hyperbolic manifolds.* J. Mod. Dyn., 19:433-453, 2023

[16] O. Guichard and A. Wienhard. *Anosov representations: Domains of discontinuity and applications.* Invent. Math. 190, Issue 2 (2012), 357-438.

[17] F. Labourie. *Anosov flows, surface groups and curves in projective space.* Invent. Math. 165 (2006), no. 1, 51-114.

[18] M. Lee and H. Oh. *Invariant measures for horospherical actions and Anosov groups.* Preprint, arXiv:2008.05296. To appear in IMRN.

[19] M. Lee and H. Oh. *Dichotomy and measures on limit sets of Anosov groups.* Preprint, arXiv:2203.06794. To appear in IMRN

[20] M. Magee. *Quantitative spectral gap for thin groups of hyperbolic isometries.* Journal of EMS, Vol 17 (2015), 151-187.

[21] H. Oh. *Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants.* Duke Math. J. 113 (2002), no. 1, 133–192.

[22] S. Patterson. *The limit set of a Fuchsian group.* Acta Math. 136 (1976), 241–273.

[23] R. Potrie and A. Sambarino. *Eigenvalues and entropy of a Hitchin representation.* Invent. Math. 209, issue 3 (2017), pp. 885-925.

[24] B. Pozzetti, A. Sambarino and A. Wienhard. *Conformality for a robust class of non-conformal attractors.* arXiv:1902.01303, To appear in Crelle.

[25] J.-F. Quint. *Divergence exponentielle des sous-groupes discrets en rang supérieur.* Comment. Math. Helv. 77 (2002), no. 3, 563-608.

[26] J.-F. Quint. *Propriété de Kazhdan et sous-groupes discrets de covolume infini.* Travaux mathématiques. Fasc. XIV, 143–151, Trav. Math., 14, Univ. Luxemb., Luxembourg, 2003.
[27] J.-F. Quint. *L'indicateur de croissance des groupes de Schottky.* Ergodic Theory Dynam. Systems 23 (2003), no. 1, 249–272.
[28] A. Sambarino. *Hyperconvex representations and exponential growth.* Ergodic Theory Dynam. Systems 34 (2014), no. 3, 986 -1010.
[29] D. Sullivan. *The density at infinity of a discrete subgroup of hyperbolic motions.* Publ. IHES (1979), 171-202
[30] D. Sullivan. *Related aspects of positivity in Riemannian geometry.* J. Diff geometry. 25 (1987), 327–351
[31] W. Thurston. *Minimal stretch maps between hyperbolic surfaces.* arXiv: 980103

Department of Mathematics, Yale University, New Haven, CT 06511
*Email address*: `dongryul.kim@yale.edu`

Department of Mathematics, Yale University, New Haven, CT 06511
*Email address*: `yair.minsky@yale.edu`

Department of Mathematics, Yale University, New Haven, CT 06511
*Email address*: `hee.oh@yale.edu`